

On Stokes Matrices in terms of Connection Coefficients

By

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Abstract. We study the classical problem of the computation of a complete system of Stokes matrices in terms of connection coefficients. Stokes matrices refer to a linear system of ODEs with Poincaré rank one and semi-simple leading matrix, while the connection coefficients connect solutions of the associated hypergeometric system of ODEs. The problem here is solved with no assumptions on the residue matrix at zero of the system of Poincaré rank one, so extending method and results of [4].

Key Words and Phrases. Stokes matrices, Connection matrices, Monodromy linear differential systems, Laplace transform.

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1. Introduction

We consider an $n \times n$ linear system of ODEs of Poincaré rank one at $z = \infty$

$$(1) \quad \frac{dY}{dz} = A(z)Y, \quad \text{where } A(z) = A_0 + \frac{A_1}{z}.$$

Here $A_0, A_1 \in GL(n, \mathbf{C})$, and A_0 has distinct eigenvalues. Up to a constant gauge, we assume that A_0 is already diagonal, namely:

$$(2) \quad A_0 = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \lambda_i \neq \lambda_j \text{ for } i \neq j, \lambda_i \in \mathbf{C}, 1 \leq i \leq n.$$

The following Fuchsian system

$$(3) \quad (A_0 - \lambda) \frac{d\Psi}{d\lambda} = (A_1 + I)\Psi, \quad I := n \times n \text{ identity matrix,}$$

is associated to (1) by Laplace transformation, as it is well known [6], [14]. Vector solutions $\vec{Y}(z)$ of (1) can be expressed in terms of convergent Laplace-type integrals of vector solutions $\vec{\Psi}(\lambda)$ of (3):

$$(4) \quad \vec{Y}(z) = \int_{\gamma} e^{\lambda z} \vec{\Psi}(\lambda) d\lambda,$$

where γ is a suitable path such that $e^{\lambda z}(\lambda - A_0)\vec{\Psi}(\lambda)|_{\gamma} = 0$.

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By means of Laplace transform, in [4] a complete system of Stokes multipliers for (1) is computed in terms of connection coefficients, connecting selected vector solutions of (3) (called associated functions) at different Fuchsian singularities. The definition of associated functions, connection coefficients, and the results of [4] depend upon the assumption that the diagonal entries of A_1 are not integers (this is called *assumption* (i) in [4]).

Special systems of the form (1) appear in the analytic theory of semisimple Frobenius manifolds [8], [9], where A_0 has distinct eigenvalues as in (2). The matrix A_1 has special form $V + cI$, where V is skew symmetric and $c \in \mathbf{C}$. The analytic description of Frobenius manifolds in terms of isomonodromic deformations of (1), with the above special A_1 , requires to consider all possible values of c . Therefore *assumption* (i) of [4] may fail, depending on the value of c . In [10] all possible values of c are analysed, obtaining the relation between Stokes matrices and connection coefficients, by means of Laplace transformation. See also [21].

Other applications to integrable systems involve systems (1) and (3): an example is the isomonodromic approach to the sixth Painlevé equation in terms of a 3×3 system of type (1) (see [18]). The computation of the relation between Stokes matrices of (1) and monodromy data of (3) may have important applications in this case, such as the understanding of the monodromy data associated to special solutions of the sixth Painlevé equation, including the solution associated to the quantum cohomology of \mathbf{CP}^2 . Monodromy data can be computed in terms of Stokes matrices, once the relation with the latter and connection coefficients is known (see Corollary 6 in this paper). Since matrix A_1 may violate *assumption* (i), the results of [4] are not enough to study this example. And indeed such study has not been done yet.

The above are motivations for our paper. Accordingly, our purpose is to extend the results of [4] when *no assumptions on A_1 are made*, by extending the same technique of [4], namely Laplace transformation. When no assumptions on A_1 are made, new types of logarithmic behaviours occur among vector solutions of (3), which come from resonances. They are not studied in [4]. As a first step, we construct selected vector solutions of (3), which include the new logarithmic behaviours (Section 3). This class of solutions extends that of [4]. As a second step, we define the connection coefficients for the selected solutions. Finally, we obtain fundamental solutions of (1) in terms of suitable Laplace integrals of the selected solutions (or their modifications). As a result, in Theorem 1 of Section 7, we obtain the explicit relation between a complete system of Stokes multipliers of (1) and connection coefficients, which holds for any A_1 . Conversely, in Corollary 6 we express traces of products of monodromy matrices of system (3) in terms of Stokes multipliers of (1). These traces

are invariant monodromy data of (3), and Corollary 6 allows their computability in terms of Stokes matrices, independently of the knowledge of connection coefficients. In order to prove Theorem 1, higher order primitives of the selected solutions need to be studied. This requires a non trivial extension of the technique of [4], and a consistent technical effort. As a side result of Theorem 1, the monodromy and connection relations of higher order primitives are obtained (see Section 8).

We stress that our purpose is to use and extend the classical technique of Laplace transformation of [4]. A more refined technique, namely the theory of summation and resurgence [12], was developed years later. In [16], this theory is applied to obtain the explicit relation between Stokes-Ramis matrices and connection constants for a general system of rank one with the assumptions of a single level equal to one (this includes the case of diagonalizable A_0 , with possibly coinciding eigenvalues). The same is done in [19] with the assumptions of an arbitrary single level. To our knowledge, the case when no assumptions *at all* are made on A_0 , possibly involving a ramified singularity at infinity (and the system of rank one may not be in Birkhoff normal form), has yet to be studied.

Finally, we remark two facts. First, that the results of the paper can be generalized to those systems of Poincaré rank $r \geq 1$ (namely $A(z) = z^{r-1}\tilde{A}(z)$, $\tilde{A}(z)$ analytic in a neighborhood of infinity), that are formally meromorphically equivalent to (1) in the sense of [3]. This can be done as in section 4 of [4] by a generalization of the definition of associated functions to include functions in the form of the $\vec{\Psi}_k(\lambda)$'s or $\vec{\Psi}_k^{(k)}(\lambda)$'s as in our Section 6.2, points 1), 2), 3) and 4). Secondly, that any system of Poincaré rank r can be reduced to a system of Poincaré rank one by enlarging the size of the system to $r \cdot n$ (see [15], and also [20], [17], [5]). In [15] the explicit relation between Stokes matrices of the initial and the reduced systems is given. In this sense, results obtained for rank one can be extended to any rank r . (I thank M. Loday-Richaud for drawing my attention to the papers [15], [16], [19]).

1.1. Organization of the paper

Section 2: We state the problem and give the main results. The example in the end of the section shows how the general result applies to the case of Frobenius manifolds.

Section 3: We construct selected vector solutions $\vec{\Psi}_k(\lambda)$, $1 \leq k \leq n$, to system (3)–(5), and define the connection coefficient, with no assumptions on A_1 .

Section 4: We construct two matrix solutions $\Psi(\lambda)$ and $\Psi^*(\lambda)$ of system (3)–(5), with no assumptions on A_1 . They generalize the matrices $Y(t)$ and

$Y^*(t)$ of [4]. We establish when they are fundamental, and compute their monodromy in terms of connection coefficients.

Section 5: We discuss the dependence of Ψ and Ψ^* on the choice of the branch cuts.

Section 6: We recall the definition of complete set of Stokes multipliers for (1). We write a fundamental matrix of system (1), having canonical asymptotics in a wide sector, as Laplace integrals of the $\vec{\Psi}_k(\lambda)$'s, $1 \leq k \leq n$ (or their modifications including singular contributions—see formula (44)). We also express the $\vec{\Psi}_k(\lambda)$'s in terms of the coefficients of the former asymptotics.

Section 7: We state the main theorem (Theorem 1), which gives Stokes matrices and Stokes factors of (1) in terms of connection coefficients of (3), and express the first monodromy invariants of system (3) in terms of Stokes matrices (Corollary 6).

Section 8: We prove Theorem 1, and find relations and monodromy for q -primitives of vector solutions of (3)–(5). It is in this section that the most technical effort is required in order to generalize the results and method of [4] when no assumptions on A_1 are made.

2. Setting and results

Let us denote the diagonal entries of A_1 as follows

$$\text{diag}(A_1) = (\lambda'_1, \dots, \lambda'_n).$$

In [4] it is assumed that $\lambda'_1, \dots, \lambda'_n$ are not integers (*assumption* (i), page 693). In this paper, we allow any values of $(\lambda'_1, \dots, \lambda'_n) \in \mathbf{C}^n$. Therefore, in order to generalize the method and the results of [4], we need to characterize the solutions of (3) for any $(\lambda'_1, \dots, \lambda'_n) \in \mathbf{C}^n$. System (3) can be rewritten as

$$(5) \quad \frac{d\Psi}{d\lambda} = \sum_{k=1}^n \frac{B_k}{\lambda - \lambda_k} \Psi, \quad B_k := -E_k(A_1 + I), \quad 1 \leq k \leq n,$$

where E_k is a $n \times n$ matrix with entries $(E_k)_{kk} = 1$ and $(E_k)_{ij} = 0$ otherwise. A vector, or a matrix solution of (5) is multivalued in $\mathbf{C} \setminus \{\lambda_1, \dots, \lambda_n\}$, with regular singularities in $\lambda_1, \dots, \lambda_n$. Let \mathcal{U} be the universal covering of $\mathbf{C} \setminus \{\lambda_1, \dots, \lambda_n\}$. Following [4], we fix parallel branch cuts L_k , oriented from λ_k to ∞ . Choose a real number η such that

$$(6) \quad \eta \neq \arg(\lambda_j - \lambda_k) \pmod{2\pi}, \quad \text{for all } 1 \leq j, k \leq n.$$

The cuts are defined as follows

$$L_k := \{\lambda \in \mathcal{U} \mid \arg(\lambda - \lambda_k) = \eta\}, \quad 1 \leq k \leq n.$$

Condition (6) means that a cut L_k does not contain other poles $\lambda_j, j \neq k$. See figure 1. Such values of η are called *admissible* in [4]. We fix the branch of any $\ln(\lambda - \lambda_k)$ by considering the following sheet of $\mathcal{P}_\eta \subset \mathcal{U}$, obtained with the above cuts:

$$\mathcal{P}_\eta := \{\lambda \in \mathcal{U} \mid \eta - 2\pi < \arg(\lambda - \lambda_k) < \eta, 1 \leq k \leq n\}.$$

We prove in Section 3 that system (3) admits a matrix solution of the form:

$$\Psi(\lambda) = [\vec{\Psi}_1(\lambda) \mid \dots \mid \vec{\Psi}_n(\lambda)], \quad \lambda \in \mathcal{P}_\eta,$$

whose columns $\vec{\Psi}_k(\lambda), k = 1, \dots, n$, are the selected vector solutions we are interested in. This matrix generalizes matrix $Y(t)$ of [4]. The $\vec{\Psi}_k(\lambda)$'s are uniquely characterized by their behaviours for λ in a neighbourhood of λ_k , as follows:

$$\vec{\Psi}_k(\lambda) = \begin{cases} (\Gamma(\lambda'_k + 1)\vec{e}_k + \sum_{l \geq 1} \vec{b}_l^{(k)}(\lambda - \lambda_k)^l)(\lambda - \lambda_k)^{-\lambda'_k - 1}, & \text{if } \lambda'_k \notin \mathbf{Z}, \\ \left(\frac{(-1)^{N_k}}{(-N_k - 1)!} \vec{e}_k + \sum_{l \geq 1} \vec{b}_l^{(k)}(\lambda - \lambda_k)^l \right) (\lambda - \lambda_k)^{-N_k - 1}, & \text{if } \lambda'_k = N_k \in \mathbf{Z}_-, \\ \vec{d}_0^{(k)} + \sum_{l \geq 1} \vec{d}_l^{(k)}(\lambda - \lambda_k)^l, & \text{if } \lambda'_k = N_k \in \mathbf{N}. \end{cases}$$

Here $\mathbf{N} = \{0, 1, 2, \dots\}$ and $\mathbf{Z}_- = \{-1, -2, -3, \dots\}$ are the non negative and negative integers respectively, and \vec{e}_k is the k -th unit column vector of \mathbf{C}^n . The Taylor series in $(\lambda - \lambda_k)$ converge in a neighbourhood of λ_k . The coefficients $\vec{b}_l^{(k)} \in \mathbf{C}^n$ are uniquely determined by the choice of the normalizations $\Gamma(\lambda'_k + 1)\vec{e}_k$ and $((-1)^{N_k}/(-N_k - 1)!) \vec{e}_k$. The coefficients $\vec{d}_l^{(k)} \in \mathbf{C}^n$ are uniquely determined by the existence of a singular vector solution at λ_k with behaviour

$$\vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \frac{N_k! \vec{e}_k + O(\lambda - \lambda_k)}{(\lambda - \lambda_k)^{N_k + 1}}, \quad N_k = \lambda'_k \in \mathbf{N}.$$

We show in Section 3 that for any j there always exist $n - 1$ vector solutions which are locally analytic at $\lambda = \lambda_j$, and there exists at most one solution which is singular at $\lambda = \lambda_j$ (see expression (16), with k replaced with j). Therefore, a solution $\vec{\Psi}_k(\lambda)$ has in general a singular contribution close to a λ_j , with $j \neq k$, which behaves as a multiple of the singular solution at λ_j . The corresponding multiplicative constant allows to define the connection coefficients, as follows. Let $P_{N_j}^{(j)}(\lambda)$ denote a vector function with polynomial entries in $(\lambda - \lambda_j)$ of degree $N_j \in \mathbf{N}$, and let $\text{reg}(\lambda - \lambda_j)$ be a vector function analytic (regular) in a

neighbourhood of λ_j . We will show (Definition 1, Section 3) that there exist unique *connection coefficients* $c_{jk} \in \mathbf{C}$ such that, in a neighbourhood of any $\lambda_j \neq \lambda_k$, $\vec{\Psi}_k(\lambda)$ behaves as follows:

$$(7) \quad \vec{\Psi}_k(\lambda) = \begin{cases} \vec{\Psi}_j(\lambda)c_{jk} + \text{reg}(\lambda - \lambda_j), & \lambda'_j \notin \mathbf{Z}, \\ \vec{\Psi}_j(\lambda) \ln(\lambda - \lambda_j)c_{jk} + \text{reg}(\lambda - \lambda_j), & \lambda'_j \in \mathbf{Z}_-, \\ \left(\vec{\Psi}_j(\lambda) \ln(\lambda - \lambda_j) + \frac{P_{N_j}^{(j)}(\lambda)}{(\lambda - \lambda_j)^{N_j+1}} \right) c_{jk} + \text{reg}(\lambda - \lambda_j), & \lambda'_j = N_j \in \mathbf{N}. \end{cases}$$

The connection coefficients multiply, in the above formula, the singular contribution at λ_j . Note that they depend on the choice of the branch cuts, namely on η (see Section 5). In particular,

$$c_{kk} = 1 \quad \text{if } \lambda'_k \notin \mathbf{Z}, \quad c_{kk} = 0 \quad \text{if } \lambda'_k \in \mathbf{Z}.$$

The connection coefficients determine the monodromy of the matrix $\Psi(\lambda)$, as follows:

Proposition I (Proposition 2). *Let the branch cuts L_1, \dots, L_n be fixed. Let A_1 be any matrix. The monodromy matrix of $\Psi(\lambda)$ for a small loop in anticlockwise direction around λ_k , not encircling all the other points $\lambda_j \neq \lambda_k$, $j = 1, \dots, n$ is the matrix $M_k = (m_{ij}^{(k)})_{i,j=1\dots n}$ with entries:*

$$\begin{aligned} m_{jj}^{(k)} &= 1 \quad 1 \leq j \leq n, j \neq k; & m_{kk}^{(k)} &= e^{-2\pi i \lambda'_k}; \\ m_{kj}^{(k)} &= \alpha_k c_{kj}, \quad 1 \leq j \leq n, j \neq k; & m_{ij}^{(k)} &= 0 \quad \text{otherwise.} \end{aligned}$$

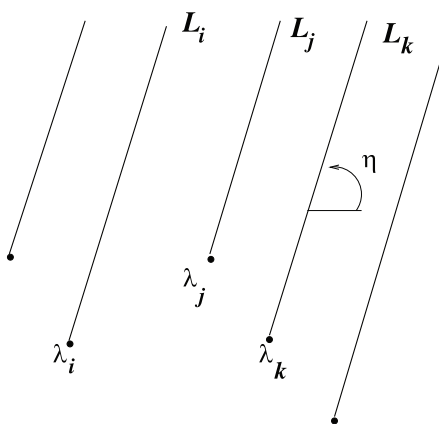


Fig. 1. The poles λ_j , $1 \leq j \leq n$ of system (3), and branch cuts L_j .

where

$$\begin{cases} \alpha_k := (e^{-2\pi i \lambda'_k} - 1), & \text{if } \lambda'_k \notin \mathbf{Z}, \\ \alpha_k := 2\pi i, & \text{if } \lambda'_k \in \mathbf{Z}. \end{cases}$$

Equivalently, the effect of the loop around λ_k is

$$\vec{\Psi}_k(\lambda) \mapsto e^{-2\pi i \lambda'_k} \vec{\Psi}_k(\lambda); \quad \vec{\Psi}_j(\lambda) \mapsto \vec{\Psi}_j(\lambda) + \alpha_k c_{kj} \vec{\Psi}_k(\lambda), \quad j \neq k.$$

The matrix solution $\Psi(\lambda)$ is not necessarily fundamental. The following gives a sufficient condition:

Proposition II (Propositions 3 and 4). *If A_1 has no integer eigenvalues, then $\Psi(\lambda)$ is a fundamental matrix and M_1, \dots, M_n generate the monodromy group of system (3). Moreover, the matrix $C := (c_{jk})$ is invertible if and only if A_1 has no integer eigenvalues.*

Remark 1. There are cases when A_1 has integer eigenvalues and Ψ is fundamental. We prove that in these cases, necessarily, some $\lambda'_k \in \mathbf{Z}$.

