MODULI OF SYMPLECTIC INSTANTON VECTOR BUNDLES OF HIGHER RANK ON PROJECTIVE SPACE $\mathbb{P}^3$

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Abstract. Symplectic instanton vector bundles on the projective space $\mathbb{P}^3$ constitute a natural generalization of mathematical instantons of rank 2. We study the moduli space $I_{n,r}$ of rank-$2r$ symplectic instanton vector bundles on $\mathbb{P}^3$ with $r \geq 2$ and second Chern class $n \geq r$, $n \equiv r(\mathrm{mod}2)$. We give an explicit construction of an irreducible component $I^*_{n,r}$ of this space for each such value of n and show that $I^*_{n,r}$ has the expected dimension $4n(r+1) - r(2r+1)$.

1. Introduction

By a *symplectic instanton vector bundle* of rank $2r$ and charge n (shortly, a *symplectic* (n,r)-*instanton*) on the 3-dimensional projective space $\mathbb{P}^3$ we understand an algebraic vector bundle $E = E_{2r}$ of rank $2r$ on $\mathbb{P}^3$ with Chern classes

$$c_1(E) = 0, \tag{1}$$

$$c_2(E) = n, \quad n \geq 1, \tag{2}$$

supplied with a symplectic structure and satisfying the vanishing conditions

$$h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0. \tag{3}$$

By a symplectic structure we mean an anti-self-dual isomorphism

$$\phi : \ E \xrightarrow{\simeq} E^\vee, \quad \phi^\vee = -\phi, \tag{4}$$

considered modulo proportionality. The vanishing of the first Chern class (1) follows from the existence of a symplectic structure (4), and if $r = 1$, then the two conditions are equivalent. We will denote the moduli space of symplectic (n,r)-instantons by $I_{n,r}$.

For $r = 1$ these bundles relate, via the so-called Atiyah-Ward correspondence, to rank-2 "physical" instantons over the 4-sphere S^4, these being anti-self-dual connections with structure group $SU(2) = \mathbf{Sp}(1)$ [AW]. Important results on the moduli spaces $I_n = I_{n,1}$ of rank-2 instantons have been obtained recently: smoothness [JV] for all n, irreducibility [T] for odd n.

Much less is known about the moduli spaces $I_{n,r}$ for $r > 1$. In fact the symplectic instantons with $r > 1$ are as natural as those with $r = 1$, for they are related, via the same Atiyah-Ward correspondence, to the anti-self-dual connections over S^4 with structure group $\mathbf{Sp}(r)$, see [A]. As far as we know, the present paper is the first one addressing the properties of the corresponding spaces $I_{n,r}$. The tool we use to construct $I_{n,r}$ is the monad method; it originates in the work of Horrocks [H] and is known as the ADHM construction of instantons since [ADHM]. It was further sharpened in the work of Barth [B], Barth and Hulek [BH] and Tyurin [Tju1], [Tju2]. This method permits to encode the instantons, usual ones or symplectic of higher rank, by hyperwebs of quadrics.

For a sample of the physical literature about symplectic instantons, see e.g. [Mc].

We fix basic terminology and notation in Section 2 and introduce the hyperwebs of quadrics in Section 3. We prove that, for any $r \geq 2$ and for any $n \geq r$ such that $n \equiv r (\mathrm{mod}\, 2)$, the moduli space $I_{n,r}$ is nonempty and is realized as a free quotient $MI_{n,r}/(GL(n)/\pm \mathrm{id})$, where $MI_{n,r}$ is a Zariski locally closed subset of an affine space (see Theorem 3.1). Thus $MI_{n,r}$ carries a natural structure of a reduced scheme, and $I_{n,r}$ is an algebraic space. In Section 4 we give an explicit construction of vector bundles from $I_{n,r}$ for the above values of n and r and introduce a component $I^*_{n,r}$ of $I_{n,r}$ characterized by a certain open condition (*), see Definition 4.6. In Section 5 we prove Theorem 5.3 on the irreducibility of $I^*_{n,r}$, the main result of this paper.

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2. Notation and conventions

In many respects, we follow the exposition of [T], and we stick to the notation introduced therein. The base field $\mathbf{k}$ is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or a scheme X, then $n\mathcal{F}$ denotes a direct sum of n copies of $\mathcal{F}$, $H^i(\mathcal{F})$ denotes the i^{th} cohomology group of $\mathcal{F}$, $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and $\mathcal{F}^\vee$ denotes the dual of $\mathcal{F}$, that is, $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $X = \mathbb{P}^r$ and t is an integer, then by $\mathcal{F}(t)$ we denote the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. $[\mathcal{F}]$ will denote the isomorphism class of a sheaf $\mathcal{F}$. For any morphism of $\mathcal{O}_X$-sheaves $f : \mathcal{F} \to \mathcal{F}'$ and any $\mathbf{k}$-vector space U (respectively, for any homomorphism $f : U \to U'$ of $\mathbf{k}$-vector spaces) we will denote, for short, by the same letter f the induced morphism of sheaves $id \otimes f : U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$ (respectively, the induced morphism $f \otimes id : U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$).

We fix an integer $n \geq 1$ and denote by H_n a fixed n-dimensional vector space over $\mathbf{k}$. Throughout the paper, V will be a fixed vector space of dimension 4 over $\mathbf{k}$, and we set $\mathbb{P}^3 := P(V)$. We reserve the letters u and v for denoting the two morphisms in the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \to 0$. For any $\mathbf{k}$-vector spaces U and W and any vector $\phi \in \mathrm{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \mathrm{Hom}(U \otimes V, W \otimes V^\vee)$ understood as a linear map $\phi : U \otimes V \to W \otimes V^\vee$ or, equivalently, as a map ${}^\sharp\phi : U \to W \otimes \wedge^2 V^\vee$, we will denote by $\widetilde{\phi}$ the composition $U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^\sharp\phi} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} W \otimes \Omega_{\mathbb{P}^3}(2)$, where ϵ is the induced morphism in the exact triple $0 \to \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 v^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} \Omega_{\mathbb{P}^3}(2) \to 0$ obtained by taking the second wedge power of the dual Euler exact sequence.

Given an integer $m \geq 1$, we denote by $\mathbf{S}_m$ (resp. $\mathbf{\Sigma}_{m+1}$) the vector space $S^2 H_m^\vee \otimes \wedge^2 V^\vee$ (resp. $\mathrm{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$). Abusing notation, we will denote by the same symbol a $\mathbf{k}$-vector space, say U, and the associated affine space $\mathbf{V}(U^\vee) = \mathrm{Spec}(Sym^* U^\vee)$.

