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Abstract. We prove the L^p-differentiability at almost every point for convolution products on $\mathbb{R}^d$ of the form $K * \mu$, where μ is bounded measure and the kernel K is homogeneous of degree $1-d$. From this result we derive the L^p-differentiability for vector fields on $\mathbb{R}^d$ whose curl and divergence are measures, and also for vector fields with bounded deformation.

Keywords: approximate differentiability, Lusin property, convolution products, singular integrals, Calderón-Zygmund decomposition, Sobolev functions, functions with bounded variation, functions with bounded deformation.

MSC (2010): 26B05, 42B20, 46E35.

1. Introduction

Let u be a convolution product on $\mathbb{R}^d$ of the form

$$u := K * \mu \tag{1.1}$$

where μ is a bounded measure and the kernel K is of class C^2 away from 0 and homogeneous of degree $1-d$. The main result of this paper (Theorem 3.4) states that u is differentiable in the L^p sense[1] at almost every point for every p with $1 \le p < \gamma(1)$, where $\gamma(q) := qd/(d-q)$ is the exponent of the Sobolev embedding for $W^{1,q}$ in dimension d.

Using this result, we show that a vector field v on $\mathbb{R}^d$ is L^p-differentiable almost everywhere for the same range of p if either of the following conditions holds (see Propositions 4.2 and 4.3):

(a) the (distributional) curl and divergence of v are measures;

(b) v belongs to the class BD of maps with bounded deformation, that is, the (distributional) *symmetric* derivative $\frac{1}{2}(\nabla v + \nabla^t v)$ is a measure.

A few comments are in place here.

Relation with Sobolev and BV functions. If the measures in the statements above were replaced by functions in L^q for some $q > 1$, then u and v would be (locally) in the Sobolev class $W^{1,q}$ (see Lemma 3.9), and it is well-known that a function in this class is $L^{\gamma(q)}$-differentiable almost everywhere when $q < d$,

[1] The definition of L^p-differentiability is recalled in §2.2.

and differentiable almost everywhere in the classical sense when $q > d$, see for instance [5, Sections 6.1.2 and 6.2].[2]

Functions in the BV class—namely those functions whose distributional derivative is a measure—share the same differentiability property of function in the class $W^{1,1}$ (see [5, Section 6.1.1]). Note, however, that this result does not apply to the functions u and v considered above, because in general they just fail to be of class BV, even locally.[3]

A Lusin-type theorem. Consider a Lipschitz function w on $\mathbb{R}^d$ whose (distributional) Laplacian is a measure. Then ∇w satisfies assumption (a) above, and therefore is L^1-differentiable almost everywhere. Using this fact we can show that w admits an L^1-Taylor expansion of order two at almost every point and has therefore the Lusin property with functions of class C^2 (see §2.4 and Proposition 4.4). This last statement is used in [1] to prove that w has the so-called weak Sard property, and was the original motivation for this paper.

Comparison with existing results. The proof of Theorem 3.4 relies on arguments and tools from the theory of singular integrals that are by now quite standard. This notwithstanding, we could not find it in the literature.

There are, however, a few results which are closely related: the approximate differentiability at almost every point of the convolution product in (1.1) was already proved in [7, Theorem 6], expanding a sketch of proof given in [3, Remark at page 129] for the spacial case $K(x) := |x|^{1-d}$ and μ replaced by a function in L^1. It should be noted that the notion of approximate differentiability (see Remark 2.3(v)) is substantially weaker than the notion of L^1-differentiability; in particular, in Remark 4.7 we show that the result in [7] cannot be used to prove the Lusin property mentioned in the previous paragraph.

Finally, the L^1-differentiability of BD functions was first proved in [2, Theorem 7.4]. As far as we can see, that proof is specific of the BD case, and cannot be adapted to the more general setting considered here.

The rest of this paper is organized as follows: in Section 2 we introduce the notation and recall a few basic fact on differentiability in the L^p sense, in Section 3 we state and prove the main result (Theorem 3.4), while in Section 4 we give a few application of this result.

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[2] In the case $q > d$ we refer to the *continuous representative* of the function.

[3] An example of $u := K * \mu$ which is not BV_{loc} is obtained by taking $K(x) := |x|^{1-d}$ and μ equal to the Dirac mass at 0. An example of v with vanishing curl and measure divergence which is not BV_{loc} is the derivative of the fundamental solutions of the Laplacian, see §4.1. Examples of vector fields v which are in the BD class but not in BV are more complicated, and are usually derived by the failure of Korn inequality for the $p = 1$ exponent, see for instance [2, Example 7.7], and [9, ***].

2. Notation and preliminary result

2.1. Notation. For the rest of this paper $d \geq 2$ is a fixed integer. Sets and functions are tacitly assumed to be Borel measurable, and measures are always defined on the Borel σ-algebra.

We use the following notation:

$\operatorname{diam}(E)$	diameter of the set E;
1_E	characteristic function of the set E (valued in $\{0,1\}$);
$\operatorname{dist}(E_1, E_2)$	distance between the sets E_1 and E_2, that is, the infimum of $\lvert x_1 - x_2\rvert$ among all $x_1 \in E_1$, $x_2 \in E_2$;
$B(x,\rho)$	open ball in $\mathbb{R}^d$ with radius ρ and center $x \in \mathbb{R}^d$;
$B(\rho)$	open ball in $\mathbb{R}^d$ with radius ρ and center 0;
S^{d-1}	$:= \{x \in \mathbb{R}^d : \lvert x\rvert = 1\}$, unit sphere in $\mathbb{R}^d$;
$\fint_E f\,d\mu$	$:= \frac{1}{\mu(E)} \int_E f\,d\mu$, average of the function f over the set E with respect to the positive measure μ;
$\rho \cdot \mu$	measure defined by the measure μ and the density function ρ, that is, $[\rho\cdot\mu](E) := \int_E \rho\,d\mu$ for every Borel set E;
$1_E \cdot \mu$	the restriction of the measure μ to the set E;
$\lvert\mu\rvert$	positive measure associated to a real- or vector-valued measure μ (total variation);
$\lVert\mu\rVert$	$:= \lvert\mu\rvert(\mathbb{R}^d)$, total mass of the measure μ;
$\mathscr{L}^d$	Lebesgue measure on $\mathbb{R}^d$;
$\mathscr{H}^k$	k-dimensional Hausdorff measure;
ω_d	$:= \mathscr{L}^d(B(1))$, Lebesgue measure of the unit ball in $\mathbb{R}^d$;
$\gamma(q)$	$:= qd/(d-q)$ for $1 \leq q < d$ and $\gamma(q) := +\infty$ for $q \geq d$; exponent of the Sobolev embedding for $W^{1,q}$ in dimension d.

When the measure is not specified, it is assumed to be the Lebesgue measure, and we often write $\int f(x)\,dx$ for the integral of f with respect to $\mathscr{L}^d$.

As usual, we denote by $o(\rho^k)$ any real- or vector-valued g function on $(0,+\infty)$ such that $\rho^{-k}g(\rho)$ tends to 0 as $\rho \to 0$, while $O(\rho^k)$ denotes any g such that $\rho^{-k}g(\rho)$ is bounded in a neighbourhood of 0.