To explain the relation between connection coefficients and Stokes multipliers of (1), recall that solutions of the latter are characterized by the Stokes phenomenon. Let $\tau = 3\pi/2 - \eta$. According to [2], there are three unique fundamental matrices of (1), say $Y_I(z)$, $Y_{II}(z)$ and $Y_{III}(z)$, with canonical asymptotic behaviour $(I + O(1/z)) \exp\{A_0 z + A_1 \ln z\}$ in the three sectors $\{z \mid \tau - \pi \leq \arg z \leq \tau\}$, $\{z \mid \tau \leq \arg z \leq \tau + \pi\}$ and $\{z \mid \tau + \pi \leq \arg z \leq \tau + 2\pi\}$ respectively. They are related by two Stokes matrices S_+ and S_- such that

$$Y_{II}(z) = Y_I(z)S_+, \quad \arg z = \tau; \quad Y_{III}(z) = Y_{II}(z)S_-, \quad \arg z = \tau + \pi.$$

Introduce in $\{1, 2, \dots, n\}$ the partial ordering $<$ given by

$$j < k \Leftrightarrow \Re(z(\lambda_j - \lambda_k)) < 0 \quad \text{for } \arg z = \tau, i \neq j, i, j \in \{1, \dots, n\}.$$

We prove the following main result:

Theorem I (Theorem 1). *The Stokes matrices of system (1), without assumptions on A_1 , and the connection coefficients c_{jk} , $1 \leq j, k \leq n$, of system (3)–(5) are related by the formulae*

$$[S_+]_{jk} = \begin{cases} e^{2\pi i \lambda'_k} \alpha_k c_{jk}, & \text{for } j < k, \\ 1, & \text{for } j = k, \\ 0, & \text{for } j > k, \end{cases} \quad [S_-^{-1}]_{jk} = \begin{cases} 0, & \text{for } j < k, \\ 1, & \text{for } j = k, \\ -e^{2\pi i (\lambda'_k - \lambda'_j)} \alpha_k c_{jk}, & \text{for } j > k. \end{cases}$$

Corollary I (Corollary 6). *Let A_1 be any matrix. The following equalities hold for the monodromy matrices of $\Psi(\lambda)$:*

$$\begin{aligned} \text{Tr}(M_k) &= n - 1 + e^{-2\pi i \lambda'_k}, \\ \text{Tr}(M_j M_k) &= \begin{cases} n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} - e^{-2\pi i \lambda'_j} [S_+]_{jk} [S_-^{-1}]_{kj}, & \text{if } j < k, \\ n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} - e^{-2\pi i \lambda'_k} [S_-^{-1}]_{jk} [S_+]_{kj}, & \text{if } j > k. \end{cases} \end{aligned}$$

Example. The general results above apply to semisimple Frobenius manifolds, where A_0 has distinct eigenvalues. In this case, the relation between Stokes matrices and connection coefficients was computed in [8] and [9] when A_1 does satisfy *assumption* (i), and in [10] when it does not. The matrix A_1 has a special form, namely it is expressed in terms of a skew symmetric matrix V as follows:

$$A_1 = V - \left(\frac{1}{2} + v\right)I, \quad v \in \mathbf{C}, \quad V^T = -V.$$

We show how our general results above apply to this case. Since $\lambda'_k = -v - 1/2$, $1 \leq k \leq n$, it follows that

$$\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha, \quad \text{where } \alpha := \begin{cases} -(1 + e^{2\pi i v}) & \text{if } v \notin \mathbf{Z} + 1/2, \\ 2\pi i & \text{if } v \in \mathbf{Z} + 1/2. \end{cases}$$

From Theorem I above (and the fact that the $c_{kk} = 0$ when $\lambda'_k \in \mathbf{Z}$), we deduce that

$$e^{2\pi i v} S_+ + S_-^{-1} = -\alpha C, \quad \text{where } C := (c_{ij}).$$

Since V is an $n \times n$ skew symmetric matrix, it can be easily verified that

$$S_+^T = S_-^{-1}.$$

Thus

$$(8) \quad e^{2\pi i v} S_+ + S_+^T = -\alpha C.$$

The above, and Proposition II, allow us to conclude that if $e^{2\pi i v} S_+ + S_+^T$ is invertible, then A_1 has no integer eigenvalues and so $\Psi(\lambda)$ is invertible. This is part of the first assertion of Theorem 4.3 of [10], namely if

$$\det(e^{2\pi i v} S_+ + S_+^T) \neq 0,$$

then system (3) has n linearly independent solutions $\vec{\Psi}_1, \dots, \vec{\Psi}_n$. From (8) and Proposition I, it follows that for an anticlockwise loop around λ_i , the monodromy of the above solutions is

$$\vec{\Psi}_i \mapsto -e^{2\pi i v} \vec{\Psi}_i, \quad \vec{\Psi}_j \mapsto \vec{\Psi}_j - e^{i\pi v} (e^{i\pi v} S_+ + e^{-i\pi v} S_+^T)_{ij} \vec{\Psi}_i, \quad j \neq i.$$

The above is formula (4.11) in Theorem 4.3 of [10].

Note. After this paper was completed, the works [13] and [23] appeared, showing that for the Frobenius manifold given by the Quantum Cohomology of Grassmannians, there may be cases (depending on the dimension) when A_0 is still diagonalizable, but with some coinciding eigenvalues.

3. Local solutions of system (5) (i.e. (3))

The matrix B_k in system (5) has zero entries, except for the k -th row. Indeed, letting $A_1 = (A_{ij})_{i,j=1,\dots,n}$, a straightforward computation yields

$$B_k = \begin{pmatrix} 0 & & & 0 & & & 0 \\ \vdots & & & \vdots & & & \vdots \\ -A_{k1} & \cdots & -A_{k,k-1} & -\lambda'_k - 1 & -A_{k,k+1} & \cdots & -A_{kn} \\ \vdots & & & \vdots & & & \vdots \\ 0 & & & 0 & & & 0 \end{pmatrix} \leftarrow k\text{-th row.}$$

A fundamental matrix solution of (5) is multivalued in $\mathbf{C} \setminus \{\lambda_1, \dots, \lambda_n\}$ and single-valued in \mathcal{D}_η , for any admissible direction η . If λ is in a neighbourhood of a λ_k not containing other poles, there exists a fundamental matrix solution

$$\Psi^{(k)}(\lambda) = [\vec{\Psi}_1^{(k)}(\lambda) | \cdots | \vec{\Psi}_n^{(k)}(\lambda)],$$

which can be computed in a standard way, depending on the value of λ'_k (see [22]). In [4], only the case $\lambda'_k \notin \mathbf{Z}$ is considered (point 1) below). Here we need to analyse also the case $\lambda'_k \in \mathbf{Z}$ (points 2), 3) and 4) below).

1) [Generic Case, as in [4]] If $\lambda'_k \notin \mathbf{Z}$, then B_k is diagonalizable, with diagonal form

$$T^{(k)} = [G^{(k)}]^{-1} B_k G^{(k)} = \text{diag}(0, \dots, 0, -\lambda'_k - 1, 0, \dots, 0),$$

where the non zero entry $-\lambda'_k - 1$ is at k -th position. The k -th column of the diagonalizing matrix $G^{(k)}$ can be chosen to be a multiple of the k -th vector \vec{e}_k of the canonical basis of \mathbf{C}^n . As in [4] we choose normalization $\Gamma(\lambda'_k + 1)\vec{e}_k$. A fundamental matrix is then

$$\begin{aligned} \Psi^{(k)}(\lambda) &= G^{(k)}(I + O(\lambda - \lambda_k))(\lambda - \lambda_k)^{T^{(k)}} \\ &= [\vec{\Psi}_1^{(k)}(\lambda) | \cdots | \vec{\Psi}_n^{(k)}(\lambda)](\lambda - \lambda_k)^{T^{(k)}} \\ &= [\vec{\Psi}_1^{(k)}(\lambda) | \cdots | \vec{\Psi}_{k-1}^{(k)}(\lambda) | \vec{\Psi}_k^{(k)}(\lambda)(\lambda - \lambda_k)^{-\lambda'_k - 1} | \vec{\Psi}_{k+1}^{(k)}(\lambda) | \cdots | \vec{\Psi}_n^{(k)}(\lambda)], \end{aligned}$$

where $O(\lambda - \lambda_k)$ is a matrix valued Taylor series, converging in the neighbourhood of λ_k and vanishing as $\lambda \rightarrow \lambda_k$. Here $[\vec{\Psi}_1^{(k)}(\lambda) | \cdots | \vec{\Psi}_n^{(k)}(\lambda)] := G^{(k)}(I + O(\lambda - \lambda_k))$ is analytic in a neighbourhood of λ_k . The columns of

$\Psi^{(k)}$ form n independent vector solutions, $n - 1$ being analytic. The k -th is singular and we denote it

$$(9) \quad \vec{\Psi}_k(\lambda) := \vec{\Psi}_k^{(k)}(\lambda) = \vec{\psi}_k^{(k)}(\lambda)(\lambda - \lambda_k)^{-\lambda'_k - 1},$$

where

$$\vec{\psi}_k^{(k)}(\lambda) = \Gamma(\lambda'_k + 1)\vec{e}_k + \sum_{l \geq 1} \vec{b}_l^{(k)}(\lambda - \lambda_k)^l.$$

The vector coefficients $\vec{b}_l^{(k)}$ can be computed rationally from the matrix coefficients B_l 's of system (5). See [22]. The above $\vec{\Psi}_k$ is called *associated function* in [4].

2) [Jordan Case] If $\lambda'_k = -1$, then B_k has Jordan form

$$J^{(k)} = [G^{(k)}]^{-1} B_k G^{(k)} = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 0 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

The entry 1 is at row $(k - 1)$ and column k . The $(k - 1)$ -th column of $G^{(k)}$ can be normalized to be $-\vec{e}_k$. There exists a fundamental matrix solution with local representation

$$\begin{aligned} \Psi^{(k)}(\lambda) &= G^{(k)}(I + O(\lambda - \lambda_k))(\lambda - \lambda_k)^{J^{(k)}} \\ &= [\vec{\psi}_1^{(k)}(\lambda) | \cdots | \vec{\psi}_n^{(k)}(\lambda)](\lambda - \lambda_k)^{J^{(k)}} \\ &= [\vec{\psi}_1^{(k)}(\lambda) | \cdots | \vec{\psi}_{k-1}^{(k)}(\lambda) | \vec{\psi}_{k-1}^{(k)}(\lambda) \ln(\lambda - \lambda_k) \\ &\quad + \vec{\psi}_k^{(k)}(\lambda) | \vec{\psi}_{k+1}^{(k)}(\lambda) | \cdots | \vec{\psi}_n^{(k)}(\lambda)], \end{aligned}$$

where the columns $\vec{\psi}_j^{(k)}$ are analytic in a neighbourhood of λ_k . The columns are n independent vector solutions, $n - 1$ being analytic and the k -th singular. We assign the symbol $\vec{\Psi}_k$ to the non-singular factor of $\ln(\lambda - \lambda_k)$, as follows

$$(10) \quad \vec{\Psi}_k(\lambda) := \vec{\psi}_{k-1}^{(k)}(\lambda) = -\vec{e}_k + \sum_{l \geq 1} \vec{b}_l^{(k)}(\lambda - \lambda_k)^l.$$

Note that this is a solution of (5). Then, the k -th column of $\Psi^{(k)}$ is

$$(11) \quad \vec{\Psi}_k^{(k)}(\lambda) = \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k),$$

where $\text{reg}(\lambda - \lambda_k)$ means an analytic (vector) function in a neighbourhood of λ_k .

3) [First Resonant Case] If $\lambda'_k = N_k \geq 0$ is integer, then B_k is diagonalizable as in case 1), but now a fundamental solution has the form

$$\Psi^{(k)}(\lambda) = G^{(k)}(I + O(\lambda - \lambda_k))(\lambda - \lambda_k)^{T^{(k)}}(\lambda - \lambda_k)^{R^{(k)}},$$

where $R^{(k)}$ is a matrix with zero entries expect for $R_{jk}^{(k)}$, $j = 1, \dots, n$, and $j \neq k$, because $(\text{Eigen}(B_k))_j - \text{Eigen}(B_k)_k = N_k + 1 > 0$. Thus, only the k -th column of $R^{(k)}$ may be non zero. Let $r_j^{(k)} := R_{jk}^{(k)}$, so that the k -th column is

$$\vec{r}^{(k)} := (r_1^{(k)}, \dots, r_{k-1}^{(k)}, 0, r_{k+1}^{(k)}, \dots, r_n^{(k)})^T,$$

where T means transposition. The entries $r_j^{(k)}$ are computed as rational functions of the entries of the matrices B_l , $l = 1, \dots, n$ (see [22]). From the above, it follows that

$$\begin{aligned} \Psi^{(k)}(\lambda) &= [\vec{\psi}_1^{(k)}(\lambda) | \dots | \vec{\psi}_n^{(k)}(\lambda)](\lambda - \lambda_k)^{T^{(k)}}(I + R^{(k)} \ln(\lambda - \lambda_k)) \\ &= [\vec{\psi}_1^{(k)}(\lambda) | \dots | \vec{\psi}_{k-1}^{(k)}(\lambda) | \Psi_k^{(k)}(\lambda) | \vec{\psi}_{k+1}^{(k)}(\lambda) | \dots | \vec{\psi}_n^{(k)}(\lambda)], \end{aligned}$$

where

$$\begin{aligned} \Psi_k^{(k)}(\lambda) &= \left\{ \sum_{j \neq k} r_j^{(k)} \vec{\psi}_j^{(k)}(\lambda) \right\} \ln(\lambda - \lambda_k) + \frac{\vec{\psi}_k^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1}}, \\ \vec{\psi}_k^{(k)}(\lambda) &= N_k! \vec{e}_k + O(\lambda - \lambda_k), \end{aligned}$$

the factor $N_k!$ coming from a chosen normalization of $G^{(k)}$. The columns are n independent vector solutions, $n - 1$ being analytic (i.e. the $\vec{\psi}_j^{(k)}$, $j \neq k$) and the k -th singular (i.e. $\Psi_k^{(k)}$). We assign the symbol $\vec{\Psi}_k$ to the non-singular factor of $\ln(\lambda - \lambda_k)$ as follows

$$\vec{\Psi}_k(\lambda) := \sum_{j \neq k} r_j^{(k)} \vec{\psi}_j^{(k)}(\lambda) = \sum_{l \geq 0} \vec{d}_l^{(k)} (\lambda - \lambda_k)^l.$$

Note that this is a solution of (5), being a linear combination of regular solutions $\vec{\psi}_j^{(k)}$. Special cases can occur when $\vec{r}^{(k)} = 0$, so that $\vec{\Psi}_k(\lambda) \equiv 0$. We conclude that the k -th column of $\Psi^{(k)}$ is

$$(12) \quad \vec{\Psi}_k^{(k)}(\lambda) = \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \frac{P_{N_k}^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1}} + \text{reg}(\lambda - \lambda_k),$$

where

$$P_{N_k}^{(k)}(\lambda) = N_k! \vec{e}_k + \sum_{l=0}^{N_k} \vec{b}_l^{(k)} (\lambda - \lambda_k)^l,$$

represents the first $N_k + 1$ terms in the expansion of $\vec{\psi}_k^{(k)}$. The vector coefficients $\vec{b}_l^{(k)}$ are computed rationally from the coefficients of (5). The solution (12) is not uniquely determined, because we can add a linear combination of regular solutions $\vec{\psi}_j^{(k)}$, but the singular part is uniquely determined by the normalization of $P^{(k)}(\lambda)$. Consequently, also $\vec{\Psi}_k(\lambda)$ is uniquely determined.

4) [Second Resonant Case] If $\lambda'_k = N_k \leq -2$ is integer, then B_k is diagonalizable as in case 1), but now a fundamental solution has the form

$$\Psi^{(k)}(\lambda) = G^{(k)}(I + O(\lambda - \lambda_k))(\lambda - \lambda_k)^{T^{(k)}}(\lambda - \lambda_k)^{R^{(k)}},$$

where $R^{(k)}$ is a matrix with zero entries expect for $R_{kj}^{(k)}$, $j = 1, \dots, n$, and $j \neq k$, because $(Eigen(B_k))_k - Eigen(B_k)_j = -N_k - 1 > 0$. Thus, only the k -th row of $R^{(k)}$ may be non zero. Let $r_j^{(k)} := R_{kj}^{(k)}$, so that the k -th row is

$$\underline{r}^{(k)} := [r_1^{(k)}, \dots, r_{k-1}^{(k)}, 0, r_{k+1}^{(k)}, \dots, r_n^{(k)}],$$

where the entries $r_j^{(k)}$ are computed as rational functions of the entries of the matrices B_l , $l = 1, \dots, n$ (see [22]). Thus,

$$\Psi^{(k)}(\lambda) = [\vec{\psi}_1^{(k)}(\lambda) | \dots | \vec{\psi}_n^{(k)}(\lambda)](\lambda - \lambda_k)^{T^{(k)}}(I + R^{(k)} \ln(\lambda - \lambda_k)),$$

where the $\vec{\psi}_j^{(k)}(\lambda)$ are analytic and Taylor expanded in a neighbourhood of λ_k . The columns of the above matrix are

$$\begin{aligned} \vec{\Psi}_j^{(k)}(\lambda) &= r_j^{(k)} \vec{\psi}_k^{(k)}(\lambda)(\lambda - \lambda_k)^{-N_k-1} \ln(\lambda - \lambda_k) + \vec{\psi}_j^{(k)}(\lambda), \quad j = 1, \dots, n, j \neq k, \\ \vec{\Psi}_k^{(k)}(\lambda) &= \vec{\psi}_k^{(k)}(\lambda)(\lambda - \lambda_k)^{-N_k-1}. \end{aligned}$$

There are at most $n - 1$ independent singular solutions at λ_k , and at least one analytic solution $\vec{\Psi}_k^{(k)}$. In special cases, it may happen that $\underline{r}^{(k)} = 0$, so that there are n independent solutions analytic at λ_k . We show below (Lemma 1) that in fact we can always find $n - 1$ independent solutions analytic at λ_k , whatever $\underline{r}^{(k)}$ is. We assign the symbol $\vec{\Psi}_k$ to the k^{th} column:

$$\vec{\Psi}_k(\lambda) := \vec{\Psi}_k^{(k)}(\lambda) = \vec{\psi}_k^{(k)}(\lambda)(\lambda - \lambda_k)^{-N_k-1},$$

with normalization

$$(13) \quad \vec{\psi}_k^{(k)}(\lambda) = \frac{(-1)^{N_k}}{(-N_k - 1)!} \vec{e}_k + \sum_{l \geq 1} b_l^{(k)}(\lambda - \lambda_k)^l,$$

where the convergent Taylor series has coefficients determined rationally by the matrices B_l 's of (5). The logarithmic solutions are rewritten as

$$\vec{\Psi}_j^{(k)}(\lambda) = r_j^{(k)} \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \vec{\psi}_j^{(k)}(\lambda), \quad j \neq k, 1 \leq j \leq n.$$

It follows that if at least one $r_j^{(k)} \neq 0$, we can pick up the singular solutions

$$(14) \quad \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k).$$

The regular part is an arbitrary linear combination of the $\vec{\psi}_j^{(k)}$'s, $1 \leq j \leq n$, $j \neq k$. The singular part is determined uniquely by the normalization (13).

Lemma 1. *Let λ'_k be an integer $N_k \leq -2$. If $r^{(k)} = 0$, system (5) has n independent solutions analytic at λ_k . If $r^{(k)} \neq 0$, system (5) has n independent solutions, of which $n - 1$ are analytic and one is log-singular at λ_k .*

Proof. Let $0 \leq s \leq n - 1$ be the number of non zero values r_{i_1}, \dots, r_{i_s} . If $r^{(k)} = 0$, then $s = 0$ and by the preceding construction there exist n independent solutions

$$\vec{\psi}_1^{(k)}, \dots, \vec{\psi}_{k-1}^{(k)}, \vec{\Psi}_k, \vec{\psi}_{k+1}^{(k)}, \dots, \vec{\psi}_n^{(k)}.$$

If $s > 0$, there are s singular (at λ_k) solutions

$$\vec{\Psi}_{i_1}^{(k)}, \vec{\Psi}_{i_2}^{(k)}, \dots, \vec{\Psi}_{i_s}^{(k)},$$

and the remaining analytic (at λ_k) solutions

$$\vec{\psi}_{j_1}^{(k)}, \vec{\psi}_{j_2}^{(k)}, \dots, \vec{\psi}_{j_l}^{(k)}.$$

Note that $\{i_1, i_2, \dots, i_s\} \cup \{j_1, j_2, \dots, j_l\}$ is a partition of $\{1, 2, \dots, k - 1, k + 1, \dots, n\}$. We construct another set of $s - 1$ independent analytic (at λ_k) solutions:

$$\begin{aligned} \varphi_{i_1}^{(k)} &:= \frac{1}{r_{i_1}^{(k)}} \vec{\Psi}_{i_1}^{(k)} - \frac{1}{r_{i_s}^{(k)}} \vec{\Psi}_{i_s}^{(k)}, \\ \varphi_{i_2}^{(k)} &:= \frac{1}{r_{i_2}^{(k)}} \vec{\Psi}_{i_2}^{(k)} - \frac{1}{r_{i_s}^{(k)}} \vec{\Psi}_{i_s}^{(k)}, \\ &\vdots \\ \varphi_{i_{s-1}}^{(k)} &:= \frac{1}{r_{i_{s-1}}^{(k)}} \vec{\Psi}_{i_{s-1}}^{(k)} - \frac{1}{r_{i_s}^{(k)}} \vec{\Psi}_{i_s}^{(k)}. \end{aligned}$$

It follows that there always exist $n - 1$ linearly independent vector solutions which are analytic at λ_k , namely

$$\varphi_{i_1}^{(k)}, \dots, \varphi_{i_{s-1}}^{(k)}; \quad \vec{\psi}_{j_1}^{(k)}, \vec{\psi}_{j_2}^{(k)}, \dots, \vec{\psi}_{j_l}^{(k)}; \quad \vec{\Psi}_k.$$

Moreover, there also exists the singular solution $\vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k)$. This proves the lemma. □

Conclusion. The four cases above are summarized below (letting $0! = 1$):

$$(15) \quad \vec{\Psi}_k(\lambda) = \begin{cases} (\Gamma(\lambda'_k + 1)\vec{e}_k + O(\lambda - \lambda_k))(\lambda - \lambda_k)^{-\lambda'_k - 1}, & \text{Case 1): } \lambda'_k \notin \mathbf{Z}, \\ \left(\frac{(-1)^{N_k}}{(-N_k - 1)!} \vec{e}_k + O(\lambda - \lambda_k) \right) (\lambda - \lambda_k)^{-N_k - 1}, & \text{Case 2), 4):} \\ \sum_{j \neq k} r_j^{(k)} \psi_j^{(k)}(\lambda) = \text{reg}(\lambda - \lambda_k), & \lambda'_k = N_k \in \mathbf{Z}_-, \\ & \text{Case 3): } \lambda'_k \in \mathbf{N}. \end{cases}$$

Moreover, there exists a singular solution given by

$$(16) \quad \vec{\Psi}_k^{(sing)}(\lambda) := \begin{cases} \vec{\Psi}_k(\lambda), & \lambda'_k \notin \mathbf{Z}, \text{ i.e. (9),} \\ \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k), & \lambda'_k = -1, \text{ i.e. (11),} \\ \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \frac{P_{N_k}^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k + 1}} + \text{reg}(\lambda - \lambda_k), & \lambda'_k \in \mathbf{N}, \text{ i.e. (12),} \\ \begin{cases} \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k), \\ \vec{\Psi}_k^{(sing)} \equiv 0, \text{ if } \underline{r}^{(k)} = 0, \end{cases} & \lambda'_k \in -\mathbf{N} - 2, \text{ i.e. (14).} \end{cases}$$

The singular part of $\vec{\Psi}_k^{(sing)}(\lambda)$ is uniquely determined. In logarithmic case of (11), (12) and (14), $\vec{\Psi}_k^{(sing)}(\lambda)$ is defined modulo the addition of a linear combination of regular solutions.

