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# MODULI OF SYMPLECTIC INSTANTON VECTOR BUNDLES OF HIGHER RANK ON PROJECTIVE SPACE $\mathbb{P}^3$

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ABSTRACT. Symplectic instanton vector bundles on the projective space  $\mathbb{P}^3$  constitute a natural generalization of mathematical instantons of rank 2. We study the moduli space  $I_{n,r}$  of rank- $2r$  symplectic instanton vector bundles on  $\mathbb{P}^3$  with  $r \geq 2$  and second Chern class  $n \geq r$ ,  $n \equiv r \pmod{2}$ . We give an explicit construction of an irreducible component  $I_{n,r}^*$  of this space for each such value of  $n$  and show that  $I_{n,r}^*$  has the expected dimension  $4n(r+1) - r(2r+1)$ .

## 1. INTRODUCTION

By a *symplectic instanton vector bundle* of rank  $2r$  and charge  $n$  (shortly, a *symplectic  $(n, r)$ -instanton*) on the 3-dimensional projective space  $\mathbb{P}^3$  we understand an algebraic vector bundle  $E = E_{2r}$  of rank  $2r$  on  $\mathbb{P}^3$  with Chern classes

$$(1) \quad c_1(E) = 0,$$

$$(2) \quad c_2(E) = n, \quad n \geq 1,$$

supplied with a symplectic structure and satisfying the vanishing conditions

$$(3) \quad h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0.$$

By a symplectic structure we mean an anti-self-dual isomorphism

$$(4) \quad \phi: E \xrightarrow{\sim} E^\vee, \quad \phi^\vee = -\phi,$$

considered modulo proportionality. The vanishing of the first Chern class (1) follows from the existence of a symplectic structure (4), and if  $r = 1$ , then the two conditions are equivalent. We will denote the moduli space of symplectic  $(n, r)$ -instantons by  $I_{n,r}$ .

For  $r = 1$  these bundles relate, via the so-called Atiyah-Ward correspondence, to rank-2 “physical” instantons over the 4-sphere  $S^4$ , these being anti-self-dual connections with structure group  $SU(2) = \mathbf{Sp}(1)$  [AW]. Important results on the moduli spaces  $I_n = I_{n,1}$  of rank-2 instantons have been obtained recently: smoothness [JV] for all  $n$ , irreducibility [T] for odd  $n$ .

Much less is known about the moduli spaces  $I_{n,r}$  for  $r > 1$ . In fact the symplectic instantons with  $r > 1$  are as natural as those with  $r = 1$ , for they are related, via the same Atiyah-Ward correspondence, to the anti-self-dual connections over  $S^4$  with structure group  $\mathbf{Sp}(r)$ , see [A]. As far as we know, the present paper is the first one addressing the properties of the corresponding spaces  $I_{n,r}$ . The tool we use to construct  $I_{n,r}$  is the monad method; it originates in the work of Horrocks [H] and is known as the ADHM construction of instantons since [ADHM]. It was further sharpened in the work of Barth [B], Barth and Hulek [BH] and Tyurin [Tju1], [Tju2]. This method permits to encode the instantons, usual ones or symplectic of higher rank, by hyperwebs of quadrics.

For a sample of the physical literature about symplectic instantons, see e.g. [Mc].

We fix basic terminology and notation in Section 2 and introduce the hyperwebs of quadrics in Section 3. We prove that, for any  $r \geq 2$  and for any  $n \geq r$  such that  $n \equiv r \pmod{2}$ , the moduli space  $I_{n,r}$  is nonempty and is realized as a free quotient  $MI_{n,r}/(GL(n)/\pm \text{id})$ , where  $MI_{n,r}$  is a Zariski locally closed subset of an affine space (see Theorem 3.1). Thus  $MI_{n,r}$  carries a natural structure of a reduced scheme, and  $I_{n,r}$  is an algebraic space. In Section 4 we give an explicit construction of vector bundles from  $I_{n,r}$  for the above values of  $n$  and  $r$  and introduce a component  $I_{n,r}^*$  of  $I_{n,r}$  characterized by a certain open condition (\*), see Definition 4.6. In Section 5 we prove Theorem 5.3 on the irreducibility of  $I_{n,r}^*$ , the main result of this paper.

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## 2. NOTATION AND CONVENTIONS

In many respects, we follow the exposition of [T], and we stick to the notation introduced therein. The base field  $\mathbf{k}$  is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If  $\mathcal{F}$  is a sheaf of  $\mathcal{O}_X$ -modules on an algebraic variety or a scheme  $X$ , then  $n\mathcal{F}$  denotes a direct sum of  $n$  copies of  $\mathcal{F}$ ,  $H^i(\mathcal{F})$  denotes the  $i^{\text{th}}$  cohomology group of  $\mathcal{F}$ ,  $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$ , and  $\mathcal{F}^\vee$  denotes the dual of  $\mathcal{F}$ , that is,  $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ . If  $X = \mathbb{P}^r$  and  $t$  is an integer, then by  $\mathcal{F}(t)$  we denote the sheaf  $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$ .  $[\mathcal{F}]$  will denote the isomorphism class of a sheaf  $\mathcal{F}$ . For any morphism of  $\mathcal{O}_X$ -sheaves  $f : \mathcal{F} \rightarrow \mathcal{F}'$  and any  $\mathbf{k}$ -vector space  $U$  (respectively, for any homomorphism  $f : U \rightarrow U'$  of  $\mathbf{k}$ -vector spaces) we will denote, for short, by the same letter  $f$  the induced morphism of sheaves  $\text{id} \otimes f : U \otimes \mathcal{F} \rightarrow U \otimes \mathcal{F}'$  (respectively, the induced morphism  $f \otimes \text{id} : U \otimes \mathcal{F} \rightarrow U' \otimes \mathcal{F}$ ).

We fix an integer  $n \geq 1$  and denote by  $H_n$  a fixed  $n$ -dimensional vector space over  $\mathbf{k}$ . Throughout the paper,  $V$  will be a fixed vector space of dimension 4 over  $\mathbf{k}$ , and we set  $\mathbb{P}^3 := P(V)$ . We reserve the letters  $u$  and  $v$  for denoting the two morphisms in the Euler exact sequence  $0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \rightarrow 0$ . For any  $\mathbf{k}$ -vector spaces  $U$  and  $W$  and any vector  $\phi \in \text{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \text{Hom}(U \otimes V, W \otimes V^\vee)$  understood as a linear map  $\phi : U \otimes V \rightarrow W \otimes V^\vee$  or, equivalently, as a map  $\sharp\phi : U \rightarrow W \otimes \wedge^2 V^\vee$ , we will denote by  $\tilde{\phi}$  the composition  $U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\sharp\phi} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} W \otimes \Omega_{\mathbb{P}^3}(2)$ , where  $\epsilon$  is the induced morphism in the exact triple  $0 \rightarrow \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 v^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} \Omega_{\mathbb{P}^3}(2) \rightarrow 0$  obtained by taking the second wedge power of the dual Euler exact sequence.

