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FROM LOG SOBOLEV TO TALAGRAND: A QUICK PROOF

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Abstract. We provide yet another proof of the Otto-Villani theorem from the log Sobolev inequality to the Talagrand transportation cost inequality valid in arbitrary metric measure spaces. The argument relies on the recent development [2] identifying gradient flows in Hilbert space and in Wassertein space, emphasizing one key step as precisely the root of the Otto-Villani theorem. The approach does not require the doubling property or the validity of the local Poincaré inequality.

1. **Introduction.** The Otto-Villani theorem [13] indicates that, on a Riemannian manifold, the validity of the log Sobolev inequality implies the Talagrand transportation cost inequality (preserving moreover optimal constant). The original proof is inspired by the Riemannian formalism on the space of probability measures established by Otto in [12]. An alternate more direct proof relying on infimum convolutions and Hamilton-Jacobi equations was presented later in [4]. Since then, the relation between the two families of inequalities, and its extension to abstract metric measure spaces, has been investigated by a number of authors, e.g. [11], [7] and the recent [9] (see also [15], [8]). In particular, the latter authors address the implication in a rather general metric space context by means of dimension free measure concentration tools [7], further investigated in the recent [9] which develops the use of Hamilton-Jacobi equations and the Herbst argument (and covers extended versions for modified log Sobolev inequalities and general transportation costs). In [11], the Otto-Villani theorem is established in metric spaces satisfying doubling property and local Poincaré inequality assumptions. These conditions actually allow the authors to rely on the existence of a weak differentiable structure on the metric measure space (built by Cheeger in [5]) which in turn gives the possibility to mimic the arguments valid in a Riemannian context.

By the dimension free nature of both the log Sobolev and the Talagrand inequalities, it is quite unnatural to impose either the doubling condition on the measure or the validity of a local Poincaré inequality, as these assumptions are finite dimensional in nature. Recent investigations [2] (see also [3], and [1] for a simplified

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exposition in the compact case) achieved to develop a differential calculus in metric measure spaces, without the doubling property and the local Poincaré inequality conditions, towards the equivalence of the heat flow generated by a suitable Dirichlet energy and the Wasserstein gradient flow of the relative entropy functional in the space of probability measures. As one key step in this identification process, the heat flow is shown to have a metric speed in the Wasserstein space controlled by the Fisher information, which is precisely the root of the Otto-Villani theorem as emphasized in the formalism of [13]. This key argument has been put forward first in [6] in the context of Alexandrov spaces, and its proof is presented below as 6 and 7 in the smooth case. It has been then extended in [2] to general metric measure spaces.

It is the purpose of this short note to use this observation to provide a simple and direct proof of the Otto-Villani theorem in a general metric measure theoretic context. While the result itself does not necessarily go strictly beyond the framework of [7] and [9], the emphasis is perhaps more on methodology, the differentiable structure on metric measure space of [2] providing the suitable and coherent circle of ideas leading to the Otto-Villani theorem as already expected by the formalism of [13].

We deal below with standard Polish metric measure spaces. Actually, the most natural setting would be that of *extended Polish spaces* as developed in [2], because this framework covers the Wiener space, which is the standard infinite dimensional space where log Sobolev inequalities hold. The approach we propose, based on the calculus tools developed in [2], covers also this general framework: proofs are verbatim the same. We however preferred to state everything in the more familiar context of complete and separable metric spaces just to avoid dealing with the uncommon technology presented in [2].

The new argument towards the Otto-Villani theorem put forward in this note is already of interest in the smooth case so that we briefly present it below in Section 3. The rest of the paper will be devoted to recall and organize the main conclusions of the recent investigation [2] linking the gradient flows in Hilbert space and in the Wasserstein space in metric measure spaces relevant to the Otto-Villani theorem. In particular, it is of importance to suitably describe the family of log Sobolev inequalities under consideration. The main result is contained in the last section. We start by collecting standard notions in metric (measure) spaces.

