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note finali coverpage

(Article begins on next page)

# INTRODUCTION TO FOURIER-MUKAI AND NAHM TRANSFORMS WITH AN APPLICATION TO COHERENT SYSTEMS ON ELLIPTIC CURVES

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These notes record, in a slightly expanded way, the lectures given by the first two authors at the School on Moduli Spaces of Vector Bundles that took place at CIMAT in Guanajuato, Mexico, from November 27th to December 8th, 2006. The School, together with the ensuing workshop on the same topic, was held in occasion of Peter Newstead's 65th anniversary. It has been a great pleasure and a privilege to contribute to celebrate Peter's outstanding achievements in algebraic geometry and his lifelong dedication to the progress of mathematical knowledge. We warmly thank the organizers of the school and workshop for inviting us, thus allowing us to participate in Peter's celebration.

The main emphasis in these notes is on the Fourier-Mukai transforms as equivalences of derived categories of coherent sheaves on algebraic varieties. For this reason, the first Section is devoted to a basic (but we hope, understandable) introduction to derived categories. In the second Section we develop the basic theory of Fourier-Mukai transforms.

Another aim of our lectures was to outline the relations between Fourier-Mukai and Nahm transforms. This is the topic of Section 3. Finally, Section 4 is devoted to the application of the theory of Fourier-Mukai transforms to the study of coherent systems.

This is a review paper. Most of the material is taken from [1] and [30], although the presentation is different in some places. We refer the reader to those works for further details and for a systematic treatment.

## 1. DERIVED CATEGORIES

**Introduction.** We start with an introduction to derived categories, especially in connection with Fourier-Mukai transforms. A more comprehensive treatment may be found in [1]. As a witness to the relevance of derived categories in this theory one may mention the title of the paper where Mukai introduced the transform now known as Fourier-Mukai's: "Duality between  $D(X)$  and  $D(\hat{X})$  with its application to Picard sheaves" [43].

Let us consider the original Mukai transform from a naive point of view. Let  $X$  be a complex abelian variety and  $\mathcal{E}$  a vector bundle on  $X$ . We consider here only algebraic (or holomorphic) vector bundles, so we can also think of  $\mathcal{E}$  as a smooth hermitian bundle  $E$  endowed with an hermitian connection  $\nabla$  which is compatible with the complex structure. We fix an index  $i$  and look for the various cohomology spaces  $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$  where  $\mathcal{P}_\xi$  varies in the space  $\hat{X}$  of all flat line bundles on  $X$  (the dual abelian variety of  $X$ ). A natural question is whether the collection of vector spaces  $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$  define a vector bundle on  $\hat{X}$ . In some cases this happens, for instance if one has  $H^j(X, \mathcal{E} \otimes \mathcal{P}_\xi) = 0$  for any  $j \neq i$  and  $\xi \in \hat{X}$ .

In general one cannot expect to be so lucky, and such a vector bundle (or more generally sheaf) may not exist. What one can do is to mimic the construction of the cohomology groups to get objects that play a similar role. On the product  $X \times \hat{X}$  there is a *universal* line bundle  $\mathcal{P}$ , called the Poincaré bundle, whose restriction to the fibre  $\hat{\pi}^{-1}(\xi)$  over  $\xi$  of the projection  $\hat{\pi}: X \times \hat{X} \rightarrow \hat{X}$  is the line bundle  $\mathcal{P}_\xi$ ; we normalise  $\mathcal{P}$  so that it restricts to the trivial line bundle on the fibre of the origin  $x_0$  of  $X$  for the other projection  $\pi: X \times \hat{X} \rightarrow X$ . In analogy with the construction of the cohomology groups of a sheaf, we take the sheaf  $\mathcal{F} = \pi^*\mathcal{E} \otimes \mathcal{P}$  (whose restriction to  $\hat{\pi}^{-1}(\xi)$  is precisely  $\mathcal{E} \otimes \mathcal{P}_\xi$ ), a resolution

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{R}^0 \rightarrow \mathcal{R}^1 \rightarrow \cdots \rightarrow \mathcal{R}^n \rightarrow \cdots$$

by injective sheaves, and define the *higher direct images* of  $\mathcal{F}$  under  $\hat{\pi}$  as the cohomology sheaves  $R^i\pi_*\mathcal{F} = \mathcal{H}^i(\pi_*\mathcal{R}^\bullet)$  of the complex

$$0 \rightarrow \pi_*\mathcal{F} \rightarrow \pi_*\mathcal{R}^0 \rightarrow \pi_*\mathcal{R}^1 \rightarrow \cdots \rightarrow \pi_*\mathcal{R}^n \rightarrow \cdots$$

