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$\mathcal{N} = 1$ geometries via M-theory

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Abstract: We provide an M-theory geometric set-up to describe four-dimensional $\mathcal{N} = 1$ gauge theories. This is realized by a generalization of Hitchin's equation. This framework encompasses a rich class of theories including superconformal and confining ones. We show how the spectral data of the generalized Hitchin's system encode the infrared properties of the gauge theory in terms of $\mathcal{N} = 1$ curves. For $\mathcal{N} = 1$ deformations of $\mathcal{N} = 2$ theories in class $\mathcal{S}$, we show how the superpotential is encoded in an appropriate choice of boundary conditions at the marked points in different S-duality frames. We elucidate our approach in a number of cases — including Argyres-Douglas points, confining phases and gaugings of T_N theories — and display new results for linear and generalized quivers.

Keywords: Brane Dynamics in Gauge Theories, Supersymmetric gauge theory, M-Theory, Anomalies in Field and String Theories

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1 Introduction

M-theory geometric description of supersymmetric gauge theories is a very powerful access key to the non-perturbative aspects of the formers and complements their perturbative Lagrangian description. In this context, a very powerful approach to four-dimensional $\mathcal{N} = 2$ gauge theories has been introduced in [1, 2]. In this paper we propose a general geometric set-up for four-dimensional $\mathcal{N} = 1$ gauge theories which naturally encompasses previous well known examples studied from the point of view of rotated M5-brane systems in [3–6].

The approach we follow consists in considering the six-dimensional $(2, 0)$ theory compactified on a Riemann surface $\mathcal{C}$ with marked points with a suitable topological twist. This is realized by wrapping an M5-brane bound system around $\mathcal{C} \times \mathbb{R}^{1,3}$, where $\mathcal{C}$ is the base of a local Calabi-Yau (CY) threefold X obtained as the total space of a rank 2 holomorphic vector bundle with normalized determinant over $\mathcal{C}$. For theories with $\mathcal{N} = 1$ superconformal invariance in the infrared, this set-up has been considered in [7, 8]. See also [9–12] for relevant field theoretical arguments.

The dynamics of the M5-branes is determined by M2-branes suspended between them. These are particles in the spacetime and therefore are effectively open strings in X ending on the holomorphic two-cycle $\mathcal{C}$. Therefore the dynamics of the M5-branes reduces to that of B-branes in X and as such it is described by the appropriate dimensional reduction of the holomorphic Chern-Simons theory on $\mathcal{C}$. Its equations of motion give rise to a generalization of the Hitchin system with suitable singular boundary conditions at the marked points.

The spectral curve of this generalized Hitchin system describes the geometry of the N M5-brane bound state as an N-branched cover of $\mathcal{C}$. This is encoded in a specific overdetermined algebraic system that we study in detail. This system describes a rich class of $\mathcal{N} = 1$ theories including superconformal and confining ones, and contains the relevant infrared data of the four-dimensional $\mathcal{N} = 1$ gauge theory. We show how to obtain the generalized Konishi anomaly equation [13], the Dijkgraaf-Vafa curve [14], the $\mathcal{N} = 1$ curves presented in [3, 4], and the curve describing the $\mathcal{N} = 1$ gauge theory coupled to T_N

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theories, related with [15, 16]. In particular the factorization of both the Seiberg-Witten curve and the Dijkgraaf-Vafa curve is elucidated and worked out in new examples.

The above geometrical objects compute the vev of chiral ring observables. The moduli space of vacua of the $\mathcal{N} = 1$ theory is described by the moduli space of solutions of the generalized Hitchin system associated to it. A particularly important class of theories is the one obtained via turning on an $\mathcal{N} = 1$ breaking superpotential of adjoint chiral fields to $\mathcal{N} = 2$ theories. This sets the boundary conditions on the generalized Hitchin field and provides an obstruction in the $\mathcal{N} = 2$ Hitchin moduli space. Therefore the generalized system has a reduced moduli space corresponding to the lifting of the flat directions in the Coulomb branch of the $\mathcal{N} = 2$ theory, induced by the superpotential. Indeed, we see that in various examples the Coulomb moduli is completely fixed, leading to isolated vacua. The dual geometric set-up to the above case is described as arising via a complex structure rotation in the target space geometry, which connects the $\mathcal{N} = 1$ breaking potential to purely geometrical data.

We show how our set-up reproduces several known examples previously studied in the literature, namely $\mathcal{N} = 1$ super Yang-Mills (SYM) theory, supersymmetric QCD (SQCD) and $\mathcal{N} = 1^*$ theories as trial cases. The new results which are obtained concern the extension of the factorization condition of the Seiberg-Witten curve for linear and generalized quivers. We also discuss a subtle difference in the interpretation of the boundary conditions which has to be taken into account when considering gauge theories with SU(N) group, rather than U(N), via a shift of the boundary condition by the meson vevs.

The use of the generalized Hitchin system requires the introduction of a vector field on $\mathcal{C}$ whose choice looks crucial in order to establish the S-duality frame of the underlying theory. For SU(2) generalized quivers we show that the factorization condition is an easy consequence of the generalized Hitchin system and provides a simple recipe for computing the chiral condensates when combined with the appropriately chosen vector field. We exemplify the relation between the choice of the vector field and the S-duality frame for the corresponding gauge theory in the case of SU(2) theory with $N_f = 4$.

This paper is organized as follows. In section 2 we consider the general geometric set-up generating four-dimensional $\mathcal{N} = 1$ gauge theories from M5-branes in the local CY curve. We discuss the generalized Hitchin system and its spectral curve data, and we focus on the theories obtained from $\mathcal{N} = 2$ ones via superpotential breaking. In section 3 we see how the framework provided in section 2 is applied to $\mathcal{N} = 1$ SU(2) gauge theories. We show that the factorization condition which follows from the generalized Hitchin system is strong enough to determine the $\mathcal{N} = 1$ curves of various SU(2) gauge theories. We also give some results for the $\mathcal{N} = 1^*$ gauge theory with SU(2) gauge group. In section 4, we study the higher rank theories including SU(N) SYM theory and the T_N theories coupled to $\mathcal{N} = 1$ SU(N) gauge groups. In section 5 the $\mathcal{N} = 1$ curve of linear quiver theory is obtained from the Konishi anomaly equations. We discuss the factorization condition of this theory in detail. Section 6 contains our final remarks and discusses some open issues. In appendix A we present some of our results by an alternative method making use of the type IIA branes picture.

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2 Generalized Hitchin system

2.1 $\mathcal{N} = 1$ M-theory engineering

$\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions can be engineered by considering the reduction of N M5-branes on the CY geometry

$$\mathbb{R}^{1,3} \times X \times \mathbb{R}$$

where the N M5-branes wrap $\mathbb{R}^{1,3} \times \mathcal{C}_{g,n}$, $\mathcal{C}_{g,n}$ being an holomorphic two-cycle with genus g and n marked points in the CY threefold X.

Let us concentrate on the geometry in the vicinity of the brane system and specify the CY threefold to be a local curve over $\mathcal{C}$,[1] namely $X = \mathrm{tot}V_{\mathcal{C}}$, where $V_{\mathcal{C}}$ is a rank two holomorphic vector bundle with $\det V_{\mathcal{C}} = K_{\mathcal{C}}$ the canonical line bundle on $\mathcal{C}$, to ensure supersymmetry by the usual CY condition.

The geometry of the N M5-branes encodes the definition of the $\mathcal{N} = 1$ gauge theory and it is described by the following generalization of the Hitchin system

$$\begin{aligned} \bar{D}\vec{\Phi} &= 0, \\ \vec{\Phi} \wedge \vec{\Phi} &= 0, \end{aligned} \tag{2.1}$$

which should be supplemented by suitable boundary conditions at the marked points. Eqs. (2.1) are the Uhlenbeck-Yau equations in $\mathrm{tot}V_{\mathcal{C}}$ dimensionally reduced to the base $\mathcal{C}$. $\vec{\Phi}$ is a section in $V_{\mathcal{C}}$ transforming in the adjoint representation of a given gauge bundle. The D-term equation is traded for the complexification of the gauge group and a stability assumption on the gauge bundle.

Notice that the BPS M5-branes, as seen from the internal CY geometry viewpoint, are B-branes of complex dimension one. The M2-branes stretching between them are particles in $\mathbb{R}^{1,3}$ and open strings in X with boundary on the B-branes. Indeed, the action functional generating the equations (2.1) is the dimensional reduction to $\mathcal{C}$ of the holomorphic Chern-Simons theory on X and reads

$$S = \frac{1}{2} \int_{\mathcal{C}} \mathrm{Tr}\vec{\Phi} \wedge \bar{D}\vec{\Phi}. \tag{2.2}$$

The geometry of the N M5-brane bound state is described as an N-branched cover of $\mathcal{C}$ in the form of the spectral curve for the commuting pair $\vec{\Phi}$. This can be easily written in components as follows. By specifying $\vec{\Phi} = (\phi_1, \phi_2)$ in components the generalized Hitchin equations (2.1) read

$$\bar{D}\phi_i = 0 \quad (i = 1, 2), \qquad [\phi_1, \phi_2] = 0 \tag{2.3}$$

whose spectral curve is given by the overdetermined algebraic system

$$\begin{aligned} \det(x_1 - \phi_1) &= 0, \\ \det(x_2 - \phi_2) &= 0, \\ \det(x_1 x_2 - \phi_1 \phi_2) &= 0, \end{aligned} \tag{2.4}$$

spanning an N-sheeted cover of $\mathcal{C}$ described by the simultaneous eigenvalues of ϕ_1 and ϕ_2.

[1] Here and in the following we drop for simplicity the subscripts: $\mathcal{C} \equiv \mathcal{C}_{g,n}$.

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The set-up we just described is seemingly quite rich and can be used to define a wide class of $\mathcal{N}=1$ supersymmetric gauge theories in four dimensions, whose moduli space of vacua is described by the moduli space of solutions of the generalized Hitchin's system. In particular maximally confining vacua correspond to isolated solutions. An important subset of these theories is $\mathcal{N}=1$ deformation of $\mathcal{N}=2$ gauge theories in class $\mathcal{S}$ by superpotential terms depending on the adjoint chiral fields. This subset includes the theories considered extensively in [14, 17–19]. We will mainly focus on this subset in this paper. Let us remark however that more general genuine $\mathcal{N}=1$ theories can be described within our approach.

In order to see the $\mathcal{N}=1$ theory as a deformation of an $\mathcal{N}=2$ theory in class $\mathcal{S}$, one should rotate the target space complex structure as

$$\operatorname{tot} V_{\mathcal{C}} \quad \to \quad \mathbb{C}\times T^*\mathcal{C}. \tag{2.5}$$

In the following, for the sake of simplicity, we will restrict to the case in which $V_{\mathcal{C}} = K_{\mathcal{C}}\mathcal{L}^{-1}\oplus\mathcal{L}$ is split in the sum of two line bundles, although more general cases can be considered. The rotation of the complex structure (2.5) can be encoded in a t-dependent coordinate redefinition $(x_1,x_2,z)\to(x,w,t)$ which has to be a canonical transformation to map (almost everywhere) the CY holomorphic top forms as

$$dx_1\wedge dx_2\wedge dz \quad \sim \quad dx\wedge dw\wedge dt,$$

where t and z are generically different coordinates on $\mathcal{C}$, x is a coordinate in the canonical bundle $K_{\mathcal{C}}$ and w in $\mathbb{C}$. The change of coordinates is then obtained from a generating potential Ψ valued in $K_{\mathcal{C}}$. For example one can have

$$w=\partial_x\Psi(x,x_2,t) \quad \text{and} \quad x_1\partial_t z=\partial_{x_2}\Psi(x,x_2,t). \tag{2.6}$$

The canonical transformation can be extended to the commuting pairs $(\phi_1,\phi_2)\to(\Phi,\varphi)$, where Φ and φ are the components of the $\vec{\Phi}$ doublet in the rotated target space geometry.

As we will describe in the following subsection the pair (Φ,φ) subject to some precise boundary conditions describes $\mathcal{N}=1$ breaking of $\mathcal{N}=2$ theories in class $\mathcal{S}$. We remark that a non trivial constraint on the canonical transformation is that it produces the correct boundary conditions. This in general cannot be done and therefore in this setting this is the condition such that the $\mathcal{N}=1$ initial theory can be seen as a deformation $\mathcal{N}=2$ one. For example, the simplest canonical transformation is the coordinate rotation induced by meromorphic sections σ_1 of $\mathcal{L}$, σ_2 of $K_{\mathcal{C}}\mathcal{L}^{-1}$ and τ of $\mathcal{L}^{-1}$

$$\Phi=\sigma_1\phi_1+\sigma_2\phi_2, \qquad \varphi=\tau\phi_2, \tag{2.7}$$

where $\Psi=\tau x_2\left(x-\frac{\sigma_2 x_2}{2}\right)$ and $\sigma_1\tau=\partial_t z$. This rotation has a non trivial effect on the behavior of the fields at the divisor dual to $\mathcal{L}$ and can induce spurious singularities in the Hitchin field. In particular, if $\mathcal{L}$ is positive we expect to have a genuine $\mathcal{N}=1$ theory. For example for $\mathcal{C}=\mathbb{P}^1$, if $\mathcal{L}=\mathcal{O}(p)$ and $p>0$, then one gets $\mathcal{N}=1$ superconformal theories studied in [8].

In what follows we focus on the study of $\mathcal{N}=1$ deformation of theories in class $\mathcal{S}$.

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2.2 $\mathcal{N} = 1$ breaking of theories in class $\mathcal{S}$

$\mathcal{N} = 2$ gauge theories are obtained as particular cases of $\mathcal{N} = 1$ theories which are constrained in their spectrum and couplings. These constraints can be implemented in the M-theory engineering by imposing the existence of higher supersymmetry in the geometry, namely by requiring the SU(2) R-symmetry to get realized as a spacetime symmetry. In concrete we need the local CY geometry threefold to have a trivial fiber and therefore to be a local K3 manifold. This produces the familiar M-theory geometric background

$$\mathbb{R}^{1,3} \times K3 \times \mathbb{R}^3$$

where the local K3 geometry is uniquely determined by the holomorphic two-cycle $\mathcal{C}$ to be the total space of its canonical bundle $K_{\mathcal{C}}$. More explicitly the holomorphic vector bundle is $V_{\mathcal{C}} = K_{\mathcal{C}} \oplus \mathbb{C}$ and $\mathbb{R} \times \mathbb{C} = \mathbb{R}^3$.

The constraints on the M5-brane system should still be consistently enforced to guarantee the overall $\mathcal{N} = 2$ supersymmetry. The generalized Hitchin equations (2.3) now read

$$\bar{D}\Phi = 0, \tag{2.8}$$

$$\bar{D}\varphi = 0, \tag{2.9}$$

$$[\varphi, \Phi] = 0, \tag{2.10}$$

where Φ is a $(1,0)$-form on $\mathcal{C}$ and φ is a scalar on $\mathcal{C}$ both transforming in the adjoint representation of the gauge bundle.

The M5-brane system should occupy a point in the $\mathbb{R}^3$ factor of the geometry and this can be enforced by setting to vanish the field φ reducing the extended Hitchin system (2.10) to the well known Hitchin equation

$$\bar{D}\Phi = 0.$$

This should be supplemented by appropriate boundary conditions at the punctures describing the M5-brane asymptotic geometry. Notice the important fact that the above procedure of eliminating φ is independent on the boundary conditions imposed on Φ.

The complete geometry is obtained by the Seiberg-Witten (SW) curve in the form of the spectral curve of the Hitchin system which follows directly from (2.4). By specifying $\varphi = 0$ and $x_2 = 0$ one gets

$$\det[x - \Phi] = 0,$$

where $x_1 \to x$ is the fiber coordinate on the local K3 geometry around $\mathcal{C}$. Notice that in order to fix $\varphi = 0$ it is enough to set to zero its boundary values at the singularities, whenever the gauge bundle is irreducible.

In conclusion, by these two steps, namely the holonomy reduction of the target *and* the assignment of vanishing boundary conditions to φ at the punctures, we could enforce the R-symmetry required to get an enhancement to an $\mathcal{N} = 2$ gauge theory.

Corresponding to the two steps above, we have therefore two seemingly different ways to break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ within the family of theories which we are considering in this paper. One is by rotating the M5-branes in a direction orthogonal to the

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local K3 by fixing suitable boundary conditions, the other one is obtained by embedding the system in a more general local CY geometry and rotating the complex structure. Actually both these steps, in the perturbative regimes, should be equivalent to switch on an $\mathcal{N}=1$ superpotential. Let us describe the two deformations in more detail.

2.2.1 Breaking via M5-branes rotation

The first way of $\mathcal{N}=1$ deformation corresponds to consider once more the extension of the Hitchin system

$$\begin{aligned} \bar{D}\Phi &= 0, \\ \bar{D}\varphi &= 0, \\ [\varphi, \Phi] &= 0, \end{aligned} \tag{2.11}$$

but now with non-vanishing boundary conditions on φ. As we noticed, if the field φ had vanishing boundary conditions at the punctures, then it vanishes everywhere. Changing the boundary conditions for φ at the punctures therefore corresponds to break to $\mathcal{N}=1$. The allowed boundary conditions for φ shall be given in terms of the field Φ itself in order to automatically satisfy the equation $[\varphi, \Phi]=0$. Henceforth the boundary conditions encode the supersymmetry breaking superpotential.

To write the boundary conditions we need to compare the Φ and φ on the Riemann surface $\mathcal{C}$ and this is obtained by introducing a (possibly singular) holomorphic vector field on $\mathcal{C}$. Let us denote this vector field by $\zeta = \zeta^t \partial_t$ and consider then the system with boundary conditions at the punctures $\{P_i\}_{i=1,\dots,n}$

$$\varphi(P) \sim W_i'(V) \quad \text{as} \quad P \sim P_i, \tag{2.12}$$

where $V = i_\zeta \Phi = \zeta^t \Phi_t$ and W_i' are polynomials in V. Let us notice that one of the boundary conditions in (2.12) can be set to zero, say $W_1'=0$, by the change of variables $\varphi \to \varphi - W_1'(V)$ unless V introduces spurious singularities by the vector field ζ.

The rotated M5-brane is then described by the spectral curve of (2.10), namely by the curve in the space $\mathbb{C} \times T^*\mathcal{C}$ spanned by the overdetermined system

$$\begin{aligned} \det[x - \Phi] &= 0, \\ \det[xw - \Phi\varphi] &= 0, \\ \det[w - \varphi] &= 0, \end{aligned} \tag{2.13}$$

where w and x are the coordinates in $\mathbb{C}$ and the fiber of $T^*\mathcal{C}$ respectively. As it will become evident in the following sections, the resulting system describes both the original Hitchin spectral curve at the singular point on the $\mathcal{N}=2$ Coulomb branch which is not lifted by the $\mathcal{N}=1$ deformation, and the $\mathcal{N}=1$ curve describing the solution of the generalized Konishi anomaly equation in a unified way. The elementary, but crucial, observation we will use is that, once fixed by the boundary conditions, the algebraic system (2.13) is solved as an algebraic relation by the fields (Φ, φ) (Hamilton-Cayley theorem for the commuting pair).

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In this framework the fixing of the $\mathcal{N}=2$ Coulomb moduli to their $\mathcal{N}=1$ vacuum values is due to the fact that the extended Hitchin system's moduli space has a reduced dimension with respect to the Hitchin system itself because of the obstructions induced by the non-vanishing boundary conditions on φ. More precisely, the equations (2.13) describe a Kähler submanifold in the Hitchin moduli space obtained by specifying the Hitchin's Hamiltonians to the $\mathcal{N}=1$ condensates. Indeed, as we will show in section 3, the second equation of (2.13) can be identified for Lagrangian theories with the generalized Konishi anomaly equations encoding the $\mathcal{N}=1$ chiral condensates.

2.2.2 Breaking via complex structure rotation

The second way to break supersymmetry down to $\mathcal{N}=1$ corresponds to rotate the target space complex structure as

$$\mathbb{C}\times T^*\mathcal{C} \quad \to \quad \mathrm{tot}V_{\mathcal{C}} \tag{2.14}$$

This can be implemented via a suitable canonical transformation, as discussed in the previous subsection as a coordinate redefinition $(x,w,t)\to(x_1,x_2,z)$ preserving (almost everywhere) the CY holomorphic top form. The canonical transformation can be extended to the commuting pairs $(\Phi,\varphi)\to(\phi_1,\phi_2)$ setting the equivalence between the descriptions in (2.10) and (2.1) *if accompanied by the induced correspondence of the boundary conditions in the appropriate way.*

The choice of the canonical transformation is indeed crucial in order to avoid a mismatch in the boundary conditions. In particular it is important to select adapted sections to match them. We will consider this issue in detail in the following examples.