All the schemes considered in the paper are Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme $\mathcal{X}$ we mean any closed point of some dense open subset of $\mathcal{X}$. An irreducible scheme is called generically reduced if it is reduced at a general point.

3. Generalities on symplectic instantons and definition of $MI_{n,r}$

In this section we enumerate some facts about symplectic instantons which are completely parallel to those for rank-2 usual instantons, see [T, Section 3].

For a given symplectic (n,r)-instanton E, the first condition (3) yields $h^0(E(-i)) = 0, i \geq 0$, which, together with the exact triple $0 \to E(-j-1) \to E(-j) \to E(-j)|_{\mathbb{P}^2} \to 0$ for $j = 0$ and (3), implies that $h^0(E(-1)|_{\mathbb{P}^2}) = 0$, hence also $h^0(E(-i)|_{\mathbb{P}^2}) = 0,\ i \geq 1$. The last equality for $i = 2$, together with (3) and the above triple for $j = 2$, gives $h^1(E(-3)) = 0$, hence also $h^1(E(-4)) = 0$. Then, from Serre duality and (4), we deduce:

(5) $h^i(E) = h^i(E(-1)) = h^{3-i}(E(-3)) = h^{3-i}(E(-4)) = 0, \ \ i \neq 1, \quad h^i(E(-2)) = 0, \ \ i \geq 0.$

By Riemann-Roch and (3), (5), we have

(6) $$h^1(E(-1)) = h^2(E(-3)) = n, \ h^1(E) = h^2(E(-4)) = 2n - 2r.$$

By tensoring the dual Euler sequence by E we also obtain

(7) $$h^1(E \otimes \Omega^1_{\mathbb{P}^3}) = h^2(E \otimes \Omega^2_{\mathbb{P}^3}) = 2n + 2r,$$

Consider a triple (E, f, ϕ) where E is a (n,r)-instanton, $f : H_n \xrightarrow{\simeq} H^2(E(-3))$ an isomorphism and $\phi : E \xrightarrow{\simeq} E^\vee$ a symplectic structure on E. Two triples (E, f, ϕ) and $(E' f', \phi')$ are called equivalent if there is an isomorphism $g : E \xrightarrow{\simeq} E'$ such that $g_* \circ f = \lambda f'$ with $\lambda \in \{1, -1\}$ and $\phi = g^\vee \circ \phi' \circ g$, where $g_* : H^2(E(-3)) \xrightarrow{\simeq} H^2(E'(-3))$ is the induced isomorphism. We denote by $[E, f, \phi]$ the equivalence class of a triple (E, f, ϕ). It follows from this definition that the set $F_{[E]}$ of all equivalence classes $[E, f, \phi]$ with given $[E]$ is a homogeneous space of the group $GL(H_n)/\{\pm \mathrm{id}\}$.

Each class $[E, f, \phi]$ defines a point

(8) $$A = A([E, f, \phi]) \in S^2 H_n^\vee \otimes \wedge^2 V^\vee$$

in the following way. Consider the exact sequences

(9) $$\begin{gathered} 0 \to \Omega^1_{\mathbb{P}^3} \xrightarrow{i_1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to 0, \\ 0 \to \Omega^2_{\mathbb{P}^3} \to \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to \Omega^1_{\mathbb{P}^3} \to 0, \\ 0 \to \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4) \to \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{i_2} \Omega^2_{\mathbb{P}^3} \to 0, \end{gathered}$$

induced by the Koszul complex of $V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{ev}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}$. Twisting these sequences by E and taking into account (3), (5)-(7), we obtain the vanishing

(10) $$h^0(E \otimes \Omega_{\mathbb{P}^3}) = h^3(E \otimes \Omega^2_{\mathbb{P}^3}) = h^2(E \otimes \Omega_{\mathbb{P}^3}) = 0$$

and the diagram with exact rows

(11) $$\begin{tikzcd} 0 \arrow[r] & H^2(E(-4)) \otimes \wedge^4 V^\vee \arrow[r] & H^2(E(-3)) \otimes \wedge^3 V^\vee \arrow[r, "i_2"] \arrow[d, "A'"] & H^2(E \otimes \Omega^2_{\mathbb{P}^3}) \arrow[r] & 0 \\ 0 & H^1(E)) \arrow[l] & H^1(E(-1)) \otimes V^\vee \arrow[l] & H^1(E \otimes \Omega_{\mathbb{P}^3}) \arrow[l, "i_1"] \arrow[u, "\partial", "\cong"'] & 0, \arrow[l] \end{tikzcd}$$

where $A' := i_1 \circ \partial^{-1} \circ i_2$. The Euler exact sequence (9) yields the canonical isomorphism $\omega_{\mathbb{P}^3} \xrightarrow{\simeq} \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau : \mathbf{k} \xrightarrow{\simeq} \wedge^4 V^\vee$ we have the isomorphisms $\tilde{\tau} : V \xrightarrow{\simeq} \wedge^3 V^\vee$ and $\hat{\tau} : \omega_{\mathbb{P}^3} \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^3}(-4)$. We define A in (8) as the composition

(12) $$A : H_n \otimes V \overset{\tilde{\tau}}{\xrightarrow{\simeq}} H_n \otimes \wedge^3 V^\vee \overset{f}{\xrightarrow{\simeq}} H^2(E(-3)) \otimes \wedge^3 V^\vee \xrightarrow{A'} H^1(E(-1)) \otimes V^\vee \overset{\phi}{\xrightarrow{\simeq}}$$

$$\overset{\phi}{\xrightarrow{\simeq}} H^1(E^\vee(-1)) \otimes V^\vee \overset{SD}{\xrightarrow{\simeq}} H^2(E(1) \otimes \omega_{\mathbb{P}^3})^\vee \otimes V^\vee \overset{\hat{\tau}}{\xrightarrow{\simeq}} H^2(E(-3))^\vee \otimes V^\vee \overset{f^\vee}{\xrightarrow{\simeq}} H_n^\vee \otimes V^\vee,$$

where SD is the Serre duality isomorphism. One can verify that A is a skew symmetric map which depends only on the class $[E, f, \phi]$, but does not depend on the choice of τ, and that $A \in \wedge^2(H_n^\vee \otimes V^\vee)$ lies in the direct summand $\mathbf{S}_n = S^2H_n^\vee \otimes \wedge^2V^\vee$ of the canonical decomposition

$$\wedge^2(H_n^\vee \otimes V^\vee) = S^2H_n^\vee \otimes \wedge^2V^\vee \oplus \wedge^2H_n^\vee \otimes S^2V^\vee. \tag{13}$$

Here $\mathbf{S}_n$ is the space of hyperwebs of quadrics in H_n. For this reason we call A the (n,r)-*instanton hyperweb of quadrics* corresponding to the data $[E, f, \phi]$.