The relationship between the sheaves  $R^i\pi_*(\pi^*\mathcal{E} \otimes \mathcal{P})$  and the cohomology groups  $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$  is given by some “cohomology base change” theorems [26, III.12]. This shows that the sheaves  $R^i\pi_*(\pi^*\mathcal{E} \otimes \mathcal{P})$  encode more information than the cohomology groups of the fibres. Another classical fact is that the higher direct images are independent of the resolution  $\mathcal{R}^\bullet$  of  $\mathcal{F}$ , that is, if  $0 \rightarrow \mathcal{F} \rightarrow \tilde{\mathcal{R}}^\bullet$  is another acyclic resolution of  $\mathcal{F}$  (meaning that the higher direct images  $R^i\hat{\pi}^i\tilde{\mathcal{R}}^j$  of the sheaves  $\tilde{\mathcal{R}}^j$  are zero for every  $i > 0, j \geq 0$ ), then the complexes of sheaves  $\pi_*\mathcal{R}^\bullet$  and  $\pi_*\tilde{\mathcal{R}}^\bullet$  have the same cohomology sheaves. If we identify two complexes of sheaves when they have the same cohomology sheaves (we say that they are *quasi-isomorphic*), and write  $\mathbf{R}\hat{\pi}_*\mathcal{F}$  for the “class” of any of the complexes  $\pi_*\mathcal{R}^\bullet$ , the information about the cohomology groups  $H^i(X, \mathcal{E} \otimes \mathcal{P}_\xi)$  is encoded in the single object  $\Phi(\mathcal{E}) = \mathbf{R}\hat{\pi}_*\mathcal{F} = \mathbf{R}\hat{\pi}_*(\pi^*\mathcal{E} \otimes \mathcal{P})$ .

To make good sense of all this, we need to construct, out of any abelian category, another category, which is called the *derived category*, where quasi-isomorphic complexes become isomorphic and we can define “derived functors” (such as  $\mathbf{R}\hat{\pi}_*$ ) and also some derived versions of the pullback functor  $\pi^*$  and of the tensor product.

**1.1. Categories of complexes.** A *complex*  $(\mathcal{K}^\bullet, d_{\mathcal{K}^\bullet})$  in an abelian category  $\mathfrak{A}$  is a sequence

$$\cdots \rightarrow \mathcal{K}^{n-1} \xrightarrow{d^{n-1}} \mathcal{K}^n \xrightarrow{d^n} \mathcal{K}^{n+1} \rightarrow \cdots$$

of morphisms in  $\mathfrak{A}$  such that  $d^{n+1} \circ d^n = 0$  for all  $n \in \mathbb{Z}$ . The family of morphisms  $d_{\mathcal{K}^\bullet}$  is called the *differential* of the complex  $\mathcal{K}^\bullet$ .

The *category of complexes*  $\mathbf{C}(\mathfrak{A})$  is the category whose objects are complexes  $(\mathcal{K}^\bullet, d_{\mathcal{K}^\bullet})$  in  $\mathfrak{A}$  and whose morphisms  $f: (\mathcal{K}^\bullet, d_{\mathcal{K}^\bullet}) \rightarrow (\mathcal{L}^\bullet, d_{\mathcal{L}^\bullet})$  are collections of morphisms  $f^n: \mathcal{K}^n \rightarrow \mathcal{L}^n$ ,  $n \in \mathbb{Z}$ , in  $\mathfrak{A}$  such that the diagrams

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{K}^{n-1} & \xrightarrow{d^{n-1}} & \mathcal{K}^n & \xrightarrow{d^n} & \mathcal{K}^{n+1} \xrightarrow{d^{n+1}} \cdots \\ & & \downarrow f^{n-1} & & \downarrow f^n & & \downarrow f^{n+1} \\ \cdots & \longrightarrow & \mathcal{L}^{n-1} & \xrightarrow{d^{n-1}} & \mathcal{L}^n & \xrightarrow{d^n} & \mathcal{L}^{n+1} \xrightarrow{d^{n+1}} \cdots \end{array}$$

commute.

The direct sum  $\mathcal{K}^\bullet \oplus \mathcal{L}^\bullet$  of two complexes  $\mathcal{K}^\bullet$  and  $\mathcal{L}^\bullet$  is defined in the obvious way. One can also describe in a natural way the kernel and the cokernel of a morphism of complexes, and readily check that the category  $\mathbf{C}(\mathfrak{A})$  of complexes of an abelian category is abelian as well.

