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Dirac's Constrained Systems: The Classification of Second-Class Constraints

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1 Introduction

Gauge theories play nowadays a very important role in modern theoretical physics and in particular in elementary particle physics. It is well known that gauge theories belong to the class of the so-called singular Lagrangian theories. A Lagrangian is called singular when the determinant of its Hessian matrix vanishes. In the Hamiltonian formalism the singular Lagrangian theories have constraints.

In [1] the classification of the constraints of a Hamiltonian system as being first- and second-class was made. A first-class constraint is defined as one having vanishing, in the constraint hypersurface, Poisson brackets with all the constraints of the system. Otherwise a constraint is called second-class.

The standard quantization methods cannot be applied to the singular Lagrangian theories directly due to the lack of a well defined theory of physical observables. One line of research concerning the gauge transformations and the theory of physical observables in the Lagrangian and Hamiltonian formalisms was developed in [1]-[14]

The basic ideas of the canonical quantization of singular Lagrangian theories were given in [1]. A development of the Feynman path integral quantization approach for systems with first-class constraints in canonical gauges was presented in [2]. In [15, 16] a generalization of this formalism to systems with second-class constraints was found. The Faddeev's procedure was improved in [17] to include covariant gauges. In [18] this relativistic version was presented in a compact form based on the BRST symmetry [19, 20].

The BRST-BFV quantization is considered at present to be the most general quantization method of systems with first-class constraints. Extensions of this approach to the case of systems having second-class constraints are given in [21]-[26].

The modern approach to quantization of gauge theories makes use of the BRST-BFV method in order to save the explicit covariance and remain within the scope of renormalizable field theories. However, the BRST-BFV approach can be criticized from several points of view:

1. The BRST-BFV method is based on the extended Hamiltonian formalism. The total Hamiltonian and the extended Hamiltonian formalisms are usually regarded as being equivalent but it can be shown that in general they are not [27, 28]. The consistency conditions on the gauge

fixings functions depend on which Hamiltonian is used [29]. Gauge degrees of freedom in the Lagrangian formalism are equal in number to the primary first-class constraints, therefore, the number of arbitrarily chosen primary gauge fixing conditions is the same as that of primary first-class constraints. A procedure for determining all the gauge fixing conditions within the scope of the total Hamiltonian approach is given in [29]. The discrepancy between the total Hamiltonian and extended Hamiltonian formalisms is discussed and it is pointed out that this discrepancy could be the reason why for some models the Faddeev's path integral approach does not reproduce the correct transition amplitude and the BRST-BFV quantization is gauge dependent [30, 31].

2. Point 1 is closely related with the search of the Dirac's physical observables through the Shanmugadhasan canonical transformation [13, 32]. The local canonical bases of observables are in general nonlinear and nonlocal in the original variables. As a result one has a probability of obtaining inequivalent quantum theories if one quantizes the observables instead of quantizing first the original variables and then making the reduction to the physical degrees of freedom at the quantum level [33]-[35]. In addition, in field theory one finds that in general the physical Hamiltonian is non polynomial in the physical degrees of freedom. Power counting methods cannot be used when looking for regularizations and renormalizations of the theory.

3. In general the Shanmugadhasan canonical transformation can be done only locally, therefore, very complicated Dirac's brackets are obtained when the gauge fixing conditions are added to a system with first-class constraints. The quantization method becomes very ambiguous and in principle geometrical methods are needed. This is related with such phenomena like the Gribov ambiguity.

4. In gauge field theories the situation is more complicated because the theorems ensuring the existence of the Shanmugadhasan canonical transformation have not been extended to the infinite-dimensional case. One of the reasons is that some of the constraints can now be interpreted as elliptic equations and they can have zero modes. The problem of the zero modes will appear as a singularity structure of the gauge foliation

of the allowed sector. It is not clear how the BRST-BFV method can take these singularities into account [33, 34]

We think that a better understanding of the classical theory of singular Lagrangians in the finite-dimensional case would clarify many unsolved problems of the modern quantization methods. Motivated by these problems and having in mind, in particular, the possible extensions of the BRST-BFV approach to systems with second-class constraints we will study here the classification of the second-class constraints in the context of the total Hamiltonian approach. In this approach there exists a clear distinction between primary and secondary, ternary, etc, constraints. The primary constraints arise only from the definition of the momenta, while the secondary, ternary, etc, constraints are direct consequences of the equations of motion.

Our classification of second-class constraints is based on a previous work [10] in which the separation property between first- and second-class constraints was demonstrated for a very wide class of systems. The main requirement for the validity of this separation property is the effectiveness [10, 36] of the constraints obtained in the Dirac's algorithm.

We will see how the maximal partition of the set of second-class constraints is achieved in the sense that each second-class constraint has weakly vanishing Poisson brackets with all the constraints of the system except with one. The chains of second-class constraints will be classified and its Poisson bracket structure will be represented graphically in a very simple way.

This work consists of four sections and three appendices. In Section 2 a review of the construction of a convenient representation [10] of the constraints is presented. In this representation the separation property between first- and second-class constraints is proven. Some new results not given in [10] are presented in this Section. In Section 3 the classification of the second-class constraints is obtained by analysing the simplest cases. Section 4 is devoted to conclusions. In it the generic structure of the second-class constraint set is investigated. In the appendices the details of some derivations used in Sections 2 and 3 are given.

2 Separation property between first and second-class constraints

We will devote this section to review the construction of a convenient representation of the Hamiltonian constraints. This representation was already given in [10], and it turned out to be very useful in proving the so-called Dirac's conjecture for a very wide class of systems. In this article we will use this representation as a basis for a more full analysis of the structure of the second class constraint set. Some new results not given in [10] will be also presented in this section.

We define the total Hamiltonian [1] of the system as:

$$H_T(q,p,\lambda) = H_c(q,p) + \lambda_\mu \Phi^{(0)}_\mu(q,p), \tag{1}$$

where H_c is any function of (q,p) satisfying,

$$H_c(q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q})) = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q,\dot{q}), \tag{2}$$

$\Phi^{(0)}_\mu$ are the primary constraints and λ_μ the corresponding Lagrangian multipliers.

The Dirac-Hamiltonian equations of motion which are equivalent to the Euler-Lagrange equations can be written as follows [1]:

$$\dot{q}_i = \{q_i, H_T\}, \tag{3}$$

$$\dot{p}_i = \{p_i, H_T\}, \tag{4}$$

$$\Phi^{(0)}_\mu(q,p) = 0, \tag{5}$$

$$i = 1, \ldots, n$$

$$\mu = 1, \ldots, m_1$$

Applying the consistency conditions to the primary constraints we can obtain either new constraints(called secondary) or canonical expressions for some of the Lagrangian multipliers, or some identities. Then we apply the consistency conditions to the secondary constraints and so on. This algorithm is very well known [1, 28] and we will not develop it here again.

Following the procedure of [28] it is possible to find the complete set of constraints in the form:

$$
\begin{array}{cccccccl}
\Phi^{(0)}_{\mu'_0} & \Phi^{(0)}_{\mu'_1} & \Phi^{(0)}_{\mu'_2} & \dots & \Phi^{(0)}_{\mu'_f} & \Phi^{(0)}_{\mu_f} & \text{–primary constraints} \\
& \Phi^{(1)}_{\mu'_1} & \Phi^{(1)}_{\mu'_2} & \dots & \Phi^{(1)}_{\mu'_f} & \Phi^{(1)}_{\mu_f} & \text{–secondary constraints} \\
& & \Phi^{(2)}_{\mu'_2} & \dots & \Phi^{(2)}_{\mu'_f} & \Phi^{(2)}_{\mu_f} & \text{–ternary constraints} \\
& & & \dots & \dots & \dots & \\
& & & & \Phi^{(f)}_{\mu'_f} & \Phi^{(f)}_{\mu_f} & \text{–f+1-ary constraints}
\end{array} \tag{6}
$$

Let us denote as $\Phi^{(i)}_{\mu_{i-1}}$ all the i+1-ary constraints. The $\Phi^{(i)}_{\mu_i}$ are all the i+1-ary constraints except $\Phi^{(i)}_{\mu'_i}$. In general, $\Phi^{(i)}_{\mu_j}$ $(j \geq i)$ are all the i+1-ary constraints except $\Phi^{(i)}_{\mu'_i}, \dots, \Phi^{(i)}_{\mu'_j}$.