2. **Metric notions.** This section recalls the necessary metric notions towards the subsequent investigation. We refer to [15], [2] for complete details. The metric spaces (X, d) will always be complete and separable. Dealing with metric measure space $(X, \mathsf{d}, \mathfrak{m})$, the measure $\mathfrak{m}$ will always be a Borel probability measure. A curve $\gamma : [0,1] \to X$ is said to be absolutely continuous if there exists $g \in L^1(0,1)$ such that

$$\mathsf{d}(\gamma_t, \gamma_s) \leq \int_t^s g(r)\,\mathrm{d}r, \qquad \forall t < s \in [0,1]. \tag{1}$$

In this case, the metric speed of γ is well-defined for a.e. t by

$$|\dot\gamma_t| := \lim_{h\to 0} \frac{\mathsf{d}(\gamma_{t+h}, \gamma_t)}{|h|}.$$

It turns out that $|\dot\gamma| \in L^1(0,1)$ and this is the minimal L^1 function for which 1 holds.

Given $f : X \to \mathbb{R}$, the local Lipschitz constant $|\nabla f| : X \to [0, \infty]$ is defined by

$$|\nabla f|(x) := \varlimsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)},$$

and the ascending/descending slopes $|\nabla^+ f|, |\nabla^- f| : X \to [0, \infty]$ by

$$|\nabla^+ f|(x) := \varlimsup_{y \to x} \frac{(f(y) - f(x))^+}{\mathsf{d}(x, y)}, \qquad |\nabla^- f|(x) := \varlimsup_{y \to x} \frac{(f(y) - f(x))^-}{\mathsf{d}(x, y)},$$

where $(\cdot)^+$, $(\cdot)^-$ are the positive and negative part, respectively. One says that $G : X \to [0, \infty]$ is an upper gradient of $f : X \to \mathbb{R}$ provided for any absolutely continuous curve $\gamma : [0, 1] \to X$ it holds

$$\big|f(\gamma_1) - f(\gamma_0)\big| \le \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t. \tag{2}$$

It is not difficult to check (see for instance Proposition 2.5 of [2] for the simple proof) that if f is Lipschitz, then $|\nabla f|$, $|\nabla^+ f|$ and $|\nabla^- f|$ are all upper gradients.

Denote by $\mathscr{P}(X)$ the space of Borel probability measures on X and by $\mathscr{P}_2(X) \subset \mathscr{P}(X)$ the space of probability measures with finite second moment. The Wasserstein distance $W_2(\mu, \nu)$ between two measures $\mu, \nu \in \mathscr{P}(X)$ is defined by

$$W_2^2(\mu, \nu) := \inf_{\boldsymbol{\gamma}} \int_{X \times X} \mathsf{d}^2(x, y) \, \mathrm{d}\boldsymbol{\gamma}(x, y)$$

where the infimum is taken among all plans $\boldsymbol{\gamma} \in \mathscr{P}(X \times X)$ such that $\pi^1_\sharp \boldsymbol{\gamma} = \mu$ and $\pi^2_\sharp \boldsymbol{\gamma} = \nu$. Notice that we are defining W_2 between arbitrary measures in $\mathscr{P}(X)$, so that it is possible that the value $+\infty$ is attained. The Wasserstein distance W_2 is a metric on $\mathscr{P}_2(X)$, and $(\mathscr{P}_2(X), W_2)$ is again complete and separable.