2.3 M-theory curve and factorization condition

In the following sections we will mainly exploit the first description involving the $\mathcal{N}=2$ Hitchin field. The curve is given by the generalized Hitchin system we discussed before, which includes the $\mathcal{N}=2$ curve and a degree N equation for the coordinate w (the spectral curve for the φ field) which has the form

$$w^N=\sum_{k=2}^{N} w^{N-k}V_k(t), \tag{2.15}$$

where $V_k(t)$ are meromorphic functions on $\mathcal{C}$ with poles at the punctures, since the coordinate w lives in $\mathcal{O}(0)$.

A curve exactly of the form written above, which encodes the matrix of gauge couplings, has been found recently for $\mathcal{N}=1$ gauge theory coupled to T_N theories [15, 16]. It is thus natural to expect that what we need is precisely the generalization of these "$\mathcal{N}=1$ Seiberg-Witten curves". We will indeed see that these curves emerge naturally for the class of theories considered in [15, 16], once we impose the correct boundary conditions at the punctures. Thus we refer to (2.15) as $\mathcal{N}=1$ curve.

These two equations must then be supplemented by a third equation, which encodes the restrictions on the Coulomb branch coordinates of the underlying $\mathcal{N}=2$ theory. From the perspective of the generalized system, this is encoded in the spectral equation for

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the product (the second equation in (2.13)). How can we understand it from the brane perspective? What we are trying to do is to construct softly broken $\mathcal{N}=2$ theories obtained by compactifying the six-dimensional $\mathcal{N}=(2,0)$ theory of type A_{N-1} on $\mathcal{C}$. Indeed, the Seiberg-Witten (SW) curve associated to the underlying $\mathcal{N}=2$ theory is an N-sheeted covering of $\mathcal{C}$ and the same should be required to hold in the present context if we want this picture to make sense. It is easy to see that this property is not satisfied for generic values of the parameters: for fixed t the SW curve gives us N possible solutions for x, and the $\mathcal{N}=1$ curve provides N solutions for w. This generically leads to N^2 possible pairs (x,w). We should then add to this system the condition that w is fixed once we have chosen x and t (and of course x is fixed once w and t are chosen), which reduces the solutions to N possible pairs (x,w). Indeed, this is what the second equation in (2.13) implies. This simply translates into the requirement that w is a rational function of x and t.

Introducing the vector field ζ, we write the SW curve in terms of $v=\lambda(\zeta)$, where $\lambda = xdt$ is the SW differential, as

$$v^N = \sum_k v^{N-k} f_k(t),$$

where $f_k(t)$'s are meromorphic functions. The factorization condition for a generic rank N theory will be of the form

$$w = \frac{P_{N-1}(v)}{Q_{N-1}(v)}, \tag{2.16}$$

(of course an analogous equation holds with v and w interchanged) where $P_{N-1}(v)$ and $Q_{N-1}(v)$ are polynomials in v of degree $N-1$, whose coefficients are meromorphic functions on $\mathcal{C}$ (rational functions for theories on the sphere, elliptic functions for genus one theories and so on). We will see in section 5 that this leads directly to the well known factorization condition for SQCD.

3 $\mathcal{N}=1$ curve in A_1 theory

In this section we consider four-dimensional $\mathcal{N}=1$ gauge theories obtained from the A_1 theory in six dimensions (or compactification of two M5-branes). The gauge groups are always SU(2) in this case. From the generalized Hitchin system viewpoint (the first description in section 2.2.1), we have two Hitchin fields Φ and φ where the spectral curves can be put in the form, in terms of $v=\lambda(\zeta)$ and w,

$$v^2 = f_2(t), \qquad w^2 = V_2(t). \tag{3.1}$$

Nota that v is identified with the coordinate $v = x_4 + ix_5$ appearing in the M-theoretic geometry if we choose $\zeta = t\partial_t$ in the theory with the Type IIA brane realization as we will see below. Since the Hitchin fields commute, they must be proportional to each other (which is true only for the A_1 case). This implies $\varphi = F(t)\Phi$ which gives us the condition $w = F(t)v$ or

$$w^2 = F(t)^2 v^2. \tag{3.2}$$

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Indeed, this condition can be obtained from the factorization condition in section 2.3. In this rank-one case the condition reads

$$w = \frac{A(t)v + B(t)}{C(t)v + D(t)}. \tag{3.3}$$

Since the two solutions of (3.1) at fixed t that we denote by $(v, \tilde{v})$ and $(w, \tilde{w})$ differ by a sign, we get

$$\frac{-A(t)v - B(t)}{C(t)v + D(t)} = -w = \tilde{w} = \frac{A(t)\tilde{v} + B(t)}{C(t)\tilde{v} + D(t)} = \frac{B(t) - A(t)v}{D(t) - C(t)v}. \tag{3.4}$$

We can consider the analogous sequence of equalities for the inverse relation $v = (B(t) - D(t)w)/(C(t) - A(t)w)$. Equating the leftmost and rightmost terms we find $B(t)D(t) = A(t)C(t)v^2$ and $A(t)B(t) = C(t)D(t)w^2$. These in turn imply that w^2/v^2 is a square of a function[2] (3.2). Thus we refer to (3.2) as factorization condition below.

In this case we can turn on only mass terms for the adjoint chiral fields (quadratic superpotentials), so the ratio w/v tends to a constant (possibly zero) at all the punctures. This is the boundary condition we have to impose for $F(t)$ at the punctures. The value of the mass parameters for the adjoint fields in the various gauge groups (in a given weakly coupled limit) will be simply encoded in the difference between the values of $F(t)$ at the punctures. Indeed, since $F(t)$ is not constant, it has poles. To avoid singularities for w away from the punctures, which indeed should not be there, the poles should be located at the zeroes of v. In order for this to be possible, $f_2(t)$ should have double zeroes. This is the way in which the restriction on the Coulomb branch coordinates appears in this description.

3.1 SU(2) SQCD

We will now show how one can use the above results to determine the curve for $\mathcal{N} = 1$ theory obtained by the mass deformation of the adjoint chiral field of $\mathcal{N} = 2$ SU(2) gauge theory with N_f fundamental hypermultiplets. These theories have a brane realization in Type IIA string theory. While the existence of the brane realization is not needed for our approach, it is helpful to identify each vacuum of $\mathcal{N} = 1$ theory with the brane configuration. Thus, we shortly review the Type IIA brane set-up [1, 20–22].

We consider D4-branes extended along $x_0, \ldots, x_3, x_6$ directions and NS5-branes extended along $x_0, \ldots, x_3, x_4, x_5$ directions. Two D4-branes stretched between two NS5-branes induce at the low energy four-dimensional $\mathcal{N} = 2$ SU(2) SYM theory on the world-volume. In the following we denote the NS5-branes as NS_1 and NS_2 below, as in figure 1. The addition of a D6-brane extended along $x_0, \ldots, x_3, x_7, x_8, x_9$ directions between the NS5-branes in the x_6-direction corresponds to the inclusion of a fundamental hypermultiplet. We always move this D6-brane to the left of NS_1- or the right of NS_2-brane, which produces the D4-branes stretched between the NS5- and D6-branes. Finally, in order to include the quadratic superpotential of the adjoint field we rotate one NS5-brane to w-direction, where $w = x_8 + ix_9$.

[2] If some of the polynomials are zero, making these two equations trivial, it is easy to see that either $B(t) = C(t) = 0$ or $A(t) = D(t) = 0$ and this leads directly to the conclusion in any case.

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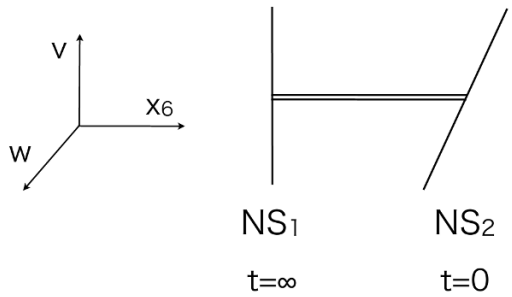

Figure 1. The rotation of the NS_2-brane. The coordinates are $v = x_4 + ix_5$, $w = x_8 + ix_9$ and $t = e^{-(x_6+ix_{10})}$.

When $N_f \geq 2$, there is a Higgs branch in the corresponding $\mathcal{N} = 2$ theory. As in [23], generically the Higgs branch is divided into the non-baryonic branches which are labeled by r with $r = 1, \ldots, [N_f/2]$ and the baryonic branch. (The baryonic branch exists when $N_f \geq N$.) These branches intersect with the Coulomb branch at the (non-)baryonic branch roots which are submanifolds in the Coulomb branch. On these submanifolds there are loci where mutually-local massless particles appear. We will see that these become vacua of $\mathcal{N} = 1$ theory after the mass deformation.

3.1.1 SU(2) SYM theory

The M-theory curve of $\mathcal{N} = 2$ SYM theory (which is identified with the SW curve [24]) is

$$\Lambda^2 t^2 - P_2(v)t + \Lambda^2 = 0, \tag{3.5}$$

where $P_2 = v^2 - u$ and u is the Coulomb moduli parameter. The SW differential is given by $\lambda = vdt/t$. The curve can be written as

$$v^2 = \Lambda^2 t + u + \frac{\Lambda^2}{t}, \tag{3.6}$$

as a double cover of the base sphere parametrized by t. The quadratic differential $\theta_2 = f_2(dt/t)^2$ has two irregular singularities of degree 3 at $t = 0$ and $t = \infty$. Note that our choice of the vector field is $\zeta = t\partial_t$ and $v = \lambda(\zeta) = xt$.

Let us rotate the NS_2-brane at $t = 0$. This means that the $\mathcal{N} = 1$ curve has the same behavior at $t = 0$ as the SW curve and subleading at infinity (in order to avoid a global rotation of the brane system). In particular we require $w \to \mu v$ at $t = 0$. This requirement leads to the curve $w^2 = \frac{(\mu\Lambda)^2}{t} + a$, where a is a constant. In principle we can allow a constant term linear in w but this can of course be eliminated by a redefinition of w and a. The curve is already in the form $w^2 = V_2(t)$, so we can use (3.2)

$$\frac{w^2}{v^2} = F^2(t) = \frac{at + (\mu\Lambda)^2}{\Lambda^2 t^2 + ut + \Lambda^2}. \tag{3.7}$$

The above expression can be the square of a rational function only if $a = 0$ and $u = \pm 2\Lambda^2$. Indeed, these are the points where the $\mathcal{N} = 2$ curve degenerates. The $\mathcal{N} = 1$ curve is

$$w^2 = \frac{(\mu\Lambda)^2}{t}, \tag{3.8}$$

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implying that $V_2(t)$ has a simple pole at $t = 0$ where we have rotated the NS5-brane. This $\mathcal{N} = 1$ curve is sensitive only to the rotated puncture. The singularity of V_2 expresses that this puncture is obtained from the $\mathcal{N} = 2$ irregular puncture with degree 3. This agrees with the one obtained in appendix (A.6) following the method in [3].

The Dijkgraaf-Vafa curve follows from these: by substituting (3.8) into the SW curve with $u = \pm 2\Lambda^2$, we obtain the curve $w^2 - \mu v w + \mu S = 0$ where $S = \pm\mu\Lambda^2$. This is identified with the DV curve: S is the gluino condensate. We can see that by the one-loop matching this is indeed the correct value $S = \pm\Lambda^3_{\mathcal{N}=1}$ where $\Lambda_{\mathcal{N}=1}$ is the dynamical scale of the theory after integrating out the massive adjoint fields.

3.1.2 SU(2) $N_f = 1$

We next consider the $N_f = 1$ case which is realized by the addition of a D6-brane. Here we put the D6-brane to the right of the NS_2-brane. The SW curve is

$$\Lambda^2 t^2 - t P_2(v) + \Lambda(v + m) = 0, \tag{3.9}$$

where m is the mass parameter of the quark field. This leads to

$$v^2 = \frac{4\Lambda^2 t^3 + 4ut^2 + 4\Lambda m t + \Lambda^2}{4t^2}. \tag{3.10}$$

Note that we have shifted v such that the linear term disappears. Thus the quadratic differential has singularities of degree 4 at $t = 0$ and of degree 3 at $t = \infty$.

Rotating the NS_2-brane at $t = 0$ as before the ansatz of the $\mathcal{N} = 1$ curve is $w^2 = \frac{(\mu\Lambda)^2}{4t^2} + \frac{b}{t} + a$, where a and b are constants. We then require

$$F^2(t) = \frac{at^2 + bt + (\mu\Lambda)^2}{4\Lambda^2 t^3 + 4ut^2 + 4\Lambda m t + \Lambda^2}.$$

This equation imposes the constraint $a = 0$, so the numerator becomes of the form $b(t + c)$ where $bc = (\mu\Lambda)^2$, and implies that the denominator factorizes as $4\Lambda^2(t + c)(t + d)^2$. We thus find $c = 1/4d^2$ and d is determined by the cubic equation $d^3 - (m/\Lambda)d + 1/2 = 0$. The Coulomb moduli is determined to be $u = \frac{\Lambda^2}{4d^2}(-3 + 8(m/\Lambda)d)$. We thus find three vacua. When the mass parameter is set to zero, the positions of the vacua in the Coulomb branch are $\mathbb{Z}_3$ symmetric, as expected. Finally the $\mathcal{N} = 1$ curve is given by

$$w^2 = \frac{(\mu\Lambda)^2}{4t^2} + \frac{4d^2(\mu\Lambda)^2}{t}, \tag{3.11}$$

which agrees with (A.10). Note that the singularity of V_2 at the rotated puncture $t = 0$ is a double pole, and is different from that of SU(2) SYM case. This is because the rotated puncture is from the $\mathcal{N} = 2$ irregular one with degree 4.

To check that these are the correct points on the Coulomb branch, we can e.g. rewrite the curve in the form [25–27]

$$y^2 = (v^2 - u)^2 - 4\Lambda^{4-N_f} \prod_i (v - m_i) \tag{3.12}$$

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(for $N_f = 3$ instead of $v^2 - u$ we have $v^2 + \Lambda v - u$) and check that these are the only points at which the discriminant vanishes (without setting $\Lambda = 0$). At these points the curve degenerates to a sphere, corresponding to singularities of the Coulomb branch of $\mathcal{N} = 2$ theory which are not lifted by the $\mathcal{N} = 1$ deformation. In the massless case for example, the discriminant is $\Lambda^6(27\Lambda^6 + 16u^3)$, which precisely vanishes at the three points found above setting $m = 0$.

Since it is not trivial in this case, let us explain how one can extract the Dijkgraaf-Vafa curve. The easiest way is to use the factorization condition to write t in terms of w and v, and then plug the result in the $\mathcal{N} = 1$ curve to get an equation in w and v only. This procedure works, provided we use the $\mathcal{N} = 2$ curve written in the parametrization as in (3.9) (and redefine w accordingly). In the present case this is accomplished by the redefinitions

$$v' = v + \frac{\Lambda}{2t}; \quad w' = w + \frac{\Lambda}{2t}.$$

We then find

$$w'^2 - w'\left(\mu v - \frac{\mu\Lambda}{2d}\right) + \mu^2\Lambda^2 d = 0, \tag{3.13}$$

which is precisely the curve we were looking for. Without this redefinition we would have found a cubic equation in w.

3.1.3 SU(2) $N_f = 2$

Let us consider SU(2) gauge theory with $N_f = 2$. Note that there is Higgs branch in the corresponding $\mathcal{N} = 2$ theory: a non-baryonic branch and a baryonic branch. The latter exists only in the massless flavor case. There are three ways to construct this theory in Type IIA: the first is to place two D6-branes to the right of the NS_2-branes; the second is to put one D6 to the right of the NS_2 and one D6 to the left of the NS_1; the third is to put two D6-branes to the left of the NS_1. Though the gauge theory and $\mathcal{N} = 1$ vacua obtained from these should be the same, the $\mathcal{N} = 1$ curve looks different. We will see these in the following.

The first realization. The SW curve is given by

$$\Lambda^2 t^2 - P_2(v)t + (v + m_1)(v + m_2) = 0. \tag{3.14}$$

This leads to

$$v^2 = \frac{m_+^2}{(t-1)^2} + \frac{\Lambda^2 t^2 + ut + m_1 m_2}{t - 1}, \tag{3.15}$$

where we defined $m_\pm = (m_1 \pm m_2)/2$. It is easy to see that the quadratic differential $\theta_2 = f_2(dt/t)^2$ has singularities of degree 2 at $t = 0, 1$ and an irregular one at $t = \infty$ of degree 3. We refer to the former singularity of degree less than or equal to 2 as regular. The SW differential is $v\frac{dt}{t}$, thus the residues at regular punctures $t = 0$ and $t = 1$ are $\pm m_-$ and $\pm m_+$ respectively.

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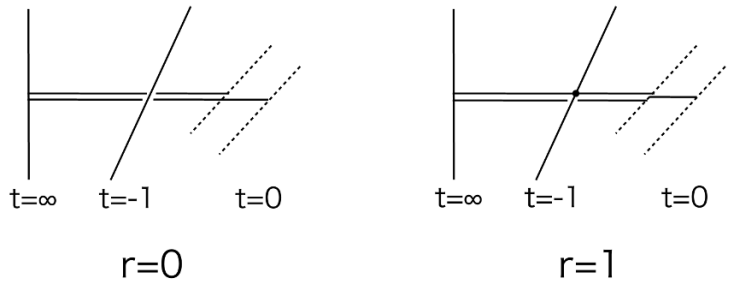

Figure 2. The first realization of SU(2) gauge theory with $N_f = 2$. *Left*: $r = 0$ case (roots of baryonic branch). The NS_2-brane is detached from the rest and is rotated to w directions. The dashed lines denote the D6-branes. *Right*: $r = 1$ case (roots of non-baryonic branch). The NS_2-brane is still attached to D4-branes.

We will consider the rotation of the NS_2-brane as above. Let us set the mass parameter m_- to be zero, namely $m_1 = m_2 = m_+ =: m$ for simplicity. The ansatz of the $\mathcal{N} = 1$ curve is $w^2 = \frac{\mu^2 m^2}{(t-1)^2} + \frac{a}{t-1} + b$. Therefore we require

$$F(t)^2 = -\frac{(t-1)^2 b + (t-1)a + \mu^2 m^2}{t(\Lambda^2 t^2(\Lambda^2 - u)t - u + m^2)}. \tag{3.16}$$

One solution is $a = \mu^2 m^2$, $b = 0$ and $u = -\Lambda^2 \pm 2\Lambda m$. The $\mathcal{N} = 1$ curve is written as

$$w^2 = \frac{\mu^2 m^2 t}{(t-1)^2}. \tag{3.17}$$

At this locus of the Coulomb branch the SW curve degenerates. Thus this is the $r = 0$ vacuum considered in [3], as reviewed in appendix A.1.3. Note that when we set $m = 0$, the $\mathcal{N} = 1$ curve is trivialized, signaling that the NS_2-brane is detached from the rest, as in figure 2. Indeed the baryonic branch of the massless theory intersects with the Coulomb branch at the locus obtained above. The detachment happens when the residue of the SW differential at $t = 1$ vanishes. We will see the same phenomenon later in the SU(2) with $N_f = 4$ case.

The other solution is $a = \mu\Lambda^2$, $b = 0$ and $u = m^2$, where $F(t) = \mu$. The $\mathcal{N} = 1$ curve is written as

$$w^2 = \frac{\mu^2 m^2}{(t-1)^2} + \frac{\mu^2 \Lambda^2}{t-1}. \tag{3.18}$$

See (A.19) for comparison. This corresponds to the $r = 1$ vacuum, namely the deformation at the root of the non-baryonic branch. Indeed, one can easily see from the $\mathcal{N} = 1$ curve that when $t \to \infty$ we get $w = 0$, when $t \to -1$ we get $w \to \infty$, and when $t \to 0$ w tends to two values $w = w_\pm$. These are explained by the figure 2. In particular, the two values $w_\pm$ are interpreted as the positions of D4-branes in w-direction at $t = 0$. Notice that the NS_2 cannot detach from the rest because of the s-rule.