2.2. Taylor expansions in the L^p sense. Let be u a real function on $\mathbb{R}^d$. Given a point $x \in \mathbb{R}^d$, a real number $p \in [1,\infty)$, and an integer $k \geq 0$, we say that u has a Taylor expansion of order k in the L^p sense at x, and we write $u \in t^{k,p}(x)$, if u can be decomposed as

$$u(x+h) = P_x^k(h) + R_x^k(h) \quad \text{for every } h \in \mathbb{R}^d, \tag{2.1}$$

where P_x^k is a polynomial on $\mathbb{R}^d$ with degree at most k and the remainder R_x^k satisfies

$$\left[\fint_{B(\rho)} \lvert R_x^k(h)\rvert^p\,dh\right]^{1/p} = o(\rho^k)\,. \tag{2.2}$$

As usual, the polynomial P_x^k is uniquely determined by (2.1) and the decay estimate (2.2).

When u belongs to $t^{0,p}(x)$ we say that it has L^p-limit at x equal to $P_x^0(0)$. When u belongs to $t^{1,p}(x)$ we say that u is L^p-differentiable at x with derivative equal to the $\nabla P_x^1(0)$.

Moreover we write $u \in T^{k+1,p}(x)$ if the term $o(\rho^k)$ in (2.2) can be replaced by $O(\rho^{k+1})$. Accordingly, we write $u \in T^{0,p}(x)$ if

$$\left[\fint_{B(\rho)} |u(x+h)|^p \, dh \right]^{1/p} = O(\rho) \, .$$

The definitions above are given for real-valued functions defined on $\mathbb{R}^n$, but are extended with the necessary modifications to vector-valued functions defined on some open neighbourhood of the point x.

Finally, it is convenient to define $t^{k,\infty}(x)$ and $T^{k,\infty}(x)$ by replacing the left-hand side of (2.2) with the L^∞ norm of $R_x^k(h)$ on $B(\rho)$. Note that u belongs to $t^{k,\infty}(x)$ if and only if it agrees almost everywhere with a function which admits a Taylor expansion of order k at x in the classical sense.

2.3. Remark. (i) The space $t^{k,p}(x)$ and $T^{k,p}(x)$ were introduced in a slightly different form in [4] (see also [10, Section 3.5]). The original definition differs from ours in that it also requires that the left-hand side of (2.2) is smaller that $c\rho^k$ for some finite constant c and for *every* $\rho > 0$ (and not just for small ρ).[4]

(ii) The function spaces $t^{k,p}(x)$ and $T^{k,p}(x)$ satisfy the obvious inclusions $T^{k,p}(x) \subset T^{k,q}(x)$ and $t^{k,p}(x) \subset t^{k,q}(x)$ whenever $p \geq q$, and $T^{k+1,p}(x) \subset t^{k,p}(x) \subset T^{k,p}(x)$.

(iii) Concerning the inclusion $t^{k,p}(x) \subset T^{k,p}(x)$, the following non-trivial converse holds: if u belongs to $T^{k,p}(x)$ for every x in the set E, then u belongs to $t^{k,p}(x)$ for *almost every* $x \in E$ [10, Theorem 3.8.1].[5]

(iv) We recall that function u on $\mathbb{R}^d$ has approximate limit $a \in \mathbb{R}$ at x if the set $\{h : |u(x+h) - a| \leq \varepsilon\}$ has density 1 at 0 for every $\varepsilon > 0$. It is immediate to check that if u has L^p-limit equal to a at x for some $p \geq 1$, then it has also approximate limit a at x.

(v) A function u on $\mathbb{R}^d$ has approximate derivative $b \in \mathbb{R}^d$ at x (and approximate limit $a \in \mathbb{R}$) if the ratio $(u(x+h) - a - b \cdot h)/|h|$ has approximate limit 0 as $h \to 0$. It is easy to check that if u has L^p-derivative b at x then it also has approximate derivative b at x.

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[4] This additional requirement is met if (and only if) the function u satisfies the growth condition $\int_{B(\rho)} |u|^p \leq c\rho^{d+kp}$ for some finite c and for sufficiently large ρ.

[5] For $k = 0$ this statement can be viewed as an L^p-version of the classical Rademacher theorem on the differentiability of Lipschitz functions. The fact that our definition of $t^{k,p}(x)$ and $T^{k,p}(x)$ differs from that considered in [10] has no consequences for the validity of this statement.

2.4. Lusin property. Let E be a set in $\mathbb{R}^d$ and u a function defined at every point of E. We say that u has the Lusin property with functions of class C^k (on E) if for every $\varepsilon > 0$ there exists a function v of class C^k on $\mathbb{R}^d$ which agrees with u in every point of E except a subset with measure at most ε.

Using the L^p-version of the Whitney extension theorem [10, Theorem 3.6.3] one easily shows that u has the Lusin property with functions of class C^k provided that $u \in t^{k,1}(x)$ for a.e. $x \in E$, or, equivalently, $u \in T^{k,1}(x)$ for a.e. $x \in E$ (recall Remark 2.3(iii)).

Assume indeed that E has finite measure: then for every $\varepsilon > 0$ we can find a compact subset D such that $\mathscr{L}^d(E \setminus D) \le \varepsilon$, u is continuous on D, and estimate (2.2) holds *uniformly* for all $x \in D$,[6] and therefore u agrees on D with a function class C^k on $\mathbb{R}^d$ by [10, Theorem 3.6.3].

3. Differentiability of convolution products

3.1. Assumptions on the kernel K. Through the rest of this paper K is a real function of class C^2 on $\mathbb{R}^d \setminus \{0\}$, homogeneous of degree $1-d$, that is, $K(\lambda x) = \lambda^{1-d} K(x)$ for every $x \neq 0$ and $\lambda > 0$.

It follows immediately that the derivative $\nabla K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ is of class C^1 and homogeneous of degree $-d$. Moreover it satisfies the cancellation property

$$\int_{S^{d-1}} \nabla K \, d\mathscr{H}^{d-1} = 0 \, . \tag{3.1}$$

Indeed, let a be the integral of ∇K over S^{k-1}, Ω the set of all $x \in \mathbb{R}^d$ such that $1 < |x| < 2$, ν the outer normal di $\partial\Omega$, and e an arbitrary vector in $\mathbb{R}^d$. By applying the divergence theorem to the vector field Ke and the domain Ω, we obtain

$$\int_{\partial\Omega} K \, e \cdot \nu \, d\mathscr{H}^{d-1} = \int_{\Omega} \frac{\partial K}{\partial e} \, d\mathscr{L}^d \, .$$

Now, using the fact that K is homogeneous of degree $1-d$ we obtain that the integral at the left-hand side is 0, while a simple computation shows that the integral at the right-hand side is equal to $\log 2 (a \cdot e)$. Hence $a \cdot e = 0$, and since e is arbitrary, $a = 0$.