Denote $W_A := H_n \otimes V/\ker A$. Using the above chain of isomorphisms we can rewrite the diagram (11) as

$$\begin{tikzcd} 0 \arrow[r] & \ker A \arrow[r] & H_n \otimes V \arrow[r, "c_A"] \arrow[d, "A"] & W_A \arrow[r] \arrow[d, "q_A", "\cong"'] & 0 \\ 0 & \ker A^\vee \arrow[l] & H_n^\vee \otimes V^\vee \arrow[l] & W_A^\vee \arrow[l, "c_A^\vee"'] & 0. \arrow[l] \end{tikzcd} \tag{14}$$

In view of (7), $\dim W_A = 2n + 2r$ and $q_A : W_A \xrightarrow{\simeq} W_A^\vee$ is a skew-symmetric isomorphism. An important property of $A = A([E, f, \phi])$ is that the induced morphism of sheaves

$$a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_A^\vee} H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{ev} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \tag{15}$$

is surjective and the composition $H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{q_A} W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ is zero. Applying Beilinson spectral sequence [Bei] to $E(-1)$, we see that $E \simeq \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A$. Thus A defines a monad

$$\mathcal{M}_A : \quad 0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee \circ q_A} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \ , \tag{16}$$

whose cohomology sheaf

$$E_{2r}(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A. \tag{17}$$

is isomorphic to E. Twisting $\mathcal{M}_A$ by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and using (17), we obtain the isomorphism $f : H_n \xrightarrow{\simeq} H^2(E(-3))$. Furthermore, the fact that q_A is symplectic implies that there is a canonical isomorphism of $\mathcal{M}_A$ with its dual which induces the symplectic isomorphism $\phi : E \xrightarrow{\simeq} E^\vee$. Thus, the data $[E, f, \phi]$ are recovered from A. This leads to the following description of the moduli space $I_{n,r}$. Consider the *set of* (n,r)-*instanton hyperwebs of quadrics*

$$MI_{n,r} := \left\{ A \in \mathbf{S}_n \;\middle|\; \begin{array}{rl} \text{(i)} & rk(A : H_n \otimes V \to H_n^\vee \otimes V^\vee) = 2n + 2r, \\ \text{(ii)} & \text{the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \text{ defined} \\ & \text{by } A \text{ in (15) is surjective,} \\ \text{(iii)} & h^0(E_{2r}(A)) = 0, \text{ where } E_{2r}(A) = \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A \\ & \text{and } q_A : W_A \xrightarrow{\simeq} W_A^\vee \text{ is a symplectic isomorphism} \\ & \text{associated to } A \text{ by (14).} \end{array} \right\} \tag{18}$$

It is a locally closed subscheme of the affine space $\mathbf{S}_n$.

Theorem 3.1. *The natural morphism*

$$\pi_{n,r} : MI_{n,r} \to I_{n,r}, \ \ A \mapsto [E_{2r}(A)], \tag{19}$$

is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology. Hence $I_{n,r}$ is a quotient stack $MI_{n,r}/(GL(H_n)/\{\pm\mathrm{id}\})$, making it an algebraic space.

Proof. See [T, Section 3]. □

Each fibre $F_{[E]} = \pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_{n,r}$ is a principal homogeneous space of the group $GL(H_n)/\{\pm\mathrm{id}\}$. Hence the irreducibility of $(I_{n,r})_{red}$ is equivalent to the irreducibility of the scheme $(MI_{n,r})_{red}$.

We can also state:

Theorem 3.2. *For each $n \geq 1$, the space $MI_{n,r}$ of (n,r)-instanton hyperwebs of quadrics is a locally closed subscheme of the vector space $\mathbf{S}_n$ given locally at any point $A \in MI_{n,r}$ by*

$$\binom{2n-2r}{2} = 2n^2 - n(4r+1) + r(2r+1) \tag{20}$$

equations obtained as the rank condition (i) in (18).

Note that from (20) it follows that

$$\dim_{[A]} MI_{n,r} \geq \dim \mathbf{S}_n - (2n^2 - n(4r+1) + r(2r+1)) = n^2 + 4n(r+1) - r(2r+1) \tag{21}$$

at any point $A \in MI_{n,r}$. Hence,

$$\dim_{[E]} I_{n,r} \geq 4n(r+1) - r(2r+1) \tag{22}$$

at any point $[E] \in I_{n,r}$, since $MI_{n,r} \to I_{n,r}$ is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology.

4. Explicit construction of symplectic instantons

4.1. Example: symplectic (n,n)-instantons. In this subsection we recall some known facts about symplectic (n,n)-instantons and their relation to usual rank-2 instantons, see [T, Sections 5-6]. We first show that each invertible hyperweb of quadrics $A \in \mathbf{S}_n$ naturally leads to a construction of a symplectic (n,n)-instanton $E_{2n}(A)$ on $\mathbb{P}^3$. Given an integer $n \geq 1$, set

$$\mathbf{S}_n^0 := \{A \in \mathbf{S}_n \mid A : H_n \otimes V \to H_n^\vee \otimes V^\vee \text{ is an invertible map}\}. \tag{23}$$

Then $\mathbf{S}_n^0$ is a dense open subset of $\mathbf{S}_n$, and it is easy to see that for any $A \in \mathbf{S}_n^0$ the following conditions are satisfied.

(1) The morphism $\widetilde{A} : H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_n^\vee \otimes \Omega_{\mathbb{P}^3}(1)$ induced by A is a subbundle embedding, and

$$E_{2n}(A) := \mathrm{coker}(\widetilde{A}) \tag{24}$$

is a symplectic (n,n)-instanton, that is,

$$[E_{2n}(A)] \in I_{n,n}. \tag{25}$$

(2) For all $i \geq 0$,

$$h^i(E_{2n}(A)) = h^i(E_{2n}(A)(-2)) = 0. \tag{26}$$

This follows from the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\widetilde{A}"] \arrow[d, "u"] & H_n^\vee \otimes \Omega_{\mathbb{P}^3}(1) \arrow[r, "e"] \arrow[d, "v^\vee"] & E_{2n}(A) \arrow[r] & 0 \\
 & H_n \otimes V \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "A", "\simeq"'] \arrow[d, "v"] & H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[d, "u^\vee"] & & \\
0 \to E_{2n}(A)^\vee \arrow[r] & H_n \otimes T_{\mathbb{P}^3}(-1) \arrow[r, "\widetilde{A}^\vee"] \arrow[d] & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{27}
$$

Thus $\mathbf{S}_n^0 \subset MI_{n,n}$. In fact, the following result is true.