Given an integer  $m \geq 1$ , we denote by  $\mathbf{S}_m$  (resp.  $\Sigma_{m+1}$ ) the vector space  $S^2 H_m^\vee \otimes \wedge^2 V^\vee$  (resp.  $\text{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$ ). Abusing notation, we will denote by the same symbol a  $\mathbf{k}$ -vector space, say  $U$ , and the associated affine space  $\mathbf{V}(U^\vee) = \text{Spec}(\text{Sym}^* U^\vee)$ .

All the schemes considered in the paper are Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme  $\mathcal{X}$  we mean any closed point of some dense open subset of  $\mathcal{X}$ . An irreducible scheme is called generically reduced if it is reduced at a general point.

3. GENERALITIES ON SYMPLECTIC INSTANTONS AND DEFINITION OF  $MI_{n,r}$ 

In this section we enumerate some facts about symplectic instantons which are completely parallel to those for rank-2 usual instantons, see [T, Section 3].

For a given symplectic  $(n, r)$ -instanton  $E$ , the first condition (3) yields  $h^0(E(-i)) = 0, i \geq 0$ , which, together with the exact triple  $0 \rightarrow E(-j-1) \rightarrow E(-j) \rightarrow E(-j)|_{\mathbb{P}^2} \rightarrow 0$  for  $j = 0$  and (3), implies that  $h^0(E(-1)|_{\mathbb{P}^2}) = 0$ , hence also  $h^0(E(-i)|_{\mathbb{P}^2}) = 0, i \geq 1$ . The last equality for  $i = 2$ , together with (3) and the above triple for  $j = 2$ , gives  $h^1(E(-3)) = 0$ , hence also  $h^1(E(-4)) = 0$ . Then, from Serre duality and (4), we deduce:

$$(5) \quad h^i(E) = h^i(E(-1)) = h^{3-i}(E(-3)) = h^{3-i}(E(-4)) = 0, \quad i \neq 1, \quad h^i(E(-2)) = 0, \quad i \geq 0.$$

By Riemann-Roch and (3), (5), we have

$$(6) \quad h^1(E(-1)) = h^2(E(-3)) = n, \quad h^1(E) = h^2(E(-4)) = 2n - 2r.$$

By tensoring the dual Euler sequence by  $E$  we also obtain

$$(7) \quad h^1(E \otimes \Omega_{\mathbb{P}^3}^1) = h^2(E \otimes \Omega_{\mathbb{P}^3}^2) = 2n + 2r,$$

Consider a triple  $(E, f, \phi)$  where  $E$  is a  $(n, r)$ -instanton,  $f : H_n \xrightarrow{\cong} H^2(E(-3))$  an isomorphism and  $\phi : E \xrightarrow{\cong} E^\vee$  a symplectic structure on  $E$ . Two triples  $(E, f, \phi)$  and  $(E', f', \phi')$  are called equivalent if there is an isomorphism  $g : E \xrightarrow{\cong} E'$  such that  $g_* \circ f = \lambda f'$  with  $\lambda \in \{1, -1\}$  and  $\phi = g^\vee \circ \phi' \circ g$ , where  $g_* : H^2(E(-3)) \xrightarrow{\cong} H^2(E'(-3))$  is the induced isomorphism. We denote by  $[E, f, \phi]$  the equivalence class of a triple  $(E, f, \phi)$ . It follows from this definition that the set  $F_{[E]}$  of all equivalence classes  $[E, f, \phi]$  with given  $[E]$  is a homogeneous space of the group  $GL(H_n)/\{\pm \text{id}\}$ .

Each class  $[E, f, \phi]$  defines a point

$$(8) \quad A = A([E, f, \phi]) \in S^2 H_n^\vee \otimes \wedge^2 V^\vee$$

in the following way. Consider the exact sequences

$$(9) \quad \begin{aligned} 0 &\rightarrow \Omega_{\mathbb{P}^3}^1 \xrightarrow{i_1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0, \\ 0 &\rightarrow \Omega_{\mathbb{P}^3}^2 \rightarrow \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \Omega_{\mathbb{P}^3}^1 \rightarrow 0, \\ 0 &\rightarrow \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{i_2} \Omega_{\mathbb{P}^3}^2 \rightarrow 0, \end{aligned}$$

induced by the Koszul complex of  $V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^3}$ . Twisting these sequences by  $E$  and taking into account (3), (5)-(7), we obtain the vanishing

$$(10) \quad h^0(E \otimes \Omega_{\mathbb{P}^3}) = h^3(E \otimes \Omega_{\mathbb{P}^3}^2) = h^2(E \otimes \Omega_{\mathbb{P}^3}) = 0$$

and the diagram with exact rows

$$(11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^2(E(-4)) \otimes \wedge^4 V^\vee & \longrightarrow & H^2(E(-3)) \otimes \wedge^3 V^\vee & \xrightarrow{i_2} & H^2(E \otimes \Omega_{\mathbb{P}^3}^2) \longrightarrow 0 \\ & & & & \downarrow A' & & \cong \uparrow \partial \\ 0 & \longleftarrow & H^1(E) & \longleftarrow & H^1(E(-1)) \otimes V^\vee & \xleftarrow{i_1} & H^1(E \otimes \Omega_{\mathbb{P}^3}) \longleftarrow 0, \end{array}$$

where  $A' := i_1 \circ \partial^{-1} \circ i_2$ . The Euler exact sequence (9) yields the canonical isomorphism  $\omega_{\mathbb{P}^3} \xrightarrow{\cong} \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4)$ , and fixing an isomorphism  $\tau : \mathbf{k} \xrightarrow{\cong} \wedge^4 V^\vee$  we have the isomorphisms  $\tilde{\tau} : V \xrightarrow{\cong} \wedge^3 V^\vee$  and  $\hat{\tau} : \omega_{\mathbb{P}^3} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^3}(-4)$ . We define  $A$  in (8) as the composition

$$(12) \quad A : H_n \otimes V \xrightarrow{\tilde{\tau}} H_n \otimes \wedge^3 V^\vee \xrightarrow{f} H^2(E(-3)) \otimes \wedge^3 V^\vee \xrightarrow{A'} H^1(E(-1)) \otimes V^\vee \xrightarrow{\phi} \tau$$

$$\xrightarrow[\cong]{\phi} H^1(E^\vee(-1)) \otimes V^\vee \xrightarrow[\cong]{SD} H^2(E(1) \otimes \omega_{\mathbb{P}^3})^\vee \otimes V^\vee \xrightarrow[\cong]{\hat{\tau}} H^2(E(-3))^\vee \otimes V^\vee \xrightarrow[\cong]{f^\vee} H_n^\vee \otimes V^\vee,$$

where  $SD$  is the Serre duality isomorphism. One can verify that  $A$  is a skew symmetric map which depends only on the class  $[E, f, \phi]$ , but does not depend on the choice of  $\tau$ , and that  $A \in \wedge^2(H_n^\vee \otimes V^\vee)$  lies in the direct summand  $\mathbf{S}_n = S^2 H_n^\vee \otimes \wedge^2 V^\vee$  of the canonical decomposition

$$(13) \quad \wedge^2(H_n^\vee \otimes V^\vee) = S^2 H_n^\vee \otimes \wedge^2 V^\vee \oplus \wedge^2 H_n^\vee \otimes S^2 V^\vee.$$

Here  $\mathbf{S}_n$  is the space of hyperwebs of quadrics in  $H_n$ . For this reason we call  $A$  the  $(n, r)$ -instanton hyperweb of quadrics corresponding to the data  $[E, f, \phi]$ .