As a consistency check, one can check that the discriminant of the curve (3.12) with $N_f = 2$ and $m_{1,2} = m$ has simple zeros at $u = -\Lambda^2 \pm 2\Lambda m$ and a double zero at $u = m^2$. These are exactly the vacua we found above.

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The second realization. Let us then consider the second realization. The SW curve is

$$\Lambda(v+m_1)t^2 + P_2(v)t + \Lambda(v+m_2) = 0. \tag{3.19}$$

We here consider the massless theory for simplicity. This leads to

$$v^2 = \frac{\Lambda^2}{4t^2}(t^4 + (2+4u/\Lambda^2)t^2 + 1). \tag{3.20}$$

There are two irregular punctures of degree 4 at $t=0$ and $t=\infty$.

We would like to rotate the brane at $t=0$. From that the boundary condition at $t=0$ is $w \to \mu v$ and $v^2 \to \frac{\Lambda^2}{4t^2}$, the $\mathcal{N}=1$ curve is of the following form $w^2 = \frac{(\mu\Lambda)^2}{4t^2} + \frac{b}{t} + a$. Thus the factorization condition is

$$F(t)^2 = \frac{4}{\Lambda^2}\frac{at^2 + bt + (\mu\Lambda)^2/4}{t^4 + (2+4u/\Lambda^2)t^2 + 1}. \tag{3.21}$$

There are two possibilities: one with the denominator factorized as $(t-1)^2(t+1)^2$, meaning $u = -\Lambda^2$, and the numerator factorized as $a(t\pm 1)^2$, meaning $a = (\mu\Lambda)^2/4$ and $b = \pm(\mu\Lambda)^2/2$. In this case,

$$w^2 = \frac{(\mu\Lambda)^2}{4}(1\pm 1/t)^2, \qquad F(t) = \frac{\mu}{(\pm t+1)}. \tag{3.22}$$

Note that the r.h.s. of the $\mathcal{N}=1$ curve is a square. The curve is identified with (A.22) (after the shift of w). Thus this solution of the factorization condition corresponds to the $r=0$ vacuum, and represents that the brane system is separated into two part as in figure 3.

The other possibility is $a=b=0$ and $u=0$ or $u=-\Lambda^2$. The latter corresponds to the previous solution after the shift of w-coordinate, thus we do not consider this. The $\mathcal{N}=1$ curve is calculated as

$$w^2 = \frac{(\mu\Lambda)^2}{4t^2}, \qquad F(t) = \frac{\mu}{t^2+1} \quad \text{or} \quad -\frac{\mu}{t^2-1}. \tag{3.23}$$

This agrees with the curve (A.23), namely the $r=1$ vacuum. See figure 3. The singularity of V_2 of both (3.22) and (3.23) at $t=0$ is indeed the same as that in section 3.1.2: the rotated puncture is of degree 4.

The third realization. The SW curve is in this case

$$(v+m_1)(v+m_2)t^2 - P_2(v)t + \Lambda^2 = 0. \tag{3.24}$$

Again we consider the massless case where

$$v^2 = -\frac{tu+\Lambda^2}{t(t-1)}. \tag{3.25}$$

There are two regular punctures at $t=1,\infty$ and an irregular one at $t=0$ of degree 3.

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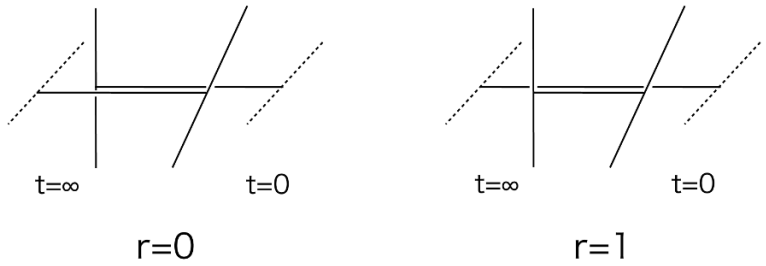

Figure 3. The second realization of SU(2) gauge theory with $N_f = 2$. *Left*: $r = 0$ case (roots of baryonic branch). The NS_2-brane (and left D6-brane stretched by a D4-brane) is detached from the rest and is rotated to w directions. *Right*: $r = 1$ case (roots of non-baryonic branch). The NS_2-brane is attached to a D4-brane.

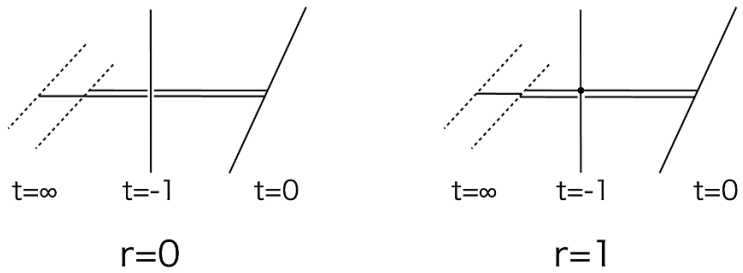

Figure 4. The third realization of SU(2) gauge theory with $N_f = 2$. *Left*: $r = 0$ case (roots of baryonic branch). *Right*: $r = 1$ case (roots of non-baryonic branch).

Let us rotate the NS_2-brane at $t = 0$. The ansatz for the $\mathcal{N} = 1$ curve is $w^2 = \frac{(\mu\Lambda)^2}{t} + a$. This is because the boundary condition at $t = 0$ is $w \to \mu v$ and $v^2 \to \Lambda^2/t$. Thus, we have

$$F^2 = \frac{(at + (\mu\Lambda)^2)(t-1)}{tu + \Lambda^2}. \tag{3.26}$$

Solving the factorization condition leads to two solutions: the one with $a = 0$ and $u = -\Lambda^2$ corresponding to the $r = 0$ vacuum

$$w^2 = \frac{(\mu\Lambda)^2}{t}, \quad F(t) = \mu. \tag{3.27}$$

the one with $a = -(\mu\Lambda)^2$ and $u = 0$ corresponding to the $r = 1$ vacuum

$$w^2 = -\frac{(\mu\Lambda)^2}{t} - (\mu\Lambda)^2, \quad F(t) = \mu(t+1). \tag{3.28}$$

By shifting w-coordinate, we can see that these two curves agree with (A.24) and (A.25) respectively. See figure 4. The singularity of V_2 at $t = 0$ is a simple pole and coincides with that of SU(2) SYM case, since the rotated puncture is of degree 3.

3.1.4 SU(2) $N_f = 3$

Let us then turn to the $N_f = 3$ case. There are two realizations of this theory in Type IIA: the first is to put two D6-branes to the right of the NS_2 and one D6-brane to the left

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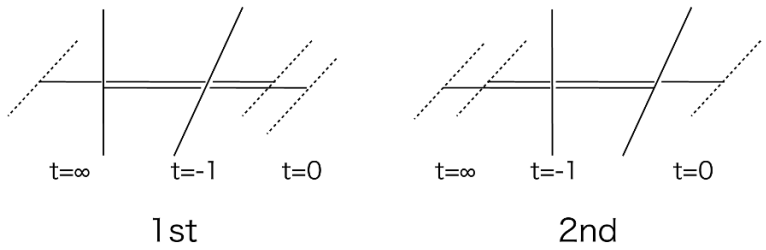

Figure 5. *Left*: the first realization of SU(2) gauge theory with $N_f = 3$. The $r = 1$ non-baryonic branch root is the same as the baryonic branch root. *Right*: the second realization.

of the NS_1, the second is the opposite to the first case, as in figure 5. We will focus only on the second option below.

Note that there is a non-baryonic branch with $r = 1$ in the corresponding $\mathcal{N} = 2$ theory. In the massless theory the root of the non-baryonic branch is the same as that of the baryonic branch.

The SW curve is given by

$$(v + m_1)(v + m_2)t^2 - P_2(v)t + \Lambda(v + m_3) = 0. \tag{3.29}$$

We will for simplicity consider the case $m_1 = -m_2 = m$ and $m_3 = 0$, where the curve can be rewritten as

$$v^2 = \frac{4m^2t^3 - 4ut^2 + \Lambda^2 t - \Lambda^2}{4t^2(t-1)}. \tag{3.30}$$

In this parametrization the mass terms lead to a double pole at infinity. There is also the irregular singularity of degree 4 at $t = 0$.

Rotating the puncture at $t = 0$, the $\mathcal{N} = 1$ curve is of the form: $w^2 = \frac{at^2+bt+c}{4t^2}$. We thus find the factorization condition

$$F^2(t) = \frac{(at^2 + bt + c)(t-1)}{4m^2t^3 - 4ut^2 + \Lambda^2 t - \Lambda^2}.$$

One solution corresponds to the fact that the denominator has the form $4m^2(t-\alpha)^2(t-\beta)$. The constants α and β are determined by $4m^2\alpha^3 - \Lambda^2\alpha + 2\Lambda^2 = 0$ and $\beta = \Lambda^2/4m^2\alpha^2$. In terms of α, we find $u = (\Lambda^2 + 8m^2\alpha^3)/4\alpha^2$. We thus get three vacua. The above factorization of the denominator in turn implies that $at^2 + bt + c = a(t-\beta)(t-1)$. By considering the asymptotics at zero as before, we can fix $a = 4m^2\mu^2\alpha$. Note that in the massless case $\alpha = 2$, $u = \Lambda^2/16$, $a = 0$, $c = -b = \mu^2\Lambda^2$. That is the $\mathcal{N} = 1$ curve is

$$w^2 = \frac{\mu^2\Lambda^2}{4t^2} - \frac{\mu^2\Lambda^2}{4t}. \tag{3.31}$$

The V_2 has a double pole at $t = 0$ representing the behavior of the rotated puncture of degree 4. This is the same as that in subsection 3.1.2. Also the curve agrees with (A.33). Therefore this denotes the $r = 0$ vacuum.

The other solution can be obtained by requiring the denominator to be a multiple of $t - 1$, which imposes the constraint $u = m^2$. $F^2(t)$ then reduces to $\frac{at^2+bt+c}{4m^2t^2+\Lambda^2}$. This can be

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the square of a rational function only if the numerator is a multiple of the denominator. Thus we obtain $a = 4m^2\mu^2$, $b = 0$ and $c = \mu^2\Lambda^2$, namely the curve is

$$w^2 = \frac{\mu^2\Lambda^2}{4t^2} + m^2\mu^2. \tag{3.32}$$

Notice that in the massless case the factorization condition and asymptotics at the NS5-branes do not allow to fix all the parameters in the $\mathcal{N} = 1$ curve. Once the mass parameters are turned on, this is no longer the case. This may corresponds to the $r = 1$ vacuum as depicted in figure 5.

As a consistency check we can rewrite the curve (3.12) as $y^2 = (v^2 + \Lambda v - u)^2 - 4\Lambda(v^3 - m^2 v)$ and check that the discriminant has degree five in u, a double zero at $u = m^2$ and other three simple zeros located precisely at the points described above. When m is sent to infinity the vacuum at $u = m^2$ disappears and the other three are at $u_i \sim \omega_3^i (m^2\Lambda)^{2/3}$, where ω_3 is the third root of unity. This matches precisely the behavior of the theory with $N_f = 1$. Note that $m^2\Lambda$ is precisely proportional to the cube of the dynamical scale of the theory with $N_f = 1$ by the one-loop matching.

3.1.5 SU(2) $N_f = 4$

The SW curve of this theory is

$$(v + m_1)(v + m_2)t^2 - (1 + q)P_2(v)t + q(v + m_3)(v + m_4) = 0, \tag{3.33}$$

where we have chosen the coefficients in such a way that the punctures are at $t = 0, q, 1$ and ∞. All the punctures are regular. For simplicity we analyze the case $m_1 = -m_2$ and $m_3 = -m_4$. We can write the SW curve as

$$v^2 = \frac{m_2^2 t^2 - (1 + q)tu + qm_4^2}{(t - 1)(t - q)}. \tag{3.34}$$

With this choice of mass parameters the quadratic differential $\theta_2 = f_2(dt/t)^2$ has simple poles at 1 and q and double poles at zero and infinity whose coefficients are m_4^2 and m_2^2. We can rotate either at q or at 1. Let us consider the first option, so the $\mathcal{N} = 1$ curve has the form $w^2 = \frac{at+b}{t-q}$. The factorization condition is

$$F^2(t) = \frac{(at + b)(t - 1)}{m_2^2 t^2 - (1 + q)tu + qm_4^2}.$$

Notice that q and m_i's are parameters and should not be tuned. The only parameter we can constrain is the coordinate on the Coulomb branch.

Requiring the denominator to be a square we find the two solutions $u = \pm 2\sqrt{q}m_2m_4/(1 + q)$. We are then forced to set $b = -a$. Imposing then $w \sim \mu v$ at $t = q$ we find

$$a = \frac{\mu^2}{(q - 1)^2}(m_2 \pm \sqrt{q}m_4)^2. \tag{3.35}$$

The $\pm$ depends on which solution we choose for u. Notice that a diverges in the limit $q \to 1$.

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We can also require that the denominator is a multiple of $t-1$. This implies then

$$u = \frac{m_2^2 + q m_4^2}{q+1}; \quad a = m_2^2 \mu^2; \quad b = -q\mu^2 m_4^2.$$

This corresponds to detaching from the brane system the brane located at $t = 1$. This is signaled by the vanishing of the residue at $t = 1$ of the quadratic differential of the SW curve.

There is another solution: $a = b = 0$. As in the previous solution, we now see that this corresponds to detaching the NS5-brane at $t = q$. Since in this case the rotated brane is detached from the rest, the coordinate w will be trivial except at $t = q$. We thus expect the $\mathcal{N} = 1$ curve to develop two branches and become of the form $(t-q)w^2 = 0$. Indeed, this is precisely what we get if we multiply the curve written before by $t - q$ on both sides and then set $a = b = 0$. However, this solution will be consistent only if the residue of the pole at $t = q$ in the quadratic differential vanishes, which is true only for

$$u = \frac{q m_2^2 + m_4^2}{q+1}.$$

This solution must be included whenever we can eliminate the corresponding pole in the differential with a suitable choice of point in the Coulomb branch. It is thus the structure of the $\mathcal{N} = 2$ curve which dictates when this solution is allowed. This phenomenon will occur in rank one theories with regular punctures and we have already encountered it in the study of the $N_f = 2$ theory.

Rewriting the SW curve in the form

$$y^2 = (v^2 - u)^2 - \frac{4q}{(q+1)^2}(v^2 - m_2^2)(v^2 - m_4^2),$$

we can test as before our result checking that the above solutions are the only points in the Coulomb branch where the discriminant of the r.h.s. vanishes. It turns out that it is a polynomial of degree six in u, whose roots are precisely the four values found above. The last two solutions are actually double roots. With a more general choice of mass parameters they split. For large m_2 and m_4 only the first two solutions remain finite and correspond to the vacua of SU(2) SYM (once we take into account the one-loop matching condition). The other four run away to infinity as all the matter fields become infinitely massive. In the massless limit all the vacua merge at $u = 0$, in agreement with the expectation that the theory flows to an IR fixed point in this limit [28].

The S-dual frame. As we have just seen, the $\mathcal{N} = 1$ curve diverges when $q \to 1$. To approach this limit the most natural procedure is to change S-duality frame and consider a description of the theory in which the gauge group becomes weakly coupled. More precisely what we want to do is to analyze the mass deformation of the $\mathcal{N} = 2$ dual theory. How can we do that in the present framework? Indeed the first step is to rewrite properly the SW curve. We can e.g. consider the reparametrization of the sphere that interchanges the punctures at $t = 1, \infty$ leaving $t = 0$ fixed by $t \to t/(t-1)$. This is accompanied by the

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redefinition $q \to q/(q-1)$. The resulting curve is

$$\theta_2 = \frac{m_2^2(1-q)t + (t-1)[qm_4^2 - (1+q)tU]}{t^2(t-1)^2(t-q)}(dt)^2. \tag{3.36}$$

Note that the q here differs from the original q. Notice also that in bringing the original curve (3.34) to this form we have redefined the Coulomb branch coordinate as

$$(1+q)U = m_4^2 q + u(1-2q) + m_2^2(q-1). \tag{3.37}$$

The mass parameters are the same but now m_2 is the residue at $t = 1$. The limit $q \to 1$ now corresponds to a weak coupling limit.

The important point is that this reparametrization should be accompanied by a change in the vector field: instead of $t\partial_t$ we have to use $\zeta = t(t-1)\partial_t$, which fixes the points $t = 0, 1$. The SW curve written in terms of t and $y = v(t-1) = \lambda(\zeta)$ reads

$$y^2 = \frac{m_2^2(1-q)t + (t-1)[qm_4^2 - (1+q)tU]}{t-q}.$$

Rotating at q as before, we find the $\mathcal{N} = 1$ curve $w^2 = \frac{a}{t-q} + b$. The factorization condition now reads

$$F^2(t) = \frac{a + b(t-q)}{m_2^2(1-q)t + (t-1)[qm_4^2 - (1+q)tU]}.$$

One can now determine the $\mathcal{N} = 1$ points as before. The analysis is very similar to the previous case and we will not repeat it. Indeed, the set of points in the Coulomb branch which are not lifted coincides with that found before once we take into account all the redefinitions, as it should be. The relevant case is the analogue of the first solution studied previously, when the denominator of $F^2(t)$ becomes a square. This implies $b = 0$ and

$$U = \frac{(\sqrt{q}m_4 \pm \sqrt{q-1}m_2)^2}{q+1}.$$

Imposing the constraint $w \sim \mu v$, we get the $\mathcal{N} = 1$ curve

$$w^2 = \mu^2 \frac{((q-1)\sqrt{q}m_4 \pm q\sqrt{q-1}m_2)^2}{t-q}. \tag{3.38}$$

Now the divergence at $q = 1$ has disappeared! The divergent limit is instead $q \to \infty$, the weak coupling limit of the original description.

Using the recipe discussed before we can now easily extract the DV curve (write t in terms of v, w using $F(t)$ and plug it in the $\mathcal{N} = 1$ curve). We find as expected $w^2 - \mu wv + \mu S = 0$, where S is the gluino condensate in the present duality frame

$$S = \mu(\sqrt{q}m_4 \pm \sqrt{q-1}m_2)((q-1)\sqrt{q}m_4 \pm q\sqrt{q-1}m_2).$$

Notice that it vanishes for $q = 1$, confirming that the gauge group becomes weakly coupled in this limit.

From this example it should be clear how to approach the $\mathcal{N} = 1$ deformation of various duality frames in more general cases as well (at least on the sphere): the recipe is simply to rewrite the $\mathcal{N} = 2$ curve in the proper way and change at the same time the location of the zeroes for the vector field. The systematic analysis of this problem will be done elsewhere.

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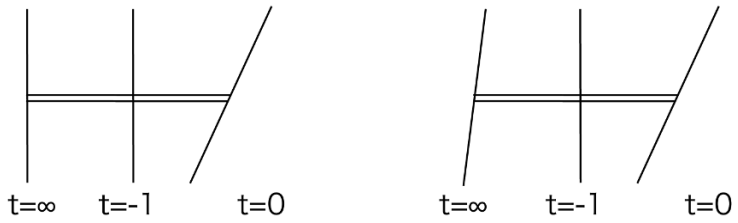

Figure 6. *Left*: the rotation of right NS5-brane (NS_3-brane). *Right*: the rotation of the left and right NS5-branes (NS_1 and NS_3-branes).

3.2 SU(2)^2 quiver

As a next example, let us consider $\mathcal{N} = 2$ SU(2) $\times$ SU(2) gauge theory coupled to a bifundamental hypermultiplet. Here we focus on the massless hypermultiplet. The Seiberg-Witten curve is

$$\Lambda_1^2 t^3 + P_{2,1}(v)t^2 + P_{2,2}(v)t + \Lambda_2^2 = 0, \tag{3.39}$$

where $P_{2,i} = v^2 - u_i$ and u_i are the Coulomb moduli parameters of two SU(2) gauge groups. The curve can be written as

$$v^2 = -\frac{\Lambda_1^2 t^3 + u_1 t^2 + u_2 t + \Lambda_2^2}{t(t+1)}. \tag{3.40}$$

There are two irregular punctures of degree 3 at $t = 0$ and $t = \infty$, and one regular puncture at $t = -1$. This theory can be realized by three NS5-branes stretched by two-D4-branes, as in figure 6. We refer to NS5-branes at $t = \infty, -1, 0$ as NS_1-, NS_2-, NS_3-branes respectively.

