Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

Homoclinic Bifurcations
And Abundance Of Strange Attractors

Thesis submitted for the degree of

"Magister Philosophiæ"

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0. INTRODUCTION

The purpose of this thesis is to present a survey of some significant results in the theory of homoclinic bifurcations and of strange attractors. The combination of these two topics is motivated by some recent work which indicates a strong connection between the two phenomena. In the course of the survey I will sketch the proof of Smale's classical homoclinic theorem and of a couple of other results. In general, however, I will limit myself to a discussion of the interelations between the various results. After some preliminaries, chapter 1 presents the essential results in the theory of homoclinic bifurcations for diffeomorphisms. This forms the core of the thesis and I go into some detail about the basic concepts developed in the theory. Chapter 2 contains a brief discussion on the notion of strange attractors and two fundamental theorems on the existence of strange attractors in particular families of diffeomorphisms. Chapter 3 is dedicated to a recent theorem of Mora and Viana which shows that the strange attractors constructed in the theorems of chapter 2 are always present when unfolding a homoclinic tangency.

My goal is to give a general idea of the kinds of results and approaches of a perticular branch of dynamical systems theory. I will not concentrate on the technical details or on the fine points of certain results, like the best estimates one can get for various parameters. Instead I will always assume the best conditions and as much differentiability as we need, taking care to emphasise the most significant aspect of the results. For this reason I will also keep to a minimum the technical definitions of which the theory of dynamical systems is full (e.g. Axiom A, strong transversality hypothesis, chain recurrent, non wandering etc.).

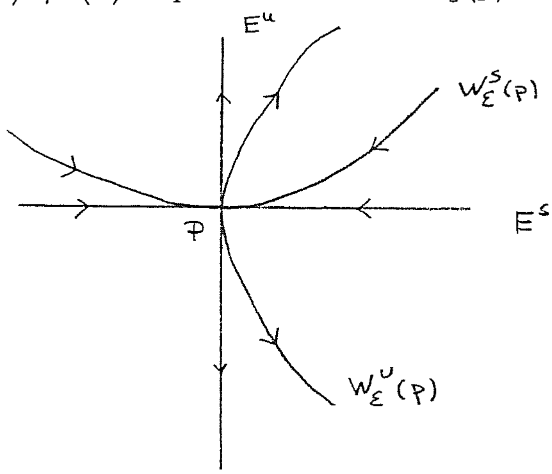
Preliminaries.

Let M be a smooth compact boundaryless surface and $\varphi : M \to M$ be a diffeomorphism of class C^3. Let p be a dissipative hyperbolic fixed saddle point of φ, i.e. $\varphi(p) = p$ and $D\varphi(p) = \begin{pmatrix} \sigma & 0 \\ 0 & \lambda \end{pmatrix}$ with $0 < \lambda < 1 < \sigma$ and $\lambda\sigma < 1$.

Then we have the fundamental:

Stable Manifold Theorem. *There exists and ε-neighbourhood of p and a C^3 curve W^s_ε containing p such that*

(1) $\varphi(W^s_\varepsilon) \subset W^s_\varepsilon$
(2) W^s_ε *is tangent to the contracting eigenspace E^s of T_pM*
(3) $\varphi^n(x) \to p$ *as* $n \to \infty \forall x \in W^s_\varepsilon(p)$.



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Applying the theorem to φ^{-1} we get a C^3 curve $W^u_\varepsilon(p)$ tangent to E^u and such that $\varphi^n(x) \to p$ as $n \to -\infty$ $\forall x \in W^u_\varepsilon(p)$. These curves are called, respectively, local stable and local unstable manifold.

The global stable manifold and the global unstable manifold can be constructed by iterating φ restricted to $W^u_\varepsilon(p)$ and φ^{-1} restricted to $W^s_\varepsilon(p)$. We get

$$W^s(p) = \bigcup_{n\geqslant 0} \varphi^n(W^u_\varepsilon)$$

$$W^u(p) = \bigcup_{n\geqslant 0} \varphi^{-n}(W^s_\varepsilon)$$

By construction $W^s(p)$ and $W^u(p)$ are C^3 immersed submanifolds; in particular they cannot self-intersect.

Dynamics in a neighbourhood of a fixed point.

The dynamics in a neighbourhood of a fixed point are fairly well understood. The Grobman-Hartman theorem [H,1964][DeP,1983] tells us that if p is a hyperbolic fixed point (i.e. no eigenvalue of the differential has modulus 1) then the dynamics in a neighbourhood of p are topologically conjugate to the dynamics induced by the differential $D\varphi$ on T_pM. Formally, there exists a homeomorphism $h : U \to V$ from a neighbourhood U of p to a neighbourhood V of the origin in T_pM which sends orbits of φ to orbits of $D\varphi$. There is also a lot of work on differentiable conjugacies [IV,1991] but the situation there is quite a bit more complicated. The differentiability class of the conjugating diffeomorphisms depends on very fine details: the ratios of the eigenvalues must satisy some non-resonance conditions related to the theory of small divisors and badly approximable irrationals.

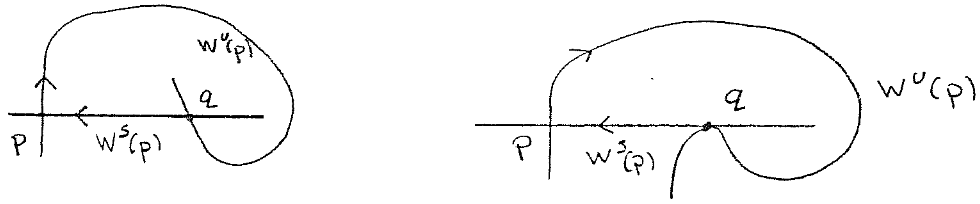
If p is a non hyperbolic fixed point, i.e. if at least one of its eigenvalues has modulus one, then it is said to be a (local) bifurcation point. For generic 1-parameter families unfolding such bifurcations there are normal forms which give us a complete description, up to topological conjugacy, of the dynamics before and after the bifurcation in a neighbourhood of p [IV,1991]. For generic k-parameter families the situation is less well understood

Remark. All the definitions and results for fixed point can easily be extended to the case of periodic points by considering the eigenvalues of $D\varphi^n$ instead of those of $D\varphi$. For simplicity I will always assume the periodic points to be fixed.

