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Strong asymptotics for Cauchy biorthogonal polynomials with application to the Cauchy two-matrix model

M. Bertola,[1] M. Gekhtman,[2] and J. Szmigielski[3]
[1] *CRM, Université de Montréal and Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec, H3G 1M8, Canada*
[2] *Department of Mathematics, University of Notre Dame, 255 Hurley, Notre Dame, Indiana 46556, USA*
[3] *Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, SK S7N 5E6, Canada*

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We apply the nonlinear steepest descent method to a class of 3 × 3 Riemann-Hilbert problems introduced in connection with the Cauchy two-matrix random model. The general case of two equilibrium measures supported on an arbitrary number of intervals is considered. In this case, we solve the Riemann-Hilbert problem for the outer parametrix in terms of sections of a spinorial line bundle on a three-sheeted Riemann surface of arbitrary genus and establish strong asymptotic results for the Cauchy biorthogonal polynomials. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4802455]

I. INTRODUCTION

In this paper, we study asymptotic behavior of a class of 3 × 3 Riemann-Hilbert problems (RHPs) motivated by the recently introduced two-matrix random model.[7] The model consists of two random Hermitian positive-definite matrices M_1, M_2 of size $n \times n$ with the probability measure

$$\mathrm{d}\mu(M_1, M_2) = \frac{1}{\mathcal{Z}_n} \frac{\mathrm{d}M_1 \mathrm{d}M_2}{\det(M_1 + M_2)^n} \mathrm{e}^{-N\mathrm{Tr}(U(M_1))} \mathrm{e}^{-N\mathrm{Tr}(V(M_2))}, \tag{1.1}$$

where U, V are scalar functions defined on $\mathbb{R}_+$. The model was termed the *Cauchy matrix model* because of the shape of the coupling term. Similarly, the case of the Hermitian one-matrix models for which the spectral statistics is expressible in terms of appropriate biorthogonal polynomials,[20] this two-matrix model is solvable with the help of a new family of biorthogonal polynomials named *the Cauchy biorthogonal polynomials* (CBOPs).[6]

The Cauchy biorthogonal polynomials are two sequences of monic polynomials $(p_j(x))_{j=0}^{\infty}, (q_j(y))_{j=0}^{\infty}$ with $\deg p_j = \deg q_j = j$ that satisfy

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} p_j(x) q_k(y) \frac{\mathrm{e}^{-N(U(x)+V(y))}}{x+y} \mathrm{d}x \mathrm{d}y = h_k \delta_{jk}\ , \quad \forall j, k \geq 0\ , h_k > 0. \tag{1.2}$$

These polynomials were studied in Ref. 9 in relation with the spectral theory of the cubic string.

In yet another application, in analogy to the moment problem approach to the Camassa-Holm peakons,[3,4] the predecessors of the Cauchy biorthogonal polynomials were used to study the peakon solutions of the Degasperis-Procesi wave equation.[19,18] More generally, the Cauchy biorthogonal polynomials are expected to play a role in a variety of inverse problems for the third order differential operators.[21]

The main features of CBOPs can summarized as follows:

1. they solve a four-term recurrence relation;
2. their zeroes are positive and simple;
3. their zeroes have the interlacing property;
4. they satisfy Christoffel-Darboux identities;

5. they can be characterized by a pair of 3×3 Riemann-Hilbert problems;
6. the solution to the corresponding Riemann-Hilbert problem yields all kernels for the correlation functions of the Cauchy two-matrix model.

Items 1–5 above have been addressed in Ref. 6 while item 6 was explained in Ref. 7. In the present paper, we apply the Deift-Zhou steepest descent method to the asymptotic analysis of the Riemann-Hilbert problem with the view towards applications to the biorthogonal polynomials and the spectral statistics of the Cauchy two-matrix model.

The paper is organized as follows. In Sec. II, we set up a Riemann-Hilbert problem that characterizes the biorthogonal polynomials and that is essential in evaluating the correlation kernels for the associated matrix model.

In Sec. III, we recall the results of Refs. 2 and 7 where a relevant potential theory problem was set up and solved. The resulting equilibrium measures are the key ingredient in the construction of the $\mathfrak{g}$-functions that pave the way for the Deift-Zhou steepest descent method.

The central section of the paper is Sec. IV, which deals with the nonlinear steepest descent analysis. We consider the general case in which the two equilibrium measures are supported on an arbitrary number of intervals. This prompts the use of a higher genus three-sheeted Riemann surface. Note that a Riemann surface of a similar structure was recently used in Ref. 22 to study Hermite-Pade approximations of pairs of functions that form generalized Nikishin systems.

We solve the Riemann-Hilbert problem for the outer parametrix in terms of sections of a spinorial line bundle in the spirit of Refs. 8 and 5. Much of the effort goes towards showing that the model problem for the outer parametrix always admits a solution (Proposition 4.4, Theorem 4.1).

It is perhaps worth mentioning that in Ref. 13 (Sec. 8), the authors approached a similar problem of solving a 4×4 RHP for multiple orthogonal polynomials that arise in the analysis of the two-matrix models with interaction $\mathrm{e}^{-N\mathrm{Tr}(M_1M_2)}$ instead of the Cauchy interaction in (1.1). Though similar, their approach differs in that they use the theorem on existence of meromorphic differentials on a Riemann surface without providing explicit formulas in terms of theta functions, contrary to the present work. Also we work with arbitrary two measures without being restricted to the choice of a quartic potential. Section V uses the asymptotic analysis in Secs. II–IV to discuss universality results for the spectral statistics of the two-matrix model. We find that individual spectral statistics exhibits the same universality phenomena as in the one-matrix model. We expect that the Cauchy two-matrix model might produce new universality classes in the case when supports of both equilibrium contain the origin. This case, however, is not considered in the present paper due to assumptions on the potentials (assumption 1). Relaxing this assumption requires a generalization of the potential theory problem studied in Ref. 2.

The Appendixes contain a discussion of a specific genus zero example (Appendix A) as well as notations and essential information regarding theta functions (Appendix B).

II. RIEMANN-HILBERT PROBLEM FOR CAUCHY BIORTHOGONAL POLYNOMIALS

In the case of ordinary orthogonal polynomials, a well-known characterization in terms of a Riemann-Hilbert problem was obtained in Ref. 15. Here, we present a similar characterization following.[6]

For symmetry reasons, we define

$$V_1(z) := U(z)\,, \qquad V_2(z) := V(-z). \tag{2.1}$$

Following Ref. 2, we consider potentials V_j that are subject to the following.

Assumption 1. The potentials $V_j(z)$ satisfy:

- *$V_j(z)$ is a* **real analytic function** *on $(-)^{j+1}\mathbb{R}_+$, $j = 1, 2$,*
- *the growth-conditions*

$$V_j(x) = -a_j \ln|x| + \mathcal{O}(1) \quad \text{as } x \to 0, \quad a_j > 1, \qquad \lim_{x\to(-1)^{j+1}\infty} \frac{V_j(x)}{\ln|x|} = +\infty, \tag{2.2}$$

- *the derivatives $V_j'(z)$ are meromorphic on a strip containing the whole real axis.*

Example 2.1. The typical examples are potentials of the form

$$V_j(x) = -a_j \ln|x| + P_j(x)\,, \quad a_j > 1, \tag{2.3}$$

where P_j are real polynomials such that $\lim_{x\to(-)^{j+1}\infty} P_j(x) = +\infty$.

The relevant Riemann-Hilbert problem that characterizes CBOPs is the following.

Problem 2.1. Find a matrix $\Gamma(z)$ such that

1. *$\Gamma(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$,*
2. *$\Gamma(z)$ satisfies the jump conditions*

$$\Gamma(z)_+ = \Gamma(z)_- \begin{bmatrix} 1 & \mathrm{e}^{-NV_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z \in \mathbb{R}_+ \tag{2.4}$$

$$\Gamma(z)_+ = \Gamma(z)_- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathrm{e}^{-NV_2(z)} \\ 0 & 0 & 1 \end{bmatrix}, \quad z \in \mathbb{R}_-, \tag{2.5}$$

where the negative axis is oriented towards $-\infty$,

3. *at $z = \infty$*

$$\Gamma(z) = \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{z^n} \end{bmatrix}; \tag{2.6}$$

4. *near $z = 0$*

$$\Gamma(z) = \left[\mathcal{O}(1),\ \mathcal{O}(\ln|z|),\ \mathcal{O}(\ln^2|z|)\right]. \tag{2.7}$$

Remark 2.1. The growth condition at $z = 0$ in Eqs. (2.7) can be replaced by $\mathcal{O}(1)$ if the densities e^{-NV_j} vanish at $x = 0$. In fact, after assumption 1 is put in place, we have $\mathrm{e}^{-NV_j(x)} = \mathcal{O}\left(|x|^{a_j N}\right)$ and this (using Plemelj formulas for the local model of Γ at $z = 0$) implies $\Gamma(0) = \mathcal{O}(1)$.

In Sec. 6.2 of Ref. 6, it was shown that (adapting the formulas to the present notation by observing that the matrix that we denote here by Γ corresponds to $\widehat{\Gamma}$ in Ref. 6),

$$\Gamma(z) = \tag{2.8}$$
$$\begin{bmatrix} p_n(z) & \dfrac{1}{2i\pi}\displaystyle\int_{\mathbb{R}_+} \frac{p_n(x)\mathrm{e}^{-NV_1(x)}\mathrm{d}x}{x-z} & \dfrac{1}{(2i\pi)^2}\displaystyle\int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \frac{p_n(x)\mathrm{e}^{-NV_1(x)-NV_2(y)}\mathrm{d}x\mathrm{d}y}{(y-z)(x-y)} \\ 2i\pi\,\widehat{p}_{n-1}(z) & 1 + \displaystyle\int_{\mathbb{R}_+} \frac{\widehat{p}_{n-1}(x)\mathrm{e}^{-NV_1(x)}\mathrm{d}x}{x-z} & \displaystyle\int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \frac{\widehat{p}_{n-1}(x)\mathrm{e}^{-NV_1(x)-NV_2(y)}\mathrm{d}x\mathrm{d}y}{2i\pi(y-z)(x-y)} + \frac{W_{\beta^*}(z)}{2i\pi} \\ \dfrac{(-)^n(2i\pi)^2}{h_{n-1}}p_{n-1}(z) & \dfrac{(-)^n 2i\pi}{h_{n-1}}\displaystyle\int_{\mathbb{R}_+} \frac{p_{n-1}(x)\mathrm{e}^{-V_1(x)}\mathrm{d}x}{x-z} & \dfrac{(-)^n}{h_{n-1}}\displaystyle\int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \frac{p_{n-1}(x)\mathrm{e}^{-NV_1(x)-NV_2(y)}\mathrm{d}x\mathrm{d}y}{(y-z)(x-y)} \end{bmatrix},$$

where $\widehat{p}_n$ are certain polynomials of exact degree n described in Ref. 6, $W_{\beta^*}(z) = \int_{\mathbb{R}_-} \frac{\mathrm{e}^{-NV_2(y)}}{y-z}\mathrm{d}y$ and all integrals are oriented integrals. We will study the asymptotic behavior of the solution of Problem 2.1 in various regions of the complex plane for

$$\mathbb{N} \ni N \to \infty\,, \quad n := N + r,\ r \in \mathbb{Z}, \tag{2.9}$$

where the integer r is bounded.

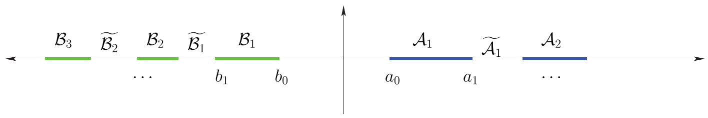

FIG. 1. The supports of the equilibrium measures ρ_1, ρ_2.

III. THE g FUNCTIONS

An asymptotic treatment for the RHPs for Γ along the lines of the nonlinear steepest descent method[10] requires that we *normalize* the problem with the use of an auxiliary matrix constructed from **equilibrium measures** that minimize a functional described below (see Refs. 2 and 7).