Let us first consider the case where only the puncture corresponding to NS_3-brane is rotated, as in figure 6. This induces the mass term of the adjoint field of one SU(2) gauge group (coming from the right two D4-branes). The boundary condition fixes $w^2 = -\frac{(\mu\Lambda_2)^2}{t} + a$. Thus the factorization condition is

$$F(t)^2 = \frac{(t+1)(-at + (\mu\Lambda_2)^2)}{\Lambda_1^2 t^3 + u_1 t^2 + u_2 t + \Lambda_2^2}. \tag{3.41}$$

The above expression will be a square only if $a = 0$ and the denominator is factorized as $(t+1)(\Lambda_1 t \pm \Lambda_2)^2$. This means $u_1 = \Lambda_1^2 \pm 2\Lambda_1\Lambda_2$ and $u_2 = \Lambda_2^2 \pm 2\Lambda_1\Lambda_2$. Therefore

$$w^2 = -\frac{(\mu\Lambda_2)^2}{t}, \qquad F(t) = \frac{\mu\Lambda_2}{\pm\Lambda_1 t + \Lambda_2}. \tag{3.42}$$

Again V_2 has a simple pole at $t = 0$ representing the rotation of the irregular puncture of degree 3.

We can easily get the DV curve from these. Due to the factorization condition the Seiberg-Witten curve can be written as $v^2 = -(\Lambda_1 t \pm \Lambda_2)^2/t$. By substituting the $\mathcal{N} = 1$ curve, we get

$$v^2 = \left(\frac{-\Lambda_1\Lambda_2^2\mu^2/w^2 \pm \Lambda_2}{\mu\Lambda_2/w}\right)^2. \tag{3.43}$$

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This implies

$$w^2(w-\mu v) \mp \mu^2 \Lambda_1 \Lambda_2 w = 0. \tag{3.44}$$

This is the form of the DV curve which we will see in section 5 with $f_0 = \mp\mu^2\Lambda_1\Lambda_2$ and $f_1 = 0$.

Next, we turn on the superpotential for both groups by rotating the NS_1 and NS_3-branes. We choose the boundary conditions $w \to \mu_1 v$ at $t = \infty$ and $w \to \mu_2 v$ at $t = 0$. This corresponds to turning on the superpotentials $W_1 = -(\mu_1/2)\mathrm{Tr}\Phi_1^2$ and $W_2 = (\mu_2/2)\mathrm{Tr}\Phi_2^2$. Thus, the ansatz of the $\mathcal{N} = 1$ curve is $w^2 = -t(\mu_1\Lambda_1)^2 - \frac{(\mu_2\Lambda_2)^2}{t} - a$. Therefore the factorization condition is

$$F^2 = \frac{(t+1)((\mu_1\Lambda_1)^2 t^2 + at + (\mu_2\Lambda_2)^2)}{\Lambda_1^2 t^3 + u_1 t^2 + u_2 t + \Lambda_2^2}. \tag{3.45}$$

One solution is that the numerator becomes $(t+1)^2((\mu_1\Lambda_1)^2 t + (\mu_2\Lambda_2)^2)$, which in turn implies that the denominator should include a factor $(\mu_1\Lambda_1)^2 t + (\mu_2\Lambda_2)^2$. We thus find

$$u_1 = \frac{\mu_2^2}{\mu_1^2}\Lambda_2^2 + 2\frac{\mu_1}{\mu_2}\Lambda_1^2, \qquad u_2 = \frac{\mu_1^2}{\mu_2^2}\Lambda_1^2 + 2\frac{\mu_2}{\mu_1}\Lambda_2^2. \tag{3.46}$$

Notice that this vacuum runs away to infinity if we set to zero one of the two mass parameters. Consequently, it exists only if we turn on both superpotentials. The $\mathcal{N} = 1$ curve is then

$$w^2 = -\frac{(t+1)(\mu_1^2\Lambda_1^2 t + \mu_2^2\Lambda_2^2)}{t}; \quad \frac{w}{v} = F(t) = \frac{\mu_1\mu_2(t+1)}{\mu_2 t + \mu_1}. \tag{3.47}$$

We can now easily extract the DV curve: using the second equation we can write t in terms of v and w and plugging this back into the $\mathcal{N} = 1$ curve we find a cubic equation for w. After the redefinition $y = w - (\mu_1 + \mu_2)v/3$, it becomes precisely the curve we will find in section 5 with $g_1(v) = g_2(v) = 0$ and

$$f_1(v) = -\mu_1 S_1 = (\mu_2 - \mu_1)\frac{\Lambda_1^2\mu_1^2}{\mu_2}; \quad f_2(v) = \mu_2 S_2 = (\mu_1 - \mu_2)\frac{\Lambda_2^2\mu_2^2}{\mu_1},$$

where S_1 and S_2 are the two gluino condensates.

When $\mu_1 = \mu_2 =: \mu$, namely the NS_1 and NS_3-branes are aligned, the $\mathcal{N} = 1$ curve becomes

$$w^2 = -t(\mu\Lambda_1)^2 - \frac{(\mu\Lambda_2)^2}{t} - \mu^2(\Lambda_1^2 + \Lambda_2^2), \qquad F(t) = \mu. \tag{3.48}$$

This solution corresponds to the locus on the Coulomb branch where the SW curve degenerates to

$$v^2 = \frac{(t+1)(\Lambda_1^2 t + \Lambda_2^2)}{t}, \tag{3.49}$$

but is still of genus-one. With this choice of mass parameters the NS_2 brane is detached.

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We have another solution imposing that the numerator factorizes as $(t+1)(\mu_1\Lambda_1 t \pm \mu_2\Lambda_2)^2$. Consequently the denominator should be $(t+1)(\Lambda_1 t \pm \Lambda_2)^2$, which means $u_1 = \Lambda_1^2 \pm 2\Lambda_1\Lambda_2$ and $u_2 = \Lambda_2^2 \pm 2\Lambda_1\Lambda_2$. The $\mathcal{N}=1$ curve is in this case

$$w^2 = -t(\mu_1\Lambda_1)^2 - \frac{(\mu_2\Lambda_2)^2}{t} \mp 2\mu_1\mu_2\Lambda_1\Lambda_2, \qquad F(t) = \frac{\mu_1\Lambda_1 t \pm \mu_2\Lambda_2}{\Lambda_1 t \pm \Lambda_2}. \tag{3.50}$$

This solution reduces to the one found previously when we just rotated one puncture if we set to zero one of the mass parameters (in our case μ_1). In this case the SW curve completely degenerates to a sphere, indicating that the theory is in the confining phase.

We saw that there are two different kinds of solutions above. In particular when we choose the mass parameters as $\mu_1 = \mu_2$, the theory below this energy scale is just $\mathcal{N}=1$ $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge theory with two bifundamental chiral multiplets which was studied in [29]. Note that by integrating out the massive adjoints we get $(B\tilde{B})^2$ terms where B is the bifundamental field. But these terms are cancelled each other by the above choice of the masses. There is a Coulomb branch parametrized by the moduli consisting of the bifundamental fields. The curve on the Coulomb branch was found in [29] to be

$$w^2 = -t\Lambda_1'^2 - \frac{\Lambda_2'^2}{t} - v_2, \tag{3.51}$$

where v_2 is the moduli parameter and $\Lambda_i' := \mu_i\Lambda_i$ $(i=1,2)$ are the dynamical scales of the theory after integrating out the adjoint chirals by one-loop matching (see [15] for convention). This clarifies what is the physical meaning of our $\mathcal{N}=1$ curve: the two solutions just described precisely coincide with the curve of Intriligator and Seiberg, for special choices of the moduli parameter as expected. We see that the second solution (3.50) is a special point on the Coulomb branch where the genus-one $\mathcal{N}=1$ curve degenerates.

3.3 SU(2) theories in class $\mathcal{S}$

Let us now give another general argument to find curves of $\mathcal{N}=1$ mass deformation of $\mathcal{N}=2$ class $\mathcal{S}$ theory, making use of the second perspective given in section 2.2.2. This will be applied to analyze $\mathcal{N}=1^*$ theory.

The Hitchin field Φ is a section of $K_{\mathcal{C}}\otimes\mathrm{End}\mathcal{E}$ with $\mathcal{E} = K_{\mathcal{C}}^{-1/2}\oplus K_{\mathcal{C}}^{1/2}$. This is specified by a meromorphic quadratic differential θ_2 on $\mathcal{C}$, with singularities at the punctures and with a suitable gauge choice reduces to

$$\Phi = \begin{pmatrix} 0 & 1 \\ \theta_2 & 0 \end{pmatrix} \quad .$$

In the $\mathcal{N}=1$ set-up, the equations (2.4) will take a quite simple universal form. Indeed, both ϕ_i are aligned in the Lie algebra. This implies that one can specify the curve

$$x_1^2 = R, \qquad x_1 x_2 = RP \tag{3.52}$$

where $R = \frac{1}{2}\mathrm{Tr}\phi_1^2$ is a section of $K_{\mathcal{C}}^2\mathcal{L}^{-2}$ and $RP = \frac{1}{2}\mathrm{Tr}\phi_1\phi_2$ is a section of $K_{\mathcal{C}}$. Let us choose then a gauge where

$$\phi_1 = \begin{pmatrix} 0 & \sigma^{-1} \\ \sigma R & 0 \end{pmatrix}$$

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is a section of $K_{\mathcal{C}}\mathcal{L}^{-1}$ in the adjoint of $\mathcal{E}$. The relation between ϕ_1 and the Hitchin field is then simply $\Phi = \sigma\phi_1$ which sets $\theta_2 = \sigma^2 R$.

The $\mathcal{N} = 2$ spectral curve has therefore the form

$$x^2 = \sigma^2 R,$$

which means that some of the zeros of the quadratic differential θ_2 get constrained to pair in couples to reproduce the square of the section σ. This implies the lowering of the genus of the SW curve and generates the $\mathcal{N} = 1$ vacua associated to the corresponding massless dyonic sectors.

The complete canonical transformation to get the rotated system would read[3]

$$x = \sigma x_1 \quad \text{and} \quad w = \tau x_2,$$

where $\sigma\tau = \partial_t z$, z being an appropriate coordinate on $\mathcal{C}$. The complete $\mathcal{N} = 1$ curve is then given by

$$x^2 = \sigma^2 R \quad \text{and} \quad xw = PR\partial_t z.$$

The usual form of the latter as a generalized Konishi anomaly equation is obtained then by introducing the scalar variable $v = i_\zeta \lambda$ — where ζ is a suitable vector field setting the S-duality frame as discussed, for a particular example, in section 3.1.5 — and by resolving the explicit t dependence using the degenerate SW curve. It is not granted that this last step can always be performed in a closed form.

3.3.1 $\mathcal{N} = 1^*$ SU(2) gauge theory

As an example of the above construction, let us consider the $\mathcal{N} = 1^*$ SU(2) gauge theory. The base curve is therefore elliptic, namely a complex one dimensional torus with a distinguished point. The canonical bundle of the torus T^2_τ is trivial and admits three square roots, namely the three spin structures. Let $\mathcal{L}_\nu$, with $\nu = 1, 2, 3$ denote the three corresponding line bundles. A meromorphic section of $\mathcal{L}_\nu$ is $\mathtt{s}_\nu(z) = \sqrt{\mathcal{P}(z) - \mathcal{P}(e_\nu)}$, where $\mathcal{P}(z)$ is the Weierstrass $\mathcal{P}$-function and $\{e_\nu\}_{(\nu=1,2,3)} = (1/2, \tau/2, (1+\tau)/2)$ are the three Weierstrass points in T^2_τ. The local CY space can be chosen therefore in three ways, being the total space of $\mathcal{L}_\nu \oplus \mathcal{L}_\nu$ on T^2_τ.

The quadratic differential of the $\mathcal{N} = 2^*$ SU(2) gauge theory is given by $\theta_2(z) = m^2\mathcal{P}(z) + u$, where m is the mass of the adjoint hypermultiplet and u is the Coulomb modulus. Due to the triviality of the canonical bundle and imposing that the rotated geometry is not more singular than the unrotated one, the $\mathcal{N} = 1$ curve is given by

$$x_1^2 = m^2 \ , \quad x_1 x_2 = \mu m,$$

where μ is a further mass scale related to the $\mathcal{N} = 1$ breaking and we fixed $R = m^2$ and $RP = \mu m$ to be constant sections of the trivial bundle on the elliptic curve.

[3] A more general one would be $x = \sigma x_1$ and $w = \tau x_2 + \rho(x_1, t)$, but the last is a global shift on φ which can we reabsorbed.

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Therefore, the equation for the quadratic differential θ_2 reads

$$\theta_2 = m^2 \mathtt{s}_\nu(z)^2, \tag{3.53}$$

which fixes the Coulomb modulus to $u = u_\nu = -m^2\mathcal{P}(e_\nu)$. These are the well known three $\mathcal{N} = 1^*$ vacua of the SU(2) gauge theory.

4 $\mathcal{N} = 1$ curve in A_{N-1} theory

In this section we consider four-dimensional $\mathcal{N} = 1$ gauge theories obtained from the A_{N-1} theory. In this case the factorization condition (2.16) cannot be simplified enough, thus we do not have a general approach to fix the moduli of the curve. We will see some examples where it is possible to find $\mathcal{N} = 1$ curves.

4.1 SU(N) SYM theory

Let us first consider SU(N) pure SYM theory. The $\mathcal{N} = 1$ breaking by brane rotation can be set for a generic choice of the superpotential, while we will discuss the breaking via complex structure rotation on a case by case basis. The SW curve is given by an N-sheeted cover of the twice punctured sphere $\mathcal{C} = \mathbb{P}^1 \setminus \{0, \infty\}$. Its equation is

$$\Lambda^N \left(t + \frac{1}{t}\right) = P_N(xt), \tag{4.1}$$

where t is a stereographic coordinate on the sphere, $v = xt$, $\lambda = xdt$ and $P_N(v) = v^N + \sum_{i=2}^N v^{N-i}u_i$. u_i are the Coulomb moduli parameters.

Let us fix the boundary condition for φ to be

$$\begin{aligned} &\varphi \sim 0 && \text{as} \quad t \sim 0, \\ &\varphi \sim W'(V) && \text{as} \quad t \sim \infty, \end{aligned} \tag{4.2}$$

where $\zeta = t\partial_t$ is the non-singular vector field fixing both the punctures and $V = i_\zeta \Phi$. Notice that ζ is non singular and therefore we can fix the boundary conditions ambiguity we discussed above by imposing vanishing boundary condition at the puncture located at $t = 0$.

We can now discuss how the curve (2.13) is affected by the above boundary conditions. From the algebraic system we can eliminate the t variable and obtain an algebraic equation involving the variables $v = tx$ and w only. This can be partially fixed by the boundary conditions as follows: Consider the function $w(w - W'(v))/W'(v)$ which is vanishing at the two punctures and independent of t. This implies that

$$w(w - W'(v)) = f(v), \tag{4.3}$$

where $f(v)$ is a polynomial of degree less than the degree of W' (independent of t by definition and of w since it would then be too divergent). This fixes the curve as the projection of the system (2.13) in the two fibers directions. To proceed further we have to specify the superpotential W.

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4.1.1 The massive deformation

Let us consider in this section the pure massive deformation $W(v) = \mu v^2/2$. In this case the equation (4.3) reads

$$w^2 - \mu w v - f_0 = 0 \tag{4.4}$$

which implies that

$$V = \frac{1}{\mu}\left(\varphi - f_0 \varphi^{-1}\right), \tag{4.5}$$

where f_0 is a constant to be determined. The last expression has to satisfy equation (4.1) and this fixes uniquely P_N to be the Chebyshev polynomial because of the boundary conditions for φ. By introducing the parameter s such that $s = w/\mu\Lambda$ and $s + s^{-1} = xt/\Lambda$, this means the SW curve is now

$$s^N + s^{-N} = t + t^{-1}, \tag{4.6}$$

The l.h.s. is equal to $2T_N(\frac{s+s^{-1}}{2})$, where T_N is the Chebyshev polynomial. This fixes the corresponding value of the condensate to $f_0 = -\Lambda^2\mu^2\omega_i$, where $\omega_i = \exp(2\pi i/N)$ $(i = 1, \ldots, N)$. Thus one finds N vacua. The $\mathcal{N} = 1$ curve is thus written as

$$w^N = \mu^N \Lambda^N t. \tag{4.7}$$

When $N = 2$ this reproduces the curve (3.8) (with $t \to 1/t$) in the previous section.

We conclude that the eigenvalues of the Hitchin field are $x_i = \frac{\Lambda}{t}\left(t^{1/N}\omega_i + t^{-1/N}\omega_i^{-1}\right)$. The SW curve is given by

$$0 = \det\left(v\mathbf{1} - V_i\right), \tag{4.8}$$

where V_i is (4.5) with f_0 being the i-th vacuum value. Correspondingly one has

$$\Phi = \frac{\Lambda}{t}\left[\Omega + \omega_i \Omega^{-1}\right], \quad \varphi = \mu\Lambda\Omega \tag{4.9}$$

with

$$\Omega = \begin{pmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ t & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

satisfying $\Omega^N = t\mathbf{1}_N$.

Let us split now the Hitchin field in the two components, namely the one which corresponds to the rotated brane and the one corresponding to the fixed one. This means that we write

$$\Phi = \sigma_1\phi_1 + \sigma_2\phi_2 \quad \text{and} \quad \varphi = \tau\phi_2,$$

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where in this case $\mathcal{L} = \mathcal{K}_{P^1}^{1/2} = \mathcal{O}(-1)$ and we choose $\sigma_2 = \Lambda/t$, $\sigma_1 = 1/\Lambda\mu t$ and $\tau = \mu\Lambda$ in the patch around $t = 0$. Therefore, by (4.9) we obtain

$$\phi_1 = \omega_i \Lambda^2 \mu \Omega^{-1}, \quad \phi_2 = \Omega, \tag{4.10}$$

and $\varphi = \Lambda\mu\phi_2$. The explicit canonical transformation is

$$x = \frac{1}{\Lambda\mu t} x_1 + \frac{\Lambda}{t} x_2, \qquad w = \Lambda\mu x_2, \tag{4.11}$$

and the z parameter satisfies $\partial_t z = \sigma_1 \tau = \frac{1}{t}$, that is $t = e^z$.

This implies that the $\mathcal{N} = 1$ curve of the rotated system is described by the equations

$$x_2^N = e^z, \qquad x_1^N = (\mu\Lambda^2)^N e^{-z} \quad \text{and} \quad x_1 x_2 = \mu\Lambda^2\omega_i \tag{4.12}$$

which follows from the fact that $\phi_1\phi_2 = \omega_i\Lambda^2\mu\mathbf{1}_N$. This is precisely equivalent to the curve given in [3]. If we now decouple the multiplet in the adjoint sending μ to infinity, we flow to $\mathcal{N} = 1$ SU(N) SYM theory provided we send to zero Λ as required by the one-loop matching keeping fixed $\mu\Lambda^2 = \Lambda^3_{\mathcal{N}=1}$. In this limit the above curve becomes the one proposed in [4] if we identify $\Lambda^3_{\mathcal{N}=1}$ with the ζ parameter appearing in that paper.

The above system of equations (4.12) is indeed equivalent to (4.1) and (4.4): using

$$v = x_2\Lambda + \frac{x_1}{\Lambda\mu},$$

the projection of the curve on the (v, t) plane gives (4.1). In particular the dependence on μ has to disappear. With our identification

$$t + \frac{1}{t} = x_2^N + \frac{x_1^N}{\mu^N\Lambda^{2N}} = x_2^N + \frac{1}{x_2^N}.$$

Using the Chebyshev polynomial it is easy to see that the r.h.s. is precisely $2T_N(v/2\Lambda)$, with v as above. We thus get the SW curve at the points where the curve can be rotated. Eq. (4.4) can be deduced as follows

$$vw = \left(x_2\Lambda + \frac{x_1}{\Lambda\mu}\right)\Lambda\mu x_2 = \frac{w^2 + \mu^2\Lambda^2\omega_i}{\mu}.$$

When v tends to infinity either x_1 or x_2 will go to infinity (while the other goes to 0 as long as μ is different from zero). In the first case we get $x_1/\mu\Lambda \to v$, so it describes the unrotated brane (and indeed $x_2 = w/\mu\Lambda \to 0$). In the second one we find $x_2\Lambda\mu = w \to \mu v$, reproducing the boundary condition (4.2) (so x_2 describes the rotated brane).