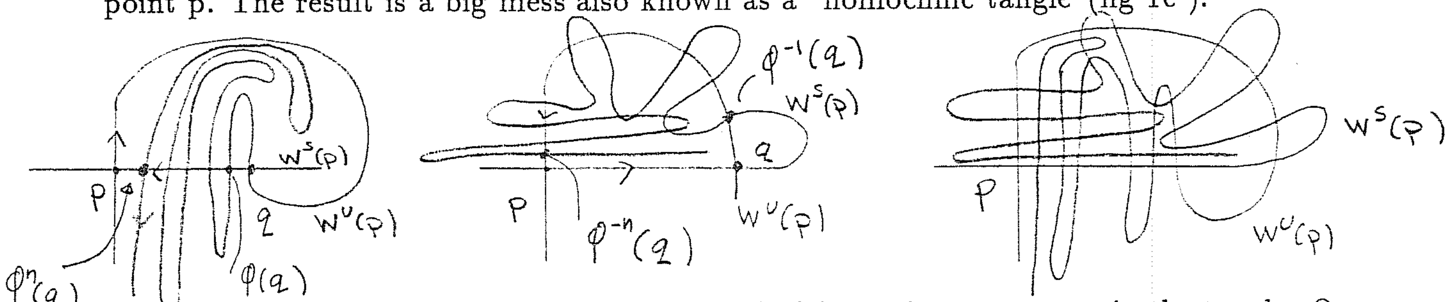
In this survey I will be interested in a more "global" kind of bifurcation which occurs when an intersection between the stable and unstable manifolds of a fixed point is created or destroyed.

1.HOMOCLINIC BIFURCATIONS

Definition. Let p be a fixed hyperbolic saddle point, and suppose that $W^s(p)$ and $W^u(p)$ intersect transversly, then we say that $q \in W^s(p) \cap W^u(p)$ is a point of *transverse homoclinic intersection*. If the intersection is not transverse then q is a point of *homoclinic tangency*.



The presence of transverse homoclinic intersections implies some very complicated dynamics. This was first noticed by Poincaré [P] and can easily be noticed by us through some simple geometric considerations. Keeping in mind that the orbit of q must tend to p both as n tends to $+\infty$ and as n tends to $-\infty$ and that $W^s(p)$ and $W^u(p)$ cannot self intersect we see (fig.1a,1b) that both manifolds accumulate on themselves near the fixed point p. The result is a big mess also known as a "homoclinic tangle"(fig 1c).



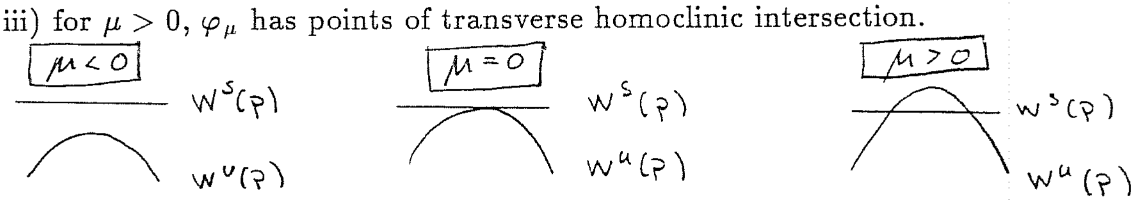
A main objective is to understand what kind of dynamics are present in the tangle. Our approach to this question will be through the study of one parameter families unfolding homoclinic tangencies.

Let $(\varphi_\mu), \mu \in I$, with I some interval containing 0, be a one parameter family of diffeomorphisms satisfying the following properties:

i) for $\mu < 0$, φ_μ has very simple stable dynamics with, say, only a finite number of periodic orbits in its limit set.

ii) φ_0 has a point of tangency between the stable and unstable manifold of some periodic point p.

iii) for $\mu > 0$, φ_μ has points of transverse homoclinic intersection.



Then the homoclinic tangency represents a "bifurcation" separating the simple dynamics of φ_μ, for $\mu < 0$ and the complicated dynamics of the homoclinic tangle. Thus homoclinic tangencies are often called *homoclinic bifurcations.*

Usually we impose some reasonable (and generic) conditions on the family (φ_μ) to make it easier to study. In particular we ask that the point of tangency be of quadratic order and that $\frac{d\varphi_\mu}{d\mu} \neq 0$ at $\mu = 0$. Under these assumptions we say that the family (φ_μ) unfolds the homoclinic tangency generically at μ=0.

Smale's horseshoe.

We begin with a landmark result of Smale [S,1967] which represents the beginning of the modern geometric approach to dynamical systems.

