SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Thesis submitted for the degree of 'Magister Philosophiae '

# REARRANGEMENTS

candidate:
Loris Faina

Supervisor:
Prof. Arrigo Cellina

Academic Year 1989/90

TRIESTE

Thesis submitted for the degree of 'Magister Philosophiae '

# REARRANGEMENTS

candidate:

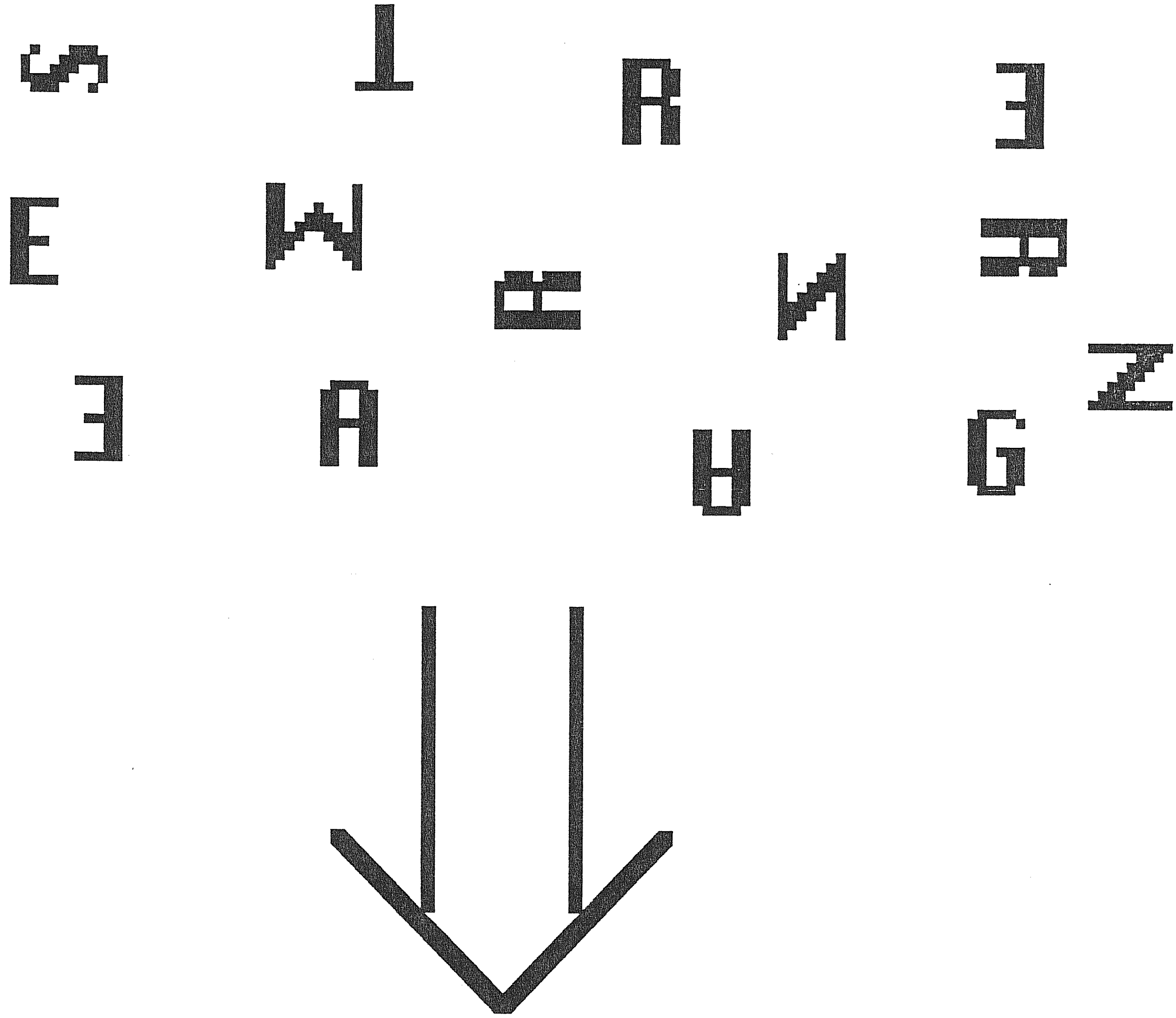
Loris Faina

Supervisor:

Prof. Arrigo Cellina

Academic Year 1989/90

# TABLE OF CONTENTS



# REARRANGEMENTS

# INTRODUCTION

Let $(T, \mathcal{F}, \mu)$ be a measure space with $\sigma$ –algebra $\mathcal{F}$ and positive measure $\mu$ . Let $L^p(T, \mu)$ be the space of all real functions $f$ on T such that $\|f\|_p = \int_T |f|^p d\mu$ is finite.

**DEFINITION 1.** *Let* $(T, \mathcal{F}, \mu)$ *and* $(T', \mathcal{F}', \mu')$ *be measure spaces with* $\mu(\Omega) = \mu'(\Omega')$ *(* $\mu, \mu'$ *are positive). Two functions* $f, g$; $f$ $\mathcal{F}$ *–measurable and* $g$ $\mathcal{F}'$ *–measurable, are called rearrangements of each other if*

$$\mu(f^{-1}([\beta, +\infty))) = \mu'(g^{-1}([\beta, +\infty))) \quad \textit{for every} \quad \beta \in \Re.$$

The commone feature of all rearrangements is that a given function $f$ is trasformed into a new function $f^*$ , which has some desired properties, like monotonicity or symmetry.

This is done by rearranging the level sets

$$E_s^f = \{t \in T : f(t) \geq s\}$$

of $f$ , and then by recostructing $f^*$ from the level sets rearranged, just like a three dimensional mountain can be recostructed from a map that shows all of its level lines or lines of constant height.

The relevence of rearrangements has been already established among the others by Zygmund [23], Hardy–Littlewood [15], Riesz [19].

Investigators have also used this concepts as a starting point for new directions in functional analysis and inequalities.

A lot of non–linear problems of Mathematical-Phisics have the following characteristic: they are posed on unbounded domains of $\Re^n$ ( like $\Re^n$ , half spaces; stripes, etc.) or are invariant under certain linear transformations, like rotations, and so they have same symmetries ( sferical, cylindrical, etc.).

The unboundedness of the domains imply, in general, the resolution of this problems with methods arising from non–linear analysis, because of the lack of compactness, coming from the fact, for example, that the Rellich's theorem does not hold in the whole $\Re^n$ .

In several different problems, it has been proved that, if one restricts oneself, in the functional space, to subspaces constituted by those functions which have the same symmetries of the problem, it is possible to obtain certain kinds of compactness. On this topic we quote:

W.A.Strauss [22], H.Berestycki–P.L.Lions [2],[3],[4], for the case of scalar camp equations;

M.J.Esteban–P.L.Lions [12], for the case of semilinear problems posed on stripes, arising from Fluid–Mechanics;

P.L.Lions [18], for the problem of the stars in rotation;

G.B.Burton [8], for the existence of a weak solution to the boundary problem for a steady vortex ring in an ideal fluid flowing along an infinite pipe of circular cross section.

In particular J.P.Lions in [17] put in evidence the results of compactness that it is possible to obtain by using the symmetries of the functions.

Actually, the aim of this thesis is to prove new theoretical properties for the set of rearrangements of a single function.

Starting from the known property of the equimeasurability of the level sets of two functions $f, g$ belonging to the same rearrangement set, we propose a procedure for interpolating at the same time all the level sets of $f, g$ , by mean of a family of measurable sets indexed on $[0,1]$ and having some desired properties.

This allows us to join $f$ and $g$ with a family of functions belonging to the same rearrangement set.

As a consequence of this procedure, we obtain fixed points, extension results and quasi–selections from upper semi–continuous maps, for the set of rearrangements of a single function.

Chapter 1 is devoted to a catalogue of rearrangements. Chapter 2 shows a procedure for interpolating a finite number of functions of a given rearrangement set that leads to a fixed point and compact extension result.

A slight different procedure for interpolating an infinite number of functions is illustrated in chapter 3. This will give us a retract property.

Finally, in chapter 4, using the interpolation procedure proved in chapter 3, we prove the existence of quasi–selection for upper semi–continuous maps with rearrangement values.

# 1 CATALOGUE OF REARRANGEMENTS.

In this chapter we will deal with $T = \Re^n$ and $\mu = m$ , where $m$ is the Lebesgue measure on $\Re^n$ .

It is clear that the rearrangement of a function $f$ is closely tied to the rearrangement of subsets of $\Re^n$ .

## 1.1 SYMMETRIC REARRANGEMENTS.

Let $D \subset \Re^n$ be a compact set, hence Lebesgue measurable. For n=1, the symmetric decreasing rearrangement $D^*$ of $D$ , is defined by

$$D^* = \begin{cases} \{t \in \Re : |t| \leq \frac{1}{2}m(D)\} & \text{if } D \neq \emptyset \\ \emptyset & \text{if } D = \emptyset. \end{cases}$$

Let $f : I \to \Re, I$ finite interval, be measurable. Then, the symmetric decreasing rearrangement of $f$ is defined by

$$f^*(t) = \sup\{c \in \Re : t \in (E_c^f)^*\}$$

for $t \in I^*$ .

Thus, each set $f^{-1}([\beta, +\infty))$ is replaced by an interval of the same measure, symmetric about zero.

**Example 1.:** Let

$$f(t) = \begin{cases} x & \text{if } -1 \leq x \leq 1 \\ 2 - x & \text{if } 1 \leq x \leq 2. \end{cases}$$

Then

$$f^*(t) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 2 - 2x & \text{if } 1 \leq |x| \leq \frac{3}{2}. \end{cases}$$

Let us introduce now rearrangements of functions of several variables.

Given $x \in \Re^n$ , we write $x = (x', y)$ with $x' \in \Re^{n-1}, y \in \Re$ . Furthermore we introduce the notation $D(x') = D \cap \{(x', y) : y \in \Re\}$ .

The Steiner symmetrization $D^*$ of a compact set $D \subset \Re^n$ with respect to the hyperplane $\{y = 0\}$ is defined sa follows:

$$D^* = \cup_{x' \in \Re^{n-1}} D^*(x'),$$

where

$$D^*(x') = \begin{cases} \{(x', y) \in \Re^n : |y| \le \frac{1}{2} m(D(x'))\} & \text{if } D(x') \ne \emptyset \\ \emptyset & \text{if } D(x') = \emptyset. \end{cases}$$

In other words, the Steiner symmetrization changes a solid into one with the same volume and at least one plane of symmetry.