3.2. First convolution operator. Take K as in the previous paragraph, and let μ be a bounded real-valued measure on $\mathbb{R}^d$. The homogeneity of K yields $|K(x)| \le c|x|^{1-d}$ for some finite constant c, and therefore a simple computation shows that we can define the convolution product $K * \mu$ by the usual formula

$$K * \mu(x) := \int_{\mathbb{R}^d} K(x-y) \, d\mu(y) \ , \tag{3.2}$$

and $K * \mu$ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for every p with $1 \le p < \gamma(1)$.

[6] That is, the functions $g_\rho(x) := \rho^{-k} \int_{B(\rho)} |R^k_x(h)| \, dh$ converge uniformly to 0 as $\rho \to 0$.

In order to give an explicit formula for the derivative of $K * \mu$ we need to give a meaning to the convolution product $\nabla K * \mu$.

3.3. Second convolution operator. Since ∇K is not summable on any neighbourhood of 0 (because of the homogeneity of degree $-d$), we cannot define $\nabla K * \mu$ by the usual formula. However, a classical result by A.P. Calderón and A. Zygmund shows that the convolution $K * \mu$ is well-defined at almost every point as a singular integral. More precisely, given the truncated kernels

$$(\nabla K)_\varepsilon(x) := \begin{cases} \nabla K(x) & \text{if } |x| \geq \varepsilon \\ 0 & \text{if } |x| < \varepsilon, \end{cases} \tag{3.3}$$

then the functions $(\nabla K)_\varepsilon * \mu$ converge almost everywhere to a limit function which we denote by $\nabla K * \mu$. Moreover the following weak L^1-estimate holds:

$$\mathscr{L}^d\big(\{x : |\nabla K * \mu(x)| \geq t\}\big) \leq \frac{c\|\mu\|}{t} \quad \text{for every } t > 0, \tag{3.4}$$

where c is a finite constant that depends only on d and K.

If μ is replaced by a function in L^1, this statement can be obtained, for example, from Theorem 4 in [8, Chapter II].[7] One easily checks that extending that theorem to bounded measures requires only minor modifications in the proof.

We can now state the main result of this section.

3.4. Theorem. *Take $u := K * \mu$ as in §3.2. Then*

(i) *u is L^p-differentiable for every p with $1 \leq p < \gamma(1)$ and almost every $x \in \mathbb{R}^d$;*

(ii) *the derivative of u is given by*

$$\nabla u = \nabla K * \mu + \beta_K f \quad \textit{a.e.}, \tag{3.5}$$

where $\nabla K * f$ is given in §3.3, f is the Radon-Nikodym density of μ with respect to the Lebesgue measure, and β_K is the vector defined by

$$\beta_K := \int_{S^{d-1}} x\, K(x)\, d\mathscr{H}^{d-1}(x)\,. \tag{3.6}$$

3.5. Remark. The range of p for which L^p-differentiability holds is optimal. Take indeed $K(x) := |x|^{1-d}$ and

$$\mu := \sum_i 2^{-i}\delta_i\,,$$

where δ_i is the Dirac mass at x_i, and the set $\{x_i\}$ is dense in $\mathbb{R}^d$.

Since $K(x)$ does not belong to $L^{\gamma(1)}(U)$ for any neighbourhood U of 0, the function $u := K * \mu$ does not belong to $L^{\gamma(1)}(U)$ in any open set U in $\mathbb{R}^d$. Hence

[7] In order to apply such theorem, the key point is that ∇K is of class C^1, homogeneous of degree $-d$, and satisfies the cancellation property (3.1).

u does not belong to $T^{0,\gamma(1)}(x)$ (and therefore not even to $t^{1,\gamma(1)}(x)$) for every $x \in \mathbb{R}^d$.

Note that the previous construction works *as is* for any nontrivial *positive* kernel K; a suitable refinement allows to remove the positivity constraint.

The rest of this section is devoted to the proof of Theorem 3.4.

The key point is to show that u is in $T^{1,p}(x)$ for all x in some "large" set (Lemma 3.11). To achieve this, the basic strategy is quite standard, and consists in writing u as sum of two functions u_g and u_b given by a suitable Calderón-Zygmund decomposition of the measure μ. Then we use Lemma 3.9 to show that u_g is a function of class $W^{1,q}$ for every $q \geq 1$, and therefore its differentiability is a well-established fact, and use Lemma 3.10 to estimate the derivative of u_b on a large set. This last lemma is the heart of the whole proof.

In the next three paragraphs we recall some classical tools of the theory of singular integrals used in the proofs of Lemmas 3.9 and 3.10.

3.6. Singular integrals: the L^q case. We have seen in §3.3 that the convolution product $\nabla K * \mu$ is well-defined at almost every point as a singular integral.

When μ is replaced by a function f in $L^q(\mathbb{R}^d)$ with $1 < q < \infty$ there holds more: taking $(\nabla K)_\varepsilon$ as in (3.3), then $\|(\nabla K)_\varepsilon * f\|_q \leq c\|f\|_q$ for every $\varepsilon > 0$ and every $f \in L^q(\mathbb{R}^d)$, where c is a finite constant that depends only on K and q. Moreover, as ε tends to 0, the functions $(\nabla K)_\varepsilon * f$ converges in the L^p-norm to some limit that we denote by $\nabla K * f$; in particular $f \mapsto \nabla K * f$ is a bounded linear operator from $L^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$.

These statements follow, for example, from [8, Chapter II, Theorem 3].

3.7. Marcinkiewicz integral. Let μ be a bounded (possibly vector-valued) measure on $\mathbb{R}^d$, and F a closed set in $\mathbb{R}^d$. Then the Marcinkiewicz integral

$$I(\mu, F, x) := \int_{\mathbb{R}^d \setminus F} \frac{\operatorname{dist}(y, F)}{|x-y|^{d+1}} \, d|\mu|(y) \tag{3.7}$$

is finite for almost every $x \in F$, and more precisely

$$\int_F I(\mu, F, x)\, dx \leq c\|\mu\| , \tag{3.8}$$

where c is a finite constant that depends only on d. This is a standard estimate, see [8, Chapter I, §2.3].

3.8. Maximal function. Let μ be a bounded (possibly vector-valued) measure on $\mathbb{R}^d$. The maximal function associated to μ is

$$M(\mu, x) := \sup_{\rho > 0} \frac{|\mu|(B(x, \rho))}{\omega_d \rho^d} . \tag{3.9}$$

Then $M(\mu, x)$ is finite for almost every x, and more precisely the following weak L^1-estimate holds:

$$\mathscr{L}^d(\{x : M(\mu, x) \geq t\}) \leq \frac{c\|\mu\|}{t} \quad \text{for every } t > 0, \tag{3.10}$$

where c is a finite constant that depends only on d.

In case μ is absolutely continuous with respect to the Lebesgue measure this statement can be found, for example, in [8, Chapter I, §1.3]; the proof for a general measure is essentially the same, cf. [8, Chapter III, §4.1].