Denote  $W_A := H_n \otimes V / \ker A$ . Using the above chain of isomorphisms we can rewrite the diagram (11) as

$$(14) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \ker A & \longrightarrow & H_n \otimes V & \xrightarrow{c_A} & W_A \longrightarrow 0 \\ & & & & \downarrow A & & \cong \downarrow q_A \\ 0 & \longleftarrow & \ker A^\vee & \longleftarrow & H_n^\vee \otimes V^\vee & \xleftarrow{c_A^\vee} & W_A^\vee \longleftarrow 0. \end{array}$$

In view of (7),  $\dim W_A = 2n + 2r$  and  $q_A : W_A \xrightarrow{\cong} W_A^\vee$  is a skew-symmetric isomorphism. An important property of  $A = A([E, f, \phi])$  is that the induced morphism of sheaves

$$(15) \quad a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_A^\vee} H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{ev} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$

is surjective and the composition  $H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{q_A} W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$  is zero. Applying Beilinson spectral sequence [Bei] to  $E(-1)$ , we see that  $E \simeq \ker(a_A^\vee \circ q_A) / \text{Im } a_A$ . Thus  $A$  defines a monad

$$(16) \quad \mathcal{M}_A : 0 \rightarrow H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee \circ q_A} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

whose cohomology sheaf

$$(17) \quad E_{2r}(A) := \ker(a_A^\vee \circ q_A) / \text{Im } a_A.$$

is isomorphic to  $E$ . Twisting  $\mathcal{M}_A$  by  $\mathcal{O}_{\mathbb{P}^3}(-3)$  and using (17), we obtain the isomorphism  $f : H_n \xrightarrow{\cong} H^2(E(-3))$ . Furthermore, the fact that  $q_A$  is symplectic implies that there is a canonical isomorphism of  $\mathcal{M}_A$  with its dual which induces the symplectic isomorphism  $\phi : E \xrightarrow{\cong} E^\vee$ . Thus, the data  $[E, f, \phi]$  are recovered from  $A$ . This leads to the following description of the moduli space  $I_{n,r}$ . Consider the set of  $(n, r)$ -instanton hyperwebs of quadrics

$$(18) \quad MI_{n,r} := \left\{ A \in \mathbf{S}_n \left| \begin{array}{l} \text{(i) } rk(A : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee) = 2n + 2r, \\ \text{(ii) the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \text{ defined} \\ \text{by } A \text{ in (15) is surjective,} \\ \text{(iii) } h^0(E_{2r}(A)) = 0, \text{ where } E_{2r}(A) = \ker(a_A^\vee \circ q_A) / \text{Im } a_A \\ \text{and } q_A : W_A \xrightarrow{\cong} W_A^\vee \text{ is a symplectic isomorphism} \\ \text{associated to } A \text{ by (14).} \end{array} \right. \right\}$$

It is a locally closed subscheme of the affine space  $\mathbf{S}_n$ .

**Theorem 3.1.** *The natural morphism*

$$(19) \quad \pi_{n,r} : MI_{n,r} \rightarrow I_{n,r}, \quad A \mapsto [E_{2r}(A)],$$

is a principal  $GL(H_n)/\{\pm \text{id}\}$ -bundle in the étale topology. Hence  $I_{n,r}$  is a quotient stack  $MI_{n,r}/(GL(H_n)/\{\pm \text{id}\})$ , making it an algebraic space.

*Proof.* See [T, Section 3].  $\square$

Each fibre  $F_{[E]} = \pi_n^{-1}([E])$  over an arbitrary point  $[E] \in I_{n,r}$  is a principal homogeneous space of the group  $GL(H_n)/\{\pm \text{id}\}$ . Hence the irreducibility of  $(I_{n,r})_{\text{red}}$  is equivalent to the irreducibility of the scheme  $(MI_{n,r})_{\text{red}}$ .

We can also state:

**Theorem 3.2.** *For each  $n \geq 1$ , the space  $MI_{n,r}$  of  $(n, r)$ -instanton hyperwebs of quadrics is a locally closed subscheme of the vector space  $\mathbf{S}_n$  given locally at any point  $A \in MI_{n,r}$  by*

$$(20) \quad \binom{2n-2r}{2} = 2n^2 - n(4r+1) + r(2r+1)$$

*equations obtained as the rank condition (i) in (18).*

Note that from (20) it follows that

$$(21) \quad \dim_{[A]} MI_{n,r} \geq \dim \mathbf{S}_n - (2n^2 - n(4r+1) + r(2r+1)) = n^2 + 4n(r+1) - r(2r+1)$$

at any point  $A \in MI_{n,r}$ . Hence,

$$(22) \quad \dim_{[E]} I_{n,r} \geq 4n(r+1) - r(2r+1)$$

at any point  $[E] \in I_{n,r}$ , since  $MI_{n,r} \rightarrow I_{n,r}$  is a principal  $GL(H_n)/\{\pm \text{id}\}$ -bundle in the étale topology.

#### 4. EXPLICIT CONSTRUCTION OF SYMPLECTIC INSTANTONS

**4.1. Example: symplectic  $(n, n)$ -instantons.** In this subsection we recall some known facts about symplectic  $(n, n)$ -instantons and their relation to usual rank-2 instantons, see [T, Sections 5-6]. We first show that each invertible hyperweb of quadrics  $A \in \mathbf{S}_n$  naturally leads to a construction of a symplectic  $(n, n)$ -instanton  $E_{2n}(A)$  on  $\mathbb{P}^3$ . Given an integer  $n \geq 1$ , set

$$(23) \quad \mathbf{S}_n^0 := \{A \in \mathbf{S}_n \mid A : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee \text{ is an invertible map}\}.$$

Then  $\mathbf{S}_n^0$  is a dense open subset of  $\mathbf{S}_n$ , and it is easy to see that for any  $A \in \mathbf{S}_n^0$  the following conditions are satisfied.

