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QUALITATIVE PROPERTIES OF INTEGRO-EXTREMAL MINIMIZERS OF NON-HOMOGENEOUS SCALAR FUNCTIONALS

Sandro Zagatti

Department of Functional Analysis
International School for Advanced Studies – ISAS (SISSA)
Via Beirut 2/4, Grignano (TS), I-34014, ITALY
e-mail: zagatti@sissa.it

Abstract: We study the properties of the integro-extremal minimizers of functionals of the form

$$\mathcal{F}(u) = \int_\Omega f(x, Du)\, dx, \quad u \in \varphi + W_0^{1,p}(\Omega).$$

We give a well posedness result and discuss the viscosity properties associated to the solution of the minimum problem in the non convex case.

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1. Introduction

We consider a functional of the form

$$\mathcal{F}(u) = \int_\Omega f(x, Du)\, dx, \qquad u \in \varphi + W_0^{1,p}(\Omega), \tag{1.1}$$

where $p \geq 1$, $\varphi \in W^{1,p}(\Omega)$ is a given datum and $f\colon \Omega \times \mathbf{R}^n \to \mathbf{R}$ is a continuous function satisfying natural growth condition ensuring the coercivity of $\mathcal{F}$ on the Sobolev space $W^{1,p}(\Omega)$.

When the map $\xi \mapsto f(x, \xi)$ is not convex the existence of minimizers is not guaranteed, hence it is convenient to introduced the relaxed functional

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$$\overline{\mathcal{F}}(u) = \int_\Omega f^{**}(x, Du)\, dx, \qquad u \in \varphi + W_0^{1,p}(\Omega), \tag{1.2}$$

where f^{**} is the lower convex envelope of f with respect to the second variable.

The aim of this paper is to study qualitative properties of the integro-extremal minimizers of $\overline{\mathcal{F}}$ introduced in papers [5], [6] and [7].

We concentrate on integro-maximal minimizers, since the treatment of integro-minimal ones is absolutely analogous, and give a well posedness result, i.e. uniqueness and continuous dependence on boundary data. In addition we discuss its use in the solution of the minimum problem for the non-convex functional $\mathcal{F}$ showing that, under suitable assumptions, the integro-maximal minimizer of $\overline{\mathcal{F}}$ is a viscosity solution of the equation

$$f^{**}(x, Du(x)) - f(x, Du(x)) = 0. \tag{1.3}$$

This fact implies, in particular, that whenever it is (classically) differentiable almost everywhere inside the set Ω, the integro-maximal minimizer of $\overline{\mathcal{F}}$ is also a minimizers of $\mathcal{F}$.

2. Notations and Preliminaries

We denote respectively by $\langle \cdot, \cdot \rangle$ and by $|\cdot|$ the inner product and the Euclidean norm in $\mathbf{R}^n$. For $x \in \mathbf{R}^n$ and $r > 0$, $B(x, r)$ is the open ball of center x and radius r. Given $E \subseteq \mathbf{R}^n$, meas (E) denotes the Lebesgue measure of E; E^c, ∂E and $\mathrm{int}(E)$ are, respectively, the complement, the boundary and the interior of E. The letter $\mathbf{N}$ denotes the set of natural numbers $\{1, 2, \dots\}$ while $\mathbf{N}_0$ is the set $\mathbf{N} \cup \{0\}$. Given a map $g : \mathbf{R}^n \to \mathbf{R}$ we define its epigraph as the set

$$\mathrm{epi}\,(g) \doteq \{(\xi, t) \in \mathbf{R}^n \times \mathbf{R} : t \geq g(\xi)\},$$

remarking that, whenever g is a convex function, epi (g) is a convex subset of $\mathbf{R}^n \times \mathbf{R}$. We denote by g^{**} the bipolar function of g and refer to [2] for its definition as well as for other basic arguments of the Calculus of Variations widely used in the paper like, for example, sequential weak lower semicontinuity of convex functionals defined on Sobolev spaces.

Throughout the paper Ω is an open bounded subset of $\mathbf{R}^n$ and we consider the spaces $C^k(\Omega)$, $C_c^k(\Omega)$ $(k = 0, 1, \dots)$, $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, for $1 \leq p \leq \infty$, with their usual (strong and weak) topologies and identify a Sobolev function with its precise representative as defined, for example, in [3].

We need the notion semidifferentials and refer to the monograph [1] (Chapter II) for proofs, general setting and for the definition of viscosity solution of

Hamilton-Jacobi equations.