3.9. Lemma. *Let f be a function in $L^1 \cap L^q(\mathbb{R}^d)$ for some q with $1 < q < +\infty$, and let $u := K * f$.*

Then u belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and the distributional derivative of u is given by

$$\nabla u = \nabla K * f + \beta_K f \tag{3.11}$$

where $\nabla K * f$ is defined in §3.6, and β_K is given in (3.6).

Since $\nabla K * f$ belongs to $L^q(\mathbb{R}^d)$, then ∇u belongs to $L^q(\mathbb{R}^d)$, and therefore u is $L^{\gamma(q)}$-differentiable almost everywhere when $q < d$, and is continuous and differentiable almost everywhere in the classical sense when $q > d$ (in both cases the pointwise derivative agrees with the distributional one almost everywhere).

Proof. We only need to prove formula (3.11); the second part of the statement follows indeed from §3.6, the standard differentiability result for Sobolev functions (see for instance [5, Sections 6.1.2 and 6.2]), and the fact that $K * f$ is continuous for $q > d$ (a matter of elementary estimates).

For every $\varepsilon > 0$ consider the truncated kernel K_ε defined as in (3.3), that is, $K_\varepsilon := 1_{\mathbb{R}^d \setminus B(\varepsilon)} K$. Then the distributional derivative of K_ε is given by

$$\nabla K_\varepsilon = (\nabla K)_\varepsilon + \sigma_\varepsilon$$

where σ_ε is the (vector-valued) measure given by the restriction of the Hausdorff measure $\mathscr{H}^{d-1}$ to the sphere $\partial B(\varepsilon)$ multiplied by the vector field $K(x)\, x/|x|$. Hence

$$\nabla(K_\varepsilon * f) = (\nabla K)_\varepsilon * f + \sigma_\varepsilon * f \,,$$

and we obtain (3.11) by passing to the limit as $\varepsilon \to 0$ in this equation.

In doing so we use the following facts:

(i) $K_\varepsilon \to K$ in the L^1-norm, and therefore $\nabla(K_\varepsilon * f) \to \nabla(K * f) = \nabla u$ in the sense of distributions;

(ii) $(\nabla K)_\varepsilon * f \to \nabla K * f$ in the L^q-norm (see §3.6);

(iii) the measures σ_ε converge in the sense of measures to β_K times the Dirac mass at 0, and then $\sigma_\varepsilon * f \to \beta_K f$ in the L^q-norm. □

3.10. Lemma. *Let F be a closed set in $\mathbb{R}^d$, $\{E_i\}$ a countable family of pairwise disjoint sets in $\mathbb{R}^d$ which do not intersect F, and μ a bounded real-valued measure on $\mathbb{R}^d$ such that*

(i) $|\mu|(\mathbb{R}^d \setminus \cup_i E_i) = 0$;

(ii) $\mu(E_i) = 0$ *for every* i;

(iii) *there exist finite and strictly positive constants* c_1 *and* c_2 *such that* $c_1 \operatorname{dist}(F, E_i^c) \leq \operatorname{diam}(E_i) \leq c_2 \operatorname{dist}(F, E_i^c)$ *for every* i, *where* E_i^c *denotes the convex hull of* E_i.

Then for every $x \in F$ *and every* p *with* $1 \leq p < \gamma(1)$ *the function* $u * K$ *satisfies*

$$\left[\fint_{B(\rho)} |u(x+h) - u(x)|^p \, dh \right]^{1/p} \leq \big[M(\mu, x) + I(\mu, F, x) \big] \, c\rho \,, \tag{3.12}$$

where $I(\mu, F, x)$ *and* $M(\mu, x)$ *are given in* (3.7) *and* (3.9), *respectively, and* c *is a finite constant that depends only on* c_1, c_2, p, d *and* K.[8]

Thus u *belongs to* $T^{1,p}(x)$ *for every* $x \in F$ *such that* $M(\mu, x)$ *and* $I(\mu, F, x)$ *are finite, that is, for almost every* $x \in F$.

Proof. For the rest of the proof we fix a point $x \in F$, $\rho > 0$, and then denote by J the set of all indexes i such that $\operatorname{dist}(x, E_i^c) < 2\rho$.

Using assumption (i) we decompose u as

$$u = \sum_i u_i \,, \tag{3.13}$$

where $u_i := K * \mu_i$ and μ_i is the restriction of the measure μ to the set E_i.

Step 1: estimate of $|u_i(x)|$ *for* $i \in J$. Choose an arbitrary point $y_i \in E_i$, and for every $s \in [0, 1]$ set

$$g(s) := \int_{E_i} K(x - (sy + (1-s)y_i)) \, d\mu(y) \,.$$

Then [9]

$$\begin{aligned} u_i(x) &= \int_{E_i} K(x - y) \, d\mu(y) \\ &= \int_{E_i} K(x - y) - K(x - y_i) \, d\mu(y) = g(1) - g(0) \,, \end{aligned}$$

and by applying the mean-value theorem to the function g we obtain that there exists $s \in [0, 1]$ such that $u_i(x) = g(1) - g(0) = \dot{g}(s)$, that is,

$$u_i(x) = \int_{E_i} \nabla K \big(\underbrace{x - (sy + (1-s)y_i)}_{z} \big) \, (y_i - y) \, d\mu(y) \,. \tag{3.14}$$

[8] When we apply this lemma later on, the constants c_1 and c_2 will depend only on d, and therefore the constant c in (3.12) will depend only on p, d and K.

[9] The second identity follows from the fact that $\mu(E_i) = 0$ by assumption (ii), and the third one follows from the definition of g.

Since ∇K is homogeneous of degree $-d$, there holds $|\nabla K(z)| \le c|z|^{-d}$,[10] and taking into account that $|z| \ge \operatorname{dist}(x, E_i^c)$ and $\operatorname{dist}(x, E_i^c) < 2\rho$ we get

$$|\nabla K(z) \cdot (y_i - y)| \le |\nabla K(z)|\,|y_i - y| \le \frac{c\operatorname{diam}(E_i)}{\operatorname{dist}(x, E_i^c)^d} \le \frac{c\rho\operatorname{diam}(E_i)}{\operatorname{dist}(x, E_i^c)^{d+1}}\,.$$

Moreover, for every $y \in E_i$ assumption (iii) implies $\operatorname{diam}(E_i) \le c\operatorname{dist}(y, F)$ and $|x - y| \le c\operatorname{dist}(x, E_i^c)$, and therefore

$$|\nabla K(z) \cdot (y_i - y)| \le \frac{c\rho\operatorname{dist}(y, F)}{|x - y|^{d+1}}\,.$$

Plugging the last estimate in (3.14) we obtain

$$|u_i(x)| \le c\rho \int_{E_i} \frac{\operatorname{dist}(y, F)}{|x - y|^{d+1}}\, d|\mu|(y)\,. \tag{3.15}$$