Theorem. *Let $\varphi : M \to M$ be a surface diffeomorphism, p a hyperbolic fixed saddle point and suppose that $W^s(p)$ and $W^u(p)$ intersect transversly. Then there exist a subset $\Lambda \subset M$ which is hyperbolic, compact, invariant, has a dense orbit and contains a dense set of periodic orbits. Moreover the map φ restricted to Λ is topologically conjugate to the full shift on two symbols.*

Definition. A compact invariant set Λ is said to be *hyperbolic* if there exists a continuous decomposition $TM = E^s \oplus E^u$ of the tangent bundle of Λ which is invariant under the action of the differential, i.e. $D\varphi(E^s) = E^s$ and $D\varphi(E^u) = E^u$ and there exists constants $C > 0$ and $\lambda > 1$ such that

$$\|D\varphi^n(x)\| \geq C\lambda^n \quad \forall x \in E^n$$

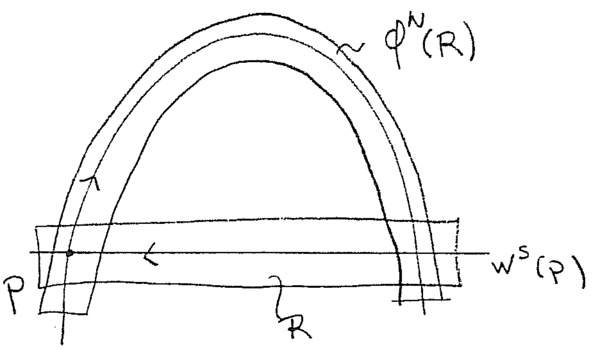
$$\|D\varphi^n(x)\| \leq C^{-1}\lambda^{-n} \quad \forall x \in E^s$$

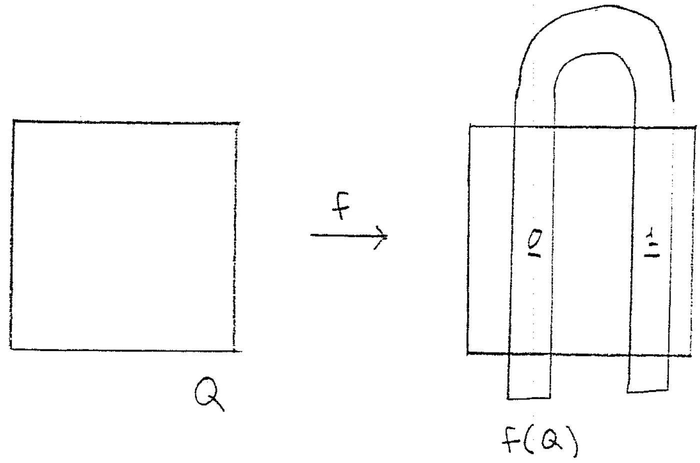
Definition. *A full shift on two symbols* is the dynamical system defined by $(\{0,1\}^{\mathbb{Z}}, \sigma)$ where $\{0,1\}^{\mathbb{Z}}$ is the space of biinfinite sequences $\underline{a} = (....a_{-1}, a_0, a_1, ...)$ with $a_i = 0, 1$ and σ is the usual shift map: $\underline{b} = \sigma\underline{a}$ is defined by $b_i = a_{i-1}$. On $\{0,1\}^{\mathbb{Z}}$ we consider the topology induced by the following metric:

$$d(\underline{a}, \underline{b}) = \sum_{i \in \mathbb{Z}} \frac{|a_i - b_i|}{2^{|i|}}$$

With this metric the (uncountable) set $\{0,1\}^{\mathbb{Z}}$ is totally disconnected and has no isolated points and is therefore homeomorphic to a Cantor set.

Sketch of proof. To achieve this result, Smale noticed that, given a transverse homoclinic intersection, one could find a thin strip in a neighbourhood of a piece of W^s which, when iterated a sufficient number of times, say N, would intersect its image in two connected components (fig.5). The action of φ^N can be schematically represented by a map f which maps a unit square Q into itself.





$Q \cap f(Q)$ is then formed by two vertical strips. It is easy to see that $Q \cap f(Q) \cap f^2(Q)$ consists of 4 vertical strips, $Q \cap f(Q) \cap f^2(Q) \cap f^3(Q)$ of 8 and so on, we get a family of nested strips. Then

$$\Lambda_+ = \bigcap_{n \geq 0} f^n(Q)$$

has the topological structure of the cartesian product of an interval and a Cantor set, $I \times C$. It is also easy to see that the inverse image of vertical strips are horizontal strips and succesive iterates of these form another family of nested strips. We have

$$\Lambda_- = \bigcap_{n \geq 0} f^{-n} \simeq C \times I$$

Now we define

$$\Lambda = \Lambda_+ \cap \Lambda_-$$

which has the topological structure of the cartesian product of two Cantor sets and which is therefore itself a Cantor set. By construction, Λ is invariant under φ. The existence of a dense subset of periodic orbits and of a dense orbit comes from the topological conjugacy of $\varphi \mid_\Lambda$ to σ and the analogous facts for the shift map. If we denote the two vertical strips by $\underline{0}$ and $\underline{1}$ and construct a sequence $\underline{x} = (....x_{-1}, x_0, x_1....)$ associated to a point $x \in \Lambda$ by the rule that

$$x_i = 0 \;\; (resp.\ 1) \longleftrightarrow f^i(x) \in \underline{0} \;\; (resp.\ \underline{1})$$

then this rule defines a bijection of the elements of $\{0,1\}^{\mathbb{Z}}$ with points of Λ. Indeed it is obvious that each point determines a unique sequence; to see that each sequence determines a unique point notice that the infinite sequence $(a_0, a_1, a_2, ...)$ determines a unique horizontal curve by the construction above and the sequence $(....a_{-3}, a_{A-2}, a_{-1})$ determines a unique vertical curve, the two lines intersect in a unique point correponding to the sequence $(...a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, ...)$. To show that this bijection is in fact a homeomorphism we can show that it sends points which are close together to points which are close together. An ε-neighbourhood of a sequence $\underline{a}$ is the set of all sequences whose middle k-terms are equal, for some k. Such sequences correspond to points contained in a small square wAich is the intersection of a vertical strip and a horizontal strip determined precisely by the first positive terms of the sequence and the first negative terms of the sequence respectively. So, we have shown that there exists a homeomorphism $H : \Lambda \to \{0,1\}^{\mathbb{Z}}$ such that the following diagram commutes:

$$(*) \qquad \begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \{0,1\}^{\mathbb{Z}} & \xrightarrow{\sigma} & \{0,1\}^{\mathbb{Z}} \end{array}$$