Correspondingly, we define the Steiner symmetrization $f^*$ of a function $f : \overline{\Omega} \to \Re$, with $\Omega$ open, relatively compact, and convex in $y$, with respect to an hyperplane $H$ as follows:

Choose Cartesian coordinates $x_1, \cdots, x_{n-1}, y$ in $\Re^n$ such that $H$ is the hyperplane $y = 0$ ; then define

$$f^*(x', y) = \sup\{c \in \Re : x \in (E_c^f)^*\}, \quad \text{for} \quad (x', y) \in \overline{\Omega}^*.$$

We can more easily define the Steiner symmetrisation of $f$ as:

$$f^*(x', y) = [f(x', \cdot)]^*$$

i.e. the symmetric decreasing rearrangement in the last variable.

For verifying that $f^*$ is a rearrangement of $f$, we apply the Fubini's theorem:

$$m(\{x : f^*(x) \ge \beta\}) = \int_{\Re^{n-1}} m(\{y : f^*(x', y) \ge \beta\}) dx' =$$

$$= \int_{\Re^{n-1}} m(\{y : f(x', y) \ge \beta\}) dx' = m(\{x : f(x) \ge \beta\}).$$

There is more than one way to extend the notion of a symmetric interval of $\Re$ to higher dimentions.

Under the Steiner symmetrization the analogue of a symmetric interval was considered to be a family of 'parallel 'symmetric intervals. If one considers an $n$ –dimentional ball as the generalization of a symmetric interval, then one obtain the notion of Schwarz symmetrization. Hence for a compact set $D \subset \Re^n$ we define the Schwarz symmetrization $D^*$ of $D$ by

$$D^* = \begin{cases} \{x \in \Re^n : |x| \le \frac{1}{2} m(D)\} & \text{if } D \ne \emptyset \\ \emptyset & \text{if } D = \emptyset. \end{cases}$$

Thus, the Schwarz symmetrization changes a solid into a ball, with the same measure, centered at the origin.

Finally, the Schwarz symmetrization of a function $f : \Omega \rightarrow \Re$ , with $\overline{\Omega}$ compact, is given by

$$f^*(x) = \sup\{c \in \Re : x \in (E_c^f)^*\}$$

for $x \in \overline{\Omega^*}$.

Thus, the Steiner symmetrization of a function $f$ , is a symmetric function with respect to a plane at least, while the Schwarz symmetrization is a symmetric function with respect to all planes.

Further, it can be proved (see [5]) that the Schwarz symmetrization can be obtain as the $L^1(\Re^n, m)$ limit of a sequence of Steiner symmetrizations with respect to different planes.

## 1.2 MONOTONE REARRANGEMENTS

A measure space $(T, \mathcal{F}, \mu)$ is called a measure interval if $\omega = \mu(\Omega)$ is finite and positive, and there exists a bijection $\sigma : T \rightarrow [0, \omega]$ such that for $A \subset T$ , we have $A \in \mathcal{F}$ if and only if $\sigma(A)$ is Lebesgue measurable, and for all $A \in \mathcal{F}$ we have $\mu(A) = m(\sigma(A))$ .

**THEOREM 1.** (Royden [20],pp. 270). *Let $\mu$ be a positive, non atomic, finite Borel measure on a complete separable metric space X, and let $\mathcal{F}$ be the $\sigma$ –algebra of open sets on X. Then, $(X, \mathcal{F}, \mu)$ is a measure interval.*

Consequenly, a bounded open set in $\Re^n$ has the same measure–theoretic structure as an interval I in $\Re$ . So, for many purpose, we may as well deal with intervals only.

Let the distribution function of $f : I \rightarrow \Re$ be denoted by

$$\alpha_f(s) = m(E_s^f).$$

Note that $\alpha_f(\cdot)$ is non negative, non increasing, and left continuous.

The *decreasing rearrangement* of $f$ on I is defined by

$$f^*(t) = \sup\{s > 0 : \alpha_f(s) \geq t\}.$$

Clearly $f^*$ is non–increasing on $I^* = \{t \in \Re : 0 \leq t \leq m(I)\}$.

Further if $\alpha_f(\cdot)$ is strictly decreasing, then $f^*$ is the inverse of $\alpha_f(\cdot)$ . In fact, it follows immediately from the definition of $f^*$ that

(1) $f^*(\alpha_f(s)) \geq s$

and since $\alpha_f(\cdot)$ is left–continuous,

(2) $\alpha_f(f^*(t)) \geq t$ ;

So $f^*(\alpha_f(s)) = s$ , since, on the contrary, it follows from (1), $\alpha_f(f^*(\alpha_f(s))) < \alpha_f(s)$ , and from (2) $\alpha_f(f^*(\alpha_f(s))) \geq \alpha_f(s)$ , that leads to a contradiction.

In an analogous way, we derive $\alpha_f(f^*(t)) = t$ .

**PROPOSITION 1.** *$f^*$ is left–continuous.*

*Proof* : Clearly $f^*(t) \leq f^*(t+h)$ for all $h > 0$ . If $f^*$ were not continuous at t, there would exist $y$ such that $f^*(t) < y < f^*(t+h)$ for all $h > 0$ . But then, (2) would imply that

$$\alpha_f(y) \geq \alpha_f(f^*(t+h)) \geq t+h \quad \text{for all} \quad h > 0.$$

Thus, $\alpha_f(y) \geq t$ and therefore $f^*(t) \geq y$ , a contradiction.

△

**PROPOSITION 2.** *$\alpha_{f^*}(s) = \alpha_f(s)$ for all $s \in \Re^+$ ,i.e. $f^*$ is a rearrangement of $f$ .*

*Proof* : Since $f^*$ is non–increasing,

(3) $\alpha_{f^*}(s) = \sup\{t > 0 : f^*(t) \geq s\}.$

Hence, $f^*(\alpha(s)) \geq s$ imply $\alpha_f(s) \leq \alpha_{f^*}(s)$ .

For the opposite inequality, note from (3) that if $t < \alpha_{f^*}(s)$ then $f^*(t) > s$ and consequently $\alpha_f(s) \geq \alpha_f(f^*(t)) \geq t$ from (2).

Thus, $\alpha_f(s) \geq \alpha_{f^*}(s)$ and the proposition is established.

△

**EXAMPLE 2:** Let

$$f(t) = \begin{cases} x & \text{if } -1 \leq x \leq 1 \\ 2-x & \text{if } 1 \leq x \leq 2; \end{cases}$$

then

$$f^*(t) = \begin{cases} 1 - \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 2-x & \text{if } 2 \leq x \leq 3. \end{cases}$$

## 1.3 COMMON PROPERTIES OF REARRANGEMENTS

In this Section, we shall assume $\overline{\Omega} = \overline{\Omega^*}$ (* denote any of the rearrangement procedures illustrated in the precedent Sections).

A foundamental property of the rearrangements is the equimeasurability, i.e. for every $\beta \in \Re$,

(1) $m(\{x \in \overline{\Omega} : f(x) \geq \beta\}) = m(\{x \in \overline{\Omega^*} : f^*(x) \geq \beta\})$ .

Another important feature of the rearrangements is that they are order preserving, i.e.

(2) $D_1 \subset D_2$ implies $D_1^* \subset D_2^*$ for compact $D_i \subset \Re^n, i = 1, 2.$

This property implies the following:

(3) the mapping $f \to f^*$ is order preserving, i.e.

$$f(x) \leq g(x) \quad \text{for} \quad x \in \overline{\Omega} \quad \text{implies} \quad f^*(x) \leq g^*(x) \quad \text{for} \quad x \in \overline{\Omega^*} .$$

If $c \in \Re$ is a constant, then

(4) $(f + c)^* = f^* + c$ ;

further

(5) the map $f \to f^*$ is positively homogeneous of degree 1; i.e.

$$\text{for every } t \geq 0 \text{, we have } (tf)^* = tf^* ;$$

(6) the mapping $f \to f^*$ is idempotent, i.e. $(f^*)^* = f^*$ ;

(7) $(|f|^\alpha)^* = |f^*|^\alpha$ for $\alpha > 0$ .

Properties (4)–(7) hold by definition.

**THEOREM 2.** *If* $f, g \in L^p(\Re^n, m), 1 \leq p \leq +\infty$ , *and* $f$ *is a rearrangement of* $g$ , *then* $\|f\|_p = \|g\|_p$ .

*Proof* : Assume first $1 \leq p < +\infty$ . It is easy to verify that $f$ is a rearrangement of $g$ if and only if $f^p$ is a rearrangement of $g^p$ . So it is enough to look at the case $p = 1$ .

Now,

$$\|f\|_1 = \int_{\Re^n} |f| dm = \int_{\Re^n} \int_0^{|f|} dy dm = \int_0^{+\infty} \alpha_f(y) dy = \int_0^{+\infty} \alpha_g(y) dy = \|g\|_1.$$

Set now $p = +\infty$ . If $\beta > \|f\|_{+\infty}$ then $m(f^{-1}([\beta, +\infty))) = 0$ , so $m(g^{-1}([\beta, +\infty))) = 0$ and $\beta > |g\|_{+\infty}$ . Hence, $\|f\|_{+\infty} \geq \|g\|_{+\infty}$ .

Similarly we can prove that $\|f\|_{+\infty} \leq \|g\|_{+\infty}$ .

△

**THEOREM 3.** *Let $f \in L^p(\overline{\Omega}, m), g \in L^q(\overline{\Omega}, m)$ , with $f, g \geq 0$ , and $p, q$ conjugate exponents. Then*

$$\int_{\overline{\Omega}} fg dx \leq \int_{\overline{\Omega^*}} f^* g^* dx.$$

**REMARK 1.** *For $f, g \in L^2(\overline{\Omega}, m)$ , this theorem is attributed to Hardy and Littlewood [15].*

*Proof of theorem* 3. We proceed by step.

(a) Let $D, E \subset \Re^n$ be compact and $f(x) = \mathcal{X}_D(x), \quad g(x) = \mathcal{X}_E(x)$ . By virtue of (1) and (2), we have

$$\int_{\Re^n} f(x)g(x)dx = m(D \cap E) = m((d \cap E)^*) \leq$$

$$\leq m(D^* \cap E^*) = \int_{\Re^n} f^*(x)g^*(x)dx.$$

(b) Let $f(x) = \sum_{j=1}^q a_j \mathcal{X}_{D_j}(x)$ and $g(x) = \sum_{k=1}^r b_k \mathcal{X}_{E_k}(x)$ , where $a_j, b_k \in \Re_0^+$ , and $D_j, E_k$ are compact. Without lost of generality we may suppose $D_1 \subset D_2 \subset \cdots \subset D_q$ and $E_1 \subset E_2 \subset \cdots \subset E_r$ . Under these special assumptions the rearrangement mapping shows linear behaviour, i.e.

$$f^*(x) = \sum_{j=1}^q a_j \mathcal{X}_{D_j^*}(x)$$

and

$$g^*(x) = \sum_{k=1}^r b_k \mathcal{X}_{E_k^*}(x).$$

Then, by virtue of step (a), we have

$$\int_{\Re^n} f(x)g(x)dx = \sum_{j=1,\cdots,q} \sum_{k=1,\cdots,r} a_j b_k m(D_j \cap E_k) \leq$$

$$\leq \sum_{j,k} a_j b_k m(D_j^* \cap E_k^*) = \int_{\Re^n} \sum_{j,k} a_j b_k \mathcal{X}_{D_j^*} \mathcal{X}_{E_K^*} dx = \int_{\Re^n} f^*(x)g^*(x)dx.$$

(c) Finally we can prove the assertion by an approximation argument.