Definition 1. Let $U \subseteq \mathbf{R}^n$ be open, $v \in C^0(U)$ and $x_0 \in U$. We set

$$D^-v(x_0) \doteq \left\{ \xi \in \mathbf{R}^n : \liminf_{x \to x_0} \frac{v(x) - v(x_0) - \langle \xi, x - x_0 \rangle}{|x - x_0|} \geq 0 \right\}, \tag{2.1}$$

$$D^+v(x_0) \doteq \left\{ \xi \in \mathbf{R}^n : \limsup_{x \to x_0} \frac{v(x) - v(x_0) - \langle \xi, x - x_0 \rangle}{|x - x_0|} \leq 0 \right\}. \tag{2.2}$$

We call these sets, respectively, *super* and *sub* differentials (or semidifferentials) of u at the point x_0 and set also

$$A^-(v) \doteq \left\{ x \in U : D^-v(x) \neq \emptyset \right\}, \tag{2.3}$$

$$A^+(v) \doteq \left\{ x \in U : D^+v(x) \neq \emptyset \right\}. \tag{2.4}$$

Lemma 1. *Let $U \subseteq \boldsymbol{R}^n$ be open, $v \in C^0(U)$ and $x_0 \in U$.*

(i) $\xi \in D^-v(x_0)$ if and only if there exists a function $\phi \in C^1(U)$ such that $D\phi(x_0) = \xi$ and and the function $x \mapsto v(x) - \phi(x)$ has local minimum at the point x_0.

(ii) $\xi \in D^+v(x_0)$ if and only if there exists a function $\phi \in C^1(U)$ such that $D\phi(x_0) = \xi$ and and the function $x \mapsto v(x) - \phi(x)$ has local maximum at the point x_0.

(iii) $D^+v(x_0)$ and $D^-v(x_0)$ are closed convex possibly empty subsets of $\boldsymbol{R}^n$.

(iv) If v is differentiable at the point x_0 then $D^+v(x_0) = D^-v(x_0) = \{Dv(x_0)\}$.

(v) If both $D^+v(x_0)$ and $D^-v(x_0)$ are nonempty then u is differentiable at x_0 and $D^+v(x_0) = D^-v(x_0) = \{Dv(x_0)\}$.

(vi) The sets $A^-(v)$ and $A^+(v)$ are dense in U.

We recall from [6], [7] and [8] the following arguments.

Lemma 2. *Let U be an open subset of $\boldsymbol{R}^n$, $p \in [1, \infty]$, $v \in W^{1,p}(U) \cap C^0(U)$, $x_0 \in A^-(v)$, $\xi \in D^-v(x_0)$, $r > 0$, and $\rho > 0$ such that $B(x_0, \rho) \subseteq U$. Then there exists a map $\hat{v} \in W^{1,p}(U) \cap C^0(U)$ with the following properties:*

(i) $\hat{v} - v \in W_0^{1,p}(U)$;

(ii) $v(x) \leq \hat{v}(x)$ *for a.e.* $x \in U$;

(iii) $\hat{\Lambda} \doteq \{x \in U : \hat{v}(x) > v(x)\}$ *is nonempty and* $\hat{\Lambda} \subseteq B(x_0, \rho)$;

(iv) $\begin{cases} |D\hat{v}(x) - \xi| = r, & \text{for a.e. } x \in \hat{\Lambda} \\ D\hat{v}(x) = Dv(x), & \text{for a.e. } x \in U \setminus \hat{\Lambda}; \end{cases}$

(v) $\displaystyle\int_U \hat{v}\,dx > \int_U v\,dx;$

(vi) $\displaystyle\int_{\hat{\Lambda}} \langle m, D\hat{v}\rangle\,dx = \int_{\hat{\Lambda}} \langle m, Dv\rangle\,dx \quad \forall m \in C^0(\Omega) : \operatorname{div} m = 0 \text{ in } \mathcal{D}'(\Omega).$

Lemma 3. *Let Ω be on open bounded subset of $\mathbf{R}^n$ with Lipschitz boundary $\partial\Omega$ and $p \in [1, +\infty[$. Let $t \in \mathbf{R}^+$ be small and $u, v \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ be such that*

$$\|u - v\|_{C(\partial\Omega)} \le t. \tag{2.5}$$

Then there exist an open subset $\Omega_t \subseteq \Omega$ and a map $w_t \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ such that

(i) $\text{meas}\,(\Omega \setminus \Omega_t) \to 0 \quad \text{as} \quad t \to 0+;$

(ii) $\|w_t\|_{W^{1,p}(\Omega)} \le \Gamma,$ *where Γ is a positive constant independent on t;*