3.1. Connection coefficients—definition

Definition 1. The connection coefficients c_{jk} , $1 \leq j, k \leq n$, are uniquely defined by

$$\begin{cases} \vec{\Psi}_k(\lambda) = \vec{\Psi}_j^{(sing)}(\lambda)c_{jk} + \text{reg}(\lambda - \lambda_j), \\ c_{jk} := 0, \quad 1 \leq k \leq n, \quad \text{when } \vec{\Psi}_j^{(sing)}(\lambda) \equiv 0 \text{ for } \lambda'_j \in -\mathbf{N} - 2. \end{cases}$$

Observe that:

- a) $c_{kk} = 1$ for $\lambda'_k \notin \mathbf{Z}$, $c_{kk} = 0$ for $\lambda'_k \in \mathbf{Z}$.
- b) In case $\lambda'_k \in \mathbf{N}$, it may happen that $\vec{\Psi}_k \equiv 0$. This occurs when $\vec{r}^{(k)} = 0$. In this case $c_{jk} = 0$ for any $j = 1, \dots, n$, namely the k -th column of the matrix $C = (c_{jk})$ is zero.
- c) In case $\lambda'_j \in -\mathbf{N} - 2$, it may happen that there is no logarithmic singularity, namely $\vec{\Psi}_j^{(sing)} \equiv 0$. This occurs if $\underline{r}^{(j)} = 0$. In such a case, we need to *define* $c_{jk} := 0$, for any k , so that the matrix $C = (c_{jk})$ has zero j -th row.
- d) Letting $c_{jk} := 0$, for any k , when $\underline{r}^{(j)} = 0$, a more explicit way to write the definition of connection coefficients is (7).

4. Matrix solutions Ψ and Ψ^* of system (3)–(5), monodromy and invertibility

In the previous section, we have constructed a matrix solution

$$(17) \quad \Psi(\lambda) := [\vec{\Psi}_1(\lambda) | \cdots | \vec{\Psi}_n(\lambda)].$$

This generalizes matrix $Y(t)$ of [4]. In Section 4.2 we will establish under which conditions it is fundamental.

Remark 2. System (3), (5) may have vector solutions that are analytic at all $\lambda_1, \dots, \lambda_n$. Such solutions must be polynomials in λ , because ∞ is a Fuchsian singularity.

The following holds:

Lemma 2. *System (3), (5) has no polynomial vector solution if and only if A_1 has no negative integer eigenvalues. Equivalently (see Remark 2), System (3), (5) has a singular solution at any λ_k , $1 \leq k \leq n$, if and only if A_1 has no negative integer eigenvalues.*

Proof. This Lemma is proved in remark 1.1 of [4]. □

In [4] it is proved, under the assumption (i) of non integer λ'_k 's, that (3) admits a matrix solution $\Psi^*(\lambda)$, whose k^{th} column, $k = 1, \dots, n$, is analytic at all poles $\lambda_j \neq \lambda_k$. This matrix is called $Y^*(t)$ in [4]. The existence of Ψ^* can be proved without any assumption on $\lambda'_1, \dots, \lambda'_n$. It has new type of logarithmic behaviours when compared to $Y^*(t)$.

Proposition 1. *Let the matrix A_1 be any (no assumptions). Then*

i) *There exists a matrix solution $\Psi^* = [\vec{\Psi}_1^*(\lambda) | \cdots | \vec{\Psi}_n^*(\lambda)]$ such that*

$$(18) \quad \vec{\Psi}_k^*(\lambda) = \text{reg}(\lambda - \lambda_j) \quad \forall j \neq k.$$

ii) *$\Psi^*(\lambda)$ is a fundamental matrix solution if and only if none of the eigenvalues of A_1 is a negative integer. In this case, $\vec{\Psi}_k^{(sing)}(\lambda) \neq 0$ for any k , and $\vec{\Psi}_k^*(\lambda)$ has the following behaviour for λ close to λ_k*

$$(19) \quad \vec{\Psi}_k^*(\lambda) = \vec{\Psi}_k^{(sing)}(\lambda) + \text{reg}(\lambda - \lambda_k) = \begin{cases} \vec{\Psi}_k(\lambda) + \text{reg}(\lambda - \lambda_k), & \text{if } \lambda'_k \notin \mathbf{Z}, \\ \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k), & \text{if } \lambda'_k \in \mathbf{Z}_-, \\ \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \frac{P_{N_k}^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1}} + \text{reg}(\lambda - \lambda_k), & \text{if } \lambda'_k \in \mathbf{N}. \end{cases}$$

$\Psi^*(\lambda)$ is uniquely defined by (18) and (19), and

$$(20) \quad \Psi(\lambda) = \Psi^*(\lambda)C, \quad C := (c_{jk}).$$

Proof. We have proved in Section 3 that at any λ_j there exist at least $n - 1$ regular solutions, whatever are $\lambda'_1, \dots, \lambda'_n$. This is enough to apply the same procedure of proof of proposition 1 in [4]. The singular behaviour of $\vec{\Psi}_k^*(\lambda)$ is directly obtained from $\vec{\Psi}_k^{(sing)}(\lambda)$. \square

Remark 3. From the above proposition it follows that $\vec{\Psi}_k^*$ is always singular at λ_k . This implies that if none of the eigenvalues of A_1 is a negative integer and $\lambda'_k \in -N - 2$, then $r^{(k)} \neq 0$, namely $\vec{\Psi}_k^{(sing)} \neq 0$.

4.1. Monodromy of Ψ and Ψ^* associated to a loop around λ_k

Consider a small loop in \mathcal{P}_η around a pole λ_k in counter-clockwise direction, not encircling the other poles; for example $\lambda - \lambda_k \mapsto (\lambda - \lambda_k)e^{2\pi i}$, $|\lambda - \lambda_k|$ small. Monodromy of $\Psi = [\vec{\Psi}_1 | \dots | \vec{\Psi}_n]$ is easily computed from (15), which immediately implies

$$\vec{\Psi}_k(\lambda) \mapsto \begin{cases} \vec{\Psi}_k(\lambda)e^{-2\pi i\lambda'_k}, & \lambda'_k \notin \mathbf{Z}, \\ \vec{\Psi}_k(\lambda), & \lambda'_k \in \mathbf{Z}. \end{cases}$$

and from (7), which implies (note that j and k are exchanged here):

$$\vec{\Psi}_j(\lambda) \mapsto \begin{cases} \vec{\Psi}_j(\lambda) + (e^{-2\pi i\lambda'_k} - 1)c_{kj}\vec{\Psi}_k(\lambda), & \lambda'_k \notin \mathbf{Z} \\ \vec{\Psi}_j(\lambda) + 2\pi ic_{kj}\vec{\Psi}_k(\lambda), & \lambda'_k \in \mathbf{Z} \end{cases}$$

These formulae make sense also when $c_{kj} = 0$ for any k in the special case $\vec{\Psi}_j = 0$, possibly occurring when $\lambda'_k \in N$, and when $c_{kj} = 0$ for any j in the special case $\vec{\Psi}_k^{(sing)} \equiv 0$, possibly occurring when $\lambda'_k \in -N - 2$.

Next, we compute the monodromy of $\Psi^* = [\vec{\Psi}_1^* | \dots | \vec{\Psi}_n^*]$, which exists when A_1 has no negative integer eigenvalues. We consider again a small loop around λ_k as above. From (18) we have

$$\vec{\Psi}_j^*(\lambda) \mapsto \vec{\Psi}_j^*(\lambda), \quad \forall j = 1, \dots, n, j \neq k.$$

Then, from the above and (20),

$$\vec{\Psi}_k^*(\lambda) \mapsto \begin{cases} e^{-2\pi i\lambda'_k}\vec{\Psi}_k^*(\lambda) + (e^{-2\pi i\lambda'_k} - 1)\sum_{j \neq k} c_{jk}\vec{\Psi}_j^*(\lambda), & \lambda'_k \notin \mathbf{Z}, \\ \vec{\Psi}_k^*(\lambda) + 2\pi i\sum_{j \neq k} c_{jk}\vec{\Psi}_j^*(\lambda), & \lambda'_k \in \mathbf{Z}. \end{cases}$$

Summarizing:

Proposition 2. *The monodromy matrices representing the monodromy of Ψ and Ψ^* for a small counter-clockwise loop around λ_k in \mathcal{P}_η are as follows.*

a) *The matrix Ψ is always defined. The monodromy is*

$$\Psi \mapsto \Psi M_k, \quad M_k = I + \alpha_k \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kk} & \cdots & c_{kn} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \quad 1 \leq k \leq n,$$

where I is the $n \times n$ identity matrix, only the k -th row in the second matrix is non zero, and

$$\begin{cases} \alpha_k := (e^{-2\pi i \lambda'_k} - 1), & \text{if } \lambda'_k \notin \mathbf{Z}, \\ \alpha_k := 2\pi i, & \text{if } \lambda'_k \in \mathbf{Z}. \end{cases}$$

b) *If A_1 has no negative integer eigenvalues, then Ψ^* exists. The monodromy is*

$$\Psi^* \mapsto \Psi^* M_k^*, \quad M_k^* = I + \alpha_k \begin{pmatrix} 0 & 0 & \cdots & c_{1k} & \cdots & 0 \\ 0 & 0 & & c_{2k} & & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_{kk} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nk} & \cdots & 0 \end{pmatrix},$$

where only the k -th column in the second matrix is non zero.

Remark 4. The matrix M_k is the matrix $(m_{ij}^{(k)})$ in Proposition I of Section 2. For a clockwise loop, we analogously find that

$$\begin{aligned} [M_k^{-1}]_{kj} &= \beta_k c_{kj}, & j \neq k; & & [M_k^{-1}]_{kj} &= 0 & \text{otherwise;} \\ [M_k^{-1}]_{jj} &= 1, & j \neq k; & & [M_k^{-1}]_{kk} &= e^{2\pi i \lambda'_k}; \end{aligned}$$

and

$$\begin{aligned} [(M_k^*)^{-1}]_{jk} &= \beta_k c_{jk}, & j \neq k; & & [(M_k^*)^{-1}]_{ji} &= 0 & \text{otherwise;} \\ [(M_k^*)^{-1}]_{jj} &= 1, & j \neq k; & & [(M_k^*)^{-1}]_{kk} &= [M_k^{-1}]_{kk}, \end{aligned}$$

where $\beta_k = -e^{2\pi i \lambda'_k} \alpha_k$.

Corollary 1. *The first invariants of the monodromy matrices in Proposition 2 are*

$$\begin{aligned}\mathrm{Tr}(M_k) &= n - 1 + e^{-2\pi i \lambda'_k}, \\ \mathrm{Tr}(M_j M_k) &= n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} + \alpha_j \alpha_k c_{jk} c_{kj} \\ &= n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} + e^{-2\pi i (\lambda'_j + \lambda'_k)} \beta_j \beta_k c_{jk} c_{kj}.\end{aligned}$$

If A_1 has no negative integer eigenvalues, then

$$\mathrm{Tr}(M_k^*) = \mathrm{Tr}(M_k), \quad \mathrm{Tr}(M_j^* M_k^*) = \mathrm{Tr}(M_j M_k).$$

From Proposition 1 we know that Ψ^* is fundamental if and only if A_1 has no negative integer eigenvalues. Thus:

Corollary 2. *Suppose that A_1 has no negative integer eigenvalues; then M_1^*, \dots, M_n^* generate the monodromy group of equation (3–5).*

4.2. On the invertibility of C and $\Psi(\lambda)$

We establish necessary and sufficient conditions for the matrices $\Psi(\lambda)$ and $C = (c_{jk})$ to be invertible. Let $\lambda \in \mathcal{P}_\eta$.

Remark 5. If $\vec{r}^{(k)} = 0$ (case $\lambda'_k \in \mathcal{N}$) then C has zero k -th column and also $\Psi(\lambda)$ has zero k -th column, so it is not a fundamental matrix. If $\underline{r}^{(k)} = 0$ (case $\lambda'_k \in -\mathcal{N} - 2$), then C has zero k -th row. In both cases, C is not invertible.

Lemma 3. i) *If A_1 has no negative integer eigenvalues and $\Psi(\lambda)$ is fundamental, then C is invertible.*

- ii) *Conversely, if C is invertible, then:*
- A_1 has no negative integer eigenvalues,
 - $\Psi(\lambda)$ is fundamental,
 - the matrix defined by $\Psi^*(\lambda) := \Psi(\lambda)C^{-1}$, is the unique fundamental solution Ψ^* of Proposition 1,
 - in case $\lambda'_k \in \mathcal{N}$ then $\vec{r}^{(k)} \neq 0$, in case $\lambda'_k \in -\mathcal{N} - 2$, then $\underline{r}^{(k)} \neq 0$.

Proof. i) If A_1 has no negative integer eigenvalues, then the fundamental matrix $\Psi^*(\lambda)$ exists from Proposition 1. If $\Psi(\lambda)$ is invertible, then $C = \Psi^*(\lambda)^{-1} \cdot \Psi(\lambda)$ is invertible.

ii) C invertible implies that $\vec{r}^{(k)} \neq 0$ and $\underline{r}^{(k)} \neq 0$, when defined (Remark 5), thus in any row and any column of C there is a $c_{ij} \neq 0$ for some $i \neq j$. Write $\Psi(\lambda)$ at λ_k :

$$\begin{aligned}\Psi &= [\vec{\Psi}_1 | \dots | \vec{\Psi}_n] = [\vec{\Psi}_k^{(sing)} c_{k1} | \dots | \vec{\Psi}_k^{(sing)} c_{kn}] + \mathrm{reg}(\lambda - \lambda_k) \\ &= \{[0 | \dots | 0 | \vec{\Psi}_k^{(sing)} | 0 | \dots | 0] + \mathrm{reg}(\lambda - \lambda_k)\} C.\end{aligned}$$

The last step is possible because existence of C^{-1} allows to write

$$\text{reg}(\lambda - \lambda_k) = \text{reg}(\lambda - \lambda_k)C^{-1}C \equiv \text{reg}(\lambda - \lambda_k)C.$$

Thus

$$\Psi C^{-1} = [0 | \dots | 0 | \vec{\Psi}_k^{(sing)} | 0 | \dots | 0] + \text{reg}(\lambda - \lambda_k).$$

This is equivalent to (18) and (19), which implies that there exist the unique fundamental matrix $\Psi^* \equiv \Psi C^{-1}$. From Proposition 1 we conclude that A_1 has no negative integer eigenvalues. Obviously, it follows also that $\Psi = \Psi^* C$ is invertible. \square

Proposition 3. C is invertible $\Leftrightarrow A_1$ has no integer eigenvalues.

Proof. The “ \Rightarrow ” is proved in the previous lemma, point ii). The proof of “ \Leftarrow ” is the following generalization of that of proposition 2 in [4]. Since A_1 has no negative integer eigenvalues, the monodromy group is generated by M_1^*, \dots, M_n^* . Enumerate the poles in such a way that the ray L_{k+1} is to the left of the ray L_k , which implies that the monodromy at infinity for an anticlockwise loop encircling all the poles is $M_\infty^* = M_n^* \dots M_1^*$. The behaviour of system (5) at ∞ implies that A_1 has no integer eigenvalues if and only if M_∞^* has no eigenvalue = 1. This is equivalent to the fact that C is invertible. Indeed, existence of an eigenvalue equal to 1 means that there exists a non zero row vector $\hat{w} = [w_1, \dots, w_n]$, such that $\hat{w}M_\infty^* = \hat{w}$. Using the explicit expression of the M_k^* in terms of the $\alpha_k c_{jk}$'s, we compute

$$\hat{w} M_n^* \dots M_1^* = \hat{w} + \sum_{j=1}^n b_j \hat{e}_j,$$

where the \hat{e}_j 's are the basis rows, and

$$b_n = \alpha_n (\hat{w}C)_n,$$

$$b_i = \alpha_i \left[(\hat{w}C)_i + \sum_{j=i+1}^n c_{ji} b_j \right], \quad i = 1, 2, \dots, n-1.$$

Since all the α_i , for $1 \leq i \leq n$, are not zero (this is the crucial point), we conclude that $\hat{w}M_\infty^* = \hat{w}$ if and only if $\hat{w}C = 0$. \square

Proposition 4. i) If A_1 has no integer eigenvalues, then $\Psi(\lambda)$ is a fundamental matrix solution.

ii) With the additional assumption that $\lambda'_k \notin \mathbf{Z}$, $\forall k = 1, \dots, n$, also the converse holds: if $\Psi(\lambda)$ is a fundamental matrix solution, then A_1 has no integer eigenvalues.