We can also define the complex of homomorphisms  $\text{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)$  by setting

$$\text{Hom}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)^n = \prod_i \text{Hom}_{\mathfrak{A}}(\mathcal{K}^i, \mathcal{L}^{i+n})$$

for each  $n \in \mathbb{Z}$ , together with a differential defined by

$$(1.1) \quad \begin{aligned} d^n: \text{Hom}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)^n &\rightarrow \text{Hom}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)^{n+1} \\ f^i &\mapsto d_{\mathcal{L}^\bullet}^{i+n} \circ f^i + (-1)^{n+1} f^{i+1} \circ d_{\mathcal{K}^\bullet}^i. \end{aligned}$$

When  $\mathfrak{A}$  has tensor products and arbitrary direct sums, we can define the tensor product of complexes by letting  $(\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet)^n = \bigoplus_{p+q=n} (\mathcal{K}^p \otimes \mathcal{L}^q)$  with the differential  $d_{\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet} = d_{\mathcal{K}^\bullet} \otimes \text{Id} + (-1)^p \text{Id} \otimes d_{\mathcal{L}^\bullet}$  over  $\mathcal{K}^p \otimes \mathcal{L}^q$ . If  $\mathfrak{A}$  has tensor products but not arbitrary direct sums,  $\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet$  is defined whenever for every  $n$  there are only a finite number of summands in  $\bigoplus_{p+q=n} (\mathcal{K}^p \otimes \mathcal{L}^q)$ .

The shift  $\mathcal{K}^\bullet[n]$  of a complex  $\mathcal{K}^\bullet$  by an integer number  $n$ , is defined by setting  $\mathcal{K}[n]^p = \mathcal{K}^{p+n}$  with the differential  $d_{\mathcal{K}^\bullet[n]} = (-1)^n d_{\mathcal{K}^\bullet}$ . A morphism of complexes  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  induces another morphism of complexes  $f[n]: \mathcal{K}^\bullet[n] \rightarrow \mathcal{L}^\bullet[n]$  given by  $f[n]^p = f^{p+n}$ . In this way,  $\mathcal{K}^\bullet \mapsto \mathcal{K}^\bullet[n]$  is an additive functor. Sometimes we shall denote  $\tau(\mathcal{K}^\bullet) = \mathcal{K}^\bullet[1]$ , so that  $\tau^n(\mathcal{K}^\bullet) = \mathcal{K}^\bullet[n]$  for any integer  $n$ . The  $n$ -th cohomology of a complex  $\mathcal{K}^\bullet$  is the object

$$\mathcal{H}^n(\mathcal{K}^\bullet) = \ker d^n / \text{Im } d^{n-1}.$$

We say that  $\mathcal{Z}^n(\mathcal{K}^\bullet) = \ker d^n$  are the  $n$ -cycles of  $\mathcal{K}^\bullet$  and  $\mathcal{B}^n(\mathcal{K}^\bullet) = \text{Im } d^{n-1}$  are the  $n$ -boundaries. A morphism of complexes  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  maps cycles to cycles and boundaries to boundaries, so that it yields for every  $n$  a morphism

$$\mathcal{H}^n(f): \mathcal{H}^n(\mathcal{K}^\bullet) \rightarrow \mathcal{H}^n(\mathcal{L}^\bullet).$$

One has  $\mathcal{H}^n(\mathcal{K}^\bullet[m]) \simeq \mathcal{H}^{n+m}(\mathcal{K}^\bullet)$  and  $\mathcal{H}^n(f[m]) \simeq \mathcal{H}^{n+m}(f)$ .

A complex  $\mathcal{K}^\bullet$  is said to be *acyclic* or *exact* if  $\mathcal{H}(\mathcal{K}^\bullet) = 0$ ; a morphism of complexes  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  is a *quasi-isomorphism* if  $\mathcal{H}(f): \mathcal{H}(\mathcal{K}^\bullet) \rightarrow \mathcal{H}(\mathcal{L}^\bullet)$  is an isomorphism. The composition of two quasi-isomorphisms is a quasi-isomorphism.

A morphism of complexes  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  is *homotopic to zero* if there is a collection of morphisms  $h^n: \mathcal{K}^n \rightarrow \mathcal{L}^{n-1}$  (a homotopy) such that  $f^n = h^{n+1} \circ d_{\mathcal{K}^\bullet}^n + d_{\mathcal{L}^\bullet}^{n-1} \circ h^n$  for every  $n$ . A complex  $\mathcal{K}^\bullet$  is *homotopic to zero* if its identity morphism is homotopic to zero. Two morphisms  $f, g: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  are *homotopic* if  $f - g$  is homotopic to zero.

The sum of two morphisms homotopic to zero is homotopic to zero as well. Furthermore,  $f \circ g$  is homotopic to zero if either  $f$  or  $g$  is homotopic to zero.