The submanifold defined in phase space by the annulation of all the constraints up to superindex k will be denoted as M_k.

The constraints (6) are defined as:

$$
\Phi^{(k)}_{\mu_{k-1}} = \{\Phi^{(k-1)}_{\mu_{k-1}}, H^{(k)}_c\} \qquad k = 1, 2, \dots, f \tag{7}
$$

where the functions $H^{(k)}_c$ are defined iteratively through the expression:

$$
H^{(k)}_c = H^{(k-1)}_c + \sigma_{\mu'_{k-1}}(q,p)\Phi^{(0)}_{\mu'_{k-1}} \qquad k = 1, \dots, f+1 \tag{8}
$$

$H^{(0)}_c \equiv H_c$ and $\sigma_{\mu'_k}$ are given by

$$
\sigma_{\mu'_k}(q,p) = (-1)^k (C^{(k)})^{-1}_{\nu'_k \mu'_k} \{H^{(k)}_c, \Phi^{(k)}_{\nu'_k}\} \tag{9}
$$

The matrix $C^{(k)}$ is defined as

$$
C^{(k)}_{\mu'_k \nu'_k} = \{\Phi^{(k)}_{\mu'_k}, \Phi^{(0)}_{\nu'_k}\} \tag{10}
$$

The functions $\sigma_{\mu'_k}$ coincide with the Lagrange multipliers $\lambda_{\mu'_k}$ $(k = 0, \dots, f)$ on the submanifold M_k ,and therefore, also on the final submanifold M_f. The Lagrangian multipliers λ_{μ_f} remain completely undetermined.

The total Hamiltonian [1] may be rewritten as

$$H_T = H' + \lambda_{\mu_f} \Phi^{(0)}_{\mu_f}, \tag{11}$$

where $H' \equiv H_c^{(f+1)}$.

As it was shown in [1] the Hamiltonian H' is a first-class function in M_f.

The following Poisson-brackets (PB) relations were obtained in [28]:

$$\{\Phi^{(i)}_{\mu'_i}, H_c^{(i+1)}\} \underset{M_i}{=} 0 \tag{12}$$

$$i = 0, 1, \ldots, f$$

$$\{\Phi^{(f)}_{\mu_f}, H_c^{(f+1)}\} \underset{M_f}{=} 0 \tag{13}$$

and

$$\{\Phi^{(k)}_{\mu_k}, \Phi^{(0)}_{\mu_{k-1}}\} \underset{M_k}{=} 0 \tag{14}$$

$$\{\Phi^{(k)}_{\mu_{k-1}}, \Phi^{(0)}_{\mu_k}\} \underset{M_k}{=} 0 \tag{15}$$

$$\det |\{\Phi^{(k)}_{\mu'_k}, \Phi^{(0)}_{\nu'_k}\}| = \det |C^{(k)}_{\mu'_k \nu'_k}| \underset{M_k}{\neq} 0 \tag{16}$$

$$k = 0, 1, \ldots, f$$

It is assumed that the determinants of all the $C^{(k)}$ matrices retain their property of being different from zero through all the stages of the Dirac algorithm.

The relations (14-16) refer to the PB of primary, secondary, ternary,..., constraints with only the primary ones. This information is not enough to carry out the full analysis of the structure of the Hamiltonian constraints.

In [10] it was constructed a new representation of the constraints which is most suitable for the studying of the PB among all the constraints. As a result of this construction it was possible to prove the separation property between the first- and second-class constraints.

The new collection of constraints [10] is defined iteratively as:

$$\chi_\mu^{(0)} \equiv \Phi_\mu^{(0)} \tag{17}$$

$$\chi_\mu^{(k)} \equiv \{\chi_\mu^{(k-1)}, H'\} \tag{18}$$

$$k = 1, \ldots$$

It was shown that the new set of constraints has also the structure:

$$\begin{array}{cccccccl}
\chi_{\mu_0'}^{(0)} & \chi_{\mu_1'}^{(0)} & \chi_{\mu_2'}^{(0)} & \ldots & \chi_{\mu_f'}^{(0)} & \chi_{\mu_f}^{(0)} & \text{–primary constraints} \\
& \chi_{\mu_1'}^{(1)} & \chi_{\mu_2'}^{(1)} & \ldots & \chi_{\mu_f'}^{(1)} & \chi_{\mu_f}^{(1)} & \text{–secondary constraints} \\
& & \chi_{\mu_2'}^{(2)} & \ldots & \chi_{\mu_f'}^{(2)} & \chi_{\mu_f}^{(2)} & \text{–ternary constraints} \\
& & & \ldots & \ldots & \ldots & \\
& & & & \chi_{\mu_f'}^{(f)} & \chi_{\mu_f}^{(f)} & \text{–f+1-ary constraints}
\end{array} \tag{19}$$

The set of constraints $\chi_{\mu_{-1}}^{(0)}, \chi_{\mu_0}^{(1)}, \ldots, \chi_{\mu_{k-1}}^{(k)}$ defines the submanifold M_k in phase space. The complete set of constraints $\chi_{\mu_{-1}}^{(0)}, \chi_{\mu_0}^{(1)}, \ldots, \chi_{\mu_{f-1}}^{(f)}$ defines the final submanifold M_f. This means that both sets Φ and χ are equivalent.

We see also that the constraint set has a chain structure. Each chain is generated by the corresponding primary constraint. The number of chains coincides, therefore, with the number of primary constraints. It may happen that a chain consists of only one (primary) constraint. We will call such a chain a 0-chain. The 1-chains are those formed by two constraints (one primary and one secondary). In general, we call a chain formed by k+1 constraints (one primary, one secondary,..., one k+1 -ary) a k-chain. We assume that the set of constraints obtained in this algorithm is effective in the exact functional way that they appear from the consistency conditions. A set of effective constraints is defined [10] as one in which all the constraints have nonvanishing linearly independent gradients at every point of M_f.

The constraints χ satisfy [10] also the following PB relations:

$$\{\chi^{(k)}_{\mu'_k}, H'\} \underset{M_k}{=} 0 \tag{20}$$

$$\{\chi^{(f)}_{\mu_f}, H'\} \underset{M_f}{=} 0 \tag{21}$$

$$\{\chi^{(k)}_{\mu_k}, \chi^{(0)}_{\mu_{k-1}}\} \underset{M_k}{=} 0 \tag{22}$$

$$\{\chi^{(k)}_{\mu_{k-1}}, \chi^{(0)}_{\mu_k}\} \underset{M_k}{=} 0 \tag{23}$$

and

$$\det|\{\chi^{(k)}_{\mu'_k}, \chi^{(0)}_{\nu'_k}\}| = \det|C^{(k)}_{\mu'_k\nu'_k}| \underset{M_k}{\neq} 0 \tag{24}$$

$$k = 0, 1, \ldots, f$$

We denote as C^k the matrix:

$$C^{(k)}_{\mu'_k\nu'_k} = \{\chi^{(k)}_{\mu'_k}, \chi^{(0)}_{\nu'_k}\} \tag{25}$$

A theorem was enunciated in [10] which allows us to prove the separation property between first- and second-class constraints. This theorem states as follows:

Theorem. All the constraints $\chi^{(k)}_{\mu_{k+i-1}}$ $(k = 0, 1, \ldots, f)$ PB commute with all the constraints $\chi^{(i)}_{\mu_{i-1}}$ except the $\chi^{(k)}_{\mu'_{k+i}}$ and $\chi^{(i)}_{\mu'_{k+i}}$ which obey,

$$\det|\{\chi^{(k)}_{\mu'_{k+i}}, \chi^{(i)}_{\mu'_{k+i}}\}| \underset{M_{k+i}}{\neq} 0 \tag{26}$$

where $i \leq k$, and $i \leq f - k$.[1]

[1] Note that a mistake was made at this point in the enunciation of this theorem in [10]. However, the proof of this theorem and its corollaries given in [10] are correct.