(iii) $w_t = u \quad \text{on} \quad \Omega_t;$

(iv) $w_t = v \quad \text{on} \quad \partial\Omega.$

3. Hypotheses

We consider a continuous function $f\colon \Omega \times \mathbf{R}^n \to \mathbf{R}$ and its lower convex envelope f^{**} with respect to the second variable. For $p \in [1, \infty[$ and given a boundary datum $\varphi \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, we introduce the functionals

$$\mathcal{F}(u) = \int_\Omega f(x, Du)\,dx, \qquad u \in \varphi + W_0^{1,p}(\Omega),$$

$$\overline{\mathcal{F}}(u) = \int_\Omega f^{**}(x, Du)\,dx, \qquad u \in \varphi + W_0^{1,p}(\Omega).$$

Hypothesis 1. We assume that the set S of minimizers of $\overline{\mathcal{F}}$ is nonempty and sequentially compact in $L^1(\Omega)$. In addition we require that any element of S belongs to $C^0(\overline{\Omega})$.

Hypothesis 2. The function $f^{**}\colon \Omega \times \mathbf{R}^n \to \mathbf{R}$ is (jointly) continuous and there exist $a, b, d > 0$ such that

$$a|\xi|^p - b \le f^{**}(x, \xi) \le d(1 + |\xi|^p) \qquad \forall x \in \Omega, \quad \forall \xi \in \mathbf{R}^n. \tag{3.1}$$

It is well known that, under this condition, there exists $c > 0$ such that

$$|f^{**}(x, \xi) - f^{**}(x, \eta)| \le c\left(1 + |\xi|^{p-1} + |\eta|^{p-1}\right)|\xi - \eta| \qquad \forall x \in \Omega, \quad \forall \xi, \eta \in \mathbf{R}^n. \tag{3.2}$$

Hypothesis 3. Define the set

$$X(x) = \{\xi \in \mathbf{R}^n \colon f(x,\xi) > f^{**}(x,\xi)\}. \tag{3.3}$$

We assume that, for every $x \in \Omega$, the set $X(x)$ is bounded (by its very definition it is convex). In addition we require the existence of a field $m \in C^0(\Omega, \mathbf{R}^n)$ such that

$$\operatorname{div} m = 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \tag{3.4}$$

and a map $q \in C^0(\Omega, \mathbf{R})$ satisfying the following conditions.

(i) Affinity on the set $\overline{X(x)}$:

$$f^{**}(x,\xi) = \langle m(x), \xi\rangle + q(x) \quad \forall \xi \in \overline{X(x)}; \tag{3.5}$$

$$f^{**}(x,\xi) \geq \langle m(x), \xi\rangle + q(x) \quad \forall \xi \in \mathbf{R}^n. \tag{3.6}$$

(ii) Extremality outside the set $\overline{X(x)}$: for every $x \in \Omega$ and for every $\xi \in (X(x))^c$, the point $(\xi, f(x,\xi))$ is an extremal point of $\operatorname{epi} f(x,\cdot)$.

As a consequence of *(ii)* we have the following implication: let $\xi, \eta \in \mathbf{R}^n$, $\xi \neq \eta$; then

$$f\left(x, \frac{\xi+\eta}{2}\right) = \frac{1}{2}\left[f(x,\xi) + f(x,\eta)\right] \quad \Longrightarrow \quad \xi, \eta \in \overline{X(x)}. \tag{3.7}$$

4. Well Posedness

Proposition 1. *Assume Hypothesis 1. Then there exists an element $\overline{u} \in S$ such that*

$$\int_\Omega \overline{u}(x)\,dx \geq \int_\Omega u(x)\,dx \quad \forall u \in S. \tag{4.1}$$

Proof. The proof follows immediately, by Weierstrass Theorem, from Hypothesis 1 and from the lower semicontinuity of the functional $\overline{\mathcal{F}}$ with respect to $L^1(\Omega)$-convergence.

Theorem 1. *Assume hypotheses 1 and 3 and let $\varphi \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$. Let $\overline{u}$ be the element of S given by Proposition 1. Then:*

(i) $\overline{u}$ is a viscosity solution of the equation

$$f^{**}(x, Du) - f(x, Du) = 0; \tag{4.2}$$

(ii) if $\overline{u}$ is differentiable almost everywhere on Ω, then $\overline{u}$ is a minimizer of $\mathcal{F}$.

(iii) the map $\overline{u}$ is unique, in the sense that if $v \in S$ and

$$\int_\Omega v(x)\, dx = \int_\Omega \overline{u}(x)\, dx\,, \tag{4.3}$$

then $v = \overline{u}$;

Proof. We prove uniqueness *(iii)*. The proof of *(i)* and *(ii)* can be found in [7].

Step 1. First of all consider the case in which the set $X(x)$ is empty for every $x \in \Omega$. This means that the map $\xi \mapsto f^{**}(x,\xi)$ is strictly convex for every $x \in \Omega$ and, consequently, by obvious computations, S is a singleton and there is nothing to prove.

