D-branes, surface operators, and ADHM quiver representations

D-BRANES, SURFACE OPERATORS, AND ADHM QUIVER REPRESENTATIONS

U. BRUZZO[1,2], W.-Y. CHUANG[3], D.-E. DIACONESCU[4], M. JARDIM[5], G. PAN[4], Y. ZHANG[4]

Abstract. A supersymmetric quantum mechanical model is constructed for BPS states bound to surface operators in five dimensional $SU(r)$ gauge theories using D-brane engineering. This model represents the effective action of a certain D2-brane configuration, and is naturally obtained by dimensional reduction of a quiver $(0,2)$ gauged linear sigma model. In a special stability chamber, the resulting moduli space of quiver representations is shown to be smooth and isomorphic to a moduli space of framed quotients on the projective plane. A precise conjecture relating a K-theoretic partition function of this moduli space to refined open string invariants of toric lagrangian branes is formulated for conifold and local $\mathbb{P}^1 \times \mathbb{P}^1$ geometries.

Contents

1. Introduction

The main goal of this paper is to construct a microscopic quantum mechanical model for BPS states bound to certain surface operators in minimally supersymmetric five dimensional $SU(r)$ gauge theories. This model is obtained employing a string theory construction of such theories consisting of IIA D-branes in a nontrivial geometric background. The BPS states are engineered in terms of D2-brane configurations, the resulting low energy effective action being naturally constructed as the dimensional reduction of a $(0,2)$ quiver gauged linear sigma model. An ADHM style theorem is proven, identifying the moduli space of quiver representations in a special stability chamber with a moduli space of decorated framed torsion free sheaves on the projective plane. The counting function of BPS states bound to surface operators is identified with a K-theoretic partition function of this moduli space. A precise conjecture is formulated, relating this partition function to refined open string invariants of toric lagrangian branes in conifold and local $\mathbb{P}^1 \times \mathbb{P}^1$ geometries. This conjecture is motivated by previous work on the subject [2, 9], where surface operators are engineered by branes wrapping such cycles. Previous papers on a similar subject also include [3, 4, 21, 20], treating various aspects of surface operators in relation with localization on affine Laumon spaces and two dimensional conformal field theory. The relation between some of these results and the present work will be explained below.

In more detail, this paper is structured as follows. Five dimensional gauge theories are constructed in section (2.1) using D6-branes wrapping exceptional cycles of a resolved ADE singularity. Surface operators are obtained by adding D4-branes wrapping certain supersymmetric cycles in this background. BPS states bound to surface operators are identified with supersymmetric ground states of a certain D2-brane system with boundary on a D4-brane. The effective action of this system is constructed in section (2.2) by dimensional reduction of a $(0,2)$ quiver gauged linear sigma model. The final result is given in the quiver diagram (2.20) and the table (2.29).

The geometry of the resulting moduli space of flat directions is studied in detail in section (3). Theorem (3.3) proves that the quantum mechanical moduli space is isomorphic to the moduli space of θ-stable representations of a quiver with relations presented in section (3.1), equation (3.5). This quiver is an enhancement of the standard ADHM quiver whose stable representations are in one-to-one correspondence to isomorphism classes of framed torsion free sheaves on the projective plane. As opposed to the standard ADHM quiver, the space of θ-stability conditions has a nontrivial chamber structure. In particular Lemma (3.1) establishes the existence of a special chamber where θ-stability is equivalent with an algebraic stability condition generalizing standard ADHM stability. Theorem (3.5) proves that the moduli space of stable quiver representation is smooth in the special chamber, and provides an explicit presentation of its tangent space. Finally, Theorem (3.6) proves that in the special stability chamber the moduli space is isomorphic to a moduli space of data (E, ξ, G, g) where E is a torsion free sheaf on the projective plane, $\xi : E \xrightarrow{\sim} \mathcal{O}_{D_\infty}^{\oplus r}$ is a framing of E along a hyperplane $D_\infty \subset \mathbb{P}^2$, and $g : E \twoheadrightarrow G$ is a skyscraper quotient of E supported (in the scheme theoretic sense) on a fixed hyperplane D. The fixed hyperplane D represents the support of the surface operator. Note that similar moduli spaces (without framing data) have been studied by Mochizuki in [22, 23]. The data (G, g) can be also interpreted as a degenerate parabolic structure of E along D, since only zero dimensional quotients of $E|_D$ are involved. In similar situations studied in the literature [3, 4, 20], surface operators are associated to affine Laumon spaces [12], which are moduli spaces of framed parabolic sheaves E on $\mathbb{P}^1 \times \mathbb{P}^1$. In those cases, the parabolic structure consists of a genuine filtration of the restriction $E|_D$, as expected from the general classification of surface operators [14]. The moduli space obtained above offers a different geometric model for surface operators, its viability being tested in section (5) by comparison with refined open string invariants. The relation between these models will become clearer below, once their connection with toric open string invariants is understood.

The counting function of BPS states is identified with a K-theoretic counting function for stable enhanced ADHM quiver representations in section (4). The moduli space of stable quiver representations is equipped by construction with a natural torus action and a determinant line bundle. The K-theoretic partition function defined in section (4.2) is a generating function for the equivariant Euler character of this determinant line bundle. From a physical point of view $(0, q)$-forms on the moduli space with values in the determinant line bundle are supersymmetric ground states in the quiver quantum mechanics constructed in section (2). The torus invariant stable quiver representations in the special chamber are classified in terms of sequences of nested partitions in Proposition (4.1). Moreover, an explicit expression for the equivariant K-theory class of the tangent space at each fixed point is also provided. This yields an explicit expression (4.17) for the equivariant Euler character of the determinant line bundle.

Section (5) consists of a detailed comparison of the $r = 1, 2$ quiver K-theoretic partition functions in the special chamber and the refined open string invariants of toric lagrangian branes in the corresponding toric threefold Z. This relation is stated in Conjecture (5.1) for $r = 1$, and Conjecture (5.3) for $r = 2$, both conjectures being supported by extensive numerical computations. Computation samples are provided in Examples (5.2), (5.4).

Some details of this relation may help elucidate the connection between the present construction and previous work [9, 4]. Note that the refined vertex formalism developed in [18] assigns to a special lagrangian cycle L three distinct refined open string partition functions corresponding to the choice of a preferred leg of the refined vertex. If the brane L is placed on one of the two ordinary legs, the resulting partition functions are related by a simple change of variables. These are cases I and II in [18, Sect 4.2]. In the third case, III, the lagrangian brane is placed on the preferred leg, resulting in a different expression for the open topological partition function. The third case has been considered in connection with surface operators in [9]. In particular the refined topological open string partition function is identified in loc. cit. with a surface operator partition function in the limit $\Lambda_{inst} \to 0$. A similar comparison was carried out in [4] for topological, non-refined open string invariants, in which case there is no distinction between the three legs. As mentioned above, the surface operator partition function is calculated in [4] by localization on affine Laumon spaces.

Conjectures (5.1) and (5.3) establish a precise relation between the K-theoretic partition function introduced in section (4) and the refined open string partition function of an external toric lagrangian brane. This means that the brane intersects only a noncompact component of the toric skeleton of the Calabi-Yau threefold Z, as discussed in detail in section (5). A similar relation is expected between the equivariant K-theory partition function of the affine Laumon space and the refined open string invariants of an internal toric lagrangian brane [9]. An internal brane intersects a compact rational component of the toric skeleton of Z, therefore such branes are naturally labelled by elements of the co-root lattice of the gauge group, in agreement with [14]. In certain situations, open string invariants of external and internal branes can be related by analytic continuation, explaining the fact that the same partition function may have different gauge theoretic constructions. In principle, the surface operators corresponding to internal branes can also be engineered as in section (2), the resulting moduli spaces of quiver representations being presumably closely related to affine Laumon spaces. This will be left for future work.

Acknowledgements. We thank Daniel Jafferis and Greg Moore for discussions. UB's work is partially supported by the PRIN project "Geometria delle varietà algebriche e dei loro spazi di moduli" and INFN project PI14 "Nonperturbative dynamics of gauge theories". WYC was partially supported by DOE grant DE-FG02-96ER40959. The work of DED was partially supported by NSF grant PHY-0854757-2009. MJ is partially supported by the CNPQ grant number 305464/2007-8 and the FAPESP grant number 2005/04558-0.

Note Added. There may be some partial overlap with [**?**], which appeared when the present work was completed.

2. Surface operators and quiver quantum mechanics

This section presents a IIA D-brane construction of BPS states in five dimensional gauge theories, in the presence of surface operators. The final outcome, presented in detail at the end of section (2.2), is a supersymmetric quiver quantum mechanical model for such states obtained as the effective action of certain D2-brane configurations with boundary.

2.1. D-brane engineering.

Minimally supersymmetric five dimensional gauge theories can be easily constructed using IIA D6-branes wrapping rational holomorphic curves in a K3 surface. More precisely, consider a K3 surface with a canonical ADE singularity. Its crepant resolution contains a configuration of (-2) rational curves whose intersection matrix is determined by the incidence matrix of the corresponding Dynkin diagram. A configuration consisting of an arbitrary number of D6-branes wrapped on each such curve yields in the low energy limit a quiver five dimensional gauge theory with eight supercharges. Moreover, BPS states in this quiver gauge theory can be obtained by wrapping D2-branes on the same holomorphic cycles. Then standard D-brane technology shows that the effective action of such a D-brane configuration is a supersymmetric quiver quantum mechanics. This is a microscopic model for such BPS states which can be effectively used in counting problems via localization on moduli spaces of stable quiver representations. It will be shown below that a similar model can be constructed for BPS states bound to a surface operator. Since toric geometry methods will be used, only K3 surfaces with A_k singularities are amenable to the approach developed below. Moreover in order to keep the technical details to a minimum, the construction will be carried out only for $k = 1$. The same basic principles apply to all $k \geq 1$, more involved computations being required.

For the present purposes it suffices to consider a noncompact K3 surface T isomorphic to the total space of the cotangent bundle $T^*\mathbb{P}^1$. The time direction will be Wick rotated to euclidean signature and assumed to be periodic. This yields a natural presentation of the BPS counting function as a finite temperature partition function. Therefore one obtains a geometric background of the form $T \times S^1 \times \mathbb{R}^5$ in IIA theory in euclidean space-time. Note that periodic time translations form a free S^1-action on the space-time manifold. In this setup, the world volume of a Dp-brane is a submanifold of space-time of real dimension $(p+1)$ preserved by the free

S^1 action. In contrast, the world-volume of a Dp-instanton is a $(p+1)$-submanifold embedded in a fixed time subspace. Dp-instantons will not be employed in the following, therefore all D-brane world-volume manifolds must be invariant under time translations. Let $(x^1, \ldots, x^5)$ be linear coordinates on $\mathbb{R}^5$.

Minimally supersymmetric five dimensional $SU(r)$ Yang Mills theory is engineered by r coincident D6-branes with world-volume $\mathbb{P}^1 \times S^1 \times \mathbb{R}^4$, where $\mathbb{P}^1$ is identified with the zero section of $T \to \mathbb{P}^1$, and $\mathbb{R}^4 \subset \mathbb{R}^5$ is a linear subspace. Let $(x^1, \ldots, x^5)$ be linear coordinates on $\mathbb{R}^5$ so that the later is the hyperplane $x^5 = 0$. BPS particles in this theory are engineered by D2-branes with world-volume $\mathbb{P}^1 \times S^1$. Therefore BPS states are identified to supersymmetric ground states in the effective action of D2-branes in the presence of D6-branes, which will be explicitly constructed later in this section.

In order to construct supersymmetric surface operators, note that there is a natural identification $T \times S^1 \times \mathbb{R}^5 \simeq T \times \mathbb{C}^\times \times \mathbb{R}^4$, where $\mathbb{R}^4 \subset \mathbb{R}^5$ is the hyperplane $x^5 = 0$. The isomorphism $S^1 \times \mathbb{R} \simeq \mathbb{C}^\times$ is given by $U = e^{x^5 + i\theta}$, where θ is an angular coordinate on S^1. The free S^1-action corresponding to euclidean time translations is $\theta \to \theta + \delta\theta$. Obviously, $T \times \mathbb{C}^\times$ is a toric Calabi-Yau threefold preserved by this action. Then surface operators will be engineered by wrapping D4-branes on $M \times \mathbb{R}^2$, where $M \subset T \times \mathbb{C}^\times$ is an S^1-invariant toric special lagrangian and $\mathbb{R}^2 \subset \mathbb{R}^4$ is the linear subspace $\{x^1 = x^2 = 0\}$.

The cycle M will be constructed employing the methods used in [1]. Note that T is a toric quotient $\left(\mathbb{C}^3 \setminus \{X_1 = X_2 = 0\}\right)/\mathbb{C}^\times$, where $(X_1, \ldots, X_3)$ are linear coordinates on $\mathbb{C}^3$ such that weights of the $\mathbb{C}^\times$ action are $(1, 1, -2)$. Alternatively, T admits a presentation as a symplectic quotient $\mathbb{C}^3//U(1)$ with respect to a hamiltonian $U(1)$ action with moment map

$$\mu(X_1, \ldots, X_3) = |X_1|^2 + |X_2|^2 - 2|X_3|^2.$$

The $U(1)$ action on the level set $\mu^{-1}(\zeta)$, $\zeta \in \mathbb{R}_{>0}$ is free and the quotient $\mu^{-1}(\zeta)/U(1)$ is isomorphic to T. Note also that there is a natural symplectic torus action $U(1)^2 \times T \to T$, the resulting moment map giving a projection $\varrho : T \to \mathbb{R}^2$. The image of ϱ is the Delzant polytope of T. In homogeneous coordinates, this map is given by

$$\varrho(X_1, X_2, X_3) = (|X_1|^2, |X_2|^2, |X_3|^2),$$

where $\mathbb{R}^2 \subset \mathbb{R}^3$ is identified with the hyperplane

$$|X_1|^2 + |X_2|^2 - 2|X_3|^2 = \zeta. \tag{2.1}$$

Obviously, there is a similar map $\widetilde{\varrho} : T \times \mathbb{C}^\times \to \mathbb{R}^3$,

$$\widetilde{\varrho}(X_1, X_2, X_3, U) = (|X_1|^2, |X_2|^2, |X_3|^2, |U|^2).$$

Using the methods of [1], the cycle M will be constructed by first specifying its image under $\widetilde{\varrho}$,

$$|X_1|^2 - |X_2|^2 = c_1, \qquad |U|^2 - |X_2|^2 = c_2, \tag{2.2}$$

where c_1, c_2 are real parameters. Suppose

$$c_1 > \zeta > 0, \qquad c_2 > 0.$$

Then, taking into account equation (2.1), it follows that any solution to (2.2) must satisfy the inequalities

$$|X_1|^2 \geq c_1, \qquad |X_2|^2 \geq 0, \qquad |X_3|^2 \geq \frac{1}{2}(c_1 - \zeta), \qquad |U|^2 \geq c_2.$$

Therefore the image of M under $\widetilde{\varrho}$ is a half real line. and X_1, X_3 are not allowed to vanish for any solution to (2.2). M is defined by specifying linear constraints on the phases of the homogeneous coordinates in addition to equations (2.2). The intersection of M with the dense open subset $X_2 \neq 0$ is a union of two two-tori defined by the equations

$$\phi_1 + \phi_2 + \phi_U = 0, \pi. \tag{2.3}$$

The intersection of M with the divisor $X_2 = 0$ is the two-torus

$$|X_1|^2 = c_1, \qquad |X_3|^2 = \frac{1}{2}(c_1 - \zeta), \qquad |U|^2 = c_2, \tag{2.4}$$

the phases of X_3, U being unconstrained, while the phase of X_1 is set to zero using $U(1)$ gauge transformation. Therefore the two branches of M defined in equation (2.3) are joined together at $X_2 = 0$, resulting in a special lagrangian cycle of the form $T^2 \times \mathbb{R}$. Taking a single branch would yield a special lagrangian cycle with boundary, $T^2 \times \mathbb{R}_{\geq 0}$.

For further reference note that there is a one parameter family of holomorphic discs in $T \times \mathbb{C}^\times$ with boundary on M cut by the equations

$$X_2 = 0, \qquad 0 \leq |X_3| \leq \frac{1}{2}(c_1 - r), \qquad U = \sqrt{c_2}e^{i\theta}. \tag{2.5}$$

Note also that M is invariant under euclidean time translations, $\phi_U \to \phi_U + \delta\phi_U$, since any such translation is compensated by a $U(1)$-gauge transformation $\phi_1 \to \phi_1 - \delta\phi_U$ in (2.3). The same S^1-action acts freely and transitively on the total space of the family of discs (2.5), identifying the parameter space of this family with the euclidean time circle.

Returning to gauge theory, surface operators are engineered by a D4-brane with world-volume $M \times \{x^1 = x^2 = 0\}$. BPS particles bound to this operator are D2-brane configurations consisting of n_1 D2-branes with world-volume

$$x^1 = \cdots = x^4 = 0, \qquad X_3 = 0, \qquad |U| = \sqrt{c_2} \tag{2.6}$$

and n_2 D2-branes with world-volume

$$x^1 = \cdots = x^4 = 0, \qquad X_2 = 0, \qquad 0 \leq |X_3| \leq \frac{1}{2}(c_1 - r), \qquad U = \sqrt{c_2}. \tag{2.7}$$

These stacks of D_2-branes will be denoted by $\mathrm{D2}_1$, $\mathrm{D2}_2$ respectively. Note that the three cycles (2.6), (2.7) are preserved by euclidean time translations, as expected. Taking quotient by this free action yields in the first case the two-cycle

$$x^1 = \cdots = x^4 = 0, \qquad X_3 = 0, \qquad x^5 = \ln\sqrt{c_2} \tag{2.8}$$

which is isomorphic to the zero section of $T \to \mathbb{P}^1$. In the second case, one obtains a holomorphic disc $\Delta \subset T$ cut by the equations

$$x^1 = \cdots = x^4 = 0, \qquad X_2 = 0, \qquad 0 \leq |X_3| \leq \frac{1}{2}(c_1 - r), \qquad x^5 = \ln\sqrt{c_2}. \tag{2.9}$$

This is obviously a vertical holomorphic disc embedded in the fiber of $T \to \mathbb{P}^1$ at $X_2 = 0$. Therefore the first stack of D2-branes is wrapped on the zero section of $\mathbb{P}^1$, while the second stack is wrapped on the disc Δ.

2.2. **D2-brane effective action via quiver $(0,2)$ models.** To summarize the construction in the previous section, five dimensional supersymmetric $SU(r)$ gauge theory is engineered by wrapping r D6-branes on the exceptional cycle of a resolved A_1 singularity T. The space-time is Wick rotated to euclidean signature, and the time direction is periodic. Surface operators in this theory are engineered by certain supersymmetric D4-brane configurations determined by equations (2.2), (2.3). BPS states bound to such operators are realized by two stacks of D2-branes with multiplicities n_1, n_2 wrapping the holomorphic cycles (2.8), (2.9), which intersect transversely at the point $X_2 = X_3 = 0$ in T.

The goal of the present section is to construct the effective action of the stacks of D2-branes in this background, including modes of D2-D4 and D2-D6 open strings. Since the D2-branes wrap compact cycles, KK reduction will yield an effective quantum mechanical action for their zero modes. In order to analyze the dynamics of this D-brane system, it is helpful to note that that it is related to the D0-D4-D8-brane configuration studied in [10]. The effective action of the D0-branes was identified in [10] with a gauged version of the $(0,4)$ ADHM sigma model action constructed in [30].

As opposed to the current case, the D-brane system analyzed in [10] is embedded in flat space. In order to understand the relation between these configurations, the complex surface $T \to \mathbb{P}^1$ must be replaced by $T' = T^2 \times \mathbb{C} \to \mathbb{C}$, allowing two flat space directions to be compact. Then consider a $D2_1$-$D2_2$-D6-brane system in the new background consisting of n_2 D2-branes wrapping a T^2 fiber of $T' \to \mathbb{C}$, n_1 D2-branes wrapping a section of $T' \to \mathbb{C}$, and r D6-brane wrapping the same section and a linear subspace $\mathbb{R}^4 \subset \mathbb{R}^5$. Obviously, the relative positions of these branes are the same as the relative positions in the $D2_1$-$D2_2$-D6 system on T. The new brane system on $T' \times \mathbb{R}^5$ is related by a T-duality transformation on T^2 to the configuration of parallel D0-D4-D8 branes studied in [10, Sect 3]. The D0-brane effective action was constructed there by dimensional reduction of a two dimensional $(0,4)$ gauged linear sigma model, obtaining a quantum mechanical action with four supercharges. In the present case, T' is replaced by T, which breaks half of the underlying thirty-two IIA supercharges, and in addition a D4-brane is added to the system. The resulting configuration preserves only two supercharges as opposed to four. Therefore by analogy with [10], the effective action will be constructed by dimensional reduction of a two dimensional $(0,2)$ gauged linear sigma model [29, Sect. 6]. Since the system is fairly complicated, it will be convenient to proceed in several stages. The $D2_1$-D6, $D2_2$-D4 configurations will be first studied separately, classifying the massless states in $(0,2)$-multiplets (reduced to one dimension), and writing down the interactions in $(0,2)$ formalism. The coupling between these two sectors via open string $D2_1$-$D2_2$ massless modes will be studied at the next stage. In the following all Chan-Paton bundles on branes will be taken topologically trivial.