In addition,

$$\{\chi^{(k)}_{\mu'_{k+s}},\chi^{(i)}_{\mu_{k+i-1}}\} \underset{M_{k+i}}{=} 0 \tag{27}$$
$$s=0,\ldots,i-1$$

On the basis of this theorem it can be shown [10] that, in fact, all the primary constraints $\chi^{(0)}_{\mu_f}$ are first-class. Since H' is a first-class function [1] we obtain that all the constraints generated from $\chi^{(0)}_{\mu_f}$:

$$\chi^{(0)}_{\mu_f}$$
$$\chi^{(k)}_{\mu_f}=\{\chi^{(k-1)}_{\mu_f},H'\} \tag{28}$$
$$k=1,\ldots,f$$

are first-class constraints. Let us remark that the chains of first-class constraints not necessarily have exactly f steps. We use this notation only for simplicity. We can have first-class 0-chains, 1-chains, etc.

It was proven also in [10] that the set (28) includes all the first-class constraints of the system. This is because the determinant of the matrix C formed by the PB among all the constraints not included in (28) is different from zero in M_f [10]. This means that no linear combination of the constraints not included in (28) can be first-class. Therefore, the constraint set generated by the primary second-class constraints is a second-class constraint set and the constraint set generated by the primary first-class constraints is a first-class constraint set.

Clearly, the matrix C formed by the PB among all the second-class constraints is antisymmetric. Since the determinant of C is different from zero in M_f, the total number of second-class constraints is always even [1].

It is not difficult to prove that (see Appendix A):

$$\{\chi^{(k-j)}_{\mu'_k},\chi^{(j)}_{\nu'_k}\} \underset{M_k}{=} (-1)^j\{\chi^{(k)}_{\mu'_k},\chi^{(0)}_{\nu'_k}\}=(-1)^jC^{(k)}_{\mu'_k\nu'_k} \tag{29}$$

for $j=0,1,\ldots,k$.

From (29) it follows that the matrix $\mathcal{C}^{(k)}$ is antisymmetric in M_k when k-even, and symmetric in M_k when k-odd.

$$\mathcal{C}^{(k)}_{\mu'_k \nu'_k} \underset{M_k}{=} (-1)^{k+1} \mathcal{C}^{(k)}_{\nu'_k \mu'_k} \tag{30}$$

Finally, on the basis of the theorem presented in this section and of the relations (29) it can be shown that:

$$\det \mathcal{C} = \det \mathcal{C}^{(0)} [\det \mathcal{C}^{(1)}]^2 [\det \mathcal{C}^{(2)}]^3 \ldots [\det \mathcal{C}^{(f)}]^{f+1} \tag{31}$$

3 Partition of the second-class constraint set

In this section we will study in detail the structure of the second-class constraint set. It will be shown in a few examples how the second-class chains separate. The sufficient conditions for the maximal separation of these chains will be also discussed. The procedure presented here is completely general and it can be applied to more complicated systems with longer second-class chains.

3.1 Systems with only primary second-class constraints

Let us study first the case of systems having only primary second-class constraints:

$$\chi^{(0)}_{\mu'_0} \qquad \mu'_0 = 1, \ldots, m_1 \tag{32}$$

Clearly, m_1 is an even number and,

$$\det |\{\chi^{(0)}_{\mu'_0}, \chi^{(0)}_{\nu'_0}\}| = \det |C^{(0)}_{\mu'_0\nu'_0}| \underset{M_0}{\neq} 0 \tag{33}$$

$$\{\chi^{(0)}_{\mu'_0}, H'\} = F^{(0)}_{\mu'_0\nu'_0}(q,p)\chi^{(0)}_{\nu'_0}(q,p) \tag{34}$$

$C^{(0)}$ is a real antisymmetric matrix at any point of M_0. Therefore, we can always find [37] a real orthogonal matrix $\alpha^{(0)}(q,p)$, such that:

$$\alpha^{(0)} C^{(0)} \alpha^{(0)\top} \underset{M_0}{=} \tilde{C}^{(0)} \tag{35}$$

where $\tilde{C}^{(0)}$ is a quasidiagonal matrix of the type:

$$\tilde{C}^{(0)} = \begin{pmatrix} 0 & \mu_1 & & & & & \\ -\mu_1 & 0 & & & & & \\ & & 0 & \mu_2 & & & \\ & & -\mu_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \mu_{\frac{m_1}{2}} \\ & & & & & -\mu_{\frac{m_1}{2}} & 0 \end{pmatrix} \tag{36}$$

Let us define a new set of primary second-class constraints $\tilde{\chi}$ as follows:

$$\tilde{\chi}_{\mu_0'} = \alpha^{(0)}_{\mu_0'\nu_0'}\chi_{\nu_0'} \tag{37}$$

We see that each primary second-class constraint $\tilde{\chi}$ PB commute with all the other primary second-class constraints except with one. The primary second-class constraint set splits into subsets. Each subset contains only one pair of primary second-class constraints which do not PB commute with each other. Each subset PB commute with all the other subsets.

This situation can be represented graphically in the following way. Let us denote each second-class constraint by a vertex. Let us draw an edge connecting two vertices when the corresponding second-class constraints do not PB commute with each other. When two second-class constraints PB commute with each other, the corresponding vertices are not connected by any edge. The case described in this subsection can be, therefore represented as it is shown in Figure 1.

3.2 Systems with only primary and secondary second-class constraints

Let us study now the case of systems having only primary and secondary second- class constraints:

$$\begin{array}{cc} \chi^{(0)}_{\mu_0'} & \chi^{(0)}_{\mu_1'} \\ & \chi^{(1)}_{\mu_1'} \end{array}$$

$$\mu_0' = 1, \ldots, m_1 - m_2$$

$$\mu_1' = 1, \ldots, m_2$$

We know (24) that

$$\det|\{\chi^{(0)}_{\mu'_0},\chi^{(0)}_{\nu'_0}\}| = \det|\mathcal{C}^{(0)}_{\mu'_0\nu'_0}| \underset{M_0}{\neq} 0 \tag{38}$$

$$\det|\{\chi^{(1)}_{\mu'_1},\chi^{(0)}_{\nu'_1}\}| = \det|\mathcal{C}^{(1)}_{\mu'_1\nu'_1}| \underset{M_1}{\neq} 0 \tag{39}$$

The matrix $\mathcal{C}^{(0)}$ is, clearly, antisymmetric in M_0, therefore $m_1 - m_2$ must be an even number. As it was shown in (30) $\mathcal{C}^{(1)}$ is symmetric in M_1.

We know (28) also that:

$$\chi^{(1)}_{\mu'_1} = \{\chi^{(0)}_{\mu'_1}, H'\} \tag{40}$$

therefore, keeping this definition (40) we see that any linear combination of the primary second-class constraints $\chi^{(0)}_{\mu'_1}$ generates exactly the same linear combination of the secondary $\chi^{(1)}_{\mu'_1}$ in M_0:

$$\{\alpha_{\mu'_1\nu'_1}\chi^{(0)}_{\nu'_1}, H'\} \underset{M_0}{=} \alpha_{\mu'_1\nu'_1}\{\chi^{(0)}_{\nu'_1}, H'\} \underset{M_0}{=} \alpha_{\mu'_1\nu'_1}\chi^{(1)}_{\nu'_1} \tag{41}$$

The matrix $\mathcal{C}$ formed by the PB among all the second-class constraints,

$$\mathcal{C} = \begin{pmatrix} \{\chi^{(0)}_{\mu'_0},\chi^{(0)}_{\nu'_0}\} & \{\chi^{(0)}_{\mu'_0},\chi^{(0)}_{\nu'_1}\} & \{\chi^{(0)}_{\mu'_0},\chi^{(1)}_{\nu'_1}\} \\ \{\chi^{(0)}_{\mu'_1},\chi^{(0)}_{\nu'_0}\} & \{\chi^{(0)}_{\mu'_1},\chi^{(0)}_{\nu'_1}\} & \{\chi^{(0)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} \\ \{\chi^{(1)}_{\mu'_1},\chi^{(0)}_{\nu'_0}\} & \{\chi^{(1)}_{\mu'_1},\chi^{(0)}_{\nu'_1}\} & \{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} \end{pmatrix} \tag{42}$$

can be represented in the form:

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}^{(0)}_{\mu'_0\nu'_0} & 0 & 0 \\ 0 & 0 & -\mathcal{C}^{(1)}_{\mu'_1\nu'_1} \\ 0 & \mathcal{C}^{(1)}_{\mu'_1\nu'_1} & \{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} \end{pmatrix} \tag{43}$$

This representation (43) is a direct consequence of the application of the theorem given in Section 2. Note that:

$$\det \mathcal{C} = \det \mathcal{C}^{(0)}[\det \mathcal{C}^{(1)}]^2 \tag{44}$$

From (43) we immediately see that the subset $\chi^{(0)}_{\mu'_0}$ splits from the subset formed by the constraints $\chi^{(0)}_{\mu'_1}$ and $\chi^{(1)}_{\mu'_1}$. All the $\chi^{(0)}_{\mu'_0}$ PB commute with all the $\chi^{(0)}_{\mu'_1}$ and $\chi^{(1)}_{\mu'_1}$. In other words, this means that the second-class 0-chains are completely disconnected from the rest.