The *homotopy category*  $K(\mathfrak{A})$  is the category whose objects are the objects of  $\mathbf{C}(\mathfrak{A})$  and whose morphisms are

$$\text{Hom}_{K(\mathfrak{A})}(\mathcal{K}^\bullet, \mathcal{L}^\bullet) = \text{Hom}_{\mathbf{C}(\mathfrak{A})}(\mathcal{K}^\bullet, \mathcal{L}^\bullet) / \text{Ht}(\mathcal{K}^\bullet, \mathcal{L}^\bullet),$$

where  $\text{Ht}(\mathcal{K}^\bullet, \mathcal{L}^\bullet)$  is the set of morphisms which are homotopic to zero.

One can see from Equation (1.1) that the  $n$ -cycles of the complex of homomorphisms  $\text{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)$  are the morphisms of complexes  $\mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet[n]$ , while the  $n$ -boundaries are the morphisms homotopic to zero. Thus,

$$\mathcal{H}^n(\text{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) = \text{Hom}_{K(\mathfrak{A})}(\mathcal{K}^\bullet, \mathcal{L}^\bullet[n]).$$

If  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  is homotopic to zero, then  $\mathcal{H}(f) = 0$ ; hence, *two homotopic morphisms induce the same morphism in cohomology and a complex  $\mathcal{K}^\bullet$  which is homotopic to zero is acyclic.*

The homotopy category  $K(\mathfrak{A})$  does not have kernels nor cokernels. This can be overcome by introducing the notion of cone of a morphism.

**Definition 1.1.** The cone of a morphism of complexes  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  is the complex  $\text{Cone}(f)$  such that  $\text{Cone}(f)^n = \mathcal{K}^{n+1} \oplus \mathcal{L}^n$ , equipped with differential

$$d_{\text{Cone}(f)}^n = \begin{pmatrix} -d_{\mathcal{K}^\bullet}^{n+1} & 0 \\ f^{n+1} & d_{\mathcal{L}^\bullet}^n \end{pmatrix}$$

△

$\text{Cone}(f)$  is not isomorphic to the direct sum  $\mathcal{K}^\bullet[1] \oplus \mathcal{L}^\bullet$  because it has another differential. There are functorial morphisms  $\beta: \text{Cone}(f) \rightarrow \mathcal{K}^\bullet[1]$ ,  $(k, l) \mapsto k$ , and  $\alpha: \mathcal{L}^\bullet \rightarrow \text{Cone}(f)$ ,  $l \mapsto (0, l)$ .

The sequence

$$\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \text{Cone } f \xrightarrow{\beta} \mathcal{K}^\bullet[1],$$

in  $K(\mathfrak{A})$  is called a *distinguished (or exact) triangle* in  $K(\mathfrak{A})$  and is also written in the form

$$\begin{array}{ccc} \mathcal{K}^\bullet & \xrightarrow{f} & \mathcal{L}^\bullet \\ & \searrow \beta & \swarrow \alpha \\ & \text{Cone } f & \end{array}$$

where the dashed arrow stands for a morphism  $\text{Cone } f \rightarrow \mathcal{K}^\bullet[1]$ . Notice that  $\alpha \circ f = 0$  and  $\beta \circ \alpha = 0$ .

**Proposition 1.2.** *Given an exact triangle  $\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \text{Cone } f \xrightarrow{\beta} \mathcal{K}^\bullet[1]$  in  $K(\mathfrak{A})$ , for every integer  $n$  there is an exact sequence of cohomology groups*

$$\mathcal{H}^n(\mathcal{K}^\bullet) \xrightarrow{\mathcal{H}^n(f)} \mathcal{H}^n(\mathcal{L}^\bullet) \xrightarrow{\mathcal{H}^n(\alpha)} \mathcal{H}^n(\text{Cone } f) \xrightarrow{\mathcal{H}^n(\beta)} \mathcal{H}^n(\mathcal{K}^\bullet[1]) \simeq \mathcal{H}^{n+1}(\mathcal{K}^\bullet).$$

□

One also obtains the so-called *cohomology long exact sequence*:

$$\dots \xrightarrow{\mathcal{H}^{n-1}(\beta)} \mathcal{H}^n(\mathcal{K}^\bullet) \xrightarrow{\mathcal{H}^n(f)} \mathcal{H}^n(\mathcal{L}^\bullet) \xrightarrow{\mathcal{H}^n(\alpha)} \mathcal{H}^n(\text{Cone } f) \xrightarrow{\mathcal{H}^n(\beta)} \mathcal{H}^{n+1}(\mathcal{K}^\bullet) \dots$$

Proposition 1.2 tells us that the functors  $\mathcal{H}^n: K(\mathfrak{A}) \rightarrow \mathfrak{A}$  are cohomological. This means the following: if  $\mathfrak{A}, \mathfrak{B}$  are abelian categories, an additive functor  $F: K(\mathfrak{A}) \rightarrow \mathfrak{B}$  is *cohomological* if for every exact triangle  $\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \text{Cone } f \xrightarrow{\beta} \mathcal{K}^\bullet[1]$  the sequence  $F(\mathcal{K}^\bullet) \xrightarrow{F(f)} F(\mathcal{L}^\bullet) \xrightarrow{F(\alpha)} F(\text{Cone } f) \xrightarrow{F(\beta)} F(\mathcal{K}^\bullet[1])$  is exact.