2.2.1. $(0,2)$ *models.* Since the massless states will be classified in $(0,2)$ multiplets reduced to one dimension, a brief review of such models is provided below, following [29, Sect 6.1]. There are three types of $(0,2)$ multiplets, the chiral multiplet, the Fermi multiplet, and the vector multiplet. The on shell $(0,2)$ chiral multiplet consists of a complex scalar field and a complex chiral fermion of positive chirality, while the $(0,2)$ Fermi multiplet consists of a complex chiral fermion of negative chirality. The $(0,2)$ gauge multiplet consists of a gauge field and an adjoint complex chiral fermion. A pair consisting of one $(0,2)$-chiral multiplet and one $(0,2)$ Fermi multiplet has the same degrees of freedom as a $(2,2)$ multiplet [29, Sect 6.1]. Chiral multiplets will be denoted by $\mathcal{A}_+$ in the following, and Fermi multiplets will be denoted by $\mathcal{Y}_-$. Each Fermi superfield $\mathcal{Y}_-$ satisfies a superspace constraint of the form

$$\overline{\mathcal{D}}_+\mathcal{Y}_- = \sqrt{2}E_{\mathcal{Y}_-}, \qquad \overline{\mathcal{D}}_+E_{\mathcal{Y}_-} = 0, \tag{2.10}$$

where $E_{\mathcal{Y}_-}$ is a holomorphic function of chiral superfields taking values in the same representation of the gauge group as $\mathcal{Y}_-$. Additional F-term like interactions can be written down in terms of some holomorphic functions $J_{\mathcal{Y}_-}$ of chiral superfields which take values in the dual representation of the gauge group. The following constraint

$$\sum_{\mathcal{Y}_-}\langle J_{\mathcal{Y}_-}\,,\, E_{\mathcal{Y}_-}\rangle = 0. \tag{2.11}$$

must be satisfied in order to obtain a $(0,2)$ supersymmetric lagrangian. Then the $(0,2)$ superspace action is [29, Sect 6.1]

$$\begin{aligned} &\frac{1}{8}\int d^2x d\theta^+ d\theta^+ \mathrm{Tr}(\mathcal{W}\mathcal{W}) - \frac{i}{2}\int d^2x d^2\theta \sum_{\mathcal{A}} \overline{\mathcal{A}}(\mathcal{D}_0 - \mathcal{D}_1)\mathcal{A} \\ &- \frac{1}{2}\int d^2x d^2\theta \sum_{\mathcal{Y}_-}\mathcal{Y}_-^\dagger \mathcal{Y}_- - \frac{1}{\sqrt{2}}\int d^2x d\theta^+ \sum_{\mathcal{Y}_-}\langle J_{\mathcal{Y}_-}\,,\, \mathcal{Y}_-\rangle|_{\overline{\theta}^+}, \end{aligned} \tag{2.12}$$

where $\mathcal{W}$ is the field strength of the vector multiplet. In addition, one can add an FI term of the form

$$\frac{\zeta}{4}\int d^2x d\theta^+ \mathrm{Tr}\mathcal{W}|_{\overline{\theta^+}=0} + h.c$$

for each simple factor of the gauge group. The total potential energy of the resulting $(0,2)$ lagrangian is

$$U_D + \sum_{\mathcal{Y}_-} |E_{\mathcal{Y}_-}|^2 + |J_{\mathcal{Y}_-}|^2 \tag{2.13}$$

where U_D is a standard D-term contribution. Moreover, assuming that $E_{\mathcal{Y}_-}, J_{\mathcal{Y}_-}$ are polynomial functions in the chiral superfields $\mathcal{A}$, the Yukawa couplings can be written as follows. Any monomial $\mathcal{A}_1\cdots\mathcal{A}_n$ in $E_{\mathcal{Y}_-}$ determines a sequence of Yukawa couplings of the form

$$\sum_{i=1}^n \langle \lambda^\dagger_{\mathcal{Y}_-}, A_1\cdots A_{i-1}\psi_{\mathcal{A}_i}\mathcal{A}_{i+1}\cdots A_n\rangle \tag{2.14}$$

and any monomial $\mathcal{A}_1\cdots\mathcal{A}_n$ in $J_{\mathcal{Y}_-}$ determines a sequence of Yukawa couplings of the form

$$\sum_{i=1}^n \langle \lambda_{\mathcal{Y}_-}, A_1\cdots A_{i-1}\psi_{\mathcal{A}_i}\mathcal{A}_{i+1}\cdots A_n\rangle. \tag{2.15}$$

Then one has to sum over all $\mathcal{Y}_-$ and over all such monomials.

2.2.2. *$D2_1$-D6 system.* Recall that this D-brane system is supported on the zero section of T, and the Chan-Paton bundles E_1, F are topologically trivial. In addition, E_1, F are equipped with hermitian structures and compatible connections, which determine in particular holomorphic structures. Since they are bundles on $\mathbb{P}^1$, E_1, F must be isomorphic to the trivial holomorphic bundles $V_1 \otimes \mathcal{O}_{\mathbb{P}^1}, W \otimes \mathcal{O}_{\mathbb{P}^1}$, where V_1, W are vector spaces of dimensions n_1, r equipped with hermitian structures. Moreover, the Chan-Paton connections are gauge equivalent to the trivial connection. The temporal component of the gauge field has a constant zero mode on $\mathbb{P}^1$.

The normal bundle to the D2-branes in $T \times \mathbb{R}^5$ is $N_1 \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{R}_{\mathbb{P}^1}$, where $\mathcal{R}_{\mathbb{P}^1}$ denotes the trivial real line bundle. The transverse fluctuations of the D2-brane are parametrized by a section $(\Phi_1, A_1, A_2, \sigma_1)$ of $N_1 \otimes End(E_1)$, the last component, σ_1, being subject to the reality condition $\sigma_1^\dagger = \sigma_1$. Then the zero modes of the transverse fluctuations are holomorphic sections of $\mathrm{End}(V_1) \otimes \big(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}\big)$. Therefore Φ_1 is identically zero, and A_1, A_2, σ_1 are constant.

In conclusion KK reduction on $\mathbb{P}^1$ yields two complex fields $A_1, A_2 \in \mathrm{End}(V_1)$, a real field $\sigma_1 \in \mathrm{End}(V_1)$, and an $U(V_1)$-gauge field. This is precisely the bosonic field content of an $D = 4$, $N = 2$ vector multiplet reduced to one dimension, which is expected since the D2-branes preserve eight supercharges. By supersymmetry the fermionic fields are obtained by dimensional reduction of the fermions in the same multiplet. The resulting massless spectrum can be organized in terms of two dimensional $(0, 2)$ multiplets reduced to one dimensions. Namely, there are two complex adjoint $(0, 2)$ chiral superfields, $\mathcal{A}_{1+}, \mathcal{A}_{2+}$ with bosonic components A_1, A_2, a $(0, 2)$ vector multiplet, and a $(0, 2)$ adjoint Fermi multiplet $\mathcal{X}_-$. The gauge fields and real adjoint bosonic field σ_1 are obtained by reduction of the two dimensional vector multiplet.

A similar analysis must be carried out for the $D2_1$-D6 fields. In flat space space, with trivial Chan-Paton bundles, and trivial gauge connections, the massless open string modes in this sector yield a $D = 3$, $N = 4$ bifundamental hypermultiplet on the D2-brane world-volume. There are two complex bosonic fields I, J, sections of $Hom(F, E_1)$, $Hom(E_1, F)$ respectively, and two bifundamental Dirac fermions $\psi, \widetilde{\psi}$, also sections of $Hom(F, E_1)$, $Hom(E_1, F)$. Note that there is an $SU(2)_R$ global symmetry group induced by transverse rotations to the D2-D6 system. The bosonic fields are $SU(2)_R$-singlets, while the fermions $(\psi, \widetilde{\psi}^\dagger)$ form a doublet. When the D-branes are wrapped on the zero section of $T \to \mathbb{P}^1$, the bosonic fields I, J are still sections I, J of $Hom(F, E_1)$, $Hom(E_1, F)$, which have constant zero modes on $\mathbb{P}^1$. Therefore KK reduction on $\mathbb{P}^1$ yields two complex bosonic fields I, J with values in $\mathrm{Hom}(W, V_1)$, $\mathrm{Hom}(V_1, W)$ respectively.

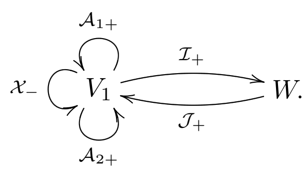
The fermions are topologically twisted as follows. The Lorentz symmetry group $Spin(3) \simeq SU(2)$ and the global symmetry group $SU(2)$ are broken to $U(1)$ subgroups identified with the spin groups of the tangent, respectively normal bundle to the zero section. Both fermions ψ, $\widetilde{\psi}$ have $U(1) \times U(1)$ charges $(1, 1) \oplus (-1, 1)$. Moreover, the normal bundle is canonically identified with the cotangent bundle of the zero section once a global holomorphic 2-form on T is chosen. Since the cotangent bundle is dual to the tangent bundle, it follows that the components of $\psi, \widetilde{\psi}$ are sections of

$$Hom(F, E_1) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)), \qquad Hom(E_1, F) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$$

respectively. Therefore dimensional reduction on $\mathbb{P}^1$ yields two chiral fermion fields with values in $Hom(W, V_1)$, $Hom(V_1, W)$, which are related by supersymmetry to the bosonic fields. In conclusion, the D2_1-D6 strings yield two $(0,2)$ chiral superfields $\mathcal{I}_+, \mathcal{J}_+$ with values in $Hom(W, V_1)$, $Hom(V_1, W)$ and no other degrees of freedom.

Finally, it is helpful to note that there is an alternative derivation of the D2_1-D6 massless spectrum, following from the observation that T is the crepant resolution of the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity. Then D-branes wrapped on the zero section with trivial Chan-Paton bundles are identified with orbifold fractional branes [11, 8] associated to the trivial representation of the orbifold group. More specifically, the D2_1-branes are identified with n_1 fractional D0-branes, while the D6-branes are identified with r fractional D4-branes. Therefore the massless open string spectrum is identified with the $\mathbb{Z}_2$-invariant part of the spectrum of a D0-D4 system transverse to the orbifold, the action of orbifold group on the Chan-Paton spaces V_1, W being trivial. Then a straightforward computation similar to [11] yields the same massless spectrum as obtained above by geometric methods. In particular, all transverse fluctuations of the D0-branes along the orbifold directions are projected out. The field content of the effective action is encoded in the following quiver diagram

(2.16)



where each arrow represents a $(0,2)$ multiplet reduced to one dimension.

As explained in section (2.2.1), the interactions are determined by two holomorphic functions $E_{\mathcal{X}_-}$, $J_{\mathcal{X}_-}$ of the chiral superfields $\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{I}_+, \mathcal{J}_+$. Their tree level values can be easily determined using the fractional brane description of the system explained in the previous paragraph. The tree level potential energy of the D2_1-D6 system is the same as the tree level potential energy of a flat space D0-D4 system, truncated to $\mathbb{Z}_2$-invariant fields. This yields the following expression

$$|[A_1, A_2] + IJ|^2 + |[A_1, A_1^\dagger] + [A_2, A_2^\dagger] + II^\dagger - J^\dagger J - \zeta_1|^2, \tag{2.17}$$

which consists of standard F-term, respectively D-term contributions. ζ_1 is an FI parameter which can be identified with a flat B-field background on the D4-brane world-volume. The F-term contribution to (2.17) determines

$$J_{\mathcal{X}_-} = [\mathcal{A}_{1+}, \mathcal{A}_{2+}] + \mathcal{I}_+\mathcal{J}_+, \qquad E_{\mathcal{X}_-} = 0 \tag{2.18}$$

up to an ambiguity exchanging $E_{\mathcal{X}_-}$ and $J_{\mathcal{X}_-}$. In the present context, exchanging $E_{\mathcal{X}_-}$ and $J_{\mathcal{X}_-}$ is equivalent to a field redefinition, hence there is no loss of generality in making the choice (2.18). One can also multiply $J_{\mathcal{X}_-}$ by an arbitrary phase, but this ambiguity can be again absorbed by a field redefinition.

2.2.3. *D2_2-D4 system.* By construction, the D4-brane world-volume is of the form $M \times \mathbb{R}^2$, where $M \subset T \times S^1 \times \mathbb{R} \simeq T \times \mathbb{C}^\times$ is the S^1-invariant special lagrangian cycle constructed in (2.2)-(2.3). The world-volume of the second stack of the D2-branes is the family of holomorphic discs (2.5) parameterized by periodic euclidean time. For fixed time, the D2-branes wrap the vertical holomorphic disc in $\Delta \subset T$ given in (2.9). The geometric background $T \times S^1 \times \mathbb{R}^5$ preserves half of the thirty-two IIA supercharges, and the D4-brane wrapped on M preserves only four. The combined $D2_2$-D4 system preserves half of the remaining four supercharges.

The D2-brane fluctuations consist of the standard gauge field, transverse fluctuations, and their superpartners. The Chan-Paton bundle E_2 is again topologically trivial, therefore it can be taken of the form $E_2 = V_2 \otimes \mathcal{O}_\Delta$, with V_2 an n_2-dimensional vector space equipped with hermitian structure. The Chan-Paton connection is gauge equivalent to the trivial connection. The temporal component of the gauge field has again a constant zero mode on Δ.

The normal bundle to $\Delta \subset T \times \mathbb{R}^5$ is trivial,

$$N_2 \simeq \mathcal{O}_\Delta \oplus \mathcal{O}_\Delta^{\oplus 2} \oplus \mathcal{R}_\Delta,$$

where the first summand is the normal bundle to Δ in T. The second and third summands correspond to the remaining five transverse directions, $\mathcal{R}_\Delta$ denoting the trivial real line bundle on Δ. The transverse fluctuations are parameterized therefore by a section $(\Phi_2, B_1, B_2, \sigma_2)$ of $\mathrm{End}(V_2) \otimes N_2$, the last component being real, $\sigma_2 = \sigma_2^\dagger$.

In order to determine the zero modes of the transverse fluctuations, boundary conditions must be specified for the fields $(\Phi_2, B_1, B_2, \sigma_2)$. The fluctuations Φ_2, B_1 are transverse to the D4-brane world-volume, therefore they have to satisfy Dirichlet boundary conditions, $\Phi_1|_{\partial\Delta} = 0$, $B_1|_{\partial\Delta} = 0$. This implies that they have no zero modes on Δ since any holomorphic function which vanishes on the boundary must vanish everywhere. The remaining fluctuations B_2, σ_2 are parallel to the D4-brane, therefore they have to satisfy Newmann boundary conditions. A holomorphic function on Δ satisfying Newmann boundary conditions must be constant, therefore B_2, σ_2 have constant zero modes on Δ.

In conclusion, KK reduction on the disc yields a spectrum of bosonic fields consisting of a complex field $B_2 \in \mathrm{End}(V_2)$, a real field $\sigma_2 \in \mathrm{End}(V_2)$ and a $U(V_2)$-gauge field. These are the bosonic components of a $(0,2)$ chiral multiplet $\mathcal{B}_{2+}$, and a $(0,2)$ vector multiplet, reduced to one dimension. Since the system $D2_2$-D4 preserves two supercharges, the zero modes of the fermionic fields must naturally provide the missing fermionic components in these multiplets. The resulting field content is summarized in the following quiver diagram

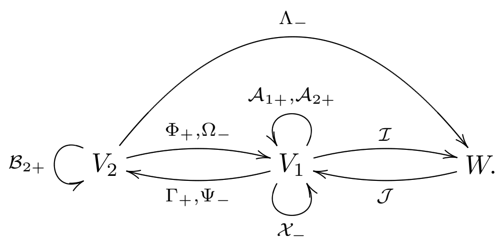
$$\mathcal{B}_{2+} \circlearrowleft V_2 \tag{2.19}$$

Since there are no Fermi superfields, the only interactions are gauge couplings and D-term interactions. This is consistent with the fact that the D2-branes are free to glide along the D4 branes with no cost in energy. Note that FI terms in the D2-brane world-volume can be obtained by turning on a flat gauge field background on the D4-brane.

2.2.4. *Coupling the two systems.* The next task is to couple the two D-brane systems analyzed above. In addition to the zero modes found in sections (2.2.2), (2.2.3), there are extra massless open string states in the $D2_1$-$D2_2$ sector and in the $D2_2$-D6-sector. In both cases the stacks of D-branes intersect transversely at a point, therefore the massless states are the same as in a similar D-brane configuration embedded in flat space. The fields in the $D2_1$-$D2_2$ sector are naturally identified with the components of a $D = 4$, $N = 2$ bifundamental hypermultiplet reduced to one dimension. In terms of $(0,2)$-superfields, there are two $(0,2)$ chiral multiplets Φ_+, Γ_+ with values in $\mathrm{Hom}(V_2, V_1)$, $\mathrm{Hom}(V_1, V_2)$ respectively, and two

Fermi superfields Ω_-, Ψ_-, also with values in in $\mathrm{Hom}(V_2, V_1)$, $\mathrm{Hom}(V_1, V_2)$. The the D2$_2$-D6-sector yields a single Fermi superfield Λ_- with values in $\mathrm{Hom}(V_2, W)$. Taking into account the previous results, the combined $(0, 2)$ spectrum is summarized in the following quiver diagram

(2.20)



Note that an arrow marked by two superfields represents in fact two distinct arrows, corresponding respectively to the two superfields. For ease of exposition, the arrows corresponding to chiral superfields will be called bosonic, while the arrows corresponding to Fermi superfields will be called fermionic. Therefore, for example, there are three arrows beginning and ending at V_1, two bosonic corresponding to $\mathcal{A}_{1+}, \mathcal{A}_{2+}$, and one fermionic, corresponding to $\mathcal{X}_-$. Similarly, there are two arrows between V_2 and V_1, one bosonic and one fermionic, and two arrows between V_1 and V_2, again, one bosonic and one fermionic.

Next one has to determine the holomorphic functions E, J for each Fermi superfield in (2.20). First note that the tree level potential energy must include quartic couplings between the fields Φ_+, Γ_+ superfields $\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{B}_2$ reflecting the fact that the D2$_1$-D2$_2$ fields become massive once the two stacks of D2-branes are displaced, their mass being proportional with the separation. Therefore, taking into account gauge invariance, the potential interactions between the bosonic components of Φ_+, Γ_+ and $\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{B}_2$ must be of the form

$$|A_1 f|^2 + |A_2 f - f B_2|^2 + |g A_1|^2 + |g A_2 - B_2 g|^2. \tag{2.21}$$

Here $f \in \mathrm{Hom}(V_2, V_1)$, $g \in \mathrm{Hom}(V_1, V_2)$ are the bosonic components of chiral superfields Φ_+, Γ_+. Note that since V_1, V_2, W are equipped with hermitian structures, any space of morphisms between any two vector spaces has an induced hermitian structure. The resulting hermitian form is denoted by $|\ \ |$ in (2.21). Such couplings are obtained by setting

$$\begin{aligned} E_{\Omega_-} &= \epsilon_1(\Phi_+ \mathcal{B}_{2+} - \mathcal{A}_{2+}\Phi_+), \qquad & J_{\Omega_-} &= \eta_1 \Gamma_+ \mathcal{A}_{1+}, \\ E_{\Psi_-} &= \epsilon_2(\mathcal{B}_{2+}\Gamma_+ - \Gamma_+ \mathcal{A}_{2+}), \qquad & J_{\Psi_-} &= \eta_2 \mathcal{A}_{1+}\Phi_+ \end{aligned} \tag{2.22}$$

where $\epsilon_1, \epsilon_2, \eta_1, \eta_2$ are phases, i.e. complex numbers with modulus 1. One can also obtain the same potential energy exchanging the ordered pairs $(E_{\Omega_-}, J_{\Omega_-})$, (J_{Ψ_-}, E_{Ψ_-}). This ambiguity is equivalent to a field redefinition, hence there is no loss of generality in making the choice (2.22).

The phases will be fixed up to field redefinitions imposing the supersymmetry condition (2.11). Since the coupling between the two sectors will not change the tree level potential energy (2.17) of the D2$_1$-D6 modes, one must have

$$J_{\mathcal{X}_-} = [\mathcal{A}_{1+}, \mathcal{A}_{2+}] + \mathcal{I}_+ \mathcal{J}_+ \tag{2.23}$$

as found in equation (2.18). The holomorphic function $E_{\mathcal{X}_-}$ is not necessarily zero, as found there, but, if nonzero, it must have nontrivial dependence on the extra chiral superfields Φ_+, Γ_+.

The supersymmetry condition (2.11) yields

$$\langle J_{\Omega_-}, E_{\Omega_-}\rangle + \langle J_{\Psi_-}, E_{\Psi_-}\rangle + \langle J_{\mathcal{X}_-}, E_{\mathcal{X}_-}\rangle + \langle J_{\Lambda_-}, E_{\Lambda_-}\rangle = 0. \tag{2.24}$$

The possible contributions to the holomorphic functions $E_{\mathcal{Y}_-}, J_{\mathcal{Y}_-}$ assigned to each Fermi superfield $\mathcal{Y}_- \in \{\mathcal{X}_-, \Omega_-, \Psi_-, \Lambda_-\}$ can be classified as follows. Let $V_{t(\mathcal{Y}_-)}$, $V_{h(\mathcal{Y}_-)}$ be the vector spaces assigned to the tail, respectively the head of the arrow corresponding to $\mathcal{Y}_-$ in the diagram (2.20). Then $\mathcal{Y}_-$ takes values in the linear space $\mathrm{Hom}(V_{t(\mathcal{Y}_-)}, V_{h(\mathcal{Y}_-)})$. The holomorphic functions

$$E_{\mathcal{Y}_-} \in \mathrm{Hom}(V_{t(\mathcal{Y}_-)}, V_{h(\mathcal{Y}_-)}), \qquad J_{\mathcal{Y}_-} \in \mathrm{Hom}(V_{h(\mathcal{Y}_-)}, V_{t(\mathcal{Y}_-)})$$

are determined by linear combinations of paths of bosonic arrows in the path algebra of the quiver (2.20).

Next note that a simple computation yields

$$\begin{aligned} &\langle J_{\Omega_-}, E_{\Omega_-}\rangle + \langle J_{\Psi_-}, E_{\Psi_-}\rangle = (\epsilon_1\eta_1 + \epsilon_2\eta_2)\mathrm{Tr}_{V_2}\big(\Gamma_+\mathcal{A}_{1+}\Phi_+\mathcal{B}_{2+}\big) \\ &- \epsilon_1\eta_1\mathrm{Tr}_{V_2}\big(\Gamma_+\mathcal{A}_{1+}\mathcal{A}_{2+}\Phi_+\big) - \epsilon_2\eta_2\mathrm{Tr}_{V_2}\big(\Gamma_+\mathcal{A}_{2+}\mathcal{A}_{1+}\Phi_+\big). \end{aligned} \tag{2.25}$$

Moreover

$$\langle J_{\mathcal{X}_-}, E_{\mathcal{X}_-}\rangle = \mathrm{Tr}_{V_1}\Big(([\mathcal{A}_{1+}, \mathcal{A}_{2+}] + \mathcal{I}_+\mathcal{J}_+)E_{\mathcal{X}_-}\Big)$$

where $E_{\mathcal{X}_-}$ must be a linear combination of paths consisting of the following building blocks

$$\Phi_+\mathcal{B}_{2+}^k\Gamma_+, \qquad \mathcal{A}_{1+}, \qquad \mathcal{A}_{2+}, \qquad \mathcal{I}_+\mathcal{J}_+,$$

with $k \in \mathbb{Z}_{\geq 0}$. Similarly, E_{Λ_-}, J_{Λ_-} must be linear combinations of paths of the form

$$\mathcal{B}_{2+}^k\Gamma_+P(\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{I}_+\mathcal{J}_+, \Phi_+\Gamma_+, \Gamma_+\Phi_+)\mathcal{I}_+,$$
$$\mathcal{J}_+Q(\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{I}_+\mathcal{J}_+, \Phi_+\Gamma_+, \Gamma_+\Phi_+)\Phi_+\mathcal{B}_{2+}^l,$$

where $k, l \in \mathbb{Z}_{\geq 0}$ and $P(\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{I}_+\mathcal{J}_+)$, $Q(\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{I}_+\mathcal{J}_+)$ are polynomial functions of $\mathcal{A}_{1+}, \mathcal{A}_{2+}, \mathcal{I}_+\mathcal{J}_+, \Phi_+\Gamma_+, \Gamma_+\Phi_+$. This implies that

$$\langle J_{\mathcal{X}_-}, E_{\mathcal{X}_-}\rangle + \langle J_{\Lambda_-}, E_{\Lambda_-}\rangle \tag{2.26}$$

cannot contain any terms proportional to

$$\mathrm{Tr}_{V_2}\big(\Gamma_+\mathcal{A}_{1+}\Phi_+\mathcal{B}_{2+}\big) = \mathrm{Tr}_{V_1}\big(\Phi_+\mathcal{B}_{2+}\Gamma_+\mathcal{A}_{1+}\big).$$

Therefore supersymmetry requires $\epsilon_1\eta_1 + \epsilon_2\eta_2 = 0$ in (2.25). Then the remaining terms in the right hand side of (2.25) can be written as

$$\epsilon_2\eta_2\mathrm{Tr}_{V_1}\big([\mathcal{A}_{1+}, \mathcal{A}_{2+}]\Phi_+\Gamma_+\big).$$

These terms must be cancelled by similar terms in the expansion of (2.26). Since all terms in the expansion of $\langle J_{\Lambda_-}, E_{\Lambda_-}\rangle$ have non-trivial dependence on $\mathcal{I}_+, \mathcal{J}_+$, the terms required by this cancellation must occur in the expansion of $\langle J_{\mathcal{X}_-}, E_{\mathcal{X}_-}\rangle$. This uniquely determines

$$E_{\mathcal{X}_-} = -\epsilon_2\eta_2\Phi_+\Gamma_+. \tag{2.27}$$

Taking into account all conditions obtained so far, the right hand side of (2.25) reduces to

$$\langle J_{\Lambda_-}, E_{\Lambda_-}\rangle - \epsilon_2\eta_2\mathrm{Tr}_{V_2}(\Gamma_+\mathcal{I}_+\mathcal{J}_+\Phi_+).$$

Given the building blocks for E_{Λ_-}, J_{Λ_-} listed above, it follows that

$$E_{\Lambda_-} = \epsilon_3\mathcal{J}_+\Phi_+, \qquad J_{\Lambda_-} = \eta_3\Gamma_+\mathcal{I}_+ \tag{2.28}$$

where ϵ_3, η_3 are phases satisfying $\epsilon_3\eta_3 - \epsilon_2\eta_2 = 0$.