(1) The morphism  $\tilde{A} : H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow H_n^\vee \otimes \Omega_{\mathbb{P}^3}(1)$  induced by  $A$  is a subbundle embedding, and

$$(24) \quad E_{2n}(A) := \text{coker}(\tilde{A})$$

is a symplectic  $(n, n)$ -instanton, that is,

$$(25) \quad [E_{2n}(A)] \in I_{n,n}.$$

(2) For all  $i \geq 0$ ,

$$(26) \quad h^i(E_{2n}(A)) = h^i(E_{2n}(A)(-2)) = 0.$$

This follows from the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 (27) \quad 0 & \longrightarrow & H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{A}} & H_n^\vee \otimes \Omega_{\mathbb{P}^3}(1) & \xrightarrow{e} & E_{2n}(A) \longrightarrow 0 \\
 & & \downarrow u & & \downarrow v^\vee & & \\
 & & H_n \otimes V \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow[\simeq]{A} & H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & & \\
 & & \downarrow v & & \downarrow u^\vee & & \\
 0 \rightarrow E_{2n}(A)^\vee & \longrightarrow & H_n \otimes T_{\mathbb{P}^3}(-1) & \xrightarrow{\tilde{A}^\vee} & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Thus  $\mathbf{S}_n^0 \subset MI_{n,n}$ . In fact, the following result is true.

**Proposition 4.1.**  $\mathbf{S}_n^0 = MI_{n,n}$ . In particular,  $MI_{n,n}$  is irreducible of dimension  $3n^2 + 3n$ , and hence  $I_{n,n}$  is irreducible of dimension  $2n^2 + 3n$ .

*Proof.* We have to show that  $MI_{n,n} \subset \mathbf{S}_n^0$ . Let  $A \in MI_{n,n}$ . Since  $n = r$ , by condition (i) from (18) the rank of the hyperweb of quadrics  $A : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$  is  $2n + 2r = 4n = \dim H_n^\vee \otimes V^\vee$ , hence  $A$  is invertible. By (23), this means that  $A \in \mathbf{S}_n^0$ .  $\square$

Now we proceed to spell out the relation between symplectic  $(n, n)$ -instantons and usual rank-2 instantons with second Chern class  $2n - 1$ . This relation is given at the level of spaces of hyperwebs of quadrics  $MI_{n,n}$  and  $MI_{2n-1,1}$  interpreted as spaces of monads.

We need some more notation. Let  $B \in \mathbf{S}_n^0$ . By definition,  $B$  is an invertible anti-self-dual map  $H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$ . Then the inverse

$$(28) \quad B^{-1} : H_n^\vee \otimes V^\vee \rightarrow H_n \otimes V$$

is also anti-self-dual. Consider the vector space  $\Sigma_n = H_n^\vee \otimes H_{n-1}^\vee \otimes \wedge^2 V^\vee$ . An element  $C \in \Sigma_n$  can be viewed as a linear map  $C : H_{n-1} \otimes V \rightarrow H_n^\vee \otimes V^\vee$ , and its transpose  $C^\vee$  as a map  $C^\vee : H_n \otimes V \rightarrow H_{n-1}^\vee \otimes V^\vee$ . As the composition  $C^\vee \circ B^{-1} \circ C$  is anti-self-dual, we can consider it as an element of  $\wedge^2(H_{n-1}^\vee \otimes V^\vee) \simeq \mathbf{S}_{n-1} \oplus \wedge^2 H_{n-1}^\vee \otimes S^2 V^\vee$  (cf. (13)). Thus the condition

$$(29) \quad C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-1}$$

makes sense.

Next, consider the upper horizontal triple in (27) with  $A = B$ . Twisting it by  $\mathcal{O}_{\mathbb{P}^3}(1)$  and passing to global sections we obtain the exact triple

$$(30) \quad 0 \rightarrow H_n \xrightarrow{\sharp B} H_n^\vee \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1)) \rightarrow 0$$

Besides, interpreting  $C \in \Sigma_n$  as a map  $\sharp C : H_{n-1} \rightarrow H_n^\vee \otimes \wedge^2 V^\vee$ , we obtain the composition  $H_{n-1} \xrightarrow{\sharp C} H_n^\vee \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1))$  which induces the morphism of sheaves

$$(31) \quad \rho_{B,C} : H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_{2n}(B).$$

Note also that the maps  $B : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$  and  $C : H_{n-1} \otimes V \rightarrow H_n^\vee \otimes V^\vee$  provide a map  $(H_n \oplus H_{n-1}) \otimes V \rightarrow H_n^\vee \otimes V^\vee$ , which induces the morphism of sheaves

$$(32) \quad \tau_{B,C} : (H_n \oplus H_{n-1}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}.$$

Now set

$$(33) \quad X_n := \left\{ (B, C) \in \mathbf{S}_n^0 \times \Sigma_n \left| \begin{array}{l} \text{(i) the condition (29) is satisfied,} \\ \text{(ii) } \rho_{B,C} \text{ in (31) is a subbundle inclusion,} \\ \text{(iii) } \tau_{B,C} \text{ in (32) is a subbundle inclusion.} \end{array} \right. \right\}$$

By definition,  $X_n$  is a locally closed subset of  $\mathbf{S}_n^0 \times \Sigma_n$ . Hence it is naturally endowed with a structure of a reduced scheme.

Now for any direct sum decomposition

$$(34) \quad \xi : H_{2n-1} \xrightarrow{\cong} H_n \oplus H_{n-1},$$

we obtain the corresponding decomposition

$$(35) \quad \tilde{\xi} : \mathbf{S}_{2n-1} \xrightarrow{\cong} \mathbf{S}_n \oplus \Sigma_n \oplus \mathbf{S}_{n-1} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).$$

Thus, considering the set  $MI_{2n-1,1}$  of  $(2n-1)$ -instanton hyperwebs of quadrics as a subset of  $\mathbf{S}_{2n-1}$ , we obtain a natural projection

$$(36) \quad f_n : MI_{2n-1,1} \rightarrow \mathbf{S}_n \oplus \Sigma_n : A \mapsto (A_1(\xi), A_2(\xi)).$$

The following result is proved in [T, Theorems 1.1, 6.1 and Remark 7.6].

**Proposition 4.2.** *For a general decomposition  $\xi$  in (34), there exists a dense open subset  $MI_{2n-1,1}(\xi)$  of  $MI_{2n-1,1}$  such that the projection  $f_n$  in (36) induces an isomorphism or integral schemes*

$$(37) \quad f_n : MI_{2n-1,1}(\xi) \xrightarrow{\cong} X_n : A \mapsto (A_1(\xi), A_2(\xi)).$$

The inverse isomorphism is given by the formula

$$(38) \quad f_n^{-1} : X_n \xrightarrow{\cong} MI_{2n-1,1}(\xi) : (B, C) \mapsto \tilde{\xi}^{-1}(B, C, -C^\vee \circ B^{-1} \circ C).$$

Besides, the projection

$$(39) \quad pr_1 : X_n \rightarrow \mathbf{S}_n^0 : (B, C) \mapsto B$$

is dominant.