Notice that a periodic point of φ corresponds to a periodic sequence. Given any sequence it is possible to find periodic sequences with an arbitrary number of terms equal to the given sequence and thus arbitrarily close to it. This proves that the periodic points are dense in Λ. To see that there exists a dense orbit notice that it is possible to construct a sequence containing all possible blocks of all lengths, where an n-block is any sequence of 0's and 1's of length n. The proof of the hyperbolicity of Λ requires many technical details and I will not go into it here. It can be found in [PT,1992]. □

Remark. The set Λ constructed above is often referred to as Smale's horseshoe. In what follows I will call horseshoe any set with the characteristics of Smale's horseshoe: compact, invariant, hyperbolic containing a dense subset of periodic orbits, containing a dense orbit and having the topological structure of a Cantor set and whose dynamics are topologically conjugate to a full shift on two symbols.

Smale's theorem tells us that in a generic 1-parameter family as above, for any given value of $\mu > 0$, φ_μ will exhibit at least the dynamical complexity of a horseshoe. The mechanism by which the horseshoe is created is, however, not at all clear. There must be some local bifurcations creating or destroying periodic points and it is completely unknown how the horseshoes for different values of the parameter are related. In this direction we have the following result by Yorke and Alligood.

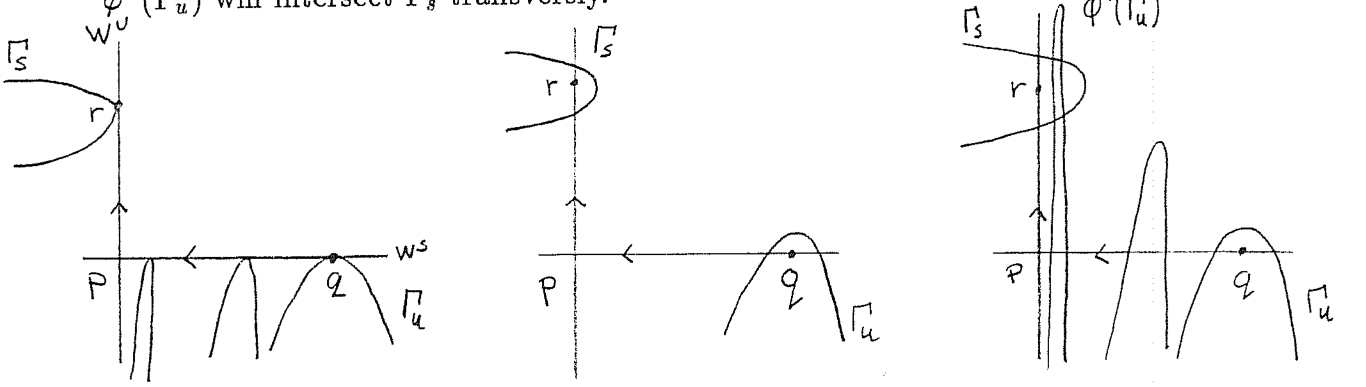
Theorem. [Y-A] *Let φ_μ be a family of diffeomorphisms unfolding generically a homoclinic tangency associated to a dissipative periodic point p. Then the creation of a horseshoe is necessarily preceeded by a cascade of period doubling bifurcations producing attracting periodic orbits of unbounded periods for different values of the parameter.*

Definition. A *cascade of period doubling bifurcations* in a one parameter family of maps is a sequence (μ_k) of parameter values, converging to some μ such that for each μ_k the corresponding map exhibits a periodic point undergoing period doubling bifurcation. A periodic point is said to undergo a period doubling bifurcation if one of the eigenvalues of the differential at the point has value -1. In that case, if the family satisfies certain generic condition, maps corresponding to values of the parameter either above or below the one at which the bifurcation occurs will exhibit periodic points with twice the period of the original point. For further details see[GH,1983] or [IY,1991].

The next result shows that diffeomorphisms exhibiting homoclinic tangencies are never isolated.

Proposition. *In a generic 1-parameter family as above, each parameter value whose corresponding diffeomorphism exhibits a homoclinic tangency is accumulated by parameter values corresponding to diffeomorphisms which also exhibit homoclinic tangencies.*

Proof. This can be seen using a simple geometric argument. We will show that $\mu = 0$ is accumulated by parameter values corresponding to diffeomorphisms exhibiting homoclinic tangencies. Given any small $\varepsilon > 0, \forall \mu \in (0, \varepsilon)$, φ_μ will have points of transverse homoclinic intersection. Call Γ_u the approximately parabolic piece of the unstable manifold near q and Γ_s the piece of the stable manifold near r. Succesive iterates of Γ_u by φ will produce other parabolas closer to p, and each time taller and more squashed (fig). For some n $\varphi^n(\Gamma_u)$ will intersect Γ_s transversly.