Proof. i) If A_1 has no integer eigenvalues, C is invertible (Proposition 3). Therefore, the statement follows from Lemma 3, point ii).

ii) Let $\Psi = [\vec{\Psi}_1 | \dots | \vec{\Psi}_n]$ be fundamental. Observe that under the hypothesis that $\lambda'_k \notin \mathbf{Z}$ for any k , the columns are singular. Namely:

$$\vec{\Psi}_k(\lambda) \equiv \vec{\Psi}_k^{(sing)}(\lambda) = (\Gamma(\lambda'_k + 1)\vec{e}_k + O(\lambda - \lambda_k))(\lambda - \lambda_k)^{-\lambda'_k - 1}, \quad 1 \leq k \leq n.$$

The monodromy of $\Psi(\lambda)$ at infinity is $M_\infty = M_n M_{n-1} \dots M_1$. Suppose that there is an integer eigenvalue of A_1 . It follows that there exists a non zero column vector $\vec{v} = (v_1, \dots, v_n)^T$ (T means transpose) such that $M_\infty \vec{v} = \vec{v}$. As in the proof of Proposition 3, making use of the explicit form of the M_k 's in terms of the c_{jk} 's, we see that $M_\infty \vec{v} = \vec{v}$ is equivalent to $C\vec{v} = \vec{0}$. Take the vector $\vec{\psi}(\lambda) = \sum_{l=1}^n v_l \vec{\Psi}_l(\lambda)$. At every λ_k it behaves like

$$\vec{\psi}(\lambda) = \sum_{l=1}^n v_l \vec{\Psi}_k c_{kl} + \text{reg}(\lambda - \lambda_k) = \left(\sum_{l=1}^n c_{kl} v_l \right) \vec{\Psi}_k + \text{reg}(\lambda - \lambda_k).$$

But $\sum_{l=1}^n c_{kl} v_l = 0$, thus

$$\vec{\psi}(\lambda) = \text{reg}(\lambda - \lambda_k), \quad \text{close to any } \lambda_k, \quad k = 1, \dots, n.$$

This implies that $\vec{\psi}(\lambda)$ is a polynomial solution. This contradicts the fact that $\vec{\Psi}_1(\lambda), \dots, \vec{\Psi}_n(\lambda)$ is a basis, each $\vec{\Psi}_k(\lambda)$ being singular at λ_k , $1 \leq k \leq n$. \square

Corollary 3. *If A_1 has no integer eigenvalues, then M_1, \dots, M_n of Proposition 2 generate the monodromy group of system (3).*

Corollary 4. *Suppose A_1 has some integer eigenvalues and $\Psi(\lambda)$ is a fundamental matrix solution (consequently, M_1, \dots, M_n generate the monodromy group). In such cases, at least some λ'_k is necessarily integer.*

5. Dependence of matrix solutions and connection coefficients on η

Following [4], we call *critical values* the inadmissible values for η , namely

$$\arg(\lambda_j - \lambda_k) \bmod 2\pi.$$

We numerate them as in [4], as follows. In the angular interval $(-\pi/2, 3\pi/2]$ there is an even number $m = 2\mu$, μ integer, of critical values, ordered as

$$\frac{3\pi}{2} \geq \eta_0 > \eta_1 > \dots > \eta_{m-1} > -\frac{\pi}{2}.$$

All the possible critical values are then

$$\eta_{v+hm} := \eta_v - 2h\pi, \quad v = 0, \dots, m-1; \quad h \in \mathbf{Z}.$$

In each interval $(\theta - 2\pi, \theta]$ lie m successive values belonging to the set of critical values $\{\eta_\nu \mid \nu \in \mathbf{Z}\}$. There is an ordering of the poles with respect to an admissible η , given by the *dominance relation* $<$ below:

Definition 2 (as in [4]). Let η be admissible. We say that $j < k$, whenever in the plane \mathcal{P}_η the cut L_j lies to the right of the cut L_k . Equivalently, choose the determinations

$$\eta_{jk} := \text{determination of } \arg(\lambda_j - \lambda_k) \text{ s.t. } \eta - 2\pi < \eta_{jk} < \eta,$$

$$j \neq k, 1 \leq j, k \leq n.$$

Then

$$(21) \quad j < k \iff -\pi + \eta < \eta_{jk} < \eta.$$

The reason for the nomenclature ‘‘dominance’’ will be explained in section 6.1.

Remark. $\lambda_1, \dots, \lambda_n$ are in *lexicographical order with respect to the admissible* η when the labelling order $j < k$ coincides with the dominance order $j < k$.

The matrices $\Psi^{(k)}(\lambda)$, $\Psi(\lambda)$ and $\Psi^*(\lambda)$ defined in the plane \mathcal{P}_η , with η admissible, and the connection matrix C , depend on η . Therefore we write

$$\Psi^{(k)}(\lambda) = \Psi^{(k)}(\lambda, \eta), \quad \Psi(\lambda) = \Psi(\lambda, \eta),$$

$$\Psi^*(\lambda) = \Psi^*(\lambda, \eta), \quad C = C(\eta).$$

For two values $\eta < \tilde{\eta}$, we consider the plane with both the cuts of \mathcal{P}_η and $\mathcal{P}_{\tilde{\eta}}$. We denote by $\mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}}$ the simply connected set of *reference points w.r.t. \mathcal{P}_η and $\mathcal{P}_{\tilde{\eta}}$* (nomenclature as in [4]), namely the points in the doubly cut plane such that $\arg(\lambda - \lambda_k) \notin [\eta, \tilde{\eta}]$, $\forall k = 1, \dots, n$. A pole λ_j is called *accessible* if it is on the boundary of $\mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}}$. See figure 2. The following generalizes propositions 3 of [4] without any assumptions on the values of $\lambda'_1, \dots, \lambda'_n$.

Proposition 5. i) Let λ_k be accessible w.r.t. some admissible η and $\tilde{\eta}$. Then

$$\vec{\Psi}_k^{(k)}(\lambda, \eta) = \vec{\Psi}_k^{(k)}(\lambda, \tilde{\eta}) \quad \text{and} \quad \vec{\Psi}_k(\lambda, \eta) = \vec{\Psi}_k(\lambda, \tilde{\eta}), \quad \forall \lambda \in \mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}}.$$

ii) Let η and $\tilde{\eta}$ lie between two consecutive critical values: namely $\eta_{\nu+1} < \eta < \tilde{\eta} < \eta_\nu$. Then

$$C(\eta) = C(\tilde{\eta}).$$

iii) Let again $\eta_{\nu+1} < \eta < \tilde{\eta} < \eta_\nu$. Then

$$\Psi^*(\lambda, \eta) = \Psi^*(\lambda, \tilde{\eta}), \quad \forall \lambda \in \mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}},$$

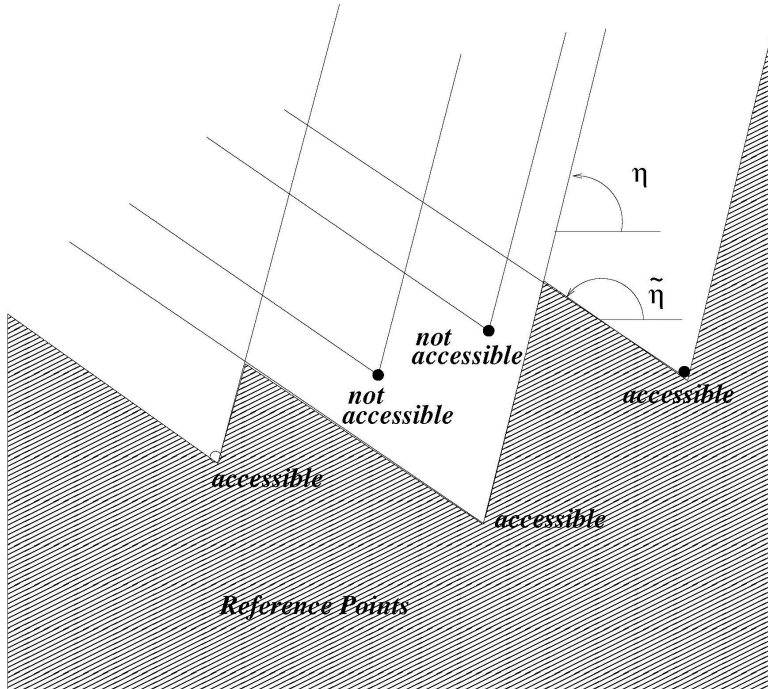


Fig. 2. Picture of $\mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}}$.

whenever Ψ^* is uniquely defined (namely, when A_1 has no negative integer eigenvalues).

Proof. i) is immediate, because λ_k is accessible and $\lambda \in \mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}}$. ii) is proved noticing that $\lambda_1, \dots, \lambda_n$ are all accessible from $\mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}}$, therefore i) holds for any k . iii) follows from i) and ii), noticing that Ψ^* is uniquely defined by the $\tilde{\Psi}_k$'s and C (Proposition 1). \square

The above implies that the dependence on η is discrete, changing only when a critical value is crossed. Therefore, if $\eta_{v+1} < \eta < \eta_v$, we follow [4] and write

$$\Psi_v(\lambda) := \Psi(\lambda, \eta), \quad \Psi_v^*(\lambda) := \Psi^*(\lambda, \eta), \quad C_v = (c_{jk}^{(v)}) := C(\eta).$$

Change of Ψ_v^* when a critical value is crossed is given by the following generalization, without assumptions on $\lambda'_1, \dots, \lambda'_n$, of proposition 4 of [4]

Proposition 6. *Suppose that A_1 has no negative integer eigenvalues (but no assumptions on $\lambda'_1, \dots, \lambda'_n$), so that the $\Psi_v^*(\lambda)$'s exist, for any $v \in \mathbf{Z}$. Let $\eta_{v+1} < \eta < \eta_v < \tilde{\eta} < \eta_{v-1}$. Then*

$$(22) \quad \Psi_{v-1}^*(\lambda) = \Psi_v^*(\lambda) W_v, \quad \forall \lambda \in \mathcal{P}_\eta \cap \mathcal{P}_{\tilde{\eta}},$$

where the invertible matrix $W_v = (W_{jk}^{(v)})$ is

$$(23) \quad W_{jk}^{(v)} = -\alpha_k c_{jk}^{(v)}, \quad \text{for } j \succ k \text{ such that } \arg(\lambda_j - \lambda_k) = \eta_v,$$

$$(24) \quad W_{jj}^{(v)} = 1, \quad j = 1, \dots, n; \quad W_{jk}^{(v)} = 0 \quad \text{otherwise,}$$

where \prec is the dominance relation w.r.t. η . In the same way,

$$\Psi_v^*(\lambda) = \Psi_{v-1}^*(\lambda) W_v^{-1}, \quad \forall \lambda \in \mathcal{P}_\eta \cap \mathcal{P}_{\bar{\eta}},$$

where W_v^{-1} has zero entries except for

$$[W_v^{-1}]_{jj} = 1, \quad j = 1, \dots, n,$$

$$[W_v^{-1}]_{jk} = -\beta_k c_{jk}^{(v-1)}, \quad \text{for } j \succ k \text{ s.t. } \arg(\lambda_j - \lambda_k) = \eta_v.$$

Proof. The proof is as for proposition 4 in [4]. Just observe that now we are using monodromy matrices (Proposition 2) and Ψ^* which are defined in more general terms. The detailed proof is in the arXiv version of this paper [11]. □

Recall that in an angular interval $(\theta - 2\pi, \theta]$, there are $m = 2\mu$ critical values. Let $\eta_{v+1} < \eta < \eta_v$ and introduce, as in [4], the matrices C_v^+ and C_v^- such that

$$(25) \quad \Psi_{v+\mu}^*(\lambda) = \Psi_v^*(\lambda) C_v^+, \quad \lambda \in \mathcal{P}_\eta \cap \mathcal{P}_{\eta-\pi},$$

$$(26) \quad \Psi_{v+\mu}^*(\lambda) = \Psi_{v+m}^*(\lambda) C_v^-, \quad \lambda \in \mathcal{P}_{\eta-\pi} \cap \mathcal{P}_{\eta-2\pi}.$$

Immediately it follows that

$$(27) \quad C_v^+ = (W_{v+\mu} \cdots W_{v+1})^{-1}, \quad C_v^- = W_{v+m} \cdots W_{v+\mu+1}.$$

Remark 6. $\mathcal{P}_\eta \cap \mathcal{P}_{\eta-\pi}$ is the half plane to the left hand side of all lines whose positive parts are the cuts of direction η , while $\mathcal{P}_{\eta-\pi} \cap \mathcal{P}_{\eta-2\pi}$ is the half plane to the right hand side of all lines whose positive parts are the cuts of direction $\eta - 2\pi$.

The following is a restatement of remark 3.3 of [4] with no assumptions on A_1 :

Lemma 4. *Let $A' := \text{diag}(\lambda'_1, \dots, \lambda'_n)$, $\lambda'_k \in \mathbb{C}$, $1 \leq k \leq n$. Then*

$$\Psi_{v+m}(\lambda) = \Psi_v(\lambda) e^{2\pi i A'},$$

for any λ in the universal covering of $\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_n\}$. Moreover

$$C_v = e^{2\pi i A'} C_{v+m} e^{-2\pi i A'}, \quad (\text{namely: } C(\eta) = e^{2\pi i A'} C(\eta - 2\pi) e^{-2\pi i A'}).$$

Proof. We write λ_η for $\lambda \in \mathcal{P}_\eta$. Then $\lambda_\eta - \lambda_k = (\lambda_{\eta-2\pi} - \lambda_k)e^{2\pi i}$. It follows from the definition of Ψ that $\Psi(\lambda, \eta - 2\pi) = \Psi(\lambda, \eta)e^{2\pi i A'}$, whatever the values $\lambda'_1, \dots, \lambda'_n$ are. From the above and the connection relations we find two expressions for $\vec{\Psi}_k(\lambda, \eta)$:

$$(28) \quad \vec{\Psi}_k(\lambda, \eta) = \vec{\Psi}_j^{(sing)}(\lambda, \eta)c_{jk}(\eta) + \text{reg}(\lambda - \lambda_j),$$

and

$$(29) \quad \begin{aligned} \vec{\Psi}_k(\lambda, \eta) &= e^{-2\pi i \lambda'_k} \vec{\Psi}_k(\lambda, \eta - 2\pi) \\ &= e^{-2\pi i \lambda'_k} \vec{\Psi}_j^{(sing)}(\lambda, \eta - 2\pi)c_{jk}(\eta - 2\pi) + \text{reg}(\lambda - \lambda_j). \end{aligned}$$

When $\lambda'_j \notin \mathbf{Z}$ we substitute in the above $\vec{\Psi}_j^{(sing)}(\lambda, \eta - 2\pi) = \vec{\Psi}_j(\lambda, \eta - 2\pi) = e^{2\pi i \lambda'_j} \vec{\Psi}_j(\lambda, \eta) \equiv e^{2\pi i \lambda'_j} \vec{\Psi}_j^{(sing)}(\lambda, \eta)$. Otherwise, we observe that

$$\begin{aligned} &\vec{\Psi}_j^{(sing)}(\lambda, \eta - 2\pi) \\ &= \vec{\Psi}_j(\lambda, \eta - 2\pi) \left[\ln(\lambda_{\eta-2\pi} - \lambda_j) + \frac{P_{N_j}^{(j)}(\lambda_{\eta-2\pi})}{(\lambda - \lambda_j)^{N_j+1}} \right] + \text{reg}(\lambda_{\eta-2\pi} - \lambda_j) \\ &= e^{2\pi i \lambda'_j} \vec{\Psi}_j(\lambda, \eta) \left\{ [\ln(\lambda_\eta - \lambda_j) - 2\pi i] + \frac{P_{N_j}^{(j)}(\lambda_\eta)}{(\lambda - \lambda_j)^{N_j+1}} \right\} + \text{reg}(\lambda_\eta - \lambda_j) \\ &= e^{2\pi i \lambda'_j} \vec{\Psi}_j^{(sing)}(\lambda, \eta) + \text{reg}(\lambda - \lambda_j). \end{aligned}$$

Therefore, confronting (28) and (29), we obtain

$$\vec{\Psi}_j^{(sing)}(\lambda, \eta)c_{jk}(\eta) + \text{reg}(\lambda - \lambda_j) = e^{2\pi i(\lambda'_j - \lambda'_k)} \vec{\Psi}_j^{(sing)}(\lambda, \eta)c_{jk}(\eta - 2\pi) + \text{reg}(\lambda - \lambda_j).$$

Finally, since $\vec{\Psi}_j^{(sing)}(\lambda, \eta)$ is singular at λ_j , the statement of the lemma for C_v follows. \square

We generalize proposition 5 of [4], with no assumptions on $\text{diag}(A_1) = (\lambda'_1, \dots, \lambda'_k)$.

Proposition 7. *Let $\eta_{v+1} < \eta < \eta_v$, and let $c_{jk}^{(v)} = c_{jk}(\eta)$. Consider the relation (25) and (26). The connection matrices C_v^+ , C_v^- are*

$$(30) \quad [C_v^+]_{jk} = \begin{cases} -\beta_k c_{jk}^{(v)} = e^{2\pi i \lambda'_k} \alpha_k c_{jk}^{(v)}, & \text{for } j < k, \\ 1, & \text{for } j = k, \\ 0, & \text{for } j > k, \end{cases}$$

$$(31) \quad [C_v^-]_{jk} = \begin{cases} 0, & \text{for } j < k, \\ 1, & \text{for } j = k, \\ e^{-2\pi i \lambda'_j} \beta_k c_{jk}^{(v)} = -e^{2\pi i (\lambda'_k - \lambda'_j)} \alpha_k c_{jk}^{(v)}, & \text{for } j > k, \end{cases}$$

where $\alpha_k = e^{-2\pi i \lambda'_k} - 1$ if $\lambda'_k \notin \mathbf{Z}$, $\alpha_k = 2\pi i$ if $\lambda'_k \in \mathbf{Z}$, $\beta_k = -e^{2\pi i \lambda'_k} \alpha_k$.

Proof. One proceeds as in the proof of proposition 5 of [4], with the more general monodromy matrices of Proposition 2 and Remark 4. The detailed proof is in the arXiv version of this paper [11]. \square

The matrices C_v^+ and C_v^- can be defined by formulae (30) and (31), independently of the fact that A_1 has no negative integer eigenvalues, namely independently of (25) and (26). The following corollary is a direct computation.

Corollary 5. *Let the matrices C_v^+ and C_v^- be defined by formulae (30) and (31). Then*

$$\begin{aligned} \text{Tr}(M_k) &= n - 1 + e^{-2\pi i \lambda'_k}, \\ \text{Tr}(M_j M_k) &= \begin{cases} n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} - e^{-2\pi i \lambda'_j} [C_v^+]_{jk} [C_v^-]_{kj}, & \text{if } j < k, \\ n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} - e^{-2\pi i \lambda'_k} [C_v^-]_{jk} [C_v^+]_{kj}, & \text{if } j > k. \end{cases} \end{aligned}$$

If moreover A_1 has no integer eigenvalues, then

$$\text{Tr}(M_k^*) = \text{Tr}(M_k), \quad \text{Tr}(M_j^* M_k^*) = \text{Tr}(M_j M_k).$$

6. Fundamental solutions of (1) as laplace integrals

6.1. Fundamental solutions of (1) and stokes matrices

The basic definitions of Stokes rays and matrices are well known [22], [2], so we just recall them

Definition 3. Stokes rays are the oriented rays from 0 to ∞ contained in the universal covering of $\mathbf{C} \setminus \{0\}$ (denoted $\widetilde{\mathbf{C} \setminus \{0\}}$) defined by the condition $\Re(z(\lambda_j - \lambda_k)) = 0$, $\Im(z(\lambda_j - \lambda_k)) < 0$ for $j \neq k$.

Let $\eta \in \mathbf{R}$ be admissible. We choose the Stokes rays

$$r_{jk} := \left\{ z \in \mathbf{C} \mid z = \rho \exp \left\{ i \left(\frac{3\pi}{2} - \eta_{jk} \right) \right\}, \rho > 0 \right\}, \quad j \neq k, 1 \leq j, k \leq n,$$

where

$$\eta_{jk} = \text{determination of } \arg(\lambda_j - \lambda_k) \text{ s.t. } \eta_{jk} \in (\eta - 2\pi, \eta].$$

It follows that $\Re(z(\lambda_j - \lambda_k)) < 0$ for z in the half plane to the right of r_{jk} . According to the definition, all the Stokes rays are characterized by $\arg z = 3\pi/2 - \eta_{jk} \bmod 2\pi$. For any (j, k) such that $\eta_{jk} < \eta < \eta_{jk} + \pi$

$$\Re(z(\lambda_j - \lambda_k)) < 0 \quad \text{if } \arg z = \frac{3\pi}{2} - \eta \bmod 2\pi.$$

This means that when an *admissible* η for system (3)–(5) is fixed, then

$$(32) \quad \Re(z(\lambda_j - \lambda_k)) < 0 \quad \text{for } \arg z = \frac{3\pi}{2} - \eta \bmod 2\pi \Leftrightarrow j < k,$$

where $<$ is the partial ordering previously defined, which therefore coincides with the dominance relation in the sense of the theory of ODE with singularity of the second kind. All the Stokes rays can be represented as $\arg z = \tau_\nu$, with $\tau_\nu := 3\pi/2 - \eta_\nu$, $\nu \in \mathbf{Z}$. It is easily seen that all Stokes rays are generated by

$$0 \leq \tau_0 < \tau_1 < \dots < \tau_{\mu-1} < \pi; \quad \tau_{\nu+h\mu} = \tau_\nu + \pi h, \quad h \in \mathbf{Z}.$$

Introduce the following notations for sectors of $\mathbf{C} \setminus \{0\}$:

$$S(\alpha, \beta) := \{z \in \mathbf{C} \setminus \{0\} \mid \alpha < \arg(z) < \beta\}, \quad \alpha < \beta \in \mathbf{R},$$

$$\mathcal{S}_\nu := S(\tau_\nu - \pi, \tau_{\nu+1}), \quad \nu \in \mathbf{Z}.$$

For any $\nu \in \mathbf{Z}$, equation (1) has a fundamental matrix solution

$$(33) \quad Y_\nu(z) = \hat{Y}_\nu(z)e^{A_0 z + A_1' \ln z}, \quad z \in \mathcal{S}_\nu,$$

where $A_1' = \text{diag}(A_1)$, and $\hat{Y}_\nu(z)$ is an invertible matrix, analytic in a neighbourhood of ∞ , with asymptotic expansion

$$(34) \quad \hat{Y}_\nu(z) \sim I + \frac{F_1}{z} + \frac{F_2}{z^2} + \dots = I + \sum_{k=1}^{\infty} \frac{F_k}{z^k}, \quad \text{for } z \rightarrow \infty \text{ in } \mathcal{S}_\nu.$$

The sector \mathcal{S}_ν is the maximal sector where the asymptotic behavior holds, and $Y_\nu(z)$ is unique, namely it is uniquely determined by its asymptotic behavior. The $n \times n$ matrices F_k are determined as rational functions of A_0 and A_1 , by formal substitution into (1) (see [22], [2]).