Proposition 4.1. $\mathbf{S}_n^0 = MI_{n,n}$. *In particular, $MI_{n,n}$ is irreducible of dimension $3n^2+3n$, and hence $I_{n,n}$ is irreducible of dimension $2n^2+3n$.*

Proof. We have to show that $MI_{n,n} \subset \mathbf{S}_n^0$. Let $A \in MI_{n,n}$. Since $n = r$, by condition (i) from (18) the rank of the hyperweb of quadrics $A : H_n \otimes V \to H_n^\vee \otimes V^\vee$ is $2n+2r = 4n = \dim H_n^\vee \otimes V^\vee$, hence A is invertible. By (23), this means that $A \in \mathbf{S}_n^0$. $\square$

Now we proceed to spell out the relation between symplectic (n,n)-instantons and usual rank-2 instantons with second Chern class $2n-1$. This relation is given at the level of spaces of hyperwebs of quadrics $MI_{n,n}$ and $MI_{2n-1,1}$ interpreted as spaces of monads.

We need some more notation. Let $B \in \mathbf{S}_n^0$. By definition, B is an invertible anti-self-dual map $H_n \otimes V \to H_n^\vee \otimes V^\vee$. Then the inverse

$$B^{-1} : H_n^\vee \otimes V^\vee \to H_n \otimes V \tag{28}$$

is also anti-self-dual. Consider the vector space $\mathbf{\Sigma}_n = H_n^\vee \otimes H_{n-1}^\vee \otimes \wedge^2 V^\vee$. An element $C \in \mathbf{\Sigma}_n$ can be viewed as a linear map $C : H_{n-1} \otimes V \to H_n^\vee \otimes V^\vee$, and its transpose $C^\vee$ as a map $C^\vee : H_n \otimes V \to H_{n-1}^\vee \otimes V^\vee$. As the composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, we can consider it as an element of $\wedge^2(H_{n-1}^\vee \otimes V^\vee) \simeq \mathbf{S}_{n-1} \oplus \wedge^2 H_{n-1}^\vee \otimes S^2 V^\vee$ (cf. (13)). Thus the condition

$$C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-1} \tag{29}$$

makes sense.

Next, consider the upper horizontal triple in (27) with $A = B$. Twisting it by $\mathcal{O}_{\mathbb{P}^3}(1)$ and passing to global sections we obtain the exact triple

$$0 \to H_n \overset{\sharp B}{\to} H_n^\vee \otimes \wedge^2 V^\vee \overset{\epsilon(B)}{\to} H^0(E_{2n}(B)(1)) \to 0 \tag{30}$$

Besides, interpreting $C \in \mathbf{\Sigma}_n$ as a map $\sharp C : H_{n-1} \to H_n^\vee \otimes \wedge^2 V^\vee$, we obtain the composition $H_{n-1} \overset{\sharp C}{\to} H_n^\vee \otimes \wedge^2 V^\vee \overset{\epsilon(B)}{\to} H^0(E_{2n}(B)(1))$ which induces the morphism of sheaves

$$\rho_{B,C} : \; H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_{2n}(B). \tag{31}$$

Note also that the maps $B : H_n \otimes V \to H_n^\vee \otimes V^\vee$ and $C : H_{n-1} \otimes V \to H_n^\vee \otimes V^\vee$ provide a map $(H_n \oplus H_{n-1}) \otimes V \to H_n^\vee \otimes V^\vee$, which induces the morphism of sheaves

$$\tau_{B,C} : \; (H_n \oplus H_{n-1}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}. \tag{32}$$

Now set

$$X_n := \left\{ (B,C) \in \mathbf{S}_n^0 \times \mathbf{\Sigma}_n \;\middle|\; \begin{array}{r l} \text{(i)} & \text{the condition (29) is satisfied,} \\ \text{(ii)} & \rho_{B,C} \text{ in (31) is a subbundle inclusion,} \\ \text{(iii)} & \tau_{B,C} \text{ in (32) is a subbundle inclusion.} \end{array} \right\} \tag{33}$$

By definition, X_n is a locally closed subset of $\mathbf{S}_n^0 \times \mathbf{\Sigma}_n$. Hence it is naturally endowed with a structure of a reduced scheme.

Now for any direct sum decomposition

$$\xi : H_{2n-1} \xrightarrow{\simeq} H_n \oplus H_{n-1}, \tag{34}$$

we obtain the corresponding decomposition

$$\tilde{\xi} : \mathbf{S}_{2n-1} \xrightarrow{\simeq} \mathbf{S}_n \oplus \mathbf{\Sigma}_n \oplus \mathbf{S}_{n-1} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{35}$$

Thus, considering the set $MI_{2n-1,1}$ of $(2n-1)$-instanton hyperwebs of quadrics as a subset of $\mathbf{S}_{2n-1}$, we obtain a natural projection

$$f_n : MI_{2n-1,1} \to \mathbf{S}_n \oplus \mathbf{\Sigma}_n : A \mapsto (A_1(\xi), A_2(\xi)). \tag{36}$$

The following result is proved in [T, Theorems 1.1, 6.1 and Remark 7.6].

Proposition 4.2. *For a general decomposition ξ in (34), there exists a dense open subset $MI_{2n-1,1}(\xi)$ of $MI_{2n-1,1}$ such that the projection f_n in (36) induces an isomorphism or integral schemes*

$$f_n : MI_{2n-1,1}(\xi) \xrightarrow{\simeq} X_n : A \mapsto (A_1(\xi), A_2(\xi)). \tag{37}$$

The inverse isomorphism is given by the formula

$$f_n^{-1} : \; X_n \xrightarrow{\simeq} MI_{2n-1,1}(\xi) : \; (B,C) \mapsto \; \tilde{\xi}^{-1}(B, \; C, \; -C^\vee \circ B^{-1} \circ C). \tag{38}$$

Besides, the projection

$$pr_1 : \; X_n \to \mathbf{S}_n^0 : \; (B,C) \mapsto B \tag{39}$$

is dominant.