The relative entropy functional $\mathrm{Ent}_{\mathfrak{m}} : \mathscr{P}(X) \to [0, \infty]$ with respect to a given $\mathfrak{m} \in \mathscr{P}(X)$ is defined by

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \displaystyle\int_X \rho \log \rho \, \mathrm{d}\mathfrak{m}, & \text{if } \mu = \rho\, \mathfrak{m}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Recall that sublevels of the entropy are tight. Indeed, using first the bound $z \log(z) \ge -\frac{1}{e}$ and then Jensen's inequality we get that for any $\mu = \rho\, \mathfrak{m}$ such that $\mathrm{Ent}_{\mathfrak{m}}(\mu) \le C$ and for any Borel set E, it holds

$$\frac{1}{e} + C \ge \frac{\mathfrak{m}(X \setminus E)}{e} + \mathrm{Ent}_{\mathfrak{m}}(\mu) \ge \int_E \rho \log \rho \, \mathrm{d}\mathfrak{m} \ge \mu(E) \log\left(\frac{\mu(E)}{\mathfrak{m}(E)}\right)$$

so that the claim follows from the tightness of $\mathfrak{m}$. We will denote by $D(\mathrm{Ent}_{\mathfrak{m}}) \subset \mathscr{P}(X)$ the domain of the entropy, i.e. the set of measures $\mu \in \mathscr{P}(X)$ such that $\mathrm{Ent}_{\mathfrak{m}}(\mu) < \infty$.

Under these notations, the Talagrand transportation cost inequality (proved in [14] in Euclidean space for $\mathfrak{m}$ the Gaussian measure) states that for some constant $C > 0$ and every $\mu \in \mathscr{P}(X)$,

$$W_2^2(\mu, \mathfrak{m}) \le C \, \mathrm{Ent}_{\mathfrak{m}}(\mu). \tag{3}$$

The Otto-Villani theorem expresses that this inequality holds under a log Sobolev inequality (to be recalled below), preserving the constants.

3. **The smooth case.** In this short section, we present the argument of [6] and the principle of proof of the Otto-Villani theorem which we would like to emphasize in the case of a smooth Riemannian manifold (X, g) with the standard differential calculus (as presented e.g. in [15]).

The log Sobolev inequality for a probability measure $\mathfrak{m}$ on the Borel sets of X indicates that there is a constant $\alpha > 0$ such that for every smooth function $f : X \to [0, \infty)$ with $\int_X f \mathrm{d}\mathfrak{m} = 1$,

$$2\alpha \int_X f \log f \, \mathrm{d}\mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \, \mathrm{d}\mathfrak{m}. \tag{4}$$

The task is thus to establish, under the log Sobolev inequality 4, the transportation cost inequality 3 (with constant $C = \frac{2}{\alpha}$; for the standard Gaussian measure in Euclidean space, $\alpha = 1$ is optimal for both the log Sobolev and the Talagrand inequalities).

Given $\mu \in \mathscr{P}(X)$ with (smooth) density f with respect to $\mathfrak{m}$, denote by (f_t) the heat flow with respect to the Dirichlet form $\int_X \langle \nabla u, \nabla v \rangle \mathrm{d}\mathfrak{m}$ starting from f. Set $\varphi(t) = \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$, $t \geq 0$, where $\mu_t = f_t \, \mathfrak{m}$. The standard calculus (Boltzmann formula) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ent}_{\mathfrak{m}}(\mu_t) = - \int_{\{f_t>0\}} \frac{|\nabla f_t|^2}{f_t} \, \mathrm{d}\mathfrak{m}, \tag{5}$$

so that the log Sobolev inequality 4 expresses equivalently that $\varphi' \leq -2\alpha\varphi$.

Following the strategy of [13] through the formal Riemannian structure on $(\mathscr{P}_2(X), W_2)$, the aim is to show that the curve $t \mapsto \mu_t = f_t \, \mathfrak{m}$ in $(\mathscr{P}_2(X), W_2)$ has a metric speed satisfying

$$|\dot{\mu}_t|^2 \leq \int_{\{f_t>0\}} \frac{|\nabla f_t|^2}{f_t} \, \mathrm{d}\mathfrak{m} \qquad (a.e.\, t). \tag{6}$$