Factorization method. Let us consider another way to get the $\mathcal{N} = 1$ curve which makes use of the factorization condition given in section 2.3. We write the SW curve in the form

$$v^N = \sum_{k=2}^{N-1} v^{N-k} u_k + \Lambda^N t + u_N + \frac{\Lambda^N}{t}. \tag{4.13}$$

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In the case with the quadratic superpotential, we impose that the $\mathcal{N}=1$ curve has the same asymptotic behavior as the SW curve at infinity and subleading at zero as in (4.2). We thus get

$$w^N - \sum_{k=2}^{N} w^{N-k} a_k = bt, \tag{4.14}$$

where a_k and b are constants. In this case the factorization condition can be written as $v = P_{N-1}(w,t)/Q_{N-1}(w,t)$, where P and Q are polynomials in w of degree (at most) $N-1$, with coefficients polynomial in t. On the other hand, the form of the $\mathcal{N}=1$ curve tells us that t can be written as a polynomial in w and substituting in the factorization condition we find that v must be a rational function of w only:

$$v = \frac{\alpha_0 w^k + \cdots + \alpha_k}{w^{k-1} + \cdots + \beta_{k-1}}, \tag{4.15}$$

for some k. Note that the exponents of the first terms in the denominator and the numerator are fixed by the boundary condition $w \to \mu v$ when $w \to \infty$.

We can now proceed as follows: the SW curve tells us that v diverges at $t=0$. This behavior is reproduced by (4.15) only if the denominator vanishes there and its roots must clearly coincide with the roots of the l.h.s. of the $\mathcal{N}=1$ curve. We can now evaluate v^N using (4.15). The resulting expression has necessarily poles of order at least N in w. Plugging instead the $\mathcal{N}=1$ curve in the SW curve we find an object with poles of order at most N in w (depending on the multiplicity of the roots of the l.h.s. of the $\mathcal{N}=1$ curve). Since the ratio of these two quantities must tend to one at $t=0$ (just because v diverges there), we conclude that both must have a pole of order exactly N. The asymptotic behavior of v at $t=0$ dictated by the SW curve and by the factorization condition are thus compatible only if $k=2$ in (4.15), so

$$v = \frac{\alpha_0 w^2 + \alpha_1 w + \alpha_2}{w + \beta_1}, \tag{4.16}$$

and the $\mathcal{N}=1$ curve reduces to $(w+\beta_1)^N = bt$. By shifting w we can then set $\beta_1 = 0$ and the boundary condition $w \sim \mu v$ at infinity implies $b = \mu^N \Lambda^N$. Plugging now the $\mathcal{N}=1$ curve in the SW curve we can easily see that the factorization condition reduces to $\mu v = w + \omega_i \mu^2 \Lambda^2 w^{-1}$ and the Coulomb branch coordinates are restricted to be at the Chebyshev points, as above.

4.1.2 Argyres-Douglas points

Let us now consider the higher order superpotential $W(v) = gv^N/N$. Eq. (4.3) now reads

$$w^2 - gwv^{N-1} - f_{N-2}(v) = 0.$$

Let us solve the boundary conditions and the commutation relation via the ansatz $\varphi = a(t) g V^{N-1}$, where a is a meromorphic function in t with a zero at $t=0$ and regular at $t \sim \infty$. The $\mathcal{N}=1$ curve then implies that

$$(a^2 - a) g^2 V^{2N-2} - f_{N-2}(V) = 0,$$

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which is compatible with (4.1) only if $V^N = k(t)\mathbf{1}$ and $(a^2 - a)k(t) = c$ is a constant. This fixes $f_{N-2} = cg^2 v^{N-2}$. The Coulomb moduli are then fixed as $P_N(v) = v^N - u_N$ and $k(t) = \Lambda^N(t + 1/t) - u_N$. The condition $(a^2 - a)k = c$ has then two solutions, namely $u_N = \mp 2\Lambda^N$ and $a = \frac{t}{t\mp 1}$ with $c = \pm\Lambda^N$ and therefore $f_{N-2}(v) = \pm\Lambda^N g^2 v^{N-2}$. This value of u_N denotes the maximal conformal fixed point [30, 31]. Notice that $a(t)$ is consistent with the requirements we put on it few lines ago. The SW curve at this point then reads

$$\Lambda^N \left(t + \frac{1}{t}\right) = v^N \pm 2\Lambda^N$$

(for definiteness we will choose the minus sign in the following).

The Hitchin field is then $\Phi = (\Lambda/t)\Omega_{ad}$, where $\Omega_{ad}^N = (t + 1/t + 2)\mathbf{1}$. More specifically we can choose

$$\Phi = \frac{\Lambda}{t} \begin{pmatrix} 0 & \dots & 0 & 1 + \frac{1}{t} \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ 0 & \dots & t+1 & 0 \end{pmatrix},$$

where the diagonal dots stand for 1's. The SW curve as written above can be seen as the spectral curve of the field $V = i_\zeta \Phi$, namely $\det(v\mathbf{1} - V) = 0$. The field φ now reads

$$\varphi = (t+1)g\Lambda^{N-1}\Omega_{ad}^{-1} = g\Lambda^{N-1} \begin{pmatrix} 0 & 1 + \frac{1}{t} & 0 & \dots \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{t} \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

where the diagonal dots stand for $1 + 1/t$'s. Plugging this in (2.4) we find the system

$$\begin{aligned} x^N &= (\Lambda/t)^N \left(t + \frac{1}{t} + 2\right), \\ w^N &= g^N \Lambda^{N(N-1)} t\,(t+1)^{N-2}, \\ xw &= g\Lambda^N \left(1 + \frac{1}{t}\right). \end{aligned} \tag{4.17}$$

The description of the vacuum at the conformal fixed point in the rotated complex structure is provided by the canonical transformation $\mathcal{O}(-2) \oplus \mathcal{O}(0) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ given by

$$x = \sigma_1 x_1 \quad \text{and} \quad w = \tau x_2,$$

where $\sigma_1 = \Lambda/t$ and $\tau = 1/\Lambda$ in the patch around $t = 0$. The $\mathbb{C}^*$ parameter is fixed by $\sigma_1\tau = \partial_t z = 1/t$. Since now $\phi_1 = \Omega_{ad}$ and $\phi_2 = (t+1)\left(\Lambda^N\right)\Omega_{ad}^{-1}$, in the new coordinates the spectral curve reads

$$x_1^N = (t+1)^2/t, \tag{4.18}$$

$$x_1 x_2 = g\Lambda^N(t+1), \tag{4.19}$$

$$x_2^N = (g\Lambda^N)^N t(t+1)^{N-2}. \tag{4.20}$$

Notice that in the rotated geometry only the combination $\hat{g} = g\Lambda^N$ appears.

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4.2 SU(2) gauge theory coupled to SCFTs of Argyres-Douglas type

We here consider the adjoint mass deformation of $\mathcal{N}=2$ SU(2) gauge theory coupled to the D_n and D_m SCFTs [32, 33], by gauging the diagonal part of two SU(2) flavor symmetries of SCFTs, though this is not higher rank theory. The D_n theory is an $\mathcal{N}=2$ SCFT of Arygres-Douglas type which can be obtained as a superconformal fixed point on the Coulomb branch of $\mathcal{N}=2$ SU$(n-1)$ gauge theory with two flavors. We will use the approach in section 4.1 to obtain the $\mathcal{N}=1$ curve.

The SW curve of this theory is

$$x^2=\frac{\Lambda^2}{t^{n+2}}+\ldots+\frac{u}{t^2}+\ldots+\Lambda^2 t^{m-2}, \tag{4.21}$$

where u and Λ are the Coulomb modulus and the dynamical scale of the SU(2) gauge group (which we consider fixed). t is the coordinate on the base sphere, and the SW differential is $\lambda=xdt$. The dots denote the moduli and the couplings of the (generically) strongly coupled theory which will not enter the $\mathcal{N}=1$ curve. In terms of the variable $v=xt$ the curve reads

$$v^2=\frac{\Lambda^2}{t^n}+\ldots+u+\ldots+\Lambda^2 t^m. \tag{4.22}$$

The only single trace superpotential is a pure massive deformation for the adjoint scalar in the vector multiplet $W(v)=\frac{\mu}{2}v^2$ and therefore, as in the case of the SU(N) SYM theory, we rotate at one puncture and stay with the curve which depends on the number and type of punctures

$$w^n(w-\mu v)^m+\sum_{i=0}^{n+m-2} w^i f_i(v)=0, \tag{4.23}$$

where $f_i(v)$ are polynomials of degree $n+m-2-i$.

The complete analysis of the vacua of this class of theories boils down to find the solutions of the factorization condition. In the $n,m=1,2$ cases the theory has a Lagrangian description and corresponds to SU(2) gauge theory with $N_f=0,1,2$ already discussed in the previous section. A class of vacua for $n=m$ can be identified as follows. The curve (4.23) can be reduced to

$$w(w-\mu v)=g_0 \tag{4.24}$$

by setting $f_i=0$ $i>0$ and $f_0=g_0^N$ with g_0 a constant. Correspondingly $V=\frac{1}{\mu}\left(\varphi+f_0\varphi^{-1}\right)$, where φ has vanishing boundary condition at $t\sim\infty$ and is diverging at $t\sim 0$. Therefore, from eq. (4.22), we get the matrix equation

$$\frac{1}{\mu^2}\left(\varphi^2+2f_0+f_0^2\varphi^{-2}\right)=\mathbf{1}\left(\frac{\Lambda^2}{t^n}+\ldots+u+\ldots+\Lambda^2 t^n\right),$$

which has to be solved under the above boundary conditions. Henceforth one finds

$$\frac{\varphi^2}{\mu^2}=\frac{\Lambda^2}{t^n}+\ldots+u_+, \tag{4.25}$$

$$\frac{f_0^2\varphi^{-2}}{\mu^2}=u_-+\ldots\Lambda^2 t^n, \tag{4.26}$$

$$2f_0=u-u_+-u_-, \tag{4.27}$$

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where $u_\pm$ are constants. In such a case one has two vacua $f_0 = \pm\mu^2\Lambda^2$ and the $\mathcal{N}=2$ moduli are fixed at

$$v^2 = \Lambda^2\left(t^n \pm 2 + t^{-n}\right).$$

Correspondingly

$$\Phi = \frac{\Lambda}{t}(t^n \pm 1)\begin{pmatrix} 0 & 1 \\ \frac{1}{t^n} & 0 \end{pmatrix}, \quad \varphi = \mu\Lambda\begin{pmatrix} 0 & 1 \\ \frac{1}{t^n} & 0 \end{pmatrix}.$$

Fixing $\sigma = \Lambda/t$ and $\tau = 1/\Lambda$ as in the case in subsection 4.1.2, we stay with

$$\phi_1 = (t^n \pm 1)\begin{pmatrix} 0 & 1 \\ \frac{1}{t^n} & 0 \end{pmatrix}, \quad \phi_2 = \mu\Lambda^2\begin{pmatrix} 0 & 1 \\ \frac{1}{t^n} & 0 \end{pmatrix}$$

and the rotated curve

$$x_1^2 = \frac{(t^n \pm 1)^2}{t^n}, \tag{4.28}$$

$$x_1 x_2 = \mu\Lambda^2\frac{(t^n \pm 1)}{t^n}, \tag{4.29}$$

$$x_2^2 = \frac{(\mu\Lambda^2)^2}{t^n}, \tag{4.30}$$

which describes the theory in the $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ complex structure.

4.3 T_N theories coupled to $\mathcal{N}=1$ vector multiplets

We will now see how our technique allows to predict the form of the $\mathcal{N}=1$ curve found in [16] for the T_N theories with two SU(N) flavor symmetries gauged. Let us start from a linear quiver of the form SU$(N)^n$ with bifundamental hypermultiplets. The SW curve can be written as

$$\Lambda_1^N t^{n+1} + t^n P_N^1(v) + \cdots + \Lambda_2^N = 0,$$

where Λ_i represent the dynamical scales of the two gauge groups at the ends of the quiver. We can bring it to the N-sheeted cover form:

$$v^N = \sum_{k=2}^{N} v^{N-k} Q_k(t),$$

where $Q_k(t)$ (for $k<N$) have simple poles at $n-1$ points (the position of the NS5-branes) and $Q_N(t)$ has two other poles at zero and infinity, associated with the irregular singularity. If we turn on a quadratic superpotential for the first and last gauge groups with the mass parameters μ_1 and μ_2, we can impose vanishing boundary conditions (for the w coordinate) at all the simple punctures. The $\mathcal{N}=1$ curve can then be parametrized as follows:

$$w^N = \sum_{k=2}^{N-1} a_k w^{N-k} + \mu_1^N\Lambda_1^N t + a_N + \alpha\frac{\mu_2^N\Lambda_2^N}{t}, \tag{4.31}$$

where α is a constant depending on the gauge coupling constants of $\mathcal{N}=2$ gauge groups in the linear quiver. This is the most general expression compatible with the absence of

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poles at the simple punctures and reproducing the asymptotics of v at zero and infinity, no matter how many simple punctures we introduce. If we have at least $N-1$ simple punctures, with a suitable choice of the coupling constants (i.e. considering the collision limit) we can degenerate the UV curve to a sphere with two irregular punctures, one maximal regular (described by a Young tableau with a single row of length N) and possibly other regular punctures. In this way we obtain the curve describing $\mathcal{N}=1$ gaugings of the T_N theory (possibly coupled to other matter sectors). Note here that the theory has adjoint multiplets with masses $\mu_{1,2}$. When we set the mass parameters to be equal this recovers the set-up in [16].

In [16] it was argued that the above curve captures the matrix of U(1) gauge couplings also for T_N theories coupled together by $\mathcal{N}=2$ vector multiplets, provided we gauge two SU(N) flavor symmetries by $\mathcal{N}=1$ vector multiplets. The associated SW curve in this case has two irregular punctures (that we can set at $t=0,\infty$) and a number of full punctures that can be thought of as arising from the collision limit of N simple punctures [34]. Since the form of the $\mathcal{N}=1$ curve derived above does not depend on the number of simple punctures, we directly recover this result.

If we now turn off the quadratic superpotential for one of the two gauge groups (sending e.g. to zero μ_2), the $\mathcal{N}=1$ curve reduces to

$$w^N=\sum_{k=2}^{N-1}a_k w^{N-k}+\mu_1^N\Lambda_1^N t+a_N.$$

Indeed this curve is still valid also when we send Λ_2 to zero, thus decoupling the second gauge group. In this way we get the $\mathcal{N}=1$ curve for the T_N theory with a single SU(N) gauged.

Notice that for $\Lambda_2\neq 0$ we can actually fix precisely the moduli of the curve: the above formula is equal to the curve we found previously when we analyzed the massive deformation of SYM theory. In that case we argued that all the parameters a_k are actually zero. This exploited the form of the factorization condition (combined with the fact that t could be written as a function of w) and the behavior of v at infinity ($v\sim t$) and at zero ($v\sim 1/t$). All these properties hold in the present case as well, (at least as long as Λ_2 is nonzero) so the conclusion is the same. We thus get the $\mathcal{N}=1$ curve $w^N=\mu_1^N\Lambda_1^N t$.

We will now provide a recipe to extract the $\mathcal{N}=1$ curve for the T_N theory coupled to three SU(N) gauge groups. The corresponding $\mathcal{N}=2$ curve is (we denote with λ the SW differential)

$$\lambda^N=\sum_{k=2}^{N}\lambda^{N-k}\theta_k,$$

where θ_k are the meromorphic differentials. In the present case they have the form

$$\theta_k=\frac{P_k(t)}{t^k(t-1)^k}(dt)^k\quad(k<N);\quad \theta_N=\frac{P_{N+3}(t)}{t^{N+1}(t-1)^{N+1}}(dt)^N, \tag{4.32}$$

where we have set the three punctures to be at $t=0,1,\infty$. So far we have mainly discussed theories with Lagrangian descriptions, whose $\mathcal{N}=1$ curve can be extracted

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just analyzing the behavior at the punctures of the SW curve written in terms of the coordinate v introduced by Witten (the SW differential reads $\lambda = vdt/t$). This can be extracted considering the vector field on the sphere that preserves the points $t = 0, \infty$: $\zeta = t\partial_t$. The v coordinate is then obtained acting on ζ with the SW differential $\lambda(\zeta) = v$. For the theory under consideration, we must use a vector field which preserves the three irregular singularities. We thus impose that it has a zero at $t = 1$ as well. This will of course introduce a pole, that we locate at $t = a$. The vector field then reads

$$\zeta' = \frac{t(t-1)}{t-a}\partial_t.$$

Acting on it with the SW differential we find the coordinate

$$y = \lambda(\zeta') = \frac{v(t-1)}{t-a}.$$

Rewriting the SW curve in terms of t and y we get

$$y^N = \sum_{k=2}^{N} y^{N-k}\vartheta_k,$$

where ϑ_k are meromorphic functions of the form

$$\vartheta_k = \frac{P_k(t)}{(t-a)^k} \quad (k < N); \quad \vartheta_N = \frac{P_{N+3}(t)}{t(t-1)(t-a)^N}.$$

The boundary conditions for the $\mathcal{N} = 1$ curve at the three punctures should be extracted from the SW curve in this set of coordinates. We thus get a curve of the form

$$w^N = \sum_{k=2}^{N-1} w^{N-k}a_k + at + a_N + \frac{b}{t} + \frac{c}{t-1}.$$

More in general, notice that a simple zero for the vector field decreases the order of the pole of the k-th differential by k. Consequently, the pole structure of the $\mathcal{N} = 1$ for all the theories considered in [16] can be recovered in the present setting imposing that the vector field has a zero at all the irregular punctures.

4.4 An example with higher degree superpotential

We will now illustrate with an example how one can determine the $\mathcal{N} = 1$ curve when higher degree superpotentials are turned on. We consider an SU(3) $\times$ SU(3) gauge theory (with a massless bifundamental hypermultiplet as the only matter field) with quadratic superpotential for one gauge group $W_1'(\Phi) = \mu\Phi$ and cubic for the second gauge group $W_2'(\Phi) = g\Phi^2 - \mu\Phi$. We can write the SW curve in the form

$$v^3 = v\frac{tu_1 - u_2}{t-1} + \frac{\Lambda_2^3 t^3 + t^2 v_1 - tv_2 + \Lambda_1^3}{t(t-1)}.$$

The $\mathcal{N} = 1$ curve has the by now familiar form

$$w^3 = wV_2(t) + V_3(t).$$

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Given the above superpotentials, the meromorphic functions $V_2(t)$ and $V_3(t)$ can be found imposing the boundary conditions

$$\frac{w}{v} \to 0 \ \ (\text{at } t=0) \quad \frac{w}{v} \to \mu \ \ (\text{at } t=1) \quad \frac{w}{gv^2} \to 1 \ \ (\text{at infinity}).$$

Let us analyze the asymptotics at each puncture:

- *At t=0:* $v^3 \sim u_2 v - \Lambda_1^3/t$ so $V_2(t)$ and $V_3(t)$ must be bounded.
- *At t=1:*

$$v^3 \sim \frac{u_1 - u_2}{t-1} v + \frac{\Lambda_1^3 + \Lambda_2^3 + v_1 - v_2}{t-1},$$

  consequently

$$V_2(t) \sim \frac{\mu^2(u_1 - u_2)}{t-1}, \quad V_3(t) \sim \frac{\mu^3(\Lambda_1^3 + \Lambda_2^3 + v_1 - v_2)}{t-1}.$$

- *At infinity:*

$$v^3 \sim u_1 v + \Lambda_2^3 t, \quad \text{so} \quad w^3 \sim (\alpha t + \beta) w + g^3 t^2 + \gamma t + \delta.$$

  We can allow a pole of degree one for $V_2(t)$, since this does not affect the asymptotic behavior of w (at leading order).