Since $\varphi_0^n(\Gamma_u) \cap \Gamma_s = \emptyset$ it follows that for some value of the parameter the two intersect tangentially creating a point of homoclinic tangency. Since this argument can be repaeated for arbitrarily small ε it follows that there is a sequence $n_k \to 0$ as $k \to \infty$ such that φ_{μ_k} exhibits homoclinic tangencies. It also follows from the proof that each of the parameter values n_k are themselves accumulated by other parameter values corresponding to diffeomorphisms exhibiting homoclinic tangencies.□

Each tangency in the above theorem is a *contact making* tangency in the sense that as the parameter value is increased the tangency is created and followed by a transverse intersection. We say that a homoclinic tangency is*contact breaking* if, as the parameter value is increased a transverse intersection turns into a tangency and then into an empty intersection. In conjunction with the above proposition we have the following recent result:

Bubble Lemma. [K-Y]-[D-G-K-K-Y] *If μ_o is a tangency value at which contact is made, then there are tangency values arbitrarily close to μ_0 at which contact is broken (and vice versa).*

Persistent tangencies.

So far we have implicitly assumed the fixed or periodic point to which the homoclinic tangency is associated to be isolated. However all the results discussed so far are valid if it belongs to a larger invariant set, in particular a "Smale horseshoe" type set. In 1979 Newhouse discovered that such a situation in fact added an entirely new level of complexity to the dynamics making it possible to obtain "persistent tangencies" and "infinitely many coexisiting periodic attractors". First we need a couple more definitions. For a hyperbolic set like the horseshoe it is possible to generalize the notion of stable and unstable manifolds for non periodic points.

Definition. Let $q \in \Lambda$, we define

$$W^s(q) = \{y \in M : d(\varphi^n(y), \varphi^n(q)) \to 0 \ as \ n \to \infty\}$$

$$W^u(q) = \{y \in m : d(\varphi^n(y), \varphi^n(q)) \to 0 \ as \ n \to -\infty\}$$

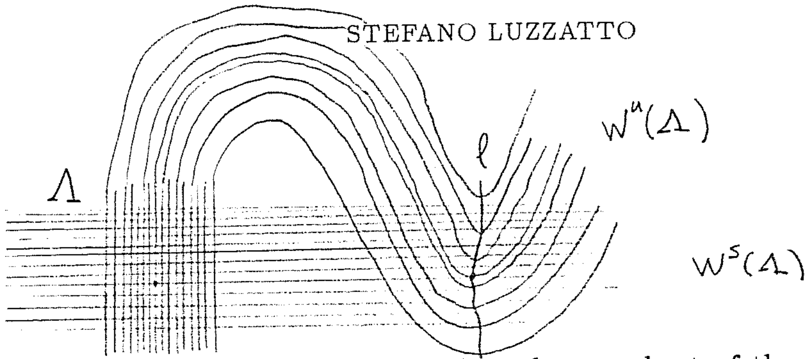
Notice that this definition is consistent with the previous one if q is periodic. Now we can define the stable and unstable sets of Λ:

Definition.

$$W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$$

$$W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x)$$

Recall that the topological structure of Λ is that of the product of two Cantor sets so the stable and unstable sets will be formed, at least locally, by a Cantor set of lines (curves)(fig.7).



Since the periodic points are dense in Λ, a dense subset of these lines will correspond to the stable and unstable manifolds of periodic points. Now consider a homoclinic intersection associated to one of these periodic Apoints. As can be seen from the figure this implies many other transverse intersections and possibly other tangencies as well between $W^s(\Lambda)$ and $W^u(\Lambda)$. These tangencies could occur along a curve ℓ which, essentially, cuts a cross section of the stable and unstable sets of Λ. Both $W^s(\Lambda)$ and $W^u(\Lambda)$ intersect ℓ in Cantor sets C^s and C^u. Thus the stable and unBstable sets of Λ will have a tangency if and only if $C^s \cap C^u \neq \emptyset$.

Newhouse tackled this problem by considering the general case of the intersection of two Cantor sets in a line. He defined a characteristic of Cantor sets called *thickness* and showed that if two Cantor sets are thick enough and if neither one is completely contained in a gap of the other then they must intersect. For completeness I give here the precise statement of this result.

Definition. Let $C \subset \mathbb{R}$ be a Cantor set. Let the connected components of $\mathbb{R} \setminus C$ be called gaps of C. Then each point $c \in C$ will be in the boundary of some gap G (see fig.). There exists a unique point $c' \in C$ such that the interval between c and c' contains no gaps of length greater than or equal to the length of G. We call the distance between c and c', D. We define the *local thickness of C at c* as

$$\tau_c(C) = \frac{\textit{length of } D}{\textit{length of } G}$$

and the *thickness of C* as

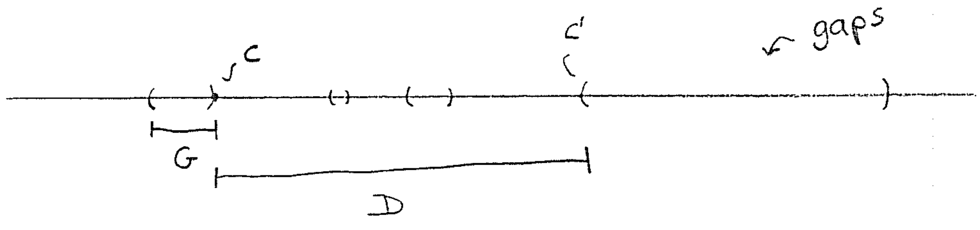
$$\tau(C) = \inf_{c \in C}\{\tau_c(C)\}$$

We have the

Gap Lemma. *Let $C_1, C_2 \subset \mathbb{R}$ be two linked Cantor sets (neither one contained in a gap of the other)If $\tau(C_1)\tau(C_2) > 1$ then $C_1 \cap C_2 \neq \emptyset$*

Notice that this lemma says that the intersection is *persistent*: we can translate one of the Cantor sets along $\mathbb{R}$ for a whole interval of values and they will continue to intersect.