Step 2: estimate of $|u_i(x + h)|$ for $i \in J$. We take p with $1 \le p < \gamma(1)$ and denote by p' the conjugate exponent of p, that is, $1/p' + 1/p = 1$. We also chose a positive test function φ on $B(\rho)$, and denote by $\|\varphi\|_{p'}$ the $L^{p'}$-norm of φ with respect to the Lebesgue measure on $B(\rho)$ renormalized to a probability measure. Then [11]

$$\begin{aligned}
&\fint_{B(\rho)} |u_i(x+h)|\,\varphi(h)\,dh \\
&\quad\le \int_{E_i} \left[\fint_{B(\rho)} |K(x+h-y)|\,\varphi(h)\,dh \right] d|\mu|(y) \\
&\quad\le \int_{E_i} \left[\fint_{B(\rho)} |\varphi(h)|^{p'}\,dh \right]^{1/p'} \left[\fint_{B(\rho)} |K(x+h-y)|^p dh \right]^{1/p} d|\mu|(y) \\
&\quad\le \int_{E_i} \|\varphi\|_{p'} \left[\frac{c}{\rho^{d/p}} \int_{B(\rho)} \frac{dh}{|x+h-y|^{p(d-1)}} \right]^{1/p} d|\mu|(y) \\
&\quad\le \frac{c}{\rho^{d/p}} \|\varphi\|_{p'} \left[\int_{B(c\rho)} \frac{dz}{|z|^{p(d-1)}} \right]^{1/p} |\mu|(E_i) \le \frac{c}{\rho^{d-1}} \|\varphi\|_{p'}\,|\mu|(E_i)\,,
\end{aligned}$$

[10] Here and in the rest of this proof we use the letter c to denote any finite and strictly positive constant that depends only on c_1, c_2, p, d, and K. Accordingly, the value of c may change at every occurrence.

[11] For first inequality we use the definition of u_i and Fubini's theorem; for second one we use Hölder inequality, for the third one we use that K is homogeneous of degree $1 - d$ and therefore $|K(x)| \le c|x|^{1-d}$; for the fourth one we use the change of variable $z = x + h - y$ and the fact that for every $y \in E_i$ assumption (iii) yields

$$|x + h - y| \le |x - y| + |h| \le \operatorname{dist}(x, E_i) + \operatorname{diam}(E_i) + \rho \le c\rho\,.$$

Note that the integral in the last line is finite only if and only if $p < \gamma(1)$. Here is the only place in the entire proof were this upper bound on p is needed.

and taking the supremum over all test function φ with $\|\varphi\|_{p'} \leq 1$ we finally get

$$\left[\fint_{B(\rho)} |u_i(x+h)|^p \, dh \right]^{1/p} \leq \frac{c}{\rho^{d-1}} |\mu|(E_i) \, . \tag{3.16}$$

Step 3. Using the estimates (3.15) and (3.16), and taking into account that E_i is contained in $B(x, c\rho)$ for every $i \in J$ (use assumption (iii)), we get

$$\begin{aligned}
&\sum_{i\in J} \left[\fint_{B(\rho)} |u_i(x+h) - u_i(x)|^p \, dh \right]^{1/p} \\
&\qquad \leq \sum_{i\in J} \left[\fint_{B(\rho)} |u_i(x+h)|^p \, dh \right]^{1/p} + |u_i(x)| \\
&\qquad \leq c \frac{|\mu|(B(x,c\rho))}{\rho^{d-1}} + c \int_{B(x,c\rho)} \frac{\operatorname{dist}(y,F)}{|x-y|^{d+1}} \, d|\mu|(y) \\
&\qquad \leq \big[M(\mu,x) + I(\mu,F,x) \big] \, c\rho \, .
\end{aligned} \tag{3.17}$$

Step 4: estimate of $|u_i(x+h) - u_i(x)|$ for $i \notin J$. Let y_i be a point in E_i. Then for every $h \in B(rho)$ there exist $t, s \in [0,1]$ such that [12]

$$\begin{aligned}
&u_i(x+h) - u_i(x) \\
&\quad = \int_{E_i} K(x+h-y) - K(x-y) \, d\mu(y) \\
&\quad = \int_{E_i} \nabla K(x+th-y) \cdot h \, d\mu(y) \\
&\quad = \int_{E_i} \big[\nabla K(x+th-y) - \nabla K(x+th-y_i) \big] \cdot h \, d\mu(y) \\
&\quad = \int_{E_i} \big[\nabla^2 K \big(\underbrace{x+th-(sy+(1-s)y_i)}_{z} \big) \, (y_i - y) \big] \cdot h \, d\mu(y) \, .
\end{aligned} \tag{3.18}$$

Now, assumption (iii) and the fact that $\operatorname{dist}(x, E_i^c) \geq 2\rho$ yield

$$|z| \geq |x - (sy + (1-s)y_i)| - t|h| \geq \operatorname{dist}(x, E_i^c) - \rho \geq \frac{1}{2} \operatorname{dist}(x, E_i^c) \, ,$$

and then, taking into account that $\nabla^2 K$ is homogeneous of degree $-d-1$,

$$\big| [\nabla^2 K(z)(y_i - y)] \cdot h \big| \leq |\nabla^2 K(z)| \, |y_i - y| \, |h| \leq \frac{c \operatorname{diam}(E_i) \, \rho}{\operatorname{dist}(x, E_i^c)^{d+1}} \, .$$

[12] The second and fourth identities are obtained by applying the mean-value theorem as in Step 1, the third one follows from assumption (ii).

Moreover assumption (iii) implies that $\mathrm{diam}(E_i) \le c\,\mathrm{dist}(y,F)$ and $|x-y| \le c\,\mathrm{dist}(x,E_i^c)$ for every $y \in E_i$, and then

$$\big|[\nabla^2 K(z)(y_i - y)]\cdot h\big| \le \frac{c\,\mathrm{dist}(y,F)\,\rho}{|x-y|^{d+1}}\,.$$

Hence (3.18) yields

$$|u_i(x+h) - u_i(x)| \le c\rho \int_{E_i} \frac{\mathrm{dist}(y,F)}{|x-y|^{d+1}}\,d|\mu|(y)\,. \tag{3.19}$$

Step 5. Inequality (3.19) yields

$$\sum_{i\notin J}\left[\fint_{B(\rho)} |u_i(x+h)-u_i(x)|^p\,dh\right]^{1/p} \le I(\mu,F,x)\,c\rho\,,$$

and recalling estimate (3.17) and formula (3.17) we finally obtain (3.12). □

3.11. Lemma. *Take u as in Theorem 3.4. Take $t>0$ and let*

$$F_t := \left\{x \in \mathbb{R}^d : \ M(\mu,x) \le t\right\}\,,$$

where $M(\mu,x)$ is the maximal function defined in (3.9).

Then u belongs to $T^{1,p}(x)$ for every p with $1 \le p < \gamma(1)$ and almost every $x \in F_t$.

Proof. *Step 1: Calderón-Zygmund decomposition of μ and u.* Since $M(\mu,x)$ is lower semicontinuous in x (being the supremum of a family of lower semicontinuous functions), the set F_t is closed.