**Corollary 1.3.** *A morphism of complexes  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  is a quasi-isomorphism if and only if  $\text{Cone}(f)$  is acyclic.* □

If  $0 \rightarrow \mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{g} \mathcal{N}^\bullet \rightarrow 0$  is an exact sequence of complexes (in  $\mathbf{C}(\mathfrak{A})$ ), then there is a morphism of complexes  $\text{Cone}(f) \rightarrow \mathcal{N}^\bullet$  defined in degree  $n$  by  $(a_{n+1}, b_n) \in \mathcal{K}^{n+1} \oplus \mathcal{L}^n \mapsto g(b_n) \in \mathcal{N}^n$ . One easily checks that it is a quasi-isomorphism. Combining this with the cohomology long exact sequence we obtain the more customary form of the latter, i.e., there exist functorial morphisms  $\delta^n: H^n(\mathcal{N}^\bullet) \rightarrow H^{n+1}(\mathcal{L}^\bullet)$  such that one has an exact sequence

$$\begin{aligned} \dots \xrightarrow{\delta^{n-1}} H^n(\mathcal{L}^\bullet) \rightarrow H^n(\mathcal{M}^\bullet) \rightarrow H^n(\mathcal{N}^\bullet) \xrightarrow{\delta^n} H^{n+1}(\mathcal{L}^\bullet) \rightarrow \\ \rightarrow H^{n+1}(\mathcal{M}^\bullet) \rightarrow H^{n+1}(\mathcal{N}^\bullet) \xrightarrow{\delta^{n+1}} \dots \end{aligned}$$

**1.2. Derived Category.** In our route toward the definition of a category where quasi-isomorphic complexes are actually isomorphic, we have first identified homotopic morphisms, and then have moved from the category of complexes  $\mathbf{C}(\mathfrak{A})$  to the homotopy category  $K(\mathfrak{A})$ . A second step is to “localise” by quasi-isomorphisms. This localisation is a fraction calculus for categories. Recall that given a ring  $A$  (e.g., the integer numbers) and  $S \subset A$  which is a multiplicative system (that is, it contains the unity and is closed under products), then one can define the localised ring  $S^{-1}A$ ; elements in  $S^{-1}A$  are equivalence classes  $a/s$  of pairs  $(a, s) \in A \times S$  where  $(a, s) \sim (a', s')$  (or  $a/s = a'/s'$ ) if there exists  $t \in S$  such that  $t(as' - a's) = 0$ . The elements  $s \in S$  become invertible in  $S^{-1}A$  because  $s/1 \cdot 1/s = 1$ . One can proceed in a similar way with morphisms of complexes, since quasi-isomorphisms verify the conditions for being a multiplicative system; the identity is a quasi-isomorphism and the composition of two quasi-isomorphisms is a quasi-isomorphism. We can then define a “fraction”  $f/\phi$  as a diagram of (homotopy classes of) morphisms of complexes

$$\begin{array}{ccc} & \mathcal{R}^\bullet & \\ \phi \swarrow & & \searrow f \\ \mathcal{K}^\bullet & & \mathcal{L}^\bullet \end{array}$$

where  $\phi$  is a quasi-isomorphism. Two diagrams  $f/\phi$  and  $g/\psi$  of the same type are said to be equivalent if there are quasi-isomorphisms  $\mathcal{R}^\bullet \leftarrow \mathcal{T}^\bullet \rightarrow \mathcal{S}^\bullet$  such that the diagram

$$\begin{array}{ccccc} & & \mathcal{T}^\bullet & & \\ & \swarrow & & \searrow & \\ \phi \swarrow & \mathcal{R}^\bullet & & \mathcal{S}^\bullet & \searrow g \\ \mathcal{K}^\bullet & \swarrow \psi & & \searrow f & \mathcal{L}^\bullet \end{array}$$

commutes in  $K(\mathfrak{A})$ . Equivalence of fractions is actually an equivalence relation; this follows from the next Proposition, whose proof is based on the properties of the cone of a morphism.