Once this holds, the conclusion easily follows. Namely, by 5, the latter amounts to $|\dot{\mu}_t| \leq \sqrt{-\varphi'(t)}$. Since by the log Sobolev inequality $\varphi' \leq -2\alpha\varphi$ and thus

$$\sqrt{-\varphi'} \leq -\sqrt{\frac{2}{\alpha}} (\sqrt{\varphi})',$$

it follows by integration (according to 1 in $(\mathscr{P}_2(X), W_2)$) that, for all $T > 0$,

$$W_2(\mu, \mu_T) \leq \int_0^T |\dot{\mu}_t| \, \mathrm{d}t \leq -\sqrt{\frac{2}{\alpha}} \int_0^T \big(\sqrt{\varphi(t)}\big)' \, \mathrm{d}t = \sqrt{\frac{2}{\alpha}} \big(\sqrt{\varphi(0)} - \sqrt{\varphi(T)}\big).$$

Since $\mathrm{Ent}_{\mathfrak{m}}(\mu_T) \to 0$ as $T \to +\infty$,

$$W_2(\mu, \mathfrak{m}) \leq \varliminf_{T\to+\infty} W_2(\mu, \mu_T) \leq \varliminf_{T\to\infty} \sqrt{\frac{2}{\alpha}} \big(\varphi(0) - \varphi(T)\big) = \sqrt{\frac{2}{\alpha}\varphi(0)} = \sqrt{\frac{2}{\alpha} \mathrm{Ent}_{\mathfrak{m}}(\mu)},$$

which amounts to the transportation cost inequality 3 with constant $C = \frac{2}{\alpha}$.

The central point of the approach is thus to establish 6. The following simple and direct proof has been thus put forward in [6] and shown in [2] to be flexible enough to extend to the case of general metric measure spaces. The argument relies on a suitable interlacing of the heat flow and the Hamilton-Jacobi equation.

For a continuous Lipschitz and bounded function g on X, and $t, s > 0$, write

$$\int_X (Q_1 g) f_{t+s} \, \mathrm{d}\mathfrak{m} - \int_X g f_t \, \mathrm{d}\mathfrak{m} = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} \bigg[\int_X (Q_r g) f_{t+rs} \, \mathrm{d}\mathfrak{m} \bigg] \mathrm{d}r$$

where

$$Q_r g(x) = \inf_{y \in X} \left[g(y) - \frac{\mathsf{d}(x,y)^2}{2r} \right]$$

is the infimum convolution of g with the quadratic cost. The semigroup $(Q_r)_{r>0}$ represents the Hopf-Lax solution of the Hamilton-Jacobi equation

$$\frac{\mathrm{d}}{\mathrm{d}r} Q_r f(x) + \frac{|\nabla Q_r f|^2(x)}{2} = 0$$

(for almost every t, x). Hence,

$$\int_X (Q_1 g) f_{t+s} \, \mathrm{d}\mathfrak{m} - \int_X g f_t \, \mathrm{d}\mathfrak{m} = \int_0^1 \int_X \left(-\frac{|\nabla Q_r g|^2}{2} \, f_{t+rs} - s \langle \nabla Q_r g, \nabla f_{t+rs} \rangle \right) \mathrm{d}\mathfrak{m} \, \mathrm{d}r.$$

By the Cauchy-Schwarz inequality

$$-\langle \nabla Q_r g, \nabla f_{t+rs} \rangle \leq \frac{1}{2s} \, |\nabla Q_r g|^2 f_{t+rs} + \frac{s}{2} \, \frac{|\nabla f_{t+rs}|^2}{f_{t+rs}}$$

so that

$$\int_X (Q_1 g) f_{t+s} \, \mathrm{d}\mathfrak{m} - \int_X g f_t \, \mathrm{d}\mathfrak{m} \leq \frac{s^2}{2} \int_0^1 \int_X \frac{|\nabla f_{t+rs}|^2}{f_{t+rs}} \, \mathrm{d}\mathfrak{m} \, \mathrm{d}r.$$

By the dual Kantorovich representation of the Wasserstein distance W_2 (cf. e.g. [15]), taking the supremum on g on the left-hand side of the preceding inequality yields, for all $t, s > 0$,

$$W_2^2(\mu_{t+s}, \mu_t) \leq s^2 \int_0^1 \int_X \frac{|\nabla f_{t+rs}|^2}{f_{t+rs}} \, \mathrm{d}\mathfrak{m} \, \mathrm{d}r, \tag{7}$$

from which 6 follows. The Otto-Villani theorem is thus established in this way.