It is not hard to check that the morphism $\rho_{B,C} : H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_{2n}(B)$ defined in (31) satisfies the condition ${}^t\rho_{B,C} \circ \rho_{B,C} = 0$, where ${}^t\rho_{B,C}$ is the composition

$${}^t\rho_{B,C} : E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi}} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$

and ϕ is a symplectic structure on $E_{2n}(B)$ (cf. [T, formulas (71)-(72)]). In other words, we obtain an anti-self-dual monad

$$0 \to H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\rho_{B,C}} E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi}} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{40}$$

with cohomology sheaf

$$E_2(A) = E_2(B,C) := \ker {}^t\rho_{B,C} / \operatorname{im} \rho_{B,C}, \quad A = f_n^{-1}(B,C). \tag{41}$$

Next, by (19) we have the natural projection

$$\pi_{2n-1,1} : MI_{2n-1,1} \to I_{2n-1,1} : \; A \mapsto [E_2(A)]. \tag{42}$$

We have the following interpretation of the isomorphism (38) on the level of vector bundles:

$$[E_2(B,C)] = \pi_{2n-1,1}(f_n^{-1}(B,C)). \tag{43}$$

Remark 4.3. Note that, according to the definitions (16)-(18) of $MI_{2n-1,1}$ and $MI_{n,n}$, for any $A \in MI_{2n-1,1}$, if $B = A_1(\xi)$ is defined by the direct sum decomposition (35), one has two other anti-self-dual monads

$$\mathcal{M}_A : \quad 0 \to H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee \circ q_A} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{44}$$

$$\mathcal{M}_B : \quad 0 \to H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_B} W_B \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_B^\vee \circ q_B} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{45}$$

with cohomology sheaves

$$E_2(A) = \ker(a_A^\vee \circ q_A)/\mathrm{im}\, a_A, \;\; E_{2n}(B) = \ker(a_B^\vee \circ q_B)/\mathrm{im}\, a_B \tag{46}$$

respectively. Moreover, (40) and (41) provide an isomorphism $w: \; W_B = H^2(E_2(B) \otimes \Omega_{\mathbb{P}^3}) \xrightarrow{\simeq} H^2(E_{2n}(A) \otimes \Omega_{\mathbb{P}^3}) = W_A$. We thus obtain a commutative anti-self-dual diagram relating these monads:
(47)

$$\begin{tikzcd}
0 \arrow[r] & H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "a_B"] \arrow[d, hook, "i_\xi"] & W_B \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "q_B", "\cong"'] \arrow[d, "w", "\cong"'] & W_B^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "a_B^\vee"] & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] & 0 \\
0 \arrow[r] & H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "a_A"] & W_A \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "q_A", "\cong"'] & W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "a_A^\vee"] \arrow[u, "w^\vee", "\cong"'] & H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[u, two heads, "i_\xi^\vee"] & 0,
\end{tikzcd}$$

where $i_\xi : H_n \hookrightarrow H_{2n-1}$ is the embedding induced by the decomposition (34). In view of (46) and the canonical isomorphism $H_{2n-1}/i_\xi(H_n) \simeq H_{n-1}$, from this diagram we obtain the monad

$$\mathcal{M}_{A,B} : \quad 0 \to H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_{A,B}} E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi}} E_{2n}(B)^\vee \xrightarrow{a_{A,B}^\vee} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{48}$$

with cohomology sheaf

$$E_2(A) = \ker(a_{A,B}^\vee \circ \phi)/\mathrm{im}\, a_A. \tag{49}$$

We call (48) the *quotient monad* of the monads (44) and (45).

Remark 4.4. Note that, by Proposition 4.2, the set of all diagrams (47) is parametrized by the irreducible variety $I_{2n-1,1}(\xi)$.

4.2. Example: a special family of symplectic (n, r)-instantons. Now assume $n \geq 2$ and, for any integer $r, \; 2 \leq r \leq n-1$, consider an inclusion

$$\tau : H_{2n-r} \hookrightarrow H_{2n-1} \tag{50}$$

such that

$$\tau(H_{2n-r}) \supset i_\xi(H_n). \tag{51}$$

We obtain a hyperweb of quadrics

$$A_\tau \in S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$$

as the image of A under the map $S^2 H_{2n-1}^\vee \otimes \wedge^2 V^\vee \to S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$ induced by τ. The corresponding monad

$$\mathcal{M}_\tau : \quad 0 \to H_{2n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_\tau} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_\tau^\vee \circ q_A} H_{2n-r}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{52}$$

has a rank-$2r$ cohomology bundle

$$E_{2r}(A_\tau) = \ker(a_\tau^\vee \circ q_A)/\mathrm{im}\, a_\tau. \tag{53}$$

where $a_\tau := a_A \circ \tau$. By construction, $E_{2r}(A_\tau)$ inherits a natural symplectic structure

$$\phi_r : \ E_{2r}(A_\tau) \xrightarrow{\simeq} E_{2r}(A_\tau)^\vee. \tag{54}$$

Besides, in view of (51), the monad (52) can be inserted as a midle row into the diagram (47), extending it to a three-row commutative anti-self-dual diagram. Arguing as in Remark 4.3 we obtain, in addition to the quotient monad (48), two more quotient monads:

$$\mathcal{M}'_\tau : \quad 0 \to H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a'_\tau} E_{2n}(B) \underset{\simeq}{\xrightarrow{\phi}} E_{2n}(B)^\vee \xrightarrow{a'^\vee_\tau} H^\vee_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{55}$$

$$E_{2r}(A_\tau) = \ker(a'^\vee_\tau \circ \phi)/\mathrm{im}\, a'_\tau,$$

$$\mathcal{M}''_\tau : \quad 0 \to H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a''_\tau} E_{2r}(B) \underset{\simeq}{\xrightarrow{\phi_\tau}} E_{2r}(B)^\vee \xrightarrow{a''^\vee_\tau} H^\vee_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{56}$$

$$E_2(A) = \ker(a'^\vee_\tau \circ \phi_\tau)/\mathrm{im}\, a_A.$$

From (26) and (55) we easily deduce:

$$h^0(E_{2r}(A_\tau)) = h^i(E_{2r}(A_\tau)(-2)) = 0, \quad i \geq 0, \quad c_2(E_{2r}(A_\tau)) = 2n - r. \tag{57}$$

By definition, this together with (52)-(54) means that

$$[E_{2r}(A_\tau)] \in I_{2n-r,r}. \tag{58}$$

Remark 4.5. Observe that, in view of (50), the maps τ belong to the set

$$N_{n,r} := \{\tau \in \mathrm{Hom}(H_{2n-r}, H_{2n-1}) | \ \tau \ \text{ is injective and } \mathrm{im}\, \tau \supset \mathrm{im}\, i_\xi\}.$$

When $A \in MI_{2n-1,1}(\xi)$ is fixed,$N_{n,r}$ parametrizes some family of hyperwebs A_τ from $MI_{2n-r,r}$. Since $N_{n,r}$ is a principal $GL(H_{2n-r})$-bundle over an open subset of the Grassmannian $Gr(n-r, n-1)$, it it is irreducible. Thus, by Remark 4.4, the family of the three-row extensions of the diagrams (47) can be parametrized by the irreducible variety $MI_{2n-1,1}(\xi) \times N_{n,r}$. Hence the family $D_{n,r}$ of isomorphism classes of symplectic rank-$2r$ bundles obtained from these diagrams by formula (53) is an irreducible locally closed subset of $I_{2n-r,r}$.