**Proposition 1.4.** *Given morphisms of complexes  $\mathcal{M}^\bullet \xrightarrow{f} \mathcal{N}^\bullet \xleftarrow{g} \mathcal{R}^\bullet$  in  $K(\mathfrak{A})$ , there are morphisms of complexes  $\mathcal{M}^\bullet \xleftarrow{g'} \mathcal{Z}^\bullet \xrightarrow{f'} \mathcal{R}^\bullet$  such that the diagram*

$$\begin{array}{ccc} \mathcal{Z}^\bullet & \xrightarrow{f'} & \mathcal{R}^\bullet \\ g' \downarrow & & \downarrow g \\ \mathcal{M}^\bullet & \xrightarrow{f} & \mathcal{N}^\bullet \end{array}$$

is commutative in  $K(\mathfrak{A})$ . Moreover,  $f'$  (respectively,  $g'$ ) is a quasi-isomorphism if and only if  $f$  (respectively,  $g$ ) is so.  $\square$

**Definition 1.5.** The derived category  $D(\mathfrak{A})$  of  $\mathfrak{A}$  is the category with the same objects as  $K(\mathfrak{A})$  (i.e., complexes of objects of  $\mathfrak{A}$ ), and whose morphisms are the equivalence classes  $[f/\phi]$  of diagrams as above.  $\triangle$

In order for this definition to make sense we need to say how to compose morphisms. Given two morphisms  $[f/\phi]$  and  $[g/\psi]$  in  $D(\mathfrak{A})$ , their composition is defined by the

diagram

$$\begin{array}{ccccc}
 & & T^\bullet & & \\
 & \swarrow \psi' & & \searrow f' & \\
 \mathcal{R}^\bullet & & & & \mathcal{S}^\bullet \\
 \swarrow \phi & & f & & \swarrow \psi \\
 \mathcal{K}^\bullet & & \mathcal{L}^\bullet & & \mathcal{M}^\bullet \\
 & & & & \searrow g
 \end{array}$$

which exists by Proposition 1.4. Hence, we set  $[g/\psi] \circ [f/\phi] = [(g \circ f')/(\phi \circ \psi')]$ ; one can readily see that this definition makes sense and that  $D(\mathfrak{A})$  is an additive category.

A morphism  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  in  $K(\mathfrak{A})$  defines the morphism  $f/\text{Id}_{\mathcal{K}^\bullet}: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  in the derived category, which we denote simply by  $f$ . This defines a functor  $K(\mathfrak{A}) \rightarrow D(\mathfrak{A})$  which is additive.

A morphism  $f/\phi: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  in the derived category induces a morphism in cohomology  $\mathcal{H}(f/\phi): \mathcal{H}(\mathcal{K}^\bullet) \xrightarrow{\mathcal{H}(\phi)^{-1}} \mathcal{H}(\mathcal{R}^\bullet) \xrightarrow{\mathcal{H}(f)} \mathcal{H}(\mathcal{L}^\bullet)$ , which is independent of the representative  $f/\phi$  of the class and is compatible with compositions.

**Definition 1.6.** Two complexes  $\mathcal{K}^\bullet$  and  $\mathcal{L}^\bullet$  are quasi-isomorphic if there is a complex  $\mathcal{Z}^\bullet$  with quasi-isomorphisms  $\mathcal{K}^\bullet \leftarrow \mathcal{Z}^\bullet \rightarrow \mathcal{L}^\bullet$ .  $\triangle$

Lemma 1.4 implies that the notion of quasi-isomorphism induces an equivalence relation between complexes. Eventually, we have the result we were looking for:

**Proposition 1.7.** *A morphism of complexes  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$  is a quasi-isomorphism if and only if the induced morphism in the derived category is an isomorphism. Moreover, two complexes are quasi-isomorphic if and only if they are isomorphic in  $D(\mathfrak{A})$ .*  $\square$

**Proposition 1.8.** *Let  $\mathfrak{C}$  be an additive category. An additive functor  $F: K(\mathfrak{A}) \rightarrow \mathfrak{C}$  factors through an additive functor  $D(\mathfrak{A}) \rightarrow \mathfrak{C}$  if and only if it maps quasi-isomorphisms to isomorphisms. If  $\mathfrak{B}$  is an abelian category, an additive functor  $G: K(\mathfrak{A}) \rightarrow K(\mathfrak{B})$  mapping quasi-isomorphisms into quasi-isomorphisms induces an additive functor  $G: D(\mathfrak{A}) \rightarrow D(\mathfrak{B})$  such that the diagram*

$$\begin{array}{ccc}
 \mathbf{C}(\mathfrak{A}) & \xrightarrow{G} & \mathbf{C}(\mathfrak{B}) \\
 \downarrow & & \downarrow \\
 D(\mathfrak{A}) & \xrightarrow{G} & D(\mathfrak{B})
 \end{array}$$