4. **The general metric measure space case.** To address the preceding proof of the Otto-Villani theorem in the context of a general metric space, we first recall the necessary differential calculus in non-smooth spaces developed in the recent [2] (and to which we refer for further details) which provides the key step 6. We repeat that metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ studied in this paper consist of a complete and separable metric space (X, d) equipped with a Borel probability measure $\mathfrak{m}$.

4.1. **The Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$.** There are several equivalent definitions of the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, here we follow the approach - and use the results - of [2]. Define the functional $\mathrm{Ch} : L^2(X, \mathfrak{m}) \to [0, \infty]$ by

$$\mathrm{Ch}(f) := \inf \varliminf_{n \to \infty} \frac{1}{2} \int_X |\nabla f_n|^2 \, \mathrm{d}\mathfrak{m},$$

where the infimum is taken among all sequences (f_n) of Lipschitz functions converging to f in $L^2(X, \mathfrak{m})$. Then the space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is defined as the set of f's such that $\mathrm{Ch}(f) < \infty$ endowed with the norm

$$\|f\|_{W^{1,2}}^2 := \|f\|_{L^2}^2 + 2\mathrm{Ch}(f).$$

It turns out that for $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ there exists a map $|\nabla f|_* \in L^2(X, \mathfrak{m})$ - which plays the role of the modulus of distributional gradient for Sobolev functions on $\mathbb{R}^d$

- which is characterized by the following two properties:

$$\mathrm{Ch}(f) = \frac{1}{2}\int_X |\nabla f|_*^2 \,\mathrm{d}\mathfrak{m},$$

$$|\nabla f|_* \leq G, \quad \mathfrak{m} - a.e. \qquad \forall G \text{ such that there exists a sequence } (f_n) \text{ of Lipschitz functions converging to } f \text{ in } L^2 \text{ such that } |\nabla f_n| \rightharpoonup G \text{ in } L^2(X,\mathfrak{m}).$$

Two theorems are worth to mention.

Theorem 4.1 (Relation with upper gradients)**.** *Let $f \in L^2(X,\mathfrak{m})$ and G an upper gradient for it as defined in 2. Assume that $G \in L^2(X,\mathfrak{m})$. Then $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ and*

$$|\nabla f|_* \leq G, \qquad \mathfrak{m} - a.e. \tag{8}$$

Theorem 4.2. *For any $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$, there exists a sequence of Lipschitz functions (f_n) converging to f in $L^2(X,\mathfrak{m})$ such that $|\nabla f_n|$ converges to $|\nabla f|_*$ in $L^2(X,\mathfrak{m})$.*

For the proof of the first theorem, see Theorem 6.2 of [2] (or [1] for the compact case). The second is obvious.

Let us briefly explain the importance of these results. The typical definition of $W^{1,2}(X,\mathsf{d},\mathfrak{m})$, see for instance [5], is to proceed by relaxing upper gradients of L^2 functions, rather than the local Lipschitz constant of Lipschitz functions. With this approach, Theorem 4.2 is totally non trivial, while Theorem 4.1 is straightforward (we refer to the survey [10] for an overview of the standard approach to Sobolev spaces over metric measure spaces). A step forward made in [2] has been to show that to relax the local Lipschitz constant produces the same space given by the relaxation of upper gradients, which therefore yields to both Theorem 4.1 and Theorem 4.2,