Note that it is a priori not clear whether the closure of $D_{n,r}$ in $I_{2n-r,r}$ is an irreducible component of $I_{2n-r,r}$.

Definition 4.6. Let $2 \leq r \leq n-1$. We say that $A \in MI_{2n-r,r}$ *satisfies property* (*) if there exists a monomorphism $i : H_n \hookrightarrow H_{2n-r}$ such that the image B of A under the surjection $\mathbf{S}_{2n-r} \twoheadrightarrow \mathbf{S}_n$ induced by i is invertible as a homomorphism $B : H_n \otimes V \to H_n^\vee \otimes V^\vee$.

The property (*) is clearly an open condition on A. Moreover, since $\pi_{2n-r,r} : MI_{2n-r,r} \to I_{2n-r,r}$ is a principal bundle (Theorem 3.1), if an element $A \in \pi^{-1}_{2n-r,r}([E_{2r}])$ satisfies (*), then any other point $A' \in \pi^{-1}_{2n-r,r}([E_{2r}])$ satisfies (*). We thus say that $[E_{2r}] \in I_{2n-r,r}$ *satisfies property* (*) if some (hence any) $A \in \pi^{-1}_{2n-r,r}([E_{2r}])$ satisfies property (*). It is obviously an open condition on $[E_{2r}] \in I_{2n-r,r}$.

Remark 4.7. By Proposition 4.2 and using (51), we see that any $[E_{2r}] \in D_{n,r}$, as well as any $A \in f_n^{-1}(D_{n,r})$ satisfies property (*). We define

$$I^*_{2n-r,r} := I_{(1)} \cup \ldots \cup I_{(k)}, \tag{59}$$

where $I_{(1)}, \ldots, I_{(k)}$ are all the irreducible components of $I_{2n-r,r}$ whose general points satisfy property (*). By definition, $D_{n,r} \subset I^*_{2n-r,r}$, hence $I^*_{2n-r,r}$ is nonempty. We also set $MI^*_{2n-r,r} = \pi^{-1}_{2n-r,r}(I^*_{2n-r,r})$, so that the map $\pi_{2n-r,r} : MI^*_{2n-r,r} \to I^*_{2n-r,r}$ is a principal bundle with structure group $GL(H_{2n-r})/\{\pm 1\}$.

5. Irreducibility of $I^*_{2n-r,r}$

5.1. A dense open subset $X_{n,r}$ of $MI^*_{2n-r,r}$. Reduction of the irreducibility of $I^*_{n,r}$ to that of $X_{n,r}$. In this section we prove the irreducibility of the component $I^*_{2n-r,r}$ of $I_{2n-r,r}$ defined in (59), see Theorem 5.3. The explicit construction of symplectic instantons in Section 4 gives us a hint to the proof. We proceed along the lines of Subsection 4.1.

Take any $B \in \mathbf{S}^0_n$ and consider it as an invertible anti-self-dual linear map $H_n \otimes V \to H_n^\vee \otimes V^\vee$. Then B^{-1} is also anti-self-dual. Let

$$\mathbf{\Sigma}_{n,r} := H^\vee_{n-r} \otimes H^\vee_n \otimes \wedge^2 V^\vee. \tag{60}$$

An element $C \in \mathbf{\Sigma}_n$ can be understood as a map $C : H_{n-r} \otimes V \to H_n^\vee \otimes V^\vee$, and its transpose $C^\vee$ is a map $H_n \otimes V \to H^\vee_{n-r} \otimes V^\vee$. The composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, i.e., it is an element of $\wedge^2(H^\vee_{n-r} \otimes V^\vee) \simeq \mathbf{S}_{n-r} \oplus \wedge^2 H^\vee_{n-r} \otimes S^2V^\vee$ (cf. (13)). We will later impose the condition

$$C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r}. \tag{61}$$

Next, as in (30), we have a well defined epimorphism $\epsilon(B) : H^\vee_n \otimes \wedge^2 V^\vee \twoheadrightarrow H^0(E_{2n}(B)(1))$. Besides, interpreting the above element $C \in \mathbf{\Sigma}_{n,r}$ as a map ${}^\sharp C : H_{n-r} \to H^\vee_n \otimes \wedge^2 V^\vee$, we obtain the composition $H_{n-r} \xrightarrow{{}^\sharp C} H^\vee_n \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1))$ which induces the morphism of sheaves

$$\rho_{B,C} : \; H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_{2n}(B). \tag{62}$$

Note also that $B : H_n \otimes V \to H^\vee_n \otimes V^\vee$ and $C : H_{n-r} \otimes V \to H^\vee_n \otimes V^\vee$ define a map $(H_n \oplus H_{n-r}) \otimes V \to H^\vee_n \otimes V^\vee$ which induces the morphism of sheaves

$$\tau_{B,C} : \; (H_n \oplus H_{n-r}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H^\vee_n \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}. \tag{63}$$

Now set

$$X_{n,r} := \left\{ (B, C) \in \mathbf{S}^0_n \times \mathbf{\Sigma}_{n,r} \;\middle|\; \begin{array}{r l} \text{(i)} & \text{the condition (61) is satisfied,} \\ \text{(ii)} & \rho_{B,C} \text{ in (62) is a subbundle inclusion,} \\ \text{(iii)} & \tau_{B,C} \text{ in (63) is a subbundle inclusion.} \end{array} \right\} \tag{64}$$

By definition, $X_{n,r}$ is a locally closed subset of $\mathbf{S}^0_n \times \mathbf{\Sigma}_{n,r}$. Hence it has a natural structure of reduced scheme.

Now for an arbitrary direct sum decomposition

$$\xi : H_{2n-r} \xrightarrow{\simeq} H_n \oplus H_{n-r} \tag{65}$$

we obtain the corresponding decomposition

$$\tilde{\xi} : \mathbf{S}_{2n-r} \xrightarrow{\simeq} \mathbf{S}_n \oplus \mathbf{\Sigma}_{n,r} \oplus \mathbf{S}_{n-r} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)). \tag{66}$$

Thus, considering the set $MI_{2n-r,r}$ of symplectic $(2n-r, r)$-instanton hyperwebs of quadrics as a subset of $\mathbf{S}_{2n-r}$, we obtain a natural projection

$$f_{n,r} : MI_{2n-r,r} \to \mathbf{S}_n \oplus \mathbf{\Sigma}_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)). \tag{67}$$

We now prove the following result parallel to Proposition 4.2.