Consider the space of pairs of probability measures μ_1, μ_2 supported on $\mathbb{R}_+$,$\mathbb{R}_-$, respectively. On this space we define the functional

$$\begin{aligned} S[\mathrm{d}\mu_1, \mathrm{d}\mu_2] := \int_{\mathbb{R}_+} V_1(x)\mathrm{d}\mu_1(x) + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \mathrm{d}\mu_1(x)\mathrm{d}\mu_1(x')\ln\frac{1}{|x-x'|} + \\ + \int_{\mathbb{R}_-} V_2(y)\mathrm{d}\mu_2(y) + \int_{\mathbb{R}_-}\int_{\mathbb{R}_-} \mathrm{d}\mu_2(y)\mathrm{d}\mu_2(y')\ln\frac{1}{|y-y'|} + \\ \int_{\mathbb{R}_+}\int_{\mathbb{R}_-} \mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y)\ln|x-y|. \end{aligned} \tag{3.1}$$

The minimization of such a functional was studied in a more general setting in Ref. 2, it is related to a similar problem for a vector of measures of Nikishin type.[24]

Theorem 3.1 (see Theorem 3.2 in Ref. 2).
Under the assumptions 1 there exists a unique pair of densities ρ_1, ρ_2 that minimizes the functional (3.1). Moreover, (see Fig. 1) the supports consist of a **finite union** *of compact intervals and the densities ρ_j are smooth on the respective supports*

$$\mathrm{supp}(\rho_1) = \bigsqcup_{\ell=1}^{L_1} \mathcal{A}_\ell \subset \mathbb{R}_+ , \qquad \mathrm{supp}(\rho_2) = \bigsqcup_{\ell=1}^{L_2} \mathcal{B}_\ell \subset \mathbb{R}_- . \tag{3.2}$$

Remark 3.1. It has been recently proven by one of the authors and A. Kuijlaars that the growth condition near the origin in (2.2) can be simply disposed of. The properties in Theorem 3.1 are still valid except that the support of equilibrium measures may contain $x = 0$, in which case $\rho_j(x) = \mathcal{O}(x^{-\frac{2}{3}})$. This behavior is crucial in deriving a new type of universality near $x = 0$ and will be part of a forthcoming publication.

Theorem 3.2 (Theorem 5.1 in Ref. 7, see also Ref. 2).
The **shifted resolvents**

$$Y^{(1)} := -W_1 + \frac{2V_1' + V_2'}{3}, \qquad Y^{(2)} := W_2 - \frac{V_1' + 2V_2'}{3}, \text{ where } W_j(z) := \int \frac{\rho_j(x)}{x-z}\mathrm{d}x,$$
$$Y^{(1)} + Y^{(2)} + Y^{(0)} = 0 \tag{3.3}$$

are the three branches of the same **cubic** *equation in the form*

$$E(y, z) := y^3 - R(z)y - D(z) = 0, \tag{3.4}$$

where $R(z)$, $D(z)$ are certain functions analytic in the common domain of V_1' and V_2'.

As a corollary of Theorem 3.2 we deduce the following.

Corollary 3.1. For generic real analytic potentials the densities of the two equilibrium measures ρ_1, ρ_2 vanish like a square root at the endpoints of each interval $\mathcal{A}_\ell, \mathcal{B}_\ell$ in the support of the spectrum.

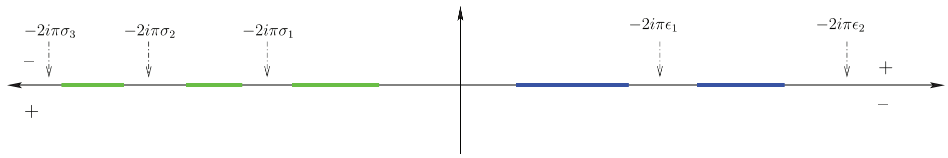

FIG. 2. The jumps of the $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ functions in the gaps.

Proof. It follows immediately from Cardano formulas, since the densities are related by Theorem 3.2 to the jump-discontinuity of the branches of the pseudo-algebraic curve (3.4) which have in general square-root type singularities corresponding to simple zeroes of the discriminant $\Delta = 4R^3(z) - 27D^2(z)$. **Q.E.D.**

Definition 3.1 ($\mathfrak{g}$-functions). The $\mathfrak{g}$-functions are defined as

$$\mathfrak{g}^{(1)}(z) := \int_{\mathbb{R}_+} \rho_1(x)\ln(z-x)\mathrm{d}x\ ;\ \mathfrak{g}^{(2)}(z) := \int_{\mathbb{R}_-} \rho_2(x)\ln(z-x)\mathrm{d}x\ ;\ \mathfrak{g}^{(0)}(z)+\mathfrak{g}^{(1)}(z)+\mathfrak{g}^{(2)}(z) \equiv 0, \tag{3.5}$$

where $\mathfrak{g}^{(1)}$ is defined as an analytic function in the domain $\mathcal{D}_1 := \mathbb{C}\setminus[a_0,\infty)$ and $\mathfrak{g}^{(2)}$ is analytic in $\mathcal{D}_2 := \mathbb{C}\setminus(-\infty, b_0]$, while $\mathfrak{g}^{(0)}$ is analytic in $\mathcal{D}_0 := \mathcal{D}_1\cap\mathcal{D}_2 = \mathbb{C}\setminus((-\infty,b_0]\cup[a_0,\infty))$ (see Fig. 2).

Remark 3.2. We remind the reader that the integral defining $\mathfrak{g}^{(2)}$ is an oriented integral. The orientation of the negative half-axis that we use may seem unusual. However, it is consistent with the one used in studies of multiple orthogonal polynomials on a collection of radial rays in the complex plane.

Definition 3.2. The right and left cumulative filling fractions are defined as

$$\epsilon_\ell := \int_{a_0}^{a_{2\ell-1}} \rho_1(w)\mathrm{d}w, \qquad \ell = 1,\ldots, \tag{3.6}$$

$$\sigma_\ell := \int_{b_0}^{b_{2\ell-1}} \rho_2(w)\mathrm{d}w, \qquad \ell = 1,\ldots. \tag{3.7}$$

Note that $\epsilon_{L_1} = 1$, while $\sigma_{L_2} = -1$.

The variational inequalities that characterize the equilibrium measures ρ_j,[23] together with the Definition 3.1 translate into the following theorem.

Theorem 3.3 (Analyticity properties for the $\mathfrak{g}$ functions). *The following properties hold:*

1. *$\mathfrak{g}^{(1)}$ is analytic in $\mathcal{D}_1$ and has the asymptotic behavior $\mathfrak{g}^{(1)} = \ln z + \mathcal{O}(z^{-1})$, $z\to\infty$;*
2. *$\mathfrak{g}^{(2)}$ is analytic in $\mathcal{D}_2$ and has the asymptotic behavior $\mathfrak{g}^{(2)} = -\ln z + \mathcal{O}(z^{-1})$, $z\to\infty$;*
3. *$\mathfrak{g}^{(0)}$ is analytic in $\mathcal{D}_0$ and has the asymptotic behavior $\mathfrak{g}^{(0)} = \mathcal{O}(z^{-1})$, $z\to\infty$;*
4. *on the **right cuts** $\mathcal{A}_\ell$ the function $\Im\left(\mathfrak{g}_+^{(1)}(x) - \mathfrak{g}_-^{(1)}(x)\right)$ is decreasing and there exists a constant γ_+ such that*

$$\mathfrak{g}_\pm^{(0)}(x) - \mathfrak{g}_\mp^{(1)}(x) + V_1(x) + \gamma_+ \equiv 0; \tag{3.8}$$

5. *on the **right gaps** $\widetilde{\mathcal{A}}_\ell$:*

$$\mathfrak{g}_+^{(1)}(x) - \mathfrak{g}_-^{(1)}(x) = -2i\pi\epsilon_\ell = \mathit{constant} \in i\mathbb{R}, \tag{3.9}$$

$$\Re\left(\mathfrak{g}_+^{(0)}(x) - \mathfrak{g}_-^{(1)}(x) + V_1(x) + \gamma_+\right) \geq 0; \tag{3.10}$$

6. *on the* **left cuts** $\mathcal{B}_\ell$ *the function* $\Im\left(\mathfrak{g}_+^{(2)}(x) - \mathfrak{g}_-^{(2)}(x)\right)$ *is decreasing and there exists a constant* γ_- *such that*

$$\mathfrak{g}_\pm^{(2)}(x) - \mathfrak{g}_\mp^{(0)}(x) + V_2(x) + \gamma_- \equiv 0; \tag{3.11}$$

7. *on the* **left gaps** $\widetilde{\mathcal{B}}_\ell$*:*

$$\mathfrak{g}_+^{(2)}(x) - \mathfrak{g}_-^{(2)}(x) = -2i\pi\sigma_\ell = constant \in i\mathbb{R}, \tag{3.12}$$

$$\Re\left(\mathfrak{g}_+^{(2)}(x) - \mathfrak{g}_-^{(0)}(x) + V_2(x) + \gamma_-\right) \geq 0; \tag{3.13}$$

8. *on the gaps* $\widetilde{\mathcal{B}}_\ell \bigcup \widetilde{\mathcal{A}}_\ell$*:*

$$\mathfrak{g}_+^{(0)}(x) - \mathfrak{g}_-^{(0)}(x) = \begin{cases} 2i\pi\epsilon_\ell & x \in \widetilde{\mathcal{A}}_\ell \\ 2i\pi\sigma_\ell & x \in \widetilde{\mathcal{B}}_\ell \end{cases}. \tag{3.14}$$

The proof is based on the facts that ρ_j are *positive* densities, the variational (in)equalities that arise from the minimization of the functional (3.1)[23,2] and elementary complex function theory. We leave the details to the reader.

Definition 3.3. If the statement that functions in items 4 and 6 above are decreasing is replaced by the requirement that their derivatives are strictly negative the inequalities in (3.10), (3.13) are strict then the potentials are said to be **regular**.

Because of the expressions (3.10), (3.13) we make the following.

Definition 3.4. Define the **effective complex potentials** *by*

$$\varphi_1(z) := V_1(z) - \mathfrak{g}^{(1)}(z) + \mathfrak{g}^{(0)}(z) + \gamma_+, \tag{3.15}$$

$$\varphi_2(z) := V_2(z) - \mathfrak{g}^{(0)}(z) + \mathfrak{g}^{(2)}(z) + \gamma_-. \tag{3.16}$$

The form of the $\mathfrak{g}$-functions (Definition 3.1) implies some important inequalities and equalities for the effective potentials that we presently explore.

Theorem 3.4. *The effective complex potentials* φ_1, φ_2 *satisfy the following properties:*

1. *For any* ℓ *there exists an open neighborhood of* $(a_{2\ell-2}, a_{2\ell-1})$ *for which the real part of* φ_1 *is negative away from the cut* $\mathcal{A}_\ell$;
2. *For any* ℓ *there exists an open neighborhood of* $(b_{2\ell-2}, b_{2\ell-1})$ *for which the real part of* φ_2 *is negative away from the cut* $\mathcal{B}_\ell$.

Proof. It suffices to consider any of the right cuts, say, $\mathcal{A}_\ell$. Then because of the identity $\sum \mathfrak{g}^{(j)} \equiv 0$ and from the properties specified in Theorem 3.3 (since $\mathfrak{g}_+^{(2)} - \mathfrak{g}_-^{(2)} = 0$) we have

$$\mathfrak{g}_+^{(1)} - \mathfrak{g}_-^{(1)} = -\mathfrak{g}_+^{(0)} + \mathfrak{g}_-^{(0)} \;\Rightarrow\; \mathfrak{g}_+^{(1)} + \mathfrak{g}_+^{(0)} = \mathfrak{g}_-^{(1)} + \mathfrak{g}_-^{(0)}. \tag{3.17}$$

More importantly, we see from (3.8) above that

$$\mathfrak{g}_+^{(1)} - \mathfrak{g}_-^{(1)} = \mathfrak{g}_+^{(1)} - \mathfrak{g}_+^{(0)} - V_1 - \gamma_+ = -\varphi_{1+} = \tag{3.18}$$

$$= -\left(\mathfrak{g}_-^{(1)} - \mathfrak{g}_-^{(0)} - V_1 - \gamma_+\right) = \varphi_{1-}. \tag{3.19}$$

Equation (3.18) implies that the jump $\Delta\mathfrak{g}^{(1)}$ is the left boundary value of the analytic function $-\varphi_1$, while Eq. (3.19) represents the same quantity as the right boundary value of the analytic function φ_1. The condition of strict decrease appearing in Theorem 3.3 then implies in view of the Cauchy-Riemann equations that $\Re\varphi_1$ strictly decreases as one moves perpendicularly away from the cut. **Q.E.D.**

Since the three $\mathfrak{g}$-functions are antiderivatives of the resolvents W_j (3.3), we deduce that there must be two constants $\gamma_{1,2}$ such that

$$\mathfrak{g}^{(1)}(z) = \gamma_1 + \frac{2V_1 + V_2}{3} - \int_{a_0}^{z} Y^{(1)}(\zeta)\mathrm{d}\zeta \ ; \ \ \mathfrak{g}^{(2)}(z) = \gamma_2 - \frac{2V_2 + V_1}{3} - \int_{b_0}^{z} Y^{(2)}(\zeta)\mathrm{d}\zeta, \qquad (3.20)$$

where the respective integrals are performed within the simply connected domains specified above and the constants $\gamma_{1,2}$ are chosen, so that asymptotically

$$\mathfrak{g}^{(1)}(z) = \ln z + \mathcal{O}(z^{-1}), \qquad \mathfrak{g}^{(2)}(z) = -\ln z + \mathcal{O}(z^{-1}). \qquad (3.21)$$

It thus appears that the three $\mathfrak{g}$-functions are intimately related to the three branches of the integral $\int y\mathrm{d}z$ over the three-sheeted Riemann surface (3.4).

IV. DEIFT-ZHOU STEEPEST DESCENT ANALYSIS

We will follow a well-established approach to asymptotic analysis of Riemann-Hilbert problems pioneered by Deift and Zhou in Ref. 11 and referred to as the *nonlinear steepest descent method*.