Since the matrix $\mathcal{C}^{(0)}$ is antisymmetric in M_0 and its determinant is different from zero (38), then, as it was shown in the preceding subsection, it is always possible to find an orthogonal matrix $\alpha^{(0)}(q,p)$ in M_0, such that:

$$\alpha^{(0)}\mathcal{C}^{(0)}\alpha^{(0)\top} \underset{M_0}{=} \tilde{\mathcal{C}}^{(0)} \tag{45}$$

where $\tilde{\mathcal{C}}^{(0)}$ is a quasidiagonal matrix (36).

We define the constraints $\tilde{\chi}_{\mu'_0}$ as follows:

$$\tilde{\chi}_{\mu'_0} = \alpha^{(0)}_{\mu'_0\nu'_0}\chi_{\nu'_0} \tag{46}$$

The matrix $\mathcal{C}^{(1)}$ is symmetric in M_1 and its determinant is different from zero, therefore, there exists an orthogonal matrix $\alpha^{(1)}(q,p)$ such that:

$$\alpha^{(1)}\mathcal{C}^{(1)}\alpha^{(1)\top} \underset{M_1}{=} \tilde{\mathcal{C}}^{(1)} \qquad \text{–diagonal matrix} \tag{47}$$

The new subset of second-class constraints equivalent to the subset formed by $\chi^{(0)}_{\mu'_1}$ and $\chi^{(1)}_{\mu'_1}$ can be expressed in the following way:

$$\tilde{\chi}^{(0)}_{\mu'_1} = \alpha^{(1)}_{\mu'_1\nu'_1}\chi^{(0)}_{\nu'_1} \tag{48}$$

$$\tilde{\chi}^{(1)}_{\mu'_1} = \alpha^{(1)}_{\mu'_1\nu'_1}\chi^{(1)}_{\nu'_1} \tag{49}$$

Let us define the orthogonal matrix α as,

$$\alpha = \begin{pmatrix} \alpha^{(0)} & 0 & 0 \\ 0 & \alpha^{(1)} & 0 \\ 0 & 0 & \alpha^{(1)} \end{pmatrix} \tag{50}$$

and,

$$\tilde{\chi} = \begin{pmatrix} \tilde{\chi}^{(0)}_{\mu'_0} \\ \tilde{\chi}^{(0)}_{\mu'_1} \\ \tilde{\chi}^{(1)}_{\mu'_1} \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi^{(0)}_{\mu'_0} \\ \chi^{(0)}_{\mu'_1} \\ \chi^{(1)}_{\mu'_1} \end{pmatrix} \tag{51}$$

We can write, in general, that,

$$\tilde{\chi} = \alpha \chi \tag{52}$$

where,

$$\alpha \mathcal{C} \alpha^{\top} \underset{M_1}{=} \tilde{\mathcal{C}} \tag{53}$$

The matrix $\tilde{\mathcal{C}}$ is of the form,

$$\tilde{\mathcal{C}} = \begin{pmatrix} \tilde{\mathcal{C}}^{(0)}_{\mu'_0 \nu'_0} & 0 & 0 \\ 0 & 0 & -\tilde{\mathcal{C}}^{(1)}_{\mu'_1 \nu'_1} \\ 0 & \tilde{\mathcal{C}}^{(1)}_{\mu'_1 \nu'_1} & \{\tilde{\chi}^{(1)}_{\mu'_1}, \tilde{\chi}^{(1)}_{\nu'_1}\} \end{pmatrix} \tag{54}$$

and clearly,

$$\{\tilde{\chi}^{(1)}_{\mu'_1}, \tilde{\chi}^{(1)}_{\nu'_1}\} = \alpha^{(1)}_{\mu'_1 \sigma'_1} \{\chi^{(1)}_{\sigma'_1}, \chi^{(1)}_{\xi'_1}\} \alpha^{(1)}_{\xi'_1 \nu'_1}{}^{\top} \tag{55}$$

Let us assume that,

$$\{\chi^{(1)}_{\mu'_1}, \chi^{(1)}_{\nu'_1}\} \underset{M_1}{=} 0 \tag{56}$$

$$\mu'_1, \nu'_1 = 1, \ldots, m_2$$

In this case, we can represent graphically the situation described in this subsection as it is shown in Figure 2.

The complete set of second-class constraints splits into subsets which are disconnected among themselves. The second-class 0-chains are disconnected from the rest and can be grouped into subsets which are disconnected among themselves. Each subset of 0-chains is connected and consists of two primary constraints which do not PB commute with each other.

The second-class 1-chains are disconnected from the 0-chains but, in general, are not disconnected among themselves. However, if the conditions (56) are satisfied, we obtain the complete separation of the 1-chains. Each second-class 1-chain will be disconnected from all the others and will consist of two constraints, the primary one and the corresponding secondary constraint generated by this primary one. It is clear from (54-55) that the necessary and sufficient conditions for the separation of the 1-chains among themselves are (56).

It can be noted that, before doing the linear transformation α on the second-class constraint set χ, we could have a PB structure different from Figure 2. This different PB structure, as we have proven, is always equivalent to Figure 2, provided that (56) hold. This equivalence can be shown in each case by finding the appropriate linear combinations of the primary second-class constraints and applying then our algorithm (28) for obtaining the secondary constraints. For example, let us focus on the second-class 1-chains. It can be shown that all the graphs represented in Figure 3 are equivalent. They can be transformed one into the other by doing linear combinations of the primary second-class constraints and by the subsequent application of our algorithm (28) for obtaining the secondary constraints.

In the same way, it can be shown that the two graphs represented in Figure 4 are equivalent.

In particular a system of the type represented in Figure 4b was discussed in [12]. We will show in Appendix B that, doing an appropriate linear transformation of the constraints, it is possible to obtain the standard structure represented in Figure 4a. This means that, in fact, for this system the second-class 1-chains separate completely.

Let us obtain now the sufficient conditions for the vanishing of the matrix $\{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\}$ in M_1 (56).