Step 2. Let now $X(x)$ be nonempty for some $x \in \Omega$. For every $u \in S$ define the set

$$\Omega_X^u \doteq \left\{ x \in \Omega : Du(x) \in \overline{X(x)} \right\}. \tag{4.4}$$

Claim. *Take any pair (u,v) of elements of S. We have*

$$meas\ (\Omega_X^u \bigtriangleup \Omega_X^v) = 0, \tag{4.5}$$

where the symbol $\bigtriangleup$ stands for the symmetric difference.

If both Ω_X^u and Ω_X^v have measure zero there is nothing to prove. Assume, by contradiction, that the set

$$G \doteq \Omega_X^u \setminus \Omega_X^v$$

has positive measure. Recalling (3.7), we have

$$f^{**}\left(x, \frac{1}{2}Du(x) + \frac{1}{2}Dv(x)\right) < \frac{1}{2}f^{**}(x,Du(x)) + \frac{1}{2}f^{**}(x,Dv(x)) \qquad \text{for a.e.} \quad x \in G. \tag{4.6}$$

Define the map

$$w \doteq \frac{1}{2}u + \frac{1}{2}v.$$

Clearly w lies in the set $\varphi + W^{1,p}(\Omega)$ and, by (4.6), we have

$$\begin{aligned}
\overline{\mathcal{F}}(w) &= \int_G f^{**}(x,Dw)\, dx + \int_{\Omega\setminus G} f^{**}(x,Dw)\, dx \\
&= \int_G f^{**}\left(x, \frac{1}{2}Du + \frac{1}{2}Dv\right)\, dx + \int_{\Omega\setminus G} f^{**}\left(x, \frac{1}{2}Du + \frac{1}{2}Dv\right)\, dx \\
&\qquad < \frac{1}{2}\int_G f^{**}\left(x,Du\right)\, dx + \frac{1}{2}\int_G f^{**}\left(x,Dv\right)\, dx
\end{aligned}$$

$$+\int_{\Omega\setminus G} f^{**}\left(x,\frac{1}{2}Du+\frac{1}{2}Dv\right)\,dx \le \frac{1}{2}\int_{\Omega} f^{**}\left(x,Du\right)\,dx+\frac{1}{2}\int_{\Omega} f^{**}\left(x,Dv\right)\,dx$$

$$=\frac{1}{2}\overline{\mathcal{F}}(u)+\frac{1}{2}\overline{\mathcal{F}}(v)=\min\overline{\mathcal{F}}. \quad (4.7)$$

Inequality (4.7) is a contradiction and this proves that $\text{meas}\,(G)=0$. Interchanging the role of u and v we prove (4.5).

Choose any $v\in S$ and set

$$\Omega_X \doteq \Omega_X^v. \quad (4.8)$$

As a consequence of (4.5) and of definition (4.8) the following properties hold:

$$Du(x)\in\overline{X(x)} \quad \text{for a.e.} \quad x\in\Omega_X \quad \forall u\in S; \quad (4.9)$$

$$Du(x)\in\left(\overline{X(x)}\right)^c \quad \text{for a.e. } x\in\Omega\setminus\Omega_X \quad \forall u\in S. \quad (4.10)$$

Claim. *Take any pair (u,v) of elements of S. We have*

$$Du(x)=Dv(x) \quad \textit{for a.e.} \quad x\in\Omega\setminus\Omega_X. \quad (4.11)$$

Assume, by contradiction, that there exists a set $G\subseteq\Omega\setminus\Omega_X$ with $\text{meas}\,(G)>0$ such that

$$Dv(x)\neq Du(x) \quad \text{for a.e.} \quad x\in G. \quad (4.12)$$

Recalling (4.4), (4.6), (4.10), (4.12) we have

$$f^{**}\left(x,\frac{1}{2}Du(x)+\frac{1}{2}Dv(x)\right)<\frac{1}{2}f^{**}(x,Du(x))+\frac{1}{2}f^{**}(x,Dv(x))$$

$$\text{for a.e.} \quad x\in G.$$

Introduce as above the map $w\doteq\frac{1}{2}u+\frac{1}{2}v$: by the same computations of (4.7) we obtain a contradiction. Hence (4.11) is proved.