Applying this result to our original problem on homoclinic tangencies we get a condition on the thickness of the cross sections of the stable and unstable sets of Λ which will guarantee that there are whole intervals of parameter values for which the stable and unstable sets of Λ have points of tangential intersections. Since a dense set of the leaves of $W^s(\Lambda)$ and $W^u(\Lambda)$ are associated to periodic points, a dense subset of this interval of parameter values will correspond to diffeomorphisms with homoclinic tangencies associated to periodic points and to which, therefore, all the previous results apply.



Newhouse also showed that when unfolding generically a homoclinic tangency there are parameter values, arbitararily close to the one for which the tangency occurs, for which the corresponding diffeomorphism has a periodic attractor. Combining this fact with the persistent tangencies he obtained residual subsets of the intervals of persistent tangencies discussed above, such that for parameter values in these residual subsets the correponding diffeonorphisms have infinitely many periodic attractors. Finally, to crown this series of spectacular results he showed that arbitrarily near a diffeomorphism exhibiting a homoclinic tangency (even if this tangency is associated to an isolated periodic point) there are diffeomorphisms exhibiting homoclinic tangencies associated to periodic points contained in a horseshoe of large thickness. This implies that arbitrarily near a diffeomorphism exhibiting a homoclinic tangency there are others exhibiting persistent tangencies and infinitely many periodic attractors. These results are summarized in:

Theorem. [N,1979][PT,1992] *Let φ_μ be a generic family of C^3 diffeomorphisms unfolding a homoclinic tangancy at $\mu = 0$. Then, given $\varepsilon > 0$ there exist intervals $N_i \subset (0, \varepsilon)$ in the parameter space such that each N_i has a dense set of parameters corresponding to diffeomorphisms exhibiting homoclinic tangencies associated to periodic orbits. Moreover, there are residual subsets $R_i \subset N_i$ corresponding to diffeomorphisms exhibiting infinitely many coexisting periodic attractors or repellors.*

Remark. It is an open problem whether an analogous result is valid for conservative systems substituting elliptic points for attractors.

Newhouse's result created a small shockwave. In the first place it cast doubts on the claims of people who were performing computer studies of various maps and who seemed to be finding non periodic "strange" attractors, like Lorenz [L,1963] and Henon [H,1976] for example. The existence of infinitely many periodic attractors necessarily implies that some of them have very high period and an orbit of very high period might easily be mistaken, in a computer study, for a non periodic attractor. If the phenomena was so common as Newhouse's theorem appeared to imply then there seemed to be a high probability that the map under investigation might indeed exhibit such periodic attractors of very high period. This reaction was partly a product of a general "philosophy" which guided research in dynamical systems theory throughout the seventies. This was the belief that the most significant and important systems to study are the *structurally stable* ones, i.e. those which are stable under small perturbations. Amongst the reasons underlying this point of view was the idea that physical systems, or at least the asymptotic behaviour o physical systems, would necessarily be structurally stable. Consistently with this *topological* notion of stability there was an accepted *topological* notion of persistence, in the sense that an open and dense subset was considered to be a huge set, pretty much as large as one could hope for, and if not open and dense, residual (countable intersection of open and dense subsets) came second best. So, a lot of research in the seventies aimed at characterizing the dynamical behaviour of open and dense or residual subsets of suitable spaces of diffoemorphisms or vector fields.

Newhouse's theorem fits perfectly into this canon. However the difficulties encountered in dealing with his results contributed to a shift in perspective which brought a different notion of persistency to be accepted and used, that of *measure theoretical persistency.* We say that a particular dynamical system is measure thoeretically persistent if it occurs for a set of parameters of positive measure in some k-parameter family. Of course if a phenomenaon occurs for an open and dense subset of parameter space it clearly occurs for a set of positive measure however it can also happen that *the complement of an open and dense subset has positive measure.* A typical case is that of a Cantor set of positive measure in an interval. In fact it is possible to have Cantor sets of arbitrarily large measure in the interval; their complement, of course, is always open and dense. Dynamical phenomena occuring for parameter values in such a Cantor set will never be structurally stable but they will occur with very large probability and so we say that they are measure theoretically persistent.

According to this new point of view then, it is interesting to estimate the measure of the set of parameter values for which infinitely many sinks coexist. There are some partial result in this direction which seem to support the conjecture that it has measure zero, but this is still an open problem. However there are some recent and relevant results on the relative measure of the intervals of hyperbolicity and those of persistent tangencies. We say that a diffeomorphism is hyperbolic if its nonwandering set is hyperbolic as a compact invariant set according to the definition above. Intuitively a hyperbolic diffeomorphism is one which is not bifurcating, i.e. which exhibits no local bifurcation orbits nor homoclinic tangencies.

Let φ_μ be a generic family of diffeomorphisms unfolding a homoclinic tangency associated to a periodic point p belonging to a horseshoe Λ. Let $HD(\Lambda)$ denote the Hausdorff dimension of Λ and define the set

$$A = \{\mu \in [-\varepsilon, \varepsilon] : \varphi_\mu \text{ is not hyperbolic}\}$$

Then we have the following results:

Theorem. [PT,1987]
If

$$HD(\Lambda) < 1$$

then

$$\lim_{\varepsilon \to 0} \frac{A \cap [-\varepsilon, \varepsilon]}{2\varepsilon} = 0$$

0 is a point of full density of hyperbolicity !

Theorem. [PY,1991] *If*

$$HD(\Lambda) > 1$$

then

$$\liminf_{\varepsilon \to 0} \frac{A \cap [-\varepsilon, \varepsilon]}{2\varepsilon} > 0$$

0 is not a point of full density of hyperbolicity !