Using the definition of the secondary constraints (40) and the Jacobi identities for the Poisson brackets we get:

$$\begin{aligned}\{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} &= \{\chi^{(1)}_{\mu'_1},\{\chi^{(0)}_{\nu'_1},H'\}\} \\ &= -\{H',\{\chi^{(1)}_{\mu'_1},\chi^{(0)}_{\nu'_1}\}\} - \{\chi^{(0)}_{\nu'_1},\{H',\chi^{(1)}_{\mu'_1}\}\}\end{aligned}$$

$$= \{\mathcal{C}^{(1)}_{\mu'_1\nu'_1}, H'\} - \{\{\chi^{(1)}_{\mu'_1}, H'\}, \chi^{(0)}_{\nu'_1}\} \tag{57}$$

Clearly, the matrix $\{\chi^{(1)}_{\mu'_1}, \chi^{(1)}_{\nu'_1}\}$ is antisymmetric and therefore, using (57) and the fact that $\mathcal{C}^{(1)}$ is symmetric in M_1, we obtain the following expression:

$$\{\chi^{(1)}_{\mu'_1}, \chi^{(1)}_{\nu'_1}\} = -\frac{1}{2}\{\{\chi^{(1)}_{\mu'_1}, H'\}, \chi^{(0)}_{\nu'_1}\} + \frac{1}{2}\{\{\chi^{(1)}_{\nu'_1}, H'\}, \chi^{(0)}_{\mu'_1}\} \tag{58}$$

We know (20) that:

$$\{\chi^{(1)}_{\mu'_1}, H'\} = F^{(0)}_{\mu'_1\nu'_0}\chi^{(0)}_{\nu'_0} + F^{(0)}_{\mu'_1\nu'_1}\chi^{(0)}_{\nu'_1} + F^{(1)}_{\mu'_1\nu'_1}\chi^{(1)}_{\nu'_1} \tag{59}$$

From (43) it follows that (58) can be finally expressed in the following way:

$$\{\chi^{(1)}_{\mu'_1}, \chi^{(1)}_{\nu'_1}\} \underset{M_1}{=} -\frac{1}{2}F^{(1)}_{\mu'_1\sigma'_1}\mathcal{C}^{(1)}_{\sigma'_1\nu'_1} + \frac{1}{2}F^{(1)}_{\nu'_1\sigma'_1}\mathcal{C}^{(1)}_{\sigma'_1\mu'_1} \tag{60}$$

As it can be seen from (60) the conditions:

$$F^{(1)}_{\sigma'_1\xi'_1} \underset{M_1}{=} 0 \tag{61}$$

$$\sigma'_1, \xi'_1 = 1, \ldots, m_1 - m_2$$

are <u>sufficient</u> for the vanishing of $\{\chi^{(1)}_{\mu'_1}, \chi^{(1)}_{\nu'_1}\}$ in M_1 (56). Therefore, (61) guarantee the maximal partition of the second-class constraint set.

Note that (61) can be reexpressed in the following equivalent way:

$$\{\chi^{(1)}_{\mu'_1}, H'\} = \begin{matrix}\text{linear combination of the} \\ \text{primary constraints.}\end{matrix} + \begin{matrix}\text{terms quadratic in the} \\ \text{constraints.}\end{matrix} \tag{62}$$

3.3 Systems with only primary, secondary and ternary second-class constraints

Let us study now the case of systems having only primary, secondary and ternary second-class constraints:

$$\begin{array}{ccc}
\chi^{(0)}_{\mu'_0} & \chi^{(0)}_{\mu'_1} & \chi^{(0)}_{\mu'_2} \\
 & \chi^{(1)}_{\mu'_1} & \chi^{(1)}_{\mu'_2} \\
 & & \chi^{(2)}_{\mu'_2}
\end{array}$$

$$\mu'_0 = 1, \ldots, m_1 - m_2$$

$$\mu'_1 = 1, \ldots, m_2 - m_3$$

$$\mu'_2 = 1, \ldots, m_3$$

The matrix $\mathcal{C}$ formed by the PB among all the second-class constraints, can be represented in the following form:

$$\mathcal{C} = \begin{pmatrix}
\mathcal{C}^{(0)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathcal{C}^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathcal{C}^{(2)} \\
0 & \mathcal{C}^{(1)} & 0 & \{\chi^{(1)}_{\mu'_1}, \chi^{(1)}_{\nu'_1}\} & 0 & \{\chi^{(1)}_{\mu'_1}, \chi^{(2)}_{\nu'_2}\} \\
0 & 0 & 0 & 0 & -\mathcal{C}^{(2)} & \{\chi^{(1)}_{\mu'_2}, \chi^{(2)}_{\nu'_2}\} \\
0 & 0 & \mathcal{C}^{(2)} & \{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_1}\} & \{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_2}\} & \{\chi^{(2)}_{\mu'_2}, \chi^{(2)}_{\nu'_2}\}
\end{pmatrix} \tag{63}$$

This representation (63) is a direct consequence of the application of the theorem given in section 2, and of the relations (29).

Note that, in this case,

$$\det \mathcal{C} = \det \mathcal{C}^{(0)} [\det \mathcal{C}^{(1)}]^2 [\det \mathcal{C}^{(2)}]^3 \tag{64}$$

As it follows from (30) the matrix $\mathcal{C}^{(0)}$ is antisymmetric in M_0, the matrix $\mathcal{C}^{(1)}$ is symmetric in M_1 and the matrix $\mathcal{C}^{(2)}$ is antisymmetric in M_2. We also know (24) that their determinants are different from zero in the corresponding submanifolds. Therefore [37], it is always possible to find three orthogonal matrices $\alpha^{(0)}, \alpha^{(1)}$, and $\alpha^{(2)}$, such that:

$$\alpha^{(0)} \mathcal{C}^{(0)} \alpha^{(0)\top} \underset{M_0}{=} \tilde{\mathcal{C}}^{(0)} \qquad \text{–quasidiagonal.} \tag{65}$$

$$\alpha^{(1)}\mathcal{C}^{(1)}\alpha^{(1)\top} \underset{M_1}{=} \tilde{\mathcal{C}}^{(1)} \qquad \text{–diagonal.} \tag{66}$$

$$\alpha^{(2)}\mathcal{C}^{(2)}\alpha^{(2)\top} \underset{M_2}{=} \tilde{\mathcal{C}}^{(2)} \qquad \text{–quasidiagonal.} \tag{67}$$

From (63) we can see also that the necessary and sufficient conditions for the maximal partition of the second-class constraint set are the following:

$$\{\chi^{(1)}_{\mu'_1}, \chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} 0 \tag{68}$$

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} 0 \tag{69}$$

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_2}\} \underset{M_2}{=} 0 \tag{70}$$

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(2)}_{\nu'_2}\} \underset{M_2}{=} 0 \tag{71}$$

If the relations (68-71) are satisfied, we obtain the matrix $\mathcal{C}$ in the following way:

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}^{(0)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathcal{C}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}^{(2)} \\ 0 & \mathcal{C}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathcal{C}^{(2)} & 0 \\ 0 & 0 & \mathcal{C}^{(2)} & 0 & 0 & 0 \end{pmatrix} \tag{72}$$

It is not difficult to prove that:

$$\alpha\mathcal{C}\alpha^{\top} = \tilde{\mathcal{C}} \tag{73}$$

where the matrix α is defined as follows:

$$\alpha = \begin{pmatrix} \alpha^{(0)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^{(2)} \end{pmatrix} \tag{74}$$

We define the new set of second-class constraints in the following way:

$$\tilde{\chi} = \alpha\chi \tag{75}$$

where,

$$\tilde{\chi} = \begin{pmatrix} \tilde{\chi}^{(0)}_{\mu'_0} \\ \tilde{\chi}^{(0)}_{\mu'_1} \\ \tilde{\chi}^{(0)}_{\mu'_2} \\ \tilde{\chi}^{(1)}_{\mu'_1} \\ \tilde{\chi}^{(1)}_{\mu'_2} \\ \tilde{\chi}^{(2)}_{\mu'_2} \end{pmatrix}, \qquad \chi = \begin{pmatrix} \chi^{(0)}_{\mu'_0} \\ \chi^{(0)}_{\mu'_1} \\ \chi^{(0)}_{\mu'_2} \\ \chi^{(1)}_{\mu'_1} \\ \chi^{(1)}_{\mu'_2} \\ \chi^{(2)}_{\mu'_2} \end{pmatrix} \tag{76}$$

Provided that the conditions (68-71) hold, we can represent graphically in our case the PB structure of the second-class constraint set as it is shown in Figure 5.

We see that the second-class 0-chains are completely disconnected from all the other chains of constraints. Each second-class 1-chain is disconnected from all the other chains and consists of two constraints (one primary and the corresponding secondary) which do not PB commute with each other. The second-class 2-chains are grouped into subsets disconnected among themselves. Each subset consists of two second-class 2-chains which are connected between themselves. We say that the partition is maximal because each second- class constraint PB commute with all the other except with one. In the case of the second-class 0- and 2-chains this one belongs to the other chain of the corresponding subset. This structure is very interesting and we will find it to be characteristic for the second-class k-chains with k-even.