Step 4. Recalling Hypothesis 3 we set

$$g(x,\xi)\doteq f^{**}(x,\xi)-\langle m(x),\xi\rangle-q(x) \quad \forall\xi\in\mathbf{R}^n \quad \forall x\in\Omega$$

and introduce the functional

$$\mathcal{G}(u)=\int_{\Omega} g(x,Du(x))dx, \quad u\in\varphi+W_0^{1,p}(\Omega).$$

Given $u\in\varphi+W_0^{1,p}(\Omega)$, write $u=\varphi+z$, with $z\in W_0^{1,p}(\Omega)$. We have, by divergence theorem and by (3.4),

$$\mathcal{G}(u)=\int_{\Omega} f^{**}(x,Du(x))dx-\int_{\Omega}\left(\langle m(x),Du\rangle+q(x)\right)\,dx$$

$$= \int_\Omega f^{**}(x, Du(x))\, dx - \int_\Omega \langle m(x), D\varphi(x) + Dz(x)\rangle\, dx - \int_\Omega q(x)\, dx$$
$$= \overline{\mathcal{F}}(u) - \int_\Omega \langle m(x), D\varphi(x)\rangle\ dx - \int_\Omega q(x)\, dx = \overline{\mathcal{F}}(u) + \gamma,$$

where γ is a constant independent on u. Hence the set of minimizers of $\mathcal{G}$ coincides with the set S of minimizers of $\overline{\mathcal{F}}$ and for this reason we may assume, replacing f^{**} by g,

$$f^{**}(x, Du(x)) = 0 \quad \text{for a.e.} \quad x \in \Omega_X \quad \forall u \in S. \tag{4.13}$$

Step 5. Take now any pair (u, v) of elements of S and define the map

$$w(x) = (u \wedge v)(x) \doteq \max\{u(x), v(x)\} \quad \text{for a.e.} \quad x \in \Omega. \tag{4.14}$$

Claim. $w \in S$.

Clearly w lies in $\varphi + W^{1,p}(\Omega)$. By (4.11) we have

$$Dw(x) = Du(x) = Dv(x) \quad \text{for a.e. } x \in \Omega \setminus \Omega_F.$$

Consequently, by Stampacchia's Theorem, we have

$$Dw(x) = \begin{cases} Du(x) & \text{for a.e.} \quad x \in \Omega \setminus (G \cap \Omega_X) \\ Dv(x) & \text{for a.e.} \quad x \in G \cap \Omega_X, \end{cases} \tag{4.15}$$

where we have set

$$G \doteq \{x \in \Omega : v(x) > u(x)\}.$$

Formula (4.15) implies that

$$\overline{\mathcal{F}}(w) = \int_{\Omega\setminus(G\cap\Omega_X)} f^{**}(x, Du)\, dx + \int_{G\cap\Omega_X} f^{**}(x, Dv)\, dx. \tag{4.16}$$

But, by (4.9) and (4.13), we have

$$f^{**}(x, Du(x)) = f^{**}(x, Dv(x)) = 0 \quad \text{for a.e.} \quad x \in G \cap \Omega_X \tag{4.17}$$

and inserting (4.17) in (4.16) we obtain that

$$\overline{\mathcal{F}}(w) = \int_\Omega f^{**}(x, Du)\, dx = \overline{\mathcal{F}}(u) = \min \overline{\mathcal{F}}.$$

This proves the claim.

Step 6. Let now (u, v) be a pair of integro-maximal element of S, that is to say:

$$\int_\Omega v\, dx = \int_\Omega u\, dx = \max\left\{\int_\Omega z\, dx, z \in S\right\}. \tag{4.18}$$

Assume, by contradiction, that there exists a set G of positive measure such that $v(x) > u(x)$ for a.e. $x \in G$. Define the map $w \doteq u \wedge v \in S$ as in (4.14) and observe that necessarily we have

$$\int_\Omega w\,dx > \int_\Omega u\,dx. \tag{4.19}$$

By Step 5 the map w lies in S and then (4.19) contradicts (4.18). Hence we have

$$v(x) \le u(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{4.20}$$

Formulas (4.18) and (4.20) imply that $v = u$ almost everywhere on Ω and this ends the proof. □

Remark 1. If $\overline{u}$ is not differentiable almost everywhere in Ω we cannot conclude that $\overline{u}$ is a minimizer of $\mathcal{F}$. However, in this case, we have the following property.

Set

$$E \doteq \{x \in \Omega\colon D\overline{u}(x) \in X(x)\}\,. \tag{4.21}$$

(i) If $\operatorname{meas}(E) = 0$, then $\overline{u}$ is a minimizer of $\mathcal{F}$.

(ii) If $\operatorname{meas}(E) > 0$ then the following phenomenon occur:

for almost every $y \in E$, for every $r > 0$ and for every $M > 0$ the set

$$B(y,r) \cap \{x \in \Omega\colon |D\overline{u}(x)| \ge M\} \tag{4.22}$$

has positive measure.