It is not hard to check that the morphism  $\rho_{B,C} : H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_{2n}(B)$  defined in (31) satisfies the condition  ${}^t\rho_{B,C} \circ \rho_{B,C} = 0$ , where  ${}^t\rho_{B,C}$  is the composition

$${}^t\rho_{B,C} : E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$

and  $\phi$  is a symplectic structure on  $E_{2n}(B)$  (cf. [T, formulas (71)-(72)]). In other words, we obtain an anti-self-dual monad

$$(40) \quad 0 \rightarrow H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\rho_{B,C}} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

with cohomology sheaf

$$(41) \quad E_2(A) = E_2(B, C) := \ker {}^t\rho_{B,C} / \operatorname{im} \rho_{B,C}, \quad A = f_n^{-1}(B, C).$$

Next, by (19) we have the natural projection

$$(42) \quad \pi_{2n-1,1} : MI_{2n-1,1} \rightarrow I_{2n-1,1} : A \mapsto [E_2(A)].$$

We have the following interpretation of the isomorphism (38) on the level of vector bundles:

$$(43) \quad [E_2(B, C)] = \pi_{2n-1,1}(f_n^{-1}(B, C)).$$



**Remark 4.3.** Note that, according to the definitions (16)-(18) of  $MI_{2n-1,1}$  and  $MI_{n,n}$ , for any  $A \in MI_{2n-1,1}$ , if  $B = A_1(\xi)$  is defined by the direct sum decomposition (35), one has two other anti-self-dual monads

$$(44) \quad \mathcal{M}_A : 0 \rightarrow H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee \circ q_A} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

$$(45) \quad \mathcal{M}_B : 0 \rightarrow H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_B} W_B \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_B^\vee \circ q_B} H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

with cohomology sheaves

$$(46) \quad E_2(A) = \ker(a_A^\vee \circ q_A) / \operatorname{im} a_A, \quad E_{2n}(B) = \ker(a_B^\vee \circ q_B) / \operatorname{im} a_B$$

respectively. Moreover, (40) and (41) provide an isomorphism  $w : W_B = H^2(E_2(B) \otimes \Omega_{\mathbb{P}^3}) \xrightarrow{\cong} H^2(E_{2n}(A) \otimes \Omega_{\mathbb{P}^3}) = W_A$ . We thus obtain a commutative anti-self-dual diagram relating these monads:

$$(47) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & H_n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_B} & W_B \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{q_B} & W_B^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_B^\vee} & H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\ & & \downarrow i_\xi & & \cong \downarrow w & & w^\vee \uparrow \cong & & \uparrow i_\xi^\vee & & \\ 0 & \longrightarrow & H_{2n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{a_A} & W_A \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{q_A} & W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{a_A^\vee} & H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0, \end{array}$$

where  $i_\xi : H_n \hookrightarrow H_{2n-1}$  is the embedding induced by the decomposition (34). In view of (46) and the canonical isomorphism  $H_{2n-1}/i_\xi(H_n) \simeq H_{n-1}$ , from this diagram we obtain the monad

$$(48) \quad \mathcal{M}_{A,B} : 0 \rightarrow H_{n-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_{A,B}} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{a_{A,B}^\vee} H_{2n-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

with cohomology sheaf

$$(49) \quad E_2(A) = \ker(a_{A,B}^\vee \circ \phi) / \operatorname{im} a_A.$$

We call (48) the *quotient monad* of the monads (44) and (45).

**Remark 4.4.** Note that, by Proposition 4.2, the set of all diagrams (47) is parametrized by the irreducible variety  $I_{2n-1,1}(\xi)$ .

**4.2. Example: a special family of symplectic  $(n, r)$ -instantons.** Now assume  $n \geq 2$  and, for any integer  $r$ ,  $2 \leq r \leq n-1$ , consider an inclusion

$$(50) \quad \tau : H_{2n-r} \hookrightarrow H_{2n-1}$$

such that

$$(51) \quad \tau(H_{2n-r}) \supset i_\xi(H_n).$$

We obtain a hyperweb of quadrics

$$A_\tau \in S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$$

as the image of  $A$  under the map  $S^2 H_{2n-1}^\vee \otimes \wedge^2 V^\vee \rightarrow S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$  induced by  $\tau$ . The corresponding monad

$$(52) \quad \mathcal{M}_\tau : 0 \rightarrow H_{2n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_\tau} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_\tau^\vee \circ q_A} H_{2n-r}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

has a rank- $2r$  cohomology bundle

$$(53) \quad E_{2r}(A_\tau) = \ker(a_\tau^\vee \circ q_A) / \operatorname{im} a_\tau.$$

where  $a_\tau := a_A \circ \tau$ . By construction,  $E_{2r}(A_\tau)$  inherits a natural symplectic structure

$$(54) \quad \phi_r : E_{2r}(A_\tau) \xrightarrow{\sim} E_{2r}(A_\tau)^\vee.$$

Besides, in view of (51), the monad (52) can be inserted as a middle row into the diagram (47), extending it to a three-row commutative anti-self-dual diagram. Arguing as in Remark 4.3 we obtain, in addition to the quotient monad (48), two more quotient monads:

$$(55) \quad \mathcal{M}'_\tau : 0 \rightarrow H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a'_\tau} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{a'^\vee_\tau} H_{n-r}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

$$E_{2r}(A_\tau) = \ker(a'^\vee_\tau \circ \phi) / \operatorname{im} a'_\tau,$$

$$(56) \quad \mathcal{M}''_\tau : 0 \rightarrow H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a''_\tau} E_{2r}(B) \xrightarrow{\phi_\tau} E_{2r}(B)^\vee \xrightarrow{a''^\vee_\tau} H_{r-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

$$E_2(A) = \ker(a'^\vee_\tau \circ \phi_\tau) / \operatorname{im} a_A.$$

From (26) and (55) we easily deduce:

$$(57) \quad h^0(E_{2r}(A_\tau)) = h^i(E_{2r}(A_\tau)(-2)) = 0, \quad i \geq 0, \quad c_2(E_{2r}(A_\tau)) = 2n - r.$$

By definition, this together with (52)-(54) means that

$$(58) \quad [E_{2r}(A_\tau)] \in I_{2n-r,r}.$$

**Remark 4.5.** Observe that, in view of (50), the maps  $\tau$  belong to the set

$$N_{n,r} := \{\tau \in \operatorname{Hom}(H_{2n-r}, H_{2n-1}) \mid \tau \text{ is injective and } \operatorname{im} \tau \supset \operatorname{im} i_\xi\}.$$

When  $A \in MI_{2n-1,1}(\xi)$  is fixed,  $N_{n,r}$  parametrizes some family of hyperwebs  $A_\tau$  from  $MI_{2n-r,r}$ . Since  $N_{n,r}$  is a principal  $GL(H_{2n-r})$ -bundle over an open subset of the Grassmannian  $Gr(n-r, n-1)$ , it is irreducible. Thus, by Remark 4.4, the family of the three-row extensions of the diagrams (47) can be parametrized by the irreducible variety  $MI_{2n-1,1}(\xi) \times N_{n,r}$ . Hence the family  $D_{n,r}$  of isomorphism classes of symplectic rank-2r bundles obtained from these diagrams by formula (53) is an irreducible locally closed subset of  $I_{2n-r,r}$ .