We take a Whitney decomposition of the open set $\mathbb{R}^d \setminus F_t$, that is, a sequence of closed cubes Q_i with pairwise disjoint interiors such that the union of all Q_i is $\mathbb{R}^d\setminus F_t$, and the distance between F_t and each Q_i is comparable to the diameter of Q_i, namely

$$c_1\,\mathrm{dist}(F_t,Q_i) \le \mathrm{diam}(Q_i) \le c_2\,\mathrm{dist}(F_t,Q_i)\,, \tag{3.20}$$

where c_1 and c_2 depend only on d (see [8, Chapter I, §3.1]).

We consider now the sets E_i obtained by removing from each Q_i part of its boundary, so that the sets E_i are pairwise disjoint and still cover $\mathbb{R}^d \setminus F_t$.

The Calderón-Zygmund decomposition of μ is $\mu = \mu_g + \mu_b$, where the “good” part μ_g is defined by

$$\mu_g := 1_{F_t}\cdot\mu + \sum_i a_i 1_{E_i}\cdot\mathscr{L}^d \quad \text{with } a_i := \frac{\mu(E_i)}{\mathscr{L}^d(E_i)}\,, \tag{3.21}$$

and the “bad” part μ_b is

$$\mu_b := \sum_i 1_{E_i}\cdot\mu - a_i 1_{E_i}\cdot\mathscr{L}^d\,. \tag{3.22}$$

From this definition and that of a_i we obtain

$$\|\mu_b\| \leq \sum_i 2|\mu|(E_i) = 2|\mu|(\mathbb{R}^d \setminus F_t)\,. \tag{3.23}$$

Finally we decompose u as

$$u = u_g + u_b\,,$$

where $u_g := K * \mu_g$ and $u_b := K * \mu_b$. To conclude the proof we need to show that u_g and u_b belong to $T^{1,p}(x)$ for every $1 \leq p < \gamma(1)$ and almost every $x \in F_t$. This will be done in the next steps.

Step 2: the measure μ_g can be written as $g \cdot \mathscr{L}^d$ with $g \in L^\infty(\mathbb{R}^d)$. It suffices to show that

(i) the measure $1_{F_t} \cdot \mu$ can be written as $\tilde{g} \cdot \mathscr{L}^d$ with $\tilde{g} \in L^\infty(\mathbb{R}^d)$;

(ii) the number a_i in (3.21) satisfy $|a_i| \leq ct$ for some finite constant c depending only on d.

Claim (i) follows by the fact that the Radon-Nikodym density of $|\mu|$ with respect to $\mathscr{L}^d$ is bounded by t at *every* point x of F_t, because $M(\mu, x) \leq t$.

To prove claim (ii), note that each E_i is contained in Q_i, which in turn is contained in a ball centered at some $x_i \in F_t$ with radius r_i comparable to $\operatorname{diam}(Q_i)$, and therefore with Lebesgue measure comparable to that of Q_i. Hence, taking into account that $M(\mu, x_i) \leq t$,

$$|\mu|(E_i) \leq |\mu|(B(x_i, r_i)) \leq t\, \mathscr{L}^d(B(x_i, r_i)) \leq ct\, \mathscr{L}^d(Q_i) = ct\, \mathscr{L}^d(E_i)\,,$$

and this implies $|a_i| \leq ct$.

Step 3: u_g is differentiable at x (and in particular belongs to $T^{1,p}(x)$ for every $1 \leq p < +\infty$) for almost every $x \in \mathbb{R}^d$. Since the measure μ_g is bounded, the function g in Step 2 belongs to $L^1 \cap L^\infty(\mathbb{R}^d)$. Then, by interpolation, g belongs to $L^1 \cap L^q(\mathbb{R}^d)$ for any $q > d$, and Lemma 3.9 implies that $u_g = K * g$ is differentiable almost everywhere.

Step 4: u_b belongs to $T^{1,p}(x)$ for almost every $x \in F_t$ and every $1 \leq p < \gamma(1)$. It suffices to apply Lemma 3.10 to the set F_t, the measure μ_b, and the sets E_i (use equations (3.20), (3.21) and (3.22) to check that the assumptions of that lemma are verified). □

Proof of Theorem 3.4, statement (i). It suffices to apply Lemma 3.11 and Remark 2.3(iii), and take into account that the sets F_t form an increasing family whose union cover almost all of $\mathbb{R}^d$ (because the maximal function $M(\mu, x)$ is finite almost everywhere). □

Proof of Theorem 3.4, statement (ii). Since we already know that u is L^p-differentiable almost everywhere, we have only to prove identity (3.5).

Moreover, by the argument used in the proof of statement (i) above, it suffices to show that (3.5) holds almost everywhere in the set F_t defined in Lemma 3.11 for every given $t > 0$.

Step 1: a decomposition of μ and u. We fix for the time being $\varepsilon > 0$, and choose a closed set C contained in $\mathbb{R}^d \setminus F_t$ such that $|\mu|(\mathbb{R}^d \setminus (F_t \cup C)) \le \varepsilon$.

We decompose μ as $\mu' + \mu''$ where μ' and μ'' are the restrictions of μ to the sets $\mathbb{R}^d \setminus C$ and C, respectively, and then we further decompose μ' as $\mu'_g + \mu'_b$ as in the proof of Lemma 3.11. Thus $\mu = \mu'_g + \mu'_b + \mu''$, and accordingly we decompose u as

$$u = u'_g + u'_b + u''$$

where $u'_g := K * \mu'_g$, $u'_b := K * \mu'_b$, and $u'' := K * \mu''$.

Using estimate (3.23) and taking into account the definition of μ' and the choice of C we get

$$\|\mu'_b\| \le 2|\mu'|(\mathbb{R}^d \setminus F_t) = 2|\mu|(\mathbb{R}^d \setminus (F_t \cup C)) \le 2\varepsilon\,. \tag{3.24}$$

Step 2: the derivative of u'_g. Going back to the proof of Lemma 3.11, we see that μ_g can be written as $g' \cdot \mathscr{L}^d$ with $g' \in L^1 \cap L^\infty(\mathbb{R}^d)$, and therefore $u'_g := K * g'$ is differentiable almost everywhere. Moreover formula (3.11) yields

$$\nabla u'_g = \nabla K * g + \beta_K g = \nabla K * \mu'_g + \beta_K g \quad \text{a.e.}$$

Next we note that the restrictions of the measures μ'_g, μ' and μ to the set F_t agree, and therefore $g = f$ almost everywhere on F_t, where f is the Radon-Nikodym density of μ with respect to $\mathscr{L}^d$. Thus the previous identity yields

$$\nabla u'_g = \nabla K * \mu'_g + \beta_K f \quad \text{a.e. in } F_t. \tag{3.25}$$

Step 3: the derivative of u'_b. Going back to the proof of Lemma 3.11 we see that we can use Lemma 3.10 to show that u'_b belongs to $T^{1,1}(x)$ for almost every $x \in F_t$. By Remark 2.3(iii) we have that u'_b is L^1-differentiable almost everywhere in F_t, and therefore estimate (3.12) in Lemma 3.10 yields

$$|\nabla u'_b(x)| \le c\, M(\mu, x) + c\, I(\mu, F_t, x) \quad \text{for a.e. } x \in F_t.\text{[13]} \tag{3.26}$$

Step 4: the derivative of u''. By construction, the support measure μ'' is contained in the closed set C and therefore the convolution $u'' := K * \mu''$ can be defined in the classical sense and is smooth in every point of the open set $\mathbb{R}^d \setminus C$, which contains F_t. Hence

$$\nabla u'' = \nabla K * \mu'' \quad \text{everywhere in } F_t.\text{[14]} \tag{3.27}$$

Step 5. Putting together equations (3.25) and (3.27), and the fact that $\mu = \mu'_g + \mu'_b + \mu''$, we obtain

$$\nabla u - (\nabla K * \mu + \beta_K f) = \nabla u'_b - \nabla K * \mu'_b \quad \text{a.e. in } F_t,$$

[13] Here and in the rest of this proof we use the letter c to denote any finite and strictly positive constant that depends only on d and K.