In conclusion, all holomorphic functions $E_{\mathcal{Y}_-}, J_{\mathcal{Y}_-}$ have been completely determined up to certain ambiguous phases which can be set to ± 1 by field redefinitions. The final results are summarized in the following table

$$\begin{array}{lll} \mathcal{Y}_- & E_{\mathcal{Y}_-} & J_{\mathcal{Y}_-} \\ \mathcal{X}_- & -\Phi_+\Gamma_+ & [\mathcal{A}_{1+}, \mathcal{A}_{2+}] + \mathcal{I}_+\mathcal{J}_+ \\ \Omega_- & \Phi_+\mathcal{B}_{2+} - \mathcal{A}_{2+}\Phi_+ & -\Gamma_+\mathcal{A}_{1+} \\ \Psi_- & \mathcal{B}_{2+}\Gamma_+ - \Gamma_+\mathcal{A}_{2+} & \mathcal{A}_{1+}\Phi_+ \\ \Lambda_- & \mathcal{J}_+\Phi_+ & \Gamma_+\mathcal{I} \end{array} \tag{2.29}$$

Then the total potential energy of the quantum mechanical effective action is

$$U = U_{gauge} + U_D + U_E + U_J \tag{2.30}$$

where U_{gauge} is the potential energy determined by gauge couplings,

$$\begin{aligned} U_{gauge} = & |[\sigma_1, A_1]|^2 + |[\sigma_1, A_2]|^2 + |[\sigma_2, B_2]|^2 + |\sigma_1 I|^2 + |J\sigma_1|^2 \\ & + |\sigma_1 f - f\sigma_2|^2 + |\sigma_2 g - g\sigma_1|^2, \end{aligned} \tag{2.31}$$

U_D is the D-term contribution

$$\begin{aligned} U_D = & \Big([A_1, A_1^\dagger] + [A_2, A_2^\dagger] + II^\dagger - J^\dagger J + ff^\dagger - g^\dagger g - \zeta_1\Big)^2 \\ & + \Big([B_2, B_2^\dagger] + gg^\dagger - f^\dagger f - \zeta_2\Big)^2, \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} U_E + U_J = & |[A_1, A_2] + IJ|^2 + |fg|^2 + |A_1 f|^2 + |gA_1|^2 \\ & + |A_2 f - fB_2|^2 + |gA_2 - B_2 g|^2 + |Jf|^2 + |gI|^2 \end{aligned} \tag{2.33}$$

are the E and J term contributions.

The supersymmetric ground states of the resulting quantum-mechanical system are obtained in the Born-Oppenheimer approximation by quantization of the moduli space of classical supersymmetric flat directions. As usual in supersymmetric theories, this approximation yields an exact count of such states. The geometry of the resulting moduli space will be studied in the next section.

3. Moduli space of flat directions and enhanced ADHM data

The main goal of this section is to analyze the geometry of the moduli space of supersymmetric flat directions of the quantum mechanical potential (2.33). It will be shown below that, for generic values of the FI parameters, such moduli space is isomorphic to the moduli space of stable representations of a quiver with relations, called the enhanced ADHM quiver. It will be also shown that, in a certain stability chamber, this moduli space admits a geometric interpretation in terms of framed torsion free sheaves on the projective plane.

Summarizing the results of the previous section, the D2-brane effective action has been constructed by dimensional reduction of a $(0,2)$ model with field content given by the quiver diagram (2.20) and interactions given by (2.29). The space of constant field configurations $(A_1, A_2, I, J, B_2, f, g, \sigma_1, \sigma_2)$ is the vector space

$$\begin{aligned} & \mathrm{End}(V_1)^{\oplus 2} \oplus \mathrm{Hom}(W, V_1) \oplus \mathrm{Hom}(V_1, W) \oplus \\ & \mathrm{End}(V_2) \oplus \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1) \oplus \mathfrak{u}(V_1) \oplus \mathfrak{u}(V_2) \end{aligned} \tag{3.1}$$

where V_1, V_2 and W are complex vector spaces equipped with hermitian inner products. The moduli space of flat directions is the moduli space of gauge equivalence classes of solutions to the zero-energy equations

$$\begin{aligned}&[A_1, A_1^\dagger] + [A_2, A_2^\dagger] + II^\dagger - J^\dagger J + ff^\dagger - g^\dagger g = \zeta_1,\\ &[B_2, B_2^\dagger] + gg^\dagger - f^\dagger f = \zeta_2,\end{aligned} \tag{3.2}$$

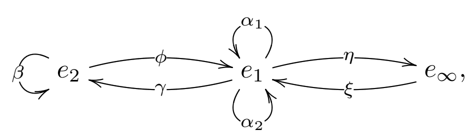
$$\begin{aligned}&[A_1, A_2] + IJ = 0, \qquad Jf = 0, \qquad gI = 0, \qquad A_1 f = 0, \qquad gA_1 = 0\\ &A_2 f - fB_2 = 0, \qquad gA_2 - B_2 g = 0, \qquad fg = 0,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&[\sigma_1, A_i] = 0, \qquad [\sigma_2, B_2] = 0, \qquad \sigma_1 I = 0, \qquad J\sigma_1 = 0,\\ &\sigma_1 f - f\sigma_2 = 0, \qquad g\sigma_1 - \sigma_2 g = 0\end{aligned} \tag{3.4}$$

derived from (2.30). Two solutions are gauge equivalent if they are related by the natural action of the gauge group $U(V_1) \times U(V_2)$ on the space (3.1). The resulting moduli space can be naturally identified with a moduli space of quiver representations, as presented below.

3.1. **Enhanced ADHM Quiver.** The enhanced ADHM quiver is the quiver with relations defined by the following diagram

(3.5)



and ideal of relations being generated by

$$\begin{aligned}&\alpha_1\alpha_2 - \alpha_2\alpha_1 + \xi\eta, \qquad \alpha_1\phi, \qquad \alpha_2\phi - \phi\beta, \qquad \eta\phi, \qquad \gamma\xi\\ &\phi\gamma, \qquad \gamma\alpha_1, \qquad \gamma\alpha_2 - \beta\gamma.\end{aligned} \tag{3.6}$$

Note that omitting the vertex e_2 and all above relations except the first one, one obtains the usual ADHM quiver.

A representation $\mathcal{R}$ of the enhanced ADHM quiver in the category of complex vector spaces is given by a triple (V_1, V_2, W) of vector spaces assigned to the vertices (e_1, e_2, e_∞) and linear maps $(A_1, A_2, I, J, B, f, g)$ assigned to the arrows $(\alpha_1, \alpha_2, \xi, \eta, \beta, \phi, \gamma)$ respectively, and satisfying the relations (3.6). The numerical type of a representation is the triple $(\dim(W), \dim(V_1), \dim(V_2)) \in (\mathbb{Z}_{\geq 0})^3$. A morphism between two such representations $\mathcal{R}$ and $\mathcal{R}'$ is a triple $(\xi_1, \xi_2, \xi_\infty)$ of linear maps between the vector spaces assigned to the nodes (e_1, e_2, e_∞), respectively, satisfying obvious compatibility conditions with the morphisms attached to the arrows. This defines an abelian category of quiver representations. Note that this abelian category contains the abelian category of representations of the ADHM quiver as the full subcategory of representations with $n_2 = 0$.

A framed representation of the enhanced ADHM quiver with type $(r, n_1, n_2) \in (\mathbb{Z}_{\geq 0})^3$ is a pair $(\mathcal{R}, h)$ consisting of a representation $\mathcal{R}$ and an isomorphism $h : W \xrightarrow{\sim} \mathbb{C}^r$. Two framed representations $(\mathcal{R}, h)$ and $(\mathcal{R}', h')$ are isomorphic if there is an isomorphism of the form $(\xi_1, \xi_2, \xi_\infty) : \mathcal{R} \xrightarrow{\sim} \mathcal{R}'$ such that $h'\xi_\infty = h$.

In order to construct moduli spaces of framed representations of the enhanced ADHM quiver, one has to introduce suitable stability conditions. By analogy with

[19], a stability condition will be defined by a triple $\theta = (\theta_1, \theta_2, \theta_\infty) \in \mathbb{Q}^3$ satisfying the relation

$$n_1\theta_1 + n_2\theta_2 + r\theta_\infty = 0. \tag{3.7}$$

A representation $\mathcal{R}$ of numerical type $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$ will be called θ-(semi)stable if the following conditions hold

(*i*) Any subrepresentation $\mathcal{R}' \subset \mathcal{R}$ of numerical type $(0, n_1', n_2')$ satisfies

$$n_1'\theta_1 + n_2'\theta_2 \ (\leq)\ 0. \tag{3.8}$$

(*ii*) Any subrepresentation $\mathcal{R}' \subset \mathcal{R}$ of numerical type (r, n_1', n_2') satisfies

$$n_1'\theta_1 + n_2'\theta_2 + r\theta_\infty \ (\leq)\ 0. \tag{3.9}$$

We emphasize that the above definition does not coincide with the one considered by King in [19, Section 3] because only subrepresentations with $r' = 0, r$ are considered in the stability condition. However, as we shall see in the next subsection, it plays essentially the same role.

Note also that θ-stability has the Harder-Narasimhan, respectively Jordan-Hölder property since the abelian category of quiver representations is noetherian and artinian. Two θ-semistable representation with identical dimension vectors will be called S-equivalent if their associated graded representations with respect to the Jordan-Hölder filtration are isomorphic.

Let $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$ be a fixed dimension vector. Note that the space of stability parameters $\theta = (\theta_1, \theta_2, \theta_\infty) \in \mathbb{Q}^3$ satisfying $n_1\theta_1 + n_2\theta_2 + r\theta_\infty = 0$ can be naturally identified with the (θ_1, θ_2)-plane $\mathbb{Q}^2$, after solving for θ_∞. Such a parameter θ will be called critical of type (r, n_1, n_2) if the set of strictly θ-semistable representations $\mathcal{R}$ with dimension vector (r, n_1, n_2) is non-empty. If this set is empty, θ will be called generic. Then it is easy to prove that, for a fixed dimension vector $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$, the set of critical stability parameters consists of of finitely many lines in the (θ_1, θ_2)-plane.

The following lemma establishes the existence of generic stability parameters for any given dimension vector (r, n_1, n_2).

Lemma 3.1. *Suppose $\theta_2 > 0$ and $\theta_1 + n_2\theta_2 < 0$ for some fixed $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$. Then a representation $\mathcal{R}$ is θ-semistable if and only if it is θ-stable and if and only if the following conditions are satisfied*

(S.1) *$f : V_2 \to V_1$ is injective and $g : V_1 \to V_2$ is identically zero.*

(S.2) *The data $\mathcal{A} = (V_1, W, A_1, A_2, I, J)$ satisfies the ADHM stability condition, that is there is no proper nontrivial subspace $0 \subset V_1' \subset V_1$ preserved by A_1, A_2 and containing the image of I.*

Proof. Under the assumptions of lemma (3.1) let $\mathcal{R}$ be a θ-semistable representation. Suppose f is not injective. Then it is straightforward to check that $\ker(f) \subset V_2$ is preserved by B_2, therefore it determines a subrepresentation of $\mathcal{R}$ with $n_1' = 0$, $r' = 0$. The semistability condition yields

$$\theta_2 \dim(\mathrm{Ker}(f)) \leq 0$$

which leads to a contradiction if $\dim(\mathrm{Ker}(f)) > 0$. Therefore f must be injective, and relation $fg = 0$ implies $g = 0$.

Similarly, if condition $(S.2)$ is not satisfied by some proper nontrivial subspace $0 \subset V_1' \subset V_1$, the data

$$\mathcal{R}' = (V_1', 0, W, A_1|_{V_1'}, A_2|_{V_1'}, I, J|_{V_1'}, 0, 0)$$

determines a proper nontrivial subrepresentation of $\mathcal{R}$ with $r' = r$ so that

$$n_1'\theta_1 + n_2'\theta_2 + r\theta_\infty = (n_1' - n_1)\theta_1 > 0.$$

This is again a contradiction.

Next let $\mathcal{R}$ be a representation satisfying conditions $(S.1)$, $(S.2)$, and suppose $\mathcal{R}' \subset \mathcal{R}$ is a nontrivial proper subrepresentation of $\mathcal{R}$. Note that $g' = 0$ since $g = 0$. There are two cases, $r' = r$ and $r' = 0$.

Suppose $r' = r$. Then $(S.2)$ implies that I is not identically zero, hence $n_1' > 0$. If $n_1' < n_1$, the data $\mathcal{A}' = (V_1', A_1', A_2', I', J')$ would violate condition $(S.2)$. Therefore $n_1 = n_1'$. Since $\mathcal{R}'$ has to be a proper subrepresentation, $n_2' < n_2$. Then

$$n_1'\theta_1 + n_2'\theta_2 + r\theta_\infty = (n_2' - n_2)\theta_2 < 0.$$

Now suppose $r' = 0$. Note that $n_1' = 0$ implies that $V_2' \subset \mathrm{Ker}(f) = 0$, hence $n_2' = 0$ as well. This is impossible since $\mathcal{R}'$ is assumed nontrivial. Therefore $n_1' \geq 1$, and

$$n_1'\theta_1 + n_2'\theta_2 \leq \theta_1 + n_2\theta_2 < 0$$

using the conditions of lemma (3.1).

□

In the following, a representation $\mathcal{R}$ of the enhanced ADHM quiver will be called stable if it satisfies conditions $(S.1)$, $(S.2)$ of lemma (3.1).

3.2. **Moduli spaces.** Moduli spaces of θ-semistable framed quiver representations will be constructed employing GIT techniques, by analogy to [19]. Since framed quiver moduli of the type considered here do not seem to be treated previously in the literature, the details will be presented below for completeness.

Let V_1, V_2, W be vector spaces of dimensions $n_1, n_2, r \in \mathbb{Z}_{>0}$ respectively. Let

$$\begin{aligned}\mathbb{X}(r, n_1, n_2) =& \mathrm{End}(V_1)^{\oplus 2} \oplus \mathrm{Hom}(W, V_1) \oplus \mathrm{Hom}(V_1, W) \oplus \\ & \mathrm{End}(V_2) \oplus \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1).\end{aligned}$$

and note that there is a natural $G = GL(V_1) \times GL(V_2)$ action on $\mathbb{X}(r, n_1, n_2)$ given by

$$\begin{aligned}(g_1, g_2) \times (A_1, A_2, & I, J, B_2, f, g) \longrightarrow \\ & (g_1 A_1 g_1^{-1}, g_1 A_2 g_1^{-1}, J g_1^{-1}, g_1 I, g_2 B_2 g_2^{-1}, g_1 f g_2^{-1}, g_2 g g_1^{-1}).\end{aligned}$$

The closed points of $\mathbb{X}(r, n_1, n_2)$ will be denoted by $\mathsf{x} = (A_1, A_2, I, J, B_2, f, g)$, and the action of $(g_1, g_2) \in G$ on a point $\mathsf{x} \in \mathbb{X}$ will be denoted by $(g_1, g_2) \cdot \mathsf{x}$. The stabilizer of a given point x will be denoted by $G_{\mathsf{x}} \subset G$. Moreover, let $\mathbb{X}_0(r, n_1, n_2) \subset \mathbb{X}$ denote the subscheme defined by the algebraic equations (3.3). Obviously, $\mathbb{X}_0(r, n_1, n_2)$ is preserved by the G-action.

Note also each representation $\mathcal{R} = (V_1, V_2, W, A_1, A_2, I, J, B_2, f, g)$ corresponds to a unique point $\mathsf{x} = (A_1, A_2, I, J, B_2, f, g)$ in $\mathbb{X}_0$; two framed representations are isomorphic if and only if the corresponding points in $\mathbb{X}_0(r, n_1, n_2)$ are in the same G-orbit.

Next, recall some standard facts on GIT quotients for a reductive algebraic group G acting on a vector space $\mathbb{X}(r, n_1, n_2)$ [19, Section 2]. Given an algebraic character $\chi : G \to \mathbb{C}^\times$ one has the following notion of χ-(semi)stability.

(*a*) A point x_0 is called χ-semistable if there exists a polynomial function $p(\mathsf{x})$ on $\mathbb{X}(r, n_1, n_2)$ satisfying $p((g_1, g_2) \cdot \mathsf{x}) = \chi(g_1, g_2)^l p(\mathsf{x})$ for some $l \in \mathbb{Z}_{\geq 1}$, so that $p(\mathsf{x}_0) \neq 0$.

(*b*) A point x_0 is called χ-stable if there exists a polynomial function $p(\mathsf{x})$ as in (*a*) above so that $\dim(G \cdot \mathsf{x}_0) = \dim(G/\Delta)$, where $\Delta \subset G$ is the subgroup acting trivially on $\mathbb{X}(r, n_1, n_2)$. and the action of G on $\{\mathsf{x} \in \mathbb{X}(r, n_1, n_2) \,|\, p(\mathsf{x}) \neq 0\}$ is closed.

This definition can be reformulated as follows. Let G act on the direct product $\mathbb{X}_0(r, n_1, n_2) \times \mathbb{C}$ by

$$(g_1, g_2) \times (\mathsf{x}, z) \to ((g_1, g_2) \cdot \mathsf{x}, \chi(g_1, g_2)^{-1} z)$$

Then according to [19, Lemma 2.2], $\mathsf{x} \in \mathbb{X}(r, n_1, n_2)$ is χ-semistable if and only if the closure of the orbit $G \cdot (\mathsf{x}, z)$ is disjoint from the zero section $\mathbb{X}(r, n_1, n_2) \times \{0\}$, for any $z \neq 0$. Moreover x is χ-stable if and only if the orbit $G \cdot (\mathsf{x}, z)$ is closed in complement of the zero section, and the stabilizer $G_{(\mathsf{x},\mathsf{z})}$ is a finite index subgroup of Δ.

One can form the quasi-projective scheme:

$$\mathcal{N}_\theta^{ss}(r, n_1, n_2) = \mathbb{X}_0(r, n_1, n_2)//_\chi G := \mathrm{Proj}\left(\oplus_{n\geq 0} A(\mathbb{X}_0(r, n_1, n_2))^{G,\chi^n}\right) ,$$

where

$$A(\mathbb{X}_0(r, n_1, n_2))^{G,\chi^n} := \{f \in A(\mathbb{X}_0(r, n_1, n_2)) \mid f(g \cdot x) = \chi(g)^n f(x) \; \forall g \in G\} .$$

Clearly, $\mathcal{N}_\theta^{ss}(r, n_1, n_2)$ is projective over $\mathrm{Spec}\left(\mathbb{X}_0(r, n_1, n_2))^G\right)$, and it is quasi-projective over $\mathbb{C}$. Geometric Invariant Theory tells us that $\mathcal{N}_\theta^{ss}(r, n_1, n_2)$ is the space of χ-semistable orbits; moreover, it contains an open subscheme $\mathcal{N}_\theta^{s}(r, n_1, n_2) \subseteq \mathcal{N}_\theta^{ss}(r, n_1, n_2)$ consisting of χ-stable orbits.

Then the following holds by analogy with [19, Prop. 3.1, Thm. 4.1]. Again the details of the proof are given below for completeness.

Proposition 3.2. *Suppose* $\theta = (\theta_1, \theta_2) \in \mathbb{Z}^2$*, and let* $\chi_\theta : G \to \mathbb{C}^\times$ *be the character*

$$\chi_\theta(g_1, g_2) = \det(g_1)^{-\theta_1} \det(g_2)^{-\theta_2}.$$

Then a representation $\mathcal{R} = (V_1, V_2, W, A_1, A_2, I, J, B_2, f, g)$ *of an enhanced ADHM quiver, of dimension vector* $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$*, is* θ*-(semi)stable if and only if the corresponding closed point* $\mathsf{x} \in \mathbb{X}_0$ *is* χ_θ*-(semi)stable.*

It follows that $\mathcal{N}_\theta^{ss}(r, n_1, n_2)$ parameterizes S-equivalence classes of θ-semistable framed representations, while $\mathcal{N}_\theta^{s}(r, n_1, n_2)$ parameterizes isomorphism classes of θ-stable framed representations.