Note that it is a priori not clear whether the closure of  $D_{n,r}$  in  $I_{2n-r,r}$  is an irreducible component of  $I_{2n-r,r}$ .

**Definition 4.6.** Let  $2 \leq r \leq n-1$ . We say that  $A \in MI_{2n-r,r}$  satisfies property (\*) if there exists a monomorphism  $i : H_n \hookrightarrow H_{2n-r}$  such that the image  $B$  of  $A$  under the surjection  $\mathbf{S}_{2n-r} \twoheadrightarrow \mathbf{S}_n$  induced by  $i$  is invertible as a homomorphism  $B : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$ .

The property (\*) is clearly an open condition on  $A$ . Moreover, since  $\pi_{2n-r,r} : MI_{2n-r,r} \rightarrow I_{2n-r,r}$  is a principal bundle (Theorem 3.1), if an element  $A \in \pi_{2n-r,r}^{-1}([E_{2r}])$  satisfies (\*), then any other point  $A' \in \pi_{2n-r,r}^{-1}([E_{2r}])$  satisfies (\*). We thus say that  $[E_{2r}] \in I_{2n-r,r}$  satisfies property (\*) if some (hence any)  $A \in \pi_{2n-r,r}^{-1}([E_{2r}])$  satisfies property (\*). It is obviously an open condition on  $[E_{2r}] \in I_{2n-r,r}$ .

**Remark 4.7.** By Proposition 4.2 and using (51), we see that any  $[E_{2r}] \in D_{n,r}$ , as well as any  $A \in f_n^{-1}(D_{n,r})$  satisfies property (\*). We define

$$(59) \quad I_{2n-r,r}^* := I_{(1)} \cup \dots \cup I_{(k)},$$

where  $I_{(1)}, \dots, I_{(k)}$  are all the irreducible components of  $I_{2n-r,r}$  whose general points satisfy property (\*). By definition,  $D_{n,r} \subset I_{2n-r,r}^*$ , hence  $I_{2n-r,r}^*$  is nonempty. We also set  $MI_{2n-r,r}^* = \pi_{2n-r,r}^{-1}(I_{2n-r,r}^*)$ , so that the map  $\pi_{2n-r,r} : MI_{2n-r,r}^* \rightarrow I_{2n-r,r}^*$  is a principal bundle with structure group  $GL(H_{2n-r})/\{\pm 1\}$ .

5. IRREDUCIBILITY OF  $I_{2n-r,r}^*$ 

**5.1. A dense open subset  $X_{n,r}$  of  $MI_{2n-r,r}^*$ . Reduction of the irreducibility of  $I_{n,r}^*$  to that of  $X_{n,r}$ .** In this section we prove the irreducibility of the component  $I_{2n-r,r}^*$  of  $I_{2n-r,r}$  defined in (59), see Theorem 5.3. The explicit construction of symplectic instantons in Section 4 gives us a hint to the proof. We proceed along the lines of Subsection 4.1.

Take any  $B \in \mathbf{S}_n^0$  and consider it as an invertible anti-self-dual linear map  $H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$ . Then  $B^{-1}$  is also anti-self-dual. Let

$$(60) \quad \Sigma_{n,r} := H_{n-r}^\vee \otimes H_n^\vee \otimes \wedge^2 V^\vee.$$

An element  $C \in \Sigma_n$  can be understood as a map  $C : H_{n-r} \otimes V \rightarrow H_n^\vee \otimes V^\vee$ , and its transpose  $C^\vee$  is a map  $H_n \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee$ . The composition  $C^\vee \circ B^{-1} \circ C$  is anti-self-dual, i.e., it is an element of  $\wedge^2(H_{n-r}^\vee \otimes V^\vee) \simeq \mathbf{S}_{n-r} \oplus \wedge^2 H_{n-r}^\vee \otimes S^2 V^\vee$  (cf. (13)). We will later impose the condition

$$(61) \quad C^\vee \circ B^{-1} \circ C \in \mathbf{S}_{n-r}.$$

Next, as in (30), we have a well defined epimorphism  $\epsilon(B) : H_n^\vee \otimes \wedge^2 V^\vee \rightarrow H^0(E_{2n}(B)(1))$ . Besides, interpreting the above element  $C \in \Sigma_{n,r}$  as a map  ${}^{\sharp}C : H_{n-r} \rightarrow H_n^\vee \otimes \wedge^2 V^\vee$ , we obtain the composition  $H_{n-r} \xrightarrow{{}^{\sharp}C} H_n^\vee \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1))$  which induces the morphism of sheaves

$$(62) \quad \rho_{B,C} : H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_{2n}(B).$$

Note also that  $B : H_n \otimes V \rightarrow H_n^\vee \otimes V^\vee$  and  $C : H_{n-r} \otimes V \rightarrow H_n^\vee \otimes V^\vee$  define a map  $(H_n \oplus H_{n-r}) \otimes V \rightarrow H_n^\vee \otimes V^\vee$  which induces the morphism of sheaves

$$(63) \quad \tau_{B,C} : (H_n \oplus H_{n-r}) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow H_n^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}.$$

Now set

$$(64) \quad X_{n,r} := \left\{ (B, C) \in \mathbf{S}_n^0 \times \Sigma_{n,r} \left| \begin{array}{l} \text{(i) the condition (61) is satisfied,} \\ \text{(ii) } \rho_{B,C} \text{ in (62) is a subbundle inclusion,} \\ \text{(iii) } \tau_{B,C} \text{ in (63) is a subbundle inclusion.} \end{array} \right. \right\}$$

By definition,  $X_{n,r}$  is a locally closed subset of  $\mathbf{S}_n^0 \times \Sigma_{n,r}$ . Hence it has a natural structure of reduced scheme.

Now for an arbitrary direct sum decomposition

$$(65) \quad \xi : H_{2n-r} \xrightarrow{\cong} H_n \oplus H_{n-r}$$

we obtain the corresponding decomposition

$$(66) \quad \tilde{\xi} : \mathbf{S}_{2n-r} \xrightarrow{\cong} \mathbf{S}_n \oplus \Sigma_{n,r} \oplus \mathbf{S}_{n-r} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).$$

Thus, considering the set  $MI_{2n-r,r}$  of symplectic  $(2n-r, r)$ -instanton hyperwebs of quadrics as a subset of  $\mathbf{S}_{2n-r}$ , we obtain a natural projection

$$(67) \quad f_{n,r} : MI_{2n-r,r} \rightarrow \mathbf{S}_n \oplus \Sigma_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)).$$

We now prove the following result parallel to Proposition 4.2.