A. Modifications of the Riemann-Hilbert problem

We proceed to introduce the contours indicated in Fig. 3: the disks $\mathbb{D}_a$ around each endpoint $a \in \{a_j, b_j\}_{j=1,\ldots}$ of the cuts have sufficiently small radii so as not to include any other end points. The exact position and shape of the upper and lower arcs joining two end points are largely irrelevant; they should lie in the region where the real parts of the effective potentials (Definition 3.4) are negative. We will name the part outside all disks and all lenses the **outer region**.

Let us define a new piecewise analytic matrix function (only nonzero entries are indicated),

$$\Gamma_0 := \begin{cases} \Gamma & \text{in the outer region} \\ \Gamma \begin{bmatrix} 1 & & \\ -\mathrm{e}^{NV_1(z)} & 1 & \\ & & 1 \end{bmatrix} & \text{in the upper half of each lens on the right cuts} \\ \Gamma \begin{bmatrix} 1 & & \\ \mathrm{e}^{NV_1(z)} & 1 & \\ & & 1 \end{bmatrix} & \text{in the lower half of each lens on the right cuts} \\ \Gamma \begin{bmatrix} 1 & & \\ & 1 & \\ & \mathrm{e}^{NV_2(z)} & 1 \end{bmatrix} & \text{in the upper half of each lens on the left cuts} \\ \Gamma \begin{bmatrix} 1 & & \\ & 1 & \\ & -\mathrm{e}^{NV_2(z)} & 1 \end{bmatrix} & \text{in the lower half of each lens on the left cuts} \end{cases} . \qquad (4.1)$$

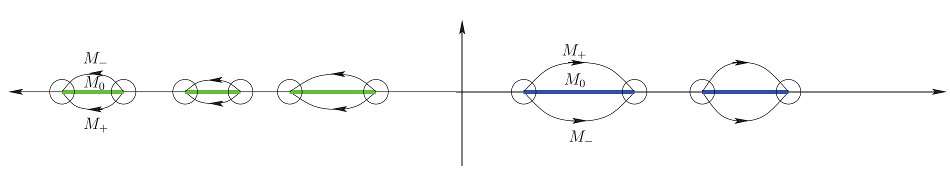

FIG. 3. The final steps in the modification of the RHP.

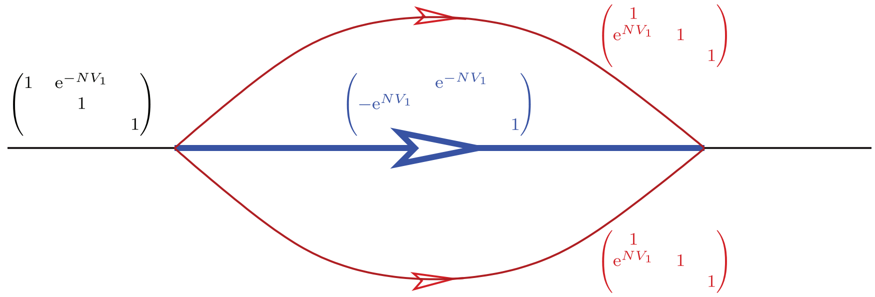

FIG. 4. The modified jumps for Γ_0 on a cut of the *right* side of the spectrum.

As a consequence the jumps of the Riemann-Hilbert problem are modified as indicated by the jump-matrices in Figs. 4 and 5. We now define

$$\mathbf{\Gamma}(z) := C_\gamma^{-1}\Gamma_0(z)G(z)C_\gamma\ , \qquad G(z) := \mathrm{diag}(e^{-N\mathfrak{g}^{(1)}(z)}, e^{-N\mathfrak{g}^{(0)}(z)}, e^{-N\mathfrak{g}^{(2)}(z)}), \tag{4.2}$$

$$C_\gamma := \begin{bmatrix} e^{N\frac{\gamma_- + 2\gamma_+}{3}} & & \\ & e^{N\frac{\gamma_- - \gamma_+}{3}} & \\ & & e^{-N\frac{\gamma_+ + 2\gamma_-}{3}} \end{bmatrix}. \tag{4.3}$$

The matrix $\mathbf{\Gamma}$ will be the main object of our interest from this point on. Recall that $\mathfrak{g}^{(j)} = (-1)^j \ln z + \mathcal{O}(z^{-1})$, for $j = 1, 2$, and that $n = N + r$. Then $\mathbf{\Gamma}$ satisfies a new Riemann-Hilbert problem

$$\mathbf{\Gamma}(z) = \left(\mathbf{1} + \mathcal{O}(z^{-1})\right) z^{\mathrm{diag}(r,0,-r)} \quad z \to \infty, \qquad \mathbf{\Gamma}_+(z) = \mathbf{\Gamma}_-(z)M(z), \tag{4.4}$$

where $M(z)$ takes different forms depending on the arc considered. Note that $\mathbf{\Gamma}$ is bounded at $z = 0$ because of assumption 1 and Remark 2.1. Recalling the definition of the effective potentials (Definition 3.4) we have (using the notation $\Delta f = f_+ - f_-$, $\mathcal{S}f = f_+ + f_-$),

$$\begin{aligned} M = M_\pm^{(right)} &:= \begin{bmatrix} 1 & & \\ e^{N\varphi_1} & 1 & \\ & & 1 \end{bmatrix}, \ \text{ on the upper/lower rims of a right lens,} \\ M = M_\pm^{(left)} &:= \begin{bmatrix} 1 & & \\ & 1 & \\ & e^{N\varphi_2} & 1 \end{bmatrix}, \ \text{ on the upper/lower rims of a left lens,} \end{aligned} \tag{4.5}$$

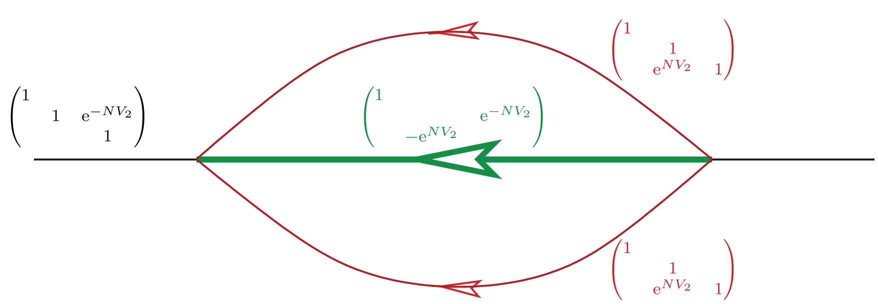

FIG. 5. The modified jumps for Γ_0 on a cut of the *left* side of the spectrum.

$$M = M_0^{(right)} := \begin{bmatrix} & e^{-N\left(V_1 - \mathfrak{g}_-^{(1)} + \mathfrak{g}_+^{(0)} + \gamma_+\right)} & \\ -e^{N\left(V_1 - \mathfrak{g}_-^{(1)} + \mathfrak{g}_+^{(0)} + \gamma_+\right)} & & \\ & & 1 \end{bmatrix} = \begin{bmatrix} & 1 & \\ -1 & & \\ & & 1 \end{bmatrix}, \quad (4.6)$$

on the right cuts,

$$M = M_0^{(left)} := \begin{bmatrix} 1 & & \\ & & e^{-N\left(V_2 - \mathfrak{g}_-^{(0)} + \mathfrak{g}_+^{(2)} + \gamma_-\right)} \\ & -e^{N\left(V_2 - \mathfrak{g}_-^{(0)} + \mathfrak{g}_+^{(2)} + \gamma_-\right)} & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & 1 \\ & -1 & \end{bmatrix}, \quad (4.7)$$

on the left cuts,

$$M = M_{gap}^{(right)} = \begin{bmatrix} e^{2i\pi N\epsilon_\ell} & e^{N\left(\mathfrak{g}_+^{(0)} - \mathfrak{g}_-^{(1)} - V_1 - \gamma_+\right)} & \\ & e^{-2i\pi N\epsilon_\ell} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} e^{\frac{N}{2}\Delta\varphi_1} & e^{-\frac{N}{2}\mathcal{S}\varphi_1} & \\ & e^{-\frac{N}{2}\Delta\varphi_1} & \\ & & 1 \end{bmatrix}, \quad (4.8)$$

on the right gaps,

$$M = M_{gap}^{(left)} = \begin{bmatrix} 1 & & \\ & e^{-2i\pi N\sigma_\ell} & e^{N\left(\mathfrak{g}_+^{(2)} - \mathfrak{g}_-^{(0)} - V_2 - \gamma_-\right)} \\ & & e^{2i\pi N\sigma_\ell} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & e^{\frac{N}{2}\Delta\varphi_2} & e^{-\frac{N}{2}\mathcal{S}\varphi_2} \\ & & e^{-\frac{N}{2}\Delta\varphi_2} \end{bmatrix} \quad (4.9)$$

on the left gaps.

Observe that the off-diagonal entries of (4.5), (4.9), and (4.8) are **exponentially small** as $N \to \infty$ because of the signs of the real parts; in particular all their L^p norms (except for $p = \infty$) are exponentially small. Yet, near the endpoints a_j, b_j of the support of the equilibrium measures, the off-diagonal entries tend to 1. However, the new RHP that effectively amounts to setting them to zero is a key ingredient in the final approximation we need. It is, in fact, the main new ingredient of the paper from the point of view of the nonlinear steepest descent method. More explicitly, we formulate the following.

Problem 4.1 (Outer parametrix). Find a 3×3 matrix $\Psi(z)$, analytic in $\mathcal{D}_0 := \mathbb{C} \setminus ((-\infty, b_0] \cup [a_0, \infty))$ and with the following properties:

(i) the jumps indicated in Figure 6;
(ii) the growth conditions at $z = \infty$ and near an endpoint $z = a$ are, respectively,

$$\Psi(z) = \left(\mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^r & & \\ & 1 & \\ & & z^{-r} \end{pmatrix}, \quad \Psi(z) = \mathcal{O}\left((z-a)^{-\frac{1}{4}}\right), \quad a \in \{a_i, b_i\}_{i=1,\ldots}. \quad (4.10)$$

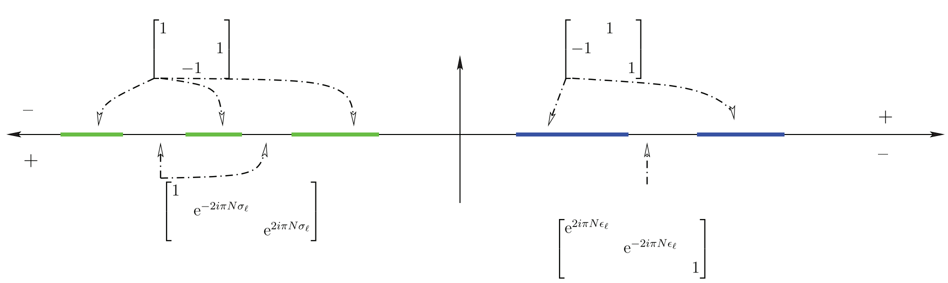

FIG. 6. The RHP for the outer parametrix.

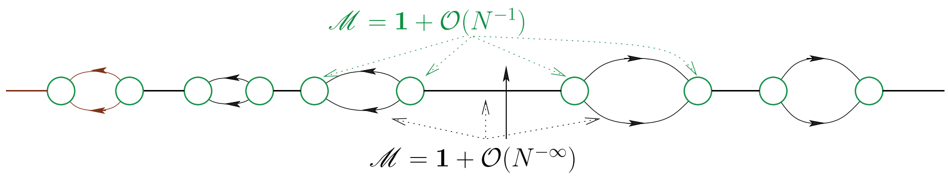

FIG. 7. The contours of the residual RHP for the error term and the orders of the jumps matrices.

The final approximation involves the solution of the exact RHP for $\mathbf{\Gamma}$ within disks around the endpoints that we have indicated above (see Figure 3). This is called the **local RHP** and is formulated below.

Problem 4.2 (Local parametrix). Let $\mathbb{D} = \mathbb{D}_a$ be a disk (previously introduced) around any of the endpoints a of the supports of ρ_1, ρ_2. Find a piecewise analytic 3×3 matrix $\mathcal{P}(z) = \mathcal{P}_a(z)$ such that

(i) $\mathcal{P}(z)$ is bounded for $z \in \mathbb{D}$ uniformly with respect to N;
(ii) within $\mathbb{D}$, $\mathcal{P}(z)$ satisfies the jump-conditions for $\mathbf{\Gamma}$ with jump-matrices (4.9), (4.8), (4.7), (4.5);
(iii) on the boundary $\partial\mathbb{D}$, $\mathcal{P}(z)\Psi^{-1}(z) = \mathbf{1} + o(1), \quad N \to \infty$, uniformly in $z \in \partial\mathbb{D}$.