Definition 4 (Stokes Matrices). Given two fundamental matrices Y_ν and $Y_{\nu'}$ as above, whose maximal sectors \mathcal{S}_ν and $\mathcal{S}_{\nu'}$ intersect in such a way that no Stokes rays are contained in $\mathcal{S}_\nu \cap \mathcal{S}_{\nu'}$, then the connection matrix S such that $Y_{\nu'}(z) = Y_\nu(z)S$, $z \in \mathcal{S}_\nu \cap \mathcal{S}_{\nu'}$, is called a *Stokes matrix*.

It is easy to see that $v' = v + \mu$, therefore Stokes matrices are the matrices S_v , $v \in \mathbf{Z}$, such that

$$Y_{v+\mu}(z) = Y_v(z)S_v, \quad z \in \mathcal{S}_v \cap \mathcal{S}_{v+\mu} = S(\tau_v, \tau_{v+1}).$$

Since $\Re(z(\lambda_j - \lambda_k)) < 0$ for $j < k$ when $z \in S(\tau_v, \tau_{v+1})$, where the dominance relation is referred to any $\eta \in (\eta_{v+1}, \eta_v)$, then from the asymptotic behaviours of $Y_{v+\mu}(z)$ and $Y_v(z)$, it follows that $\delta_{jk} \sim e^{(\lambda_j - \lambda_k)z} (S_v)_{jk}$ and thus

$$(S_v)_{jj} = 1, \quad (S_v)_{jk} = 0 \quad \text{for } j \succ k.$$

Definition 5 (Stokes Factors). The Stokes factors are the connection matrices V_v such that

$$Y_{v-1}(z) = Y_v(z)V_v, \quad z \in \mathcal{S}_{v-1} \cap \mathcal{S}_v = S(\tau_v - \pi, \tau_v).$$

Asymptotic behaviours and dominance relations in $S(\tau_v - \pi, \tau_v)$ yield: $(V_v)_{jj} = 1$, and $(V_v)_{jk} = 0$ for all $j \neq k$, except possibly for (j, k) s.t. $\arg(\lambda_j - \lambda_k) = 3\pi/2 - \tau_v$ and $j \succ k$ with respect to $\eta \in (\eta_{v+1}, \eta_v)$. From the definitions above, it follows that

$$(35) \quad S_v = (V_{v+\mu} \dots V_{v+1})^{-1}.$$

The monodromy of $Y_v(z)$ is completely described by S_v , $S_{v+\mu}$ and A' , because the following holds

$$(36) \quad Y_v(ze^{2\pi i}) = Y_v(z)e^{2\pi i A'}(S_v S_{v+\mu})^{-1}, \quad z \in \mathcal{S}_v.$$

Definition 6. S_v and $S_{v+\mu}$ are a *complete set of Stokes multipliers*, because any other Stokes matrix can be expressed in terms of entries of S_v , $S_{v+\mu}$ and A' (see [2], [3]), by iterations of $S_{v+2\mu} = e^{-2\pi i A'} S_v e^{2\pi i A'}$.

6.2. Solutions of (1) as laplace integrals

We consider a path $\gamma_k(\eta)$ which comes from infinity along the left side of the cut L_k of direction η , encircles λ_k with a small loop excluding all the other poles, and goes back to infinity along the right side of L_k (where L_k is oriented from λ_k to ∞). See figure 3.

1) Case of $\lambda'_k \notin \mathbf{Z}$ We define

$$(37) \quad \vec{Y}_k(z, \eta) := \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k(\lambda, \eta) d\lambda \equiv \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k^{(k)}(\lambda, \eta) d\lambda.$$

Since $\lambda = \infty$ is a regular singularity of $\vec{\Psi}_k(\lambda, \eta)$, the exponential ensures that the integral converges in the sector

$$(38) \quad \mathcal{S}(\eta) := \{z \in \mathbf{C} \setminus \{0\} \mid \Re(ze^{i\eta}) < 0\} \Rightarrow \frac{\pi}{2} - \eta < \arg z < \frac{3\pi}{2} - \eta.$$

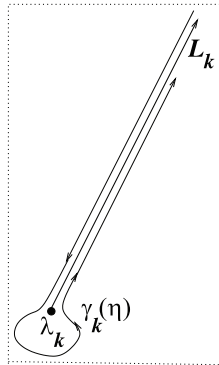


Fig. 3. The path $\gamma_k(\eta)$.

The asymptotic behaviour of (37) can be computed by expanding the integrand $\vec{\Psi}_k(\lambda)$ in series at λ_k and then formally exchanging integration and series (see [7]). Namely, for any $\mathcal{N} > 0$ integer,

$$\vec{Y}_k(z, \eta) = \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \left[\Gamma(\lambda'_k + 1) \vec{e}_k + \sum_{l \geq 1} \vec{b}_l^{(k)} (\lambda - \lambda_k)^l \right] (\lambda - \lambda_k)^{-\lambda'_k - 1} d\lambda = (*).$$

Now write $\sum_{l \geq 1} = \sum_{l=1}^{\mathcal{N}} + \sum_{l > \mathcal{N}}$. The known formula $\int_{\gamma_k(\eta)} (\lambda - \lambda_k)^{-a} e^{z\lambda} d\lambda = z^{a-1} e^{\lambda_k z} / \Gamma(a)$ yields

$$(39) \quad (*) = \left(\vec{e}_k + \sum_{l=1}^{\mathcal{N}} \frac{\vec{b}_l^{(k)}}{\Gamma(\lambda'_k + 1 - l)} \frac{1}{z^l} + \mathcal{R}(z) \right) e^{\lambda_k z} z^{\lambda'_k},$$

where $\mathcal{R}(z)$ is the integral of $\sum_{l > \mathcal{N}}$. It is standard computation to show that $\mathcal{R}(z) = O(z^{-\mathcal{N}})$. Thus, formula (39) allows us to write the asymptotic expansion

$$\vec{Y}_k(z, \eta) e^{-\lambda_k z} z^{-\lambda'_k} \sim \vec{e}_k + \sum_{l=1}^{\infty} \frac{\vec{b}_l^{(k)}}{\Gamma(\lambda'_k + 1 - l)} \frac{1}{z^l}, \quad z \rightarrow \infty, z \in \mathcal{S}(\eta).$$

Lemma 5. Assume $\lambda'_k \notin \mathbf{Z}$. Let $\eta \in (\eta_{v+1}, \eta_v)$, and $\tau_v := 3\pi/2 - \eta_v$. Then, $\vec{Y}_k(z, \eta)$ defined by (37) is the k -th column of the unique fundamental solution of (1) identified by the asymptotic behavior (33), (34) in the sector

$$\mathcal{S}_v = \mathcal{S}(\tau_v - \pi, \tau_{v+1}).$$

Proof. If $\eta_{v+1} < \eta < \tilde{\eta} < \eta_v$, then $Y_k(z, \eta) = Y_k(z, \tilde{\eta})$. This defines the analytic continuation of (37) to

$$\mathcal{S}(\tau_v - \pi, \tau_{v+1}) = \bigcup_{\eta_{v+1} < \eta < \eta_v} \mathcal{S}(\eta),$$

with the required asymptotic behaviour. It remains to prove that $Y_k(z, \eta)$ is a vector solution of (1). This follows from integration by parts, as shown in the Introduction, since $\gamma_k(\eta)$ is such that $e^{\lambda z}(\lambda - A_0)\vec{\Psi}_k(\lambda)|_{\gamma_k} = 0$. \square

If we write coefficients in (34) as

$$F_k = [\vec{f}_1^{(k)} | \dots | \vec{f}_n^{(k)}],$$

then, for $\eta_{v+1} < \eta < \eta_v$, solution (9) reads

$$\vec{\Psi}_k(\lambda, \eta) \equiv \vec{\Psi}_k^{(k)}(\lambda) = \sum_{l \geq 0} \Gamma(\lambda'_k + 1 - l) \vec{f}_l^{(k)} (\lambda - \lambda_k)^{l - \lambda'_k - 1}, \quad \vec{f}_0^{(k)} = \vec{e}_k.$$

2) Case of $\lambda'_k = -1$ We define

$$(40) \quad \vec{Y}_k(z, \eta) := \int_{L_k} e^{z\lambda} \vec{\Psi}_k(\lambda, \eta) d\lambda = - \int_{-L_k} e^{z\lambda} \vec{\Psi}_k(\lambda, \eta) d\lambda,$$

along the cut L_k from λ_k to infinity. This is convergent in $\mathcal{S}(\eta)$ as before. Its asymptotic behaviour is obtained as before by expanding $\vec{\Psi}_k$ in the convergent series (10), and then exchanging integration and series:

$$\begin{aligned} Y_k(z, \eta) &\sim \vec{e}_k \int_{-L_k} e^{z\lambda} d\lambda - \sum_{l \geq 1} \vec{b}_l^{(k)} \int_{-L_k} (\lambda - \lambda_k)^l e^{z\lambda} d\lambda \\ &= \frac{e^{\lambda_k z}}{z} \left[\vec{e}_k + \sum_{l=1}^{\infty} (-1)^{l+1} l! \vec{b}_l^{(k)} \frac{1}{z^l} \right], \end{aligned}$$

where we have used the fact that

$$\int_{-L_k} (\lambda - \lambda_k)^l e^{\lambda z} d\lambda = \frac{e^{\lambda_k z}}{z^{l+1}} \int_{+\infty e^{i\phi}}^0 \zeta^l e^{\zeta} d\zeta = \frac{e^{\lambda_k z}}{z^{l+1}} l! (-1)^l, \quad \frac{\pi}{2} < \phi < \frac{3\pi}{2}.$$

The same proof of Lemma 5 yields the following

Lemma 6. Assume $\lambda'_k = -1$. Let $\eta \in (\eta_{v+1}, \eta_v)$, and $\tau_v := 3\pi/2 - \eta_v$. Then, $Y_k(z, \eta)$ defined by (40) is the k -th column of the unique fundamental solution of (1) identified by the asymptotic behavior (33), (34) in the sector

$$\mathcal{S}_v = S(\tau_v - \pi, \tau_{v+1}).$$

By virtue of the lemma, we rewrite (10), for $\eta_{v+1} < \eta < \eta_v$, as follows

$$\vec{\Psi}_k(\lambda, \eta) = -\vec{e}_k + \sum_{l \geq 1} \frac{(-1)^{l+1}}{l!} \vec{f}_l^{(k)} (\lambda - \lambda_k)^l.$$

Lemma 7. *In case $\lambda'_k = -1$, the solution (40) has also the representation*

$$(41) \quad \vec{Y}_k(z, \eta) = \int_{L_k} e^{z\lambda} \vec{\Psi}_k(\lambda, \eta) d\lambda \equiv \frac{1}{2\pi i} \int_{\gamma_k(\eta)} \vec{\Psi}_k^{(k)}(\lambda) e^{z\lambda} d\lambda,$$

where $\gamma_k(\eta)$ is the same as (37).

Proof. Recall that $\vec{\Psi}_k^{(k)} = \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k)$. Since $\int_{\gamma_k} \text{reg}(\lambda - \lambda_k) e^{z\lambda} d\lambda = 0$, we have

$$\int_{\gamma_k(\eta)} \vec{\Psi}_k^{(k)}(\lambda) e^{z\lambda} d\lambda = \int_{\gamma_k(\eta)} \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) e^{z\lambda} d\lambda.$$

Indicate with L_k^L and L_k^R the left and right sides of L_k (oriented from λ_k to ∞), and with $(\lambda - \lambda_k)_{R/L}$ the branch of $(\lambda - \lambda_k)$ to the right/left of L_k . Then

$$\int_{\gamma_k(\eta)} = \int_{-L_k^L} + \int_{L_k^R} = \int_{L_k^R} - \int_{L_k^L} = (*).$$

Moreover $(\lambda - \lambda_k)_L = e^{-2\pi i}(\lambda - \lambda_k)_R$, where $\arg((\lambda - \lambda_k)_R) = \eta$. Therefore

$$\begin{aligned} (*) &= \int_{L_k^R} \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k)_R e^{z\lambda} d\lambda \\ &+ \left\{ 2\pi i \int_{L_k} \vec{\Psi}_k(\lambda) e^{z\lambda} d\lambda - \int_{L_k^R} \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k)_R e^{z\lambda} d\lambda \right\} \\ &\equiv 2\pi i \int_{L_k} \vec{\Psi}_k(\lambda) e^{z\lambda} d\lambda. \quad \square \end{aligned}$$

3) Case of $\lambda'_k \in \mathbb{N}$ Define the convergent in $\mathcal{S}(\eta)$ integral

$$(42) \quad \vec{Y}_k(z, \eta) := \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k^{(k)}(\lambda, \eta) d\lambda,$$

where $\vec{\Psi}_k^{(k)}(\lambda)$ is (12). For $z \rightarrow \infty$, the logarithmic part yields

$$\frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) d\lambda = \int_{L_k} e^{z\lambda} \vec{\Psi}_k(\lambda) d\lambda \sim \sum_{l=0}^{\infty} (-1)^{l+1} l! d_l^{(k)} \frac{1}{z^{l+1}} e^{z\lambda_k}.$$

On the other hand, by Cauchy theorem, the terms with poles yield

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \frac{P^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1}} \\ &= \frac{1}{N_k!} \frac{d^{N_k}}{d\lambda^{N_k}} (P^{(k)}(\lambda) e^{z\lambda})|_{\lambda=\lambda_k} \\ &= e^{\lambda_k z} \sum_{q=0}^{N_k} \frac{\vec{b}_{N_k-q}^{(k)}}{q!} z^q = \left[\vec{e}_k + \dots + \frac{\vec{b}_{N_k}^{(k)}}{z^{N_k}} \right] z^{N_k} e^{\lambda_k z}, \quad \vec{b}_0^{(k)} = N_k! \vec{e}_k. \end{aligned}$$

We conclude that $\vec{Y}_k(z, \eta)$ has the correct asymptotics. The same proof of Lemma 5 yields the following

Lemma 8. Assume $\lambda'_k \in \mathbb{N}$. Let $\eta \in (\eta_{v+1}, \eta_v)$, and $\tau_v := 3\pi/2 - \eta_v$. Then, $\vec{Y}_k(z, \eta)$ defined by (42) is the k -th column of the unique fundamental solution of (1) identified by the asymptotic behavior (33), (34) in the sector

$$\mathcal{S}_v = S(\tau_v - \pi, \tau_{v+1}).$$

Accordingly, we rewrite for $\eta_{v+1} < \eta < \eta_v$:

$$\vec{\Psi}_k^{(k)}(\lambda, \eta) = \sum_{l=0}^{N_k} (N_k - l)! \vec{f}_l^{(k)} u^{l-N_k-1} + \left[\sum_{l=0}^{\infty} \frac{(-1)^{l+1} \vec{f}_{N_k+l+1}^{(k)}}{l!} u^l \right] \ln(u) + \text{reg}(u),$$

where $u := \lambda - \lambda_k$.

4) Case of $\lambda'_k = N_k \in -N - 2$ We define

$$(43) \quad \vec{Y}_k(z, \eta) := \int_{L_k} e^{z\lambda} \vec{\Psi}_k(\lambda) d\lambda.$$

The asymptotic behaviour of the above is readily computed:

$$\begin{aligned} \int_{L_k} e^{z\lambda} \vec{\Psi}_k(\lambda) d\lambda &= \int_{L_k} e^{z\lambda} \sum_{l \geq 0} \vec{b}_l^{(k)} (\lambda - \lambda_k)^{l-N_k-1} d\lambda \\ &\sim e^{\lambda_k z} (-1)^{N_k} \sum_{l \geq 0} \frac{(-1)^l (l - N_k - 1)! \vec{b}_l^{(k)}}{z^{l-N_k}} \\ &= \left[\vec{e}_k + (-1)^{N_k} \sum_{l \geq 1} (-1)^l (l - N_k - 1)! \vec{b}_l^{(k)} \frac{1}{z^l} \right] z^{N_k} e^{\lambda_k z}. \end{aligned}$$

Here we have used the normalization $\vec{b}_0^{(k)} = (-1)^{N_k} \vec{e}_k / (-N_k - 1)!$. We conclude that $\vec{Y}_k(z, \eta)$ has the correct asymptotics. The same proof of Lemma 5 yields the following

Lemma 9. Assume $\lambda'_k = N_k \in -N - 2$. Let $\eta \in (\eta_{v+1}, \eta_v)$, and $\tau_v := 3\pi/2 - \eta_v$. Then, $\vec{Y}_k(z, \eta)$ defined by (43) is the k -th column of the unique

fundamental solution of (1) identified by the asymptotic behaviour (33), (34) in the sector

$$\mathcal{S}_v = \mathcal{S}(\tau_v - \pi, \tau_{v+1}).$$

Accordingly:

$$\vec{\Psi}_k(\lambda) = \sum_{l \geq 0} \frac{(-1)^{l-N_k}}{(l - N_k - 1)!} \vec{J}_l^{(k)} (\lambda - \lambda_k)^{l-N_k-1}.$$

Also in this case we have

$$\frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) d\lambda = \int_{L_k} e^{z\lambda} \vec{\Psi}_k(\lambda) d\lambda.$$

Therefore, when the singular solution $\vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k)$ exists, we have

$$\vec{Y}_k(z, \eta) = \int_{\gamma_k(\eta)} e^{z\lambda} (\vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k)) d\lambda.$$

Proposition 8. *The following are the fundamental matrix solutions of (1) uniquely identified by the asymptotic behaviour (33), (34) in \mathcal{S}_v , $v \in \mathbf{Z}$:*

$$(44) \quad Y_v(z) = [\vec{Y}_1(z, \eta) | \dots | \vec{Y}_n(z, \eta)],$$

$$\vec{Y}_k(z, \eta) = \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k^{(sing)}(\lambda, \eta) d\lambda, \quad 1 \leq k \leq n, \eta_v < \eta < \eta_{v+1},$$

where $\vec{\Psi}_k^{(sing)}$ is defined in (16). In case $\vec{\Psi}_k^{(sing)} = 0$, for $\lambda'_k \in -N - 2$, then (44) is replaced by (43).

Proof. The above is a consequence of the preceding discussion. Linear independence of the columns of $Y_v(z)$ follows from the independence of the first term of the asymptotic behaviour of each column. Uniqueness follows from the maximality of the sector. \square

Lemma 10. *If A_1 has no negative integer eigenvalues, then, $\vec{\Psi}_k^{(sing)}$ in the integral (44) can be replaced by $\vec{\Psi}_k^*$. Namely:*

$$\vec{Y}_k(z, \eta) = \frac{1}{2\pi i} \int_{\gamma_k(\eta)} e^{z\lambda} \vec{\Psi}_k^*(\lambda, \eta) d\lambda, \quad 1 \leq k \leq n, \eta_v < \eta < \eta_{v+1}.$$

Proof. Recall that if A_1 has no negative integer eigenvalues, then $\vec{\Psi}_k^{(sing)} \neq 0$ also for $\lambda'_k \in -N - 2$. Since $\vec{\Psi}_k^{(sing)} - \vec{\Psi}_k^* = \text{reg}(\lambda - \lambda_k)$, we have

$$\int_{\gamma_k(\eta)} (\vec{\Psi}_k^{(sing)}(\lambda) - \vec{\Psi}_k^*(\lambda)) e^{z\lambda} d\lambda = 0. \quad \square$$

7. Stokes factors and matrices in terms of C —main theorem (Th. 1)

In this section we state the main result of the paper, which is Theorem 1 and Corollary 6. Consider as in [4] a new path of integration $\gamma(\eta)$ homotopic to the product $\gamma_{k_n}(\eta) \dots \gamma_{k_1}(\eta)$, $k_1 < k_2 < \dots < k_n$, namely a path coming from ∞ in direction η to the left of all the poles $\lambda_1, \dots, \lambda_n$, encircling all the poles, and going back to ∞ in direction η to the right of all the poles. The following Proposition is the generalization of Theorem 2' of [4] when no assumptions are made on $\lambda'_1, \dots, \lambda'_n$.