Proof. First, we prove that if $\mathsf{x} \in \mathbb{X}$ is χ_θ-semistable, then the corresponding representation $\mathcal{R}$ is θ-semistable. Suppose that there exists a nontrivial proper subrepresentation $0 \subset \mathcal{R}' \subset \mathcal{R}$ with either $r' = 0$ or $r' = r$ so that

$$n_1'\theta_1 + n_2'\theta_2 + r'\theta_\infty > 0.$$

Let us first consider the case $r' = 0$. Since $\mathcal{R}' = (V_1', V_2', \{0\}, A_1', A_2', I', J', B_2', f', g')$ is a subrepresentation of $\mathcal{R}$, then V_1' and V_2' can be regarded as subspaces of V_1 and

V_2, respectively, and it follows that

$$f(V_2') \subseteq V_2', \qquad g(V_2') \subseteq V_1', \qquad A_i(V_1') \subseteq V_1', \\ B_2(V_2') \subseteq V_2', \qquad J(V_1') = 0, \tag{3.10}$$

for $i = 1, 2$. Then there exist direct sum decompositions $V_1 \simeq V_1' \oplus V_1''$, $V_2 \simeq V_2' \oplus V_2''$ such that the linear maps A_1, A_2, B_2, f, and g have block decomposition of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \tag{3.11}$$

while I, J have block decompositions of the form

$$I = \begin{bmatrix} * \\ * \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & * \end{bmatrix}. \tag{3.12}$$

Consider a one-parameter subgroup of G of the form

$$g_1(t) = \begin{bmatrix} t1_{V_1'} & 0 \\ 0 & 1_{V_1''} \end{bmatrix}, \qquad g_2(t) = \begin{bmatrix} t1_{V_2'} & 0 \\ 0 & 1_{V_2''} \end{bmatrix}.$$

It follows that the linear maps $(A_1(t), A_2(t), I(t), J(t), B_2(t), f(t), g(t)) = (g_1(t), g_2(t))\cdot$ x have block decompositions of the form

$$\begin{bmatrix} * & t* \\ 0 & * \end{bmatrix}, \tag{3.13}$$

and

$$I^t = \begin{bmatrix} t* \\ * \end{bmatrix}, \qquad J^t = \begin{bmatrix} 0 & * \end{bmatrix}. \tag{3.14}$$

At the same time, $\chi_\theta(g_1(t), g_2(t))^{-1} z = t^{n_1'\theta_1 + n_2'\theta_2} z$, with $n_1'\theta_1 + n_2'\theta_2 > 0$. Therefore the limit of $(g_1(t), g_2(t)) \cdot (\mathsf{x}, z)$ as $t \to 0$ is a point on the zero section, which contradicts χ_θ-semistability.

Suppose x is χ_θ-stable but $\mathcal{R}$ is not θ-stable. Then the previous argument shows that $\mathcal{R}$ must be θ-semistable, therefore there must exist a nontrivial proper subrepresentation $0 \subset \mathcal{R}' \subset \mathcal{R}$, $r' = 0$ or $r' = r$, so that

$$n_1'\theta_1 + n_2'\theta_2 + r'\theta_\infty = 0.$$

Since the orbit $G \cdot (\mathsf{x}, z)$ must be closed in the complement of the zero section for any $z \neq 0$ it follows that the block decompositions (3.11) must be diagonal, and the upper block in the decomposition of I in (3.12) must be trivial. Otherwise the limit of $(g_1(t), g_2(t)) \cdot (\mathsf{x}, z)$ exists, but does not belong to the G-orbit through (x, z). However, this implies that the one-parameter subgroup $(g_1(t), g_2(t))$ stabilizes (x, z). Since the kernel Δ of the representation of G on $\mathbb{X}$ is trivial, this contradicts the χ_θ-stability assumption. Therefore $\mathcal{R}$ must be θ-stable.

Next, consider the case $r' = r$. As in the previous case, it follows that

$$f(V_2') \subseteq V_2', \qquad g(V_2') \subseteq V_1', \qquad A_i(V_1') \subseteq V_1', \\ B_2(V_2') \subseteq V_2', \qquad I(W) \subseteq V_1', \tag{3.15}$$

for $i = 1, 2$. Therefore there exist direct sum decompositions $V_1 \simeq V_1' \oplus V_1''$, $V_2 \simeq V_2' \oplus V_2''$ such that the linear maps (A_1, A_2, B_2, f, g) have block decomposition of the form (3.11) while I, J have block form decompositions of the form

$$I = \begin{bmatrix} * \\ 0 \end{bmatrix}, \qquad J = \begin{bmatrix} * & * \end{bmatrix}. \tag{3.16}$$

Consider a one-parameter subgroup of G of the form

$$g_1(t) = \begin{bmatrix} 1_{V_1'} & 0 \\ 0 & t^{-1}1_{V_1''} \end{bmatrix}, \qquad g_2(t) = \begin{bmatrix} t1_{V_2'} & 0 \\ 0 & t^{-1}1_{V_2''} \end{bmatrix}.$$

Then the linear maps $(A_1^t, A_2^t, B_2^t, f^t, g^t)$ in $(g_1(t), g_2(t)) \cdot \mathsf{x}$ have block decompositions of the form (3.13) and (I^t, J^t) have block decompositions

$$I^t = \begin{bmatrix} * \\ 0 \end{bmatrix}, \qquad J^t = \begin{bmatrix} * & t* \end{bmatrix}. \tag{3.17}$$

Since $\chi_\theta(g_1(t), g_2(t))^{-1} z = t^{(n_1'-n_1)\theta_1 + (n_2'-n_2)\theta_2} z$, this leads again to a contradiction.

Suppose x is χ_θ-stable, but $\mathcal{R}$ is not θ-stable. Then, as above, it follows that the block decompositions (3.11) must be diagonal, and the left block in the decomposition of J in (3.14) must be trivial. This again implies that x has nontrivial stabilizer, leading to a contradiction.

The proof of the converse statement is very similar, the details being left to the reader.

□

As observed above Lemma 3.1, for fixed dimension vector $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$, the space of stability parameters θ can be naturally identified with the (θ_1, θ_2)-plane and there is a critical set of lines through the origin dividing it into finitely many stability chambers. All moduli spaces associated to stability parameters within a chamber are canonically isomorphic and do not contain strictly semi-stable points.

Lemma 3.1 shows that there is a special stability chamber, determined by the inequalities $\theta_2 > 0$, $\theta_1 + n_2\theta_2 < 0$, within which θ-semistability is equivalent to θ-stability and to conditions $(S.1)$, $(S.2)$ stated in Lemma 3.1. Framed representations of the enhanced ADHM quiver satisfying conditions $(S.1)$, $(S.2)$ will simply be called stable, and their moduli space will be denoted by $\mathcal{N}(r, n_1, n_2)$.

Theorem 3.3. *Let $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$ be a fixed dimension vector and $\theta = (\theta_1, \theta_2, \theta_\infty) \in \mathbb{Z}^2 \times \mathbb{Q}$ be a generic stability parameter. Then the set of gauge equivalence classes of solutions to equations (3.2)-(3.4) with $\zeta_1 = \theta_1$ and $\zeta_2 = -\theta_2$ is a complex quasi-projective scheme isomorphic to $\mathcal{N}_\theta^s(r, n_1, n_2)$.*

Proof. The two equations in (3.2) are obviously moment map equations for the natural hamiltonian $U(V_1) \times U(V_2)$-action on the vector space

$$\begin{aligned} \mathbb{X}(r, n_1, n_2) =& \mathrm{End}(V_1)^{\oplus 2} \oplus \mathrm{Hom}(W, V_1) \oplus \mathrm{Hom}(V_1, W) \oplus \\ & \mathrm{End}(V_2) \oplus \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1). \end{aligned}$$

The parameters (ζ_1, ζ_2) determine the level of the moment map $\mu : \mathbb{X}(r, n_1, n_2) \to \mathfrak{u}(V_1)^* \oplus \mathfrak{u}(V_2)^*$. Standard results imply that for generic $(\theta_1, \theta_2) \in \mathbb{Z}^2$, the symplectic Kähler quotient $\mu^{-1}(-\theta_1, -\theta_2)/U(V_1) \times U(V_2)$, is isomorphic to the GIT quotient $\mathbb{X}_0(r, n_1, n_2)//_\chi G$, where $\chi : G \to \mathbb{C}^\times$ is a character of the form

$$\chi(g_1, g_2) = \det(g_1)^{-\theta_1} \det(g_2)^{-\theta_2}.$$

As it was observed below Proposition 3.2, the GIT quotient $\mathbb{X}_0(r, n_1, n_2)//_\chi G$ is isomorphic to the moduli space of S-equivalence classes of θ-semistable quiver representations $\mathcal{N}_\theta^{ss}(r, n_1, n_2)$. For generic θ there are no strictly semistable representations by Lemma 3.1, hence $\mathcal{N}_\theta^{ss}(r, n_1, n_2) = \mathcal{N}^s(r, n_1, n_2)$. In conclusion, the symplectic quotient $\mu^{-1}(-\theta_1, -\theta_2)/U(V_1) \times U(V_2)$ is isomorphic to the moduli space $\mathcal{N}_\theta^s(r, n_1, n_2)$.

Finally, note that equations (3.4) imply that the triple $(\exp(\sigma_1), \exp(\sigma_2), 1_W)$ is an endomorphism of the enhanced ADHM quiver representation $\mathcal{R} = (V_1, V_2, W, A_1, A_2, I, J, B_2, f)$ preserving the framing $h : W \xrightarrow{\sim} \mathbb{C}^r$. However, the proof of Proposition (3.2) implies that a nontrivial endomorphism of a stable framed representation must be the identity. In conclusion, σ_1, σ_2 must be identically 0 for generic θ.

□

In particular, it follows from the proof above and from Lemma 3.1 that if $\zeta_2 < 0$ and $\zeta_1 + n_2\zeta_2 > 0$, then the moduli space of flat directions is isomorphic to $\mathcal{N}(r, n_1, n_2)$.

For further reference, note that if $\mathcal{R} = (V_1, V_2, W, A_1, A_2, I, J, B_2, f)$ is a stable framed representation of type $(r, n_1, n_2) \in \mathbb{Z}^3_{>0}$ with $n_1 > n_2$, the linear maps (A_1, A_2, I, J) yield linear maps

$$\widetilde{A}_i : V_1/\mathrm{Im}(f) \to V_1/\mathrm{Im}(f), \qquad \widetilde{I} : W \to V_1/\mathrm{Im}(f), \qquad \widetilde{J} : V_1/\mathrm{Im}(f) \to W$$

with $i = 1, 2$, which satisfy the ADHM relation

$$[\widetilde{A}_1, \widetilde{A}_2] + \widetilde{I}\widetilde{J} = 0.$$

Moreover, it is not difficult to check that the resulting ADHM data $(V, W, \widetilde{A}_1, \widetilde{A}_2, \widetilde{I}, \widetilde{J})$, where $V = V_1/\mathrm{Im}(f)$ satisfies the ADHM stability condition $(S.2)$.

Lemma 3.4. *Suppose $n = n_1 - n_2 > 0$ and let V_2 be a complex vector space of dimension $n_2 \in \mathbb{Z}_{>0}$. Let also $\mathcal{M}(r, n)$ denote the moduli space of stable ADHM data of type $(n, r) \in (\mathbb{Z}_{>0})^2$. Then there is a surjective morphism $\mathsf{q} : \mathcal{N}(r, n_1, n_2) \to \mathcal{M}(r, n)$ mapping a the isomorphism class of the stable framed representation $\mathcal{R} = (V_1, V_2, W, A_1, A_2, I, J, B_2, f)$ to isomorphism class of the ADHM data $(V, W, \widetilde{A}_1, \widetilde{A}_2, \widetilde{I}, \widetilde{J})$ constructed above.*

Proof. The existence of the morphism q of moduli spaces follows from repeating the above construction for flat families of quiver representations.

In order to prove its surjectivity, start with a stable ADHM data $(V, W, \widetilde{A}_1, \widetilde{A}_2, \widetilde{I}, \widetilde{J})$ of type (n, r) and $B_2 \in \mathrm{End}(V_2)$, and set

$$V_1 = V_2 \oplus V, \text{ and } \qquad f = \begin{bmatrix} 1_{V_2} \\ 0 \end{bmatrix}.$$

Now let $A_1, A_2 \in \mathrm{End}(V_1)$, $I \in \mathrm{Hom}(W, V_1)$, and $J \in \mathrm{Hom}(V_1, W)$ be of the following form

$$A_1 = \begin{bmatrix} 0 & A_1' \\ 0 & \widetilde{A}_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} B_2 & A_2' \\ 0 & \widetilde{A}_2 \end{bmatrix}$$

$$I = \begin{bmatrix} I' \\ \widetilde{I} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & \widetilde{J} \end{bmatrix},$$

according to the decomposition $V_1 = V_2 \oplus V$. To be precise, one has $A_1', A_2' \in \mathrm{Hom}(V, V_2)$ and $I' \in \mathrm{Hom}(W, V_2)$.

One immediately sees that $A_1 f = A_2 f - f B_2 = J f = 0$, while $[A_1, A_2] + IJ = 0$ if and only if the following auxiliary equation is satisfied:

$$A_1' \widetilde{A}_2 - A_2' \widetilde{A}_1 - B_2 A_1' + I' \widetilde{J} = 0. \tag{3.18}$$

Clearly, $f : V_2 \to V_1$ is injective, and note that (V_1, W, A_1, A_2, I, J) defined above is stable if and only if the following conditions hold:

(i) at least one of the linear maps A_1', A_2', I' is nontrivial;
(ii) there is no proper subspace $S \subsetneq V_2$ such that $A_1'(V), A_2'(V), I'(W) \subset S$ and $B_2(S) \subseteq S$.

Indeed, if $A_1' = A_2' = I' = 0$, then V is a subspace of V_1 which violates the ADHM stability condition. As for the second condition, (V_1, W, A_1, A_2, I, J) is not stable if and only if there is a subspace $\widetilde{S} \subsetneq V_1$ which is invariant under A_1 and A_2, and contains the image of I. Since $(V, W, \widetilde{A}_1, \widetilde{A}_2, \widetilde{I}, \widetilde{J})$ is stable, $\widetilde{S}$ must be of the form $S \oplus V$, with $S \subsetneq V_2$ nontrivial, $A_1'(V), A_2'(V), I'(W) \subset S$ and $B_2(S) \subseteq S$.

Therefore, in order to prove the surjectivity of the morphism q it is sufficient to prove that there exist nontrivial solutions of the auxiliary equation (3.18), so that linear subspaces $0 \subsetneq S \subsetneq V_2$ as in the previous paragraph do not exist.

Choose a basis $\{v_1, \ldots, v_{n_2}\}$ of V_2 and let B_2 be a diagonal matrix with distinct eigenvalues, $B_2 = \mathrm{diag}(\beta_1, \ldots, \beta_{n_2})$, $\beta_i \neq \beta_j$ for all $i, j = 1, \ldots, n_2$, $i \neq j$. Let $I' : W \to V_2$ be a rank one linear map so that its image is generated by a vector $v = \sum_{i=1}^{n_2} v_i$. Note that the set $\{v, B(v), \ldots, B^{n_2-1}(v)\}$ is a basis of V_2. Otherwise there would exist a nontrivial linear relation of the form

$$\sum_{i=1}^{n_2} x_i B^i(v) = 0.$$

Given the above choice for B_2, this would imply that the x_i are a solution of the linear system

$$\sum_{i=1}^{n_2} \beta_j^i x_i = 0$$

where $j = 1, \ldots, n_2$. However the discriminant of this system is the Vandermonde determinant $\Delta(\beta_1, \ldots, \beta_{n_2}) = \prod_{i<j}(\beta_j - \beta_i)$, which is nonzero since the β_i are assumed to be distinct. Therefore all x_i would have to vanish, leading to a contradiction. In conclusion, $\{v, B(v), \ldots, B^{n_2-1}(v)\}$ is a basis of V_2. In particular there are no nontrivial proper subspaces $0 \subsetneq S \subsetneq V_2$ preserved by B_2 and containing $\mathrm{Im}(I')$.

Having fixed B_2, I' as in the previous paragraph, equation (3.18) is a linear system of $n_2(n_1 - n_2)$ linear equations in the $2n_2(n_1 - n_2)$ variables A_1', A_2'. Such a system has a $n_2(n_1 - n_2)$ dimensional space of solutions. Any nontrivial solution determines a set (V_1, W, A_1, A_2, I, J) of stable ADHM data.

$\square$

3.3. **Smoothness.**

The main result of this subsection is the following.

Theorem 3.5. *The moduli space $\mathcal{N}(r, n_1, n_2)$ of stable framed representations of an enhanced ADHM quiver with fixed numerical invariants $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$ is a smooth, quasi-projective variety of dimension $(2n_1 - n_2)r$. Moreover, the tangent space to $\mathcal{N}(r, n_1, n_2)$ at a closed point $[\mathcal{R}] = [(A_1, A_2, I, J, B_2, f)]$ is isomorphic to* *the first cohomology group of a complex $\mathcal{C}(\mathcal{R})$ of the form*

$$
\begin{array}{ccccccc}
 & & \mathrm{End}(V_1)^{\oplus 2} & & & & \\
 & & \oplus & & & & \\
 & & \mathrm{Hom}(W,V_1) & & \mathrm{End}(V_1) & & \\
\mathrm{End}(V_1) & & \oplus & & \oplus & & \\
\oplus & \xrightarrow{d_0} & \mathrm{Hom}(V_1,W) & \xrightarrow{d_1} & \mathrm{Hom}(V_2,V_1)^{\oplus 2} & \xrightarrow{d_2} & \mathrm{Hom}(V_2,V_1) \\
\mathrm{End}(V_2) & & \oplus & & \oplus & & \\
 & & \mathrm{End}(V_2) & & \mathrm{Hom}(V_2,W) & & \\
 & & \oplus & & & & \\
 & & \mathrm{Hom}(V_2,V_1) & & & &
\end{array}
\tag{3.19}
$$

where the four terms have degrees $0,\ldots,3$*, and the differentials are given by*

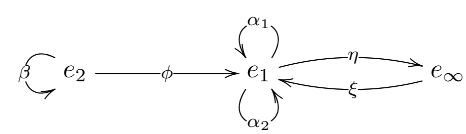
$$d_0(\alpha_1,\alpha_2)^t = ([\alpha_1,A_1],[\alpha_1,A_2],\alpha_1 I,-J\alpha_1,[\alpha_2,B_2],\alpha_1 f - f\alpha_2)^t$$

$$d_1(a_1,a_2,i,j,b_2,\phi)^t = ([a_1,A_2]+[A_1,a_2]+Ij+iJ, A_1\phi+a_1 f, A_2\phi+a_2 f-fb_2-\phi B_2, jf+J\phi)^t$$

$$d_2(c_1,c_2,c_3,c_4)^t = c_1 f + A_2 c_2 - c_2 B_2 - A_1 c_3 - I c_4.$$

Proof. First note that the moduli space of stable framed representations of the enhanced ADHM quiver (3.5) can be canonically identified with the moduli space of stable framed representations of the following simpler quiver

(3.20)



with relations

$$\alpha_1\alpha_2 - \alpha_2\alpha_1 + \xi\eta, \qquad \alpha_1\phi, \qquad \alpha_2\phi - \phi\beta, \qquad \eta\phi. \tag{3.21}$$

For further reference, let $(\rho_1,\ldots,\rho_4)$ denote the generators (3.21) respectively.

The moduli space $\widetilde{\mathcal{N}}(r,n_1,n_2)$ of stable framed representations of numerical type $(r,n_1,n_2) \in (\mathbb{Z}_{>0})^3$ is defined in complete analogy with the moduli space of similar representations of the enhanced ADHM quiver (3.5). In particular, a result analogous to Lemma (3.1) also holds for θ-stable framed representations of (3.20). Namely, if $\theta_1 < 0$, $\theta_2 > 0$, $\theta_1 + n_2\theta_2 < 0$, a framed representation $(V_1,V_2,W,A_1,A_2,I,J,B_2,f)$ of (3.20) is θ-semistable if and only if it is θ-stable and if and only if f is injective and the data (V_1,W,A_1,A_2,I,J) satisfies the ADHM stability condition (S.2). Finally, there is an obvious morphism $\widetilde{\mathcal{N}}(r,n_1,n_2) \to \mathcal{N}(r,n_1,n_2)$, which is an isomorphism according to Lemma (3.1). This isomorphism will be used implicitly in the following, making no distinction between stable framed representations of (3.5) and (3.20).

The truncated cotangent complex of the moduli space $\widetilde{\mathcal{N}}(r,n_1,n_2)$ can be determined by a standard computation in deformation theory. Such an explicit computation has been carried out in a similar context, see [7, Sect. 4.1]. To be more precise, the differential d_0 comes from the linearization of the action of G on $\mathbb{X}$, while the differential d_1 is just the linearization of the relations (3.21). The only new element in the present case is the fact that the generators $(\rho_1,\ldots,\rho_4)$ in (3.21) satisfy the relation

$$\rho_1\phi + \alpha_2\rho_2 - \rho_2\beta - \alpha_1\rho_3 - \xi\rho_4 = 0.$$

This "relation on relations" yields an extra term in the deformation complex of a framed representation $\mathcal{R} = (V_1, V_2, W, A_1, A_2, I, J, B_2, f)$ of the quiver (3.20), and the differential d_2 is the precisely its linearization.

We conclude that the infinitesimal deformation space of $\mathcal{R}$ is the first cohomology group $H^1(\mathcal{C}(\mathcal{R}))$ and the obstruction space is $H^2(\mathcal{C}(\mathcal{R}))$. In order to prove theorem (3.5), it suffices to show that $H^i(\mathcal{C}(\mathcal{R})) = 0$, for $i = 0, 2, 3$, for any stable framed representation $\mathcal{R}$. A helpful observation is that $\mathcal{C}(\mathcal{R})$ can be presented as a cone of a morphism between simpler complexes as follows.

Let $\mathcal{A} = (V_1, A_1, A_2, I, J)$ and $\mathcal{B} = (V_2, B_2)$, and construct the following complexes of vector spaces:

- $\mathcal{C}(\mathcal{A})$ is the three term complex

$$\mathrm{Hom}(V_1, V_1) \xrightarrow{d_0} \begin{matrix} \mathrm{End}(V_1, V_1)^{\oplus 2} \\ \oplus \\ \mathrm{Hom}(W, V_1) \\ \oplus \\ \mathrm{Hom}(V_1, W) \end{matrix} \xrightarrow{d_1} \mathrm{End}(V_1, V_1) \tag{3.22}$$

where the terms have degrees $0, 1, 2$, and the differentials are given by

$$d_0(\alpha_1) = ([\alpha_1, A_1], [\alpha_1, A_2], \alpha_1 I, -J\alpha_1)^t$$

$$d_1(a_1, a_2, i, j)^t = ([a_1, A_2] + [A_1, a_2] + Ij + iJ);$$

- $\mathcal{C}(\mathcal{B})$ is the two-term complex

$$\mathrm{Hom}(V_2, V_2) \xrightarrow{d_0} \mathrm{Hom}(V_2, V_2) \tag{3.23}$$

with differential

$$d_0(\alpha_2) = [\alpha_2, B_2]$$

and terms in degrees $0, 1$;

- $\mathcal{C}(\mathcal{A}, \mathcal{B})$ is the three term complex

$$\mathrm{Hom}(V_2, V_1) \xrightarrow{d_0} \begin{matrix} \mathrm{Hom}(V_2, V_1)^{\oplus 2} \\ \oplus \\ \mathrm{Hom}(V_2, W) \end{matrix} \xrightarrow{d_1} \mathrm{Hom}(V_2, V_1) \tag{3.24}$$

where the terms have degrees $0, 1, 2$ and the differentials are

$$d_0(\phi) = -(A_1\phi, A_2\phi - \phi B_2, J\phi)^t$$

$$d_1(c_2, c_3, c_4)^t = -(A_2 c_2 - c_2 B_2 - A_1 c_3 - I c_4).$$

Abusing notation, the differentials of the above three complexes have been denoted by the same symbols d_0, d_1. The distinction will be clear from the context. Note that $\mathcal{C}(\mathcal{A})$ is the deformation complex of the representation $\mathcal{A}$ of the standard ADHM quiver.