Let us now discuss the sufficient conditions for the validity of the relations (68-71). They will be, of course, the sufficient conditions for the maximal partition of the second-class constraint set.

It is not difficult to prove that (see Appendix C):

$$\{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} -\frac{1}{2}F^{(1)}_{\mu'_1\sigma'_1}C^{(1)}_{\sigma'_1\nu'_1}+\frac{1}{2}F^{(1)}_{\nu'_1\sigma'_1}C^{(1)}_{\sigma'_1\mu'_1} \tag{77}$$

$$\{\chi^{(2)}_{\mu'_2},\chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} -F^{(1)}_{\mu'_2\sigma'_1}C^{(1)}_{\sigma'_1\nu'_1} \tag{78}$$

$$\{\chi^{(2)}_{\mu'_2},\chi^{(1)}_{\nu'_2}\} \underset{M_2}{=} -\frac{2}{3}F^{(2)}_{\mu'_2\sigma'_2}C^{(2)}_{\sigma'_2\nu'_2}-\frac{1}{3}F^{(2)}_{\nu'_2\sigma'_2}C^{(2)}_{\sigma'_2\mu'_2} \tag{79}$$

$$\{\chi^{(2)}_{\mu'_2},\chi^{(2)}_{\nu'_2}\} \underset{M_2}{=} \{\{\chi^{(2)}_{\mu'_2},\chi^{(1)}_{\nu'_2}\},H'\}-F^{(2)}_{\mu'_2\sigma'_2}\{\chi^{(2)}_{\sigma'_2},\chi^{(1)}_{\nu'_2}\}+F^{(1)}_{\mu'_2\sigma'_2}C^{(2)}_{\sigma'_2\nu'_2} \tag{80}$$

The coefficient functions F are defined from the following expressions (20):

$$\begin{aligned}\{\chi^{(1)}_{\mu'_1},H'\} &= F^{(0)}_{\mu'_1\sigma'_0}\chi^{(0)}_{\sigma'_0}+F^{(0)}_{\mu'_1\sigma'_1}\chi^{(0)}_{\sigma'_1}+F^{(0)}_{\mu'_1\sigma'_2}\chi^{(0)}_{\sigma'_2}+\\ &\quad +F^{(1)}_{\mu'_1\sigma'_1}\chi^{(1)}_{\sigma'_1}+F^{(1)}_{\mu'_1\sigma'_2}\chi^{(1)}_{\sigma'_2}\end{aligned} \tag{81}$$

$$\begin{aligned}\{\chi^{(2)}_{\mu'_2},H'\} &= F^{(0)}_{\mu'_2\sigma'_0}\chi^{(0)}_{\sigma'_0}+F^{(0)}_{\mu'_2\sigma'_1}\chi^{(0)}_{\sigma'_1}+F^{(0)}_{\mu'_2\sigma'_2}\chi^{(0)}_{\sigma'_2}+\\ &\quad +F^{(1)}_{\mu'_2\sigma'_1}\chi^{(1)}_{\sigma'_1}+F^{(1)}_{\mu'_2\sigma'_2}\chi^{(1)}_{\sigma'_2}+F^{(2)}_{\mu'_2\sigma'_2}\chi^{(2)}_{\sigma'_2}\end{aligned} \tag{82}$$

We can see from (77-80) that the following conditions:

$$F^{(1)} \underset{M_2}{=} 0, \qquad F^{(2)} \underset{M_2}{=} 0 \tag{83}$$

are sufficient for the validity of (68-71).

The conditions (83) can be also expressed in the following way:

$$\{\chi^{(k)}_{\mu'_k}, H'\} = \begin{array}{l}\text{linear combination of the}\\ \text{primary second-class}\\ \text{constraints.}\end{array} + \begin{array}{l}\text{terms quadratic in the}\\ \text{constraints (up to k+1-}\\ \text{ary).}\end{array} \qquad (84)$$

$$k = 1, 2$$

4 Conclusions

As it was stated at the beginning of Section 3 the procedure that we have used here is very general and it can be applied to the cases of systems with longer second-class chains. We think that, in the generic case, the sufficient conditions for the maximal partition of the second-class constraint set are the following:

$$\{\chi^{(k)}_{\mu'_k}, H'\} = \text{linear combination of the primary second-class constraints.} + \text{terms quadratic in the constraints (up to k+1-ary).} \quad (85)$$

$$k = 1, \ldots, f$$

We are working at present on the rigorous proof of this fact and our results will be given in [38].

So far in Section 3 we have discussed the systems with only second-class constraints. This was done in order to simplify all the expressions. However, it is obvious that the presence of the first-class constraints do not affect at all our results. This is, precisely, because the first-class constraints by definition PB commute with all the constraints in the final submanifold M_f. So, for example, in the presence of first-class constraints the conditions (85) are generalized as follows:

$$\{\chi^{(k)}_{\mu'_k}, H'\} = \text{linear combination of the first-class constraints.} + \text{linear combination of the primary second-class constraints.} + \text{terms quadratic in the constraints (up to k+1-ary).} \quad (86)$$

Assuming that the conditions (86) hold, let us present now some other examples in order to give an idea of the structure of the second-class constraint set in the generic case. The procedure which allows us to obtain the following representation is exactly the same as that of the preceding subsections. The point is to realize an appropriate linear orthogonal transformation α on the set χ of second-class primary constraints and then apply our algorithm (28) for obtaining the secondary, ternary,... , f+1-ary constraints. This is done to get the matrix C formed by the PB among all the second-class constraints in the simplest way. This representation is convenient because in it all the matrices $C^{(k)}$ (30) are diagonal for k-odd and quasidiagonal for k-even. Note that the relations (29) are also crucial.

Examples are represented in Figures 6-11.

We believe that having seen these examples the reader could draw the corresponding graphs for the second-class 6-chains, 7-chains, etc. It can be noticed that each second-class k-chain (for k-odd) separates completely from all the other chains. On the other hand, the second-class k-chains (for k-even) are grouped into pairs. Each pair is disconnected from all the others. We say that the partition is maximal because each second-class constraint of the system PB commute in M_f with all the constraints except with one.

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Appendix A

In this appendix we will present the proof of the relations (29) given in section 2. The proof will be by induction.

Clearly (29) hold for $j = 0$. Let us assume that (29) hold for $j = 0, 1, \ldots, l-1 \qquad (l-1 < k)$. We have to prove that:

$$\{\chi^{(k-l)}_{\mu'_k}, \chi^{(l)}_{\nu'_k}\} = (-1)^l \{\chi^{(k)}_{\mu'_k}, \chi^{(0)}_{\nu'_k}\} \tag{87}$$

Using the definition (18) and the Jacobi identities for the PB we can write:

$$\begin{aligned} \{\chi^{(k-l)}_{\mu'_k}, \chi^{(l)}_{\nu'_k}\} &= \{\chi^{(k-l)}_{\mu'_k}, \{\chi^{(l-1)}_{\nu'_k}, H'\}\} \\ &= -\{H', \{\chi^{(k-l)}_{\mu'_k}, \chi^{(l-1)}_{\nu'_k}\}\} - \{\chi^{(l-1)}_{\nu'_k}, \{H', \chi^{(k-l)}_{\mu'_k}\}\} \end{aligned}$$

It follows from the theorem of Section 2 that

$$\{\chi^{(k-l)}_{\mu'_k}, \chi^{(l-1)}_{\nu'_k}\} \underset{M_{k-1}}{=} 0 \tag{88}$$

and therefore,

$$\{\{\chi^{(k-l)}_{\mu'_k}, \chi^{(l-1)}_{\nu'_k}\}, H'\} \underset{M_k}{=} 0 \tag{89}$$

Using (89) and the definition (18) we get that:

$$\{\chi^{(k-l)}_{\mu'_k}, \chi^{(l)}_{\nu'_k}\} = -\{\chi^{(k-l+1)}_{\mu'_k}, \chi^{(l-1)}_{\nu'_k}\} \tag{90}$$

By hypothesis (29) hold for $j = 0, 1, \ldots, l-1$ and therefore we finally get:

$$\begin{aligned} \{\chi^{(k-l)}_{\mu'_k}, \chi^{(l)}_{\nu'_k}\} &= (-1)(-1)^{l-1}\{\chi^{(k)}_{\mu'_k}, \chi^{(0)}_{\nu'_k}\} \\ &= (-1)^l \{\chi^{(k)}_{\mu'_k}, \chi^{(0)}_{\nu'_k}\} \end{aligned} \tag{91}$$

which is the expression that we wanted to prove.