Proposition 9. *If A_1 has no negative integer eigenvalues (but no assumptions on $\lambda'_1, \dots, \lambda'_n$) the fundamental matrix of Proposition 8 is*

$$Y_v(z) = \frac{1}{2\pi i} \int_{\gamma(\eta)} e^{z\lambda} \Psi^*(\lambda, \eta) d\lambda, \quad \eta_{v+1} < \eta < \eta_v,$$

and

$$Y_{v-1}(z) = Y_v(z) W_v, \quad z \in \mathcal{L}_{v-1} \cap \mathcal{L}_v = \mathcal{S}(\tau_v - \pi, \tau_v),$$

where the W_v 's are given in Proposition 6. The Stokes factors and matrices are

$$V_v = W_v, \quad S_v = C_v^+, \quad S_{v+\mu}^{-1} = C_v^-.$$

Proof. The proof is technically as in Theorem 2' of [4], so for brevity we do not repeat it. The crucial point is that now the matrices Ψ , Ψ^* , W_v , and C_v^\pm have been defined—as in the construction of previous sections—for any $\lambda'_1, \dots, \lambda'_n$. So the proof holds for any $\lambda'_1, \dots, \lambda'_n$. \square

With much more technical effort—which requires a non trivial generalization of the technique of [4]—we are now going to prove that the statement of the above Proposition holds without any assumptions on A_1 , namely:

Theorem 1. *Let A_1 be any $n \times n$ matrix. The (complete set of) Stokes multipliers and matrices of system (1) are given in terms of the connection coefficients $c_{jk}^{(v)}$ of system (3) according to the formulae*

$$V_v = W_v, \quad S_v = C_v^+, \quad S_{v+\mu}^{-1} = C_v^-, \quad \forall v \in \mathbf{Z},$$

where W_v is defined by formulae (23), (24), and C_v^+ and C_v^- are defined by formulae (30) and (31).

Remark. Here formulae (23), (24), (30) and (31) are taken as the definitions of W_v , C_v^+ and C_v^- , independently of the existence of $\Psi^*(\lambda)$.

Corollary 6. *Let A_1 be any $n \times n$ matrix. The following equalities hold for the monodromy matrices of $\Psi(\lambda)$ of system (3)–(5), defined in (17):*

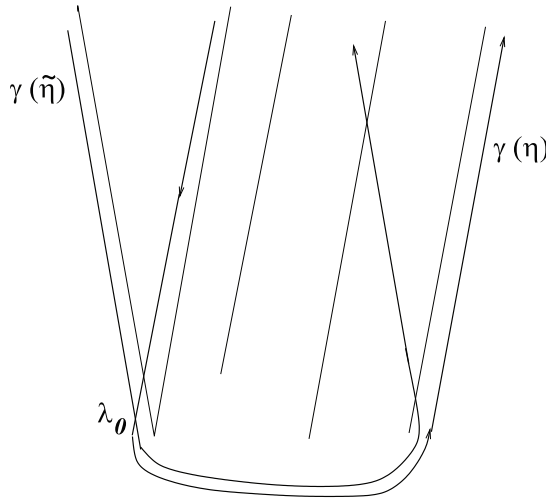


Fig. 4. The paths $\gamma(\eta)$ and $\gamma(\tilde{\eta})$.

$$\begin{aligned} \text{Tr}(M_k) &= n - 1 + e^{-2\pi i \lambda'_k}, \\ \text{Tr}(M_j M_k) &= \begin{cases} n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} - e^{-2\pi i \lambda'_j} [S_v]_{jk} [S_{v+\mu}^{-1}]_{kj} & \text{if } j < k, \\ n - 2 + e^{-2\pi i \lambda'_j} + e^{-2\pi i \lambda'_k} - e^{-2\pi i \lambda'_k} [S_{v+\mu}^{-1}]_{jk} [S_v]_{kj} & \text{if } j > k. \end{cases} \end{aligned}$$

The corollary above is a restatement of Corollary 5. We prove Theorem 1 in a few steps.

8. Proof of Theorem 1

We define

$${}_y Y(z) := z^{-y} Y(z),$$

which yields a gauge transformation of the linear systems (1):

$$(45) \quad \frac{d}{dz}({}_y Y) = \left(A_0 + \frac{A_1 - y}{z} \right) {}_y Y.$$

The fundamental solutions ${}_y Y_v(z) = z^{-y} Y_v(z)$, have the same Stokes multiplier and Stokes matrices than $Y_v(z)$, and their columns are obtained as Laplace transforms of solutions of

$$(46) \quad (A_0 - \lambda) \frac{d}{d\lambda}({}_y \Psi) = (A_1 - y + I) {}_y \Psi.$$

If A_1 has diagonal entries $\lambda'_1, \dots, \lambda'_n$, some of which may be integers, then we can always find a sufficiently small $\gamma_0 > 0$ such that, for any $0 < \gamma < \gamma_0$, $A_1 - \gamma$ has diagonal entries $\lambda'_1 - \gamma, \dots, \lambda'_n - \gamma$ which are not integers, and moreover has no integer eigenvalues, so that Ψ^* exists. In the following, we assume that γ has this property.

For system (3), the matrix $C_v = (c_{jk}^{(v)})$ is defined by (7). The matrices C_v^+ and C_v^- can always be defined by the formulae of Proposition 7, independently of the existence of Ψ^* and formulae (25) and (26). On the other hand, for system (46) the matrices C_v^+ and C_v^- (which depend on γ , so we write $C_v^+[\gamma]$ and $C_v^-[\gamma]$), are well defined by formulae (25) and (26). According to Proposition 7, their entries are again given in terms of γ -dependent connection coefficients $c_{jk}^{(v)} = c_{jk}^{(v)}[\gamma]$'s. The latter are defined by the first equality of (7) applied to the solutions ${}_\gamma \vec{\Psi}_k$, namely:

$$(47) \quad {}_\gamma \vec{\Psi}_k(\lambda) = {}_\gamma \vec{\Psi}_j(\lambda) c_{jk}^{(v)}[\gamma] + \text{reg}(\lambda - \lambda_j).$$

The following Proposition is the key step to prove Theorem 1

Proposition 10. *Let $\gamma_0 > 0$ be small enough such that the diagonal part of $A_1 - \gamma I$ has no integer entries and A_1 has no integer eigenvalues for any $0 < \gamma < \gamma_0$. Let $\eta_{v+1} < \eta < \eta_v$ be fixed. Let $c_{jk}^{(v)}$ be the corresponding connection coefficients of system (3), defined by (7), and $c_{jk}^{(v)}[\gamma]$ be the connection coefficients of (46), defined by (47). Finally, let*

$$\alpha_k = \begin{cases} e^{-2\pi i \lambda'_k} - 1, & \lambda'_k \notin \mathbf{Z}, \\ 2\pi i, & \lambda'_k \in \mathbf{Z}, \end{cases} \quad \alpha_k[\gamma] = e^{-2\pi i(\lambda'_k - \gamma)} - 1.$$

Then, the following equalities hold

$$\begin{aligned} \alpha_k c_{jk}^{(v)} &= e^{-2\pi i \gamma} \alpha_k[\gamma] c_{jk}^{(v)}[\gamma], & \text{if } k \succ j, \\ \alpha_k c_{jk}^{(v)} &= \alpha_k[\gamma] c_{jk}^{(v)}[\gamma], & \text{if } k \prec j, \end{aligned}$$

where the partial ordering \prec refers to η .

Corollary 7. *Let γ be as in Proposition 10. Let $C_v^+[\gamma]$ and $C_v^-[\gamma]$ be the connection matrices defined in (25) and (26) for system (46). Let C_v^+ and C_v^- be the matrices for system (3) defined by (30) and (31), where the $c_{jk}^{(v)}$ are defined by (7). Then*

$$C_v^+ = C_v^+[\gamma], \quad C_v^- = C_v^-[\gamma], \quad \forall v \in \mathbf{Z}.$$

Also, let W_v be defined by (23) and (24) for system (3), and $W_v[\gamma]$ be the matrix defined by (22) for system (46). Then

$$W_v = W_v[\gamma], \quad \forall v \in \mathbf{Z}.$$

Proof of Corollary 7. It is enough to compare the formulae of Proposition 10 with those of Propositions 7 and 6. \square

Before proving Proposition 10, we give the proof of Theorem 1.

Proof of Theorem 1. The S_v 's are unchanged by the gauge ${}_\gamma Y(z) = z^{-\gamma} Y(z)$. Moreover, Proposition 9 applies to the system (46), therefore

$$S_v = C_v^+[\gamma], \quad S_{v+\mu}^{-1} = C_v^-[\gamma], \quad V_v = W_v[\gamma].$$

Thus, Corollary 7 implies Theorem 1. \square

8.1. Proof of Proposition 10, by steps

Proposition 10 is the analogous of Lemma 2' in [4], but it requires much more technical efforts. The proof is based on the properties of higher order primitives of vector solutions of (3)–(5). Since $\lambda'_1, \dots, \lambda'_n$ are any complex numbers, including integers, the description and the proof of these properties require a highly non trivial extension of the classical techniques used in [4]. We proceed in a few steps, namely Proposition 11, Lemmas 11, 12, and Proposition 12. First, we introduce higher order primitives of vector solutions of system (5).

For $\lambda'_k \in \mathbb{C} \setminus \mathbb{N}$, we recall that there are solutions

$$\begin{aligned} \vec{\Psi}_k(\lambda) &= \sum_{l=0}^{\infty} \Gamma(\lambda'_k + 1 - l) \vec{f}_l^{(k)}(\lambda - \lambda_k)^{l - \lambda'_k - 1}, \quad \lambda'_k \notin \mathbb{Z}, \\ \vec{\Psi}_k(\lambda) &= \sum_{l=0}^{\infty} \frac{(-1)^{l - N_k}}{(l - N_k - 1)!} \vec{f}_l^{(k)}(\lambda - \lambda_k)^{l - N_k - 1}, \quad \lambda'_k = N_k \in \mathbb{Z}_-. \end{aligned}$$

We define the q primitive of $\vec{\Psi}_k$. This is the function $(\vec{\Psi}_k)^{[-q]}(\lambda)$, $q \in \mathbb{N}$, given by analytic continuation of the series obtained by q -fold term wise integration of the corresponding term in $\vec{\Psi}_k(\lambda)$. Namely:

$$(48) \quad (\vec{\Psi}_k)^{[-q]}(\lambda) := (-1)^q \sum_{l=q}^{\infty} \Gamma(\lambda'_k + 1 - l) \vec{f}_{l-q}^{(k)}(\lambda - \lambda_k)^{l - \lambda'_k - 1}, \quad \lambda'_k \notin \mathbb{Z},$$

$$(49) \quad (\vec{\Psi}_k)^{[-q]}(\lambda) := (-1)^q \sum_{l=q}^{\infty} \frac{(-1)^{l - N_k}}{(l - N_k - 1)!} \vec{f}_{l-q}^{(k)}(\lambda - \lambda_k)^{l - N_k - 1}, \quad \lambda'_k = N_k \leq -1.$$

They above converge in a neighbourhood of λ_k , contained in \mathcal{P}_η , where $\vec{\Psi}_k$ has convergent series. Indeed, if $\lambda_0 \neq \lambda_k$ is in the neighbourhood, then

$$(50) \quad \int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{q-1}} ds_q \vec{\Psi}_k(s_q) = (\vec{\Psi}_k)^{[-q]}(\lambda) - \mathcal{Q}_{q-1}(\lambda - \lambda_0),$$

where

$$Q_{q-1}(\lambda - \lambda_0) = (\vec{\Psi}_k)^{[-q]}(\lambda_0) + (\vec{\Psi}_k)^{[1-q]}(\lambda_0)(\lambda - \lambda_0) + \frac{(\vec{\Psi}_k)^{[2-q]}(\lambda_0)}{2!}(\lambda - \lambda_0)^2 + \dots + \frac{(\vec{\Psi}_k)^{[-1]}(\lambda_0)}{(q-1)!}(\lambda - \lambda_0)^{q-1},$$

is a polynomial in $(\lambda - \lambda_0)$ of degree $q - 1$. The path of integration is any in \mathcal{P}_η , such that $|\lambda - \lambda_0|$ is small enough for the series of $\vec{\Psi}_k$ to converge. Once $(\vec{\Psi}_k)^{[-q]}(\lambda)$ is defined by the convergent series, then it is analytically continued to \mathcal{P}_η .

For $\lambda'_k = N_k \in N$ integer, we recall that we have the solution

$$\begin{aligned} \vec{\Psi}_k^{(k)}(\lambda) &= \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \frac{P_{N_k}^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1}} + \text{reg}(\lambda - \lambda_k), \\ P_{N_k}^{(k)}(\lambda) &= \sum_{l=0}^{N_k} (N_k - l)! \vec{f}_l^{(k)}(\lambda - \lambda_k)^l, \\ \vec{\Psi}_k(\lambda) &= \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} \vec{f}_{N_k+l+1}^{(k)}(\lambda - \lambda_k)^l. \end{aligned}$$

The series is convergent in a neighbourhood of λ_k contained in \mathcal{P}_η . Let λ_0 belong to the neighbourhood. Let $q \geq 0$ integer, and compute q times the integral of $\vec{\Psi}_k^{(k)}(\lambda)$. Due to convergence of the series, we can take integration term by term. We obtain:

i) For $q \leq N_k$:

$$\begin{aligned} &\int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{q-1}} ds_q \vec{\Psi}_k^{(k)}(s_q) \\ &= \sum_{l=0}^{\infty} \frac{(-1)^{l+1-q}}{l!} \vec{f}_{N_k+1+l-q}^{(k)}(\lambda - \lambda_k)^l \ln(\lambda - \lambda_k) + \frac{P_{N_k-q}^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1-q}} + \text{reg}(\lambda - \lambda_k), \\ P_{N_k-q}^{(k)}(\lambda) &= (-1)^q \sum_{l=0}^{N_k-q} (N_k - l - q)! \vec{f}_l^{(k)}(\lambda - \lambda_k)^l. \end{aligned}$$

ii) For $q = N_k + 1$:

$$\int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{N_k}} ds_{N_k+1} \vec{\Psi}_k^{(k)}(s_{N_k+1}) = \hat{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k),$$

where we defined

$$(51) \quad \hat{\Psi}_k(\lambda) := \sum_{l=0}^{\infty} \frac{(-1)^{l+N_k}}{l!} \vec{f}_l^{(k)}(\lambda - \lambda_k)^l.$$

The function $\hat{\Psi}_k(\lambda)$ is defined by the series, that converges in the neighbourhood of λ_k in \mathcal{P}_η , where the series of $\vec{\Psi}_k^{(k)}$ converges. Then it is analytically continued in \mathcal{P}_η . Note that if all $r_j^{(k)} = 0, \forall j$, namely when there is no logarithmic term in $\vec{\Psi}_k^{(k)}$, then the sum in $\hat{\Psi}_k(\lambda)$ is truncated to $\sum_{l=0}^{N_k}$, giving a polynomial of degree N_k .

iii) For $q = N_k + 1 + \tilde{q}$, with $\tilde{q} \geq 0$:

$$\int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{q-1}} ds_q \vec{\Psi}_k^{(k)}(s_q) = (\hat{\Psi}_k)^{[-\tilde{q}]}(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k),$$

$$(\hat{\Psi}_k)^{[-\tilde{q}]}(\lambda) = (-1)^{\tilde{q}} \sum_{l=\tilde{q}}^{\infty} \frac{(-1)^{l+N_k}}{l!} \vec{f}_{l-\tilde{q}}^{(k)}(\lambda - \lambda_k)^l.$$

The function $(\hat{\Psi}_k)^{[-q]}(\lambda)$ is the q primitive of $\hat{\Psi}_k(\lambda)$, and the same computation of (50) yields

$$(52) \quad \int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{q-1}} ds_q \hat{\Psi}_k(s_q) = (\hat{\Psi}_k)^{[-q]}(\lambda) - \mathcal{Q}_{q-1}(\lambda - \lambda_0),$$

where \mathcal{Q}_{q-1} is as in (50) with substitution $\vec{\Psi}_k \mapsto \hat{\Psi}_k$.

Remark 7. Let

$$(\hat{\Psi}_k)^{[r]}(\lambda) := \frac{d^r}{d\lambda^r} (\hat{\Psi}_k(\lambda)) = \sum_{l=0}^{\infty} \frac{(-1)^{l+N_k-r}}{l!} \vec{f}_{l+r}^{(k)}(\lambda - \lambda_k)^l, \quad 0 \leq r \leq N_k + 1.$$

In particular, $(\hat{\Psi}_k)^{[N_k+1]}(\lambda) = \vec{\Psi}_k(\lambda)$. Then

$$\begin{aligned} & \frac{d^r}{d\lambda^r} (\hat{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k)) \\ &= (\hat{\Psi}_k)^{[r]}(\lambda) \ln(\lambda - \lambda_k) + \frac{P_{r-1}^{(k)}(\lambda)}{(\lambda - \lambda_k)^r} + \text{reg}(\lambda - \lambda_k), \end{aligned}$$

with $P_{r-1}^{(k)}(\lambda) = (-1)^{N_k+1-r} \sum_{l=0}^{r-1} (r-1-l)! \vec{f}_l^{(k)}(\lambda - \lambda_k)^l$.

We summarize (50) and (52) and the computations involving logarithmic solutions in the following

Proposition 11. *Let $\lambda_0 \neq \lambda_j$ for any $j = 1, 2, \dots, n$. For a given $k \in \{1, \dots, n\}$ define*

$$(53) \quad \phi_k(\lambda) := \begin{cases} \vec{\Psi}_k(\lambda), & \text{if } \lambda'_k \in \mathbf{C} \setminus \mathbf{N}, \\ \hat{\Psi}_k(\lambda), & \text{if } \lambda'_k \in \mathbf{N}, \end{cases}$$

$$\Phi_k^{[-q]}(\lambda, \lambda_0) := \int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{q-1}} ds_q \phi_k(s_q),$$

where $\vec{\Psi}_k$ are defined in (15), and $\hat{\Psi}$ is defined in (51). Then

$$(54) \quad \Phi_k^{[-q]}(\lambda, \lambda_0) = (\phi_k)^{[-q]}(\lambda) - Q_{q-1}^{(k)}(\lambda - \lambda_0),$$

where $Q_{q-1}^{(k)}$ is a polynomial of degree $q-1$ in $(\lambda - \lambda_0)$. It follows from the definition that

$$(55) \quad \int_{\lambda_0}^{\lambda} ds \Phi_k^{[-q]}(s, \lambda_0) = \Phi_k^{[-q-1]}(\lambda, \lambda_0).$$

For $\lambda'_k \in \mathbf{Z}$, consider the singular solutions $\vec{\Psi}_k^{(sing)}$ of system (3):

$$\begin{aligned} \varepsilon_k \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k), \quad \lambda'_k \in \mathbf{Z}_-, \\ \vec{\Psi}_k(\lambda) \ln(\lambda - \lambda_k) + \frac{P^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1}} + \text{reg}(\lambda - \lambda_k), \quad \lambda'_k = N_k \in \mathbf{N}. \end{aligned}$$

In the above, $\varepsilon_k = 0$ if $\vec{\Psi}_k^{(sing)} \equiv 0$, otherwise $\varepsilon_k = 1$. Then, for $\lambda'_k \in \mathbf{Z}_-$:

$$(56) \quad \int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{q-1}} ds_q (\varepsilon_k \vec{\Psi}_k(s_q) \ln(s_q - \lambda_k) + \text{reg}(s_q - \lambda_k)) \\ = \varepsilon_k (\vec{\Psi}_k)^{[-q]}(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k), \quad q \geq 0,$$

and for $\lambda'_k = N_k \in \mathbf{N}$:

$$(57) \quad \int_{\lambda_0}^{\lambda} ds_1 \int_{\lambda_0}^{s_1} ds_2 \dots \int_{\lambda_0}^{s_{q-1}} ds_q \left(\vec{\Psi}_k(s_q) \ln(s_q - \lambda_k) + \frac{P_{N_k}^{(k)}(s_q)}{(s_q - \lambda_k)^{N_k+1}} + \text{reg}(s_q - \lambda_k) \right) \\ = \begin{cases} (\hat{\Psi}_k)^{[N_k+1-q]}(\lambda) \ln(\lambda - \lambda_k) + \frac{P_{N_k-q}^{(k)}(\lambda)}{(\lambda - \lambda_k)^{N_k+1-q}} + \text{reg}(\lambda - \lambda_k), & 0 \leq q \leq N_k, \\ (\hat{\Psi}_k)^{[-q+N_k+1]}(\lambda) \ln(\lambda - \lambda_k) + \text{reg}(\lambda - \lambda_k), & q \geq N_k + 1. \end{cases}$$

The above expressions hold by analytic continuation for $\lambda \in \mathcal{P}_\eta$.