It is then straightforward to check that the complex $\mathcal{C}(\mathcal{R})[1]$ is the cone of the morphism of complexes

$$\varrho : \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \longrightarrow \mathcal{C}(\mathcal{A}, \mathcal{B})$$

$$\varrho_0(\alpha_1, \alpha_2)^t = -(\alpha_1 f - f\alpha_2)$$

$$\varrho_1(a_1, a_2, i, j, b_2)^t = -(a_1 f, a_2 f - f b_2, jf)^t$$

$$\varrho_2(c_1) = -c_1 f.$$

In particular, there is an exact triangle

$$\mathcal{C}(\mathcal{R}) \longrightarrow \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \longrightarrow \mathcal{C}(\mathcal{A},\mathcal{B}). \tag{3.25}$$

Next, note that the following vanishing results hold

$$H^0(\mathcal{C}(\mathcal{A})) = 0 \qquad H^2(\mathcal{C}(\mathcal{A})) = 0 \qquad H^2(\mathcal{C}(\mathcal{A},\mathcal{B})) = 0. \tag{3.26}$$

if $\mathcal{R}$ is stable. The first two follow from observing that $\mathcal{C}(\mathcal{A})$ is just the deformation complex of a stable ADHM data; the vanishing of H^0 and H^2 in this case is a well-known result.

The last vanishing in (3.26) follows from considering the dual of the differential $d_1 : \mathcal{C}^1(\mathcal{A},\mathcal{B}) \to \mathcal{C}^2(\mathcal{A},\mathcal{B})$. It reads

$$d_1^\vee : \mathrm{Hom}(V_1, V_2) \to \begin{array}{c} \mathrm{Hom}(V_1,V_2)^{\oplus 2} \\ \oplus \\ \mathrm{Hom}(W,V_2) \end{array}$$

$$d_1^\vee(\psi) = (B_2\psi - \psi A_2, \psi A_1, \psi I)^t.$$

Suppose $d_1^\vee(\psi) = 0$. Then it is straightforward to check that $\mathrm{Ker}(\psi)$ is preserved by A_1, A_2 and contains the image of I, which implies that $\mathrm{Ker}(\psi)$ is either 0 or V_1. If ψ is injective, then $A_1 = 0$ and $I = 0$, leading to a contradiction. Therefore $\psi = 0$, and d_1 is surjective.

Using a similar argument, it is also straightforward to prove that the morphism

$$H^0(\mathcal{C}(\mathcal{A})) \oplus H^0(\mathcal{C}(\mathcal{B})) \stackrel{H^0(\varrho)}{\longrightarrow} H^0(\mathcal{C}(\mathcal{A},\mathcal{B}))$$

is injective if the stability conditions are satisfied. Then the long exact cohomology sequence of the exact triangle (3.25) implies that

$$H^0(\mathcal{C}(\mathcal{R})) = 0, \qquad H^3(\mathcal{C}(\mathcal{R})) = 0, \tag{3.27}$$

and there is a short exact sequence of cohomology groups

$$H^1(\mathcal{C}(\mathcal{A})) \oplus H^1(\mathcal{C}(\mathcal{B})) \stackrel{H^1(\varrho)}{\longrightarrow} H^1(\mathcal{C}(\mathcal{A},\mathcal{B})) \longrightarrow H^2(\mathcal{C}(R)) \longrightarrow 0.$$

Therefore, in order to prove that $H^2(\mathcal{C}(\mathcal{R})) = 0$, it suffices to prove that $h^1(\varrho)$ is surjective. Then, denoting by $Z^1(\mathcal{C})$ the kernel of $d_1 : \mathcal{C}^1 \to \mathcal{C}^2$ for any complex $\mathcal{C}$, it suffices to prove that the induced map

$$z^1(\varrho) : Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B})) \longrightarrow Z^1(\mathcal{C}(\mathcal{A},\mathcal{B}))$$

is surjective. The vanishing results (3.26) imply that there is a commutative diagram of linear maps with exact rows of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B})) & \longrightarrow & \mathcal{C}^1(\mathcal{A}) \oplus \mathcal{C}^1(\mathcal{B}) & \stackrel{d_1}{\longrightarrow} & \mathcal{C}^2(\mathcal{A}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z^1(\varrho)} & & \downarrow{\scriptstyle \varrho_1} & & \downarrow{\scriptstyle \varrho_2} & & \\
0 & \longrightarrow & Z^1(\mathcal{C}(\mathcal{A},\mathcal{B})) & \longrightarrow & \mathcal{C}^1(\mathcal{A},\mathcal{B}) & \stackrel{d_1}{\longrightarrow} & \mathcal{C}^2(\mathcal{A},\mathcal{B}) & \longrightarrow & 0.
\end{array}$$

Since $f : V_2 \to V_1$ is injective, it follows trivially that the maps ϱ_1, ϱ_2 are surjective. In order to prove that $z^1(\varrho)$ is surjective, it suffices to prove that the fiber of ϱ_1 over any point in $Z^1(\mathcal{C}(\mathcal{A},\mathcal{B}))$ intersects $Z^1(\mathcal{C}(\mathcal{A})) \oplus Z^1(\mathcal{C}(\mathcal{B}))$ nontrivially in $\mathcal{C}^1(\mathcal{A}) \oplus \mathcal{C}^1(\mathcal{B})$. Since ϱ_1 is a surjective linear map, its fiber over any point in

$\mathcal{C}^1(\mathcal{A},\mathcal{B})$ is a torsor over the linear space $\mathrm{Ker}(\varrho_1)$. Since $Z^1(\mathcal{C}(\mathcal{A}))\oplus Z^1(\mathcal{C}(\mathcal{B}))$ is a linear subspace of $\mathcal{C}^1(\mathcal{A})\oplus\mathcal{C}^1(\mathcal{B})$, it suffices to check that

$$\dim\left(\mathrm{Ker}(\varrho_1)\right)+\dim\left(Z^1(\mathcal{C}(\mathcal{A}))\oplus Z^1(\mathcal{C}(\mathcal{B}))\right)-\dim\left(\mathcal{C}^1(\mathcal{A})\oplus\mathcal{C}^1(\mathcal{B})\right)\geq 0.$$

This follows by an elementary computation, given that the stability conditions imply $\dim(V_2)\leq\dim(V_1)$.

□

Finally, we also conclude that the morphism $\mathsf{q}:\mathcal{N}(r,n_1,n_2)\to\mathcal{M}(n,r)$ introduced in Lemma 3.4 is a submersion, and that its fibers have dimension n_1r.

3.4. **Geometric interpretation in terms of framed sheaves.** Let S be a smooth projective surface and D, D_∞ smooth irreducible divisors on S with transverse intersection. According to [6], if D_∞ is big and nef, and $c\in A^\bullet(S)\otimes\mathbb{Q}$ (the Chow group of S), there is a quasi-projective fine moduli scheme $\mathcal{M}(c)$ parametrizing isomorphism classes of pairs (E,ξ), where

- E is a torsion free sheaf on S with numerical invariants $\mathrm{ch}(E)=c$;
- $\xi: E|_{D_\infty}\xrightarrow{\sim}\mathcal{O}_{D_\infty}^{\oplus r}$ is an isomorphism of $\mathcal{O}_{D_\infty}$-modules.

In particular there exists a universal framed torsion free sheaf $(\mathcal{U},\varepsilon)$ on $\mathcal{M}(c)\times S$, flat over $\mathcal{M}(c)$. The class $c=(r,c_1,\mathrm{ch}_2)$ will satisfy the constraint $c_1\cdot D_\infty=0$. Under some additional assumptions (e.g., if the condition $(K_S+D_\infty)\cdot D_\infty<0$ holds), the moduli scheme $\mathcal{M}(c)$ is smooth.

We shall consider the functor $\mathbf{F}_{r,n,d}\colon \mathit{Sch}_{/\mathbb{C}}^{op}\to\mathit{Sets}$ which to any scheme T associates the isomorphism classes of quadruples (E_T,ξ_T,G_T,g_T), where

- E_T is a coherent sheaf on $T\times S$, flat over T, such that for all closed points $t\in T$ the sheaf $E_{T,t}=E_{|\{t\}\times S}$ is torsion-free and has fixed Chern character $\mathrm{ch}_0=r$, $\mathrm{ch}_1=0$, $\mathrm{ch}_2=-n$;
- $\xi_T\colon E_{T\times D_\infty}\to\mathcal{O}_{T\times D_\infty}^{\oplus r}$ is an isomorphism of $\mathcal{O}_{D_\infty\times T}$-modules;
- G_T is a coherent sheaf on $T\times S$, supported on $T\times D$ and flat over T, such that for all closed points $t\in T$, the sheaf $G_{T,t}$ is a skyscraper of fixed length $d\geq 1$, whose support is disjoint from $T\times(D\cap D_\infty)$;
- $g_T\colon E_T\twoheadrightarrow G_T$ is a surjective morphism of $\mathcal{O}_{T\times S}$-modules.

Two such quadruples (E_T,ξ_T,G_T,g_T) and (E'_T,ξ'_T,G'_T,g'_T) are considered to be isomorphic if there exist an isomorphism of $\mathcal{O}_{T\times D}$-modules $\phi_T\colon E_T\xrightarrow{\sim}E'_T$ and an isomorphism of $\mathcal{O}_{T\times D}$-modules $\psi_T\colon G_T\xrightarrow{\sim}G'_T$ such that the diagrams

$$\begin{array}{ccc} E_{T|T\times D_\infty} & \xrightarrow[\xi_T]{\sim} & \mathcal{O}_{T\times D_\infty}^{\oplus r} \\ {\scriptstyle \phi_{T|T\times D_\infty}}\downarrow & \nearrow{\scriptstyle \sim\ \xi'_T} & \\ E'_{T|T\times D_\infty} & & \end{array}\qquad\qquad \begin{array}{ccc} E_T & \xrightarrow{g_T} & G_T \\ {\scriptstyle\phi_T}\downarrow & & \downarrow{\scriptstyle\psi_T} \\ E'_T & \xrightarrow{g'_T} & G'_T \end{array}$$

commute. There is a forgetful natural transformation from $\mathbf{F}_{r,n,d}$ to the moduli functor represented by $\mathcal{M}(r,n)$, which simply forgets the data G_T and g_T.

The steps leading to the construction of the moduli space $\mathcal{M}(r,n)$ [6, 17, 16] can be easily generalized to get a moduli scheme $\mathcal{M}_D(r,n,d)$ which universally represents the functor $\mathbf{F}_{r,n,d}$. Moreover the above-mentioned forgetful functor induces a projective morphism $\mathcal{M}_D(r,n,d)\to\mathcal{M}(r,n)$. However these results can be obtained in a more economical way by noting that $\mathbf{F}_{r,n,d}$ is isomorphic to a Quot functor, which is representable by general theory. For any $d\geq 1$, let $\mathbf{Q}_{r,n,d}$ be the functor $Sch^{op}_{/\mathcal{M}(r,n)} \to Sets$ which associates to a scheme $T \to \mathcal{M}(r,n)$ over $\mathcal{M}(r,n)$ an isomorphism class of pairs (F_T, f_T) where

- F_T is a flat coherent $\mathcal{O}_{T\times D}$-module with finite support over T of relative length d, disjoint from $T \times (D \cap D_\infty)$, and
- $f_T : (\mathcal{U}_D)_T \to F_T$ is a surjective morphism of $\mathcal{O}_{D\times T}$-modules.

Two such quotients (F_T, g_T) (F'_T, g'_T) are isomorphic if there exists an isomorphism $\eta_T : F_T \to F'_T$ such that $f'_T = \eta_T \circ f_T$. In accordance with Grothendieck's general theory of the Quot scheme, there exists a relative $\mathcal{M}(r,n)$-scheme $\pi : \mathcal{Q}(\mathcal{U}_D, d) \to \mathcal{M}(r,n)$ that universally represents the functor $\mathbf{Q}_{r,n,d}$.

The previously mentioned natural transformation $\mathbf{F}_{r,n,d} \to \mathbf{Q}_{r,n,d}$ is defined by $(E_T, \xi_T, G_T, g_T) \to (G_T, g_T)$. The inverse transformation is obtained by taking $E_T = \ker(g_T)$ and noting that, as consequence of the condition on the support of G_T, the framing of the universal sheaf $\mathcal{U}$ induces a framing ξ_T on E_T. As a consequence, we have an isomorphism of $\mathcal{M}(r,n)$-schemes $\mathcal{M}_D(r,n,d) \simeq \mathcal{Q}(E_D, d)$.

Next let $S = \mathbb{P}^2$ with homogeneous coordinates $[z_0, z_1, z_2]$ and let D, D_∞ be the hyperplanes defined by $z_1 = 0$ and $z_0 = 0$, respectively. Then the moduli space $\mathcal{M}(r,n)$ is isomorphic to the moduli space of stable ADHM data of type (r,n) [24, Thm. 2.1]. Let $\mathcal{N}(r, n+d, d)$ denote the moduli space of stable representations of an enhanced ADHM quiver of type $(n+d, d, r)$. Recall also that Lemma (3.4) proves the existence of a surjective morphism $\mathsf{q} : \mathcal{N}(r, n+d, d) \to \mathcal{M}(r,n)$. Then the following holds.

Theorem 3.6. *There is an isomorphism $\mathcal{M}_D(r,n,d) \simeq \mathcal{N}(r, n+d, d)$ of schemes over $\mathcal{M}(r,n)$.*

Proof. The proof relies on the Beilinson spectral sequence, by analogy with the proof of the ADHM correspondence [24, Thm 2.1]. Detailed computations has been carried out in a similar context in [7, Sect. 7.1],[15], therefore it suffices here to outline the main steps, omitting many details.

Now recall that the Beilinson spectral sequence yields an isomorphism [24, Thm 2.1] between the moduli stack of framed torsion free sheaves on S with fixed numerical invariants (r,n) and a moduli stack of three-term locally free monad complexes on S. The same correspondence exists for families of framed sheaves; this has been worked out in [15] when S is a blowup of the complex plane, but it can be easily adapted to the case of $\mathbb{P}^2$. More specifically, let (E_T, ξ_T) be a flat family of framed torsion free sheaves on S parameterized by a scheme T of finite type over $\mathbb{C}$. Let $p_T : T \times S \to T$, $p_S : T \times S \to S$ denote the canonical projections and for any coherent sheaf F_T on $T \times S$, $F_T(k) = F_T \otimes p_S^*\mathcal{O}_S(k)$ for any $k \in \mathbb{Z}$. One can check that the direct images $R^i p_{T*}(E_T(-1))$ vanish for $i = 0, 2$, and $R^1 p_{T*}(E_T(-1))$ is a locally free sheaf $\mathcal{V}_T$ of rank n on T. Then the relative Beilinson spectral spectral sequence for the projective bundle $T \times S \to T$ collapses to a monad complex $F(E_T, \xi_T)$ of the form

$$p_T^*\mathcal{V}_T(-1) \xrightarrow{a_T} p_T^*\mathcal{V}_T^{\oplus 2} \oplus p_T^*\mathcal{W}_T \xrightarrow{b_T} p_T^*\mathcal{V}_T(1). \tag{3.28}$$

where

$$\mathcal{W}_T = R^0 p_{T*}E \otimes \mathcal{O}_{T\times D_\infty} \simeq \mathcal{O}_T^{\oplus r}.$$

The differentials a_T, b_T are of the form

$$a_T = \begin{bmatrix} z_1 - z_0 A_{T,1} \\ z_2 - z_0 A_{T,2} \\ z_0 J_T \end{bmatrix} \qquad b_T = \begin{bmatrix} -z_2 + z_2 A_{T,2} z_1 - z_0 A_{T,1} & z_0 I_T \end{bmatrix}$$

where

$$(A_{T,1}, A_{T,2}, I_T, J_T) \in \mathrm{End}(\mathcal{V}_T)^{\oplus 2} \oplus \mathrm{Hom}(\mathcal{W}_T, \mathcal{V}_T) \oplus \mathrm{Hom}(\mathcal{V}_T, \mathcal{W}_T)$$

is a flat family of stable representations of the ADHM quiver. Recall that the monad complex $F(E_T, \xi_T)$ is exact at both ends, and its middle cohomology sheaf is isomorphic to E_T. The three terms have degrees $0, 1, 2$ respectively. Recall also that

$$I_T : \mathcal{W}_T = R^0 p_{T*} E \otimes \mathcal{O}_{T \times D_\infty} \to \mathcal{V}_T = R^1 p_{T*} E(-1)$$

is the natural coboundary morphism.

There is a similar isomorphism between the moduli stack of degree d skyscraper sheaves G on D with support disjoint from D_∞ and a moduli stack of locally-free two-term complexes on $D = \mathbb{P}^1$. Given a flat family G_T of such objects parameterized by a scheme T, the corresponding two-term monad complex $F(G_T)$ is

$$p_T^* \mathcal{V}_{T,2}(-1) \xrightarrow{b_{T,2}} p_T^* \mathcal{V}_{T,2} \tag{3.29}$$

where $\mathcal{V}_{T,2} = R^0 p_{T*} G_T$ is a locally free $\mathcal{O}_T$-module, and the terms have degrees $-1, 0$ respectively. The differential is of the form

$$b_{T2} = \begin{bmatrix} z_2 - z_0 B_{T,2} \end{bmatrix}$$

where $B_{T,2} \in \mathrm{End}(\mathcal{V}_{T,2})$ is an endomorphism of $\mathcal{V}_{T,2}$.

Let $g_T : E_T \to G_T$ be a surjective morphism of $\mathcal{O}_{T \times S}$-modules, and let $\tilde{E}_T = \mathrm{Ker}(g_T)$; $\tilde{E}_T$ is a flat family of torsion free $\mathcal{O}_S$-modules. Since the support of G_T is disjoint from $T \times D$, there is a canonical isomorphism $E_T \otimes \mathcal{O}_{T \times D_\infty} \simeq \tilde{E}_T \otimes \mathcal{O}_{T \times D_\infty}$. Therefore the framing of E_T along $T \times D_\infty$ yields a framing ξ'_T of $\tilde{E}_T$. Therefore the Beilinson spectral sequence of $\tilde{E}_T$ is again a monad complex $\mathcal{F}(\tilde{E}_T, \xi'_T)$. Let $\mathcal{V}_{T,1} = R^1 p_{T*} \tilde{E}_T$. Since the Beilinson spectral sequence is functorial, the exact sequence

$$0 \to \tilde{E}_T \to E_T \to G_T \to 0 \tag{3.30}$$

yields an exact triangle of the form

$$\mathcal{F}(G_T)[-1] \xrightarrow{\varphi} \mathcal{F}(\tilde{E}_T, \xi'_T) \to \mathcal{F}(E_T, \xi_T). \tag{3.31}$$

Proceeding by analogy with [7, Sect. 7.1], it follows that the morphism $\varphi : \mathcal{F}(G_T)[-1] \to \mathcal{F}(\tilde{E}_T, \xi'_T)$ is a morphism of monad complexes determined by the natural injective morphism of sheaves

$$f_T : \mathcal{V}_{T,2} = R^0 p_{T*} G_T \to \mathcal{V}_{T,1} = R^1 p_{T*} \tilde{E}(-1),$$

which satisfies

$$A_{T,1} f_T = 0, \qquad A_{T,2} f_T = f_T B_{2,T}, \qquad J_T f_T = 0. \tag{3.32}$$

The details are very similar to those in loc.cit., hence will be omitted. In conclusion, there is a morphism of stacks between the stack of data $((E, \xi), G, g)$ on S and the moduli stack of stable framed representations of the enhanced ADHM quiver.

Conversely, suppose $\mathcal{R}_T = (\mathcal{V}_{T,1}, \mathcal{V}_{T,2}, A_{T,1}, A_{T,2}, I_T, J_T, B_{T,2}, f_T)$ is a flat family of stable framed quiver representations parameterized by T with $\mathcal{W}_T = \mathcal{O}_T^{\oplus r}$. Since the relations (3.32) are satisfied and $\mathrm{Im}(f_T) \cap \mathrm{Im}(I_T) = 0$, the data $(A_{T,1}, A_{T,2}, I_T, J_T)$ induce ADHM data $(\widetilde{A}_{T,1}, \widetilde{A}_{T,2}, \widetilde{I}_T, \widetilde{J}_T)$ on the quotient sheaf $\mathcal{V}_{T,1}/\mathrm{Im}(f_T)$ as in lemma (3.4). Note that this quotient is locally free since the restriction of f_T to any point $t \in T$ is injective. Moreover, it is straightforward to check that the resulting flat family of ADHM data is a flat family of stable ADHM data. Given this data, one can easily construct an exact sequence of monad complexes of the form (3.31), which in turns yields an exact sequence of framed shaves of the form (3.30).

□

4. The Quiver Partition Function

Summarizing the results obtained so far, a quiver quantum mechanical model for BPS states bound to surface operators has been constructed in section (2). The geometry of the moduli space of supersymmetric vacua has been studied in detail in section (3). In particular, according to Theorems (3.3), (3.5), in a special chamber in the space of FI parameters, the moduli space $\mathcal{N}(r, n_1, n_2)$ is a smooth quasi-projective variety. An important application of these results is a rigorous mathematical construction of a counting function for such BPS states, which is the main focus of this section.

From a physics point of view, the BPS counting function is the Witten index of the supersymmetric quantum mechanics obtained in section (2). This index can be computed exactly in the Born-Oppenheimer low energy approximation. In this limit the gauged linear quantum mechanical model reduces to a one dimensional sigma model on the moduli space of supersymmetric vacua, by analogy with the two dimensional situation [29]. A complete description of this one dimensional sigma model requires an explicit computation of the space of fermion zero modes, at any point in the moduli space. The zero modes of the fermionic components of chiral multiplets are in one-to-one correspondence with the zero modes of the bosonic components, by supersymmetry. The zero modes of the fermionic components of Fermi multiplets are determined by a system of linear equations following from the Yukawa couplings (2.14), (2.15). A slightly tedious linear algebra computation shows that in the special stability chamber all these fermionic fields are in fact massive at any point in the moduli space. Therefore the only fermion zero modes in the low energy effective action belong to the chiral multiplets. By supersymmetry, they must take values in the holomorphic tangent space to the moduli space. In particular, there are no fermion zero modes with values in the anti-holomorphic tangent bundle. This implies that the supersymmetric ground states are in one-to-one correspondence with cohomology classes in $\oplus_i H^{0,i}(\mathcal{N}(r, n_1, n_2))$. In conclusion, the Witten index is given in this case by the holomorphic Euler character $\chi(\mathcal{O}_{\mathcal{N}(r,n_1,n_2)})$ of the trivial line bundle on the moduli space $\mathcal{N}(r, n_1, n_2)$.