Suppose now that we have managed to solve Problems 4.1 and 4.2 and let us define the matrix

$$\widehat{\mathbf{\Gamma}}(z) := \begin{cases} \Psi(z) & z \notin \bigcup_a \mathbb{D}_a \\ \mathcal{P}_a(z) & z \in \mathbb{D}_a \end{cases}. \tag{4.11}$$

Then the "error" matrix $\mathcal{E}(z) := \mathbf{\Gamma}(z)\widehat{\mathbf{\Gamma}}^{-1}(z)$ solves the RHP,

$$\mathcal{E}_+ = \mathcal{E}_-(\mathbf{1} + G)\,, \quad \mathcal{E}(z) = \mathbf{1} + \mathcal{O}(z^{-1}), \tag{4.12}$$

where the jumps are supported on the boundaries $\partial\mathbb{D}_a$, on the rims of the lenses and gaps *outside* the local disks. A direct standard inspection (see, e.g., Ref. 10) shows that G tends to zero as $N \to \infty$ in all L^p norms $p \in [1, \infty]$ and hence $\mathcal{E}(z)$ is close to the identity matrix $\mathbf{1}$ uniformly on $\mathbb{P}^1$. If the equilibrium problem is *regular* (in the sense of Remark 3.3), then the bounds on the jumps of the error term $\mathcal{E}$ (and hence on the error itself) are as depicted in Fig. 7.

B. Outer parametrix

The construction of the solution to the RHP 4.1 uses theta-functions associated with an algebraic curve studied below. Solutions to problems of this kind can be derived in several ways.[17,8] While we will eventually write down "explicit" formulas involving theta functions, these formulas are not numerically effective unless the underlying Riemann surface has genus zero. However, we do need to prove *existence* of a solution. Thus, our strategy will be to produce theta-functional expressions for the solution to Problem 4.1 and subsequently use results on bordered Riemann surfaces from Ref. 14 to ensure solvability of the RHP in terms of the proposed expressions. As an example, we will also provide the corresponding (explicit) expressions for the case of genus zero.

1. *The abstract Riemann surface*

The title of this section refers not to the Riemann surface defined by the pseudo-algebraic equation $E(y, z) = 0$ (3.4), but to an abstract Riemann surface $\mathcal{L}$ described below.

Let L_1, L_2 be the number of intervals of the supports of the measures ρ_1, ρ_2; note that endpoints of cuts correspond to zeroes of odd multiplicity of the discriminant equation $\Delta(z) := 4R^3(z) - 27D^2(z)$, and the assumption of regularity of the potentials translates into the requirement that all zeroes of odd multiplicity of $\Delta(z)$ are real and simple.

We consider now the abstract Riemann surface obtained by gluing together three Riemann spheres parametrized by the variable z; the Riemann sphere labelled 1 is slit along the support of ρ_1

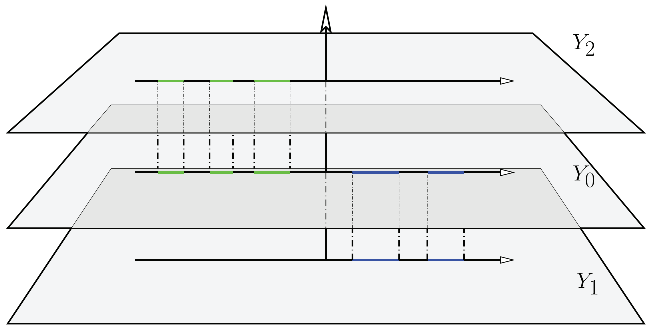

FIG. 8. The Hurwitz diagram of the abstract spectral curve $\mathcal{L}$: the vertical dotted lines represent the identification of points.

and glued there with the middle Riemann sphere labelled 0. The latter is subsequently slit also along the support of ρ_2 and glued across it with the Riemann sphere labelled 2, cut along the support of ρ_2 (see Fig. 8).

The resulting Riemann surface $\mathcal{L}$ is a compact surface of genus

$$g = genus(\mathcal{L}) = L_1 + L_2 - 2 \tag{4.13}$$

as follows from the Riemann-Hurwitz formula. The endpoints of the intervals $\mathcal{A}_\ell, \ell = 1, \dots L_1$ and $\mathcal{B}_\ell,\ \ell = 1, \dots, L_2$ are branch points of order 2, i.e., the local coordinate is $\sqrt{z - x_0}$, with x_0 one of the endpoints. The local coordinate around the three points at infinity ∞_1, ∞_0, ∞_2 is $1/z$.

This point of view allows one to think of $Y^{(0,\,1,\,2)}$ appearing in Theorem 3.2 as the three branches of a locally analytic function y over the Riemann surface $\mathcal{L}$; indeed near a branch point x_0, due to the assumption about Δ and due to Cardano formulas, the function y has a square-root Puiseux expansion in $z - x_0$, and thus is a well-defined function on the curve $\mathcal{L}$.

If V_1', V_2' are meromorphic functions, then y is a meromorphic function on $\mathcal{L}$, otherwise y will in general have other isolated singularities or will be defined only in a strip around the three-copies of the real axis in $\mathcal{L}$.

We will use the basis of the canonical homology of $\mathcal{L}$ indicated in Fig. 9.

Proposition 4.1. The curve $\mathcal{L}$ possesses a natural antiholomorphic involution *defined as the complex conjugation of each of the three sheets. The finite gaps provide $g = genus(\mathcal{L})$ pointwise invariant nontrivial cycles. Thus $\mathcal{L}$ can be realized as a double of the bordered Riemann surface obtained by identifying the upper half of sheet 2 with the lower half of sheet 0 along the left cuts, and the lower half of sheet 0 with the upper half of sheet 1 along the right cuts (see Chap. 6 in Ref. 14).*

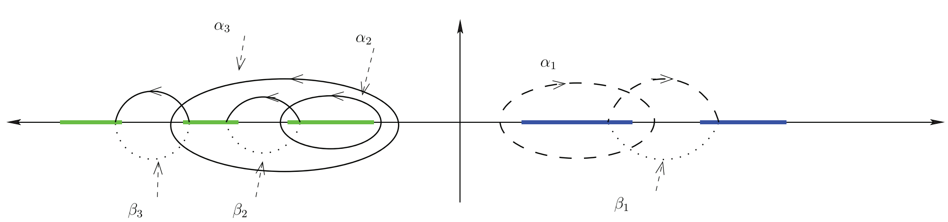

FIG. 9. Our choice of the canonical homology basis on the abstract Riemann surface $\mathcal{L}$; the solid lines represent arcs on Sheet 2, the dotted ones are arcs on Sheet 0, and the dashed ones are arcs on Sheet 1. In the example there are 5 total cuts and hence the genus is 3. We can declare that the curves lying on the same sheet are the α-cycles, whereas the curves lying on two different sheets are the corresponding β-cycles.

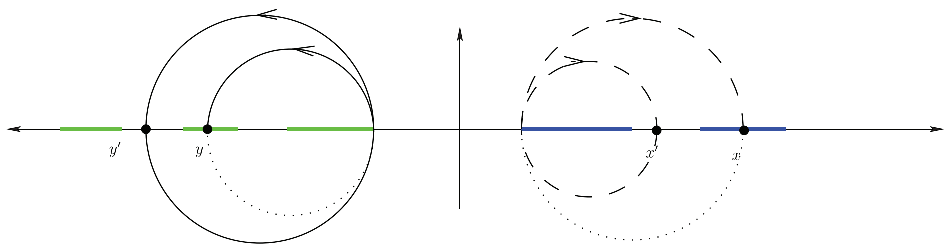

FIG. 10. Depiction of the lifted contours $\gamma_x, \gamma_{x'}, \gamma_y, \gamma_{y'}$. Their X-projection is just a closed path (in the picture we have chosen circles) intersecting the real axis only at x and one of b_0 or a_0. The points x', y' belong to the gaps and by definition the curves $\gamma_{x'}, \gamma_{y'}$ lie on one sheet and are actually homologic to the α-cycles chosen before (see Fig. 9 and its caption).

Proposition 4.1 shows that the properties of $\mathcal{L}$ are very similar to the properties of a hyperelliptic curve defined as $w^2 = P(z)$ for a *real* polynomial P of degree $2g + 2$. This fact will be used to obtain information on the theta divisor of the Jacobian of $\mathcal{L}$ when constructing the outer parametrix.

2. Multiplier system χ and spinors

Recall the definition of domains $\mathcal{D}_{0,1,2}$ (Definition 3.1). By lifting points in the complex plane z one can construct three sections $p_j : \mathcal{D}_j \to \mathcal{L}$ such that: p_1 is analytic in $\mathcal{D}_1 = \mathbb{C} \setminus \sqcup \mathcal{A}_\ell$, p_2 is analytic in $\mathcal{D}_2 = \mathbb{C} \setminus \sqcup \mathcal{B}_\ell$ and p_0 is a analytic in $\mathcal{D}_0 = \mathcal{D}_1 \cap \mathcal{D}_2$ with the following boundary values:

$$\begin{aligned} p_1(x)_\pm &= p_0(x)_\mp\,, \qquad x \in \mathcal{A} := \sqcup_{\ell=1}^{L_1} \mathcal{A}_\ell \\ p_2(x)_\pm &= p_0(x)_\mp\,, \qquad x \in \mathcal{B} := \sqcup_{\ell=1}^{L_2} \mathcal{B}_\ell. \end{aligned} \tag{4.14}$$

Definition 4.1. Let $x \in \mathbb{R}_+$ ($x \in \mathbb{R}_-$); the contour $\gamma_x : S^1 \to \mathcal{L}$ is defined as the unique lift to $\mathcal{L}$ of the closed path on $\mathbb{C}$ that contains the leftmost (rightmost) point $a_0 \in \mathcal{A}$ ($b_0 \in \mathcal{B}$) and intersecting the real axis only at a_0 and x (see Fig. 10). The lift is accomplished by using the map $p_1 : \mathcal{D}_1 \to \mathcal{L}$ for the part in the upper half-plane (or $p_2 : \mathcal{D}_2 \to \mathcal{L}$ if $x \in \mathbb{R}_-$). The lower part of the path is lifted using p_0 if x belongs to a cut or p_1 (p_2) when x belongs to a gap.

By this definition γ_x defines a closed cycle in the homology of $\mathcal{L}$. Moreover, γ_x span the whole homology as x ranges through $\mathbb{R}$.

We define the following vectors (characteristics) in the Jacobian of the curve $\mathcal{L}$:

$$\mathbf{A} = 2i\pi N \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{L_1-1} \\ \sigma_1 \\ \vdots \\ \sigma_{L_2-1} \end{bmatrix}, \qquad \mathbf{B} = 0 \in \mathbb{C}^g \tag{4.15}$$

where the cumulative filling fraction have been introduced in (3.6) and (3.7). Note the linear dependence of the vector $\mathbf{A}$ on N.

Given any two vectors $\mathbf{A} = (A_j)$, $\mathbf{B} = (B_j) \in \mathbb{C}^g$ one can define a **character**, namely, a homomorphism $\chi : \pi_1(\mathcal{L}) \to \mathbb{C}^*$ (with π_1 the fundamental group of the Riemann surface) by extending the values it takes on the basis of the homology, $\chi(\alpha_j) = \mathrm{e}^{A_j}$, $\chi(\beta_j) = \mathrm{e}^{B_j}$. For $\mathbf{A}, \mathbf{B}$ in (4.15) we have

$$\chi(\alpha_j) = \begin{cases} \mathrm{e}^{\mathrm{N2i\pi\epsilon_j}} & j \le L_1 - 1 \\ \mathrm{e}^{N2i\pi\sigma_{j-L_1+1}} & j > L_1 - 1 \end{cases}, \qquad \chi(\beta_j) = 1. \tag{4.16}$$

A **meromorphic spinor** ψ for the character χ is a multivalued half-differential that acquires the factor $\chi(\gamma)$ under analytic continuation along a closed contour γ. Any half-differential ϕ at a point p can be expressed in a local coordinate ζ as $\phi(p) = f(\zeta)\sqrt{\mathrm{d}\zeta}$. In particular, since z is a local coordinate away from the branch points, we can write $\psi(p) = f(z)\sqrt{\mathrm{d}z}$, where $p \in \mathcal{L}$ is a preimage of z. Recalling the definition of the three sections $p_j(z)$ given above we have

$$\psi(p_1(x))_\pm = \chi(\gamma_x)\psi(p_0(x))_\mp\,, \quad x \in \mathcal{A}\,, \quad \psi(p_1(x))_+ = \chi(\gamma_x)\psi(p_1(x))_-\,, \quad x \in \mathbb{R}_+ \setminus \mathcal{A}$$

$$\psi(p_2(x))_\pm = \chi(\gamma_x)\psi(p_0(x))_\mp\,, \quad x \in \mathcal{B}\,, \quad \psi(p_2(x))_+ = \chi(\gamma_x)\psi(p_2(x))_-\,, \quad x \in \mathbb{R}_- \setminus \mathcal{B} \tag{4.17}$$

$$\psi(p_0(x))_+ = \chi(\gamma_x)^{-1}\psi(p_0(x))_-\,, \quad x \in \mathbb{R}_+ \setminus (\mathcal{A} \cup \mathcal{B}). \tag{4.18}$$