To prove this statement assume, by contradiction, that there exists a point $x_0 \in E$, $r_0 > 0$ and $M_0 > 0$ such that

$$|D\overline{u}(x)| \le M_0 \quad \text{for a.e.} \quad x \in B(x_0, r_0). \tag{4.23}$$

It follows that $\overline{u}$ is Lipschitz continuous in the ball $B(x_0, r_0)$ and then it is differentiable almost everywhere in it. Then, invoking Lemma 2 and by a contradictory argument analogous to the one used in the proof of existence theorems in [6] and [7], we obtain an element $\hat{u}$ of S such that

$$\int_\Omega \hat{u}(x)\,dx > \int_\Omega \overline{u}(x)\,dx. \tag{4.24}$$

This inequality is absurd and the assertion is proved.

We turn now our attention to the dependence on the boundary data and formulate the following hypothesis.

Hypothesis 4. Let Ω be an open bounded subset of $\mathbf{R}^n$ with Lipschitz boundary. Let $\{\varphi_k, k = 0, 1, \dots\}$ be a sequence in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, bounded in $W^{1,p}(\Omega)$. We assume that

$$\varphi_k\lfloor_{\partial\Omega} \to \varphi_0\lfloor_{\partial\Omega} \quad \text{as} \quad k \to \infty \quad \text{in} \quad C^0(\partial\Omega). \tag{4.25}$$

For every $k \in \mathbf{N}_0$ introduce the set

$$\mathcal{W}_k \doteq \varphi_k + W^{1,p}(\Omega),$$

the variational problem

$$\text{Minimize} \quad \overline{\mathcal{F}}(u) = \int_\Omega f^{**}(x, Du)\, dx; \quad u \in \mathcal{W}_k,$$

the minimum

$$m_k \doteq \min \left\{ \overline{\mathcal{F}}(u); u \in \mathcal{W}_k \right\},$$

and the corresponding set of minimizers

$$S_k \doteq \left\{ u \in \mathcal{W}_k : \overline{\mathcal{F}}(u) = m_k \right\}.$$

We assume that the set S_k is compact in $L^1(\Omega)$ and that any element of S_k belong to $C^0(\overline{\Omega})$ and is differentiable almost everywhere in Ω for every $k \in \mathbf{N}_0$ (see [7] for conditions ensuring that these properties hold true).

Assuming that the hypotheses of Theorem 1 hold, for every $k \in \mathbf{N}_0$ we denote by $\overline{u}_k$ the unique integro-extremal elements of S_k as defined in Proposition 1. We have the following

Theorem 2. *Assume Hypotheses 2, 3 and 4 and call $\overline{u}_k$ the unique integro-maximal element of S_k given by Proposition 1. Then*

$$\overline{u}_k \to \overline{u}_0 \quad \textit{weakly in} \quad W^{1,p}(\Omega). \tag{4.26}$$

Proof. Step 1. For every $\epsilon > 0$ and for every $k \in \mathbf{N}_0$ we define the set

$$S_k^\epsilon \doteq \left\{ u \in \mathcal{W}_k : \overline{\mathcal{F}}(u) \leq m_k + \epsilon \right\}. \tag{4.27}$$

Let (u_j) be a sequence in S_k^ϵ; by the coercivity of $\overline{\mathcal{F}}$ (see (3.1)) we have that there exists $M > 0$ such that

$$\|Du_j\|_{W^{1,p}(\Omega)} \leq M \quad \forall j \in \mathbf{N}.$$

It follows that (u_j) admits a subsequence, still denoted by (u_j), weakly converging in $W^{1,p}(\Omega)$ to $u \in \mathcal{W}_k$. By sequential weak lower semicontinuity of $\overline{\mathcal{F}}$, we have

$$\overline{\mathcal{F}}(u) \leq \liminf \overline{\mathcal{F}}(u_j) \leq m_k + \epsilon;$$

hence u lies in S_k^ϵ and, in particular, by Rellich Theorem, such set turns out to be sequentially compact in $L^1(\Omega)$. Invoking Proposition 1 and Hypothesis 4, for every $k \in \mathbf{N}_0$ and for every $\epsilon > 0$ we may select an element $\overline{u}_k^\epsilon \in S_k^\epsilon$ such that

$$\int_\Omega \overline{u}_k^\epsilon\, dx \geq \int_\Omega u\, dx \quad \forall u \in S_k^\epsilon. \tag{4.28}$$

Claim. *For every $k \in \mathbf{N}_0$ we have*

$$\overline{u}_k^\epsilon \rightharpoonup \overline{u}_k \quad \text{weakly in} \quad W^{1,p}(\Omega) \quad \text{as} \quad \epsilon \to 0+\,. \tag{4.29}$$