4.2. **Properties of the gradient flow of** Ch**.** Notice that the functional $\mathrm{Ch} : L^2(X,\mathfrak{m}) \to [0,\infty]$ is convex, lower semicontinuous and with dense domain, so that the standard theory of gradient flows on Hilbert space applies. This means that for any $f \in L^2(X,\mathfrak{m})$ there exists a unique locally absolutely continuous curve $(f_t) \subset L^2(X,\mathfrak{m})$ on $[0,\infty)$ such that $f_0 = f$ and it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t \in -\partial\mathrm{Ch}(f_t), \qquad a.e.\ t.$$

In [2] (with an approach inspired by [6]) properties of this gradient flow have been studied extensively, especially in connection with concepts coming from optimal transport theory like Wasserstein distance and relative entropy. In particular, the following main results of [2] are critical in the identification of the gradient flows for the Cheeger energy in $L^2(X,\mathfrak{m})$ and for the entropy functional in $(\mathscr{P}_2(X), W_2)$. As announced and illustrated in Section 3, they actually contain the key step towards the Otto-Villani theorem (described here in item *(ii)* corresponding to 6).

Theorem 4.3. *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space, $\mu \in D(\mathrm{Ent}_\mathfrak{m})$ and let f be its density with respect to $\mathfrak{m}$. Let (f_t) be the gradient flow of* Ch *starting from f. Then the following are true:*

(i) $f_t\,\mathfrak{m} \in \mathscr{P}(X)$ *for any* $t \geq 0$ *;*

(ii) *the curve $t \mapsto \mu_t := f_t\,\mathfrak{m}$ is locally absolutely continuous on $(0,\infty)$ with respect to the Wasserstein distance W_2 and for its metric speed $|\dot\mu_t|$ it holds*

$$|\dot\mu_t|^2 \le \int_{\{f_t>0\}} \frac{|\nabla f_t|_*^2}{f_t}\,\mathrm{d}\mathfrak{m}, \qquad \text{a.e. } t;$$

(iii) *the map $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t) = \int_X f_t \log f_t\,\mathrm{d}\mathfrak{m}$ is locally absolutely continuous and it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}_{\mathfrak{m}}(\mu_t) = -\int_{\{f_t>0\}} \frac{|\nabla f_t|_*^2}{f_t}\,\mathrm{d}\mathfrak{m}, \qquad \text{a.e. } t.$$

Items (i) and (iii) are proved in $(c),(d)$ of Theorem 4.16 in [2] while (ii) is the content of Lemma 6.1 therein. For the simplified approach in the case of compact spaces, we refer to [1].

5. **The Otto-Villani theorem.** On the basis of the previous differential calculus and the main Theorem 4.3, we address in this section the Otto-Villani theorem in the general context of a metric measure space with the scheme of proof put forward in the smooth case (Section 3).

We start by noticing that various typical formulations of the log Sobolev inequality are all equivalent. The result of the following proposition is already well-known if $\mathfrak{m}$ is doubling and $(X,\mathsf{d},\mathfrak{m})$ supports a local Poincaré inequality (see [11]). However, as remarked in the introduction of [3], there is no need for these hypotheses, and the statement below is just an exemplification of what is expressed in [3] for the particular case of the log Sobolev inequality. Technically speaking, the proof is based on the density of Lipschitz functions in $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ proved in [2].

Proposition 1 (Equivalent formulations of the log Sobolev inequality)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space and let $\alpha > 0$. Then the following are equivalent:*

(i) *For any Lipschitz function $f : X \to [0,\infty)$ with $\int_X f\,\mathrm{d}\mathfrak{m} = 1$, it holds*

$$2\alpha \int_X f \log f\,\mathrm{d}\mathfrak{m} \le \int_{\{f>0\}} \frac{|\nabla f|^2}{f}\,\mathrm{d}\mathfrak{m}.$$