Corollary 8. Let $\lambda'_k = N_k \in \mathbf{N}$. Let c_{jk} denote $c_{jk}(\eta) = c_{jk}^{(v)}$. The vector function $\hat{\Psi}_k(\lambda)$ in (51), $\lambda \in \mathcal{P}_\eta$, has the following behaviours at $\lambda_j \neq \lambda_k$.

For $\lambda'_j \notin \mathbf{Z}$:

$$(58) \quad \hat{\Psi}_k(\lambda) = \vec{\Psi}_j^{[-N_k-1]}(\lambda) c_{jk} + \text{reg}(\lambda - \lambda_j).$$

For $\lambda'_j = N_j \in \mathbf{N}$:

$$(59) \quad \hat{\Psi}_k(\lambda) = \begin{cases} \left(\hat{\Psi}_j^{[N_j-N_k]}(\lambda) \log(\lambda - \lambda_j) + \frac{P_{N_j-N_k-1}(\lambda)}{(\lambda - \lambda_j)^{N_j-N_k}} \right) c_{jk} & N_j \geq N_k + 1, \\ + \operatorname{reg}(\lambda - \lambda_j), \\ \hat{\Psi}_j^{[-N_k+N_j]}(\lambda) \log(\lambda - \lambda_j) c_{jk} + \operatorname{reg}(\lambda - \lambda_j), & N_k \geq N_j. \end{cases}$$

For $\lambda'_j = N_j \in \mathbf{Z}_-$:

$$(60) \quad \hat{\Psi}_k(\lambda) = \vec{\Psi}_j^{[-N_k-1]} \ln(\lambda - \lambda_j) c_{jk} + \operatorname{reg}(\lambda - \lambda_j).$$

Note that (60) always makes sense, because $c_{jk} = 0$ for any $k = 1, \dots, n$ when $\vec{\Psi}_j^{(\text{sing})} \equiv 0$.

Proof of Corollary 8. We use the formulae of Proposition 11. We observe that

$$(61) \quad \int_{\lambda_0}^{\lambda} d\xi_{N_k+1} \int_{\lambda_0}^{\xi_{N_k}} \dots \int d\xi_1 \hat{\Psi}_k^{[N_k+1]}(\xi_1) \\ = \hat{\Psi}_k(\lambda) - \sum_{l=0}^{N_k} \frac{(-1)^{l+N_k}}{l!} \vec{f}_l^{(k)}(\lambda - \lambda_k)^l - \mathcal{Q}_{N_k}(\lambda - \lambda_0) = \hat{\Psi}_k(\lambda) + \operatorname{reg}(\lambda),$$

where \mathcal{Q}_{N_k} is a polynomial in $(\lambda - \lambda_0)$ of degree N_k and $\operatorname{reg}(\lambda)$ is analytic of $\lambda \in \mathbf{C}$. Now recall that $\hat{\Psi}_k^{[N_k+1]} = \vec{\Psi}_k$. Using (7), we have

$$\hat{\Psi}_k^{[N_k+1]}(\xi_1) = \begin{cases} \vec{\Psi}_j(\xi_1) c_{jk} + \operatorname{reg}(\xi_1 - \lambda_j), & \lambda'_j \notin \mathbf{Z}, \\ \left(\vec{\Psi}_j(\xi_1) \log(\xi_1 - \lambda_j) + \frac{P^{(j)}(\xi_1)}{(\xi_1 - \lambda_j)^{N_j+1}} \right) c_{jk} & \lambda'_j = N_j \in \mathbf{N}, \\ + \operatorname{reg}(\xi_1 - \lambda_j), \\ \vec{\Psi}_j(\xi_1) \log(\xi_1 - \lambda_j) c_{jk} + \operatorname{reg}(\xi_1 - \lambda_j), & \lambda'_j \in \mathbf{Z}_-. \end{cases}$$

When $\lambda'_j \notin \mathbf{Z}$, from (7) and (61), we have

$$\hat{\Psi}_k(\lambda) = \operatorname{reg}(\lambda) + \int_{\lambda_0}^{\lambda} d\xi_{N_k+1} \int_{\lambda_0}^{\xi_{N_k}} \dots \int d\xi_1 [\vec{\Psi}_j(\xi_1) c_{jk} + \operatorname{reg}(\xi_1 - \lambda_j)] \\ = \operatorname{reg}(\lambda) + \hat{\Psi}_j^{[-N_k-1]}(\lambda) c_{jk} + \operatorname{reg}(\lambda - \lambda_j) \\ = \operatorname{reg}(\lambda) + (\vec{\Psi}_j^{[-N_k-1]}(\lambda) - \mathcal{Q}_{N_k}(\lambda - \lambda_0)) c_{jk} + \operatorname{reg}(\lambda - \lambda_j) \\ = \vec{\Psi}_j^{[-N_k-1]}(\lambda) c_{jk} + \operatorname{reg}(\lambda - \lambda_j).$$

When $\lambda'_j = N_j \in \mathbf{N}$,

$$\begin{aligned} \hat{\Psi}_k(\lambda) &= \text{reg}(\lambda) + \int_{\lambda_0}^{\lambda} d\xi_{N_k+1} \int_{\lambda_0}^{\xi_{N_k}} \dots \int d\xi_1 \\ &\quad \times \left[\left(\vec{\Psi}_j(\xi_1) \ln(\xi_1 - \lambda_j) + \frac{P_{N_j}^{(j)}(\xi_1)}{(\xi_1 - \lambda_j)^{N_j+1}} \right) c_{jk} + \text{reg}(\xi_1 - \lambda_j) \right] \\ &= \begin{cases} \left(\hat{\Psi}_j^{[N_j-N_k]}(\lambda) \log(\lambda - \lambda_j) + \frac{P_{N_j-N_k-1}(\lambda)}{(\lambda - \lambda_j)^{N_j-N_k}} \right) c_{jk} & N_j \geq N_k + 1, \\ + \text{reg}(\lambda - \lambda_j), \\ \hat{\Psi}_j^{[-N_k+N_j]}(\lambda) \log(\lambda - \lambda_j) c_{jk} + \text{reg}(\lambda - \lambda_j), & N_k \geq N_j, \end{cases} \end{aligned}$$

where the last step follows from Proposition 11.

When $\lambda'_j = N_j \in \mathbf{Z}_-$, again from Proposition 11 we have:

$$\begin{aligned} \hat{\Psi}_k(\lambda) &= \text{reg}(\lambda) + \int_{\lambda_0}^{\lambda} d\xi_{N_k+1} \int_{\lambda_0}^{\xi_{N_k}} \dots \int d\xi_1 (\vec{\Psi}_j(\xi_1) \ln(\xi_1 - \lambda_j) c_{jk} + \text{reg}(\xi_1 - \lambda_j)) \\ &= \vec{\Psi}_j^{[-N_k-1]} \ln(\lambda - \lambda_j) c_{jk} + \text{reg}(\lambda - \lambda_j). \quad \square \end{aligned}$$

Next, we introduce the “ γ deformed” series corresponding to ${}_{\gamma}\vec{\Psi}_k$. For $\gamma_0 > 0$ sufficiently small and $0 < \gamma < \gamma_0$, $A_1 - \gamma$ has non integer diagonal entries and no integer eigenvalues, therefore:

$$({}_{\gamma}\vec{\Psi}_k)^{[-q]}(\lambda) = (-1)^q \sum_{l=q}^{\infty} \Gamma(\lambda'_k - \gamma + 1 - l) \vec{f}_{l-q}^{(k)}(\lambda - \lambda_k)^{l-\lambda'_k+\gamma-1}, \quad \forall q \geq 0,$$

and in particular $({}_{\gamma}\vec{\Psi}_k)^{[0]}(\lambda) = {}_{\gamma}\vec{\Psi}_k(\lambda)$. Recall that the coefficients $\vec{f}_{l-q}^{(k)}$ are the same for any $\gamma \in \mathbf{C}$.

Lemma 11. *Let $0 < \gamma < \gamma_0$ be such that $(A_1 - \gamma)$ has no integer diagonal entries and no integer eigenvalues. Let $q_1, q_2 \in \mathbf{N}$. Then*

$$\begin{aligned} &\int_{\lambda_k}^{\lambda} ds (\lambda - s)^{q_1+\gamma-1} ({}_{\gamma}\vec{\Psi}_k)^{[-q_2]}(s) \\ &= \frac{\Gamma(q_1 + \gamma) \sin \pi(\lambda'_k - \gamma)}{\sin \pi \lambda'_k} ({}_{\gamma}\vec{\Psi}_k)^{[-q_1-q_2]}(\lambda), \quad \lambda'_k \notin \mathbf{Z}, \\ &\int_{\lambda_k}^{\lambda} ds (\lambda - s)^{q_1+\gamma-1} ({}_{\gamma}\vec{\Psi}_k)^{[-q_2]}(s) \\ &= \frac{\Gamma(q_1 + \gamma) \sin \pi \gamma}{\pi} ({}_{\gamma}\vec{\Psi}_k)^{[-q_1-q_2]}(\lambda), \quad \lambda'_k = N_k \in \mathbf{Z}_-. \end{aligned}$$

The branch of $(\lambda - s)^\gamma$ in the integrals, for $\lambda \in \mathcal{P}_\eta$, is given by $\eta - 2\pi < \arg(\lambda - s)|_{s=\lambda_k} < \eta$, and the continuous change along the path of integration. The integrals are well defined for $0 < \gamma < \gamma_0$, $q_1 \geq 0$ and q_2 sufficiently big.

Proof. If $\lambda'_k \notin \mathbf{Z}$ the statement is proved in [4], Lemma 2'. Analogous computations bring the result for $\lambda'_k = N_k \in \mathbf{Z}_-$. It is enough to integrate expressions (48) and (49) term by term (where $|\lambda - \lambda_k|$ is small enough to make the series converge). In each term, the following integral appears

$$\int_{\lambda_k}^{\lambda} (\lambda - s)^{q_1 + \gamma - 1} (s - \lambda_k)^{l - \lambda'_k - 1} ds = (*).$$

Since one can integrate along a line from λ_k to λ , we parametrize the line with parameter $x \in [0, 1]$ as follows: $s = \lambda_k + x(\lambda - \lambda_k)$. This yealds the integral representation of the Beta function $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$. Indeed

$$\begin{aligned} (*) &= (\lambda - \lambda_k)^{q_1 + \gamma + l - \lambda'_k - 1} \int_0^1 (1 - x)^{q_1 + \gamma - 1} x^{l - \lambda'_k - 1} dx \\ &= (\lambda - \lambda_k)^{q_1 + \gamma + l - \lambda'_k - 1} \frac{\Gamma(q_1 + \gamma)\Gamma(l - \lambda'_k)}{\Gamma(q_1 + \gamma + l - \lambda'_k)}. \end{aligned}$$

The formula holds for any value of λ'_k . Note that $l \geq q_2$, thus if q_1 and q_2 are big enough, the integrals converge. Note also that for $\lambda'_k = N_k \leq -1$, the integrals converge for $q_2 \geq 0$. Moreover, since we have assumed $\gamma > 0$, the integrals converge for any $q_1 \geq 0$. Finally, some manipulations using $\Gamma(x)\Gamma(1 - x) = \pi/\sin(\pi x)$, yield the result. For example, in case $\lambda'_k = N_k \leq -1$, we have

$$\begin{aligned} &\int_{\lambda_k}^{\lambda} ds (\lambda - s)^{q_1 + \gamma - 1} (\vec{\Psi}_k)^{[-q_2]}(s) \\ &= (-1)^{q_2} \sum_{l \geq q_2} \frac{(-1)^{l - N_k}}{(l - N_k - 1)!} \vec{f}_{l - q_2}^{(k)} (\lambda - \lambda_k)^{q_1 + \gamma + l - N_k - 1} \frac{\Gamma(q_1 + \gamma)\Gamma(l - N_k)}{\Gamma(q_1 + \gamma + l - N_k)} = (**). \end{aligned}$$

We use $\Gamma(l - N_k) = (l - N_k - 1)!$,

$$\frac{1}{\Gamma(q_1 + \gamma + l - N_k)} = \frac{\Gamma(N_k + 1 - \gamma - l - q_1) \sin(q_1 + l - N_k + \gamma)}{\pi},$$

and change $l \mapsto l - q_1$. We get

$$\begin{aligned} (***) &= (-1)^{q_1 + q_2} \frac{\Gamma(q_1 + \gamma) \sin \pi \gamma}{\pi} \sum_{l \geq q_1 + q_2} \Gamma(N_k - \gamma_1 - l) \vec{f}_{l - q_1 - q_2}^{(k)} (\lambda - \lambda_k)^{l - (N_k - \gamma) - 1} \\ &= \frac{\Gamma(q_1 + \gamma) \sin \pi \gamma}{\pi} (\gamma \vec{\Psi}_k)^{[-q_1 - q_2]}(\lambda). \quad \square \end{aligned}$$

Lemma 12. *Let $0 < \gamma < \gamma_0$ be such that $(A_1 - \gamma)$ has no integer diagonal entries and no integer eigenvalues. Then*

$$\int_{\lambda_k}^{\lambda} ds (\lambda - s)^{\gamma-1} (\hat{\Psi}_k)^{[-q]}(s) = \frac{\Gamma(\gamma) \sin \pi\gamma}{\pi} ({}_{\gamma}\vec{\Psi}_k)^{[-N_k-1-q]}(\lambda), \quad \lambda'_k = N_k \in \mathbf{N}.$$

The integral is well defined for $0 < \gamma < \gamma_0$ and $q \geq 0$ integer. The branch of $(\lambda - s)^\gamma$ is defined in the same way as in Lemma 11.

Proof. Integration term by term yields

$$\begin{aligned} (\hat{\Psi}_k)^{[-q]}(s) &= (-1)^{N_k+1+q} \sum_{l \geq q} \frac{(-1)^{l+1}}{l!} \vec{f}_{l-q}^{(k)}(\lambda - \lambda_k)^l, \\ ({}_{\gamma}\vec{\Psi}_k)^{[-N_k-1-q]}(\lambda) &= (-1)^{N_k+1+q} \sum_{l \geq N_k+1+q} \Gamma(N_k - \gamma + 1 - l) \vec{f}_{l-N_k-1-q}^{(k)}(\lambda - \lambda_k)^{l-N_k+\gamma-1}. \end{aligned}$$

As in the previous lemma, we compute

$$\int_{\lambda_k}^{\lambda} ds (\lambda - s)^{\gamma-1} (s - \lambda_k)^l = (\lambda - \lambda_k)^{\gamma+l} \frac{\Gamma(\gamma)\Gamma(l+1)}{\Gamma(\gamma+l+1)},$$

and use $1/\Gamma(\gamma+l+1) = (-1)^{l+1} \sin(\pi\gamma)\Gamma(-\gamma-l)/\pi$. This implies that

$$\int_{\lambda_k}^{\lambda} ds (\lambda - s)^{\gamma-1} (\hat{\Psi}_k)^{[-q]}(s) = (-1)^{N_k+1+q} \frac{\Gamma(\gamma) \sin \pi\gamma}{\pi} \sum_{l \geq q} \Gamma(-l-\gamma) \vec{f}_{l-q}^{(k)}(\lambda - \lambda_k)^{l+\gamma}.$$

After redefining $l' = l + N_k + 1$ we obtain the final result. \square

We establish the monodromy of $\vec{\Psi}_k^{[-q]}$ and $\hat{\Psi}_k^{[-q]}$ in the following

Proposition 12. *Let $\lambda \in \mathcal{P}_\eta$. Let $q \geq 0$ be an integer. Let $\alpha_j = 2\pi i$ when $\lambda'_j \in \mathbf{Z}$, and $\alpha_j = e^{-2\pi i \lambda'_j} - 1$ when $\lambda'_j \notin \mathbf{Z}$. The following transformations hold for a loop γ_j around a pole λ_j .*

a) *If $\lambda'_k \notin \mathbf{Z}$ or $\lambda'_k \in \mathbf{Z}_-$:*

$$\vec{\Psi}_k^{[-q]}(\lambda) \mapsto \vec{\Psi}_k^{[-q]}(\lambda) + \begin{cases} \alpha_j c_{jk} \vec{\Psi}_j^{[-q]}(\lambda), & \lambda'_j \notin \mathbf{Z} \text{ or } \lambda'_j \in \mathbf{Z}_-, \\ \alpha_j c_{jk} \vec{\Psi}_j^{[-q+N_j+1]}(\lambda), & \lambda'_j = N_j \in \mathbf{N}. \end{cases}$$

b) *If $\lambda'_k \in \mathbf{N}$:*

$$\hat{\Psi}_k^{[-q]}(\lambda) \mapsto \hat{\Psi}_k^{[-q]}(\lambda) + \begin{cases} \alpha_j c_{jk} \vec{\Psi}_j^{[-q-1-N_k]}(\lambda), & \lambda'_j \notin \mathbf{Z} \text{ or } \lambda'_j \in \mathbf{Z}_-, \\ \alpha_j c_{jk} \vec{\Psi}_j^{[-q+N_j-N_k]}(\lambda), & \lambda'_j = N_j \in \mathbf{N}. \end{cases}$$

Proof. We consider the function $\Phi_k^{[-q]}(\lambda, \lambda_0)$ defined in (53) for $\lambda, \lambda_0 \in \mathcal{P}_\eta$. For simplicity of notation, we omit λ_0 , namely we write

$$\Phi_k^{[-q]}(\lambda) = \int_{\lambda_0}^{\lambda} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 \phi_k(\xi_1),$$

$$\Phi_k^{[-q]}(\lambda) = \phi_k(\lambda), \quad \text{if } q = 0.$$

In the cut-plane \mathcal{P}_η , we consider λ close to $\lambda_j \neq \lambda_k$ in such a way that the series representations of $(\phi_j)^{[-q]}(\lambda)$ converge. We also consider a loop γ_j around λ_j in counter-clockwise direction, represented by $(\lambda - \lambda_j) \mapsto (\lambda - \lambda_j)e^{2\pi i}$. The new path of integration from λ_0 to λ is represented in figure 5. We have the following transformation after the loop

$$\Phi_k^{[-q]}(\lambda) \mapsto \Phi_k^{[-q]}(\lambda) + \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 \phi_k(\xi_1).$$

By formula (54) it follows that the analytic continuation of $(\phi_k)^{[-q]}(\lambda)$ along the loop γ_j is

$$(\phi_k)^{[-q]}(\lambda) \mapsto (\phi_k)^{[-q]}(\lambda) + \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 \phi_k(\xi_1), \quad \forall q \geq 0.$$

Next, we express $\phi_k(\xi_1)$ in terms of the solutions ϕ_j at λ_j . We distinguish the two cases in the proposition.

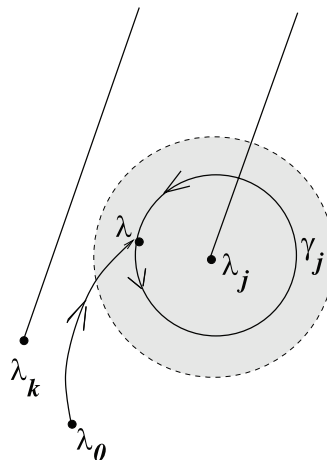


Fig. 5. The paths of integration after the analytic continuation, starting from λ_0 , going around a loop γ_j around λ_j and ending in λ .