Fix $k \in \mathbf{N}_0$. By the coercivity of $\overline{\mathcal{F}}$, and with the same argument used above, we deduce that there exists a positive constant M such that

$$\|\overline{u}_k^\epsilon\|_{W^{1,p}(\Omega)} \leq M, \quad \forall \epsilon > 0.$$

Take any sequences (ϵ_j) and $(\overline{u}_k^{\epsilon_j})$ such that $\epsilon_j \to 0+$ and

$$\overline{u}_k^{\epsilon_j} \rightharpoonup v \quad \text{weakly in} \quad W^{1,p}(\Omega) \quad \text{as} \quad j \to \infty \tag{4.30}$$

for some $v \in \mathcal{W}_k$. By the sequential weak lower semicontinuity of $\overline{\mathcal{F}}$ we have

$$\overline{\mathcal{F}}(v) \leq \liminf_{j\to\infty} \overline{\mathcal{F}}(\overline{u}_k^{\epsilon_j})$$

and, since

$$\overline{\mathcal{F}}(\overline{u}_k^{\epsilon_j}) \leq m_k + \epsilon_j\,, \quad \forall j \in \mathbf{N}\,, \tag{4.31}$$

and $\epsilon_j \to 0+$, we obtain that

$$\overline{\mathcal{F}}(v) \leq m_k + \epsilon_j\,, \quad \forall j \in \mathbf{N}. \tag{4.32}$$

It follows that $\overline{\mathcal{F}}(v) \leq m_k$ and then

$$v \in S_k. \tag{4.33}$$

On the other hand, since $S_k \subseteq S_k^\epsilon$ for every $\epsilon > 0$, we have

$$\int_\Omega \overline{u}_k^\epsilon\, dx \geq \int_\Omega \overline{u}_k\, dx \quad \forall \epsilon > 0 \tag{4.34}$$

and consequently, by the continuity of the integral operator with respect to weak convergence in $W^{1,p}$, (4.30) and (4.34) imply that

$$\int_\Omega v\, dx \geq \int_\Omega \overline{u}_k\, dx. \tag{4.35}$$

Inequality (4.35), inclusion (4.33) and the uniqueness result of Theorem 1 imply that $v = \overline{u}_k$. Then the arbitrariness of (ϵ_j) and $(\overline{u}_k^{\epsilon_j})$ proves (4.29) and, in particular, by Rellich Theorem, we have

$$\int_\Omega \overline{u}_k^\epsilon\, dx \to \int_\Omega \overline{u}_k\, dx \quad \text{as} \quad \epsilon \to 0+\,. \tag{4.36}$$

Step 2. Consider the sequence $(m_k)_{k\in\mathbf{N}}$.

Claim.

$$m_k \to m_0 \quad \text{as} \quad k \to \infty. \tag{4.37}$$

First of all remark that Hypotheses 2 and 4 imply that (m_k) is bounded in

R. Take any converging subsequence $(\tilde{m}_k)$ and call $\tilde{m}$ its limit. Assume first $\tilde{m} < m_0$: there exist some positive δ, an index $k_\delta \in \mathbf{N}$ and elements $v_k \in S_k$ such that

$$\overline{\mathcal{F}}(v_k) = \tilde{m}_k \leq m_0 - \delta, \quad \forall k > k_\delta. \tag{4.38}$$

By coercivity of $\overline{\mathcal{F}}$ we may extract from (v_k) a subsequence, still denoted (v_k), weakly converging in $W^{1,p}(\Omega)$ to v. By convergence (4.25) of Hypothesis 4 we have that v lies in $\mathcal{W}_0$ and, in addition, by weak lower semicontinuity of $\overline{\mathcal{F}}$ and by (4.38),

$$\overline{\mathcal{F}}(v) \leq \liminf \overline{\mathcal{F}}(v_k) \leq m_0 - \delta < \min\left\{\overline{\mathcal{F}}(u); u \in \mathcal{W}_0\right\}. \tag{4.39}$$

Inequality (4.39) is absurd and then

$$\tilde{m} \geq m_0. \tag{4.40}$$

Recalling from Hypothesis 4 the properties of the sequence (φ_k) we apply Lemma 3, defining a family (Ω_k) of open subsets of Ω and a sequence $(u_k)_{k\in\mathbf{N}}$ in $W^{1,p}(\Omega)$ such that for every $k \in \mathbf{N}$ and for a suitable positive M independent on k, the following properties hold:

$$\|u_k\|_{W^{1,p}(\Omega)} \leq M, \tag{4.41}$$

$$u_k = \overline{u}_0 \quad \text{on} \quad \Omega_k, \tag{4.42}$$

$$u_k = \varphi_k \quad \text{on} \quad \partial\Omega \tag{4.43}$$

and, in addition,

$$\text{meas}\,(\Omega \setminus \Omega_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{4.44}$$