In addition, we will use the spinor $\sqrt{\mathrm{d}z}$ defined on $\mathcal{L}$ slit along the top of the cuts $\mathcal{A}$ in sheet 1 and bottom on sheet 0, and along the top of $\mathcal{B}$ in sheet 2 and bottom in sheet 0. By definition $\sqrt{\mathrm{d}z}$ satisfies the boundary conditions

$$\sqrt{\mathrm{d}z}(p_1(x))_\pm = \pm\sqrt{\mathrm{d}z}(p_0(x))_\mp\,, \quad x \in \mathcal{A}, \tag{4.19}$$

$$\sqrt{\mathrm{d}z}(p_1(x))_+ = \sqrt{\mathrm{d}z}(p_1(x))_-\,, \quad x \in \mathbb{R} \setminus \mathcal{A}, \tag{4.20}$$

$$\sqrt{\mathrm{d}z}(p_2(x))_\pm = \pm\sqrt{\mathrm{d}z}(p_0(x))_\mp\,, \quad x \in \mathcal{B}, \tag{4.21}$$

$$\sqrt{\mathrm{d}z}(p_2(x))_+ = \sqrt{\mathrm{d}z}(p_2(x))_-\,, \quad x \in \mathbb{R} \setminus \mathcal{B}. \tag{4.22}$$

Note that the jump relations (4.19) and (4.21) imply that $\sqrt{\mathrm{d}z}$ is globally defined on a double cover of $\mathcal{L}$ branched at the *ramification* points (to be distinguished from the branch points) since a 2π–loop around one such point (i.e., a 4π-loop around $x = c$ for c a branch point) yields a transformation $\sqrt{\mathrm{d}z} \to -\sqrt{\mathrm{d}z}$. If we define a multivalued function $f(p) := \frac{\psi(p)}{\sqrt{\mathrm{d}X(p)}}$ it follows from the above that its composition with the three sections p_j (Sec. IV B 2) satisfies the following boundary-value conditions:

$$\begin{aligned} f(p_1(x))_\pm &= \pm\chi(\gamma_x)f(p_0(x))_\mp\,, && x \in \mathcal{A}\\ f(p_1(x))_+ &= \chi(\gamma_x)f(p_1(x))_-\,, && x \in \mathbb{R}_+ \setminus \mathcal{A}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} f(p_2(x))_\pm &= \mp\chi(\gamma_x)f(p_0(x))_\mp\,, && x \in \mathcal{B}\\ f(p_2(x))_+ &= \chi(\gamma_x)f(p_2(x))_-\,, && x \in \mathbb{R}_- \setminus \mathcal{B}. \end{aligned} \tag{4.24}$$

For our choice (4.16) of χ the above relations take the form

$$\begin{aligned} f(p_1(x))_+ &= f(p_0(x))_-\,, \quad f(p_0(x))_+ = -f(p_1(x))_-\,, && x \in \mathcal{A}\\ f(p_1(x))_+ &= \mathrm{e}^{2i\pi N\epsilon_j}f(p_1(x))_-\,, && x \in (a_{2j-1}, a_{2j}) \end{aligned} \tag{4.25}$$

$$\begin{aligned} f(p_2(x))_+ &= -f(p_0(x))_-\,, \quad f(p_0(x))_+ = f(p_2(x))_-\,, && x \in \mathcal{B}\\ f(p_2(x))_+ &= \mathrm{e}^{2i\pi N\sigma_\ell}f(p_2(x))_-\,, && x \in (b_{2\ell-1}, b_{2\ell}). \end{aligned} \tag{4.26}$$

Moreover, near a branch point $x = a$, since the local coordinate on the curve $\mathcal{L}$ is $\sqrt{z-a}$, the functions $f_j(z) := f(p_j(z))$ behave as

$$f_j(z) = \mathcal{O}((z-a)^{-\frac{1}{4}}). \tag{4.27}$$

All the above discussion amounts to the following.

Proposition 4.2. Let ψ be a meromorphic spinor for the character χ and let $f_j(\zeta) := \frac{\psi(p_j(\zeta))}{\sqrt{\mathrm{d}z(p_j(\zeta))}}$. Then the vector

$$\mathbf{F}(z) := [f_1(z), f_0(z), f_2(z)]\,, \quad z \in \mathbb{C} \setminus (-\infty, b_0] \cup [a_0, \infty) \tag{4.28}$$

has the properties

$$\mathbf{F}(x)_+ = \mathbf{F}(x)_- \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix}, \ x \in \mathcal{A}; \mathbf{F}(x)_+ = \mathbf{F}(x)_- \begin{pmatrix} \mathrm{e}^{2i\pi N\epsilon_j} & & \\ & \mathrm{e}^{-2i\pi N\epsilon_j} & \\ & & 1 \end{pmatrix}, \ x \in (a_{2j-1}, a_{2j}),$$

$$\mathbf{F}(x)_+ = \mathbf{F}(x)_- \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix}, \ x \in \mathcal{B}; \quad \mathbf{F}(x)_+ = \mathbf{F}_- \begin{pmatrix} 1 & & \\ & \mathrm{e}^{-2i\pi N\sigma_\ell} & \\ & & \mathrm{e}^{2i\pi N\sigma_\ell} \end{pmatrix}, \ x \in (b_{2\ell-1}, b_{2\ell}) \tag{4.29}$$

as well as $\mathbf{F}(x) = \mathcal{O}((x-a)^{-\frac{1}{4}})$ *for any endpoint* a.

This means that the jump conditions for Problem 4.1 are already satisfied. The complete solution amounts to finding a spinor ψ for each row such that it has an appropriate growth at the points above $z = \infty$. Generically, a spinor for a character χ is uniquely determined, up to a multiplicative constant, by choosing a divisor of degree -1. This is a consequence of Riemann–Roch–Serre theorem[1,16] for line-bundles (our line bundle is the tensor product of a flat line bundle defined by the character χ and a spinor bundle, namely, a line bundle whose square is the canonical bundle) We have the following.

Proposition 4.3. Consider the two sequences of spinors $\psi_r, \widehat{\psi}_r, \ r \in \mathbb{Z}$ *for the same character* χ *(4.16) satisfying the following divisor properties (see (B15) in Appendix B),*

$$(\psi_r) \ge -(r+1)\infty_1 + r\infty_2\,; \qquad (\widehat{\psi}_r) \ge -(r+1)\infty_1 - \infty_0 + (r+1)\infty_2. \tag{4.30}$$

Consider the matrix

$$\mathfrak{V}(z) := \begin{pmatrix} f_r(p_1) & f_r(p_0) & f_r(p_2) \\ \widehat{f}_{r-1}(p_1) & \widehat{f}_{r-1}(p_0) & \widehat{f}_{r-1}(p_2) \\ f_{r-1}(p_1) & f_{r-1}(p_0) & f_{r-1}(p_2) \end{pmatrix}, \qquad f_r := \frac{\psi_r}{\sqrt{\mathrm{d}z}}\,, \quad \widehat{f}_r := \frac{\widehat{\psi}_r}{\sqrt{\mathrm{d}z}}. \tag{4.31}$$

Then $\mathfrak{V}(z)$ *solves a RHP with the jumps (4.29), the asymptotic behavior*

$$\mathfrak{V}(z) = \mathrm{diag}(C_1, C_2, C_3)(\mathbf{1} + \mathcal{O}(z^{-1}))\mathrm{diag}(z^r, 1, z^{-r}) \tag{4.32}$$

and Problem 4.1 admits a solution if and only if $C_1C_2C_3 \neq 0$.

Proof. By Proposition 4.2 the jumps conditions are automatically fulfilled. As $z \to \infty$, $p_j(z) \to \infty_j \in \mathcal{L}$ by construction. Then

$$f_r(p_1) = K_r^{(1)} z^r (1 + \mathcal{O}(z^{-1}))\,, \, f_r(p_0) = K_r^{(0)} \tfrac{1}{z}(1 + \mathcal{O}(z^{-1}))\,, \, f_r(p_2) = K_r^{(2)} z^{-r-1}(1 + \mathcal{O}(z^{-1})),$$

$$\widehat{f}_{r-1}(p_1) = \widehat{K}_{r-1}^{(1)} z^{r-1}(1 + \mathcal{O}(z^{-1}))\,, \, \widehat{f}_{r-1}(p_0) = \widehat{K}_{r-1}^{(0)}(1 + \mathcal{O}(z^{-1}))\,, \, \widehat{f}_{r-1}(p_2) = \widehat{K}_{r-1}^{(2)} z^{-r-1}(1 + \mathcal{O}(z^{-1}))\,.$$

This asymptotic behaviour follows from the divisor properties of $\psi_r, \widehat{\psi}_r$; for example, since ψ_r has a pole at most of degree $r+1$ at ∞_1 where $\sqrt{\mathrm{d}z}$ has a simple pole, f_r must have at most a pole of order r at infinity.

The last claim follows from the fact that Problem 4.1 requires that the constants $K_r^{(1)}, K_{r-1}^{(2)}$, and $\widehat{K}_{r-1}^{(0)}$ should not vanish. It is only in this case that the matrix $\mathfrak{V}$ can be normalized by a left multiplication to behave like $(\mathbf{1} + \mathcal{O}(z^{-1}))\mathrm{diag}(z^r, 1, z^{-r})$. To prove the necessity of the last claim we have to show that if any of the constants $K_r^{(1)}, K_{r-1}^{(2)}, \widehat{K}_{r-1}^{(0)}$ vanish, then Problem 4.1 is unsolvable. To see this we first point out that standard arguments show that if a solution to Problem 4.1 exists, then that solution is unique. Therefore, the solution of Problem 4.1 exists *if and only if* the only row-vector solution $\mathbf{F}(z)$ of the RHP with the same jumps as 4.1 but with the asymptotic condition

(4.10) at $z = \infty$ replaced by

$$\mathbf{F}(z) = [\mathcal{O}(z^{r-1}), \mathcal{O}(z^{-1}), \mathcal{O}(z^{-r-1})] \tag{4.33}$$

is the trivial solution $\mathbf{F}(z) \equiv 0$. If any of the constants above is zero, the corresponding row, then provides a nontrivial solution precisely to this latter problem. This proves the necessity of the last claim. **Q.E.D.**

C. Theta-functional expressions

The notation below is borrowed from Ref. 8 and the definitions can be found in Ref. 14 and are reviewed in Appendix B.

The spinors $\psi_r, \widehat{\psi}_r$ in Proposition 4.3 can be written "explicitly" in terms of theta functions as follows:

$$\psi_r = \frac{\Theta_\Delta(p-\infty_2)^r}{\Theta_\Delta(p-\infty_1)^{r+1}} \Theta \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} (p + r\infty_2 - (r+1)\infty_1) h_\Delta(p), \tag{4.34}$$

$$\widehat{\psi}_{r-1} = \frac{\Theta_\Delta(p-\infty_2)^r}{\Theta_\Delta(p-\infty_1)^r \Theta_\Delta(p-\infty_0)} \Theta \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} (p + r\infty_2 - \infty_0 - r\infty_1) h_\Delta(p). \tag{4.35}$$

Here, h_Δ is a certain fixed holomorphic spinor defined in (B10). We now recall that $\Theta_\Delta(p-q)$, as a function of a point p, has a simple zero at $p = q$ and at other $g - 1$ points whose positions are independent of q and depend solely on the choice of odd characteristic Δ. Also, by construction,[14] the spinor h_Δ has zeroes precisely at the same $g - 1$ points (and of the same multiplicity). It thus appears that ψ_r has a zero of multiplicity r at ∞_2 and a pole of order $r + 1$ at ∞_1. The order of the pole at ∞_1 may be smaller if the term $\mathcal{T}_r(p) := \Theta\left[\begin{smallmatrix} \mathbf{A} \\ \mathbf{B} \end{smallmatrix}\right](p + r\infty_2 - (r+1)\infty_1)$ vanishes there. If this happens, the constant C_1 in Proposition 4.3 would be zero and the RHP unsolvable. Similarly, if $\mathcal{T}_{r-1}(\infty_2) = 0$, then ψ_{r-1} has a zero of multiplicity *higher* than r, and consequently $C_3 = 0$. Finally, if $\widehat{\mathcal{T}}_{r-1}(p) := \Theta\left[\begin{smallmatrix} \mathbf{A} \\ \mathbf{B} \end{smallmatrix}\right](p + r\inf ty_2 - \infty_0 - r\infty_1)$ vanishes at ∞_0, then the spinor $\widehat{\psi}_{r-1}$ does not have a pole at ∞_0 and thus C_2 in Proposition 4.3 vanishes. Note that

$$\mathcal{T}_r(\infty_1) = \mathcal{T}_{r-1}(\infty_2) = \widehat{\mathcal{T}}_{r-1}(\infty_0) = \Theta \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} (r\infty_2 - r\infty_1).$$

Therefore, we have the proposition below.