△

**THEOREM 5.** *The mapping* $f \to f^*$ *is non–expansive in* $L^2(\Omega)$ .

*Proof* : Let $f, g \in L^2(\Omega)$ . We have from theorem 4,

$$\|f^* - g^*\|_2^2 = \|f\|_2^2 + \|g\|_2^2 - 2\int_\Omega f^* g^* dx \leq \|f - g\|_2^2.$$

△

Mc Crandall and L.Tartar [10] showed that

$$(*) \qquad \int_{\Omega^*} J(|f^* - g^*|)dx \leq \int_\Omega J(|f - g|)dx$$

for every convex lower–semicontinuous function $J : \Re_0^+ \to \Re_0^+$ , with $J(0) = 0$ . Therefore, in particular, $(*)$ proves that $f \to f^*$ is nonexpansive in $L^p(\Omega)$ for every $p$ , $1 \leq p \leq +\infty$ .

## 1.4 MORE ON DECREASING REARRANGEMENTS.

In the whole section, I will denote the intervall $[0, \omega]$ , and $*$ will denote the decreasing rearrangement.

A function $\sigma : I \to I$ is said measure–preserving trasformation if for every measurable $A \subset [0, \omega], \sigma^{-1}(A)$ is measurable and $m(\sigma^{-1}(A)) = m(A)$ .

**THEOREM 6.** ( J.V.Ryff [21]) *Let* $f$ *be a measurable function on I, and define* $\sigma(x) = m(\{t : f(t) \geq f(x)\}) + m(\{t : f(t) = f(x) \quad \text{and} \quad t \leq x\})$ . *Then,* $\sigma$ *is measure–preserving on I and* $f^* \circ \sigma = f$ .

**EXAMPLE 3.** Let

$$f(t) = \begin{cases} 1 - 2t & t \in [0, \frac{1}{2}] \\ 2t - 1 & \text{if } t \in [\frac{1}{2}, 1]; \end{cases}$$

then, $f^*(t) = 1 - t$ and

$$\sigma(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2 - 2t & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Although $\sigma$ is measure preserving, in general, it is not invertible. In fact, in the last example, $\sigma$ is not injective and $m(\sigma(A)) \neq m(A)$ , in general; indeed $\sigma([0, \frac{1}{2}]) = [0,1]$ . Moreover there is no measure preserving trasformation $\psi$ satisfying $f^* = f \circ \psi$ .

Let $R(f)$ denotes the set of all rearrangements of $f$ on $[0,\omega]$ .

**THEOREM 7.** *Let* $f_0 \in L^p([0,\omega]), 1 \leq p \leq +\infty$ . *Then, every bounded linear functional on* $L^p$ *attains its supremum relative to* $R(f_0)$ .

*Proof* : Any bounded linear functional on $L^p$ can be represented as $(\cdot, g)$ for some $g \in L^q$ ( $(\cdot,\cdot)$ denotes the duality between $L^p$ and $L^q$ ).

We have

$$(f,g) \leq (f^*, g^*) = (f_0^*, g^*) \quad \text{for every} \quad f \in R(f_0).$$

It will be sufficient to find an $f$ for which equality holds. Now $g = g^* \circ \sigma$ for some measure–preserving transformation $\sigma$ . Take $f = f_0^* \circ \sigma$ . Then $f_0$ is a rearrangement of $f$ , and

$$(f,g) = \int_0^\omega (f_0^* \circ \sigma)(g^* \circ \sigma) dm = \int_0^\omega (f_0^* g^*) \circ \sigma dm =$$

$$= \int_0^\omega f_0^* g^* dm,$$

since $(f_0^* g^*) \circ \sigma$ is a rearrangement of $f_0^* g^*$ .

△

**THEOREM 8.** *Let* $f_0 \in L^p([0,\omega]), 1 \leq p < +\infty$ , *and let* $\Phi : L^p \to \Re$ *be convex and weakly sequentially continuous. Then* $\Phi$ *attains its supremum relative to* $R(f_0)$ .

*Proof* : We claim that $R(f_0)$ is weakly relatively compact in $L^p$ . This is clear if $p > 1$ , since $R(f_0)$ is bounded and $L^p$ is reflexive. In the case $p = 1$ , the equiintegrability of the rearrangements of $f$ ensures that $R(f_0)$ is weakly relatively compact in $L_1$ .

Write $M = \sup_{f \in R(f_0)} \Phi(f)$ , and let $u$ be the weak limit of a maximising sequence for $\Phi$ . Then $\Phi(u) = M$ . Weak sequential continuity implies strong continuity; together with convexity, this implies subdifferentiability of $\Phi$ .

Choose $h \in \partial\Phi(u) \subset L^q$ , we have

$$(u,h) \leq k = \sup\{(f,h) : f \in R(f_0)\}$$

by the weak continuity of $(\cdot, h)$ .

By virtue of theorem 7, let $\tilde{f}$ be such that $(\tilde{f}, h) = k$ .

Then,

$$\Phi(\tilde{f}) \geq \Phi(u) + (\tilde{f} - u, h) = M + k - (u, h) \geq M.$$

Hence $\tilde{f}$ maximises $\Phi$ relative to $R(f_0)$ .

△

## 1.5 AN APPLICATION TO VARIATIONAL PROBLEMS.

Let $\Omega \subset \Re^n$ be open, let $H_0^{1,2}(\Omega)$ be the Sobolev space

$$H_0^{1,2}(\Omega) = \{f \in L^2(\Omega) : \exists h \in (L^2(\Omega))^n \quad \text{with}$$

$$\int_\Omega f D\varphi = -\int_\Omega h\varphi \quad \text{for every} \quad \varphi \in C_0^{+\infty}(\Omega)\}$$

(denote $Df = h$ ). Given $f \in H_0^{1,2}(\Omega)$ define $\|f\|_{H_0^{1,2}}(\Omega) = (\int_\Omega |f|^2 dx + \int_\Omega |Df|^2 dx)^{\frac{1}{2}}$ .

Consider the problem

$$(\mathcal{P}) \qquad \min_{\{f \in H_0^{1,2}(\Re): \int_{-\infty}^{+\infty} f^3 dx = 1\}} \int_\Re (f^2 + |Df|^2) dx.$$

If $f \in H_0^{1,2}(\Re)$ is any given function, then $f_t(x) = f(x - t)$ satisfies, $f_t \to 0$ weakly in $H_0^{1,2}(\Re)$ as $t \to +\infty$ , and $\|f\|_{H_0^{1,2}}, \|f\|_{L^3}$ are unchanged.

More in general, if $(f^n)_n$ is any bounded sequence in $H_0^{1,2}(\Re)$ , then it is easy to show that there exists a sequence $(t_n)_n$ with $f_{t_n}^n \to 0$ weakly in $H_0^{1,2}(\Re)$ as $n \to +\infty$ .

Hence, any minimizing sequence for problem $(\mathcal{P})$ can be replaced by a minimizing sequence that converge weakly to 0. There is thus no hope of proving that *every* minimizing sequence has a subsequence that converge to a minimiser.

A possibly remedy is to seek a procedure that replaced minimizing sequences by minimizing sequences that have better convergence properties.

The rearrangement procedures are effective for this purpose. The following theorem is a semplified form of a result due to P.L.Lions [17].

**THEOREM 9.** *Let* $\Omega = U \times \Re$ , *where* $U$ *is a bounded open set in* $\Re$ . *The closed convex cone of all Steiner symmetric functions in* $H_0^{1,2}(\Omega)$ *is compactly embedded in* $L^p(\Omega)$ , *for* $2 \leq p < +\infty$ . *Further,* *if* $f \in H_0^{1,2}(\Omega), f \geq 0$ , *then* $f^* \in H_0^{1,2}(\Omega)$ *and* $\|f^*\|_{H_0^{1,2}} \leq \|f\|_{H_0^{1,2}}$ [(1)] .

Now we present an example that shows the opportunity of rearrangement procedures.

**EXAMPLE 5.** *Let* $\Omega = \{(x,y) \in \Re^2 : 0 < y < 1\}$ . *We consider the problem to seek for a non–trivial solution in* $H_0^{1,2}(\Omega)$ *for*

$$-\Delta f + f = g(x)|f|^{\sigma-2} f$$

*where* $g$ *is a Cauchy symmetric* $L^\infty$ *function,* $g \not\equiv 0, g \geq 0,$ and $2 < \sigma < +\infty$ .

*Consider the variational formulation*

$$(\mathcal{P}) \qquad \min_{\{f \in H_0^{1,2}(\Omega): F(f)=1\}} \frac{1}{2}\|f\|^2$$

*where* $F(f) = \frac{1}{\sigma}\int_\Omega g(x)|f|^\sigma dx$.

*Observe that* $F(\alpha f) = \alpha^\sigma f(f)$ *for* $\alpha > 0$; *so* $g \not\equiv 0$ *ensures that there exists* $f \in H_0^{1,2}(\Omega)$ with $F(f) > 0$ ; *now* $F(\alpha f) = 1$ *for some* $\alpha$ . *So, the constraint set is non empty.*

*Since* $F(|f|) = F(f)$ *and* $\||f|\|_{H_0^{1,2}} = \|f\|_{H_0^{1,2}}$ , *the problem* $(\mathcal{P})$ *has a minimizing sequence of non–negative functions.*

*Let* $f \in H_0^{1,2}(\Omega)$ . *By virtue of theorem 9, we have*

$f^* \in H_0^{1,2}(\Omega)$

$\|f^*\|_{H_0^{1,2}} \leq \|f\|_{H_0^{1,2}}$

$F(f^*) = \frac{1}{\sigma}\int_\Omega g(x)|f^*|^\sigma dx = \frac{1}{\sigma}\int_\Omega g^*(x)(|f|^\sigma)^* dx \geq \frac{1}{\sigma}\int_\Omega g(x)|f|^\sigma dx = F(f)$ .