Since the moduli space is non-compact, this Euler character is ill-defined, as the cohomology groups are infinite dimensional. However, in instanton computations one is interested in an equivariant Euler character with respect to a natural torus action on the moduli space [26]. In this case, $\mathbf{T} = \mathbb{C}^\times \times \mathbb{C}^\times \times (\mathbb{C}^\times)^r$ and the action on the moduli space $\mathcal{N}(r,n_1,n_2)$ is given by

$$(t_1,t_2,z)\times(V_1,V_2,W,A_1,A_2,I,J,B_2,f) \longrightarrow \\ (V_1,V_2,W,t_1A_1,t_2A_2,Iz^{-1},zt_1t_2J,t_2B_2,f) \tag{4.1}$$

where $z=(z_1,\ldots,z_r)\in(\mathbb{C}^\times)^r$. From the point of view of (topologically twisted) supersymmetric quantum mechanics, the equivariant Euler character can still be interpreted as an Witten index employing a deformation of the nilpotent BRST operator [5]. This solves the non-compactness problem because a direct application of a standard fixed point theorem shows that the equivariant Euler character is an element of the quotient field of the representation ring of $\mathbf{T}$.

Finally, note that there is in fact a natural family of equivariant partition functions depending on two integers $(p_1,p_2)\in\mathbb{Z}^2$. These are obtained by coupling the quantum mechanical system with a line bundle on $\mathcal{N}(r,n_1,n_2)$ as in [28]. Since $\mathcal{N}(r,n_1,n_2)$ is a fine moduli space of quiver representations, it is equipped with a universal locally free quiver sheaf/ In particular there are three tautological bundles $\mathcal{V}_1,\mathcal{V}_2,\mathcal{W}$ on the moduli space corresponding to the nodes e_1,e_2,e_∞ of the enhanced ADHM quiver. By construction, $\mathcal{W}\simeq\mathcal{O}^{\oplus r}_{\mathcal{N}(r,n_1,n_2)}$. Let $\mathcal{L}_1=\det(\mathcal{V}_1)$, $\mathcal{L}_2=\det(\mathcal{V}_2)$ be the determinant line bundles of $\mathcal{V}_1,\mathcal{V}_2$. For any pair of integers $(p_1,p_2)\in\mathbb{Z}^2$ let $\mathcal{L}_{(p_1,p_2)}=\mathcal{L}_1^{\otimes p_1}\otimes\mathcal{L}_2^{\otimes p_2}$. Then the partition function of the quantum mechanical system coupled to the line bundle $\mathcal{L}_{(p_1,p_2)}$ is the equivariant Euler character $\chi_T(\mathcal{N}(r,n_1,n_2),\mathcal{L}_{(p_1,p_2)})$. Note that $\mathcal{L}_{(p_1,p_2)}$ has by construction a canonical $\mathbf{T}$-linearization. In principle one can consider more general partition functions twisting the linearization of $\mathcal{L}_{(p_1,p_2)}$ by an arbitrary irreducible representation S of T. Therefore the most general quiver partition function is an equivariant Euler character of the form $\chi_{\mathbf{T}}(\mathcal{N}(r,n_1,n_2),S\otimes\mathcal{L}_{(p_1,p_2)})$.

Next let the discrete data $r,d\in\mathbb{Z}_{>0}$, $(p_1,p_2)\in\mathbb{Z}_2$ and S be fixed. Let (Q_1,Q_2,R_a), $a=1,\ldots,r$, denote the canonical generators of the representation ring of $\mathbf{T}$ and (q_1,q_2,ρ_a), $a=1,\ldots,r$ denote their characters. Let T be a formal variable. Then define a generating function

$$\mathcal{Z}^{(r,d,p_1,p_2,S)}_{quiv}(q_1,q_2,\rho_a,T)=\sum_{n\geq 0}\mathrm{ch}_{\mathbf{T}}\,\chi_{\mathbf{T}}(\mathcal{N}(r,n+d,d),S\otimes\mathcal{L}_{(p_1,p_2)})T^n \tag{4.2}$$

where $\mathrm{ch}_T(R)$ denotes the character of the representation R of $\mathbf{T}$. A combinatorial formula for this counting function will be derived in the following by equivariant localization. This requires an explicit classification of the $\mathbf{T}$-fixed loci in the moduli space $\mathcal{N}(r,n+d,d)$, and a computation of the equivariant normal bundles to the fixed loci.

4.1. **T-fixed loci and nested Young diagrams.** The $\mathbf{T}$-fixed loci in $\mathcal{N}(r,n+d,d)$ will be classified in terms of pairs of nested Young diagrams, which are defined as follows.

Recall that a Young diagram is a finite set μ of integral points $(i,j)\in(\mathbb{R}_{\geq 1})^2$ with the property that if μ contains a point $(i,j)\in(\mathbb{R}_{\geq 1})^2$, then it contains all integral points $(i',j')\in(\mathbb{R}_{\geq 1})^2$ so that $1\leq i'\leq i$ and $1\leq j'\leq j$. To fix conventions, the number of columns of a (nonempty) Young diagram μ will be denoted by $c_\mu\in\mathbb{Z}_{\geq 1}$, the columns being labelled by $i=1,\ldots,c_\mu$. The number of rows will be denoted by $l_\mu\in\mathbb{Z}_{\geq 1}$, the rows being labelled by $j=1,\ldots,l_\mu$. The number of points in the i-th column of μ will be denoted by μ_i. Note that the number of points in the j-th row equals the number of points μ^t_j in the j-th column of the transpose diagram μ^t.

Obviously, $\mu_i = 0$ unless $1 \leq i \leq c_\mu$, $h_0 \geq h_1 \cdots \geq \mu_{c_\mu}$, and $\mu_1 + \cdots + \mu_{c_\mu} = |\mu|$. If μ is empty, by convention $c_\mu = 0$ and $\mu_i = 0$ for all $i \in \mathbb{Z}$.

A pair (μ, ν) of Young diagrams will be called a pair of nested Young diagrams if there is an inclusion $\nu \subseteq \mu$ so that the complement $\mu \setminus \nu$ satisfies the following condition

(N) If $(i,j) \in \mu \setminus \nu$, then $(i+1,j) \notin \mu$.

Ordered sequence of $r \geq 1$ Young diagrams will be denoted by $\underline{\mu} = (\mu^a)_{1\leq a\leq r}$ and r will be called the length of the sequence. The size of the sequence if defined as

$$|\underline{\mu}| = \sum_{a=1}^{r} |\mu^a|.$$

A pair (μ, ν) of ordered sequences of equal length will be called nested if (μ^a, ν^a) is a pair of nested Young diagrams for all $1 \leq a \leq r$. Given such a pair (μ, ν) of nested sequences, the number of columns of μ^a, ν^a will be denoted by $c^a \in \mathbb{Z}_{\geq 0}$, $e^a \in \mathbb{Z}_{\geq 0}$ respectively, for $a = 1, \ldots, r$. The height of the i-th column of μ^a will be denoted by μ^a_i, and the height of the i-th column of ν^a will be denoted by ν^a_i, for $a = 1, \ldots, r$. The pair $(|\underline{\mu}|, |\underline{\nu}|) \in (\mathbb{Z}_{\geq 0})^2$ will be called the numerical type of the pair of nested sequences.

Note that condition (N) implies that no two points in the complement $\mu \setminus \nu$ are allowed to be in the same row. Then it is easy to check that the following inequalities must hold

$$0 \leq c^a - e^a \leq 1, \qquad 0 \leq \mu^a_i - \nu^a_i \leq \nu^a_{i-1} - \nu^a_i \tag{4.3}$$

for any $a = 1, \ldots, r$, and any $i \geq 0$. If any partition μ^a or ν^a is empty, by convention, $c^a = 0$, respectively $e^a = 0$. Recall also that by convention $\mu^a_i = 0$, $\nu^a_i = 0$ if $i > c^a$, respectively $i > e^a$. Moreover, Q_1, Q_2, R_a denote the one dimensional representations of $\mathbf{T}$ with characters t_1, t_2, z_a, $a = 1, \ldots, r$, respectively.

The classification of $\mathbf{T}$-fixed loci in $\mathcal{N}(r, n_+d, d)$ will be facilitated by the existence of the projection morphism $\mathfrak{q} : \mathcal{N}(r, n+d, d) \to \mathcal{M}(r, n)$ constructed in lemma (3.4). There is an analogous $\mathbf{T}$-action on the moduli space $\mathcal{M}(n, r)$, the fixed loci being classified in [24] for $r = 1$, and [25] for all $r \geq 1$. According to [25, Prop. 2.9], the fixed locus $\mathcal{M}(r, n)^{\mathbf{T}}$ is a finite set of points in one-to-one correspondence with length r sequences $\underline{\nu} = (\nu^a)_{1\leq a\leq r}$ of Young diagrams so that $|\underline{\nu}| = n$. Moreover, according to [**?**], [25, Thm. 4.2], the tangent space $T_{\underline{\nu}}\mathcal{M}(r, n)$, regarded as an element of the representation ring of $\mathbf{T}$, is given by the following formula

$$T_{\underline{\nu}}\mathcal{M}(r,n) = \sum_{a,b=1}^{r} R_a^{-1} R_b \Bigg(\sum_{(i,j)\in\nu^a} Q_1^{i-(\nu^b)^t_j} Q_2^{\nu^a_i - j+1} + \sum_{(i,j)\in\nu^b} Q_1^{(\nu^a)^t_j - i+1} Q_2^{j-\nu^b_i} \Bigg) \tag{4.4}$$

The analogous result for $\mathcal{N}(r, n+d, d)$ is given below.

Proposition 4.1. *The $\mathbf{T}$-fixed locus $\mathcal{N}(r, n+d, d)^{\mathbf{T}}$ is a finite set of points in one-to-one correspondence with pairs of nested length r sequences $(\underline{\mu}, \underline{\nu}) = (\mu^a, \nu^a)_{1\leq a\leq r}$ of Young diagrams of type $(|\underline{\mu}|, |\underline{\nu}|) = (n+d, n)$. The tangent space to the moduli space at a $\mathbf{T}$-fixed point $(\underline{\mu}, \underline{\nu})$, regarded as an element of the representation ring of*

$\mathbf{T}$, *is given by the following formula*

$$
\begin{aligned}
&T_{(\underline{\mu},\underline{\nu})}\mathcal{N}(r,n+d,d) = \\
&T_{\underline{\nu}}\mathcal{M}(n,r) + \sum_{a,b=1}^{r}\sum_{i=2}^{e^a+1}\sum_{j=1}^{c^b}\sum_{s=1}^{\mu_j^b-\nu_j^b} R_a^{-1}R_bQ_1^{i-j}\big(Q_2^{\mu_i^a-\nu_j^b-s+1} - Q_2^{\nu_{i-1}^a-\nu_j^b-s+1}\big) \\
&\qquad + \sum_{a,b=1}^{r}\sum_{j=1}^{c^b}\sum_{s=1}^{\mu_j^b-\nu_j^b} R_a^{-1}R_bQ_1^{-j+1}Q_2^{\mu_1^a-\nu_j^b-s+1}.
\end{aligned} \tag{4.5}
$$

Proof. Using lemma (3.4) the moduli space of stable framed representations $\mathcal{N}(r,n+d,d)$ can be alternatively characterized as the moduli space of pairs $\mathcal{A} = (V_1,W,A_1,A_2,I,J)$, $\widetilde{\mathcal{A}} = (V,W,\widetilde{A}_1,\widetilde{A}_2,\widetilde{I},\widetilde{J})$ of stable ADHM data of type $(n+d,r)$, (n,r) respectively, and a surjective morphism $\widetilde{f} : V_1 \to V$ of ADHM data such that $A_1|_{\mathrm{Ker}(\widetilde{f})}$ is identically zero. Then the correspondence between the $\mathbf{T}$-fixed loci in $\mathcal{N}(r,n+d,d)$ and r-collections of pairs of nested Young diagrams is a direct consequence of the classification of $\mathbf{T}$-fixed loci in the moduli spaces of stable ADHM data $\mathcal{M}(r,n+d)$, $\mathcal{M}(r,n)$ [25, Prop. 2.9].

In order to prove equation (4.5), recall that the tangent space at a closed point $[\mathcal{R}] \in \mathcal{N}(r,n+d,d)$ is isomorphic to the first cohomology group of the complex $\mathcal{C}(\mathcal{R})$ constructed in theorem (3.5), equation (3.19). Moreover, in the proof of theorem (3.5) it has been proven that there is an exact triangle

$$\mathcal{C}(\mathcal{R}) \longrightarrow \mathcal{C}(\mathcal{A}) \oplus \mathcal{C}(\mathcal{B}) \longrightarrow \mathcal{C}(\mathcal{A},\mathcal{B}), \tag{4.6}$$

where $\mathcal{A} = (V_1,A_1,A_2,I,J)$, $\mathcal{B} = (V_2,B_2)$ and the complexes $\mathcal{C}(\mathcal{A})$, $\mathcal{C}(\mathcal{B})$, $\mathcal{C}(\mathcal{A},\mathcal{B})$, are given in equations (3.22), (3.23),(3.24) respectively. Note that there is a natural $\mathbf{T}$-equivariant structure on the restrictions $\mathcal{C}(\mathcal{A})|_{(\underline{\mu},\underline{\nu})}$, $\mathcal{C}(\mathcal{B})|_{(\underline{\mu},\underline{\nu})}$, $\mathcal{C}(\mathcal{A},\mathcal{B})|_{(\underline{\mu},\underline{\nu})}$ to a $\mathbf{T}$-fixed point $(\underline{\mu},\underline{\nu}) = (\mu^a,\nu^a)_{1\le a\le r}$ induced by the action of $\mathbf{T}$ on the moduli space. The resulting $\mathbf{T}$-equivariant structures are given below.

$$
\mathcal{C}(\mathcal{A}) : \quad \mathrm{End}(V_1) \xrightarrow{d_0} \begin{array}{c} Q_1\otimes\mathrm{End}(V_1) \\ \oplus \\ Q_2\otimes\mathrm{End}(V_1) \\ \oplus \\ \mathrm{Hom}(W,V_1) \\ \oplus \\ Q_1\otimes Q_2\otimes\mathrm{Hom}(V_1,W) \end{array} \xrightarrow{d_1} Q_1\otimes Q_2\otimes\mathrm{End}(V_1) \tag{4.7}
$$

$$\mathcal{C}(\mathcal{B}) : \quad Q_2\otimes\mathrm{End}(V_2)\xrightarrow{d_0}\mathrm{End}(V_2) \tag{4.8}$$

(4.9)

$$
\mathcal{C}(\mathcal{A},\mathcal{B}) : \quad \mathrm{Hom}(V_2,V_1) \xrightarrow{d_0} \begin{array}{c} Q_1\otimes\mathrm{Hom}(V_2,V_1) \\ \oplus \\ Q_2\otimes\mathrm{Hom}(V_2,V_1) \\ \oplus \\ Q_1\otimes Q_2\otimes\mathrm{Hom}(V_2,W) \end{array} \xrightarrow{d_1} Q_1\otimes Q_2\otimes\mathrm{Hom}(V_2,V_1)
$$

where V_1, V_2, W have the following expressions in the representation ring of $\mathbf{T}$
(4.10)
$$V_1 = \sum_{a=1}^{r} \sum_{(i,j)\in\mu^a} R_a Q_1^{1-i} Q_2^{1-j}, \qquad V_2 = \sum_{a=1}^{r} \sum_{(i,j)\in\mu^a\setminus\nu^a} R_a Q_1^{1-i} Q_2^{1-j}, \qquad W = \sum_{a=1}^{r} R_a.$$

Note that $\mathcal{C}(\mathcal{A})$ is the equivariant deformation complex of the $\mathbf{T}$-fixed ADHM data $\mathcal{A}$. The underlying vector space $V \simeq V_1/V_2$ of the quotient ADHM data $\widetilde{\mathcal{A}}$ has a similar expression,

$$V = \sum_{a=1}^{r} \sum_{(i,j)\in\nu^a} R_a Q_1^{1-i} Q_2^{1-j}. \tag{4.11}$$

Obviously, $V_1 = V + V_2$. Then the exact triangle (4.6) yields the following identity in the representation ring of $\mathbf{T}$

$$\begin{aligned} T_{(\underline{\mu},\underline{\nu})}\mathcal{N}(n+d,d,r) =& -(1-Q_1)(1-Q_2)V_1^\vee V_1 + W^\vee V_1 + Q_1Q_2V_1^\vee W \\ & -(1-Q_2)V_2^\vee V_2 \\ & +(1-Q_1)(1-Q_2)V_2^\vee V_1 - Q_1Q_2V_2^\vee W \\ =& \, T_{\underline{\nu}}\mathcal{M}(n,r) + (1-Q_2)(Q_1V^\vee V_2 - V_1^\vee V_2) + W^\vee V_2. \end{aligned} \tag{4.12}$$

Next note that

$$\begin{aligned} (1-Q_2)V_1^\vee =& \sum_{a=1}^{r} R_a^{-1} \sum_{(i,j)\in\mu^a} (1-Q_2)Q_1^{i-1}Q_2^{j-1} \\ =& \sum_{a=1}^{r}\sum_{i=1}^{c^a}\sum_{j=1}^{\mu_i^a} R_a^{-1}Q_1^{i-1}(Q_2^{j-1}-Q_2^j) \\ =& \sum_{a=1}^{r}\sum_{i=1}^{c^a} R_a^{-1}Q_1^{i-1}(1-Q_2^{\mu_i^a}). \end{aligned}$$

Similarly,

$$(1-Q_2)V^\vee = \sum_{a=1}^{r}\sum_{i=0}^{e^a-1} R_a^{-1}Q_1^i(1-Q_2^{\nu_i^a}).$$

Moreover,

$$V_2 = \sum_{a=1}^{r}\sum_{i=1}^{c^a}\sum_{s=1}^{\mu_i^a-\nu_i^a} R_aQ_1^{1-i}Q_2^{-\nu_i^a-s+1}.$$

Therefore,

$$\begin{aligned} (1-Q_2)Q_1V^\vee V_2 =& \sum_{a,b=1}^{r}\sum_{i=1}^{e^a}\sum_{l=1}^{c^b}\sum_{s=1}^{\mu_l^b-\nu_l^b} R_a^{-1}R_bQ_1^{i-l+1}(1-Q_2^{\nu_i^a})Q_2^{-\nu_l^b-s+1} \\ =& \sum_{a,b=1}^{r}\sum_{i=2}^{e^a+1}\sum_{l=1}^{c^b}\sum_{s=1}^{\mu_l^b-\nu_l^b} R_a^{-1}R_bQ_1^{i-l}Q_2^{-\nu_l^b-s+1}(1-Q_2^{\nu_{i-1}^a}), \end{aligned} \tag{4.13}$$

$$-(1-Q_2)V_1^\vee V_2 = -\sum_{a,b=1}^{r}\sum_{i=1}^{c^a}\sum_{l=1}^{c^b}\sum_{s=1}^{\mu_l^b-\nu_l^b} R_a^{-1}R_bQ_1^{i-l}Q_2^{-\nu_l^b-s+1}(1-Q_2^{\mu_i^a}), \tag{4.14}$$

$$W^\vee V_2 = \sum_{a,b=1}^{r}\sum_{l=1}^{c^b}\sum_{s=1}^{\mu_l^b-\nu_l^b} R_a^{-1}R_bQ_1^{1-l}Q_2^{-\nu_l^b-s+1}. \tag{4.15}$$

Given inequalities (4.3), it follows that the sum over $i = 1,\ldots,c^a$ can be written as a sum over $i = 1,\ldots,e^a + 1$ employing the convention that $h_i^a = 0$ for $i \geq c^a + 1$. Then (4.5) follows from (4.12) adding the right hand sides of equations (4.13)-(4.14). □

4.2. **Equivariant Euler character.** Given Proposition (4.1), the computation of the equivariant Euler character $\chi_{\mathbf{T}}(\mathcal{N}(r,n_1,n_2), S\otimes\mathcal{L}_{(p_1,p_2)})$ is a straightforward exercise. Explicit formulas will be given below only for $(p_1,p_2) = (0,1)$, which is the relevant case for comparison with toric open string invariants. For simplicity, let $\mathcal{L}$ denote $\mathcal{L}_{(0,1)}$ below. Note that the restriction of $\mathcal{L}$ to the $\mathbf{T}$-fixed point $(\underline{\mu},\underline{\nu})$ is given by

$$\mathcal{L}_{(\underline{\mu},\underline{\nu})} = \prod_{a=1}^{r}\prod_{i=1}^{c^a}\prod_{s=1}^{\mu_i^a-\nu_i^a} R_aQ_1^{1-i}Q_2^{-\nu_i^a-s+1}. \tag{4.16}$$

Then the localization theorem yields the following formula for the equivariant Euler character of $\mathcal{L}$.