*is commutative.*  $\square$

We can also define derived categories out of some subcategories of  $\mathbf{C}(\mathfrak{A})$ ; the only condition we need is that all the operations we have done can be performed in the new situation. More precisely, we should be able to construct the corresponding homotopy category and to localise by quasi-isomorphisms; to this end one needs to define the cone of a morphism inside the new category. Some examples are the following:

*Example 1.9.* A complex  $\mathcal{K}^\bullet$  is *bounded below* (resp. *bounded above*) if there is an integer  $n_0$  such that  $\mathcal{K}^n = 0$  for all  $n \leq n_0$  (resp.  $n \geq n_0$ ). A complex is *bounded* if it is bounded on both sides.

Bounded below complexes form a category  $\mathbf{C}^+(\mathfrak{A})$ . We can define its homotopy category  $K^+(\mathfrak{A})$  and a “derived” category  $D^+(\mathfrak{A})$  as we did before. By Proposition 1.8, the natural functor  $K^+(\mathfrak{A}) \rightarrow D(\mathfrak{A})$  induces a functor  $\gamma: D^+(\mathfrak{A}) \rightarrow D(\mathfrak{A})$ . The latter is fully faithful, that is,

$$\text{Hom}_{D^+(\mathfrak{A})}(\mathcal{K}^\bullet, \mathcal{M}^\bullet) \simeq \text{Hom}_{D(\mathfrak{A})}(\gamma(\mathcal{K}^\bullet), \gamma(\mathcal{M}^\bullet))$$

for any pair of objects  $\mathcal{K}^\bullet, \mathcal{M}^\bullet$  in  $D^+(\mathfrak{A})$ , and its essential image is the faithful subcategory of  $D(\mathfrak{A})$  consisting of complexes in  $\mathfrak{A}$  with bounded below cohomology. (The essential image of the functor  $\gamma$  is the subcategory of objects which are isomorphic to objects of the form  $\gamma(\mathcal{K}^\bullet)$  for some  $\mathcal{K}^\bullet$  in  $D^+(\mathfrak{A})$ ). One can also define the categories  $\mathbf{C}^-(\mathfrak{A})$  of *bounded above* complexes and  $\mathbf{C}^b(\mathfrak{A})$  of complexes *bounded* on both sides, giving rise to “derived” categories  $D^-(\mathfrak{A})$  and  $D^b(\mathfrak{A})$ . These are characterised as faithful subcategories of  $D(\mathfrak{A})$  as above.  $\triangle$

*Example 1.10.* An abelian subcategory  $\mathfrak{A}'$  of  $\mathfrak{A}$  is *thick* if any extension in  $\mathfrak{A}$  of two objects of  $\mathfrak{A}'$  is also in  $\mathfrak{A}'$ . If  $\mathfrak{A}'$  is a thick abelian subcategory of  $\mathfrak{A}$ , we denote by  $\mathbf{C}_{\mathfrak{A}'}(\mathfrak{A})$  the category of complexes whose cohomology objects are in  $\mathfrak{A}'$ . We can construct its homotopy category  $K_{\mathfrak{A}'}(\mathfrak{A})$  and its derived category  $D_{\mathfrak{A}'}(\mathfrak{A})$ . The functor  $K_{\mathfrak{A}'}(\mathfrak{A}) \rightarrow D(\mathfrak{A})$  induces a fully faithful functor  $D_{\mathfrak{A}'}(\mathfrak{A}) \rightarrow D(\mathfrak{A})$  (cf. Proposition 1.8), whose essential image is the subcategory of  $D(\mathfrak{A})$  whose objects are the complexes with cohomology objects in  $\mathfrak{A}'$ .  $\triangle$

*Example 1.11.* We can also introduce the homotopy categories  $K_{\mathfrak{A}'}^+(\mathfrak{A})$ ,  $K_{\mathfrak{A}'}^-(\mathfrak{A})$  and  $K_{\mathfrak{A}'}^b(\mathfrak{A})$  of complexes bounded below, above and on both sides, respectively, whose cohomology objects are in the subcategory  $\mathfrak{A}'$  of  $\mathfrak{A}$ . The corresponding derived categories  $D_{\mathfrak{A}'}^+(\mathfrak{A})$ ,  $D_{\mathfrak{A}'}^-(\mathfrak{A})$  and  $D_{\mathfrak{A}'}^b(\mathfrak{A})$  can be defined as well.  $\triangle$