**Theorem 5.1.** *Let  $n \geq 3$  and  $2 \leq r \leq n - 1$ .*

(i) *For a general decomposition  $\xi$  in (65) there is an open dense subset  $MI_{2n-r,r}^*(\xi)$  of  $MI_{2n-r,r}^*$  and an isomorphism of reduced schemes*

$$(68) \quad f_{n,r} : MI_{2n-r,r}^*(\xi) \xrightarrow{\cong} X_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)),$$

where  $A_1(\xi)$  and  $A_2(\xi)$  are defined by (66).

(ii) *The inverse isomorphism is given by the formula*

$$(69) \quad f_{n,r}^{-1} : X_{n,r} \xrightarrow{\cong} MI_{2n-r,r}^*(\xi) : (B, C) \mapsto \tilde{\xi}^{-1}(B, C, -C^\vee \circ B^{-1} \circ C),$$

where  $\tilde{\xi}$  is defined by (66).

*Proof.* Set  $MI_{2n-r,r}^*(\xi) := \{A \in MI_{2n-r,r}^* \mid A \text{ satisfies property } (*) \text{ for the monomorphism } i : H_n \hookrightarrow H_{2n-r} \text{ defined by } \xi\}$ . It follows from Definition 4.6 and Remark 4.7 that, for a general decomposition  $\xi$  in (65),  $MI_{2n-r,r}^*(\xi)$  is a dense open subset of  $MI_{2n-r,r}^*$ . Then, for this choice of  $\xi$ , the proof of this Theorem essentially mimics the proof of [T, Proposition 6.1] in which we make the substitution  $m + 1 \mapsto n$ ,  $m \mapsto n - r$  and change the notation accordingly.  $\square$

The proof of the following theorem will be given in Subsection 5.2.

**Theorem 5.2.**  *$X_{n,r}$  is irreducible of dimension  $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$ .*

From Theorems 5.1 and 5.2 it follows that  $MI_{2n-r,r}^*$  is irreducible of dimension  $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$  for any  $n \leq 3$  and  $2 \leq r \leq n - 1$ . Hence  $I_{2n-r,r}^*$  is irreducible of dimension  $4(2n - r)(r + 1) - r(2r + 1)$  for these values of  $n$  and  $r$ . Note that the irreducibility of  $I_{2n-r,r}^*$  is also true when  $r = n$ , and in this case  $I_{n,n}^*$  coincides with  $I_{n,n}$ . Substituting  $2n - 1 \mapsto n$ , we obtain the following main result of the paper.

**Theorem 5.3.** *For any integer  $r \geq 2$  and for any integer  $n \geq r$  such that  $n \equiv r \pmod{2}$ ,  $I_{n,r}^*$  is an irreducible component of  $I_{n,r}$  of dimension  $4n(r + 1) - r(2r + 1)$ .*

**5.2. Proof of the irreducibility of  $X_{n,r}$ .** In this subsection we give the proof of Theorem 5.2. Define

$$(70) \quad \tilde{X}_{n,r} := \{(D, C) \in (\mathbf{S}_n^\vee)^0 \times \Sigma_{n,r} \mid (C^\vee \circ D \circ C : H_{n-r} \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee) \in \mathbf{S}_{n-r}\},$$

a closed subscheme of  $(\mathbf{S}_m^\vee)^0 \times \Sigma_{n,r}$  defined by the equations

$$(71) \quad C^\vee \circ D \circ C \in \mathbf{S}_{n-r}.$$

Since the conditions (ii) and (iii) in the definition (33) of  $X_{n,r}$  are open and  $X_{n,r}$  is nonempty (see Theorem 5.1), the isomorphism

$$\mathbf{S}_n^0 \xrightarrow{\cong} (\mathbf{S}_n^\vee)^0 : B \mapsto B^{-1}$$

implies that  $X_{n,r}$  is a nonempty open subset of  $(\tilde{X}_{n,r})_{\text{red}}$ ,

$$(72) \quad \emptyset \neq X_{n,r} \xrightarrow{\text{open}} (\tilde{X}_{n,r})_{\text{red}}.$$

Fix a direct sum decomposition

$$H_n \xrightarrow{\cong} H_{n-r} \oplus H_r.$$

Then any linear map

$$(73) \quad C \in \Sigma_{n,r} = \text{Hom}(H_{n-r}, H_n^\vee \otimes \wedge^2 V^\vee), \quad C : H_{n-r} \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee,$$

can be represented as a map

$$(74) \quad C : H_{n-r} \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee \oplus H_r^\vee \otimes V^\vee,$$

or else as a block matrix

$$(75) \quad C = \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

where

$$(76) \quad \phi \in \text{Hom}(H_{n-r}, H_{n-r}^\vee) \otimes \wedge^2 V^\vee = \Phi_{n-r}, \quad \psi \in \Psi_{n,r} := \text{Hom}(H_{n-r}, H_r^\vee) \otimes \wedge^2 V^\vee.$$

Similarly, any  $D \in (\mathbf{S}_n^\vee)^0 \subset \mathbf{S}_n^\vee = S^2 H_n \otimes \wedge^2 V \subset \text{Hom}(H_n^\vee \otimes V^\vee, H_n \otimes V)$  can be represented in the form

$$(77) \quad D = \begin{pmatrix} D_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix},$$

where

$$(78) \quad D_1 \in \mathbf{S}_{n-r}^\vee \subset \text{Hom}(H_{n-r}^\vee \otimes V^\vee, H_{n-r} \otimes V),$$

$$\lambda \in \mathbf{L}_{n,r} := \text{Hom}(H_r^\vee, H_{n-r}) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_r := S^2 H_r \otimes \wedge^2 V.$$

By (75) and (77) the composition

$$C^\vee \circ D \circ C : H_{n-r} \otimes V \rightarrow H_{n-r}^\vee \otimes V^\vee \quad (C^\vee \circ D \circ C \in \wedge^2(H_{n-r}^\vee \otimes V^\vee))$$

can be written in the form

$$(79) \quad C^\vee \circ D \circ C = \phi^\vee \circ D_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda^\vee \circ \phi + \psi^\vee \circ \mu \circ \psi.$$