In this way the proof of (29) is concluded.

Appendix B

In this Appendix we will discuss one example presented in [12, 13] in order to show that also in this case it is possible to obtain the maximal partition of the second-class constraint set. This can be achieved by doing an appropriate linear orthogonal transformation of the primary second-class constraints, and applying then our algorithm for obtaining the secondary constraints.

The Lagrangian of the system in consideration is the following:

$$L = q_1 q_2 + q_2 q_3 + q_3 q_1$$

It is not difficult to obtain in this case the Hamiltonian H_c as,

$$H_c = -q_1 q_2 - q_2 q_3 - q_3 q_1$$

and the primary constraints as,

$$\chi_1^{(0)} = p_1, \qquad \chi_2^{(0)} = p_2, \qquad \chi_3^{(0)} = p_3 \tag{92}$$

The total Hamiltonian of the system can be written as follows:

$$H_T = -q_1 q_2 - q_2 q_3 - q_3 q_1 + \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$$

Applying the consistency conditions on the primary constraints (92) we get the following secondary constraints:

$$\chi_1^{(1)} = q_2 + q_3, \qquad \chi_2^{(1)} = q_1 + q_3, \qquad \chi_3^{(1)} = q_1 + q_2 \tag{93}$$

If we apply now the consistency conditions on (93) we obtain that

$$\lambda_1 \underset{M_1}{=} 0, \qquad \lambda_2 \underset{M_1}{=} 0, \qquad \lambda_3 \underset{M_1}{=} 0 \tag{94}$$

Notice that all the constraints (92) and (93) form a second-class set.

We see that the first-class Hamiltonian H' is in this case equal to H_c

The matrix C formed by the Poisson brackets among all the second-class constraints is of the following form:

$$
\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{95}
$$

From (95) it follows that our case can be graphically described as in Figure 4b.

Since the matrix $\mathcal{C}^{(1)}$ is symmetric,

$$
\mathcal{C}^{(1)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{96}
$$

we can find an orthogonal matrix $\alpha^{(1)}$ such that,

$$
\alpha^{(1)}\mathcal{C}^{(1)}\alpha^{(1)^{\top}} = \tilde{\mathcal{C}}^{(1)} \qquad \text{–diagonal matrix} \tag{97}
$$

In our case,

$$
\alpha^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{98}
$$

and,

$$
\tilde{\mathcal{C}}^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{99}
$$

Therefore, doing the following linear combination of the primary second-class constraints:

$$
\tilde{\chi}_1^{(0)} = \frac{1}{\sqrt{2}}p_1 - \frac{1}{\sqrt{2}}p_2
$$

$$
\tilde{\chi}_2^{(0)} = -\frac{1}{\sqrt{6}}p_1 - \frac{1}{\sqrt{6}}p_2 + \frac{2}{\sqrt{6}}p_3
$$

$$\tilde{\chi}_3^{(0)} = \frac{1}{\sqrt{3}}p_1 + \frac{1}{\sqrt{3}}p_2 + \frac{1}{\sqrt{3}}p_3$$

and applying our algorithm for obtaining the secondary second-class constraints:

$$\tilde{\chi}_1^{(1)} = \frac{1}{\sqrt{2}}(q_2 + q_3) - \frac{1}{\sqrt{2}}(q_1 + q_3)$$

$$\tilde{\chi}_2^{(1)} = -\frac{1}{\sqrt{6}}(q_2 + q_3) - \frac{1}{\sqrt{6}}(q_1 + q_3) + \frac{2}{\sqrt{6}}(q_1 + q_2)$$

$$\tilde{\chi}_3^{(1)} = \frac{1}{\sqrt{3}}(q_2 + q_3) + \frac{1}{\sqrt{3}}(q_1 + q_3) + \frac{1}{\sqrt{3}}(q_1 + q_2)$$

we see that the Poisson bracket structure of the constraint set $\tilde{\chi}$ can be represented graphically as it is shown in Figure 4a.

Appendix C

Let us start by proving the relations (77). Using the definitions (18) and (25), and the Jacobi identities we can write that:

$$\{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} = \{\mathcal{C}^{(1)}_{\mu'_1\nu'_1}, H'\} - \{\{\chi^{(1)}_{\mu'_1}, H'\},\chi^{(0)}_{\nu'_1}\} \tag{100}$$

Obviously, the matrix $\{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\}$ is antisymmetric and since $\mathcal{C}^{(1)}$ is symmetric in M_1 we can obtain from (100) that:

$$\{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} -\frac{1}{2}\{\{\chi^{(1)}_{\mu'_1}, H'\},\chi^{(0)}_{\nu'_1}\} + \frac{1}{2}\{\{\chi^{(1)}_{\nu'_1}, H'\},\chi^{(0)}_{\mu'_1}\} \tag{101}$$

Using (81) and (63) we immediately find that:

$$\{\chi^{(1)}_{\mu'_1},\chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} -\frac{1}{2}F^{(1)}_{\mu'_1\sigma'_1}\mathcal{C}^{(1)}_{\sigma'_1\nu'_1} + \frac{1}{2}F^{(1)}_{\nu'_1\sigma'_1}\mathcal{C}^{(1)}_{\sigma'_1\mu'_1} \tag{102}$$

Let us present now the proof of (78). Using (18) and the Jacobi identities for the Poisson brackets we get:

$$\{\chi^{(2)}_{\mu'_2},\chi^{(1)}_{\nu'_1}\} = \{\{\chi^{(2)}_{\mu'_2},\chi^{(0)}_{\nu'_1}\}, H'\} - \{\{\chi^{(2)}_{\mu'_2}, H'\},\chi^{(0)}_{\nu'_1}\} \tag{103}$$

From (63) we know that

$$\{\chi^{(2)}_{\mu'_2},\chi^{(0)}_{\nu'_1}\} \underset{M_2}{=} 0 \tag{104}$$

and, therefore, we obtain that:

$$\{\chi^{(2)}_{\mu'_2},\chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} -\{\{\chi^{(2)}_{\mu'_2}, H'\},\chi^{(0)}_{\nu'_1}\} \tag{105}$$

Finally, from (82) and (63) it follows that:

$$\{\chi^{(2)}_{\mu'_2},\chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} -F^{(1)}_{\mu'_2\sigma'_1}\mathcal{C}^{(1)}_{\sigma'_1\nu'_1} \tag{106}$$

In an analogous way it can be proven that:

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} -F^{(1)}_{\nu'_1\sigma'_2}\mathcal{C}^{(2)}_{\sigma'_2\mu'_2} \tag{107}$$

In order to prove (107) notice that $\chi^{(2)}_{\mu'_2} = \{\chi^{(1)}_{\mu'_2}, H'\}$ and that

$$\{\chi^{(1)}_{\mu'_2}, \chi^{(1)}_{\nu'_1}\} \underset{M_2}{=} 0$$

Let us prove now (79). Since the idea of the proof is similar to the preceding ones in this Appendix, we will present only the main intermediary results.

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_2}\} \underset{M_2}{=} -\{\mathcal{C}^{(2)}_{\mu'_2\nu'_2}, H'\} + \{\chi^{(2)}_{\nu'_2}, \chi^{(1)}_{\mu'_2}\} \tag{108}$$

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_2}\} \underset{M_2}{=} \{\mathcal{C}^{(2)}_{\mu'_2\nu'_2}, H'\} - \{\{\chi^{(2)}_{\mu'_2}, H'\}, \chi^{(0)}_{\nu'_2}\} \tag{109}$$

$$\{\chi^{(2)}_{\nu'_2}, \chi^{(1)}_{\mu'_2}\} \underset{M_2}{=} -\{\mathcal{C}^{(2)}_{\mu'_2\nu'_2}, H'\} - \{\{\chi^{(2)}_{\nu'_2}, H'\}, \chi^{(0)}_{\mu'_2}\} \tag{110}$$

For writing (110) we have used the fact that $\mathcal{C}^{(2)}$ is antisymmetric in M_2. The relation (79) follows directly from (108-110) and (63).