2.STRANGE ATTRACTORS

The term *strange attractor* was coined by Ruelle and Takens in 1971 [RT,1971] to indicate, in a loose way, the concept of a non-periodic attractor with some curious properties. There is still no universally accepted meaning for it. Here we adopt the following working definitions:

Definition. A compact invariant set Λ is called an *attractor* if there exists a set of positive Lebesgue measure U such that:

$$d(\varphi^n(U), \Lambda) \to 0 \quad as \quad n \to \infty$$

and if Λ is minimal in this respect, i.e. it does not contain any proper subsets which are also attractors.

Definition. An attractor Λ is a *strange attractor* if it contains a point z which has a dense orbit and positive Lyapunov exponent, i.e. there exist constants $C > o$ and $\lambda > 1$ such that

$$||D\varphi^n(z)|| > C\lambda^n \quad \forall n \geqslant 0$$

Remarks.

1) The definition of attractor is consistent with fixed or periodic attracting orbits. The positive Lyapunov exponent indicates a kind of expansivity on and near the attractor. Points get closer and closer to the attractor, but the closer they get the more influence the expansivity of the attractor has and nearby points tend to get pulled apart producing the characteristic sensitivity to initial conditions which is noticeable in computer studies.

2) It is desirable to define on the attractor an invariant measure absolutely continuous with respect to Lebesue measure. This is in general difficult but it could be that the existence of such a measure might become a requirement in the definition of some class of attractors.

3) So far, all the strange attractors we know are structurally unstable but measure theoretically persistent. It does not seem unreasonable to wonder whether these characteristic are in some sense intrinsic and intimately related to the dynamical structure of strange attractors.

The most well known strange attractors occur in 1-dimensional quadratic maps. Already in 1981, Jakobson proved the following:

Theorem. [J,1981] *Let*

$$P_a(x) = x^2 + a \quad a \in \mathbb{R}$$

For $a \leq \frac{1}{4}$ let β_a denote the largest fixed point of P_a. Let $1 < \lambda < 2$, $\delta > 0$ Then there exists an $a_0 \in (-2, \frac{1}{4})$ and a set $A \subset [-2, a_0]$ such that

(1) $m(A) \geqslant (1-\delta)m[-2, a_0]$

(2) *for $a \in A$, we have: the interval $[-\beta_a, \beta_a]$ is a strange attractor with a uniqe invariant ergodic measure absolutely continuous with respect to Lebesgue measure.*

We also have the following very recent result of Swiątek:

Theorem. [S] *Let $P_a(x) = x^2 + a$ be the quadratic family as in the previous theorem. Then the hyperbolic (structurally stable) maps form and open and dense subset of the parameter space*

This is exactly the kind of situation mentioned above: hyperbolic behaviour in an open and dense subset with positive measure in the complement. It is an interesting and open question whether strange attractors form a set of full measure in the complement of hyperbolicity.

In 1989, Benedicks and Carleson proved, using completely different methods, a generalization of Jakobson's result to a special family of plane diffeomorphisms: The Henon family. These maps are defined by:

$$P_{a,b}(x,y) = (x^2 + a + y, bx)$$

Henon studied this family numerically for values of (a,b) around $(1.4, 0.3)$ and conjectured the existence of a strange attractor. The result of Benedicks and Carleson concerns the dynamics for parameter values near $(a,b) = (-2,0)$ under condition of extremely strong dissipativeness. Notice first that for $b = 0$ the map degenerates into the non invertible quadratic family $P_a(x,0) = (x^2 + a, 0)$ on the real axis for which we have Jakobson's result. Essentially, Benedicks and Carleson proved the theorem in the one dimensional case using a method which allowed them to prove an analogous theorem for small perturbations of the degenerate case. We have the following:

Theorem. [B-C] *Let $1 < \lambda < 2$ and $\delta > 0$. There exists an $a_o \in [-2,0]$ and a $b_0 > 0$ and, for each $0 < b < b_0$ a set $A_b \subset [-2, a_0]$ with the following properties:*

(1) $m(A_b) \geqslant (1-\delta)m(-2, a_0)$
(2) *For $0 < b < b_o$, $a_0 \in A_b$, denote by $\Lambda = \Lambda_{a,b}$ the closure of the unstable manifold of the hyperbolic fixed point of $P_{a,b}$, then Λ is a strange attractor.*

It has been announced by Benedicks and Young that they have constructed absolutely continuous invariant measures for these attractors[BY,1991].

3.STRANGE ATTRACTORS WHEN UNFOLDING HOMOCLINIC TANGENCIES

In 1985, while Benedicks' and Carleson's work was already in development, Palis suggested an extension of their method to more general perturbations of the quadratic family on the real line. One could then use the fact that a generic family unfolding a homoclinic tangency admits a renormalization which is Hénon-like to apply the results on the existence of strange attractors to unfoldings of homoclinic tangencies.

The Heénon-like renormalization of a generic family unfolding a homoclinic tangency is guaranteed by the following result:

Theorem. [PT,1992] *Let* $(\varphi_m u$ *be a generic one parameter family as above with* q *a point on the orbit of tangency for* $\mu = 0$. *Then there exists a constant* N *and, for each positive integer* n, *reparametrizations* $/mu = M_n(\tilde{\mu})$ *of the parameter and* $\tilde{\mu}$ *dependent coordinate transformations*

$$\Psi_{n,\tilde{\mu}}(\tilde{x}, \tilde{y}) = (x, y)$$

such that

(1) *for each compact set* K *in the* $\tilde{\mu}, \tilde{x}, \tilde{y}$ *space, the images of* K *under the maps*

$$(\tilde{\mu}, \tilde{x}, \tilde{y}) \to (M_n(\tilde{\mu}), \Psi_{n,\tilde{\mu}}(\tilde{x}, \tilde{y}))$$