Proposition 4.4. The Riemann-Hilbert Problem 4.1 is solvable if and only if

$$\Theta \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} (r\infty_2 - r\infty_1) \neq 0. \tag{4.36}$$

We now need to establish the nonvanishing property required by Proposition 4.4. We recall that in our case $\mathbf{A} \in i\mathbb{R}^g$ and $\mathbf{B} = 0$ (4.15). We then can use the results on bordered Riemann surfaces contained in Ref. 14.

Theorem 4.1. *For any* $\mathbf{A} \in 2i\pi\mathbb{R}^g,\ r \in \mathbb{Z}$,

$$\Theta \begin{bmatrix} \mathbf{A} \\ 0 \end{bmatrix} (r\infty_2 - r\infty_1) \neq 0. \tag{4.37}$$

Therefore, Problem 4.1 is always solvable.

Remark 4.1. The theorem could be rephrased as stating that the monodromy data for the outer parametrix do not lie on the Malgrange divisor. In this case the Malgrange divisor is actually identifiable with the (Θ) *divisor in the Jacobian of the curve* $\mathcal{L}$.[17]

Proof. Recall from Proposition 4.1 that $\mathcal{L}$ has an antiholomorphic involution and can be represented as the *double of a bordered Riemann surface* (Chap. VI of Ref. 14).

The β-cycles are homologous to cycles fixed by the antiholomorphic involution, since we can realize each of them as a path following the gap in $\mathbb{R}$ on one of the outer sheets and the same segment on the middle sheet. There are in total $g + 1$ cycles fixed by the involution; aside from the g β-cycles, there is also the cycle which we denote by β_{g+1} that covers the union of the segment $[b_0, a_0]$ on the middle sheet with all the unbounded gaps in the three sheets (see Fig. 1).

We now use Corollary 6.5 on page 114 of Ref. 14 which describes the intersection of the Θ-divisor with a g-dimensional real torus in the Jacobian of the form $\left[\begin{smallmatrix} i\mathbb{R}^g \\ \mu \end{smallmatrix}\right]$ (with $\mu = (\mu_j) \in \frac{1}{2}\mathbb{Z}^g$ a half-period). It consists of the image in the Jacobian of the set of divisors of degree zero that have a form $\mathcal{D}_{g-1} - \mathcal{D}^0_{g-1}$, where $\mathcal{D}^0_{g-1}$ is the divisor of degree $g - 1$, such that $2\mathcal{D}^0_{g-1}$ is linearly equivalent to the canonical divisor and $\mathcal{D}_{g-1}$ is any positive divisor of degree $g - 1$ invariant under the involution and such that $\mathcal{D}_{g-1}$ has $(1 + 2\mu_j) \mod 2$ points in β_j for $j = 1, \dots, g$. This means that if there existed a point $\mathbf{A} \in i\mathbb{R}^g$ such that $\Theta\left[\begin{smallmatrix} \mathbf{A} \\ 0 \end{smallmatrix}\right](0) = 0$, then it would be the image of a positive divisor of degree $g - 1$ with exactly one point in each of the g gaps, a clear contradiction (see also Proposition 6.16 in Ref. 14). This proves that $\Theta\left[\begin{smallmatrix} \mathbf{A} \\ 0 \end{smallmatrix}\right](0)$ cannot vanish for any value of $\mathbf{A} \in i\mathbb{R}^g$.

To complete the proof, recall that $\Theta\left[\begin{smallmatrix} \mathbf{A} \\ 0 \end{smallmatrix}\right](r\infty_2 - r\infty_1)$ is proportional to $\Theta(\mathbf{A} + r\infty_2 - r\infty_1)$. To prove that the latter expression is nonzero we only need to show that $\mathfrak{u}(r\infty_2 - r\infty_1) \in i\mathbb{R}^g$, where $\mathfrak{u}$ is the Abel map. By our choice of α-cycles and normalization (B1), holomorphic differentials ω_j satisfy $\bar{\omega}_j(\bar{z}) = -\omega_j(z)$. Therefore, an image under the Abel map of any divisor of degree 0 contained in β_{g+1} is purely imaginary. Since $\infty_{1,2} \in \beta_{g+1}$, the proof is complete. **Q.E.D.**

The expressions in terms of theta functions for the columns of the outer parametrix Ψ possess other interesting relations that were investigated in Ref. 5.

1. Genus 0 case

A particularly simple situation is the case in which the surface $\mathcal{L}$ is of genus 0, namely, there are only two cuts, one in $\mathbb{R}_+$ and one in $\mathbb{R}_-$. In this case we can write quite explicit algebraic expressions for all the objects above (see Fig. 11 for an example). First, we introduce a uniformizing parameter $t : \mathbb{C}P^1 \to \mathcal{L}$ in terms of which the meromorphic function $z : \mathcal{L} \to \mathbb{C}P^1$ is written as

$$z := u_0 t + C + \frac{u_1}{t-1} + \frac{u_2}{t+1}. \tag{4.38}$$

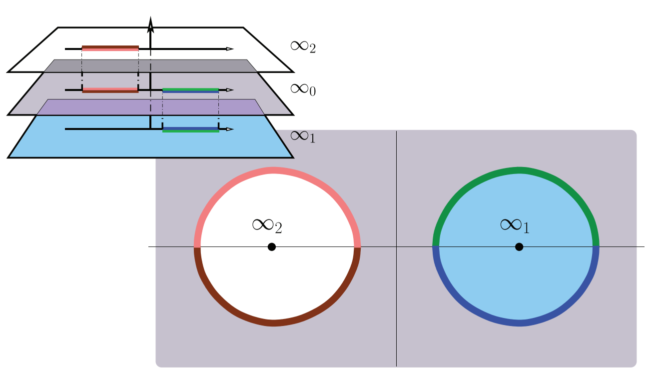

FIG. 11. Depiction of the three sheets and how they are mapped onto three disjoint regions of the t-plane. The real x-axis is mapped to the real t-axis and the two sides of each cut are mapped to the boundaries of the oval-shaped regions. The intersection of the ovals with the real t-axis are the four ramification points of the map $z(t)$.

Here, we have identified the three poles $\infty_{0,1,2}$ with $t = \infty, 1, -1$, respectively. Once the uniformization has been fixed, the expressions for the spinors above are extremely simple

$$
\begin{aligned}
f_r &:= \frac{\psi_r}{\sqrt{\mathrm{d}z}} = \frac{(t+1)^r}{(t-1)^{r+1}}\sqrt{\frac{\mathrm{d}t}{\mathrm{d}z}}\,, \qquad & f_r &= \frac{(t+1)^{r+1}}{(t-1)^r\sqrt{u_0(t^2-1)^2 - u_1(t+1)^2 - u_2(t-1)^2}}\,,\\
\widehat{f}_r &:= \frac{\widehat{\psi}_r}{\sqrt{\mathrm{d}z}} = \frac{(t+1)^{r+1}}{(t-1)^{r+1}}\sqrt{\frac{\mathrm{d}t}{\mathrm{d}z}}\,, \qquad & \widehat{f}_r &= \frac{(t+1)^{r+2}}{(t-1)^r\sqrt{u_0(t^2-1)^2 - u_1(t+1)^2 - u_2(t-1)^2}}\,.
\end{aligned}
\tag{4.39}
$$

The normalization is obtained by expanding near the three points at infinity

$$
f_r \sim \begin{cases} z^r \dfrac{2^r}{(-u_1)^{r+\frac12}} & t\sim 1\\[2mm] \dfrac{i}{z^{r+1}}\dfrac{u_2^{r+\frac12}}{2^{r+1}} & t \sim -1\\[2mm] \dfrac{\sqrt{u_0}}{z} & t\sim 0\end{cases}\,,\qquad
\widehat{f}_{r-1} \sim \begin{cases} z^{r-1} \dfrac{2^r}{(-u_1)^{r-\frac12}} & t\sim 1\\[2mm] \dfrac{-i}{z^{r+1}}\dfrac{u_2^{r+\frac12}}{2^{r}} & t \sim -1\\[2mm] \dfrac{1}{\sqrt{u_0}} & t\sim 0\end{cases}\,.
\tag{4.40}
$$

The solution to the RHP problem Problem 4.1 is

$$
\Psi(z) = \mathrm{diag}((-u_1)^{r-\frac12}2^{-r}, i\sqrt{u_0}, iu_2^{-r+\frac12})\begin{pmatrix} f_r(t_1(z)) & f_r(t_0(z)) & f_r(t_2(z))\\ \widehat{f}_{r-1}(t_1(z)) & \widehat{f}_{r-1}(t_0(z)) & \widehat{f}_{r-1}(t_2(z))\\ f_{r-1}(t_1(z)) & f_{r-1}(t_0(z)) & f_{r-1}(t_2(z))\end{pmatrix}.
\tag{4.41}
$$

Here, $t_{1,2,3}(z)$ are t-coordinates of sections $p_{1,2,3}$ discussed in Sec. IV B 2.

D. Local parametrix (solution of Problem 4.2)

For the sake of completeness we spell out the form of the parametrix near a endpoint of a cut where the density $\rho_{1,2}$ vanishes like a square root (see Corollary 3.1).

1. The rank-two parametrix

We need to solve the exact RHP in Fig. 12. This part is essentially identical to the established results in Refs. 12 and 10. We consider only the previously defined neighborhood $\mathbb{D}_a$ of the right endpoint $a := a_{2\ell-1}$ of one of the intervals of ρ_1. The modifications for the other cases are straightforward. Near $x = a$ the effective potential φ_1 is a piecewise analytic function with the jump

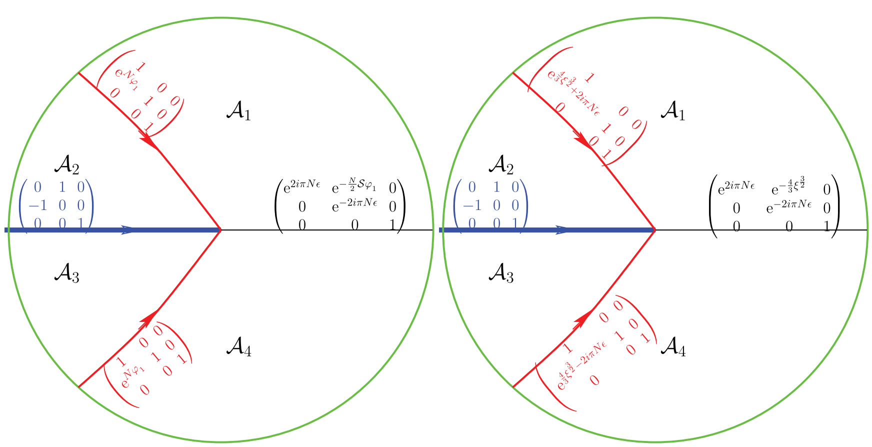

FIG. 12. The exact Riemann-Hilbert problem near a right endpoint in $\mathbb{R}_+$, and in terms of the zooming local coordinate ξ.

$\varphi_{1+} - \varphi_{1-} = 4i\pi\epsilon_\ell$, where ℓ is the number of the gap that starts on the right of a and ϵ_ℓ is the cumulative filling fraction (3.6).

Let φ be a function defined by

$$\varphi(z) = \begin{cases} \varphi_1(z) + 2i\pi(1-\epsilon_\ell)\,, & \Im z > 0 \\ \varphi_1(z) + 2i\pi(1+\epsilon_\ell)\,, & \Im z \le 0 \end{cases}.$$

Then φ has an analytic expansion in Puiseux series of the form described below.

Lemma 4.1. The locally analytic function φ has an expansion

$$\varphi(x) = \frac{4}{3}C(x-a)^{\frac{3}{2}}(1+\mathcal{O}(x-a))$$
$$C = \lim_{x\to a^-}\frac{\pi\rho_1(x)}{\sqrt{a-x}} > 0,$$

where the cut of the root extends on the left of a and the term $\mathcal{O}(x-a)$ is analytic at a (i.e., has a convergent Taylor series).

Proof. Recalling the representation (3.20) for the $\mathfrak{g}$ functions we have

$$\varphi(z) = \int_a^z \frac{\mathrm{d}\varphi_1(\xi)}{\mathrm{d}\xi}\mathrm{d}\xi = \int_a^z \left(Y^{(1)}(\xi) - Y^{(0)}(\xi)\right)\mathrm{d}\xi. \tag{4.42}$$

Now a small loop around a interchanges $Y^{(1)} \leftrightarrow Y^{(0)}$, which means that

$$Y^{(1)}(\xi) - Y^{(0)}(\xi) = \sqrt{\xi - a}\,(2C + \mathcal{O}(\xi - a)) \tag{4.43}$$

with the term $\mathcal{O}(\xi - a)$ analytic. The expression for C is obtained by recalling that by Definition 3.4,

$$\varphi'(z) = \varphi_1'(z) = V_1'(z) - 2\int_0^{+\infty}\frac{\rho_1(s)\mathrm{d}s}{z-s} - \int_0^{-\infty}\frac{\rho_2(s)\mathrm{d}s}{z-s}. \tag{4.44}$$

Now $\rho_1(s) \sim C\sqrt{a-s}$ near $s = a$ (with $C > 0$) due to (4.43) and $\varphi'_+ - \varphi'_- = 4\pi i\rho_1$. The sign of $C > 0$ is due to the fact that $\Re\varphi_1 = \Re\varphi$ and it must be positive in the gap ($x > a$). **Q.E.D.**

Remark 4.2. In a non-generic situation, one can only conclude that

$$\varphi(z) = (z-a)^{k+\frac{3}{2}}(\widetilde{C} + \mathcal{O}(z-a))\,,\quad \widetilde{C} > 0 \tag{4.45}$$

for some nonnegative integer k. This case is called nonregular *and it is treated in Ref. 10. The construction there applies identically in this case, but we prefer to concentrate on a detailed discussion for the most generic case.*

We define the **zooming local parameter** by the equation

$$\frac{4}{3}\xi^{\frac{3}{2}} = N\varphi(z), \tag{4.46}$$

so that the fixed-size neighborhood of a in the z-plane is mapped conformally (one-to-one) to an homothetically expanding neighborhood of the origin in the ξ plane with a diameter growing like $N^{\frac{2}{3}}$. The choice of the root in (4.46) is such that the cut is mapped to $\mathbb{R}_-$ of ξ-plane.