[14] Note that this "classical" convolution agrees (a.e) with the singular integral.

and using estimate (3.26),

$$|\nabla u - (\nabla K * \mu + \beta_K f)| \le c\, M(\mu'_b, \cdot) + c\, I(\mu'_b, \cdot) + |\nabla K * \mu'_b| \quad \text{a.e. in } F_t. \quad (3.28)$$

Finally, using the fact that $\|\mu'_b\| \le 2\varepsilon$ (see (3.23)) and estimates (3.10), (3.8), and (3.4) we obtain that each term at the right-hand side of (3.28) is smaller than $\sqrt{\varepsilon}$ outside an exceptional set with measure at most $c\sqrt{\varepsilon}$.

Since ε is arbitrary, we deduce that $\nabla u = \nabla K * \mu + \beta_K f$ almost everywhere in F_t, and the proof is complete. □

4. Further differentiability results

4.1. The kernels K_h. Let $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ be the fundamental solution of the laplacian $(-\Delta)$ on $\mathbb{R}^d$, that is,

$$G(x) := \begin{cases} \dfrac{1}{d(d-2)\omega_d} |x|^{2-d} & \text{if } d > 2 \\[2ex] \dfrac{1}{2\pi} \log |x| & \text{if } d = 2, \end{cases}$$

and for every $h = 1, \dots, d$ we set

$$K_h(x) := -\partial_h G(x) = \frac{1}{d\omega_d} |x|^{-d} x_h \,.$$

We can now state the main results of this section; proofs will be given after Remark 4.5.

4.2. Proposition. *Let v be a vector field in $L^1(\mathbb{R}^d)$ whose distributional curl and divergence are bounded measures, and denote by μ_0 and μ_{hk}, with $1 \le h, k \le d$, the following measures:*

$$\mu_0 := \operatorname{div} v \quad \textit{and} \quad \mu_{hk} := (\operatorname{curl} v)_{hk} = \partial_h u_k - \partial_k u_h \,. \quad (4.1)$$

Then, for every $k = 1, \dots, d$, there holds

$$u_k = K_k * \mu_0 + \sum_{h=1}^{d} K_h * \mu_{hk} \quad \textit{a.e.}, \quad (4.2)$$

and therefore v_k is L^p-differentiable at almost every $x \in \mathbb{R}^d$ and for every p with $1 \le p < \gamma(1)$.

4.3. Proposition. *Let v be a vector field in $L^1(\mathbb{R}^d)$ with bounded deformation, that is, the distributional symmetric derivative $\frac{1}{2}(\nabla v + \nabla^t v)$ is a bounded measure, and denote by λ_{hk}, with $1 \le h, k \le d$ the following measures:*

$$\lambda_{hk} := \frac{1}{2}(\partial_h u_k + \partial_k u_h) \,. \quad (4.3)$$

Then for every $k = 1, \dots, d$ there holds

$$v_k = \sum_{h=1}^{d} \bigl(2K_h * \lambda_{hk} - K_k * \lambda_{hh}\bigr) \quad a.e., \tag{4.4}$$

and therefore v_k is L^p-differentiable at almost every $x \in \mathbb{R}^d$ and for every p with $1 \le p < \gamma(1)$.

4.4. Proposition. *Let Ω be an open set in $\mathbb{R}^d$, and w a real function in $L^1_{\mathrm{loc}}(\Omega)$ whose distributional Laplacian is a locally bounded measure. Then w admits an L^p-Taylor expansion of order two for a.e. $x \in \mathbb{R}^d$ and every $1 \le p < \gamma(\gamma(1))$.*

In particular w has the Lusin property with functions of class C^2.

4.5. Remark. (i) Using statement (ii) in Theorem 3.4 we can write an explicit formula for the (pointwise) derivatives of the vector fields v considered in Propositions 4.2 and 4.3.

(ii) Let Ω be any open set in $\mathbb{R}^d$. The differentiability property stated in Proposition 4.2 holds also for vector fields v in $L^1_{\mathrm{loc}}(\Omega)$ whose curl and divergence are locally bounded measures. The key observation is that given a smooth cutoff function φ on $\mathbb{R}^d$ with support contained in Ω, then φv is a vector field in $L^1(\mathbb{R}^d)$ and its curl and divergence are bounded measures.

The same argument applies to Proposition 4.3.

(iii) The range of p in Proposition 4.4 is optimal, and this shown by taking $\Omega = \mathbb{R}^d$ and $w := G * \mu$ where G is given in §4.1 and μ is given in Remark 3.5. Indeed $-\Delta w = \mu$ and one easily checks that w does not belong to $L^{\gamma(\gamma(1))}$ on any open set of $\mathbb{R}^d$. Hence w does not belong to $T^{0,\gamma(\gamma(1))}(x)$, and therefore not even to $t^{2,\gamma(\gamma(1))}(x)$, for any $x \in \mathbb{R}^d$.

(iv) The range of p in Proposition 4.2 is also optimal. Let indeed $v := \nabla w$ where w is the function constructed above: then the curl of w vanishes and the divergence agrees with the measure $-\mu$, and v does not belong to $L^{\gamma(1)}$ for any open set in $\mathbb{R}^d$ (otherwise the Sobolev embedding would imply that w belongs to $L^{\gamma(\gamma(1))}$ for some open set).

(v) We do not know whether the range of p in Proposition 4.3 is optimal, and more precisely whether a map in BD belongs to $t^{1,\gamma(1)}(x)$ for almost every x. This possibility cannot be ruled out as above because the space BD embeds in $L^{\gamma(1)}$ for regular domains [9, ***].

Proof of Proposition 4.2. By applying the Fourier transform to the identities in (4.1) we obtain

$$\sum_h i\xi_h \hat{u}_h = \hat{\mu}_0 \quad \text{and} \quad i\xi_h \hat{u}_k = i\xi_k \hat{u}_h + \hat{\mu}_{hk} \,, \tag{4.5}$$

where $i = \sqrt{-1}$ and ξ denotes the Fourier variable.