Theorem 5.1. *Let $n \geq 3$ and $2 \leq r \leq n-1$.*

(i) For a general decomposition ξ in (65) there is an open dense subset $MI^*_{2n-r,r}(\xi)$ of $MI^*_{2n-r,r}$ and an isomorphism of reduced schemes

$$f_{n,r}:\ MI^*_{2n-r,r}(\xi) \xrightarrow{\simeq} X_{n,r}:\ A \mapsto (A_1(\xi), A_2(\xi)), \tag{68}$$

where $A_1(\xi)$ and $A_2(\xi)$ are defined by (66).

(ii) The inverse isomorphism is given by the formula

$$f^{-1}_{n,r}:\ X_{n,r} \xrightarrow{\simeq} MI^*_{2n-r,r}(\xi):\ (B,C) \mapsto\ \widetilde{\xi}^{-1}(B,\ C,\ -C^\vee \circ B^{-1} \circ C), \tag{69}$$

where $\widetilde{\xi}$ is defined by (66).

Proof. Set $MI^*_{2n-r,r}(\xi) := \{A \in MI^*_{2n-r,r} \mid A$ satisfies property (*) for the monomorphism $i: H_n \hookrightarrow H_{2n-r}$ defined by $\xi\}$. It follows from Definition 4.6and Remark 4.7 that, for a general decomposition ξ in (65), $MI^*_{2n-r,r}(\xi)$ is a dense open subset of $MI^*_{2n-r,r}$. Then, for this choice of ξ, the proof of this Theorem essentially mimics the proof of [T, Proposition 6.1] in which we make the substitution $m+1 \mapsto n,\ m \mapsto n-r$ and change the notation accordingly. □

The proof of the following theorem will be given in Subsection 5.2.

Theorem 5.2. *$X_{n,r}$ is irreducible of dimension $(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1)$.*

From Theorems 5.1 and 5.2 it follows that $MI^*_{2n-r,r}$ is irreducible of dimension $(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1)$ for any $n \leq 3$ and $2 \leq r \leq n-1$. Hence $I^*_{2n-r,r}$ is irreducible of dimension $4(2n-r)(r+1)-r(2r+1)$ for these values of n and r. Note that the irreducibility of $I^*_{2n-r,r}$ is also true when $r = n$, and in this case $I^*_{n,n}$ coincides with $I_{n,n}$. Substituting $2n-1 \mapsto n$, we obtain the following main result of the paper.

Theorem 5.3. *For any integer $r \geq 2$ and for any integer $n \geq r$ such that $n \equiv r(\mathrm{mod}2)$, $I^*_{n,r}$ is an irreducible component of $I_{n,r}$ of dimension $4n(r+1) - r(2r+1)$.*

5.2. Proof of the irreducibility of $X_{n,r}$. In this subsection we give the proof of Theorem 5.2. Define

$$\widetilde{X}_{n,r} := \{(D,C) \in (\mathbf{S}_n^\vee)^0 \times \mathbf{\Sigma}_{n,r} \mid\ (C^\vee \circ D \circ C : H_{n-r} \otimes V \to H^\vee_{n-r} \otimes V^\vee) \in \mathbf{S}_{n-r}\}, \tag{70}$$

a closed subscheme of $(\mathbf{S}_m^\vee)^0 \times \mathbf{\Sigma}_{n,r}$ defined by the equations

$$C^\vee \circ D \circ C \in \mathbf{S}_{n-r}. \tag{71}$$

Since the conditions (ii) and (iii) in the definition (33) of $X_{n,r}$ are open and $X_{n,r}$ is nonempty (see Theorem 5.1), the isomorphism

$$\mathbf{S}_n^0 \xrightarrow{\simeq} (\mathbf{S}_n^\vee)^0:\ B \mapsto B^{-1}$$

implies that $X_{n,r}$ is a nonempty open subset of $(\widetilde{X}_{n,r})_{red}$,

$$\varnothing \neq X_{n,r} \xhookrightarrow{\text{open}} (\widetilde{X}_{n,r})_{red}. \tag{72}$$

Fix a direct sum decomposition

$$H_n \xrightarrow{\simeq} H_{n-r} \oplus H_r.$$

Then any linear map

$$C \in \mathbf{\Sigma}_{n,r} = \mathrm{Hom}(H_{n-r}, H_n^\vee \otimes \wedge^2 V^\vee), \quad C: H_{n-r} \otimes V \to H_n^\vee \otimes V^\vee, \tag{73}$$

can be represented as a map

$$C : H_{n-r} \otimes V \to H_{n-r}^\vee \otimes V^\vee \ \oplus \ H_r^\vee \otimes V^\vee, \tag{74}$$

or else as a block matrix

$$C = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{75}$$

where

$$\phi \in \mathrm{Hom}(H_{n-r}, H_{n-r}^\vee) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_{n-r}, \quad \psi \in \mathbf{\Psi}_{n,r} := \mathrm{Hom}(H_{n-r}, H_r^\vee) \otimes \wedge^2 V^\vee. \tag{76}$$

Similarly, any $D \in (\mathbf{S}_n^\vee)^0 \subset \mathbf{S}_n^\vee = S^2 H_n \otimes \wedge^2 V \subset \mathrm{Hom}(H_n^\vee \otimes V^\vee, H_n \otimes V)$ can be represented in the form

$$D = \begin{pmatrix} D_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix}, \tag{77}$$

where

$$D_1 \in \mathbf{S}_{n-r}^\vee \subset \mathrm{Hom}(H_{n-r}^\vee \otimes V^\vee, H_{n-r} \otimes V), \tag{78}$$

$$\lambda \in \mathbf{L}_{n,r} := \mathrm{Hom}(H_r^\vee, H_{n-r}) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_r := S^2 H_r \otimes \wedge^2 V.$$

By (75) and (77) the composition

$$C^\vee \circ D \circ C : H_{n-r} \otimes V \to H_{n-r}^\vee \otimes V^\vee \ \ (C^\vee \circ D \circ C \in \wedge^2(H_{n-r}^\vee \otimes V^\vee))$$

can be written in the form

$$C^\vee \circ D \circ C = \phi^\vee \circ D_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda^\vee \circ \phi + \psi^\vee \circ \mu \circ \psi. \tag{79}$$