We then introduce the standard rank-two Airy parametrix $\mathcal{R}^0(\xi)$[10,12] as the piecewise defined matrix $\mathcal{R}^0_j$ constructed in terms of the Airy function $\mathrm{Ai}(x)$ as follows. If $\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, define

$$\mathcal{R}^0_k(\xi) := \sqrt{2\pi}\mathrm{e}^{-\frac{i\pi}{4}} \begin{cases} \begin{pmatrix} y_0 & -y_2 \\ y_0' & -y_2' \end{pmatrix} \mathrm{e}^{\left(\frac{2}{3}\xi^{\frac{3}{2}} + i\pi N\epsilon_\ell\right)\boldsymbol{\sigma}_3} = \begin{pmatrix} y_0 & -y_2 \\ y_0' & -y_2' \end{pmatrix} \mathrm{e}^{\frac{N}{2}\varphi_1\boldsymbol{\sigma}_3} & k=1 \\ \begin{pmatrix} -y_1 & -y_2 \\ -y_1' & -y_2 \end{pmatrix} \mathrm{e}^{\left(\frac{2}{3}\xi^{\frac{3}{2}} + i\pi N\epsilon_\ell\right)\boldsymbol{\sigma}_3} = \begin{pmatrix} -y_1 & -y_2 \\ -y_1' & -y_2 \end{pmatrix} \mathrm{e}^{\frac{N}{2}\varphi_1\boldsymbol{\sigma}_3} & k=2 \\ \begin{pmatrix} -y_2 & y_1 \\ -y_2' & y_1' \end{pmatrix} \mathrm{e}^{\left(\frac{2}{3}\xi^{\frac{3}{2}} - i\pi N\epsilon_\ell\right)\boldsymbol{\sigma}_3} = \begin{pmatrix} -y_2 & y_1 \\ -y_2' & y_1' \end{pmatrix} \mathrm{e}^{\frac{N}{2}\varphi_1\boldsymbol{\sigma}_3} = & k=3 \\ \begin{pmatrix} y_0 & y_1 \\ y_0' & y_1' \end{pmatrix} \mathrm{e}^{\left(\frac{2}{3}\xi^{\frac{3}{2}} - i\pi N\epsilon_\ell\right)\boldsymbol{\sigma}_3} = \begin{pmatrix} y_0 & y_1 \\ y_0' & y_1' \end{pmatrix} \mathrm{e}^{\frac{N}{2}\varphi_1\boldsymbol{\sigma}_3} & k=4 \end{cases}, \quad (4.47)$$

where index k refers to regions in Fig. 12 and, for $s = 0, 1, 2$,

$$y_s := \omega^s \mathrm{Ai}(\omega^s \xi), \quad \omega = \mathrm{e}^{2i\pi/3}. \quad (4.48)$$

Each $\mathcal{R}^0_k$ has the following uniform asymptotic behavior near $\xi = \infty$,

$$\mathcal{R}^0_k(\xi) = \xi^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \mathrm{e}^{\left(-\frac{i\pi}{4} \pm i\pi N\epsilon_\ell\right)\boldsymbol{\sigma}_3}(\mathbf{1} + \mathcal{O}(\xi^{-3/2})), \quad (4.49)$$

where the $\pm$ depends on the half-plane (upper/lower) in which the asymptotics is considered.

Thus, the final form of the local parametrix is defined as a matrix $\mathcal{R}$ whose restrictions to the four regions in Fig. 12 are

$$\mathcal{R}_k(\xi) := \overbrace{\mathrm{e}^{-\left(-\frac{i\pi}{4} \pm i\pi N\epsilon_\ell\right)\boldsymbol{\sigma}_3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xi^{\frac{\sigma_3}{4}}}^{:=F(z)} \mathcal{R}^0_k(\xi). \quad (4.50)$$

Here,

(i) the prefactor $F(z)$ solves a RHP on the left

$$\begin{aligned} F(z)_+ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} F(z)_- , \qquad \xi(z) \in \mathbb{R}_- \\ F(z)_+ &= \mathrm{e}^{-2i\pi N\epsilon_\ell\sigma_3} F(z)_- , \qquad \xi(z) \in \mathbb{R}_+ . \end{aligned} \quad (4.51)$$

(ii) $\mathcal{R}$ satisfies the exact jump conditions of the RHP in Fig. 12, except for the jump condition on the cut:

$$\begin{aligned} \mathcal{R}_+ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathcal{R}_- \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \xi \in \mathbb{R}_- \\ \mathcal{R}_+ &= \mathrm{e}^{-2i\pi N\epsilon_\ell\boldsymbol{\sigma}_3} \mathcal{R}_- \mathrm{e}^{2i\pi N\epsilon_\ell\boldsymbol{\sigma}_3} \quad \xi \in \mathbb{R}_+ \end{aligned} . \quad (4.52)$$

(iii) $\mathcal{R}(\xi) = \mathbf{1} + \mathcal{O}(N^{-1})$ uniformly on the boundary of the neighborhood $\mathbb{D}_a$.

E. Rank-three parametrix

Now we embed the rank-two parametrix from Sec. IV D into a matrix

$$\mathcal{R}_a(x) = \mathcal{R}(x) \oplus 1 =: \left[\begin{array}{c|c} \mathcal{R}(x) & \\ \hline & 1 \end{array}\right]. \tag{4.53}$$

Let define the parametrix $\mathcal{P}_a(z)$ within the disk $\mathbb{D}_a$ as

$$\mathcal{P}_a(z) := \Psi(z)\mathcal{R}_a(z), \tag{4.54}$$

where Ψ is the outer parametrix constructed earlier. The only point to address now is whether $\Psi(z)\mathcal{R}_a(z)$ is *bounded* inside the disk, since near a,

$$\Psi(z) = \mathcal{O}((z-a)^{-\frac{1}{4}}), \quad F(z) = \mathcal{O}((z-a)^{-\frac{1}{4}}). \tag{4.55}$$

Thus, the product $\Psi(z)(F(z) \oplus 1)$ may at most have square root singularities: however, comparing the RHPs that Ψ and F solve, we see that the product is a single-valued matrix, thus it must be analytic since at worst it may have singularities of type $(z-a)^{-\frac{1}{2}}$. Being single-valued, however, it must have a Laurent series expansion that must, in fact, be a Taylor series by the previous *a priori* estimate of its growth. Therefore, the product is actually analytic. Since $F(z)$ is solely responsible for the unboundedness of $\mathcal{R}$, this proves that $\mathcal{P}_a(z)$ is bounded.

F. Asymptotics of the biorthogonal polynomials

Let us go back to modifications discussed in Sec. 4.1 of the RHP Problem 2.1 for Cauchy biorthogonal polynomials. It follows from (4.2), that within the half-lenses depicted in Figs. 4 and 5 (on the $\pm$ sides of the cuts) the original matrix $\Gamma(z)$ reads

$$\Gamma_\pm(x) = C_\gamma \boldsymbol{\Gamma}(x)_\pm G_\pm^{-1} C_\gamma^{-1} \begin{bmatrix} 1 & & \\ \pm e^{NV_1} & 1 & \\ & & 1 \end{bmatrix}, \ x \in \mathcal{A}, \tag{4.56}$$

$$\Gamma_\pm(x) = C_\gamma \boldsymbol{\Gamma}(x)_\pm G_\pm^{-1} C_\gamma^{-1} \begin{bmatrix} 1 & & \\ & 1 & \\ & \pm e^{NV_2} & 1 \end{bmatrix}, \ x \in \mathcal{B}. \tag{4.57}$$

According to (2.8), the degree n monic biorthogonal polynomial is

$$\begin{aligned} p_n(x) = \Gamma_{11}(x) &= e^{\frac{N}{2}\left(V_1 + \mathfrak{g}_\pm^{(0)} + \mathfrak{g}_\pm^{(1)} + \gamma_+\right)} \left[\boldsymbol{\Gamma}_{11\pm} e^{-\frac{N}{2}\varphi_{1\pm}} \pm \boldsymbol{\Gamma}_{12\pm} e^{\frac{N}{2}\varphi_{1\pm}}\right] = \\ &= e^{\frac{N}{2}\left(V_1 - \mathfrak{g}^{(2)} + \gamma_+\right)} \left[\boldsymbol{\Gamma}_{11\pm} e^{-\frac{N}{2}\varphi_{1\pm}} \pm \boldsymbol{\Gamma}_{12\pm} e^{\frac{N}{2}\varphi_{1\pm}}\right] \end{aligned} \quad (x \in \mathcal{A}). \tag{4.58}$$

Note that for $x > 0$, $\mathfrak{g}^{(2)}(x) \in \mathbb{R}$ and has no jumps, and for $x \in \mathcal{A}$ we have $\varphi_{1\pm} = -2i\Im \mathfrak{g}_\pm^{(1)}(x) = \pm 2i\pi \left(\int_{a_0}^x \rho_1(s)\mathrm{d}s - 1\right)$. From (4.2)–(4.9),

$$\boldsymbol{\Gamma}(z) = \mathrm{diag}(1,-1,1)\overline{\boldsymbol{\Gamma}(\overline{z})}\mathrm{diag}(1,-1,1) \ (z \in \mathbb{C}), \quad \boldsymbol{\Gamma}_{12\pm}(x) = \pm\boldsymbol{\Gamma}_{11\mp}(x) = \pm\bar{\boldsymbol{\Gamma}}_{11\pm}(x) \ (x \in \mathcal{A}), \tag{4.59}$$

and thus

$$p_n(x) = e^{\frac{N}{2}\left(V_1 - \mathfrak{g}^{(2)} + \gamma_+\right)} 2\Re\left[\boldsymbol{\Gamma}_{11} e^{-iN\pi \int_{a_0}^x \rho_1(s)\mathrm{d}s}\right], \qquad x \in \mathcal{A}. \tag{4.60}$$

Vice versa, for $z \notin \mathcal{A}$ (and outside of the right lenses) one has

$$p_n(z) = e^{N\mathfrak{g}^{(1)}} \boldsymbol{\Gamma}_{11}(z). \tag{4.61}$$

Recall that $\mathbf{\Gamma} = \left(\mathbf{1} + \mathcal{O}(N^{-1})\right)\widehat{\mathbf{\Gamma}}$, where $\widehat{\mathbf{\Gamma}}$ is defined in (4.11). Then one can obtain uniform asymptotic information on the behaviour of p_n in any compact set of the complex plane. In particular, away from the endpoints, the asymptotics is expressible in terms of theta functions (genus ≥ 1) or algebraic expressions (genus 0).

To obtain asymptotic information for the biorthogonal companions q_n, one simply interchanges the roles of the measures.

V. ASYMPTOTIC SPECTRAL STATISTICS AND UNIVERSALITY

Once we have obtained a uniform asymptotic control of the biorthogonal polynomials, it is natural to investigate, in parallel to what has been done for the Hermitian matrix model (see, e.g., Ref. 10), the large N behavior of the correlation functions for the Cauchy two-matrix model studied in Ref. 7. We recall that the finite-size correlation functions for the spectra of M_1, M_2 distributed according to (1.1) correspond to a multi-level determinantal point process with the kernel

$$\mathbb{K}_N(x, y) := \sum_{j=0}^{N-1} \frac{p_j(x) q_j(y)}{h_j}, \tag{5.1}$$

$$\iint_{\mathbb{R}_+^2} p_j(x) q_i(y) \frac{\mathrm{e}^{-N(V_1(x)+V_2(-y))}}{x+y} = h_j \delta_{ij}, \tag{5.2}$$

where $\{p_j(x), q_j(y)\}_{j\in\mathbb{N}}$ are the *monic* biorthogonal polynomials.

Remark 5.1. Normalizing factors h_j appear in the definition of $\mathbb{K}_N(x, y)$ in contrast to the formula (3.31) in Ref. 7 since here we use monic rather than orthonormal biorthogonal polynomials.