*Now, we suppose* $F(f) = 1$ ; *then we may choose* $\alpha, 0 < \alpha \leq 1$ , *such that*

$$F(\alpha f^*) = 1 \text{ ; so, } \|\alpha f^*\|_{H_0^{1,2}} = \alpha\|f^*\|_{H_0^{1,2}} \leq \|f^*\|_{H_0^{1,2}} \leq \|f\|_{H_0^{1,2}} .$$

*It follows that problem* $(\mathcal{P})$ *has a minimizing sequence of non negative, Steiner symmetric functions.*

*Let* $(f_n)_n$ *be a minimizing sequence of Steiner symmetric functions for problem* $(\mathcal{P})$ . *The sequence* $(f_n)_n$ *is therefore bounded and, unless passing to subsequences, we can suppose* $f_n \to f_0$ *weakly in* $H_0^{1,2}(\Omega)$ . *Notice that* $f_0$ *is Steiner symmetric, since the Steiner symmetric functions form a closed convex set in* $H_0^{1,2}(\Omega)$ , *which is therefore*

---

(1) *here* $*$ *denotes the Steiner symmetrization*

*weakly closed. Again, by virtue of theorem 9, we have* $f_n \to f_0$ *strongly in* $L^\sigma$ *. Hence* $F(f_0) = 1$ *.*

*By the weak lower semicontinuity of* $\|\cdot\|$ *, we have*

$$\frac{1}{2}\|f_0\|^2_{H_0^{1,2}} \le \lim_n \frac{1}{2}\|f_n\|^2_{H_0^{1,2}} = \inf \mathcal{P}.$$

*Thus* $f_0$ *is the desired minimiser; observe also that* $f_0 \not\equiv 0$ *, since* $F(f_0) = 1$ *.*

*Now, by the Lagrange multiplier rule, there exists* $\zeta \in \Re$ *with*

$$-\Delta f_0 + f_0 = \zeta |f_0|^{\sigma-2} f_0 g(x).$$

*Multiplying by* $f_0$ *and integrating by parts we get*

$$\int_\Omega |Df_0|^2 dx + \int_\Omega f_0^2 dx = \zeta \int_\Omega g(x)|f_0|^\sigma dx,$$

*i.e.*

$$2\|f_0\|^2_{H_0^{1,2}} = \sigma\zeta F(f_0) = \sigma\zeta.$$

*Therefore,* $\zeta = 2\sigma^{-1}\|f_0\|^2_{H_0^{1,2}} > 0.$
*Now, write* $v = \frac{f_0}{\alpha}$ *, where* $\alpha > 0$ *is to be determined. Then*

$$\alpha(-\Delta v + v) = \zeta\alpha^{\sigma-1} g(x)|v|^{\sigma-2} v.$$

*If we choose* $\alpha$ *so that* $\alpha^{\sigma-2}\zeta = 1$ *, then*

$$-\Delta v + v = g(x)|v|^{\sigma-2} v;$$

*moreover v is non trivial, non negative and Steiner–symmetric.*

## 2. COMPACT RETRACT AND COMPACT FIXED POINT PROPERTY FOR SETS OF REARRANGEMENTS.

Any continuous function which maps a closed subset A of a metric space X into a totally bounded set of a normed space E can be extended to the whole space X keeping the value in a totally bounded set [11]. In fact the range of the extension, the convex hull of a totally bounded set of a normed space, is totally bounded.

Such a kind of result has been proved in [13], for maps into $L^1(T,E)$, using the concept of decomposable hull instead of that of a convex hull.

Purpose of this section, is to present a similar result for maps into $L^1(T,E)$ using the concept of rearrangement.

From this point on, $(T,\mathcal{F},\mu)$ will denote a measure space, with $\mu$ positive and non–atomic measure.

Given a function $f \in L^1(T,\Re)$, $f \cdot \mu$ denotes the measure having density $f$ with respect to $\mu$.

Let $\nu : \mathcal{F} \to \Re^n$ be a vector measure, whose component have no atoms.
A family $(A_\alpha)_{\alpha\in[0,1]}, A_\alpha \in \mathcal{F}$, is called increasing if $A_\alpha \subset A_\beta$ when $\alpha \leq \beta$. An increasing family is called refining $C \in \mathcal{F}$ with respect to the measure $\nu$, if

$$A_0 = \emptyset, A_1 = C, \quad \text{and} \quad \nu(A_\alpha) = \alpha\nu(C) \text{ for every } \alpha \in [0,1] .$$

**LEMMA 1.** *Let* $g_1, g_2, \cdots, g_n \in L^1(T,\Re)$ *and let* $\nu$ *be the vector measure whose component* $\nu_i$ *are the measure* $g_i \cdot \nu$.

*Then, there exists a family* $(A_\alpha)_{\alpha\in[0,1]}$ *refining* $T$ *with respect to* $(\nu,\mu)$.

For this result one can refer to [14].

Given a simple function $\psi \in L^1(T,\Re)$, we denote with $R(\psi)$ the set of all rearrangements of $\psi$ on T.

**THEOREM 10.** *The set* $R(\psi)$ *is closed in* $L^1(T,\Re)$.

*Proof* : Let $(g_n)_{n\geq 0} \subset R(\psi)$, with $g_n \to g_0$ in $L^1(T,\Re)$.

We shall prove that

$$\mu(\{x : g_0(x) \geq \alpha\}) = \mu(\{x : \psi(x) \geq \alpha\}) \quad \text{for every} \quad \alpha \in \Re.$$

Fix $\alpha \in \Re$. Note that $\{x : g_0(x) \geq \alpha\}) = \{x : (\alpha - g_0)^+ = 0\}$.

By virtue of the Lebesgue dominate convergence theorem, we have

$$(\alpha - g_n)^+ \to (\alpha - g_0)^+ \quad \text{in} \quad L^1(T,\Re).$$

Since the functional $F(u) = \int_T \mathcal{X}_{\{0\}}(u(t))d\mu(t)$ is upper semicontinuous on $L^1(T,\Re)$, we have

(1) $$\mu(\{x : g_0(x) \geq \alpha\}) \geq \mu(\{x : \psi(x) \geq \alpha\}).$$

Now, observe that $\{x : g_0(x) < \alpha\} = \cup_{k=1}^{\infty}\{x : g_0(x) \leq \alpha - \frac{1}{k}\}$ .

Applying the precedent reasoning, one prove that

$$\mu(\{x : g_0(x) \leq \alpha - \frac{1}{k}\}) \geq \mu(\{x : \psi(x) \leq \alpha - \frac{1}{k}\}) \quad \text{for every} \quad k \in \mathbb{N}.$$

Therefore

$$(2) \qquad \mu(\{x : g_0(x) < \alpha\}) = \lim_{k\to+\infty} \mu(\{x : g_0(x) \leq \alpha - \frac{1}{k}\}) \geq$$

$$\geq \mu(\{x : \psi(x) < \alpha\}).$$

Putting together (1) and (2), we obtain the result.

**THEOREM 11.** (Compact extension result) *Let $A$ be a closed subset of a metric space (X,d), and let $f : A \to R(\psi)$ be a continuous map whose image is relatively compact. Then, there exists a totally bounded set B, with $f(A) \subset B \subset R(\psi)$ , and a continuous function $\tilde{f} : X \to B$ such that $\tilde{f}_{|A} = f$.*

The foundamental argument for the proof of this theorem is the following interpolation result on $R(\psi)$ . More precisely,

**LEMMA 2.** (Interpolation result) *Let $\varphi_1, \varphi_2, \cdots, \varphi_p \in R(\psi)$ . Then, there exists a family of functions $(\eta_\lambda)_\lambda$ , where $\lambda = (\lambda_1, \cdots, \lambda_p), \lambda_i \geq 0$ and $\sum_{i=1}^{p} \lambda_i = 1$ , with the properties:*

($b_1$) $\eta_\lambda \in R(\psi)$ *for every* $\lambda \in S^p$ (1) ;

($b_2$) $\eta_{\lambda^j} = \varphi_j$ *for every* $j = 1, \cdots, p,$ *where*

$\lambda^j$ *is the m–ple of all zeros and with 1 in the j–th position;*

($b_3$) $\lambda \to \eta_\lambda$ *is continuous from* $S^p$ *into* $L^1(T, \Re)$ ;

($b_4$) $\|\varphi_j - \eta_\lambda\|_1 \leq \beta \sup_{\{i:\lambda_i \neq 0\}} \|\varphi_j - \varphi_i\|_1,$

*for every* $\lambda \in S^p$, *where* $\beta \in \Re^+$ *is a positive constant* .

*Proof* : Rearrange the range of $\psi$ into a finite monotone sequence $a_1, a_2, \cdots, a_M$ $(a_i < a_{j+1}$ for every $i = 1, \cdots, M-1)$ .

(1) $S^p$ *denotes the* $p$ *–dimentional simplex*

Let $c(p,k)$ be the set of all k–ples $(c_1, c_2, \cdots, c_k)$ with $c_i \in [1,p] \cap \mathbb{N}$ and $c_j < c_{j+1}$ for every $j = 1, \cdots, k-1$ .

Given $\sigma_k \in C(p,k)$ we write $i \in \sigma_k = (c_1, \cdots, c_k)$ if there exists $j$, with $c_j = i$ . Order the set $\cup_{k=1}^{p} C(p,k)$ in the following way:

$$\sigma_k \leq \delta_{k'} \quad \text{if} \quad k \geq k'$$

or

$$k = k' \quad \text{and} \quad \sigma_k \leq \delta_k \quad \text{with respect to the lexicographic order} \quad .$$

Then, rearrange the set $\cup_{k=1}^{p} C(p,k)$ into a finite sequence of indices $d_1, \cdots, d_N$ .

Let $(A^l_{a_j}(\lambda))_\lambda; l = 1, \cdots, N; j = 1, \cdots, M$ , be an increasing family of measurable sets refining the set

$$D^l_{a_j} = (\cup_{i \in d_l} I^i_{a_j}) \setminus (\cup_{i \in d_l} I^i_{a_j})$$

with respect to $\mu$ and the measures generated by the densities of the form:

$$|\varphi_i - \varphi_j|(t) \quad \text{for every} \quad i,j = 1, \cdots, M,$$

where $I^i_{a_j} = \{t : \varphi_i(t) \geq a_j\}$ .

Now, we are able to construct a family of measurable sets $(E_{a_j}(\lambda))_\lambda$ which interpolates the level sets $(\varphi^{-1}([a_j, +\infty)))_{i=1,\cdots,M}$ :

$$E_{a_j}(\lambda) = \cup_{l=1}^{N} A^l_{a_M}(y_{b_l}(\lambda)))$$

where $y_{b_l}(\lambda) = \sum_{i \in b_l} \lambda_i$ ;

suppose we have already defined $(E_{a_j}(\lambda))_\lambda$ for $k+1 \leq j \leq M$ .