We thus find the following answer:

$$\begin{aligned} V_3(t) &= \frac{g^3(t-1)^3 + a(t-1)^2 + b(t-1) + \mu^3(\Lambda_1^3 + \Lambda_2^3 + v_1 - v_2)}{t-1}, \\ V_2(t) &= c(t-1) + d + \frac{\mu^2(u_1 - u_2)}{t-1}. \end{aligned} \tag{4.33}$$

The parameters a, b, c and d and the Coulomb branch coordinates can then be fixed imposing the factorization condition.

In general, in order to write down the $\mathcal{N}=1$ curve if we have to impose the boundary condition $w \sim v^k$ at a puncture, the most convenient procedure is to consider the Newton polygon associated to the puncture. In [2] it is explained how to construct it for the $\mathcal{N}=2$ theory: one marks the point in the plane with coordinates (k,i) if the k-th differential has a pole of degree i at the puncture. One then considers the convex envelope of all the marked points. The slope of the edges of the polygon encode the leading behavior of the N roots at the puncture. One has to repeat the same procedure in this case, but using the pole structure of the meromorphic functions appearing in the SW curve written in terms of the coordinate identified by the choice of vector field.

The above boundary condition simply means that we have to multiply by k the slope of all the edges, or equivalently that we have to multiply by k the "height" of all the vertices. All the dots contained in the interior of the new Newton polygon are allowed and the corresponding term in the meromorphic functions $V_k(t)$'s should be included. In the above example the $\mathcal{N}=2$ curve has a simple pole at infinity, so the associated Newton polygon

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is the triangle with vertices $(0,0)$, $(3,0)$ and $(3,1)$. The leading behavior of the three roots at infinity is $v \sim t^{1/3}$. Since we want to impose the boundary condition $w \sim v^2$, the three roots of the $\mathcal{N}=1$ curve should be $w \sim t^{2/3}$. This is done multiplying by 2 the height of the third vertex. The $\mathcal{N}=1$ Newton polygon is thus the triangle with edges $(0,0)$, $(3,0)$ and $(3,2)$. We can easily see that the point $(2,1)$ lies in the interior of the triangle and is thus allowed. In fact we included in $V_2(t)$ a term linear in t. This algorithm can be used in principle to write down the curve in the general case.

5 Linear quivers and the Dijkgraaf-Vafa curve

In this section we will provide another method to write down the curve for $\mathcal{N}=1$ theories with Lagrangian descriptions that admit a brane construction in Type IIA string theory, focusing especially on linear quivers. For this class of theories there is a globally well defined projection of the curve on the (w,v) plane, that can be identified with the Dijkgraaf-Vafa curve. In this section we will make use of the generalized Konishi anomaly equations to determine the curve explicitly in some examples.

5.1 SU(N) SYM theory

In [35] it was recognized that for $\mathcal{N}=1$ SYM theory the M-theory curve can be written in terms of the Dijkgraaf-Vafa curve. We will now review that argument and see how one can fix all the parameters of the curve using the generalized anomaly equations, once the superpotential and the point in the Coulomb branch are given.

The SW is written as (4.1). If we break extended supersymmetry adding a polynomial superpotential (of degree $n+1$) that we denote by $W(\Phi)$ for the chiral multiplet in the adjoint representation, the curve describing our $\mathcal{N}=1$ theory will now be embedded in a threefold. The system defining the $\mathcal{N}=1$ curve can be conveniently written as [35]

$$\begin{aligned} \Lambda^N \left(t+\frac{1}{t}\right) &= P_N(v), \\ w^2 - wW'(v) + f_{n-1}(v) &= 0, \end{aligned} \tag{5.1}$$

where $f_{n-1}(v)$ is a polynomial of degree $n-1$. The second equation is implied by the boundary conditions at the punctures [35]:[4]

$$\text{for} \quad t \to 0 \qquad w \to W'(v), \qquad v \simeq t^{-1/N}, \tag{5.2}$$

$$\text{for} \quad t \to \infty \qquad w \to 0, \qquad v \simeq t^{1/N}, \tag{5.3}$$

and is referred to as DV curve. As seen in the previous sections, the brane cannot be rotated at a generic point of the moduli space. The submanifold at which the curve can be rotated is defined by a certain factorization equation (we will see in the next subsection

[4]The analysis of [35] applies to U(N) SYM theory. In the present context this statement is slightly modified. This will not be important in this section and we postpone a detailed discussion of this point to the next section, where the $\mathcal{N}=1$ deformation of SQCD is considered.

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that it precisely coincides with the factorization condition introduced in this paper), which implies that the curve degenerates to a genus (at most) $n-1$ surface [35].

So far we have determined the general form of the DV curve. However, we still do not know the precise form of f_{n-1} in a given vacuum. It can actually be determined explicitly in several cases, as we will see. In order to determine the polynomial f, let us recall that the chiral condensates in such a theory can be computed explicitly using the generalized Konishi anomaly studied in [13]: if we introduce the generating functions

$$T(v) = \left\langle \mathrm{Tr}\frac{1}{v-\Phi} \right\rangle, \; R(v) = \frac{-1}{32\pi^2}\left\langle \mathrm{Tr}\frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{v-\Phi} \right\rangle,$$

where $\mathcal{W}_\alpha$ denotes the field strength superfield, we can determine all chiral condensates simply expanding $R(v)$ and $T(v)$ in powers of $1/v$, once we know them explicitly. These can in turn be determined recognizing that they satisfy the following Ward-identities [13]:

$$\begin{aligned} W'(v)T(v) + c_{n-1}(v) &= 2R(v)T(v), \\ W'(v)R(v) - f_{n-1}(v) &= R^2(v). \end{aligned} \tag{5.4}$$

Notice that $R(v)$ satisfies the same equation as w in (5.1). This motivates the identification $R(v) = w$ proposed in [35]. Furthermore, it was shown in [36] that $T(v)dv = dlog(2t) = dt/t$.

The f polynomial appearing in the above equation is precisely the f_{n-1} entering in the DV curve and c is another polynomial in v. From the second equation we get immediately

$$R(v) = \frac{1}{2}\left(W'(v) - \sqrt{W'(v)^2 - 4f_{n-1}(v)}\right). \tag{5.5}$$

Let us first of all determine $c_{n-1}(v)$: from the first Ward identity in (5.4) we find

$$c_{n-1}(v) = \left(W'(v)T(v)\right)_+,$$

where $(s)_+$ stands for the polynomial part of s. Since $T(v) = N/v + O(v^{-3})$, for quadratic or cubic superpotentials (or whenever $T(v) = N/v + O(v^{-n-1})$, which will be true only on a submanifold of the Coulomb branch) we can keep only the leading term

$$c_{n-1}(v) = -N\frac{W'(v)}{v}. \tag{5.6}$$

In this section we will discuss the cases of quadratic superpotential and $\mathcal{N}=1$ deformation of Arygres-Douglas points only, so this approximation can be used.

With a similar argument we can also determine $f_{n-1}(v)$: since $T(v)dv = dt/t$ we have $T(v) = \frac{1}{t}\frac{dt}{dv}$. Plugging this into the first Ward-identity in (5.4) and using (5.6) we get

$$R(v) = \frac{1}{2}\left(W'(v) - N\frac{t}{v}\frac{dW(v)}{dt}\right).$$

Comparing now with (5.5) we then find

$$f_{n-1}(v) = \frac{1}{4}\left[W'(v)^2 - N^2\frac{t^2}{v^2}\left(\frac{dW(v)}{dt}\right)^2\right], \tag{5.7}$$

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thinking of $W(v)$ as a function of t, determined by the SW curve. At a generic point in the Coulomb branch this formula will not give a polynomial of degree $n-1$. However, this will be true at all points at which the NS5-brane can be rotated.

Let us analyze explicitly the two cases we have studied before: SU(N) SYM at the Arygres-Douglas point and the case of quadratic superpotential. We will now see that using (5.7) we can recover the same result found before with the help of the Hitchin field.

SU(N) SYM at the Argyres-Douglas point. The SW curve at the Argyres-Douglas point is (4.1) with $P_N(v) = v^N \pm 2\Lambda^N$, thus $v^N = \Lambda^N\left(t + \frac{1}{t} \mp 2\right)$. It is known that this point of the moduli space is not lifted if we turn on a superpotential of the form v^N. Let us set $W(v) = (g/N)v^N$. From (5.7) we immediately find

$$f_{N-2}(v) = \frac{g^2}{4}\left[v^{2N-2} - \frac{t^2}{v^2}\left(\frac{dv^N}{dt}\right)^2\right].$$

Using the SW curve we can rewrite v^N in terms of t. We then find

$$\frac{t^2}{v^2}\left(\frac{dv^N}{dt}\right)^2 = \frac{\Lambda^{2N}}{v^2}\left(t - \frac{1}{t}\right)^2 = v^{2N-2} \pm 4\Lambda^N v^{N-2},$$

where we have used the identity $\left(t - \frac{1}{t}\right)^2 = \left(t + \frac{1}{t} \mp 2\right)^2 \pm 4\left(t + \frac{1}{t} \mp 2\right)$. The final result is then $f_{N-2}(v) = \mp g^2\Lambda^N v^{N-2}$. The second equation defining the $\mathcal{N} = 1$ curve consequently is

$$w^2 - gv^{N-1}w \mp g^2\Lambda^N v^{N-2} = 0. \tag{5.8}$$

This agrees with the equation obtained in section 4.1.2.

We can make a simple consistency check on our formula for f_{N-2}: using (5.4) and (5.5) it is straightforward to determine $T(v)$ explicitly

$$T(v) \equiv \sum_i \frac{\langle \mathrm{Tr}\Phi^{\mathrm{i}}\rangle}{v^{i+1}} = \frac{N}{v\sqrt{1 \pm 4(\Lambda/v)^N}} = \frac{N}{v} \mp \frac{2N\Lambda^N}{v^{N+1}} + \dots$$

We can now compute $P_N(v)$ using this formula and check that we get precisely our starting point. In order to do this the simplest way is to introduce $U_i = \frac{1}{i}\langle \mathrm{Tr}\Phi^{\mathrm{i}}\rangle$ and then use the formula given in [36] which relates the SW curve to these quantities:

$$P_N(x) = \left(x^N e^{-\sum_{i\geq 1}\frac{U_i}{x^i}}\right)_+, \tag{5.9}$$

It is easy to check that this equation gives precisely $P_N(v) = v^N \pm 2\Lambda^N$.

Quadratic superpotential. We now turn to the analysis of the massive deformation $W(v) = (\mu/2)v^2$. In this case the points which are not lifted can be determined using Chebyshev polynomials: $P_N(v) = 2\Lambda^N T_N(v/2\Lambda)$. If we introduce the variable s defined as $v/\Lambda = s + 1/s$, then $2T_N(v/2\Lambda) = s^N + s^{-N}$. In this case f_{n-1} is simply a constant and from (5.7) we find

$$f_0 = \frac{\mu^2}{4}\left[v^2 - N^2t^2\left(\frac{dv}{dt}\right)^2\right] = \frac{\mu^2\Lambda^2}{4}\left[(s+s^{-1})^2 - N^2t^2\left(\frac{d(s+s^{-1})}{dt}\right)^2\right].$$

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In order to evaluate the last term we exploit the equation coming from the SW curve: $s^N + s^{-N} = t + t^{-1}$. If we differentiate with respect to t this equation we get

$$1 - \frac{1}{t^2} = \frac{N}{s}(s^N - s^{-N})\frac{ds}{dt}.$$

Taking the square and multiplying by t^2 the l.h.s. becomes $\left(t - \frac{1}{t}\right)^2 = \left(t + \frac{1}{t}\right)^2 - 4 = s^{2N} + s^{-2N} - 2 = (s^N - s^{-N})^2$. The r.h.s. becomes instead $(s^N - s^{-N})^2 \left(\frac{Nt}{s}\frac{ds}{dt}\right)^2$. Equating these two we finally get $\left(\frac{Nt}{s}\frac{ds}{dt}\right)^2 = 1$. Using this formula the equation for f_0 reduces to

$$f_0 = \frac{\mu^2\Lambda^2}{4}\left[(s + s^{-1})^2 - (s - s^{-1})^2\right] = \mu^2\Lambda^2.$$

We thus find the $\mathcal{N} = 1$ curve

$$w^2 - \mu v w + \mu^2\Lambda^2 = 0. \tag{5.10}$$

This correctly reproduces the result found previously.

5.2 The curve for SQCD

We have seen that the $\mathcal{N} = 1$ deformation of SU(N) SYM theory can be described by a curve whose form is dictated by the SW curve of the parent $\mathcal{N} = 2$ theory and the DV curve, as noticed in [35]. This recipe can be easily extended to the theory with matter fields in the fundamental. The matrix model curve can be easily derived from the generalized anomaly equations: considering the variation $\delta\Phi = \frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{z-\Phi}$, we obtain the Ward identity which is the same as the one in (5.4) where $W(v)$ is now the superpotential of the theory with the matter fields [37].

The above variation indeed changes the trace part of Φ, which is not a problem if the gauge group is U(N). Since we are dealing with SU(N) gauge theories, we have to slightly modify the above variation in order to preserve the traceless condition. This can be done simply setting [38]

$$\delta\Phi = \frac{\mathcal{W}_\alpha\mathcal{W}^\alpha}{v - \Phi} + \frac{32\pi^2}{N}R(v)I.$$

The resulting Ward identity is then

$$R^2(v) = \left(W'(v) - \langle W'_\Phi\rangle\right)R(v) - f_{n-1}(v), \tag{5.11}$$

where W'_Φ is the trace of the derivative of the superpotential with respect to Φ (times the prefactor $1/N$). This is automatically zero for U(N) gauge theories because of the equations of motion, but it is not zero in the present case. The DV curve is then obtained just identifying $R(v)$ with the complex coordinate w as before. Notice that this procedure is equivalent to using the formula valid for U(N) gauge theories with a modified superpotential which includes a linear term whose coefficient is equal to W'_Φ.

We should note that the shift in the variation in principle changes the result of the SYM theory in the previous subsection. However, at least for the vacua considered there, the effect of the shift is irrelevant: for the quadratic superpotential $\operatorname{tr} W'(\Phi) = \operatorname{tr}\Phi = 0$; at the Argyres-Douglas points, $\langle \operatorname{tr}\Phi^{N-1}\rangle = 0$.

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Note also that we are not forced to restrict to deformations of the $\mathcal{N}=2$ theory that include only a polynomial superpotential for Φ. Superpotential terms of the form $\tilde{Q}P(\Phi)Q$ with P a generic polynomial can be easily incorporated. The only way in which they affect the curve is by modifying the term $\langle W'_\Phi \rangle$. The curve (5.11) can be derived as in the previous sections from the boundary conditions for w: at the punctures corresponding to the presence of the NS5-branes the v coordinate[5] diverges and the presence of the superpotential $W(\Phi)$ is encoded in the difference between the boundary conditions imposed at the two punctures. We are free to impose the condition $w \to 0$ at one puncture and consequently $w \to W'(v) - \langle W'_\Phi \rangle$ at the second one. At this puncture w tends to $W'(v)$ in the sense that their ratio tends to one, the difference being a constant given by the vev of the chiral operator W'_Φ.

Let us comment further on the boundary conditions for w. As is well known and seen also in subsection 3.1, the Riemann surface associated to SU(N) SQCD with N_f flavors, in which the M5-branes are compactified, is a sphere with two, three or four punctures depending on the number of flavors and on the Type IIA realization of the theory, that is on the number of semi-infinite D4 branes inserted on each side. Whenever there are less than N D4-branes on both sides the theory is described by a sphere with two irregular punctures, where the v coordinate tends to infinity. In this case the situation is very similar to the SYM case analyzed before, so there is no need to discuss it further. If $N_f \geq N$, we can indeed put N semi-infinite D4 branes (or D4-branes whose ends are attached to D6-branes) on the same side. In this case there are one irregular puncture at let's say $t=\infty$ (provided $N_f < 2N$), one maximal regular puncture at $t=0$ and one minimal puncture at $t=1$. In this case the punctures associated to the "transverse" NS5-branes are those at $t=1$ and infinity, where v diverges. The puncture at the origin is instead related to the N D4-branes and the v coordinate remains finite in this limit. The N asymptotic values for v just correspond to the mass parameters of the associated flavors [1]. The boundary conditions for w, which describe the rotation of the NS5-branes, should then be imposed at $t=1$ and $t=\infty$. The third puncture simply does not play any role in this discussion.

If $N_f = 2N$ the $\mathcal{N}=2$ theory is conformal and the underlying base sphere has four regular punctures: two minimal and two maximal. The punctures associated to the NS5-branes are the two minimal ones and this is where the boundary conditions for w should be imposed. Again, at the maximal punctures v does not diverge and its limit can be identified with the various mass parameters. This discussion generalizes in a straightforward way to quiver theories: a linear quiver with n gauge groups is associated to a sphere with $n+1$ minimal punctures and this is where we have to impose the boundary conditions for w. At one of them we can impose the condition $w=0$, since an overall rotation of the brane system just corresponds to a reparametrization of the curve. The other n boundary conditions precisely encode the information about the various superpotentials for the chiral superfields in the adjoint. We thus see how the general picture gradually emerges. We will discuss in more detail quiver theories in the following subsection.

[5] In this section when we mention the SW curve we always refer to the curve in the parametrization with v and t.

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5.2.1 The factorization condition

Our proposal for the M-theory curve associated to softly broken $\mathcal{N}=2$ SQCD is then

$$\begin{aligned} t^2 - tP_N(v) + \Lambda^{2N-N_f}\prod_i (v-m_i) &= 0, \\ w^2 - w\widetilde{W}'(v) + f_{n-1}(v) &= 0, \end{aligned} \tag{5.12}$$

where $\widetilde{W}'(v)$ stands for $W'(v) - \langle W'_\Phi \rangle$. As we mentioned before, the restriction on the Coulomb branch coordinates is encoded in a factorization condition which relates the above two equations. It proves convenient in order to define it, to introduce the variables $y = 2t - P_N(v)$ and $x = 2w - \widetilde{W}'(v)$. The above equations then become

$$y^2 = P_N^2(v) - 4\Lambda^{2N-N_f}\prod_i (v-m_i); \quad x^2 = (\widetilde{W}'(v))^2 - 4f_{n-1}(v).$$

The factorization condition then reads as follows:

$$y^2 = H_{N-k}^2(v)F_{2k}(v); \quad x^2 = Q_{n-k}^2(v)F_{2k}(v),$$

where F has only roots of multiplicity one. This equation encodes the well-known fact that a point in the $\mathcal{N}=2$ Coulomb branch is not lifted only if there are at least $N-n$ massless monopoles.

We will now see how this factorization condition is related to our constraint (2.16). In the special case of SQCD, we can rewrite (2.16) as follows:

$$w = \frac{A(v)t + B(v)}{C(v)t + D(v)}, \tag{5.13}$$

where A, B, C and D are polynomials in v (any higher power of t can be eliminated using the SW curve which in the present case is quadratic in t). This is the equation that should be added to (5.12). Clearly, modulo a redefinition of the four polynomials, the same equation holds for the variables y and x introduced before. It is convenient to work in terms of them because the two solutions at fixed v (that we indicate with x, $\tilde{x}$, y and $\tilde{y}$) differ just by a sign. Now the argument proceeds as in the analysis of the rank one case around (3.4) which leads to $B(v)D(v) = A(v)C(v)y^2$ and $A(v)B(v) = C(v)D(v)x^2$. These in turn imply[6]

$$\frac{A^2(v)}{D^2(v)} = \frac{x^2}{y^2}; \quad x^2y^2 = \frac{B^2(v)}{C^2(v)}. \tag{5.14}$$

It is easy to see that both these equations are equivalent to the factorization condition. We have thus established that from the brane perspective the constraint which selects the points in the moduli space which are not lifted by the $\mathcal{N}=1$ deformation is equivalent to the requirement that the M-theory curve is an N-sheeted cover of the base sphere.

[6] If some of the polynomials are zero, making these two equations trivial, it is easy to see that either $B(v) = C(v) = 0$ or $A(v) = D(v) = 0$ and this leads directly to (5.14) in any case.