$$\begin{aligned} \mathrm{ch}_{\mathbf{T}}(\chi_{\mathbf{T}}(\mathcal{L})) &= \sum_{\substack{(\underline{\mu},\underline{\nu})\\ (|\underline{\mu}|,|\underline{\nu}|)=(n+d,n)}} \frac{\mathrm{ch}_{\mathbf{T}}(\mathcal{L}_{(\underline{\mu},\underline{\nu})})}{\Lambda_{-1}(T^\vee_{(\underline{\mu},\underline{\nu})}\mathcal{N}(r,n+d,d))} \\ &= \sum_{\substack{(\underline{\mu},\underline{\nu})\\ (|\underline{\mu}|,|\underline{\nu}|)=(n+d,n)}} \frac{\mathcal{W}_{(\underline{\mu},\underline{\nu})}(q_1,q_2,\rho_a)}{\Lambda_{-1}(T^\vee_{\underline{\nu}}\mathcal{M}(r,n))}, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \mathcal{W}_{(\underline{\mu},\underline{\nu})}(q_1,q_2,\rho_a) = &\frac{\prod_{a=1}^r\prod_{i=1}^{c^a}\prod_{s=1}^{\mu_i^a-\nu_i^a}\rho_aq_1^{1-i}q_2^{-\nu_i^a-s+1}}{\prod_{a,b=1}^r\prod_{i=2}^{e^a+1}\prod_{j=1}^{c^b}\prod_{s=1}^{\mu_j^b-\nu_j^b}(1-\rho_a\rho_b^{-1}q_1^{j-i}q_2^{\nu_j^b+s-\mu_i^a-1})} \\ &\frac{\prod_{a,b=1}^r\prod_{i=2}^{e^a+1}\prod_{j=1}^{c^b}\prod_{s=1}^{\mu_j^b-\nu_j^b}(1-\rho_a\rho_b^{-1}q_1^{j-i}q_2^{\nu_j^b+s-\nu_{i-1}^a-1})}{\prod_{a,b=1}^r\prod_{j=1}^{c^b}\prod_{s=1}^{\mu_j^b-\nu_j^b}(1-\rho_a\rho_b^{-1}q_1^{j-1}q_2^{\nu_j^b+s-\mu_1^a-1})}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\frac{1}{\Lambda_{-1}(T^\vee_{\underline{\nu}}\mathcal{M}(r,n))} = \\ &\frac{1}{\prod_{a,b=1}^r\prod_{(i,j)\in\nu^a}(1-\rho_a\rho_b^{-1}q_1^{(\nu^b)^t_j-i}q_2^{j-\nu_i^a-1})\prod_{(i,j)\in\nu^b}(1-\rho_a\rho_b^{-1}q_1^{i-(\nu^a)^t_j-1}q_2^{\nu_i^b-j})} \end{aligned} \tag{4.19}$$

and

$$q_1 = \mathrm{ch}_T(Q_1), \qquad q_2 = \mathrm{ch}_2(Q_2), \qquad \rho_a = \mathrm{ch}_{\mathbf{T}}(R_a),\ a = 1,\ldots,r.$$

Given any collection of r Young diagrams $\underline{\nu}$ and an positive integer $d\in\mathbb{Z}_{\geq 1}$, set

$$\mathcal{W}_{\underline{\nu},d}(q_1,q_2,\rho_a) = \sum_{\substack{(\underline{\mu},\underline{\nu})\\ |\underline{\mu}|=|\underline{\nu}|+d}} \mathcal{W}_{(\underline{\mu},\underline{\nu})}(q_1,q_2,\rho_a). \tag{4.20}$$

where the sum is over all nested sequences $(\underline{\mu}, \underline{\nu})$ of r Young diagrams with fixed $\underline{\nu}$. Then, obviously,

$$\mathrm{ch}_{\mathbf{T}}\chi_{\mathbf{T}}(\mathcal{L}) = \sum_{\underline{\nu}} \frac{1}{\Lambda_{-1}(T^\vee_{\underline{\nu}}\mathcal{M}(r,n))}\mathcal{W}_{\underline{\nu},d}(q_1,q_2,\rho_a).$$

In conclusion, for fixed $r, d \in \mathbb{Z}_{\geq 1}$, $(p_1,p_2) = (0,1)$ and S, the quiver partition function (4.2) is given by

(4.21)

$$\mathcal{Z}^{(r,d,S)}_{quiv}(q_1,q_2,\rho_a,T) = \sum_n T^n \mathrm{ch}_{\mathbf{T}}(S) \sum_{|\underline{\nu}|=n} \frac{1}{\Lambda_{-1}(T^\vee_{\underline{\nu}}\mathcal{M}(r,n))}\mathcal{W}_{\underline{\nu},d}(q_1,q_2,\rho_a).$$

5. Comparison with refined open string invariants

The goal of this section is to formulate a precise conjecture relating the quiver partition functions (4.21) , with $r = 1, 2$, to refined open string invariants of special lagrangian branes in toric Calabi-Yau threefolds. According to [2, 9], M5-branes wrapping such cycles yield surface operators in the five dimensional gauge theory effective action. Therefore a direct comparison between the quiver partition (4.21) and refined open string invariants is an important test for the models constructed in this paper.

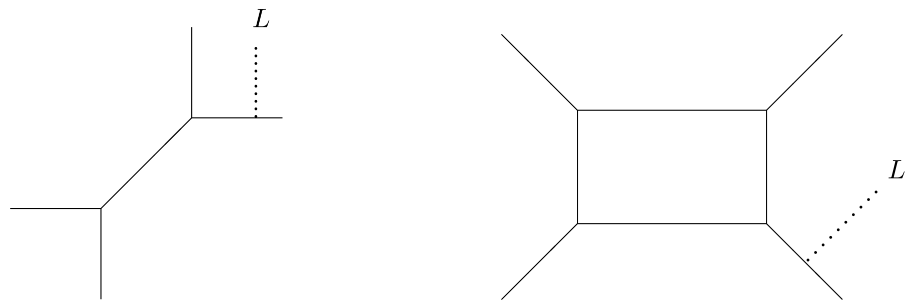
Five dimensional pure gauge theories with eight supercharges and gauge group $SU(r)$, $r \geq 2$ are engineered by toric Calabi-Yau threefolds constructed as follows. Let Y be a resolved conifold geometry, that is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$. Note that the finite group Γ_r of r-th roots of unity acts fiberwise on Y by

$$\omega \times (s_1, s_2) \to (\omega s_1, \omega^{-1}s_2)$$

where $\omega = e^{2i\pi/r}$ and s_1, s_2 are linear coordinates along the fibers. The quotient Z_0 is a local Calabi-Yau threefold with a curve $X \simeq \mathbb{P}^1$ of $\mathbb{C}^2/\Gamma_r$ singularities. Let $Z \to Z_0$ be the natural crepant resolution; Z is a smooth Calabi-Yau threefold containing a reducible exceptional divisor with $r-1$ components $S_1, \ldots, S_{r-1}$. Each component S_i is a geometrically ruled surface over X with smooth $\mathbb{P}^1$-fibers. One can formally allow $r = 1$ in this construction, in which case Γ_r is trivial, and $Z \simeq Z_0 \simeq Y$.

Note that the threefolds Z are toric, therefore they are equipped with canonical symplectic $U(1)^3$ actions. The resulting moment map, $\rho_Z : Z \to \mathbb{R}^3$ maps Z surjectively onto its Delzant polytope. The boundary of the Delzant polytope consists of a collection of 2-dimensional faces linearly embedded in $\mathbb{R}^3$, which intersect along 1-faces. The 1-faces form trivalent a trivalent graph Δ_Z in $\mathbb{R}^3$, which is the image of the toric skeleton of Z under the moment map ρ_Z. The toric skeleton of Z is the union of all rational holomorphic curves in Z, both compact and noncompact, preserved by the $U(1)^3$-action. The compact components of the toric skeleton are mapped to finite 1-faces while the non-compact components are mapped to semi-infinite 1-faces.

Toric special lagrangian cycles $L \subset Z$ can be constructed applying the methods of [1], as in section (2.1). They are essentially classified by their image under the moment map ρ_Z, which has to be a half real line embedded in the Delzant polytope of Z. There is a special class of cycles L such that $\rho_Z(L)$ intersects a 1-face of the graph Δ_Z. These cycles have topology $\mathbb{R}^2 \times S^1$ and intersect the toric skeleton of Z along a one dimensional orbit of the canonical $U(1)$ action. They are naturally classified in external lagrangian cycles, in which case L intersects a non-compact component of the toric skeleton, and internal cycles, in which case L L intersects a compact component of the toric skeleton. Equivalently, $\rho_Z(L)$ intersects a semi-infinite 1-face, respectively a finite 1-face of Δ_Z. The lagrangian cycles of primary interest in the following will be external cycles as shown below for $r = 1, 2$.



$r = 1$ $r = 2$

The refined open string partition function for an external toric special lagrangian cycle $L \subset Z$ is constructed using the refined topological vertex of [18], which will be briefly reviewed below.

Given three (possibly empty) Young diagrams (λ, μ, ν), the refined vertex is a formal series of two variables (t, q) of the form

$$
\begin{aligned}
C_{\lambda\,\mu\,\nu}(t,q) = {} & \left(\frac{t}{q}\right)^{\frac{||\mu||^2}{2}} q^{\frac{\kappa(\mu)+||\nu||^2}{2}} \widetilde{Z}_\nu(t,q) \\
& \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})
\end{aligned}
\tag{5.1}
$$

where $s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu})$, $s_{\mu/\eta}(t^{-\nu^t}q^{-\rho})$ are skew Schur functions of the infinite set of variables $t^{-\rho}q^{-\nu} = (t^{\frac{1}{2}}q^{-\nu_1}, t^{\frac{3}{2}}q^{-\nu_2}, t^{\frac{5}{2}}q^{-\nu_3}, \ldots)$ defined in [27],

$$
\widetilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu} (1 - q_1^{\nu_j^t - i + 1} q_2^{\nu_i - j - 1}),
$$

and for any partition λ,

$$
|\lambda| = \sum_i \lambda_i, \qquad ||\lambda|| = \sum_i \lambda_i^2, \qquad \kappa(\lambda) = ||\lambda||^2 - ||\lambda^t||^2.
$$

Note that the expression (5.1) differs from [18, Eqn. 24] by the choice of normalization, which is closely related to the normalization chosen in [13, Sect. 5]. Detailed computations will show below that (5.1) yields the same results as [18] for refined closed string invariants.

The gluing algorithm developed in [18], assigns to any triple (Z, L, λ) a formal series $\mathcal{Z}_\lambda(q, t, Q)$, which is an expansion in the formal variables $Q = (Q_1, \ldots, Q_M)$ associated to the Mori cone generators of X. $\mathcal{Z}_\lambda(q, t, Q)$ is constructed assigning an expression of the form (5.1) to each trivalent vertex of the dual toric polytope of Z, the partitions (λ, μ, ν) being assigned to the edges meeting at the given vertex. Then one has to specify gluing rules along edges, eventually including certain framing factors, and sum over all partitions associated to finite edges. Toric lagrangian branes correspond to infinite edges, and the corresponding partitions are not summed over. The details are somewhat intricate and easier to explain in concrete examples as shown in sections (5.1), (5.2) below.

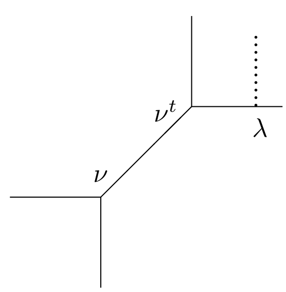
Suppose there is a stack of m D3-branes wrapped on L, the holonomy of the flat $U(m)$ gauge field around S^1 being in the conjugacy class of an element $(\alpha_1, \ldots, \alpha_m)$ of the maximal torus. In order to compute the refined open topological **A**-model partition function of such a D3-brane system, let $y = (y_1, y_2, \ldots,)$ be an infinite set of formal variables and let

$$\mathcal{Z}^{ref}_{open}(t, q, Q; y) = \sum_{\lambda} \mathcal{Z}_\lambda(t, q, Q) s_\lambda(y) \tag{5.2}$$

Then the refined open topological partition function of m D3-branes on L with holonomy in the conjugacy class of the diagonal matrix $(\alpha_1, \ldots, \alpha_m)$ is obtained by evaluating (5.2) at $\underline{y} = (\alpha_1, \ldots, \alpha_m, 0, 0, \ldots)$. Note that only Young diagrams λ with $|\lambda| \leq m$ contribute to this truncation.

Using this formalism, the quiver partition function (4.21) will be related to the corresponding refined open string partition function for $r = 1, 2$. For $r = 1$, the threefold Z is isomorphic to the crepant resolution of a conifold singularity, while for $r = 1$, Z is isomorphic to the total space of the canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.

5.1. **Conifold.** The resolved conifold is the toric threefold Y isomorphic to the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. Note that there is only one formal variable Q assigned to the class of the zero section. The toric polytope and projection of special lagrangian cycle are represented below.



Then, applying the refined vertex construction, one obtains

$$\begin{aligned} \mathcal{Z}_\lambda(t, q, Q) &= \sum_\nu (-Q)^{|\nu|} C_{\emptyset,\emptyset,\nu}(t, q) C_{\lambda,\emptyset,\nu^t}(q, t) \\ &\sum_\nu (-Q)^{|\nu|} q^{||\nu||^2/2} t^{||\nu^t||^2/2} \left(\frac{t}{q}\right)^{|\lambda|/2} \\ &\qquad \widetilde{Z}_\nu(t, q) \widetilde{Z}_{\nu^t}(q, t) s_{\lambda^t}(q^{-\rho} t^{-\nu^t}) \end{aligned} \tag{5.3}$$

Note that under the change of variables

$$t = q_1, \qquad q = q_2^{-1}, \qquad Q = T(q_1 q_2)^{1/2} \tag{5.4}$$

the expression

$$(-Q)^{|\nu|} q^{||\nu||^2/2} t^{||\nu^t||^2/2} \widetilde{Z}_\nu(t, q) \widetilde{Z}_{\nu^t}(q, t)$$

becomes

$$T^{|\nu|} \frac{1}{\Lambda_{-1} T^*_\nu(\mathcal{M}(|\nu|, 1))}.$$

Then (5.3) becomes

(5.5)

$$\begin{aligned}\mathcal{Z}_\lambda(q_1, q_2^{-1}, T(q_1q_2)^{1/2}) = & \sum_\nu T^{|\nu|} \frac{1}{\Lambda_{-1}T_\nu^*\mathcal{M}(|\nu|,1)} (q_1q_2)^{|\lambda|/2} s_{\lambda^t}(q_2^{-\rho}q_1^{-\nu^t}) \\ & \sum_\nu T^{|\nu|} \frac{1}{\Lambda_{-1}T_\nu^*\mathcal{M}(|\nu|,1)} q_1^{|\lambda|/2} q_2^{|\lambda|} s_{\lambda^t}(q_2^{-1/2}q_2^{-\rho}q_1^{-\nu^t})\end{aligned}$$

Redefining the formal variables y_i by

$$y_i = q_1^{-1/2}q_2^{-1}x_i$$

for all $i \geq 1$, it follows that

$$\begin{aligned}&\mathcal{Z}^{ref}_{open}(q_1, q_2^{-1}, (q_1q_2)^{1/2}T; q_1^{1/2}q_2x) = \\ &\sum_\lambda\sum_\nu T^{|\nu|}\frac{1}{\Lambda_{-1}T_\nu^*\mathcal{M}(|\nu|,1)} s_{\lambda^t}(q_2^{-1/2}q_2^{-\rho}q_1^{-\nu^t})s_\lambda(x) = \\ &\sum_\nu T^{|\nu|}\frac{1}{\Lambda_{-1}T_\nu^*\mathcal{M}(|\nu|,1)}\prod_{i=1}^\infty\prod_{j=1}^\infty(1+q_2^{1-i}q_1^{-\nu_i^t}x_j)\end{aligned} \tag{5.6}$$

The right hand side of equation (5.6) can be expanded in terms of the monomial basis $M_\eta(x)$ in the space of symmetric functions, which is labelled by partitions η. Note that for any positive integer $d \in \mathbb{Z}_{>0}$, $M_{(d,0,0,\ldots)}(x) = x_1^d + x_2^d + \cdots$. Let $\mathcal{Z}^{ref}_{open,d}(q_1, q_2, T)$ be the coefficient of $M_{(d,0,0,\ldots)}(x)$ in this expansion, which can be computed as follows.

Let $E_k(x)$, $k \in \mathbb{Z}_{\geq 0}$ be the degree k elementary symmetric function in the variables $x = (x_1, x_2, \ldots)$. Then

$$\begin{aligned}\ln\prod_{i=1}^\infty\prod_{j=1}^\infty(1+q_2^{1-i}q_1^{-\nu_i^t}x_j) &= \ln\prod_{i=1}^\infty\left(\sum_{k=0}^\infty q_2^{k(1-i)}q_1^{-k\nu_i^t}E_k(x)\right). \\ &= \sum_{i=1}^\infty \ln\left(\sum_{k=0}^\infty q_2^{k(1-i)}q_1^{-k\nu_i^t}E_k(x)\right) \\ &= \sum_{i=1}^\infty\sum_{l=1}^\infty\frac{(-1)^{l-1}}{l}\left(\sum_{k=1}^\infty q_2^{k(1-i)}q_1^{-k\nu_i^t}E_k(x)\right)^l.\end{aligned}$$

Therefore

$$\prod_{i=1}^\infty\prod_{j=1}^\infty(1+q_2^{1-i}q_1^{-\nu_i^t}x_j) = \exp\left[\sum_{i=1}^\infty\sum_{l=1}^\infty\frac{(-1)^{l-1}}{l}\left(\sum_{k=1}^\infty q_2^{k(1-i)}q_1^{-k\nu_i^t}E_k(x)\right)^l\right] \tag{5.7}$$

In order to compute the coefficients of $M_{(d,0,0,\ldots)}(x) = x_1^d + x_2^d + \cdots$ in the expansion, it suffices to truncate the argument of the exponential function in right hand side of (5.7) to $k = 1$ terms,

$$\exp\left[\sum_{i=1}^\infty\sum_{l=1}^\infty\frac{(-1)^{l-1}}{l}\left(q_2^{1-i}q_1^{-\nu_i^t}E_1(x)\right)^l\right].$$

Let

$$F_\nu(q_1,q_2) = \sum_{i=1}^\infty q_2^{1-i}q_1^{-\nu_i^t} = \sum_{i=1}^{l_\nu} q_2^{1-i}q_1^{-\nu_i^t} + \frac{q_2^{-l_\nu}}{1-q_2^{-1}}.$$

Then one has to identify the coefficients of the monomial functions $M_{(d,0,0,\ldots)}(x)$ in the expansion of

$$\exp\left[\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} E_1(x)^l F_\nu(q_1^l, q_2^l)\right],$$

which is the same as the coefficient of x_1^d in the expansion of

$$\exp\left[\sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} x_1^l F_\nu(q_1^l, q_2^l)\right].$$

Expanding the exponential function and collecting all relevant terms, it follows that the coefficient of $M_{(d,0,0,\ldots)}(x)$, $d \geq 1$ is

$$\frac{1}{d!} \sum_{\eta=(1^{d_1},2^{d_2},\ldots)} \frac{d!}{\prod_{k=1}^d d_k!} \prod_{k=1}^d \left(\frac{(-1)^{k-1}}{k} F_\nu(q_1^k, q_2^k)\right)^{d_k} \tag{5.8}$$

where the sum is over all partitions $\eta = (1^{d_1}, 2^{d_2}, \ldots)$ of d.

In conclusion the coefficient of $M_{(d,0,0,\ldots)}(x)$ in the right hand side of (5.6) is (5.9)

$$\mathcal{Z}^{ref}_{open,d}(q_1, q_2, T) = \sum_\nu T^{|\nu|} \frac{1}{\Lambda_{-1} T^*_\nu \mathcal{M}(|\nu|, 1)} \sum_{\eta=(1^{d_1},2^{d_2},\ldots)} \frac{(-1)^{d-\sum_{k=1}^d d_k}}{\prod_{k=1}^d (d_k!\, k^{d_k})} \prod_{k=1}^d F_\nu(q_1^k, q_2^k)^{d_k}.$$

For any ν and $d \geq 1$ let

$$\mathcal{Z}_{\nu,d}(q_1, q_2) = \sum_{\eta=(1^{d_1},2^{d_2},\ldots)} \frac{(-1)^{d-\sum_{k=1}^d d_k}}{\prod_{k=1}^d (d_k!\, k^{d_k})} \prod_{k=1}^d F_\nu(q_1^k, q_2^k)^{d_k}.$$

Then the relation between the quiver partition function (4.21) and the refined open topological string partition function (5.5) is given by:

Conjecture 5.1. *The following identity holds for any Young diagram* ν *and any* $d \in \mathbb{Z}_{\geq 1}$.

$$\mathcal{W}_{\nu,d}(q_1, q_2) = \mathcal{Z}_{\nu,d}(q_1, q_2), \tag{5.10}$$

where $\mathcal{W}_{\nu,d}(q_1, q_2)$ *is defined in equation* (4.20). *In particular*

$$\mathcal{Z}^{(1,d,1)}_{quiv}(q_1, q_2, T) = \mathcal{Z}^{ref}_{open,d}(q_1, q_2, T). \tag{5.11}$$

Extensive numerical computations show that conjecture (5.1) holds for all Young diagrams ν with $|\nu| \leq 10$ and all $1 \leq d \leq 10$. A sample computation is presented below.

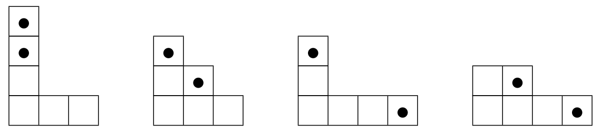
Example 5.2. *Let* $\nu =$  *and* $d = 2$. *Then*

$$F_\nu(q_1, q_2) = q_1^{-3} + q_1^{-1} q_2^{-1} + q_2^{-2}(1 - q_2^{-1})^{-1}$$

and

$$\begin{aligned} \mathcal{Z}_{\nu,2}(q_1, q_2) &= \frac{1}{2} F_\nu(q_1, q_2)^2 - \frac{1}{2} F_\nu(q_1^2, q_2^2) \\ &= \frac{q_1^4 + q_1^3 q_2^2 - q_1^3 + q_1 q_2^3 - q_2 q_1 - q_2^3 + q_2 + q_2^4 - q_2^2}{q_1^4 q_2^2 (1 - q_2)(1 - q_2^2)} \end{aligned}$$

The set of all nested pairs (μ,ν) with $|\mu| = |\nu| + 2$ consists of the four elements $(\mu_1,\nu),\ldots,(\mu_4,\nu)$ represented below.



The boxes in the complement $\mu \setminus \nu$ are marked with $\bullet$. Then equation (4.18) specializes to

$$\mathcal{W}_{(\mu_1,\nu),2}(q_1,q_2) = \frac{q_1^4 - q_2q_1 - q_1^3 + q_2}{q_2^2(1-q_2^2)(1-q_2)(q_1-q_2^2)(q_1^3-q_2^3)}$$

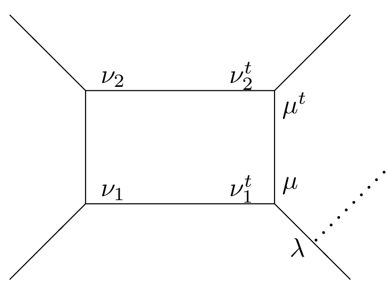
$$\mathcal{W}_{(\mu_2,\nu),2}(q_1,q_2) = \frac{q_1^5 - q_1^3 - q_2q_1^2 + q_2}{q_1(1-q_2)(q_1^2-q_2)(q_1-q_2^2)(q_1^3-q_2^2)}$$

$$\mathcal{W}_{(\mu_3,\nu),2}(q_1,q_2) = \frac{(q_1^2 + q_2q_1 - q_1 - q_2)q - 2}{q_1^3(q_1^2-q_2)(q_1-q_2)(q_1^2+q_2q_1+q_2^2)(1-q_2)}$$

$$\mathcal{W}_{(\mu_4,\nu),2}(q_1,q_2) = \frac{q_2^2}{q_1^4(q_1-q_2)(q_1^3-q_2^2)}$$

Adding the above expressions, it follows that indeed $\mathcal{W}_{\nu,2}(q_1,q_2) = \mathcal{Z}_{\nu,2}(q_1,q_2)$.