Define four auxiliary kernels

$$H_{00}(x, y) := \mathbb{K}_N(x, y), \tag{5.3}$$

$$H_{10}(y', y) := \int_{\mathbb{R}_+} \frac{\mathrm{e}^{-NV_1(x)}\mathrm{d}x}{y'+x} H_{00}(x, y), \tag{5.4}$$

$$H_{01}(x, x') := \int_{\mathbb{R}_+} \frac{\mathrm{e}^{-NV_2(-y)}\mathrm{d}y}{y+x'} H_{00}(x, y), \tag{5.5}$$

$$H_{11}(y, x) := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} H_{00}(x', y') \frac{\mathrm{e}^{-N(V_1(x')+V_2(-y'))}\mathrm{d}x'\mathrm{d}y'}{(y+x')(x+y')} - \frac{1}{x+y}. \tag{5.6}$$

Remark 5.2. To obtain kernels for the correlation functions auxiliary kernels need to be multiplied by appropriate exponentials involving the potentials V_1, V_2.

Thus, using results from Ref. 7 with notational changes indicated in Remark 5.1 and before Eq. (2.8), we have the following.

Proposition 5.1 (Proposition 3.2 in Ref. 7). *The auxiliary kernels are given in terms of the solution of the RHP in Proposition 2.1 as follows (for $x, x', y, y' \geq 0$),*

$$\begin{aligned}
H_{00}(x, y) &= -\frac{1}{(2i\pi)^2} \frac{\left[\Gamma^{-1}(-y)\Gamma(x)\right]_{3,1}}{x+y}, \qquad H_{10}(y', y) = \frac{1}{2i\pi} \frac{\left[\Gamma^{-1}(-y)\Gamma(-y')\right]_{3,2}}{y'-y},\\
H_{01}(x, x') &= \frac{1}{2i\pi} \frac{\left[\Gamma^{-1}(x')\Gamma(x)\right]_{2,1}}{x-x'}, \qquad H_{11}(y, x) = \frac{\left[\Gamma^{-1}(x)\Gamma(-y)\right]_{2,2}}{x+y}.
\end{aligned} \tag{5.7}$$

One should not expect that the kernels $H_{00}(x, y)$, $H_{11}(y, x)$, $x, y > 0$ display any universal behavior, even in the scaling regime $x = \frac{\xi}{N^\alpha}$, $y = \frac{\eta}{N^\alpha}$. The only "interaction" between the two matrices M_1 and M_2 that can lead to a universality class is near the zero eigenvalue.

On the other hand, if one considers separately statistics of the eigenvalues of M_1 or M_2, no new universality phenomena will appear, as we briefly explain below. It is sufficient to consider $H_{01}(x, x')$ since the computation for $H_{10}(y, y')$ is completely analogous. Again, we consider only the regular case.

Universality in the bulk. Let c belong to the interior of some cut. For simplicity we will consider only $\mathcal{A}$, but the result extends to the other case with obvious modifications in the roles of the potentials. Due to regularity assumptions, at c we have $\rho_1(c) = C > 0$. Consider

$$x = c + \frac{\xi}{CN}, \qquad x' = c + \frac{\eta}{CN}. \tag{5.8}$$

Then a straightforward computation using Proposition 5.1 yields

$$\frac{1}{\rho_1(c)N} H_{01}(x, x') \mathrm{e}^{-\frac{N}{2}(V_1(x)+V_1(x'))} = \frac{\mathrm{e}^{-\frac{N}{2}\left(\mathfrak{g}^{(2)}(x)-\mathfrak{g}^{(2)}(x')\right)}}{2\pi i(\xi-\eta)} \left(\mathrm{e}^{\frac{N}{2}\left(\varphi_{1+}(x)-\varphi_{1+}(x')\right)} - \mathrm{e}^{-\frac{N}{2}\left(\varphi_{1+}(x)-\varphi_{1+}(x')\right)}\right)(1+\mathcal{O}(N^{-1})). \tag{5.9}$$

Recall that

$$\varphi_{1+}(x) = i\Im\varphi_{1+}(x) = -2i\pi \int_x^\infty \rho_1(s)\mathrm{d}s, \tag{5.10}$$

so that

$$\lim_{N\to\infty} \frac{1}{\rho_1(c)N} H_{01}(x, x') \mathrm{e}^{-\frac{N}{2}(V_1(x)+V_1(x'))} = \mathrm{e}^{-\frac{\mathfrak{g}^{(2)\prime}(c)}{2\rho_1(c)}(\xi-\eta)} \underbrace{\frac{\sin(\pi(\xi-\eta))}{\pi(\xi-\eta)}}_{K_{\sin}(\xi,\eta)} =: \widehat{K}_{\sin}(\xi, \eta). \tag{5.11}$$

The prefactor above to the usual sine-kernel $K_{\sin}(\xi, \eta)$ is not universal (it depends on the densities); however, since $\widehat{K}_{\sin}$ is conjugate to $K_{\sin}$,

$$\det \widehat{K}_{\sin}(\xi_i, \xi_j) = \det K_{\sin}(\xi_i, \xi_j). \tag{5.12}$$

Therefore all spectral statistics, gap probabilities, etc., for M_1 or M_2 separately will follow the standard universality results for the Hermitian matrix model.

Universality at the edge. Similarly, one finds near the edge $x = a$,

$$x = a + \frac{\xi}{CN^{\frac{2}{3}}}, \qquad x' = a + \frac{\eta}{CN^{\frac{2}{3}}}, \qquad C := \lim_{s\to a^-} \frac{\pi\rho_1(s)}{\sqrt{a-s}}, \tag{5.13}$$

$$\lim_{N\to\infty} \frac{1}{CN^{\frac{2}{3}}} H_{01}(x, x') \mathrm{e}^{-\frac{N}{2}(V_1(x)+V_1(x'))} = \frac{Ai(\eta)Ai'(\xi) - Ai'(\eta)Ai(\xi)}{\xi-\eta}. \tag{5.14}$$

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APPENDIX A: AN EXAMPLE: "DOUBLE LAGUERRE" BIORTHOGONAL POLYNOMIALS

We consider the case where the two potentials are the same $V_1(x) = V_2(-x)$ and are of the simplest possible form

$$\mathrm{e}^{-V_1(x)} = x^a \mathrm{e}^{-bx}, \tag{A1}$$

where both $a, b > 0$.

We can rescale the axis and set $b = 1$ without loss of generality.

The curve (3.4) appearing in Theorem 3.2 was computed in Refs. 2 and 7,

$$y^3 - \left(\frac{1}{3} + \frac{a^2}{z^2}\right) y - \left(\frac{2a^2 + 6a + 1}{3z^2} - \frac{2}{27}\right) = 0. \tag{A2}$$

According to Sec. IV C 1, we find a rational uniformization of this curve as

$$z = (1 + a)t + \frac{2a + 1}{2a + 2}\left(\frac{1}{t - 1} + \frac{1}{t + 1}\right), \tag{A3}$$

$$y = \frac{2a + 1}{2a}\left(\frac{1}{(a + 1)t - a} - \frac{1}{(a + 1)t + a}\right) - \frac{2}{3}. \tag{A4}$$

For $a > 0$ there are four symmetric branch points on the real axis and the inner ones tend to zero as $a \to 0$, whereas all four tend to infinity as $\pm(a \pm 2\sqrt{a}) + \mathcal{O}(1)$ as $a \to \infty$. Explicit formulas for the parametrix can be obtained by substituting $u_0 = 1 + a,\ u_1 = u_2 = \frac{2a+1}{2a+2}$ into (4.41).

APPENDIX B: NOTATION AND MAIN TOOLS

For a given smooth compact curve $\mathcal{L}$ of genus g with a fixed choice of symplectic homology basis of α and β-cycles, we denote by ω_ℓ the normalized basis of holomorphic differentials

$$\oint_{\alpha_j} \omega_\ell = \delta_{j\ell}\ , \qquad \oint_{\beta_j} \omega_\ell = \tau_{j\ell} = \tau_{\ell j}. \tag{B1}$$

We will denote by Θ the theta function

$$\Theta(\mathbf{z}) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathrm{e}^{i\pi \vec{n}\cdot\tau\vec{n} - 2i\pi \mathbf{z}\cdot\vec{n}}. \tag{B2}$$

The Abel map $\mathfrak{u} : \mathcal{L} \to \mathbb{C}^g$ with a base-point p_0 is

$$\mathfrak{u}(p) = \left[\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g\right]^t \tag{B3}$$

and is defined up to the *period lattice* $\mathbb{Z} + \tau \cdot \mathbb{Z}$. For brevity we will omit any symbolic reference to the Abel map when it appears as argument of a theta function: namely, if $p \in \mathcal{L}$ is a point and it appears as an argument of a theta function, the Abel map will be implied, meaning that

$$\Theta(p - q) \text{ stands for } \Theta(\mathfrak{u}(p) - \mathfrak{u}(q)).$$

We denote by $\mathcal{K}$ the vector of Riemann constants

$$\mathcal{K}_j = -\sum_{\ell=1}^{g}\left[\oint_{a_\ell} \omega_\ell(p) \int_{p_0}^{p} \omega_j(q) - \delta_{j\ell}\frac{\tau_{jj}}{2}\right], \tag{B4}$$

where the cycles α_j are realized as loops with basepoint p_0 and the inner integration is done along a path lying in the canonical dissection of the surface along the chosen representatives of the basis in the homology of the curve.

The Riemann constants have the crucial property that for a nonspecial divisor $\mathcal{D}$ of degree g, $\mathcal{D} = \sum_{j=1}^{g} p_j$, then the function

$$f(p) = \Theta(p - \mathcal{D} - \mathcal{K}) \tag{B5}$$

has zeroes precisely and only at $p = p_j, j = 1 \dots g$.

We also use theta functions with (complex) characteristics: for any two complex vectors $\vec{\epsilon}, \vec{\delta}$ the theta function with half-characteristics $\vec{\epsilon}, \vec{\delta}$ is defined via

$$\Theta\begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}(\mathbf{z}) := \exp\left(2i\pi\left(\frac{\vec{\epsilon}\cdot\tau\cdot\vec{\epsilon}}{8} + \frac{1}{2}\vec{\epsilon}\cdot\mathbf{z} + \frac{1}{4}\vec{\epsilon}\cdot\vec{\delta}\right)\right)\Theta\left(\mathbf{z} + \frac{\vec{\delta}}{2} + \tau\frac{\vec{\epsilon}}{2}\right). \tag{B6}$$

Here the half-characteristics of a point are defined by

$$2\mathbf{z} = \vec{\delta} + \tau\vec{\epsilon}. \tag{B7}$$

This modified theta function has the following periodicity properties, for $\lambda, \mu \in \mathbb{Z}^g$,

$$\Theta\begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}(\mathbf{z} + \lambda + \tau\mu) = \exp\left[i\pi(\vec{\epsilon}\cdot\lambda - \vec{\delta}\cdot\mu) - i\pi\mu\cdot\tau\cdot\mu - 2i\pi\mathbf{z}\cdot\mu\right]\Theta\begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}(\mathbf{z}). \tag{B8}$$

Definition B.1. The prime form $E(p, q)$ is the $(-1/2, -1/2)$ bi-differential on $\mathcal{L} \times \mathcal{L}$,

$$E(p,q) = \frac{\Theta_\Delta(p-q)}{h_\Delta(p)h_\Delta(q)}, \tag{B9}$$

$$h_\Delta(p)^2 := \sum_{k=1}^{g} \partial_{\mathrm{u}_k} \ln\Theta_\Delta\bigg|_{\mathrm{u}=0}\omega_k(p) =: \omega_\Delta(p), \tag{B10}$$

where $\Delta = \begin{bmatrix}\vec{\epsilon}\\ \vec{\delta}\end{bmatrix}$ is a half-integer odd characteristic (i.e., $\vec{\epsilon}\cdot\vec{\delta}$ is odd). The prime form does not depend on a choice of Δ.

The prime form $E(p, q)$ is antisymmetric in the argument and it is a section of an appropriate line bundle, i.e., it is multiplicatively multivalued on $\mathcal{L} \times \mathcal{L}$,

$$E(p + \alpha_j, q) = E(p, q), \tag{B11}$$

$$E(p + \beta_j, q) = E(p, q)\exp\left(-\frac{\tau_{jj}}{2} - \int_p^q \omega_j\right). \tag{B12}$$

In our notation for the half-characteristics, the vectors $\vec{\epsilon}, \vec{\delta}$ appearing in the definition of the prime form are actually integer valued. We also note that the half order differential h_Δ is in fact also multivalued according to

$$h_\Delta(p + \alpha_j) = \mathrm{e}^{i\pi\epsilon_j}h_\Delta(p), \tag{B13}$$

$$h_\Delta(p + \beta_j) = \mathrm{e}^{-i\pi\delta_j}h_\Delta(p). \tag{B14}$$

Given a meromorphic function F (or a section of a line bundle) we will use the notation (F) for its divisor of zeroes/poles. For example,

$$(F) \geq -kp + mq \tag{B15}$$

means that F has at most a pole of order k at p and a zero of multiplicity at least m at q.

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