Then, define

$$E_{a_k}(\lambda) = \cup_{l=1}^{N} \{(E_{a_{k+1}}(\lambda) \cap D^l_{a_k}) \cup A^l_{a_k}(p^k_l(\lambda))\}$$

where $p^k_l(\lambda) \in [0,1], p^k_l(\lambda) \leq y_{b_l}(\lambda)$ satisfies

$$p^k_l(\lambda) = 0 \quad \text{if} \quad \mu(E_{a_{k+1}}(\lambda) \cup \cup_{j=1}^{l-1} A^l_{a_k}(p^k_l(\lambda))) = \mu(\{t : \psi(t) \geq a_k\}),$$

or

$$\mu(E_{a_k}(\lambda) \cup \cup_{j=1}^{l} A_{a_k}^{l}(p_l^k(\lambda))) \leq \mu(\{t : \psi(t) \geq a_k\}).$$

It is easy to verify the following properties:

(1) $E_{a_k}(\lambda^i) = \varphi_i^{-1}([a_k, +\infty))$ ;

(2) $E_{a_{k+1}}(\lambda) \subset E_{a_k}(\lambda)$ ,

for every $i = 1, \cdots, p; k = 1, \cdots, M-1; \lambda \in S^p$ .

Since $\mu(\cup_{l=1}^{N} A_{a_j}^{l}(y_{b_l}(\lambda))) = \sum_{i=1}^{p} \lambda_i \mu(I_{a_j}^{i}) = \mu(\{t : \psi(t) \geq a_j\})$ for every $j = 1, \cdots, M$ , it is clear that

(3) $\mu(E_{a_j})) = \mu(\{t : \psi(t) \geq a_j\})$ for every $\lambda \in S^p; j = 1, \cdots, M.$

Property (2) allows us to construct a family of functions $(\eta_\lambda(t))_{\lambda \in S^p}$ having the family $(E_{a_j}(\lambda))$ as level sets.

More precisely, define

$$(*) \qquad \eta_\lambda(t) = \sum_{j=1}^{M} a_j \mathcal{X}_{E_{a_j}(\lambda) \setminus E_{a_{j+1}}(\lambda)} \quad (E_{a_{M+1}}(\lambda) = \emptyset).$$

By virtue of (1) and (3) follows

(4) $\eta_{\lambda^i} = \varphi_i$ ;

(5) $\mu(\eta_\lambda^{-1}([a_j, +\infty))) = \mu(\psi^{-1}([a_j, +\infty)))$ for every $\lambda \in S^p; j = 1, \cdots, M; i = 1, \cdots, p$ .

Now, we prove that $\lambda \rightarrow \eta_\lambda$ is continuous from $S^p$ into $L^1(T, \Re)$ . It results

$$\mu(E_{a_M}(\lambda) \Delta E_{a_M}(\lambda')) \leq \sum_{l=1}^{N} \mu(A_{a_M}^{l}(y_{b_l}(\lambda)) \Delta A_{a_M}^{l}(y_{b_l}(\lambda'))) =$$

$$= \sum_{l=1}^{N} |y_{b_l}(\lambda) - y_{b_l}(\lambda')| \mu(D_{a_M}^{l}) \leq \|\lambda - \lambda'\| \mu(T).$$

Analogously we can deduce the extimates

$$\mu(E_{a_j}(\lambda) \Delta E_{a_j}(\lambda')) \leq \|\lambda - \lambda'\| (M - j + 1) \mu(T).$$

This, of course, ensures the continuity of $\lambda \rightarrow \eta_\lambda$.

It remains to prove $(b_4)$ .

By the definition of $D^l_{a_j}$ , $(*)$ can be rewrite as

$$\eta_\lambda(t) = \varphi_i(t) \quad \text{if} \quad t \in D^l_{a_j} \quad \text{for some} \quad l,j \quad \text{with} \quad i \in b_l.$$

Therefore,

$$\|\eta_\lambda - \varphi_i\|_1 \leq \sum_{k=1}^{M} \sum_{l=1}^{N} \int_{A^l_{a_k}(p^k_l(\lambda))} |\varphi_j - \varphi_i| d\mu \leq\leq M \sup_{\{h:\lambda_h \neq 0\}} \|\varphi_h - \varphi_i\|_1$$

where $j \in b_l$ , and this conclude the proof.

△

*Proof of theorem* 11 .

Let $(A_n)_{n \geq 1}$ be the open sets defined by

$A_1 = \{x \in X : d(x,A) > 1\}$

$A_2 = \{x \in X : \frac{1}{2} < d(x,A) < \frac{3}{2}\}$

$\dots$

$A_n = \{x \in X : \frac{1}{2^{n-1}} < d(x,A) < \frac{3}{2^{n-1}}\}$

$\dots$

We have: $X \setminus A = \cup_{n \geq 1} A_n$ .

Set $\varepsilon_n = \frac{1}{2^n}, n \geq 1$ , and let $N_n = \{g^n_1, \cdots, g^n_{j_n}\}$ be an $\varepsilon_n$ –net of f(A). Let $\pi : X \to A$ be a function such that $d(x, \pi x) = d(x,A)$ ( $\pi$ is any selection of the projection of minimal distance). Put

$$\mathcal{U}^n_j = A_n \cap (\pi^{-1}(f^{-1}(g^n_j + \varepsilon_n B_1))).$$

Consider the pairs $(n,j); n \geq 1, j = 1, \cdots, j_n$ , in the lexicographic order; the pair $(n,j)$ is identified with a natural l by the relation $l = \sum_{i=1}^{n-1} j_i + j$ . If $l$ corresponds to the pair $(n,j)$ , $g_l$ e $\mathcal{U}^l$ will denote respectively $g^n_j$ and $\mathcal{U}^n_j$ .

Let $\{q^l(x)\}$ be a continuous partition of unity subordinate to $\{\mathcal{U}^l\}$ .

Apply Lemma 2 to the functions of the set $N_1 \cup N_2$ , i.e.

$$\varphi_i = g_i \qquad i = 1, \cdots, p; p = j_1 + j_2,$$

and denote with $(f^1_\lambda)_\lambda$ the corrispondent interpolator functions.

Define a continuous function $\tilde{f}_1$ on $A_1$ by setting

$$\tilde{f}_1(x) = f^1_{\lambda(x)},$$

where $\lambda(x) = (\lambda_1(x), \cdots, \lambda_p(x)), \lambda_i(x) = q^i(x)$ .

Let $R_1 = \{b^1_m\}_{m=0,\cdots,m_1}$ be an $\varepsilon_3$ –net of the totally bounded set $\tilde{f}_1(A_1)$ . Let $\theta_1$ be a function mapping each x belonging to $A_1$ into an element of $R_1$ , whose distance from $\tilde{f}_1(x)$ is less than $\varepsilon_3$ .

Define the open sets

$$\mathcal{V}^1_{j,m} = \mathcal{U}^1_j \cap (\theta_1^{-1}(b^1_m) + \varepsilon_4 B_1), \quad j = 1, \cdots, j_1; m = 0, \cdots, m_1$$
$$\mathcal{V}^n_{j,0} = \mathcal{U}^n_j, \quad j = 1, \cdots, j_n; n \geq 2.$$

Consider the triples $(n, j, m); n \geq 1, j = 1, \cdots, j_n, m = 0, \cdots, m_n$ (set $m_n = 0$ if $n \neq 1$) , in the lexicografic order; the triple $(n, j, m)$ is identified with a natural l by the relation $l = \sum_{i=1}^{n-1} j_i(m_i + 1) + j \cdot (m + 1)$. Denote with $l_n$ the index corresponding to the triple $(n, j_n, m_n)$ . If $l$ corresponds with the triple $(n, j, m)$ ; $g_l$ , $\mathcal{V}^l$ will denote respectively $g^n_j$ and $\mathcal{V}^n_{j,m}$ .

Let $\{q^l(x)\}$ be a continuos partition of unity subordinate to $\{\mathcal{V}^l\}$ .

Apply Lemma 2 to the functions of the set $R_1 \cup N_1 \cup N_2 \cup N_3$ , i.e.

$$\varphi_i = b^1_i \qquad i = 0, \cdots, m_1;$$

$$\varphi_i = g_{\tau(i)} \quad i = m_1 + 2, \cdots, l_3,$$

where $\tau(i) = i - (m_1 + 1)$ , and denote with $(f^2_\lambda)_\lambda$ the corrispondent interpolator functions.

Define a continuous function $\tilde{f}_2$ on $A_1 \cup A_2$ by setting

$$\tilde{f}_2 = f^2_{\lambda(x)},$$

where $\lambda(x) = (\lambda_i(x), \cdots, \lambda_{l_n}(x)), \lambda_i(x) = q^i(x)$ .

Further, from the definition of $R_1$ ,for every $x \in A_1 \setminus \overline{A_2}$ , we have

$$\|\tilde{f}_1(x) - \tilde{f}_2(x)\|_1 \leq \|\tilde{f}_1(x) - b^1_{\overline{m}}\| + \|b^1_{\overline{m}} - \tilde{f}_2(x)\| \leq 4\varepsilon_3 = \varepsilon_1.$$

Let us proceed by induction. Suppose that we have defined continuous functions $\tilde{f}_j$ on $\cup_{l\leq j} A_l$ such that

(@) $\qquad \|\tilde{f}_{j-1}(x) - \tilde{f}_j(x)\|_1 \leq \varepsilon_{j-1}$ on $(\cup_{l\leq j-1} A_l) \setminus \overline{A}_j$ for $j = 2, \cdots, n-1$.

Then there exist $\tilde{f}_n$ such that (@) holds for $j = n$. In fact, let $R_n = \{b_m^n\}_{m=0,\cdots,m_n}$ be an $\varepsilon_{n+2}$ –net of the totally bounded set $\tilde{f}_{n-1}(\cup_{m\leq n-1} A_m \setminus \overline{A}_n)$ .

Let $\theta_n$ be a function that maps each x belonging to $\cup_{l\leq n} A_l$ into an element of $R_n$ , whose distance from $\tilde{f}_{i-1}(x)$ is less than $\varepsilon_{n+2}$ . Then, define the open sets

$\mathcal{V}_{j,m}^k = \mathcal{U}_j^k \cap \left(\theta_n^{-1}(b_m^n) + \varepsilon_{n+3} B_1\right); j = 1, \cdots, j_k; m = 0, \cdots, m_n; k = 1, \cdots, n-1$ ;

$\mathcal{V}_{j,0}^k = \mathcal{U}_j^k, \quad j = 1, \cdots, j_k; k \geq n.$

Let $\{q^l(x)\}$ be a continuous partition of unity subordinate to $\{\mathcal{V}^l\}$ . Apply Lemma 2 to the functions of the set $R_n \cup \cup_{j=1}^{n+1} N_j$ , i.e.

$$\varphi_i = b_i^n, \qquad i = 0, \cdots, m_n;$$

$$\varphi_i = g_{\tau(i)}, \qquad i = m_n + 1, \cdots, l_{n+1}$$

where $\tau(i) = i - (m_n + 1)$ , and denote with $(f_\lambda^n)_\lambda$ the corrispondent interpolator functions.

Then, define a continuous function $\tilde{f}_n$ on $\cup_{j=1}^n A_j$ , by setting

$$\tilde{f}_n(x) = f_{\lambda(x)}^n$$

where $\lambda(x) = (\lambda_1(x), \cdots, \lambda_{l_{n+1}}(x)), \lambda_i(x) = q^i(x)$ .