We multiply the second identity in (4.5) by $-i\xi_h$ and sum over all h; taking into account the first identity in (4.5) we get

$$|\xi|^2 \hat{u}_k = \xi_k \sum_h \xi_h \hat{u}_h - \sum_h i\xi_h \hat{\mu}_{hk} = -i\xi_k \hat{\mu}_0 - \sum_h i\xi_h \hat{\mu}_{hk} \,.$$

Now $-\Delta G = \delta_0$ implies $\hat{G} = |\xi|^{-2}$ and then $\hat{K}_h = -i\xi_h \hat{G} = -i\xi_h |\xi|^{-2}$ (see §4.1). Thus the previous identity yields

$$\hat{u}_k = \frac{-i\xi_k}{|\xi|^2}\hat{\mu}_0 + \sum_h \frac{-i\xi_h}{|\xi|^2}\hat{\mu}_{hk} = \hat{K}_k \hat{\mu}_0 + \sum_h \hat{K}_h \hat{\mu}_{hk} \,,$$

and (4.2) follows by taking the inverse Fourier transform. The rest of Proposition 4.2 follows from Theorem 3.4. □

Proposition 4.3 can be proved in the same way as Proposition 4.2; we omit the details.

4.6. Lemma. *Let $k \geq 0$ be an integer, and $p \geq 1$ a real number. Let u be a function in $W^{1,1}(\Omega)$ where Ω is a bounded open set in $\mathbb{R}^d$, and assume that the distributional derivative ∇u belongs to $t^{k,p}(x)$ (respectively, $T^{k,p}(x)$) for some point $x \in \Omega$. Then u belongs to $t^{k+1,\gamma(p)}(x)$ (respectively, $T^{k+1,\gamma(p)}(x)$).*

This lemma is contained in [4, Theorem 11], at least in the case $\Omega = \mathbb{R}^d$ and u with compact support (keep in mind Remark 2.3(i)). Note that we can always reduces to this case by multiplying u by suitable cutoff functions.

Proof of Proposition 4.4. Apply Proposition 4.2 to the vector field ∇w and then use Lemma 4.6 (and recall §2.4). □

We conclude this section with a comment on the last proof.

4.7. Remark. The key step in the proof of the Lusin property for the functions w considered in Proposition 4.4 is the L^p-differentiability for ∇w in in Proposition 4.2. Here we argue that the approximate differentiability of ∇w in the sense of Remark 2.3(v), which follows from the result in [7], would have not been sufficient.

We claim indeed that even in dimension $d = 1$, the approximate differentiability of the derivative of a function w at almost every point of a set E is not enough to prove that w has the Lusin property with functions of class C^2 on E. More precisely, there exists a function $w : \mathbb{R} \to \mathbb{R}$ of class C^1 such that $\dot{w} = 0$ on some set E with positive measure—and therefore $\dot{w}$ is approximately differentiable with derivative equal to 0 at almost every point of E—but w does not have the Lusin property with functions of class C^2 on E.

The construction of such a function is briefly sketched in the next paragraph.

4.8. Example. We fix λ such that $1/4 < \lambda < 1/2$ and consider the following variant of the Cantor set: E is the intersection of the set E_n with $n = 0, 1, 2 \dots$, where each E_n is the union of 2^n closed interval $I_{n,k}$, $k = 1 \dots, 2^n$, all with the same length and obtained as follows: $I_{0,1} = E_0$ is a closed interval with length 2, and the intervals $I_{n+1,k}$ are obtained by removing from each $I_{n,k}$ a concentric open interval $J_{n,k}$ with length $(1-2\lambda)\lambda^n$.

Thus E is a compact set with empty interior such that $\mathscr{L}^1(E) = 1$.

Next we construct a non-negative continuous function $v : \mathbb{R} \to \mathbb{R}$ such that $v = 0$ outside the union of the intervals $J_{n,k}$ over all n and k, and the integral of v over each $J_{n,k}$ is equal to $(1-2\lambda)\lambda^n$.

Finally we take w so that $\dot{w} = v$.

It is easy to verify that for every n the set $E'_n := w(E_n)$ is the union of the disjoint intervals $I'_{n,k} := w(I_{n,k})$, $k = 1, \dots, 2^n$, all with length λ^n. Moreover E'_{n+1} can be written as the union of two disjoint copies of E'_n scaled by a factor λ. Therefore the set $E' := w(E)$ can be written as the union of two disjoint copies of itself scaled by a factor λ. In other words, E' is a self-similar fractal determined by two homoteties with scaling factor λ: it is then well-known that E' has Hausdorff dimension $d := \log 2/\log(1/\lambda)$ (see [6, Section 8.3]).

Moreover, denoting by μ the push-forward according to w of the restriction of the Lebesgue measure to E, one easily checks that μ is supported on E' and satisfies $\mu(I'_{n,k}) = 2^{-n}$ for every n and k. Therefore μ agrees with the canonical probability measure associated to the fractal E', which in turn agrees, up to a constant factor, with the restriction of $\mathscr{H}^d$ to E'. In particular, since $d > 1/2$ we have that $\mu(A) = 0$ for every set A which is σ-finite with respect to $\mathscr{H}^{1/2}$.

To show that w does not have the Lusin property with functions of class C^2 on E it is now sufficient to recall the following elementary fact: *let $u : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that u has the Lusin property with functions of class C^2 on E and $\dot{u} = 0$ on E; then the push-forward of $1_E \cdot \mathscr{L}^1$ according to w is supported on a set A which is σ-finite with respect to $\mathscr{H}^{1/2}$.*[15]

References

[1] G. Alberti, S. Bianchini, G. Crippa: A uniqueness result for the continuity equation in two dimensions. Preprint, 2011. Available at `http://cvgmt.sns.it/`

[2] L. Ambrosio, A. Coscia, G. Dal Maso: Fine properties of functions with bounded deformation. *Arch. Rational Mech. Anal.*, **139** (1997), 201–238.

[3] A.P. Calderón, A. Zygmund: On the existence of certain singular integrals. *Acta Math.*, **88** (1952), 85–139.

[4] A.P. Calderón, A. Zygmund: Local properties of solutions of elliptic partial differential equations. *Studia Math.*, **20** (1961), 171–225.

[5] L.C. Evans, R.F. Gariepy: *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, 1992.

[6] K.J. Falconer:: *The geometry of fractal sets.* Cambridge Tracts in Mathematics, 85. Cambridge University Press, Cambridge 1985.

[15] This is an immediate consequence of another elementary fact, namely that a function $u : \mathbb{R} \to \mathbb{R}$ of class C^2 maps any bounded set where $\dot{u} = 0$ into an $\mathscr{H}^{1/2}$-finite set.

[7] P. Hajłasz: On approximate differentiability of functions with bounded deformation. *Manuscripta Math.*, **91** (1996), 61–72.

[8] E.M. Stein: *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, 30. Princeton University Press, Princeton, 1970.

[9] R. Temam, G. Strang: Functions of bounded deformation. *Arch. Rational Mech. Anal.*, **75** (1980/81), 7–21.

[10] W.P. Ziemer: *Weakly differentiable functions. Sobolev spaces and functions of bounded variation.* Graduate texts in mathematics, 120. Springer, New York, 1989.

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