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5.2.2 Comparison with previous results

Specializing this construction to the case of quadratic superpotential $W = \frac{\mu}{2}\mathrm{Tr}\Phi^2 + \sum_{i=1}^{N_f}\sqrt{2}\widetilde{Q}^i\Phi Q_i + \sum_{i=1}^{N_f} m_i\widetilde{Q}^i Q_i$, we recover the class of models studied in [3]. We will see that our curve allows to reproduce the results presented in that paper. Our curve is given by (5.12) with

$$\widetilde{W}'(v) = \mu v - M; \qquad M \equiv \frac{\sqrt{2}}{N}\langle\widetilde{Q}^i Q_i\rangle.$$

The polynomial $f_{n-1}(v)$ in this case is just a constant, equal to μ times the gluino condensate S. We then immediately see that we can rewrite the matrix model curve as follows

$$v = \frac{w^2 + wM + \mu S}{\mu w} = \frac{(w - w_+)(w - w_-)}{\mu w}. \tag{5.15}$$

We have seen before that the coordinate w can be identified with $R(v)$ which encodes all the chiral condensates $\langle\mathrm{Tr}\Phi^n\mathcal{W}_\alpha\mathcal{W}^\alpha\rangle$. If one considers as in [36] the transformation $\delta Q^i = \frac{1}{v-\Phi}Q^i$, the corresponding anomalous Ward identity reads as (for fixed flavor index i)

$$\sqrt{2}\left\langle\widetilde{Q}^i\frac{1}{v-\Phi}Q_i\right\rangle = \frac{R(v)}{v+m_i} - \frac{R(v_I)}{v+m_i},$$

where with v_I $(I = 1, 2)$ we indicate the two preimages on the DV curve of the point $v = -m_i$ (see [36] for a detailed explanation of this point). We thus see that $R(v_I)$ (or equivalently the two solutions for w at $v = -m_i$) is naturally interpreted as $-\sqrt{2}\langle\widetilde{Q}^i Q_i\rangle$. It was shown in [3] that the matrix of meson vev's is diagonal with at most two different eigenvalues $M_\pm$. This is explained in the present framework by the possibility of choosing v_I in the above formula (clearly different choices correspond to different vacua of the theory).

Let us now focus for simplicity on the theory with equal bare masses ($m_i = m\ \forall i$). From the above argument we can compute the meson condensates simply setting $v = -m$ in the matrix model curve. This in turn implies that the parameters $w_\pm$ become precisely proportional to the meson vevs ($w_\pm = -\sqrt{2}M_\pm$), after a simple redefinition which brings the curve to the form

$$v + m = \frac{w^2 + wM + \mu S}{\mu w} = \frac{(w - w_+)(w - w_-)}{\mu w}.$$

This coincides with the one found in [3].

Another consequence of the analysis performed in [3] is that w is a global coordinate on the M-theory curve describing the massive deformation of $\mathcal{N} = 2$ SQCD. This fact is a simple consequence of the factorization condition: we can invert (5.13) to obtain

$$t = \frac{B(v) - D(v)w}{C(v)w - A(v)}.$$

Since we have just seen that v is a function of w only, it is enough to substitute (5.15) in the previous expression. The explicit form of the function $t(w)$ clearly depends on the boundary conditions we choose to impose for w. Notice that the DV curve only tells us that $W'(v)$ can be expressed as a function of w. We thus see that for a generic choice of superpotential it will not be true that t and v are functions of w only.

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5.3 Linear quivers

In order to extend what we have done so far to quiver gauge theories, we will again use the generalized Konishi anomaly equations as a guidance. We expect the curve to be described by the SW curve of the underlying $\mathcal{N} = 2$ theory, together with the DV curve which encodes the boundary conditions for the coordinate w (or equivalently which carries the information associated to the various superpotentials) and a constraint similar to (5.13), encoding the restrictions on the Coulomb branch coordinates.

5.3.1 Generalized anomaly equations for linear quivers

The analysis of [13, 36, 37] has been generalized to linear unitary quivers with gauge group $\prod_{k=1}^{n} \mathrm{SU}(N_k)$ in [39–41]. For each chiral multiplet in the adjoint representation Φ_k we can define the generating functional

$$R_k(v) = -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{\mathcal{W}_{k\alpha}\mathcal{W}_k^\alpha}{v-\Phi_k}\right\rangle .$$

Considering the variations

$$\delta\Phi_k = \frac{\mathcal{W}_{k\alpha}\mathcal{W}_k^\alpha}{v-\Phi_k} + \frac{32\pi^2 R_k(v)}{N_k} I,$$

we find the Ward identities (compare with equations (5.2)-(5.4) of [40]):

$$\begin{aligned}
R_k^2(v) &= \left(W_k'(v) - \langle W'_{\Phi_k}\rangle\right) R_k(v) - f_k(v) - \frac{1}{32\pi^2}\left\langle \mathrm{Tr}\widetilde{X}\frac{\mathcal{W}_{k\alpha}\mathcal{W}_k^\alpha}{v-\Phi_k}X\right\rangle \\
&\qquad + \frac{1}{32\pi^2}\left\langle \mathrm{Tr}X\frac{\mathcal{W}_{k\alpha}\mathcal{W}_k^\alpha}{v-\Phi_k}\widetilde{X}\right\rangle \qquad (k=2,\dots,n-1), \\
R_1^2(v) &= \left(W_1'(v) - \langle W'_{\Phi_1}\rangle\right) R_1(v) - f_1(v) - \frac{1}{32\pi^2}\left\langle \mathrm{Tr}\widetilde{X}\frac{\mathcal{W}_{1\alpha}\mathcal{W}_1^\alpha}{v-\Phi_1}X\right\rangle , \\
R_n^2(v) &= \left(W_n'(v) - \langle W'_{\Phi_n}\rangle\right) R_n(v) - f_n(v) - \frac{1}{32\pi^2}\left\langle \mathrm{Tr}X\frac{\mathcal{W}_{n\alpha}\mathcal{W}_n^\alpha}{v-\Phi_n}\widetilde{X}\right\rangle ,
\end{aligned} \tag{5.16}$$

where X denotes the bifundamentals, $\langle W'_{\Phi_k}\rangle$ is the vev of the $\mathrm{U}(N_k)$ equation of motion as explained in the previous section (divided by N_k as before) and

$$f_k(v) = \frac{1}{32\pi^2}\left\langle \frac{W_k'(\Phi_k) - W_k'(v)}{v-\Phi_k}\mathcal{W}_{k\alpha}\mathcal{W}_k^\alpha\right\rangle .$$

The further complication is the presence of the generating functionals $\left\langle \mathrm{Tr}\widetilde{X}\frac{\mathcal{W}_{k\alpha}\mathcal{W}_k^\alpha}{v-\Phi_k}X\right\rangle$, which can be evaluated considering more general variations as explained in detail in [40]. Combining all the chiral ring equations one can show that the $R_k(v)$'s functionals satisfy constraints of degree $k = 2, \dots, n+1$ (where n is the number of gauge groups), which in principle allow to determine them. In [40] the quadratic and cubic constraints have been determined explicitly. From now on we will denote the quantity $W_k'(v) - \langle W'_{\Phi_k}\rangle$ simply with $W_k'(v)$, hoping that no confusion arises. This is the analogue of the redefinition of the superpotential discussed in the previous section. With this modification the equations

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given in [40] apply in our case as well. For example, in the present notation the quadratic constraint reads:

$$\sum_{k=1}^{n-1} R_k R_{k+1} = \sum_{k=1}^{n} (R_k^2 - W_k' R_k + f_k). \tag{5.17}$$

5.3.2 The $\mathcal{N} = 1$ curve

From the analysis of SQCD in the previous section, we expect the right boundary condition for w at the i-th puncture to be of the form

$$w \to \sum_{k<i} W_k'(v),$$

where $i = 1, \ldots, n+1$ and $W_k'(\Phi_k)$ stands for the derivative of the "modified" superpotential associated to the k-th gauge group. Clearly we are free to impose vanishing boundary conditions at one puncture. This leads to the equation

$$\frac{w(w - W_1'(v))(w - W_1'(v) - W_2'(v)) \ldots (w - \sum_{i=1}^n W_i'(v))}{(\sum_{i=1}^n W_i'(v))^n} = 0,$$

or equivalently

$$w(w - W_1'(v))(w - W_1'(v) - W_2'(v)) \ldots \left(w - \sum_{i=1}^{n} W_i'(v)\right) + \sum_{j=2}^{n+1} w^{n+1-j} f_{(j-1)N-1}(v) = 0, \tag{5.18}$$

for some polynomials $f_{(j-1)N-1}(v)$ of degree $(j-1)N-1$ (notice that by assuming they are polynomials we impose the constraint that w does not diverge for v finite, which is indeed the expected behavior). In the above formula $N = \max_i\{\deg W_i'\}$.

In order to make contact with the generalized anomaly equations, we should now understand the relationship between the w coordinate and the generating functionals $R_k(v)$. Let us then reconsider the SQCD case: as we have seen, the boundary conditions lead to the curve $w(w - W'(v)) + f(v) = 0$. We then get the two solutions $w_1(v)$ and $w_2(v) = W'(v) - w_1(v)$. One of them coincides with $R(v) = \langle \mathrm{Tr} \frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{v - \Phi} \rangle$ (which one clearly depends on our choice of boundary conditions). With a shift in w we can eliminate the linear term finding the equation

$$y^2 - \frac{W'(v)^2}{4} + f(v) = 0. \tag{5.19}$$

The fact that $R(v)$ (defined as a power series in $1/v$) coincides with one of the two roots $w_{1,2}$ can then be rewritten as follows

$$\left(y - R(v) + \frac{W'(v)}{2}\right)\left(y + R(v) - \frac{W'(v)}{2}\right) = 0. \tag{5.20}$$

Equating (5.19) and (5.20), we then get a quadratic constraint for $R(v)$: $R^2(v) - W'(v)R(v) + f(v) = 0$, which reproduces precisely the anomalous Ward identity introduced previously. Our goal is then to generalize (5.20).

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We can easily eliminate from (5.18) the term of degree n in w with a shift. This leads to the equation

$$\prod_{i=1}^{n+1}\left(y+\sum_{k=i}^{n}\frac{n+1-k}{n+1}W_k'(v)-\sum_{k=1}^{i-1}\frac{k}{n+1}W_k'(v)\right)+\sum_{j=2}^{n+1}y^{n+1-j}g_{(j-1)N-1}(v)=0. \quad (5.21)$$

By analogy with the SQCD case, we expect the $n+1$ solutions in y (at fixed v) to be proportional to various linear combinations of $R_k(v)$'s. More precisely, we expect to be able to rewrite (5.21) in the form

$$\prod_{i=1}^{n+1}\left(y+\sum_{j=1}^{n}c_{ij}R_j(v)+\sum_{k=i}^{n}\frac{n+1-k}{n+1}W_k'(v)-\sum_{k=1}^{i-1}\frac{k}{n+1}W_k'(v)\right)=0. \quad (5.22)$$

Equating the terms with the same power of y in (5.21) and (5.22), leads to n equations in the $R_k(v)$'s which should reproduce the various constraints coming from the generalized Konishi anomaly equations.

Imposing that the terms proportional to y^{n-1} reproduce the quadratic constraint (5.17), we find the solution (setting $R_0(v)=R_{n+1}(v)=0$)

$$\sum_{j=1}^{n}c_{ij}R_j(v)=R_{i-1}(v)-R_i(v).$$

The curve (5.22) with this choice of c_{ij} already appeared in the literature: it coincides with the matrix model curve for linear quivers found in [42]. This naturally generalizes our findings for SQCD. As shown in [40], one can check that the above curve correctly reproduces the anomaly cubic constraint as well.

The above discussion shows that the coefficients of the polynomials $g_{jN-1}(v)$'s appearing in (5.21) are proportional to the various chiral condensates of the theory, exactly as for SQCD. In the simplest case of two gauge groups we can be completely explicit: in this case (5.21) reduces to

$$\left(y+\frac{2}{3}W_1'(v)+\frac{1}{3}W_2'(v)\right)\left(y-\frac{1}{3}W_1'(v)+\frac{1}{3}W_2'(v)\right)\left(y-\frac{1}{3}W_1'(v)-\frac{2}{3}W_2'(v)\right)$$
$$+g_{N-1}(v)y+g_{2N-1}(v)=0.$$

Using (5.22) and comparing with the explicit expressions for the quadratic and cubic constraints given in [40], we find

$$\begin{aligned}g_{N-1}(v)&=\sum_{k=1,2}f_k(v)=\frac{1}{32\pi^2}\sum_{k=1,2}\left\langle \mathrm{Tr}\frac{W_k'(\Phi_k)-W_k'(v)}{v-\Phi_k}\mathcal{W}_{k\alpha}\mathcal{W}_k^{\alpha}\right\rangle,\\ g_{2N-1}(v)&=f_2(v)\left(\frac{2W_1'}{3}+\frac{W_2'}{3}\right)-f_1(v)\left(\frac{W_1'}{3}+\frac{2W_2'}{3}\right)+g_1(v)+g_2(v),\end{aligned} \quad (5.23)$$

where

$$\begin{aligned}g_1(v)&=\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{W_1'(\Phi_1)-W_1'(v)}{v-\Phi_1}\mathcal{W}_{1\alpha}\mathcal{W}_1^{\alpha}X\widetilde{X}\right\rangle,\\ g_2(v)&=\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\frac{W_2'(\Phi_2)-W_2'(v)}{v-\Phi_2}\mathcal{W}_{2\alpha}\mathcal{W}_2^{\alpha}\widetilde{X}X\right\rangle.\end{aligned}$$

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The final point we have to discuss is the factorization condition. In the case of linear quivers the $\mathcal{N}=2$ curve is a polynomial equations of degree $n+1$ in t. The argument of the previous section for SQCD then implies that the factorization condition reads:

$$w = \frac{A_1(v)t^n + \cdots + A_{n+1}(v)}{B_1(v)t^n + \cdots + B_{n+1}(v)}.$$

6 Discussion

In this paper we have proposed a general framework to study the $\mathcal{N}=1$ breaking of class $\mathcal{S}$ theories. We provided for a large class of models a recipe for constructing the M-theory curve of the theory and we have seen that the condition determining whether a point on the Coulomb branch is lifted or not arises in a simple way from the geometric set-up. Our construction exploits the choice of a vector field and we have seen in the case of SU(2) SQCD that changing this choice allows to study the $\mathcal{N}=1$ breaking in S-duality frames. There are several directions worth to be further pursued.

In this paper we focused on A_{N-1} theories, it would be interesting to analyze along the same lines other classical groups.

Another aspect which deserves to be deepened is the analysis of the mathematical properties of the generalized Hitchin system, with particular emphasis on its moduli space and its possible relation with integrable systems and their quantization [43–50].

An intriguing related question is whether one can find a two-dimensional field theory description encoding the infrared properties of the $\mathcal{N}=1$ four dimensional gauge theory. For the subset of theories obtained via breaking of theories in $\mathcal{S}$-class the chiral condensates can be computed via instanton counting by fixing the Coulomb moduli to the vacua of the superpotential [51]. From the AGT viewpoint [52], this amounts to fix the momentum of the internal channel to some particular values, which correspond to the stationary points of the Hitchin system's Hamiltonians [53].

This has a resemblance with the insertion of surface operators in the four-dimensional gauge theory and the supersymmetry breaking induced by these defects. For example a relation between surface operators and DV curves has been recently studied in [54].

The geometrical framework we studied should help to elucidate and deepen our understanding of $\mathcal{N}=1$ dualities. There are a number of questions that should be investigated with respect to this issue, for example to which deformation the marginal parameters correspond to and how to read the superpotential of the dual theory. Presumably the choice of vector field plays an important role in this respect.

An interesting class of $\mathcal{N}=1$ four-dimensional gauge theories has been introduced recently in [55–57]. These are associated to bypartite graphs on bordered Riemann surfaces. It would be interesting to investigate whether these models have a connection with the ones described by the generalized Hitchin system.

Finally the analysis of $\mathcal{N}=1$ theories on more general four-dimensional manifolds could be considered in an extension of the geometrical framework considered in this paper.

Note added. While finalizing this paper, it appeared in the arXiv [58] where a similar geometric set-up is considered.

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A Comparison with IIA brane theory perspective

In this appendix we follow the method in [3, 59] and compute the $\mathcal{N} = 1$ curve. We see the agreement with the results in section 3 using the factorization condition from the generalized Hitchin viewpoint.

A.1 SU(2) SQCD

Let us begin with SU(2) gauge theory with $N_f \leq 4$ flavors. By the mass deformation of the adjoint chiral field, the loci on the Coulomb branch of corresponding $\mathcal{N} = 2$ theory where massless particles appear become the vacua of the $\mathcal{N} = 1$ theory. In this case it was argued in [3] that only if the $\mathcal{N} = 2$ curve is completely factorized to a sphere, the curve can be rotated to $x^{7,8,9}$ direction. Since w and v are related by $w = \mu v$ $(v \to \infty)$, we have [3]

$$v = \frac{(w - w_+)(w - w_-)}{\mu w}. \tag{A.1}$$

Writing in the form

$$w^2 - (\mu v w + c) + f_0 = 0, \tag{A.2}$$

which is identified with the DV curve, this implies $c = w_+ + w_-$ and $f_0 = w_+ w_-$. The curve is always of this form in SU(2) gauge theory with N_f flavors. This is also true in SU(N) gauge theory with N_f flavors.

A.1.1 Pure SYM

The brane configuration of this theory is described by two D4-branes suspended between two NS5-branes. Again we denote the NS5-branes as NS_1 and NS_2 below, as in figure 1. The $\mathcal{N} = 2$ Seiberg-Witten curve is (3.5). Let us consider the rotation of the $\mathrm{NS5}_2$-brane (at $t = 0$) which corresponds to the inclusion of the mass term of the adjoint chiral field. The boundary conditions are

$$\begin{aligned} \mathrm{NS}_1 :\ & w \to 0 \quad \text{as} \quad t = v^2/\Lambda^2 \quad (v \to \infty), \\ \mathrm{NS}_2 :\ & w \to \mu v \quad \text{as} \quad t = \Lambda^2/v^2 \quad (v \to \infty). \end{aligned} \tag{A.3}$$

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This completely fixes the $\mathcal{N}=1$ curve as follows. By using the DV curve we have $v \sim w/\mu$ when $w \to \infty$ and $v \sim (w_+ w_-/\mu\Lambda) w^{-1}$ when $w \to 0$. Thus the boundary conditions can be written as

$$\begin{aligned} \mathrm{NS}_1 :\ & w \to 0 \quad \text{as} \quad t = (w_+ w_-/\mu\Lambda)^2\, w^{-2}, \\ \mathrm{NS}_2 :\ & w \to \mu v \quad \text{as} \quad t = (\mu\Lambda)^2 w^{-2}. \end{aligned} \tag{A.4}$$

This means simply

$$t = \frac{(\mu\Lambda)^2}{w^2}, \tag{A.5}$$

with $w_+ w_- = (\mu\Lambda)^2$. In other words, this is expressed as

$$w^2 = \frac{(\mu\Lambda)^2}{t}, \tag{A.6}$$

Notice that we have unknown constants w_+ and w_- in the curve. This is determined by substituting (A.2) and (A.5) into the $\mathcal{N}=2$ curve (3.5) and expanding by w, which leads to $w_+ + w_- = 0$ and $w_+ w_- = (\mu\Lambda)^2$.

A.1.2 $N_f = 1$

There are two ways to construct $\mathcal{N}=2$ SU(2) gauge theory with one flavors in the IIA and M-theory. One is to place the D6-brane to the right of the NS_2-brane. The other is to the left of the NS_1-brane. Let us consider these in order.

For the first case, the $\mathcal{N}=2$ curve is (3.9). Let us below focus on the massless case $m=0$. The massive case does not change the form of the final curve. Then, we consider the rotation of the NS_2-brane. The boundary conditions are

$$\begin{aligned} \mathrm{NS}_1 :\ & w \to 0 \quad \text{as} \quad t = v^2/\Lambda^2 = (w_+ w_-/\mu\Lambda)^2 w^{-2} \quad (v \to \infty), \\ \mathrm{NS}_2 :\ & w \to \mu v \quad \text{as} \quad t = \Lambda/v = \mu\Lambda/w \quad (v \to \infty). \end{aligned} \tag{A.7}$$

where we have used the DV curve as in (A.4). These determine

$$t = (\mu\Lambda)\frac{w - w_-}{w^2}, \tag{A.8}$$

where the root of the numerator was obtained by the fact that $t=0$ leads to $v=0$. This can be rewritten as

$$w^2 - \frac{\mu\Lambda}{t} w + \frac{\mu\Lambda w_-}{t} = 0. \tag{A.9}$$

By eliminating the linear term in v, we get

$$w^2 = \frac{(\mu\Lambda)^2}{4t^2} - \frac{\mu\Lambda w_-}{t}, \tag{A.10}$$

implying that V_2 has double poles at $t=0$. By checking the consistency with the $\mathcal{N}=2$ curve as above we get $w_+ + 2w_- = 0$ and $w_+^2 w_- = -(\mu\Lambda)^3$.