By (75)-(78) we have

$$\mathbf{S}_n^\vee \times \Sigma_{n,r} = \mathbf{S}_{n-r}^\vee \times \Phi_{n-r} \times \Psi_{n,r} \times \mathbf{L}_{n,r} \times \mathbf{M}_r,$$

and there are well defined morphisms

$$\tilde{p} : \tilde{X}_{n,r} \rightarrow \mathbf{L}_{n,r} \times \mathbf{M}_r : (D_1, \phi, \psi, \lambda, \mu) \mapsto (\lambda, \mu).$$

and

$$p := \tilde{p}|_{\overline{X}_{n,r}} : \overline{X}_{n,r} \rightarrow \mathbf{L}_{n,r} \oplus \mathbf{M}_r,$$

where  $\overline{X}_{n,r}$  is the closure of  $X_{n,r}$  in  $(\mathbf{S}_n^\vee)^0 \times \Sigma_{n,r}$ . We now invoke the following result from [T]:

**Proposition 5.4.** *Let  $n \geq 2$ . Then for any  $D \in (\mathbf{S}_n^\vee)^0$  and for a general choice of the decomposition  $H_n \xrightarrow{\sim} H_{n-r} \oplus H_r$ , the block  $D_1$  of  $D$  in (77) is nondegenerate.*

*Proof.* See [T, Proposition 7.3]. By repeatedly applying this proposition  $r$  times, we can find a decomposition  $H_n \xrightarrow{\sim} H_{n-r} \oplus H_r$  such that  $D_1 : H_{n-r}^\vee \otimes V^\vee \rightarrow H_{n-r} \otimes V$  in (77) is nondegenerate, i.e.,  $D_1 \in (\mathbf{S}_{n-r}^\vee)^0$ .  $\square$

Let  $\mathcal{X}$  be any irreducible component of  $X_{n,r}$  and let  $\overline{\mathcal{X}}$  be its closure in  $\overline{X}_{n,r}$ . Fix a point  $z = (D_1, \phi, \psi, \lambda, \mu) \in \mathcal{X}$  not lying in the components of  $X_{n,r}$  different from  $\mathcal{X}$ . Consider the morphism

$$(80) \quad f : \mathbb{A}^1 \rightarrow \overline{\mathcal{X}} : t \mapsto (D_1, t^2 \phi, t \psi, t \lambda, t^2 \mu), \quad f(1) = z,$$

which is well defined by (79). By definition, the point  $f(0) = (D_1, 0, 0, 0, 0)$  lies in the fibre  $p^{-1}(0, 0)$ . Hence,  $p^{-1}(0, 0) \cap \overline{\mathcal{X}} \neq \emptyset$ . In other words,

$$(81) \quad \rho^{-1}(0, 0) \neq \emptyset, \quad \text{where } \rho := p|_{\overline{\mathcal{X}}}.$$

Now, it follows from (79) and the definition of  $\tilde{X}_{n,r}$  that

$$(82) \quad \tilde{p}^{-1}(0, 0) = \{(D_1, \phi, \psi) \in (\mathbf{S}_{n-r}^\vee)^0 \times \Phi_{n-r} \times \Psi_{n,r} \mid \phi^\vee \circ D_1 \circ \phi \in \mathbf{S}_{n-r}\}.$$

Consider the set

$$Z_{n-r} = \{(D, \phi) \in (\mathbf{S}_{n-r}^\vee)^0 \times \Phi_{n-r} \mid \phi^\vee \circ D \circ \phi \in \mathbf{S}_{n-r}\}.$$

It carries a natural scheme structure, where it is a closed subscheme of  $(\mathbf{S}_{n-r}^\vee)^0 \times \Phi_{n-r}$ . Comparing the definition of  $Z_{n-r}$  with (82) we see that there are scheme-theoretic inclusions of schemes

$$(83) \quad \rho^{-1}(0, 0) \subset p^{-1}(0, 0) \subset \tilde{p}^{-1}(0, 0) = Z_{n-r} \times \Psi_{n,r}.$$

By [T, Theorem 7.2],  $Z_{n-r}$  is an integral scheme of dimension  $4(n-r)(n-r+2)$ . This together with (83) implies that

$$(84) \quad \dim \rho^{-1}(0, 0) \leq \dim p^{-1}(0, 0) \leq \dim Z_{n-r} + \dim \Psi_{n,r} = 4(n-r)(n-r+2) + 6r(n-r) = \\ = (n-r)(4n+2r+8).$$

Hence in view of (81)

$$(85) \quad \dim \overline{\mathcal{X}} \leq \dim \rho^{-1}(0, 0) + \dim \mathbf{L}_{n,r} + \dim \mathbf{M}_r \leq (n-r)(4n+2r+8) + 6r(n-r) + 3r(r+1) = \\ = (2n-r)^2 + 4(2n-r)(r+1) - r(2r+1).$$

On the other hand, formula (21), with  $2n-r$  substituted for  $n$ , and Theorem 5.1(ii) show that, for any point  $x \in \mathcal{X}$  such that  $A := f_{n,r}^{-1}(x) \in MI_{2n-r,r}^0(\xi)$ ,

$$(86) \quad (2n-r)^2 + 4(2n-r)(r+1) - r(2r+1) \leq \dim_A MI_{2n-r,r}^0(\xi) = \dim \overline{\mathcal{X}}.$$

Comparing (85) with (86), we see that all the inequalities in (84)-(86) are equalities. In particular,

$$(87) \quad \dim \rho^{-1}(0, 0) = \dim(Z_{n-r} \times \Psi_{n,r}) = \dim \overline{\mathcal{X}} - \dim(\mathbf{L}_{n,r} \times \mathbf{M}_r).$$

Since by Theorem [T, Theorem 7.2] the scheme  $Z_{n-r}$  is integral and so  $Z_{n-r} \times \Psi_{n,r}$  is integral as well, (83) and (87) yield the equalities of integral schemes

$$(88) \quad \rho^{-1}(0, 0) = p^{-1}(0, 0) = \tilde{p}^{-1}(0, 0) = Z_{n-r} \times \Psi_{n,r}.$$

Now we invoke one auxiliary result from [T].

**Lemma 5.5.** *Let  $f : X \rightarrow Y$  be a morphism of reduced schemes, where  $Y$  is a smooth integral scheme. Assume that there exists a closed point  $y \in Y$  such that for any irreducible component  $X'$  of  $X$  the following conditions are satisfied:*

(a)  $\dim f^{-1}(y) = \dim X' - \dim Y$ ,

(b) *the scheme-theoretic inclusion of fibres  $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$  is an isomorphism of integral schemes.*

Then

(i) *there exists an open subset  $U$  of  $Y$  containing the point  $y$  such that the morphism  $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$  is flat, and*

(ii)  *$X$  is integral.*

*Proof.* See [T, Lemma 7.4]. □

Applying assertions (i)-(ii) of this lemma to  $X = X_{n,r}$ ,  $X' = \mathcal{X}$ ,  $Y = \mathbf{L}_{n,r} \times \mathbf{M}_r$ ,  $y = (0, 0)$ ,  $f = p$ , and using (87) and (88), we obtain that  $X_{n,r}$  is integral of dimension  $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$ . Theorem 5.2 is proved.

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