By (75)-(78) we have

$$\mathbf{S}_n^\vee \times \mathbf{\Sigma}_{n,r} = \mathbf{S}_{n-r}^\vee \times \mathbf{\Phi}_{n-r} \times \mathbf{\Psi}_{n,r} \times \mathbf{L}_{n,r} \times \mathbf{M}_r,$$

and there are well defined morphisms

$$\tilde{p} : \widetilde{X}_{n,r} \to \mathbf{L}_{n,r} \times \mathbf{M}_r : (D_1, \phi, \psi, \lambda, \mu) \mapsto (\lambda, \mu).$$

and

$$p := \tilde{p} | \overline{X}_{n,r} : \overline{X}_{n,r} \to \mathbf{L}_{n,r} \oplus \mathbf{M}_r,$$

where $\overline{X}_{n,r}$ is the closure of $X_{n,r}$ in $(\mathbf{S}_n^\vee)^0 \times \mathbf{\Sigma}_{n,r}$. We now invoke the following result from [T]:

Proposition 5.4. *Let $n \geq 2$. Then for any $D \in (\mathbf{S}_n^\vee)^0$ and for a general choice of the decomposition $H_n \xrightarrow{\sim} H_{n-r} \oplus H_r$, the block D_1 of D in (77) is nondegenerate.*

Proof. See [T, Proposition 7.3]. By repeatedly applying this proposition r times, we can find a decomposition $H_n \xrightarrow{\sim} H_{n-r} \oplus H_r$ such that $D_1 : H_{n-r}^\vee \otimes V^\vee \to H_{n-r} \otimes V$ in (77) is nondegenerate, i.e., $D_1 \in (\mathbf{S}_{n-r}^\vee)^0$. □

Let $\mathcal{X}$ be any irreducible component of $X_{n,r}$ and let $\overline{\mathcal{X}}$ be its closure in $\overline{X}_{n,r}$. Fix a point $z = (D_1, \phi, \psi, \lambda, \mu) \in \mathcal{X}$ not lying in the components of $X_{n,r}$ different from $\mathcal{X}$. Consider the morphism

$$f : \ \mathbb{A}^1 \to \overline{\mathcal{X}} : \ t \mapsto (D_1, t^2\phi, t\psi, t\lambda, t^2\mu), \quad f(1) = z, \tag{80}$$

which is well defined by (79). By definition, the point $f(0) = (D_1, 0, 0, 0, 0)$ lies in the fibre $p^{-1}(0,0)$. Hence, $p^{-1}(0,0) \cap \overline{\mathcal{X}} \neq \varnothing$. In other words,

$$\rho^{-1}(0,0) \neq \varnothing, \quad \text{where} \quad \rho := p|\overline{\mathcal{X}}. \tag{81}$$

Now, it follows from (79) and the definition of $\widetilde{X}_{n,r}$ that

$$\tilde{p}^{-1}(0,0) = \{(D_1, \phi, \psi) \in (\mathbf{S}^\vee_{n-r})^0 \times \mathbf{\Phi}_{n-r} \times \mathbf{\Psi}_{n,r} \mid \phi^\vee \circ D_1 \circ \phi \in \mathbf{S}_{n-r}\}. \tag{82}$$

Consider the set

$$Z_{n-r} = \{(D, \phi) \in (\mathbf{S}^\vee_{n-r})^0 \times \mathbf{\Phi}_{n-r} \mid \phi^\vee \circ D \circ \phi \in \mathbf{S}_{n-r}\}.$$

It carries a natural scheme structure, where it is a closed subscheme of $(\mathbf{S}^\vee_{n-r})^0 \times \mathbf{\Phi}_{n-r}$. Comparing the definition of Z_{n-r} with (82) we see that there are scheme-theoretic inclusions of schemes

$$\rho^{-1}(0,0) \subset p^{-1}(0,0) \subset \tilde{p}^{-1}(0,0) = Z_{n-r} \times \mathbf{\Psi}_{n,r}. \tag{83}$$

By [T, Theorem 7.2], Z_{n-r} is an integral scheme of dimension $4(n-r)(n-r+2)$. This together with (83) implies that

$$\begin{aligned}\dim \rho^{-1}(0,0) \le \dim p^{-1}(0,0) \le \dim Z_{n-r} + \dim \mathbf{\Psi}_{n,r} = 4(n-r)(n-r+2) + 6r(n-r) = \\ = (n-r)(4n+2r+8).\end{aligned} \tag{84}$$

Hence in view of (81)

$$\begin{aligned}\dim \overline{\mathcal{X}} \le \dim \rho^{-1}(0,0) + \dim \mathbf{L}_{n,r} + \dim \mathbf{M}_r \le (n-r)(4n+2r+8) + 6r(n-r) + 3r(r+1) = \\ = (2n-r)^2 + 4(2n-r)(r+1) - r(2r+1).\end{aligned} \tag{85}$$

On the other hand, formula (21), with $2n-r$ substituted for n, and Theorem 5.1(ii) show that, for any point $x \in \mathcal{X}$ such that $A := f_{n,r}^{-1}(x) \in MI^0_{2n-r,r}(\xi)$,

$$(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1) \le \dim_A MI^0_{2n-r,r}(\xi) = \dim \overline{\mathcal{X}}. \tag{86}$$

Comparing (85) with (86), we see that all the inequalities in (84)-(86) are equalities. In particular,

$$\dim \rho^{-1}(0,0) = \dim(Z_{n-r} \times \mathbf{\Psi}_{n,r}) = \dim \overline{\mathcal{X}} - \dim(\mathbf{L}_{n,r} \times \mathbf{M}_r). \tag{87}$$

Since by Theorem [T, Theorem 7.2] the scheme Z_{n-r} is integral and so $Z_{n-r} \times \mathbf{\Psi}_{n,r}$ is integral as well, (83) and (87) yield the equalities of integral schemes

$$\rho^{-1}(0,0) = p^{-1}(0,0) = \tilde{p}^{-1}(0,0) = Z_{n-r} \times \mathbf{\Psi}_{n,r}. \tag{88}$$

Now we invoke one auxiliary result from [T].

Lemma 5.5. *Let $f : X \to Y$ be a morphism of reduced schemes, where Y is a smooth integral scheme. Assume that there exists a closed point $y \in Y$ such that for any irreducible component X' of X the following conditions are satisfied:*

(a) $\dim f^{-1}(y) = \dim X' - \dim Y$,

(b) the scheme-theoretic inclusion of fibres $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$ is an isomorphism of integral schemes.

Then

(i) there exists an open subset U of Y containing the point y such that the morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is flat, and

(ii) X is integral.

Proof. See [T, Lemma 7.4]. □

Applying assertions (i)-(ii) of this lemma to $X = X_{n,r}$, $X' = \mathcal{X}$, $Y = \mathbf{L}_{n,r} \times \mathbf{M}_r$, $y = (0,0)$, $f = p$, and using (87) and (88), we obtain that $X_{n,r}$ is integral of dimension $(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1)$. Theorem 5.2 is proved.

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