Case a). $\lambda'_k \notin \mathbf{Z}$ or $\lambda'_k \in \mathbf{Z}_-$ We have $\phi_k(\xi_1) = \vec{\Psi}_k(\xi_1)$, therefore we use (7), namely

$$\phi_k(\xi_1) = \begin{cases} \vec{\Psi}_j(\xi_1) c_{jk} + \text{reg}(\xi_1 - \lambda_j), & \lambda'_j \notin \mathbf{Z}, \\ \vec{\Psi}_j(\xi_1) \ln(\xi_1 - \lambda_j) c_{jk} + \text{reg}(\xi_1 - \lambda_j), & \lambda'_j \in \mathbf{Z}_-, \\ \left(\vec{\Psi}_j(\xi_1) \ln(\xi_1 - \lambda_j) + \frac{P^{(j)}(\xi_1)}{(\xi_1 - \lambda_j)^{N_j+1}} \right) c_{jk} + \text{reg}(\xi_1 - \lambda_j), & \lambda'_j = N_j \in \mathbf{N}. \end{cases}$$

a.1) When $\lambda'_j \notin \mathbf{Z}$, we have

$$\begin{aligned} & \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 (\vec{\Psi}_j(\xi_1) c_{jk} + \text{reg}(\xi_1 - \lambda_j)) \\ &= \oint_{\gamma_j} d\xi_q \Phi_j^{[-(q-1)]}(\xi_q) c_{jk} + \oint_{\gamma_j} d\xi_q \text{reg}(\xi_q - \lambda_j) \equiv \oint_{\gamma_j} d\xi_q \Phi_j^{[-(q-1)]}(\xi_q) c_{jk} + 0, \end{aligned}$$

because the loop integral of regular terms at λ_j vanishes. Now, by (55) and (54), we have

$$\begin{aligned} \oint_{\gamma_j} d\xi_q \Phi_j^{[-(q-1)]}(\xi_q) c_{jk} &= c_{jk} \Phi_j^{[-q]}(\xi_q) \Big|_{\lambda-\lambda_j}^{(\lambda-\lambda_j)e^{2\pi i}} \\ &\equiv c_{jk} \vec{\Psi}_j^{[-q]}(\xi_q) \Big|_{\lambda-\lambda_j}^{(\lambda-\lambda_j)e^{2\pi i}} = \vec{\Psi}_j^{[-q]}(\lambda) (e^{-2\pi i \lambda'_j} - 1) c_{jk}, \quad q \geq 0. \end{aligned}$$

The last step follows from the series representation (48). This proves the Proposition in case a.1).

a.2) When $\lambda'_j \in \mathbf{Z}_-$, we use (56) and compute

$$\begin{aligned} & \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 (\vec{\Psi}_j(\xi_1) \ln(\xi_1 - \lambda_j) c_{jk} + \text{reg}(\xi_1 - \lambda_j)) \\ &= c_{jk} (\vec{\Psi}_j^{[-q]}(\xi_q) \ln(\xi_q - \lambda_j)) \Big|_{\lambda-\lambda_j}^{(\lambda-\lambda_j)e^{2\pi i}} = 2\pi i c_{jk} (\vec{\Psi}_j^{[-q]}(\lambda)), \quad q \geq 0. \end{aligned}$$

This implies the Proposition in case a.2).

a.3) When $\lambda'_j = N_j \in \mathbf{N}$, we use (57) and compute

$$\begin{aligned} & \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 \left[\left(\vec{\Psi}_j(\xi_1) \ln(\xi_1 - \lambda_j) + \frac{P^{(j)}(\xi_1)}{(\xi_1 - \lambda_j)^{N_j+1}} \right) c_{jk} + \text{reg}(\xi_1 - \lambda_j) \right] \\ &= c_{jk} \hat{\Psi}_j^{[-q+N_j+1]}(\xi_q) \ln(\xi_q - \lambda_j) \Big|_{\lambda-\lambda_j}^{(\lambda-\lambda_j)e^{2\pi i}} = 2\pi i c_{jk} \hat{\Psi}_j^{[-q+N_j+1]}(\lambda), \quad q \geq N_j + 1. \end{aligned}$$

This proves the Proposition in case a.3).

Case b) when $\lambda'_k = N_k \in \mathbf{N}$, we need to compute

$$(62) \quad \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 \hat{\Psi}_k(\xi_1).$$

b.1) In the case of $\lambda'_j \notin \mathbf{Z}$, we use (58), and find

$$(62) = \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 (\vec{\Psi}_j^{[-N_k-1]}(\xi_1) c_{jk} + \text{reg}(\xi_1 - \lambda_j)) \\ = (c_{jk} \vec{\Psi}_j^{[-N_k-1-q]}(\xi_q) + \text{reg}(\xi_q - \lambda_j)) \Big|_{(\lambda-\lambda_j)}^{e^{2\pi i}(\lambda-\lambda_j)} = c_{jk} (e^{-2\pi i \lambda'_j} - 1) \vec{\Psi}_j^{[-N_k-1-q]}(\lambda).$$

b.2) In the case of $\lambda'_j \in \mathbf{N}$, we use (59) and Proposition 11, and find

$$(62) = \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 \left[\left(\hat{\Psi}_j^{[N_j-N_k]}(\xi_1) \ln(\xi_1 - \lambda_j) + \frac{P_{N_j-N_k-1}(\xi_1)}{(\xi_1 - \lambda_j)^{N_j-N_k}} \right) c_{jk} \right. \\ \left. + \text{reg}(\xi_1 - \lambda_j) \right],$$

where $P_{N_j-N_k-1} = 0$ for $N_k \geq N_j$.

For $0 \leq q \leq N_j - N_k - 1$, the integral is

$$\left[c_{jk} \left(\vec{\Psi}_j^{[N_j-N_k-q]}(\xi_q) \ln(\xi_q - \lambda_j) + \frac{P_{N_j-N_k-1-q}(\xi_q)}{(\xi_q - \lambda_j)^{N_j-N_k-q}} \right) + \text{reg}(\xi_q - \lambda_j) \right] \Big|_{(\lambda-\lambda_j)}^{e^{2\pi i}(\lambda-\lambda_j)}.$$

For $q \geq N_j - N_k \geq 0$, the integral is

$$[c_{jk} \vec{\Psi}_j^{[N_j-N_k-q]}(\xi_q) \ln(\xi_q - \lambda_j) + \text{reg}(\xi_q - \lambda_j)] \Big|_{(\lambda-\lambda_j)}^{e^{2\pi i}(\lambda-\lambda_j)}.$$

In both cases, the above expressions yield

$$(62) = 2\pi i c_{jk} \vec{\Psi}_j^{[N_j-N_k-q]}(\lambda).$$

b.3) In case $\lambda'_j \in \mathbf{Z}_-$, we use (60) and Proposition 11, and find

$$(62) = \oint_{\gamma_j} d\xi_q \int_{\lambda_0}^{\xi_q} d\xi_{q-1} \dots \int_{\lambda_0}^{\xi_2} d\xi_1 (\vec{\Psi}_j^{[-N_k-1]}(\xi_1) \ln(\xi_1 - \lambda_j) c_{jk} + \text{reg}(\xi_1 - \lambda_j)) \\ = [\vec{\Psi}_j^{[-N_k-1-q]}(\xi_q) \ln(\xi_q - \lambda_j) c_{jk} + \text{reg}(\xi_q - \lambda_j)] \Big|_{(\lambda-\lambda_j)}^{e^{2\pi i}(\lambda-\lambda_j)} = 2\pi i c_{jk} \vec{\Psi}_j^{[-N_k-1-q]}(\lambda).$$

The above computations prove the Proposition in case b). \square

Proof of Proposition 10. Given a function $f(\lambda)$, $\lambda \in \mathcal{P}_\eta$, we denote with $f_+(\lambda)$ the value on the left side of L_j , where $\arg(\lambda - \lambda_j) = \eta - 2\pi$. We denote

with $f_-(\lambda)$ the value on the right side, where $\arg(\lambda - \lambda_j) = \eta$. By Lemmas 11 and 12 we have:

$$({}_\gamma \vec{\Psi}_k)_\pm^{[-q(\lambda'_k)]}(\lambda) = F(\lambda'_k) \left\{ \int_{\lambda_k}^{\lambda_j} (\lambda - s)^{\gamma-1} \phi_k^{[-q]}(s) ds + \int_{\lambda_j}^{\lambda} (\lambda - s)^{\gamma-1} (\phi_k^{[-q]})_\pm(s) ds \right\},$$

where

$$q(\lambda'_k) = \begin{cases} q, & \text{if } \lambda'_k \notin \mathbf{Z} \text{ or } \lambda'_k \in \mathbf{Z}_-, \\ q + N_k + 1, & \text{if } \lambda'_k = N_k \in \mathbf{N}, \end{cases}$$

$$\phi_k = \begin{cases} \vec{\Psi}_k, & \text{if } \lambda'_k \notin \mathbf{Z} \text{ or } \lambda'_k \in \mathbf{Z}_-, \\ \vec{\Psi}_k, & \text{if } \lambda'_k \in \mathbf{N}, \end{cases}$$

$$F(\lambda'_k) = \begin{cases} \frac{\sin \pi \lambda'_k}{\Gamma(\gamma) \sin \pi(\lambda'_k - \gamma)}, & \text{if } \lambda'_k \notin \mathbf{Z}, \\ \frac{\pi}{\Gamma(\gamma) \sin \pi \gamma}, & \text{if } \lambda'_k \in \mathbf{Z}. \end{cases}$$

In the integral, $\arg(\lambda - s)$, $s \in \mathcal{D}_\eta$, has the value obtained by the continuous change along the path of integration from λ_k up to s belonging to L_j . Change from f_- to f_+ is obtained along a small loop encircling only λ_j . Therefore, Proposition 12 yields

$$(63) \quad ({}_\gamma \vec{\Psi}_k)_-^{[-q(\lambda'_k)]}(\lambda) - ({}_\gamma \vec{\Psi}_k)_+^{[-q(\lambda'_k)]}(\lambda) = \alpha_j[\gamma] c_{jk}[\gamma] ({}_\gamma \vec{\Psi}_j)_+^{[-q(\lambda'_k)]}(\lambda).$$

By Lemmas 11 and 12 we write

$$(64) \quad ({}_\gamma \vec{\Psi}_k)_-^{[-q(\lambda'_k)]}(\lambda) - ({}_\gamma \vec{\Psi}_k)_+^{[-q(\lambda'_k)]}(\lambda) = F(\lambda'_k) \int_{\lambda_j}^{\lambda} (\lambda - s)^{\gamma-1} [(\phi_k^{[-q]})_-(s) - (\phi_k^{[-q]})_+(s)] ds.$$

We need to distinguish two cases.

1) $\lambda'_k \notin \mathbf{Z}$, or $\lambda'_k \in \mathbf{Z}_-$ In this case (63), (64) and Proposition 12 applied to the integrand yield the following equalities.

1.a) for $\lambda'_j \notin \mathbf{Z}$ or $\lambda'_j \in \mathbf{Z}_-$:

$$(65) \quad \alpha_j[\gamma] c_{jk}[\gamma] ({}_\gamma \vec{\Psi}_j)_+^{[-q]}(\lambda) = F(\lambda'_k) \int_{\lambda_j}^{\lambda} ds (\lambda - s)^{\gamma-1} \alpha_j c_{jk} ({}_\gamma \vec{\Psi}_j)_+^{[-q]}(s).$$

1.b) for $\lambda'_j \in \mathbf{N}$:

$$(66) \quad \alpha_j[\gamma] c_{jk}[\gamma] ({}_\gamma \vec{\Psi}_j)_+^{[-q]}(\lambda) = F(\lambda'_k) \int_{\lambda_j}^{\lambda} ds (\lambda - s)^{\gamma-1} \alpha_j c_{jk} ({}_\gamma \vec{\Psi}_j)_+^{[-q+N_j+1]}(s).$$

We apply again Lemmas 11 and 12 to express the r.h.s. of the above equalities. To this end, we need $(\lambda - s)_+$ in the integrand. Observe that

$$(\lambda - s)^{\gamma-1} \text{ in the integrand} = \begin{cases} (\lambda - s)_-^{\gamma-1} = [e^{2\pi i}(\lambda - s)_+]^{\gamma-1}, & \text{when } k \succ j, \\ (\lambda - s)_+^{\gamma-1}, & \text{when } k < j. \end{cases}$$

Indeed, when λ belongs to the left side of L_j and $s \in \mathcal{P}_\eta$, then $\eta - 2\pi < \arg(\lambda - s) < \eta$. When s reaches L_j from the left, then $\arg(\lambda - s) \rightarrow \eta$ if L_k is to the left of L_j , namely $k \succ j$: in this case we obtain $(\lambda - s)_-$. On the other hand, $\arg(\lambda - s) \rightarrow \eta - 2\pi$ if L_k is to the right of L_j , namely $k < j$: in this case we obtain $(\lambda - s)_+$. See figure 6. Applying Lemmas 11 and 12 we find

$$\text{r.h.s. of (65) and (66)} = \begin{cases} \frac{F(\lambda'_k)}{F(\lambda'_j)} e^{2\pi i \gamma} \alpha_j c_{jk}(\gamma, \vec{\Psi}_j)^{[-q]}(\lambda), & k \succ j, \\ \frac{F(\lambda'_k)}{F(\lambda'_j)} \alpha_j c_{jk}(\gamma, \vec{\Psi}_j)^{[-q]}(\lambda), & k < j. \end{cases}$$

Namely:

$$(67) \quad \alpha_j[\gamma] c_{jk}[\gamma] = \begin{cases} \frac{F(\lambda'_k)}{F(\lambda'_j)} e^{2\pi i \gamma} \alpha_j c_{jk}, & k \succ j, \\ \frac{F(\lambda'_k)}{F(\lambda'_j)} \alpha_j c_{jk}, & k < j. \end{cases}$$

Finally, we compute the ratio $F(\lambda'_k)/F(\lambda'_j)$. For $\lambda'_j \notin \mathbf{Z}$:

$$\frac{F(\lambda'_k)}{F(\lambda'_j)} = \frac{\sin \pi \lambda'_k \sin \pi(\lambda'_j - \gamma)}{\sin \pi \lambda'_j \sin \pi(\lambda'_k - \gamma)} = \frac{(1 - e^{-2\pi i \lambda'_k})(1 - e^{-2\pi i(\lambda'_j - \gamma)})}{(1 - e^{-2\pi i \lambda'_j})(1 - e^{-2\pi i(\lambda'_k - \gamma)})} = \frac{\alpha_k \alpha_j[\gamma]}{\alpha_j \alpha_k[\gamma]}.$$

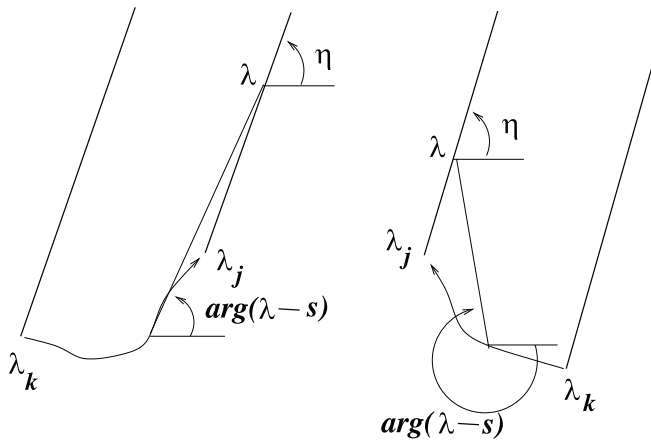


Fig. 6. The figure shows $\arg(\lambda - s)$ as $s \rightarrow \lambda_j$.

For $\lambda'_j \in \mathbf{Z}$:

$$\frac{F(\lambda'_k)}{F(\lambda'_j)} = \frac{\sin \pi\gamma \sin \pi\lambda'_k}{\pi \sin \pi(\lambda'_k - \gamma)} = \frac{e^{2\pi i\gamma} - 1}{2\pi i} \frac{e^{-2\pi i\lambda'_k} - 1}{e^{-2\pi i(\lambda'_k - \gamma)} - 1} = \frac{\alpha_j[\gamma]}{\alpha_j} \frac{\alpha_k}{\alpha_k[\gamma]},$$

where we have used the fact that $\alpha_j = 2\pi i$ and $\alpha_j[\gamma] = e^{2\pi i\gamma} - 1$.

The above computations imply the statement of Proposition 10 when $\lambda'_k \notin \mathbf{Z}$ and $\lambda'_k \in \mathbf{Z}_-$.

2) $\lambda'_k \in N$ In this case (63), (64) and Proposition 12 applied to the integrand yield the following equalities.

2.a) For $\lambda'_j \notin \mathbf{Z}$ or $\lambda'_j \in \mathbf{Z}_-$:

$$(68) \quad \alpha_j[\gamma] c_{jk}[\gamma](\gamma \vec{\Psi}_j)_+^{[-q-N_k-1]}(\lambda) = F(\lambda'_k) \int_{\lambda_j}^{\lambda} ds (\lambda - s)^{\gamma-1} \alpha_j c_{jk}(\vec{\Psi}_j)_+^{[-q-N_k-1]}(s).$$

2.b) For $\lambda'_j \in N$:

$$(69) \quad \alpha_j[\gamma] c_{jk}[\gamma](\gamma \vec{\Psi}_j)_+^{[-q-N_k-1]}(\lambda) = F(\lambda'_k) \int_{\lambda_j}^{\lambda} ds (\lambda - s)^{\gamma-1} \alpha_j c_{jk}(\hat{\Psi}_j)_+^{[-q-N_k-1+N_j+1]}(s).$$

We apply again Lemmas 11 and 12 to express the r.h.s. of the above equalities, keeping into account the branch of $(\lambda - s)^{\gamma-1}$ as before. We find

$$\text{r.h.s. of (68) and (69)} = \begin{cases} \frac{F(\lambda'_k)}{F(\lambda'_j)} e^{2\pi i\gamma} \alpha_j c_{jk}(\gamma \vec{\Psi}_j)_+^{[-q-N_k-1]}(\lambda), & k \succ j, \\ \frac{F(\lambda'_k)}{F(\lambda'_j)} \alpha_j c_{jk}(\gamma \vec{\Psi}_j)_+^{[-q-N_k-1]}(\lambda), & k < j. \end{cases}$$

Namely, we obtain again

$$(70) \quad \alpha_j[\gamma] c_{jk}[\gamma] = \begin{cases} \frac{F(\lambda'_k)}{F(\lambda'_j)} e^{2\pi i\gamma} \alpha_j c_{jk}, & k \succ j, \\ \frac{F(\lambda'_k)}{F(\lambda'_j)} \alpha_j c_{jk}, & k < j. \end{cases}$$

Finally, we compute the ratio $F(\lambda'_k)/F(\lambda'_j)$. For $\lambda'_j \notin \mathbf{Z}$:

$$\frac{F(\lambda'_k)}{F(\lambda'_j)} = \frac{\pi}{\sin \pi\gamma} \frac{\sin \pi(\lambda'_j - 1)}{\sin \pi\lambda'_j} = \frac{2\pi i}{e^{2\pi i\gamma} - 1} \frac{e^{-2\pi i(\lambda'_j - \gamma)} - 1}{e^{-2\pi i\lambda'_j} - 1} = \frac{\alpha_k}{\alpha_k[\gamma]} \frac{\alpha_j[\gamma]}{\alpha_j},$$

where we have used the fact that $\alpha_k = 2\pi i$ and $\alpha_k[\gamma] = e^{2\pi i\gamma} - 1$.

For $\lambda'_j \in \mathbf{Z}$:

$$\frac{F(\lambda'_k)}{F(\lambda'_j)} = 1.$$

In this last case, observe that $\alpha_k = \alpha_j = 2\pi i$ and $\alpha_k[\gamma] = \alpha_j[\gamma] = e^{2\pi i\gamma} - 1$.

The above computations imply the statement of Proposition 10 when $\lambda'_k \in N$. \square

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