converge, for $n \to \infty$ *in the* $(\mu, x, y,)$ *space to* $(\mu, 0, q)$

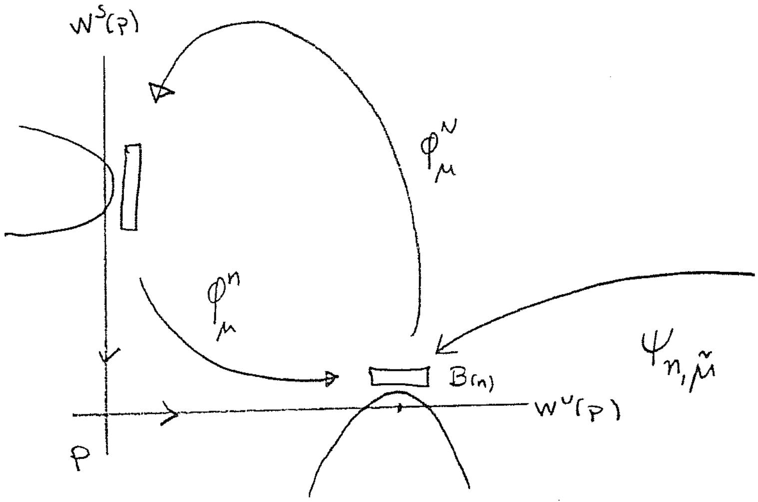
(2) *the domains of the maps*

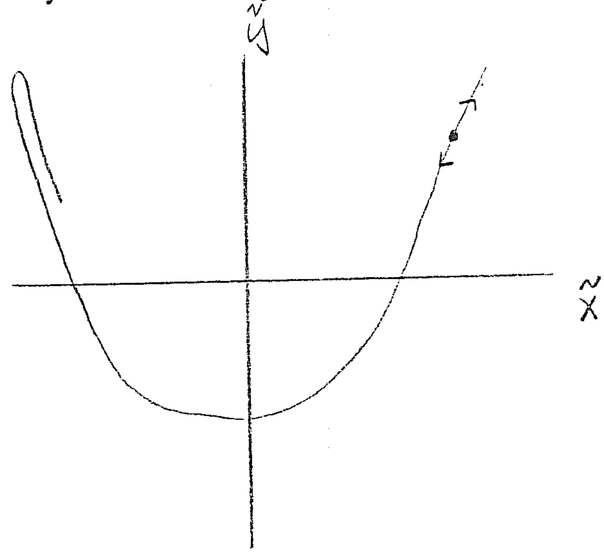
$$(\tilde{\mu}, \tilde{x}, \tilde{y}) \to (\tilde{\mu}, (\Psi^{-1}_{n,\tilde{\mu}} \circ \varphi^{n+N}_{M_n(\tilde{\mu})} \circ \Psi_{n,\tilde{\mu}}))$$

converge, for $n \to \infty$ *to all of* $\mathbb{R}^3$ *and the maps converge in the* C^2 *topology to the map*

$$(\tilde{\mu}, \tilde{x}, \tilde{y}) \to (\tilde{\mu}, \tilde{y}, \tilde{y}^2 + \tilde{\mu})$$

Essentially, this theorem says that φ restricted to a *very small domain* near the orbit of tangency and *for a very high number of iterates* is very close to the one dimensional quadratic family (fig 8).





The program proposed by Palis was carried out in detail by Mora and Viana who proved the following:

Theorem. [MV,1992] *Let φ_μ be a C^∞ generic one-parameter family of diffeomorphisms on a surface and suppose that φ_0 has a homoclinic tangency associated to some periodic point p_0. Then, under generic assumptions, there is a positive Lebesgue measure set E of parameter values near $\mu = 0$ such that for $\mu \in E$, $\varphi_m u$ exhibits a strange attractor, or repellor, near the orbit of tangency.*

Further results, open problems and conjectures.

From the results presented above it appears that homoclinic bifurcations are a main source of rich and complicated behaviour. Indeed their presence implies the "nearby" presence of most of the "complicated" dynamical phenomena which we know today: e.g. infinitely many sinks, Hénon-like strange attractors and the so-called Feigenbaum attractors -accumulation points of cascades of period doubling bifurcations-. A most significant question is whether the converse is true:

Question. [PT,1992] *Do infinitely many sinks, Hénon-like strange attractors and Feigenbaum attractors imply the existence "nearby" of diffeomorphisms exhibiting homoclinic tangencies ?*

There are very few results in this direction. A partial result due to Ures[U,1992], states that *for the Hénon-like strange attractors constructed in the proof of Mora-Viana*, the question above has an affirmative answer.

Palis has proposed the following

Conjecture. *The set of all hyperbolic diffeomorphisms and all diffeomorphisms exhibiting homoclinic tangencies is dense in $Diff^k(M)$, the space of all C^k diffeomorphisms.*

An affirmative answer to the first question above would be a big step in the direction of proving this conjecture. Araujo and Mañe have proved the conjecture for C^1 diffeomorphisms in the C^2 topology.

Other interesting questions concern the *relative density of strange attractors* at $\mu = 0$. By the result of Palis and Takens above on the relative density of hyperbolicity, strange attractors cannot have positive density in general. However we can ask whether, in a generic family as above, $\mu = 0$ can be a point of positive density of strange attractors if $HD(\Lambda_0) > 1$ where Λ_0 is the horsehoe containing the periodic point to which the tangency is associated. The same question can be asked for infinitely many sinks. Moreover it is conceivable that there might exist diffeomorphism exhibiting infinitely many coexisting strange attractors. Palis has proposed the following:

Conjecture. *The set of parameters for which φ_μ exhibits infinitely many coexisting sinks or infinitely many coexisting Hénon-like strange attractors has Lebesgue measure 0*

However, Diaz, Rocha and Viana [DRV,1992] have recently found parameter values of surface diffeomorphism which are points of *positive density* of Hénon-like strange attractors.

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