The last part of this Appendix is devoted to obtaining the relations (80). Using (18) and the Jacobi identities we have:

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(2)}_{\nu'_2}\} = \{\{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_2}\}, H'\} - \{\{\chi^{(2)}_{\mu'_2}, H'\}, \chi^{(1)}_{\nu'_2}\} \tag{111}$$

From (82) and (63) it finally follows that:

$$\{\chi^{(2)}_{\mu'_2}, \chi^{(2)}_{\nu'_2}\} \underset{M_2}{=} \{\{\chi^{(2)}_{\mu'_2}, \chi^{(1)}_{\nu'_2}\}, H'\} - F^{(2)}_{\mu'_2\sigma'_2}\{\chi^{(2)}_{\sigma'_2}, \chi^{(1)}_{\nu'_2}\} + F^{(1)}_{\mu'_2\sigma'_2}\mathcal{C}^{(2)}_{\sigma'_2\nu'_2} \tag{112}$$

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Figure 1: Representation of the primary second-class constraints

Figure 2: Representation of the primary and secondary second-class constraints

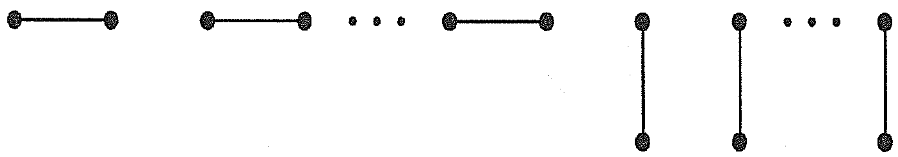

Figure 3: Equivalent graphical representations of two second-class 1-chains

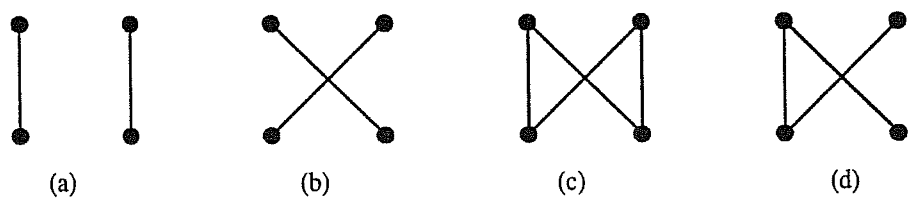

Figure 4: Equivalent graphical representations of three second-class 1-chains

Figure 5: Representation of the primary, secondary and ternary second-class constraints

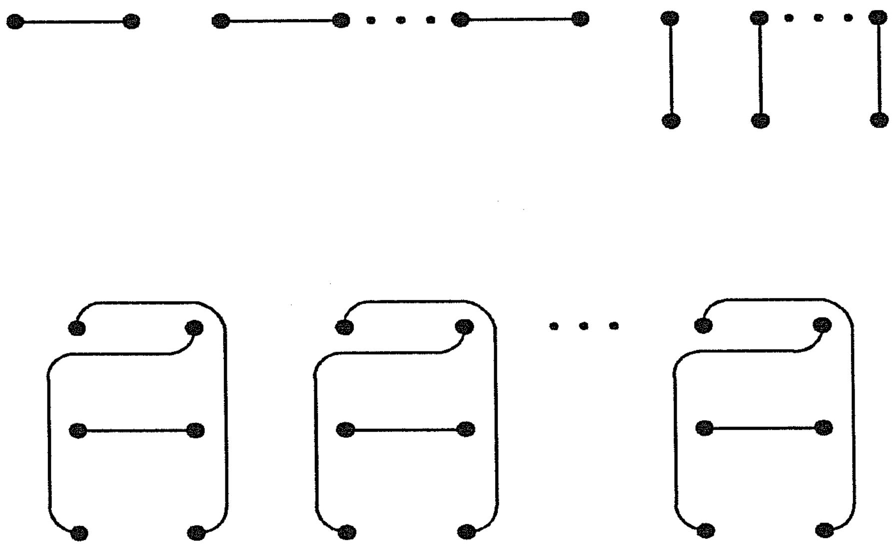

Figure 6: 0-chains

Figure 7: 1-chains

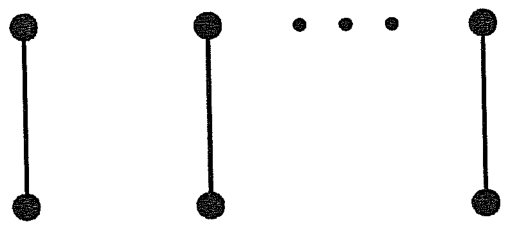

Figure 8: 2-chains

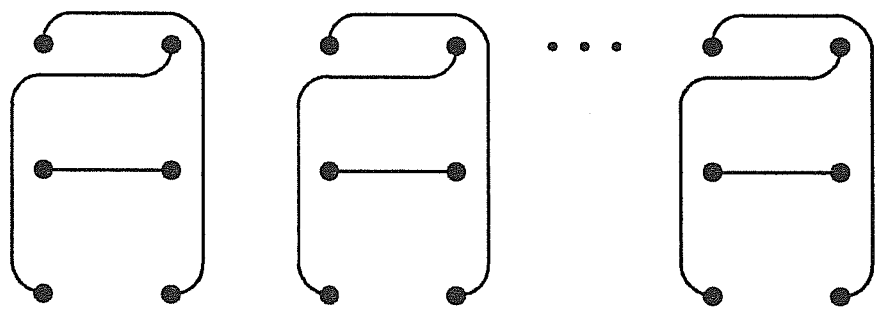

Figure 9: 3-chains

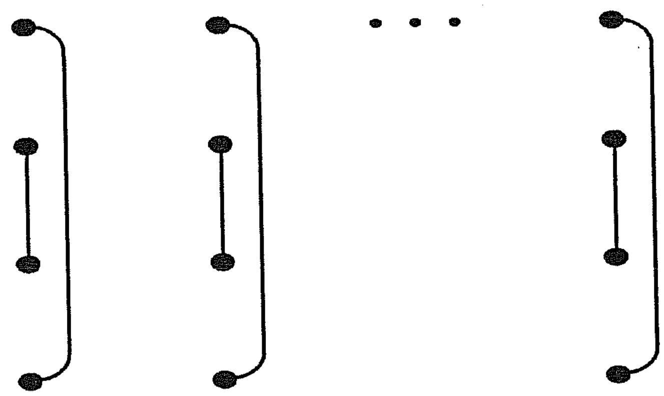

Figure 10: 4-chains

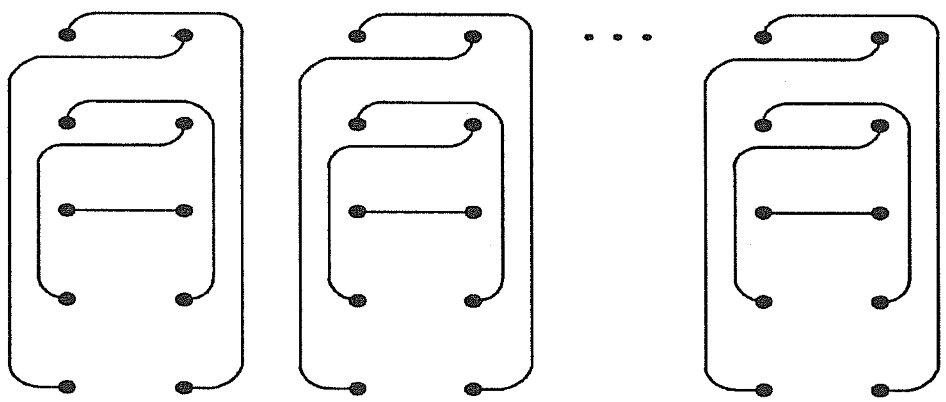

Figure 11: 5-chains