5.2. **Local $\mathbb{P}^1 \times \mathbb{P}^1$.** In this case Z is isomorphic to the total space of the canonical bundle $\mathcal{O}(-2,-2)$ of $\mathbb{P}^1 \times \mathbb{P}^1$. The Mori cone of Z is generated by the two curve classes associated to the two obvious rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. The corresponding formal variables will be denoted by Q_f, Q_b. The toric polytope and projection of the special lagrangian cycle L are represented below.



By analogy with [18, Sect. 5.5], the refined open string partition function is

(5.12)
$$\mathcal{Z}_\lambda(t,q,Q_f,Q_b) = \sum_{\nu_1,\nu_2} (-Q_b)^{|\nu_1|+|\nu_2|} \widetilde{f}_{\nu_1^t}(q,t)\widetilde{f}_{\nu_2}(t,q) Z_{\nu_1^t,\nu_2^t,\emptyset}(t,q,Q_f) Z_{\nu_1,\nu_2,\lambda}(q,t,Q_f)$$

where

$$\mathcal{Z}_{\nu_1,\nu_2,\lambda}(q,t,Q_f) = \sum_{\nu_1,\nu_2,\mu} (-Q_f)^{|\mu|} C_{\lambda,\mu,\nu_1^t}(q,t) C_{\mu^t,\emptyset,\nu_2^t}(q,t) f_\mu(t,q) \tag{5.13}$$

and $f_\eta(t,q)$ $\widetilde{f}_\eta(t,q)$ are framing factors of the form

$$f_\eta(t,q) = (-1)^{|\eta|} t^{||\eta^t||^2/2 - |\eta|/2} q^{-||\eta||^2/2 + |\eta|/2}, \qquad \widetilde{f}_\eta(t,q) = (-1)^{|\eta|} \left(\frac{t}{q}\right)^{|\eta|/2} f_\eta(t,q)$$

Substituting (5.1) in (5.13) yields

(5.14)

$$\mathcal{Z}_{\nu_1,\nu_2,\lambda}(q,t,Q_f) = t^{\frac{||\nu_1^t||^2+||\nu_2^t||^2}{2}} \sum_{\nu_1,\nu_2,\mu} Q_f^{|\mu|} \sum_{\eta} \left(\frac{t}{q}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(q^{-\rho}t^{-\nu_1^t}) s_{\mu/\eta}(q^{-\nu_1}t^{-\rho}) s_{\mu}(q^{-\rho}t^{-\nu_2^t}).$$

Using the skew Schur function identities

$$\sum_{\alpha} s_{\alpha/\eta_1}(x)s_{\alpha/\eta_2}(y) = \prod_{i,j}(1-x_iy_j)^{-1}\sum_{\kappa} s_{\eta_2/\kappa}(x)s_{\eta_1/\kappa}(y)$$

$$\sum_{\alpha} s_{\alpha^t/\eta_1}(x)s_{\alpha/\eta_2}(y) = \prod_{i,j}(1+x_iy_j)\sum_{\kappa} s_{\eta_2^t/\kappa^t}(x)s_{\eta_1^t/\kappa}(y),$$

it follows that

$$\sum_{\mu} Q_f^{|\mu|}\left(\frac{q}{t}\right)^{|\mu|/2} s_{\mu/\eta}(q^{-\nu_1}t^{-\rho})s_{\mu}(q^{-\rho}t^{-\nu_2^t}) = \prod_{i,j\geq 1}(1-Q_fq^{j-\nu_{1,i}}t^{i-1-\nu_{2,j}^t})^{-1} s_{\eta}(Q_fq^{-\rho+1/2}t^{-\nu_2^t-1/2})$$

$$\sum_{\lambda}\left(\frac{t}{q}\right)^{|\lambda|/2} s_{\lambda^t/\eta}(q^{-\rho}t^{-\nu_1^t})s_{\lambda}(y) = \prod_{i,j\geq 1}(1+q^{i-1}t^{-\nu_{1,j}^t+1/2}y_j)s_{\eta^t}(t^{1/2}q^{-1/2}y)$$

Then

$$\begin{aligned}
&\sum_{\lambda}\mathcal{Z}_{\nu_1,\nu_2,\lambda}(q,t,Q_f)s_{\lambda}(y) = \\
&t^{\frac{||\nu_1^t||^2+||\nu_2^t||^2}{2}}\prod_{i,j\geq 1}(1-Q_fq^{j-\nu_{1,i}}t^{i-1-\nu_{2,j}^t})^{-1}\prod_{i,j\geq 1}(1+q^{i-1}t^{-\nu_{1,j}^t+1/2}y_j) \\
&\sum_{\eta}\left(\frac{t}{q}\right)^{|\eta|/2} s_{\eta}(Q_fq^{-\rho+1/2}t^{-\nu_2^t-1/2})s_{\eta^t}(t^{1/2}q^{-1/2}y) = \\
&t^{\frac{||\nu_1^t||^2+||\nu_2^t||^2}{2}}\prod_{i,j\geq 1}(1-Q_fq^{j-\nu_{1,i}}t^{i-1-\nu_{2,j}^t})^{-1}\prod_{i,j\geq 1}(1+q^{i-1}t^{-\nu_{1,j}^t+1/2}y_j) \\
&\prod_{i,j\geq 1}(1+Q_fq^{i-1}t^{-\nu_{2,j}^t+1/2}y_j)
\end{aligned} \tag{5.15}$$

Taking into account the framing factors in (5.12) and redefining $y_j = t^{1/2}x_j$, it follows that

$$\begin{aligned}
&\mathcal{Z}^{ref}_{open}(t,q,Q_f,Q_b;t^{1/2}x) = \\
&\sum_{\nu_1,\nu_2}(-Q_b)^{|\nu_1|+|\nu_2|}q^{||\nu_1||^2}t^{||\nu_2^t||^2}\widetilde{Z}_{\nu_1}(t,q)\widetilde{Z}_{\nu_1^t}(q,t)\widetilde{Z}_{\nu_2}(t,q)\widetilde{Z}_{\nu_2^t}(q,t) \\
&\qquad P_{(\nu_1,\nu_2)}(t,q,Q_f)\prod_{i,j\geq 1}(1+q^{i-1}t^{-\nu_{1,j}^t}x_j)(1+Q_fq^{i-1}t^{-\nu_{2,j}^t}x_j)
\end{aligned} \tag{5.16}$$

where

$$P_{\nu_1,\nu_2}(t,q,Q_f) = \prod_{i,j\geq 1}(1-Q_fq^{j-\nu_{1,i}}t^{i-1-\nu_{2,j}^t})^{-1}\prod_{i,j\geq 1}(1-Q_ft^{i-\nu_{1,j}^t}q^{j-1-\nu_{2,i}})^{-1}$$

For the purpose of comparison with the quiver partition function, one has to consider the normalized partition function $\widetilde{\mathcal{Z}}^{ref}_{open}(t,q,Q_f,Q_b;t^{1/2}x)$ obtained by replacing $P_{\nu_1,\nu_2}(t,q,Q_f)$ in equation (5.15) by

$$\frac{P_{\nu_1,\nu_2}(t,q,Q_f)}{P_{\emptyset,\emptyset}(t,q,Q_f)} = \prod_{i,j\geq 1} \frac{(1-Q_ft^{i-1}q^j)(1-Q_fq^{i-1}t^j)}{(1-Q_fq^{j-\nu_{1,i}}t^{i-1-\nu^t_{2,j}})(1-Q_ft^{i-\nu^t_{1,j}}q^{j-1-\nu^t_{2,i}})}.$$

Proceeding by analogy with [18, Sect. 5.5.1] it follows that

$$\frac{1}{\Lambda_{-1}(T_{\nu_1,\nu_2}\mathcal{M}(|\nu_1|+|\nu_2|,2))} =$$

$$q^{||\nu_1||^2}t^{||\nu_2^t||^2}\Big(-Q_f\frac{t}{q}\Big)^{|\nu_1|+|\nu_2|}\widetilde{Z}_{\nu_1}(t,q)\widetilde{Z}_{\nu_1^t}(q,t)\widetilde{Z}_{\nu_2}(t,q)\widetilde{Z}_{\nu_2^t}(q,t)\frac{P_{\nu_1,\nu_2}(t,q,Q_f)}{P_{\emptyset,\emptyset}(t,q,Q_f)}\Bigg|_{\substack{t=q_1,\ q=q_2^{-1}\\ Q_f=\rho_1^{-1}\rho_2}}$$

Therefore

(5.17)

$$\widetilde{\mathcal{Z}}^{ref}_{open}(q_1,q_2^{-1},\rho_1^{-1}\rho_2,q_1q_2\rho_1^{-1}\rho_2T;t^{1/2}x) =$$

$$\sum_{\nu_1,\nu_2}\frac{T^{|\nu_1|+|\nu_2|}}{\Lambda_{-1}(T_{\nu_1,\nu_2}\mathcal{M}(|\nu_1|+|\nu_2|,2))}\prod_{i,j\geq 1}(1+q_2^{1-i}q_1^{-\nu^t_{1,j}}x_j)(1+\rho_1^{-1}\rho_2q_2^{1-i}q_1^{-\nu^t_{2,j}}x_j)$$

Let $\rho_{12}=\rho_1^{-1}\rho_2$. Proceeding by analogy with section (5.1), (5.6) – (5.9), the coefficient of $M_{(d,0,\ldots)}(x_1,x_2,\ldots)$ in the expansion of the right hand side of (5.17) is

$$\begin{aligned}&\widetilde{\mathcal{Z}}^{ref}_{open,d}(q_1,q_2,\rho_{12},T) =\\ &\sum_{\nu_1,\nu_2}\frac{T^{|\nu_1|+|\nu_2|}}{\Lambda_{-1}(T_{\nu_1,\nu_2}\mathcal{M}(|\nu_1|+|\nu_2|,2))}\mathcal{Z}_{(\nu_1,\nu_2),d}(q_1,q_2,\rho_{12})\end{aligned} \tag{5.18}$$

where

$$\mathcal{Z}_{(\nu_1,\nu_2),d}(q_1,q_2,\rho_1^{-1}\rho_2) = \sum_{\eta=(1^{d_1},2^{d_2},\ldots)}\frac{(-1)^{d-\sum_{k=1}^d d_k}}{\prod_{k=1}^d(d_k!\,k^{d_k})}\prod_{k=1}^d F_{(\nu_1,\nu_2)}(q_1^k,q_2^k,\rho_{12}^k)^{d_k}$$

and

$$F_{(\nu_1,\nu_2)}(q_1,q_2,\rho_{12}) = F_{\nu_1}(q_1,q_2)+\rho_{12}F_{\nu_2}(q_1,q_2) =$$

$$\sum_{i=1}^{l_{\nu_1}}q_2^{1-i}q_1^{-\nu^t_{1,i}}+\frac{q_2^{-l_{\nu_1}}}{1-q_2^{-1}}+\rho_{12}\Big(\sum_{i=1}^{l_{\nu_2}}q_2^{1-i}q_1^{-\nu^t_{2,i}}+\frac{q_2^{-l_{\nu_2}}}{1-q_2^{-1}}\Big)$$

Then the relation between the quiver partition function (4.21) and the refined open topological string partition function (5.17) is given by:

Conjecture 5.3. *The following identity holds for any pair of Young diagrams* (ν_1,ν_2) *and any* $d\in\mathbb{Z}_{\geq 1}$.

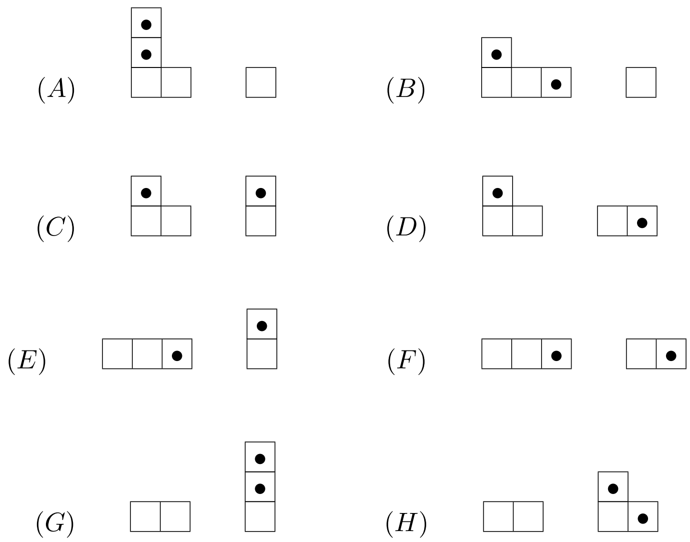
$$\rho_1^{-d}\mathcal{W}_{(\nu_1,\nu_2),d}(q_1,q_2,\rho_1,\rho_2) = \mathcal{Z}_{(\nu_1,\nu_2),d}(q_1,q_2,\rho_{12}), \tag{5.19}$$

where $\mathcal{W}_{\nu,d}(q_1,q_2,\rho_1,\rho_2)$ *is defined in equation* (4.20). *In particular*

$$\mathcal{Z}^{(2,d,R_1^{-d})}_{quiv}(q_1,q_2,\rho_1,\rho_2,T) = \widetilde{\mathcal{Z}}^{ref}_{open,d}(q_1,q_2,\rho_{12},T). \tag{5.20}$$

Again, extensive numerical computations show that conjecture (5.3) holds for all pairs of Young diagrams (ν_1,ν_2) with $|\nu_1|+|\nu_2|\leq 10$ and all $1\leq d\leq 10$. A sample computation is presented below.

Example 5.4. *Let* $(\nu_1,\nu_2) = (\square\square, \square)$ *and* $d = 2$. *Then there are eight sequences of nested pairs* $((\mu_1,\mu_2),(\nu_1,\nu_2))$ *with* $|\mu_1|+|\mu_2| = 5$. *The partitions* (μ_1,μ_2) *are listed below for all these cases.*



Then

$$F_{(\nu_1,\nu_2)}(q_1,q_2,\rho_{12})^2 = q_1^{-2} + \frac{q_2^{-1}}{1-q_2^{-1}} + \rho_{12}\left(q_1^{-1} + \frac{q_2^{-1}}{1-q_2^{-1}}\right)$$

and

$$\mathcal{Z}_{(\nu_1,\nu_2)}(q_1,q_2,\rho_{12}) = \frac{1}{2}F_{(\nu_1,\nu_2)}(q_1,q_2,\rho_{12})^2 - \frac{1}{2}F_{(\nu_1,\nu_2)}(q_1^2,q_2^2,\rho_{12}^2) =$$
$$\frac{q_2^2q_1+q_1^3-q_1}{(1-q_2^2)(1-q_2)q_1^3} + \rho_{12}\frac{q_2^3+q_1^2q_2^2+q_2^2q_1-q_2^2+q_2q_1^3-q_2+1-q_1^2-q_1+q_1^3}{(1-q_2^2)(1-q_2)q_1^3}$$
$$+\rho_{12}^2\frac{q_1^2q_2^2+q_1^3-q_1^2}{(1-q_2^2)(1-q_2)q_1^3}$$

Equation (4.18) *specializes respectively to*

$$\mathcal{W}^{(A)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = \frac{(q_1^2-1)(q_1-\rho_{12})}{(-1+q_2)^2(1+q_2)(q_1^2-q_2^2)(-1+\rho_{12})(-1+q_2\rho_{12})(q_1-q_2^2\rho_{12})}$$

$$\mathcal{W}^{(B)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = \frac{q_2^2(q_1-\rho_{12})(q_2-q_1\rho_{12})}{q_1^2(q_2-1)(q_1^2-q_2^2)(\rho_{12}-1)(q_1\rho_{12}-1)(q_1^2\rho_{12}-q_2)(q_1-q_2\rho_{12})}$$

$$\mathcal{W}^{(C)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = \frac{(q_1-1)^2(1+q_1)q_2^2(q_1-\rho_{12})\rho_{12}^2(q_1^2\rho_{12}-1)}{(q_1-q_2)(q_1^2-q_2)(q_2-1)^2(q_2-\rho_{12})(q_1^2\rho_{12}-q_2)(q_1-q_2\rho_{12})(q_2\rho_{12}-1)}$$

$$\mathcal{W}^{(D)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = -\frac{(q_1^2-1)q_2^2\rho_{12}^2(q_1q_2\rho_{12}-1)}{q_1(q_1-q_2)(q_1^2-q_2)(q_2-1)(\rho_{12}-1)(q_1\rho_{12}-1)(q_1-q_2^2\rho_{12})}$$

$$\mathcal{W}^{(E)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = \frac{(q_1-1)q_2^2\rho_{12}^2(q_1\rho_{12}-q_2)}{q_1^2(q_1-q_2)(q_1^2-q_2)(q_2-1)(\rho_{12}-1)(q_1\rho_{12}-1)(q_1^2\rho_{12}-q_2^2)}$$

$$\mathcal{W}^{(F)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = \frac{q_2^4\rho_{12}^2}{q_1^3(q_1-q_2)(q_1^2-q_2)(q_1^2\rho_{12}-q_2)(q_1-q_2\rho_{12})}$$

$$\mathcal{W}^{(G)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = -\frac{(q_1-1)\rho_{12}^4(q_1^2\rho_{12}-1)}{(q_2-1)^2(q_2+1)(q_2^2-q_1)(q_2-\rho_{12})(\rho_{12}-1)(q_2^2-q_1^2\rho_{12})}$$

$$\mathcal{W}^{(H)}_{(\underline{\mu},\underline{\nu}),2}(q_1,q_2,\rho_1,\rho_2) = \frac{q_2^2\rho_{12}^4(q_1^2\rho_{12}-1)(q_1q_2\rho_{12}-1)}{q_1(q_2-1)(q_1-q_2^2)(\rho_{12}-1)(q_1\rho_{12}-1)(q_1^2\rho_{12}-q_2)(q_1-q_2\rho_{12})}$$

Adding all above expressions confirms identity (5.19) *in this case.*

References

[1] M. Aganagic and C. Vafa. Mirror symmetry, D-branes and counting holomorphic discs. 2000. hep-th/0012041.

[2] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa, and H. Verlinde. Loop and surface operators in N=2 gauge theory and Liouville modular geometry. *JHEP*, 01:113, 2010.

[3] L. F. Alday and Y. Tachikawa. Affine SL(2) conformal blocks from 4d gauge theories. 2010. arXiv:1005.4469.

[4] H. Awata, H. Fuji, H. Kanno, M. Manabe, and Y. Yamada. Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String. 2010. arXiv:1008.0574.

[5] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini. Multi-instanton calculus and equivariant cohomology. *JHEP*, 05:054, 2003.

[6] U. Bruzzo and D. Markushevich. Moduli of framed sheaves on projective surfaces. arXiv:math/0906.1436.

[7] D. E. Diaconescu. Moduli of ADHM sheaves and local Donaldson-Thomas theory. arXiv.org:0801.0820.

[8] D.-E. Diaconescu, J. Gomis, and M. R. Douglas. Fractional branes and wrapped branes. *J. High Energy Phys.*, (2):Paper 13, 9 pp. (electronic), 1998.

[9] T. Dimofte, S. Gukov, and L. Hollands. Vortex Counting and Lagrangian 3-manifolds. 2010. arXiv:1006.0977.

[10] M. R. Douglas. Gauge Fields and D-branes. *J. Geom. Phys.*, 28:255–262, 1998.

[11] M. R. Douglas and G. W. Moore. D-branes, Quivers, and ALE Instantons. 1996. hep-th/9603167.

[12] M. Finkelberg, D. Gaitsgory, and A. Kuznetsov. Uhlenbeck spaces for $\mathbb{A}^2$ and affine Lie algebra $\widehat{sl}_n$. *Publ. Res. Inst. Math. Sci.*, 39(4):721–766, 2003.

[13] S. Gukov, A. Iqbal, C. Kozcaz, and C. Vafa. Link homologies and the refined topological vertex. *Commun. Math. Phys.*, 298:757–785, 2010.

[14] S. Gukov and E. Witten. Gauge theory, ramification, and the geometric Langlands program. In *Current developments in mathematics, 2006*, pages 35–180. Int. Press, Somerville, MA, 2008.

[15] A. Henni. Monads for torsion-free sheaves on multi-blow-ups of the projective plane. 2009. arXiv:0903.3190.

[16] D. Huybrechts and M. Lehn. Framed modules and their moduli. *Internat. J. Math.*, 6(2):297–324, 1995.

[17] D. Huybrechts and M. Lehn. Stable pairs on curves and surfaces. *J. Algebraic Geom.*, 4(1):67–104, 1995.

[18] A. Iqbal, C. Kozcaz, and C. Vafa. The refined topological vertex. *JHEP*, 10:069, 2009.

[19] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.

[20] C. Kozcaz, S. Pasquetti, F. Passerini, and N. Wyllard. Affine sl(N) conformal blocks from N=2 SU(N) gauge theories. 2010. arXiv:1008.1412.

[21] C. Kozcaz, S. Pasquetti, and N. Wyllard. A and B model approaches to surface operators and Toda theories. *JHEP*, 08:042, 2010. arXiv:1004.2025.

[22] T. Mochizuki. The geometry of the parabolic Hilbert scheme. arXiv:math/0210210.

[23] T. Mochizuki. A theory of the invariants obtained from the moduli stacks of stable objects on a smooth polarized surface. arXiv:math/0210211.

[24] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[25] H. Nakajima and K. Yoshioka. Instanton counting on blowup. I. 4-dimensional pure gauge theory. *Invent. Math.*, 162(2):313–355, 2005.

[26] N. A. Nekrasov. Seiberg-Witten Prepotential From Instanton Counting. *Adv. Theor. Math. Phys.*, 7:831–864, 2004.

[27] A. Okounkov, N. Reshetikhin, and C. Vafa. Quantum Calabi-Yau and Classical Crystals. hep-th/0309208.

[28] Y. Tachikawa. Five-dimensional Chern-Simons terms and Nekrasov's instanton counting. *JHEP*, 02:050, 2004.
[29] E. Witten. Phases of N = 2 theories in two dimensions. *Nucl. Phys.*, B403:159–222, 1993.
[30] E. Witten. Sigma models and the ADHM construction of instantons. *J. Geom. Phys.*, 15:215–226, 1995.

[1] Department of Mathematics, University of Pennsylvania, David Rittenhouse Laboratory, 209 South 33rd Street, Philadelphia, PA 19104, USA[†]

[2] Istituto Nazionale di Fisica Nucleare, Sezione di Trieste

[3] Department of Mathematics, National Taiwan University, Taipei, Taiwan

[4] NHETC, Rutgers University, Piscataway, NJ 08854-0849 USA

[5] IMECC - UNICAMP Department of Mathematics Rua Sérgio Buarque de Holanda, 651 Baro Geraldo, Campinas, SP - Brazil CEP 13083-859

[†] On leave of absence from International School for Advanced Studies, Trieste, Italy