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Let us then consider the second realization of this theory. The SW curve is

$$\Lambda(v+m)t^2 - P_2(v)t + \Lambda^2 = 0. \tag{A.11}$$

Again we set $m = 0$ for simplicity. The boundary conditions are

$$\begin{aligned} w &\to 0 \quad \text{as} \quad t = v/\Lambda = (w_+ w_-/\mu\Lambda)w^{-1} \quad (v \to \infty), \\ w &\to \mu v \quad \text{as} \quad t = \Lambda^2/v^2 = (\mu\Lambda)^2 w^{-2} \quad (v \to \infty). \end{aligned} \tag{A.12}$$

This gives the $\mathcal{N} = 1$ curve:

$$t = (\mu\Lambda)^2 \frac{1}{w(w - w_+)}. \tag{A.13}$$

The constants w_+ and w_- are determined by the same condition as the first case. The $\mathcal{N} = 1$ curve is written as

$$w^2 - w_- w - \frac{(\mu\Lambda)^2}{t} = 0. \tag{A.14}$$

The singularity structure is the same as that of the pure case in subsection A.1.1. Indeed, this should be the case since we have rotated the $\mathcal{N} = 2$ puncture of degree 3 at $t = 0$.

A.1.3 $N_f = 2$

As in subsection 3.1.3, we see three different realizations of this theory.

The first realization. The SW curve is (3.14). We would like to rotate the NS_2-brane corresponding to the regular puncture at $t = 1$. The boundary conditions are

$$\begin{aligned} w &\to 0 \quad \text{as} \quad t = v^2/\Lambda^2 = (w_+ w_-/\mu\Lambda)^2 w^{-2} \quad (v \to \infty), \\ w &\to \mu v \quad \text{as} \quad t = 1 \quad (v \to \infty). \end{aligned} \tag{A.15}$$

Thus the $\mathcal{N} = 1$ curve is

$$t = \frac{(w - w_+)^{2-r}(w - w_-)^r}{w^2} \tag{A.16}$$

with $r = 0, 1$. $w_\pm$ are determined by $w_+^r w_-^{2-r} = (\mu\Lambda)^2$ and $(2-r)w_+ + r w_- = 0$, which are the same as eq. (4.7) and (4.8) in [3]. Thus the interpretation of "r" is the same as in [3]: $r = 1$ corresponds to non-baryonic branch and $r = 0$ corresponds to a locus where the A-cycle collapses. In this SU(2) with $N_f = 2$ example, the $r = 0$ locus is just the root of the baryonic branch.

When $r = 0$, $w_+ = 0$. Therefore the DV and the $\mathcal{N} = 1$ curves get simplified to

$$w = \mu v + w_-, \quad t = 1. \tag{A.17}$$

Indeed, on the root of the baryonic branch, the NS_2-brane is detached from the other branes. Thus the NS_2 is rotatable, meaning a flat NS5-brane with $t = 1$ with $w = \mu v$. (See figure 2.) Let us notice that the $\mathcal{N} = 2$ curve is factorized as

$$0 = \Lambda^2 t^2 - P_2(v)t + v^2 = -(t-1)(-\Lambda^2 t + v^2). \tag{A.18}$$

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Thus, it is easy to see that the $\mathcal{N}=2$ Coulomb moduli is fixed to be $u=-\Lambda^2$.

When $r=1$, the curves are

$$w^2-\mu v w+(\mu\Lambda)^2=0, \quad w^2=\frac{(\mu\Lambda)^2}{t-1}. \tag{A.19}$$

The $\mathcal{N}=1$ curve agrees with (3.18).

The second realization. The $\mathcal{N}=2$ curve is (3.19). The boundary conditions are

$$\begin{aligned} &\mathrm{NS}_1: \ w\to 0 \quad \text{as} \ \ t=(w_+w_-/\mu\Lambda)w^{-1} \ \ (v\to\infty), \\ &\mathrm{NS}_2: \ w\to \mu v \ \ \text{as} \ \ t=(\mu\Lambda)w^{-1} \ \ (v\to\infty). \end{aligned} \tag{A.20}$$

Thus, we get

$$t=(\mu\Lambda)\frac{(w-w_-)^{r-1}}{w(w-w_+)^{r-1}}, \tag{A.21}$$

with $r=0,1$. $w_\pm$ are determined by the same equations as above.

When $r=0$, the curves again get simplified:

$$w=\mu v+w_-, \quad w-w_-+\frac{\mu\Lambda}{t}=0. \tag{A.22}$$

The $\mathcal{N}=1$ curve denotes that when $t\to\infty$ we have $w=w_-$, and when $t\to 0$ we have $w=\infty$. This precisely describes the rotation of the NS_2 on the root of the baryonic branch as depicted in figure 3. We interpret w_- as the position in w-direction of the D4-brane stretched between the NS_2 and D6-branes. This agrees with the result obtained in (3.22). Indeed the $\mathcal{N}=1$ curve (A.22) can be written as $w^2-\mu\Lambda(1/t\pm 1)w=0$, since $w_-=\pm\mu\Lambda$. Thus by shifting w-coordinate, we have $\tilde{w}^2=(\mu\Lambda/2)^2(1/t\pm 1)^2$, which is (3.22).

When $r=1$,

$$w^2-\mu v w+(\mu\Lambda)^2=0, \quad w-\frac{(\mu\Lambda)}{t}=0. \tag{A.23}$$

The $\mathcal{N}=1$ curve simply denotes that when $t\to\infty$, $w=0$ and when $t\to 0$, $w\to\infty$, which is depicted in figure 3. The argument as above shows that this agrees with (3.23).

The third realization. The $\mathcal{N}=2$ curve is (3.24). We only show the results here. The DV curves for $r=0,1$ are the same as above. The $\mathcal{N}=1$ curves are as follows: when $r=0$

$$w^2-2w_-w+w_-^2-\frac{(\mu\Lambda)^2}{t}=0, \tag{A.24}$$

with $w_-=\pm\mu\Lambda$, meaning that when $t\to\infty$ the equation has double zero at w_+ which is the position of D4-branes at $t=\infty$ in w-direction (see figure 4); when $r=1$

$$w^2+w_+w_--\frac{(\mu\Lambda)^2}{t}=0, \tag{A.25}$$

meaning that when $t\to\infty$ the equation has two zeros at $w_\pm$ ($w_++w_-=0$). This denotes the non-baryonic branch as in figure 4. The NS_2-brane cannot detach from the D4-branes because of the s-rule.

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A.1.4 $N_f = 3$

We consider two realizations depicted in figure 5 in order.

The first realization. The SW curve is

$$\Lambda(v+m_1)t^2 - P_2(v)t + (v+m_2)(v+m_3) = 0. \tag{A.26}$$

There are two regular punctures at $t=0$ and $t=-1$ and one irregular puncture of degree 4 at $t=\infty$. As before we consider the massless case.

The boundary conditions after the rotation of the NS_2-brane are

$$\begin{aligned} &\mathrm{NS}_1: \ w \to 0 \quad \text{as} \ \ t = v/\Lambda = (w_+ w_-/\mu\Lambda)w^{-1} \ \ (v\to\infty), \\ &\mathrm{NS}_2: \ w \to \mu v \ \ \text{as} \ \ t = 1 \ \ (v\to\infty). \end{aligned} \tag{A.27}$$

Thus,

$$t = \frac{(w-w_+)^{2-r}}{(w-w_-)^{1-r}} \tag{A.28}$$

with $r=0,1$. It is easy to see $w_+^{-1+r}w_-^{2-r} = -\mu\Lambda$ and $(2-r)w_+ + (-1+r)w_- = 0$.

When $r=0$, the curves are

$$w^2 - (3w_+ + \mu v) + w_+ w_- = 0, \qquad w^2 + \frac{\mu\Lambda}{2}w - \frac{(\mu\Lambda)^2}{4(t-1)} = 0. \tag{A.29}$$

This corresponds to the locus in the Coulomb branch where the A-cycle collapses. This also expresses the behavior of the rotation of the $\mathcal{N}=2$ regular puncture: the behavior at singularity is just the same as that of the irregular puncture of degree 3.

When $r=1$, the curves are

$$w = \mu v + w_-, \qquad t = 1. \tag{A.30}$$

This describes the root of the baryonic branch, as can be seen in figure 5. Notice that the configuration in the figure can also be considered as the root of the non-braryonic branch by reconnecting the NS_2-branes to the D4-brane stretched between the NS_1 and D6-branes. Contrary to the first and the third realizations of the $N_f=2$ case, no s-rule is applied.

The second realization. The SW curve is (3.29). Let us present only the result of the massless case here. The DV curves for $r=0,1$ are the same as above. The $\mathcal{N}=1$ curves are

$$w^2 - \left(\frac{\mu\Lambda}{t} + 2w_-\right)w + w_-^2 + \frac{\mu\Lambda w_+}{t} = 0 \tag{A.31}$$

for $r=0$, and

$$w - w_- - \frac{\mu\Lambda}{t} = 0 \tag{A.32}$$

for $r=1$. In the former expression, we can shift w to get

$$w^2 = \frac{\mu^2\Lambda^2}{4t^2} - \frac{\mu\Lambda w_+}{t}. \tag{A.33}$$

where $w_- = -\mu\Lambda/4$. This agrees with (3.31). The latter denotes the brane system consisting of the NS_2, D4 and D6-branes (figure 5).

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A.1.5 $N_f = 4$

As the final example of SU(2) SQCD we consider the $N_f = 4$ case. The SW curve is (3.33). Again we consider the massless case. The boundary conditions are when $v \to \infty$, $t = q, 1$; $w \to \mu v$ as $v \to \infty$, $t = q$. The solution is

$$t = q(w - w_+)^{r-2}(w - w_-)^{2-r}, \tag{A.34}$$

where $r = 0, 1, 2$. By checking the consistency with the SW curve we see that $w_+ = w_-$. Therefore the $\mathcal{N} = 1$ curve is trivially $t = q$. In this theory all the vacua with $r = 0, 1, 2$ are at $u = 0$. Indeed, only the locus where the SW curve factorizes is at $u = 0$. However, this denotes the point where superconformal symmetry is restored. The adjoint chiral mass deformation of this point leads to $\mathcal{N} = 1$ superconformal fixed point in the IR [28]. Therefore, our curve system is related to this $\mathcal{N} = 1$ fixed point.

A.2 SU(2)^2 quiver

Let us then consider SU(2)^2 quiver gauge theory. The SW curve is (3.39).

A.2.1 Rotation of an NS5-brane

The $\mathcal{N} = 1$ deformation of this quiver has been studied in [59]. Let us first study the case where one NS5-brane is rotated.

Rotation of the NS$_3$. Let us first consider the mass deformation of the adjoint chiral of the second SU(2) vector multiplet by μ. This is described by the rotation of the right NS5-branes as in figure 7. We refer to three NS5-branes as NS$_1$, NS$_2$ and NS$_3$ from the left. The boundary conditions are

$$\begin{aligned} &\mathrm{NS}_1 : w \to 0 \quad \text{as} \quad t = -v^2/\Lambda_1^2 \quad (v \to \infty), \\ &\mathrm{NS}_2 : w \to w_2 \quad \text{as} \quad t = -1 \quad (v \to \infty), \\ &\mathrm{NS}_3 : w \to \mu v \quad \text{as} \quad t = -\Lambda_2^2/v^2 \quad (v \to \infty), \end{aligned} \tag{A.35}$$

where w_2 is an unknown constant. The DV curve is easily determined: since $w \to \mu v$, v is written as $v = \frac{P_3(v)}{\mu R_2(v)}$ where P_3 and R_2 are polynomials of degree 3 and 2. The boundary conditions at NS$_1$ and NS$_2$ means $R_2 = w(w - w_2)$. Thus

$$v = \frac{P_3(v)}{\mu w(w - w_2)}. \tag{A.36}$$

We will later see the relation between this and the DV curve obtained in the previous note.

Let us then write the $\mathcal{N} = 1$ curve. Let P_3 be $w^3 + p_1 w^2 + p_2 w + p_3$. Since the DV curve behaves $v \sim w/\mu$ at $w \to \infty$ and $v \sim -p_3/\mu w_2 w$ at $w = 0$, so $t = -(p_3/\mu\Lambda_1 w_2)^2 w^{-2}$ and $t = -(\mu\Lambda_2)^2 w^{-2}$ at these points. Thus, the curve is simply

$$w^2 + \frac{(\mu\Lambda_2)^2}{t} = 0 \tag{A.37}$$

with $p_3 = \mu^2\Lambda_1\Lambda_2 w_2$. This also determines $w_2 = \pm\mu\Lambda_2$. Indeed, this curve has the expected singularity of the rotated puncture at $t = 0$ of $\mathcal{N} = 2$ irregular one of degree 3,

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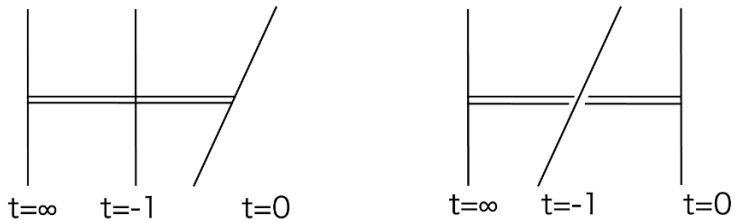

Figure 7. *Left*: the rotation of NS_3. *Right*: the rotation of NS_2. The NS5-branes can be detached from the rest.

seen in subsection A.1.1. This suggests that this behavior around the puncture is also true in the more generic theories and will help the analysis drastically.

The other coefficients in P_3 are obtained by substituting these two equations into $\mathcal{N}=2$ curve and by checking the orders w^{-1}, w^{-5} and w^{-6} terms

$$p_1 = -w_2, \quad p_2 = -p_3/w_2. \tag{A.38}$$

Now the DV curve can be written as, after the shift $w \to w + w_2/2$,

$$w^2(w - (\mu v - w_2/2)) + w f_0 + f_1(v) = 0, \tag{A.39}$$

where

$$f_0 = -\mu^2 \Lambda_1 \Lambda_2, \quad f_1 = \frac{\mu w_2^2}{4} v + \frac{\mu^2 w_2}{2}(\Lambda_1\Lambda_2 - \Lambda_2^2/4). \tag{A.40}$$

This is indeed the form of the DV curve obtained in the previous note with $W_1' = 0$ and $W_2' = \mu v - w_2/2$.

Rotation of the NS_2. This corresponds to the introduction of the masses μ and $-\mu$ of the adjoint chirals of $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$ vector multiplets. Again the boundary conditions are

$$\begin{aligned} &\mathrm{NS}_1 : w \to 0 \quad \text{as} \quad t = -v^2/\Lambda_1^2 \quad (v \to \infty), \\ &\mathrm{NS}_2 : w \to \mu v \quad \text{as} \quad t = -1 \quad (v \to \infty), \\ &\mathrm{NS}_3 : w \to w_3 \quad \text{as} \quad t = -\Lambda_2^2/v^2 \quad (v \to \infty), \end{aligned} \tag{A.41}$$

The DV curve is not changed except that w_2 is now w_3

$$v = \frac{P_3(w)}{\mu w(w - w_3)}. \tag{A.42}$$

So the boundary conditions are $t = -(\frac{P_3(0)}{\mu\Lambda_1 w_3})^2 w^{-2}$ at $w = 0$ and $t = -(\frac{\mu\Lambda_2 w_3}{P_3(w_R)})^2 (w - w_3)^2$ at $w = w_3$. Thus

$$t = -\frac{(w - w_3)^2}{w^2}. \tag{A.43}$$

However, the w_3 is determined to vanish by checking the w^1 order terms of the $\mathcal{N}=2$ Seiberg-Witten curve (by substituting the above equations). Thus $t = -1$ which means

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that the NS_2-brane is detached from the rest. The rest part does not depend on the w-coordinate, namely on the fixed position of w, as can be seen in figure 7. This implies that the polynomial in the DV curve is $P_3 = w^3$, and the DV curve is $v = w/\mu$.

The detachment observed above means that the $\mathcal{N} = 2$ Seiberg-Witten curve is factorized to

$$0 = \Lambda_1^2 t^3 + P_{2,1}(v)t^2 + P_{2,2}(v)t + \Lambda_2^2 = (t+1)(\Lambda_1^2 t^2 + (v^2+f)t + \Lambda_2^2), \tag{A.44}$$

where $f = u_1 - \Lambda_1^2 = u_2 - \Lambda_2^2$, and $u_{1,2}$ are the Coulomb moduli of $\mathrm{SU}(2)_{1,2}$. Note that the moduli are related in these $\mathcal{N} = 1$ vacua, but still there is one (complex-)dimensional moduli space. Let us rewrite the latter part of the curve as

$$v^2 + f + \Lambda_1^2 t + \frac{\Lambda_2^2}{t} = 0. \tag{A.45}$$

As noticed in section 3.2, the theory with masses μ and $-\mu$ is $\mathcal{N} = 1$ $\mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory with two bifundamental chiral multiplets. This choice was exactly the two NS_1 and NS_3 branes are parallel. And after the detachment of the NS_2 brane, the two D4-branes freely move describing the moduli of this theory.

A.3 SU(N) SQCD

Let us consider $\mathrm{SU}(N)$ SQCD with N_f flavors. The $\mathcal{N} = 2$ curve is

$$\Lambda^N t^2 + P_N(v)t + \Lambda^{N-N_f} \prod_{f=1}^{N_f} (v - m_f) = 0, \tag{A.46}$$

where P_N is a polynomial of degree N.

The boundary conditions are

$$\begin{aligned} &\mathrm{NS}_1 : w \to 0 \quad \text{as} \quad t = -v^N/\Lambda^N \quad (v \to \infty), \\ &\mathrm{NS}_2 : w \to \mu v \quad \text{as} \quad t = -v^{N_f - N}/\Lambda^{N-N_f} \quad (v \to \infty). \end{aligned} \tag{A.47}$$

The DV curve is again

$$v = \frac{(w - w_+)(w - w_-)}{\mu w}. \tag{A.48}$$

The $\mathcal{N} = 1$ curve is obtained by observing that when $w \to \infty$, $t = -(\mu\Lambda/w)^{N-N_f}$, and when $w \to 0$, $t = -(w_+ w_-/\mu\Lambda)^N w^{-N}$:

$$t = -(\mu\Lambda)^{N-N_f} \frac{(w - w_+)^r (w - w_-)^{N_f - r}}{w^N}, \tag{A.49}$$

where $r = 0, 1, \ldots, [N_f/2]$. $w_\pm$ are determined by

$$\begin{aligned} w_+^{N-r} w_-^{N-N_f+r} &= (-1)^{N_f} (\mu\Lambda)^{2N-N_f}, \\ (N - N_f + r)w_+ + (N - r)w_- &= 0. \end{aligned} \tag{A.50}$$

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The $\mathcal{N}=1$ curve in this vacuum can be written as

$$w^N + \sum_k V_k(t) w^{N-k} = 0, \tag{A.51}$$

where

$$\begin{aligned} V_1 &= V_2 = \ldots = V_{N-N_f-1} = 0, \\ V_{N-N_f} &= \frac{(\mu\Lambda)^{N-N_f}}{t}, \qquad V_{N-N_f+1} = \frac{(\mu\Lambda)^{N-N_f}(-rw_+ - (N_f-r)w_-)}{t}, \ldots, \\ V_N &= \frac{(\mu\Lambda)^{N-N_f}(-1)^{N_f} w_+^r w_-^{N_f-r}}{t}. \end{aligned} \tag{A.52}$$

In particular, for the pure SYM theory, we obtain the $\mathcal{N}=1$ curve (A.51) with

$$V_1 = V_2 = \ldots = V_{N-1} = 0, \qquad V_N = \frac{(\mu\Lambda)^N}{t}. \tag{A.53}$$

Namely, $w^N = (\mu\Lambda)^N/t$.

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