(ii) *For any Lipschitz function $f : X \to [0,\infty)$ with $\int_X f\,\mathrm{d}\mathfrak{m} = 1$, it holds*

$$2\alpha \int_X f \log f\,\mathrm{d}\mathfrak{m} \le \int_{\{f>0\}} \frac{|\nabla^- f|^2}{f}\,\mathrm{d}\mathfrak{m}.$$

(iii) *For any continuous $f : X \to [0,\infty)$ with $\int_X f\,\mathrm{d}\mathfrak{m} = 1$ and any upper gradient G of f it holds*

$$2\alpha \int_X f \log f\,\mathrm{d}\mathfrak{m} \le \int_{\{f>0\}} \frac{G^2}{f}\,\mathrm{d}\mathfrak{m}.$$

(iv) *For any non-negative $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ with $\int_X f\,\mathrm{d}\mathfrak{m} = 1$, it holds*

$$2\alpha \int_X f \log f\,\mathrm{d}\mathfrak{m} \le \int_{\{f>0\}} \frac{|\nabla f|_*^2}{f}\,\mathrm{d}\mathfrak{m}.$$

Proof. $(ii) \Rightarrow (i)$ is trivial, and given that for Lipschitz functions slopes are upper gradients $(iii) \Rightarrow (ii)$ is also trivial. The property of weak gradients expressed in inequality 8 shows that also $(iv) \Rightarrow (iii)$ is true.

Thus everything boils down in proving that (i) implies (iv). For this we use the density Theorem 4.2. Pick $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ and assume without loss of generality that $\int_{\{f>0\}} \frac{|\nabla f|_*^2}{f} \,\mathrm{d}\mathfrak{m} < \infty$. For $c > 0$, let $f_c := a_c \min\{c^{-1}, \max\{f, c\}\}$, where a_c is such that $\int_X f_c \,\mathrm{d}\mathfrak{m} = 1$. Notice that $a_c \to 1$ as $c \downarrow 0$. Also, observe that $\int_X \frac{|\nabla f_c|_*^2}{f_c} = 4\mathrm{Ch}(\sqrt{f_c}) < \infty$ and use Theorem 4.2 to find a sequence (g_c^n) of Lipschitz functions such that $g_c^n \to \sqrt{f_c}$ and $|\nabla g_c^n| \to |\nabla\sqrt{f_c}|_*$ in $L^2(X, \mathfrak{m})$. Up to passing to a subsequence we can also assume that the convergence of (g_c^n) to $\sqrt{f_c}$ is also $\mathfrak{m}$-a.e.. Then the Lipschitz functions $f_c^n := (g_c^n)^2$ converge to f_c pointwise a.e. and thus by dominated convergence it holds

$$\int_X f_c \log f_c \,\mathrm{d}\mathfrak{m} = \lim_{n\to\infty} \int_X f_c^n \log f_c^n \,\mathrm{d}\mathfrak{m}.$$

Since f_c^n is Lipschitz for any n, c by hypothesis we know that

$$2\alpha \int_X f_c^n \log f_c^n \,\mathrm{d}\mathfrak{m} \le \int_{\{f_c^n>0\}} \frac{|\nabla f_c^n|^2}{f_c^n} \,\mathrm{d}\mathfrak{m} = 4 \int_X |\nabla g_c^n|^2 \,\mathrm{d}\mathfrak{m}.$$

The right-hand side of this equality converges, by construction, to $4\mathrm{Ch}(\sqrt{f_c}) = \int_X \frac{|\nabla f_c|_*^2}{f_c} \,\mathrm{d}\mathfrak{m}$. Hence we proved that

$$2\alpha \int_X f_c \log f_c \,\mathrm{d}\mathfrak{m} \le \int_X \frac{|\nabla f_c|_*^2}{f_c} \,\mathrm{d}\mathfrak{m},$$

for any $c > 0$. Letting $c \downarrow 0$ we complete the proof. □

As the proof shows it would be equivalent to state (ii) with the ascending slope $|\nabla^+ f|$ in place of $|\nabla^- f|$. The formulation here is the way the log Sobolev inequality is usually written (see e.g. Theorem 3.22 of [15]). Also, we point out that the equivalence between (i) and (iv) relies only on Theorem 4.2 (and not on the delicate Theorem 4.1), which in turn is a direct consequence of the way $|\nabla f|_*$ is defined.