Further, from the definition of $R_n$ , for every $x \in \cup_{j=1}^{n-1} A_j \setminus \overline{A_n}$ , we have

$$\|\tilde{f}_n(x) - \tilde{f}_{n-1}(x)\|_1 \leq \|\tilde{f}_n(x) - b_{\overline{m}}^n\|_1 + \|b_{\overline{m}}^n - \tilde{f}_{n-1}(x)\|_1 \leq 4\varepsilon_{n+2} = \varepsilon_n.$$

Define a function $\tilde{f} : X \to X$ by setting, for every $x \in A_n$ ,

$$\tilde{f}(x) = \lim_{m \geq n} \tilde{f}_m(x)$$

and $\tilde{f}(x) = f(x)$ for every $x \in A$ . Since the image of each $\tilde{f}_m$ is contained in $R(\psi)$ , then also $\tilde{f}(X) \subset R(\psi)$ (cfr. theorem 10).

From the relation

$$\|\tilde{f}_p(x) - \tilde{f}_q(x)\|_1 \leq \sum_{j=p}^{q} \varepsilon_j, \quad p \leq q, \quad x \in \cup_{h=1}^{p} A_h \setminus \overline{A}_{p+1}$$

it is easy to verify that $\tilde{f}$ is continuous on $X \setminus A$ . Let us check the continuity on A. Fix $\varepsilon > 0$ and $a \in A$ ; there exists a $\delta > 0, \delta < \varepsilon$ ,such that if $b \in A$ with $d(a,b) < \delta$ then $\|\tilde{f}(a) - \tilde{f}(b)\|_1 < \frac{\varepsilon}{8\beta}$. Now, if $x \in X \setminus A$ and $d(x,a) < \frac{\delta}{2}$ , then x belongs to some $\mathcal{U}_{j_0}^n \setminus \overline{A}_{n+1}$ , with n sufficently large (we can also suppose that $\delta$ is small enough to verify $\varepsilon_n < \frac{\varepsilon}{8\beta}$) . Indeed, $d(\pi x, a) \leq d(\pi x, x) + d(x,a) < \delta$ , and so $\|\tilde{f}(a) - \tilde{f}(\pi x)\|_1 < \frac{\varepsilon}{8\beta}$ .

Therefore, if $q_j^n(x) \neq 0$ , $\|\tilde{f}(a) - g_j^n)\|_1 < \|\tilde{f}(a) - f(\pi x)\|_1 + \|f(\pi x) - g_j^n\|_1 \leq \frac{\varepsilon}{8\beta} + \varepsilon_n < \frac{\varepsilon}{4\beta}$ .

Then, by virtue of lemma 2 part $(b_4)$ , we have

$$\|\tilde{f}_n(x) - g_{j_0}^n\|_1 \leq \beta \sup_{\{j:q_j^n(x)\neq 0\}} \|g_j^n - g_{j_0}^n\|_1 \leq \frac{\varepsilon}{2},$$

and so

$$\|\tilde{f}_n(x) - \tilde{f}(a)|_1 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4}.$$

Because of the relation

(**) $\|\tilde{f}(x) - \tilde{f}_n(x)\|_1 < \sum_{j=n}^{\infty} \varepsilon_j \leq \varepsilon_n < \frac{\varepsilon}{8}$ , for every $x \in \cup_{l=1}^{n} A_l \setminus \overline{A}_{n+1}$ ,

we have, $\|\tilde{f}(x) - \tilde{f}(a)\|_1 < \varepsilon$ , for every $x \in X$ with $d(x,a) < \frac{\delta}{2}$ .

It is left to show that $\tilde{f}(X)$ is totally bounded. Fix $\varepsilon > 0$ . Since $\tilde{f}$ is continuous, and $\overline{f(A)}$ is compact, there exists $\delta > 0$ such that $\tilde{f}(A + \delta B_1) \subset \tilde{f}(A) + \frac{\varepsilon}{2} B_1$. Since $f(A)$ is totally bounded, then $\tilde{f}(A + \delta B_1)$ can be covered by a finite number of balls of radius $\varepsilon$ . Choose m so that $\{A_j : j = 1, \cdots, m\}$ cover $X \setminus [A + \delta B_1]$ while $A_{m+1}$ has empty intersection with it. Since each $\tilde{f}_j(\cup_{l=1}^{m} A_l), j \geq m$ is totally bounded, and (**) holds, we have that whenever $j$ satisfies $\varepsilon_j < \frac{\varepsilon}{2}$ , an $\frac{\varepsilon}{2}$ -net of $\tilde{f}_j(\cup_{l=1}^{m} A_l)$ is also an $\varepsilon$ -net of $\tilde{f}(\cup_{l=1}^{m} A_l)$ .

Hence we have found a finite $\varepsilon$ -net for the set $\tilde{f}(X)$ .

△

As a consequence of theorem 11, the set of rearrangements $R(\psi)$ has a relatively compact retract property.

Another consequence of theorem 11, is the following compact fixed point property.

**COROLLARY 1** *Let $F : R(\psi) \to R(\psi)$ be a continuous function with $F(R(\psi))$ totally bounded. Then F has a fixed point in $R(\psi)$ .*

*Proof* : Set $A = \overline{F(R(\psi))}$ . Following the notations of theorem 11, let $i : A \to A$ be the identity map on A, and let $\tilde{i} : L^1(T, \Re) \to B$ be the continuous function with $A \subset B \subset R(\psi)$ , B totally bounded and $\tilde{i}_{|A} = i$ .

For every $x \in L_1(T, \Re)$ , define the function $\hat{F}(x) = F(\tilde{i}(x))$ .

For every $x \in L_1(T, E)$, $\hat{F}(x) \subset F(B) \subset A$ ; in particular $\hat{F}$ maps $\overline{co}(A)$ into itself.

Let $x^*$ be a fixed point of $\hat{F}$ . Then $x^* = F(\tilde{i}(x^*)) \in A$ , hence, $\hat{F}(x^*) = F(x^*)$.

△

## 3. THE REARRANGEMENTS ARE RETRACTS

As it is well known the notion of an absolute retract [11] offers a general setting for several problems of analysis, namely the existence of extensions and fixed points. Up to recently, only few examples of such sets were known, mainly akin to convexity.

The progress of non linear analysis has provided new examples of absolute retracts: decomposable sets [6] or sets of solutions to differential inclusion [7].

Purpose of this section is to show that such set of rearrangements are absolute retracts, hence providing a new framework for their use.

**LEMMA 3.** (Lyapunov extended theorem). *Let $(T, \mathcal{F}, \mu)$ be a measure space, with $\mu$ positive, non–atomic measure.*

*Let $(g_n)_{n \geq 0}$ be a sequence of non–negative functions in $L^1(T, \Re)$ with $g_0 \equiv 1$ , Then, there exists a map $\gamma : \Re_0^+ \times [0,1] \to \mathcal{F}$ with the following properties:*

$(a_1)$ $\gamma(\tau, \lambda_1) \subset \gamma(\tau, \lambda_2)$ *if* $\lambda_1 \leq \lambda_2$ ;

$(a_2)$ $\mu(\gamma(\tau_1, \lambda_1) \Delta \gamma(\tau_2, \lambda_2)) \leq 2|\tau_1 - \tau_2| + |\lambda_1 - \lambda_2|$ ;

$(a_3)$ $\int_{\gamma)\tau,\lambda)} g_n d\mu = \lambda \int_T g_n d\mu$ *for every* $n \leq \tau$,

*for all* $\lambda, \lambda_1, \lambda_2 \in [0,1], \tau, \tau_1, \tau_2 \geq 0$ .

For this last result we refer to [6].

**THEOREM 12.** *Assume that* $L^1(T,\Re)$ *is separable. Then, the set* $R(\psi)$ *is a retract of the whole space* $L^1(T,\Re)$ .

The foundamental argument of the proof is an interpolation result on $R(\psi)$ . More precisely,

**LEMMA 4.** *Let* $(\varphi_n)_{n\in\mathbb{N}} \subset R(\psi)$ , *and let* $\{p_i(x)\}_{i\in\mathbb{N}}$ *be a continuous partition of unity subordinate to a locally finite covering* $\{\mathcal{U}_i\}_{i\in\mathbb{N}}$ *of a metric space (X,d).*

*Then, there exists a family of functions* $(\eta_x)_{x\in X}$ , *with the following properties:*

$(b_1)$ $\eta_x \in R(\psi)$ *for every* $x \in X$ ;

$(b_2)$ $\eta_{\overline{x}^i} = \varphi_i$ *if* $\overline{x}^i \in X$ *is such that* $p_j(\overline{x}^i) = 0$ *for all* $j \in \mathbb{N}$ *except* $j = i$ ;

$(b_3)$ $x \to \eta_x$ *is continuous from X into* $L^1(T,\Re)$ ;

$(b_4)$ $\|\eta_x - \varphi_i\|_1 \le \sup_{\{j:p_j(x)\neq 0\}} \|\varphi_j - \varphi_i\|_1,$ *for every* $x \in X, i \in \mathbb{N}$ .

*Proof* : Rearrange the range of $\psi$ into a finite monotone sequence of indices $a_1, a_2, \cdots, a_M$ .

Apply lemma 3 to the functions $(g_k)_{k\ge 0}$ defined by:

$$g_k(t) = \begin{cases} \mathcal{X}_{D^h_{a_j}}(t) & \text{if } k = 5^h \cdot 7^j, \\ |\varphi_i - \varphi_l|(t) \cdot \mathcal{X}_{D^i_{a_j} \setminus D^i_{a_{j+1}}}(t) & \text{if } k = 2^i \cdot 3^l \cdot 11^j, \\ 1 & \text{otherwise.} \end{cases}$$

where $D^h_{a_j} = \{t : \varphi_h(t) \ge a_j\}$ , and let $(\gamma(\tau,\lambda))_{\tau,\lambda}$ be the corrispondent measurable sets coming from lemma 3.

Now, we are able to construct a family of measurable sets $(E_{a_j}(x))_{x\in X}$ which interpolates the level sets

$$\varphi_i^{-1}([a_j, +\infty)))_{i\in\mathbb{N}} :$$

$$E_{a_j}(x) = \cup_{l=1}^{+\infty}\{[\gamma(\tau(x),\theta_l(x)) \setminus \gamma(\tau(x),\theta_{l-1}(x))] \cap D^l_{a_j}\},$$

where

$$\theta_l(x) = \sum_{i=1}^{l} p_i(x),$$

$$\tau(x) = \sum_{l,h,i,j} q_l(x)q_h(x)q_i(x)q_j(x)2^l 3^h 5^i 7^M 11^M,$$

and $(q_n(x))_n$ are continuous functions with *supp* $q_n \subset \mathcal{U}_n$ and $q_n \equiv 1$ on *supp* $p_n$ .