We stress that $\overline{u}_0$ is the integro-maximal minimizer of the variational problem with boundary datum φ_0. Conditions (4.41)-(4.44) imply that $u_k \in \mathcal{W}_k$ for every k and that

$$u_k \to \overline{u}_0 \quad \text{strongly in} \quad W^{1,p}(\Omega) \quad \text{as} \quad k \to \infty. \tag{4.45}$$

Then, using (3.2) of Hypothesis 2, we estimate

$$\begin{aligned} \left|\overline{\mathcal{F}}(u_k) - \overline{\mathcal{F}}(\overline{u}_0)\right| &= \left|\int_\Omega (f^{**}(x, Du_k) - f^{**}(x, D\overline{u}_0))\,dx\right| \\ &\leq c\int_\Omega (1 + |Du_k|^{p-1} + |D\overline{u}_0|^{p-1})|Du_k - D\overline{u}_0|\,dx \\ &\leq c\left(1 + M^{p-1} + \|D\overline{u}_0\|^{p-1}_{L^p(\Omega)}\right)\|Du_k - D\overline{u}_0\|_{L^p(\Omega)} \stackrel{k\to\infty}{\longrightarrow} 0, \end{aligned}$$

where we have used Hölder inequality and (4.45). Hence we have

$$\overline{\mathcal{F}}(u_k) \to \overline{\mathcal{F}}(\overline{u}_0) \quad \text{as} \quad k \to \infty \tag{4.46}$$

and, as a consequence of (4.46), $\forall\delta > 0$ there exists $k_\delta \in \mathbf{N}$ such that

$$\tilde{m}_k = \overline{\mathcal{F}}(\overline{u}_k) \leq \overline{\mathcal{F}}(u_k) \leq m_0 + \delta, \quad \forall k > k_\delta. \tag{4.47}$$

By the arbitrariness of δ we deduce from (4.47) that

$$\tilde{m} \leq m_0. \tag{4.48}$$

Inequalities (4.40) and (4.48) imply that $\tilde{m} = m_0$ and we conclude that any converging subsequence of (m_k) tends to m_0. This proves (4.37).

Step 3. Consider now the sequence $(\overline{u}_k)_{k\in\mathbf{N}}$.

Claim.

$$\overline{u}_k \rightharpoonup \overline{u}_0 \quad \textit{weakly in} \quad W^{1,p}(\Omega) \quad \textit{as} \quad k \to \infty. \tag{4.49}$$

First of all remark that, by the boundedness of the sequence (φ_k) in $W^{1,p}(\Omega)$ (Hypothesis 4) and by the coercivity of $\overline{\mathcal{F}}$, the sequence $(\overline{u}_k)$ is bounded in $W^{1,p}(\Omega)$. Take any weakly converging (in $W^{1,p}(\Omega)$) subsequence, still denoted $(\overline{u}_k)$, and call v its limit. Convergence (4.25) in Hypothesis 4 implies that $v \in \mathcal{W}_0$ and, by weak lower semicontinuity of $\overline{\mathcal{F}}$ and by (4.37), we have

$$\overline{\mathcal{F}}(v) \leq \liminf \overline{\mathcal{F}}(\overline{u}_k) = m_0.$$

Hence

$$v \in S_0. \tag{4.50}$$

Consider the sequence (u_k) defined in Step 2 with properties (4.41)-(4.44). By (4.37) and (4.46) we obtain that $\left[\overline{\mathcal{F}}(\overline{u}_k) - \overline{\mathcal{F}}(u_k)\right] \to 0$ as $k \to \infty$; recalling definition (4.27) it follows that for every $\epsilon > 0$ there exists $k_\epsilon \in \mathbf{N}$ such that $u_k \in S_k^\epsilon$ for all $k > k_\epsilon$. Hence, by (4.28), we have

$$\int_\Omega u_k \, dx \leq \int_\Omega \overline{u}_k^\epsilon \, dx \quad \forall k > k_\epsilon. \tag{4.51}$$

Recalling (4.36) and (4.45) we deduce from (4.51) and from $\overline{u}_k \rightharpoonup v$ in $W^{1,p}(\Omega)$ that

$$\int_\Omega \overline{u}_0 \, dx \leq \int_\Omega v \, dx;$$

then the definition of $\overline{u}_0$ and (4.50) imply that

$$\int_\Omega \overline{u}_0 \, dx = \int_\Omega v \, dx. \tag{4.52}$$

By Theorem 1 we deduce from (4.52) that $v = \overline{u}_0$. Hence any weakly converging subsequence of $(\overline{u}_k)$ tends weakly to $\overline{u}_0$ in $W^{1,p}(\Omega)$ and this proves (4.49).

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