We then say that $(X, \mathsf{d}, \mathfrak{m})$ supports the log Sobolev inequality with constant α provided any of the inequalities in the Proposition 1 is true.

Lemma 5.1. *Assume that $(X, \mathsf{d}, \mathfrak{m})$ supports the log Sobolev inequality with constant $\alpha > 0$ and let $\mu \in D(\mathrm{Ent}_\mathfrak{m})$. Let f be the density of μ with respect to $\mathfrak{m}$ and (f_t) be the gradient flow of* Ch *starting from f. Set $\mu_t = f_t \,\mathfrak{m}$, $t \ge 0$. Then as $t \to +\infty$, we have $\mathrm{Ent}_\mathfrak{m}(\mu_t) \to 0$ and $\mu_t \rightharpoonup \mathfrak{m}$ in duality with $C_b(X)$.*

Proof. Let $\varphi(t) := \mathrm{Ent}_\mathfrak{m}(\mu_t)$, $t \ge 0$, and recall that (iii) of Theorem 4.3 yields that φ is locally absolutely continuous with $\varphi'(t) = -\int_X \frac{|\nabla f_t|_*^2}{f_t} \,\mathrm{d}\mathfrak{m}$. Then the fact that $(X, \mathsf{d}, \mathfrak{m})$ supports the log Sobolev inequality with constant α implies (see formulation (iv) of Proposition 1) that $\varphi' \le -2\alpha\varphi$, which by the Gronwall lemma further gives

$$\varphi(t) \le e^{-2\alpha t}\varphi(0) = e^{-2\alpha t}\,\mathrm{Ent}_\mathfrak{m}(\mu), \tag{9}$$

so that the first claim is proved. To prove the second one, recall that the sublevels of $\mathrm{Ent}_\mathfrak{m}$ are tight, and so for any sequence $t_n \to \infty$ there exists a subsequence, not relabeled, such that (μ_{t_n}) weakly converges to some ν as $n \to \infty$ in duality with $C_b(X)$. By 9 we then have

$$\mathrm{Ent}_\mathfrak{m}(\nu) \le \varliminf_{n\to\infty} \mathrm{Ent}_\mathfrak{m}(\mu_{t_n}) = 0,$$

hence $\nu = \mathfrak{m}$. Since this is independent on the chosen sequence (t_n), the claim is proved. □

Provided with these results, the Otto-Villani theorem is now established exactly as in the smooth case (Section 3).

Theorem 5.2 (Otto-Villani theorem)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space supporting the log Sobolev inequality with constant $\alpha > 0$ (Proposition 1). Then it also supports the Talagrand inequality with constant $\frac{2}{\alpha}$, i.e. it holds*

$$W_2^2(\mu, \mathfrak{m}) \leq \frac{2}{\alpha} \operatorname{Ent}_{\mathfrak{m}}(\mu)$$

for all $\mu \in \mathscr{P}(X)$.

The theorem immediately leads to the following consequence.

Corollary 1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space supporting the log Sobolev inequality for some $\alpha > 0$. Then $\mathfrak{m}$ has finite second moment.*

Proof. Thanks to the Talagrand inequality, it is sufficient to prove that $D(\operatorname{Ent}_{\mathfrak{m}}) \cap \mathscr{P}_2(X)$ is non empty. But this is obvious, as any measure of the form $\mathfrak{m}(A)^{-1}\mathfrak{m}_{|A}$, with $A \subset X$ Borel, bounded and with positive $\mathfrak{m}$-measure is in both $D(\operatorname{Ent}_{\mathfrak{m}})$ and $\mathscr{P}_2(X)$. □

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