It is easy to verify the following properties:

(1) $E_{a_j}(\overline{x}^i) = \varphi_i^{-1}([a_j, +\infty))$ ,

(2) $E_{a_{j+1}}(x) \subset E_{a_j}(x)$ ,

(3) $\mu(E_{a_j}(x)) = \mu(\{t : \psi(t) \geq a_j\})$ ,

for every $j = 1, \cdots, M-1; i \in \mathbb{N}, x \in X$ .

Property (2) allows us to construct a family of functions $(\eta_x(t))_{x \in X}$ having the family $(E_{a_j})(x)_{x \in X; j=1,\cdots,M}$ as level sets.

More precisely,define

$$(*) \qquad \eta_x(t) = \sum_{j=1}^{M} a_j \mathcal{X}_{E_{a_j}(x) \setminus E_{a_{j+1}}(x)}.$$

By virtue of (1) and (3), follows

(4) $\eta_{\overline{x}^i} = \varphi_i$,

(5) $\mu(\eta_x^{-1}([a_j, +\infty))) = \mu(\psi^{-1}([a_j, +\infty)))$,

for every $x \in X; j = 1, \cdots, M; i \in \mathbb{N}$ .

Now, we prove that $x \to \eta_x$ is continuous from X into $L^1(T, \Re)$ .

Set $J(x) = \{j : p_j(x) \neq 0\}$ and $N(x) = |J(x)|$ .

It results, be lemma 3 part $(a_3)$ ,

$$\mu(E_{a_j}(x) \Delta E_{a_j}(x_0)) \leq \mu(\cup_{l=1}^{+\infty} \{[(\gamma(\tau(x), \theta_l(x)) \setminus \gamma(\tau(x), \theta_{l-1}(x)) \cap D^l_{a_j}] \Delta$$

$$\Delta[(\gamma(\tau(x_0), \theta_l(x_0)) \setminus \gamma(\tau(x_0), \theta_{l-1}(x_0))) \cap D^l_{a_j}]\}) \leq$$

$$\sum_{l=1}^{+\infty} \mu(D^l_{a_j})[\mu(\gamma(\tau(x), \theta_l(x)) \Delta \gamma(\tau(x_0), \theta_l(x_0))) + \mu(\gamma(\tau(x), \theta_{l-1}(x)) \Delta \gamma(\tau(x_0), \theta_{l-1}(x_0)))] \leq$$

$$\leq \mu(\{t : \psi(t) \geq a_j\}) \cdot \sum_{l \in J_x \cup J_{x_0}} [4|\tau(x) - \tau(x_0)| + |\theta_{l-1}(x) - \theta_{l-1}(x_0)| + |\theta_l(x) - \theta_l(x_0)|] \leq$$

$$\leq \mu(\{t : \psi(t) \geq a_j\})(N(x) + N(x_0))(4|\tau(x) - \tau(x_0)| + 2|p(x) - p(x_0)|),$$

where $|p(x) - p(x_0)| = \sum_i |p_i(x) - p_i(x_0)|$ ; and this, of course, ensures the continuity of $x \to \eta_x$ .

it remains to prove only $(b_4)$ .

By definition of $D^i_{a_j}$ , $(*)$ can be rewrite as

$$(**) \qquad \eta_x(t) = \varphi_i(t) \quad \text{if} \quad t \in D^i_{a_j} \quad \text{for some} \quad i \in \mathbb{N}, j \in [1,m] \cap \mathbb{N}.$$

Therefore, by virtue of lemma 3 part $(a_3)$ ,

$$\|\eta_x - \varphi_i\|_i \leq \sum_{l=1}^{\infty} \int_{\gamma(\tau(x),\theta_l(x)) \setminus \gamma(\tau(x),\theta_{l-1}(x))} \mathcal{X}_{D^l_{a_j} \setminus D^l_{a_{j+1}}} |\varphi_l - \varphi_i| d\mu =$$

$$= \sum_{l=1}^{\infty} p_l(x) \int_T |\varphi_l - \varphi_i| d\mu \leq \sup_{\{l: p_l(x) \neq 0\}} \int_T |\varphi_l - \varphi_i| \mu.$$

△

*Proof of theorem* 12. We have to prove that, given a closed subset A of a metric space (X,d), and a continuous function $f : A \to R(\psi)$ , there exists a continuous function $\tilde{f} : X \to R(\psi)$ , with $\tilde{f}_{|A} = f$ .

To this end, for each $x \in X \setminus A$ , take an open ball $B(x, r_x)$ with radius $r_x < d(x, A)$ . The family $\{B(x, r_x) : x \in X \setminus A\}$ is an open covering of the paracompact space $X \setminus A$ , hence it admits a locally finite open refinement $\{V_i : i \in I\}$ . Here I is a possibly uncountable set of indices. For each $i$ , choose two points $x_i \in V_i$ and $y_i \in A$ such that $d(x_i, y_i) < 2d(x_i, A)$ .

Select a countable subset $D = \{f_n : n \geq 1\}$ of $f(A)$ which is dense in $f(A)$ . For each $i \in I$ , choose $f_{\nu(i)} \in D$ such that

$$\|f_{\nu(i)} - f(y_i)\|_1 < d(x_i, y_i).$$

Let $\{s_i(\cdot) : i \in I\}$ be a continuous partition of unity subordinate to the covering $\{V_i\}$ .

For every $n \geq 1$ , define the open set $W_n = \cup\{V_i : \nu(i) = n\}$ and let

$$p_n(x) = \textstyle\sum_{\nu(i)=n} s_i(x).$$

Clearly, $\{p_n(\cdot) : n \geq 1\}$ is a continuous partition of unity subordinate to the locally finite open covering $\{W_n : n \in \mathbb{N}\}$ .

Taking into account lemma 4 applied to the functions of D and the partition of unity $\{p_n(\cdot) : n \geq 1\}$ , we can extend the map $f$ to the whole space X by setting

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ f_x & \text{if } x \in X \setminus A. \end{cases}$$

The function $\tilde{f}$ maps X into $R(\psi)$ and moreover it is continuous on $X \setminus A$ (cfr. lemma 4, part $(b_1), (b_3)$ ).

In order to prove that $\tilde{f}$ is continuous on A, let $a \in A$ and $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $\delta < \frac{\varepsilon}{12}$ and $\|f(y) - f(a)\|_1 < \frac{\varepsilon}{2}$ whenever $y \in A, d(y, a) < 12\delta$ .

If $d(x, a) < \delta$ and $x \in V_i$ for some $i \in I$ , then $diam(V_i) < 2\delta$ , $d(x_i, A) < 3\delta$ and $d(x_i, y_i) < 6\delta$ .

Therefore, $p_i(x) \neq 0$ implies that $d(y_i, a) < 9\delta, \|f(y_i) - f(a)\|_1 < \frac{\varepsilon}{2}$ and $\|f_{\nu(i)} - f(a)\|_1 < \varepsilon$ .

From the last inequality, it follows that

$$(@) \qquad \|f_n - f(a)\|_1 < \varepsilon \quad \text{for every} \quad n, \quad \text{with} \quad p_n(x) \neq 0.$$

For any $x \in X \setminus A$ with $d(x, a) < \delta$ , fix an integer $j$ for which $p_j(x) \neq 0$ .

From lemma 4 part $(b_3)$ and $(@)$ , we have

$$\|f(a) - \tilde{f}(x)\|_1 \leq \|f(a) - f_j\|_1 + \|f_j - \tilde{f}(x)\|_1 \leq 3\varepsilon.$$

Since $\varepsilon$ was arbitrary, this prove the continuity of $\tilde{f}$ on A.

△

Theorem 12 allows us to give a different proof of the fixed point result stated in corollary 1.

**COROLLARY 2.** *The set $R(\psi)$ has the compact fixed point property.*

*Proof* : Let $f : R(\psi) \to R(\psi)$ be a continuous map whose image is relatively compact, and let X be the closure of the convex hull of $f(R(\psi))$ . Since X is compact, it is obviously separable. Using theorem 12 , extend the identity map i on $X \cap R(\psi)$ to a continuous map $\tilde{i} : X \to R(\psi)$ . The composition $f \circ \tilde{i}$ maps X into $X \cap R(\psi)$ . By Schauder's theorem, it has a fixed point $x^* \in X \cap R(\psi)$ , which is also a fixed point of $f$ .

△

## 4. APPROXIMATE SELECTIONS FOR UPPER SEMICONTINUOUS MAPS WITH REARRANGEMENT VALUES.

Let X be a metric space and Y a Banach space. A multifunction $F : X \to 2^Y$ is Hausdorff upper semicontinuous if, for every $x_0 \in X$ and every $\varepsilon > 0$, there exists a neighbourhood V of $x_0$ such that $F(x) \subset B(F(x_0), \varepsilon)$ for all $x \in V$.

As it is well known, Hausdorff upper semicontinuous maps admit, in general, only approximate selections. The classical result is the following;

**THEOREM 13.** (cfr.[1],pp.84) *Let $F : X \to 2^Y$ be an Hausdorff upper semicontinuous map from X into the convex subsets of Y. Then, for every $\varepsilon > 0$ there exists a locally Lipschitzean map $f_\varepsilon : X \to Y$ such that*

$$graph(f_\varepsilon) \subset graph F + \varepsilon \cdot B_1,$$

*i.e. $f_\varepsilon$ is an $\varepsilon$-approximate selection of F, and $f_\varepsilon \subset coF(X)$.*

Cellina,Colombo and Fonda [9], proved an analogous result to theorem 13, by replacing the convexity assumption with the decomposability, and with some compactness assumptions on the domain of F. Bressan–Colombo [6] removed the compactness assumptions by using lemma 4.

Here, we present an analogous result to those of [6] replacing the decomposability concept with the rearrangement one.

First of all we need a technical lemma concerning paracompact spaces.

**LEMMA 5.**( cfr [6]) *Let X be a paracompact topological space. For every $x \in X$, let $U_x$ be an open neighborhood of $x$ and let $L(x)$ be an integer number.*

*Then, there exists a continuous function $\tau : X \to \Re^+$ such that*

$$\tau(x) \geq \min\{L(x') : x \in U_{x'}\} \quad \text{for every} \quad x \in X.$$

A set $B \subset R(\psi)$ is said closed–rearrangement if lemma 4 holds with $R(\psi)$ replaced by B.

**THEOREM 14.** *Let $X$ be a metric space and let $F : X \to R(\psi)$ be an Haurdorff upper semicontinuous map with closed rearrangement value. If either $X$ or $L^1(T,\Re)$ is separable, then for every $\varepsilon > 0$ there exists a continuous map $f_\varepsilon : X \to L^1(T,\Re)$ such that*

$$graph f_\varepsilon \subset B(graph F, \varepsilon).$$

*Moreover $f_\varepsilon(X) \subset R(\psi)$ .*

*Proof* : Assume first that $L^1(T,\Re)$ is separable.

Fix $\varepsilon > 0$ . For every $x \in X$ , choose a number $\delta(x) \in ]0, \frac{\varepsilon}{6}[$ such that $F(x') \subset B(F(x), \frac{\varepsilon}{6})$ whenever $x' \in B(x, \delta(x))$ .

Let $\{V_i : i \in I\}$ be an open locally finite refinement of the covering $\{B(x, \frac{\delta(x)}{2}) : x \in X\}$ of X.

For each $i$ , choose $x_i \in X$ such that $V_i \subset B(x_i, \frac{\delta(x_i)}{2})$ and select $u_i \in F(x_i)$ . For $i,j \in I$ , choose also $v_{i,j} \in F(x_j)$ such that

$$\|u_i - v_{i,j}\|_1 \leq \frac{\varepsilon}{6} + \inf\{\|u_i - v\|_1 : v \in F(x_j)\}. \tag{1}$$

Let $D = \{f_n : n \geq 1\}$ be a countable dense subset of $F(x)$ .

For every $i \in I$ select a $f_{v(i)} \in D$ for which $\|u_i - f_{v(i)}\|_1 < \frac{\varepsilon}{6}$ .

Let $\{s_i(\cdot) : i \in I\}$ be a continuous partition of unity subordinate to the covering $\{V_i\}$ . For every $n \geq 1$ , define the open set $W_n = \cup\{V_i : \nu(i) = n\}$ and let $p_n(x) = \sum_{\nu(i)=n} s_i(x)$ .

Clearly, $\{p_n(\cdot) : n \geq 1\}$ is a continuous partition of unity, subordinate to the locally finite open covering $\{W_n\}$ .

Let $\{q_n(\cdot)\}$ be continuous functions such that $supp\ q_n \subset W_n$ and $q_n \equiv 1$ on $supp\ p_n$ .

For every $x \in X$ , take an open neighborghood $U_x$ di $x$ which intersects finitely many sets $V_i$ . Setting $I(U_x) = \{i \in I : U_x \cap V_i \neq \emptyset\}$ , this means that $N(x) = |I(U_x)|$ is a finite integer.

For every couple of indices $i,j \in I(U_x)$ , choose a $f_{\nu(i,j,x)} \in D$ such that

$$\|f_{\nu(i,j,x)} - v_{i,j}\|_1 < \frac{\varepsilon}{6N(x)}. \tag{2}$$

Let $L(x)$ be an integer greater that $2^{\nu(i)} \cdot 3^{\nu(i,j,x)} \cdot 11^M$ for every $i, j \in I(U_x)$ and also greater than $\sum_{n,m,l,h} q_n(x) q_m(x) q_l(x) q_h(x) 2^n \cdot 3^m \cdot 5^l \cdot 7^M \cdot 11^M$.

Applying lemma 5 to the collection of neighborhoods $\{U_x : x \in X\}$ and integer $L(x)$ , we get the existence of a continuous function $\tau : X \to \Re^+$ such that

$$\tau(x) \geq \min\{L(x') : x \in U_{x'}\}.$$

Apply lemma 4 to $(f_n)_n, (p_n)_n, (W_n)_n$ using the continuous function $\tau(x)$ found before to get the level sets $(E_{a_j}(x))_{x \in X; j=1,\cdots,M}$ , and denote with $(f_x)_{x \in X}$ the corrispondent interpolator functions.

Then, the map $f_\varepsilon : X \to L_1(T, \Re)$ can be defined by setting

$$(*) \qquad f_\varepsilon(x) = f_x.$$

Clearly $f_\varepsilon(\cdot)$ is continuous and takes values inside $R(\psi)$ . To show that $f_\varepsilon(\cdot)$ is an $\varepsilon$ -appoximate selection of F, fix $x \in X$ and define $I(x) = \{i \in I : s_i(x) \neq 0\}$, $J(x) = \{n \geq 1 : p_n(x) \neq 0\}$ .

Notice that $|J(x)| \leq |I(x)| < +\infty$ . Since $I(x)$ is finite, there exist an $\hat{\imath} \in I(x)$ such that $\hat{\delta} = \delta(x, \hat{\imath}) = \max\{\delta(x_i) : i \in I(x)\}$ .

For every $i \in I(x)$ we have that $x_i \in B(x_{\hat{\imath}}, \hat{\delta})$ , hence

$$(3) \qquad F(x_i) \subset B(F(x_{\hat{\imath}}; \frac{\varepsilon}{6}).$$

Take a point $z \in X$ such that $x \in U_z$ and $L(z) = \min\{L(x') : x \in U_{x'}\}$ .

For every $n \in J(x)$ , select an index $i_n \in I(x) \subset I(U_z)$ such that $\nu(i_n) = n$ .

Apply two times lemma 4 respectively to the sequence of functions, $(f_{\nu(i_n,\hat{\imath},z)})_n$ and $(v_{i_n,\hat{\imath}})_n$ (of course with respect to $(W_n)_n$ and $(p_n(\cdot))_n$) , and denote with $(f_x^1)_{x \in X}$ and $(f_x^2)_{x \in X}$ the corrispondent interpolator functions.

Therefore, the functions $w(x) = f_x^1$ and $w^*(x) = f_x^2$ are continuous and $w^*(\cdot) \in F(x_{\hat{\imath}})$ . For every $n \in J(x)$ , using (1),(2), and (3), we obtain

$$(4) \qquad \|f_n - f_{\nu(i_n,\hat{\imath},z)}\|_1 \leq \|f_n - u_{i_n}\|_1 + \|u_{i_n} - v_{i_n,\hat{\imath}}\|_1 + +\|v_{i_n,\hat{\imath}} + f_{\nu(i_n,\hat{\imath},z)}\|_1 \leq$$

$$\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6N(x)} + \inf\{\|u_{i_n} - v\|_1 : v \in F(x_{\hat{i}})\} \leq \frac{2\varepsilon}{3M}.$$

Taking into account property $(b_4)$ of lemma 4, and notice that we can define the interpolator functions $(f_x^1)_x, (f_x^2)_x$ using the same family of measurable sets $(\gamma(\tau,\lambda))_{\tau,\lambda}$ occurring in the definition of $(f_x)_x$ , from (4) we deduce the extimates

(5) $$\|f_\varepsilon(x) - w\|_1 = \|f_x - f_x^1\|_1 \leq \sup_l \|f_l - f_{\nu(i_l,\hat{i},z)}\|_1 \leq \frac{2\varepsilon}{3},$$

(6) $$\|w - w^*\|_1 = \|f_x^1 - f_x^2\|_1 \leq \sum_{n \in J(x)} \|f_{\nu(i_n,\hat{i},z)} - v_{i_n,\hat{i}}\|_1 \leq$$

$$\leq \frac{|J(x)|\varepsilon}{6N(z)M} \leq \frac{|I(x)|\varepsilon}{6N(z)M} = \frac{\varepsilon}{6M} < \frac{\varepsilon}{6}.$$

Putting together (5) and (6), one has

$$d_{X\times L^1}((x, f_\varepsilon(x)), (x_{\hat{i}}, w^*)) \leq d_X(x, x_{\hat{i}}) +$$

$$+\|f_\varepsilon(x) - w\|_1 + \|w - w^*\|_1 < \frac{\varepsilon}{6} + \frac{2\varepsilon}{3} + \frac{\varepsilon}{6} = \varepsilon.$$

Hence $(x, f_\varepsilon(x)) \in B(graphF, \varepsilon)$ .

This complete the proof in the case where $L^1(T, \Re)$ is separable.

When X is separable, a slight modification of the above arguments is needed. The locally finite open covering $\{V_i : i \in I\}$ of X can be considered countable, because ot the separability assumption. It is therefore possible to define the countable set $D = \{u_i : i \in I\} \cup \{v_{i,j} : i,j \in I\}$ and arrange it into a sequence. After this choice of set D, the rest of the proof goes exactly as in the previous case.

△

## REFERENCES


[1] J.P.Aubin–A.Cellina, Differential Inclusion, Springer–Verlag, Berlin 1984.

[2] H.Berestycki–P.L.Lions, Existence d'ondes solitaires dans des probléms non linéaires du type Klein–Gordon, C.R. Acad. Sci. Paris 287, 503–506, 1978.

[3] ————————, Existence d'ondes solitaires dans des problémes non linéaires du type Klein–Gordon, C.R. Acad. Sci. Paris 288, 395–398, 1979.

[4] ————————, Non linear scalar field equations, I and II, Arch. Rat. Mech. Anal.

[5] H.J.Brascamp–E.H.Lieb–J.M.Luttinger, A general rearrangement inequality for multiple integrals, J.Func.Anal. 17, 227–237, 1974.

[6] A.Bressan–G.Colombo, Extensions and selections of maps with decomposable values,Studia Math. vol. 90, 69–86, 1988.

[7] A.Bressan–A.Cellina–A.Fryszkowski, A class of absolute retracts in spaces of integrable functions, to appear on Proc. Am. Math. Soc.

[8] G.B.Burton, Vortex rings in a cylinder and rearrangements, J. Diff. Eq., 333–348, 1987.

[9] A.Cellina–G.Colombo–A.Fonda, Approximate selections and fixed points for upper semicontinuous maps with decomposable values, Proc. Am. Math.Soc. 98 1–4, 1986.

[10] M.F.Crandal–L.Tartar, Some relations between non expansive and order preserving mappings, Proc.Am.Math.Soc. 78, 385–390, 1980.

[11] J.Dugundji,Topology, Allyn and Bacon, Boston 1966.

[12] M.J.Esteban–P.L.Lions, A compactness lemma, Non Linear Anal.

[13] L.Faina, Extensions of compact continuous maps into decomposable sets, to appear on Rend.Sem. Math. Padova.

[14] A.Fryszkowski, Continuous selections for a class of non convex multivalued maps, Studia Math. 76, 163–174, 1983.

[15] G.H.Hardy–J.E.Littlewood–G.Polya, Inequalities, Cambridge Univ. Press, Cambridge 1934.

[16] B.Kawoll, Rearrangements and convexity of level sets in PDE, Lecture Notes 1150, 1980.

[17] P.L.Lions, Symmetrie et compacite dans les espaces de Sobolev, J. Func. Anal. 49, 315–334, 1982.

[18] P.L.Lions. Minimization problems in $L^1(\Re^n)$ , J.Funct. Anal. 41, 236–275, 1981.

[19] F.Riesz, Sur une inégalite intégrale, J. Lond. Math. Soc. 5, 162–168, 1930.

[20] H.L.Royden, Real Analysis, Macmillan, 1963.

[21] J.V.Ryff, Measure preserving transformations and rearrangements, J.Math. Anal. Appl. 31, 449–458, 1970.

[22] W.A.Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55, 149–162, 1977.

[23] A.Zygmund, Trigonometryc Series, vol. II, Cambridge University Press, London/New York, 1959.