Let us write  $\star$  for any of the symbols  $+$ ,  $-$ ,  $b$ , or for no symbol at all. The natural functor  $K^\star(\mathfrak{A}') \rightarrow D(\mathfrak{A})$  maps quasi-isomorphisms to isomorphisms, so that induces a functor  $D^\star(\mathfrak{A}') \rightarrow D_{\mathfrak{A}'}^\star(\mathfrak{A})$ . In general, it may fail to be an equivalence of categories. There are special notations for the derived categories we are most interested in:

- If  $\mathfrak{A}$  is the category of modules over a commutative ring  $A$ , we simply write  $D(A)$ ,  $D^+(A)$ ,  $D^-(A)$ , and  $D^b(A)$ .
- If  $\mathfrak{A} = \mathfrak{Mod}(X)$  is the category of sheaves of  $\mathcal{O}_X$ -modules on an algebraic variety  $X$ , we write  $D(X)$ ,  $D^+(X)$ ,  $D^-(X)$ , and  $D^b(X)$ .
- If  $\mathfrak{A} = \mathfrak{Mod}(X)$  and  $\mathfrak{A}' = \mathfrak{Qco}(X)$  is the category of quasi-coherent sheaves of  $\mathcal{O}_X$ -modules on  $X$ , the derived category  $D_{\mathfrak{A}'}(\mathfrak{A})$  of complexes of  $\mathcal{O}_X$ -modules with quasi-coherent cohomology sheaves is denoted  $D_{qc}(X)$ . In a similar way we have the categories  $D_{qc}^+(X)$ ,  $D_{qc}^-(X)$  and  $D_{qc}^b(X)$ .
- If  $\mathfrak{A} = \mathfrak{Mod}(X)$  and  $\mathfrak{A}' = \mathfrak{Coh}(X)$  is the category of coherent sheaves of  $\mathcal{O}_X$ -modules on  $X$ , the derived category  $D_{\mathfrak{A}'}(\mathfrak{A})$  of complexes of  $\mathcal{O}_X$ -modules with coherent cohomology sheaves is denoted  $D_c(X)$ . One also has the derived categories  $D_c^+(X)$ ,  $D_c^-(X)$  and  $D_c^b(X)$ .
- If  $\mathfrak{A} = \mathfrak{Qco}(X)$  and  $\mathfrak{A}' = \mathfrak{Coh}(X)$ , we have the derived categories  $D_c(\mathfrak{Qco}(X))$ ,  $D_c^+(\mathfrak{Qco}(X))$ ,  $D_c^-(\mathfrak{Qco}(X))$  and  $D_c^b(\mathfrak{Qco}(X))$ .

One has equivalences of categories  $D_{qc}^+(X) \simeq D^+(\mathfrak{Qco}(X))$  and  $D_{qc}^b(X) \simeq D^b(\mathfrak{Qco}(X))$ . The first equivalence is a consequence of the fact that every quasi-coherent sheaf on an algebraic variety can be embedded as a subsheaf of an *injective* quasi-coherent sheaf. One also has  $D_c^+(X) \simeq D_c^+(\mathfrak{Qco}(X))$  and  $D_c^b(X) \simeq D_c^b(\mathfrak{Qco}(X))$ . When  $X$  is smooth, the same is true for unbounded complexes as well, so that  $D_{qc}^\star(X) \simeq D^\star(\mathfrak{Qco}(X))$  and  $D^\star(\mathfrak{Coh}(X)) \simeq D_c^\star(\mathfrak{Qco}(X)) \simeq D_c^\star(X)$  for any value of  $\star$ .

**1.2.1. The derived category as a triangulated category.** Derived categories are examples of triangulated categories. We shall not give here a formal definition but shall just point out some of the features of the derived category that make it into a triangulated category.



The first is the existence of a shift functor  $\tau: D(\mathfrak{A}) \rightarrow D(\mathfrak{A})$ ,  $\tau(\mathcal{K}^\bullet) = \mathcal{K}^\bullet[1]$ , which is an equivalence of categories. The second is the existence of “triangles”, and among them a class of “distinguished triangles” satisfying some properties we do not describe here. A *triangle* in  $D(\mathfrak{A})$  is a sequence of morphisms

$$\mathcal{K}^\bullet \xrightarrow{u} \mathcal{L}^\bullet \xrightarrow{v} \mathcal{M}^\bullet \xrightarrow{w} \mathcal{K}^\bullet[1]$$

which we also write in the form

$$\begin{array}{ccc} \mathcal{K}^\bullet & \xrightarrow{u} & \mathcal{L}^\bullet \\ & \swarrow w \quad \searrow v & \\ & \mathcal{M}^\bullet & \end{array}$$

where the dashed arrow stands for the morphism  $\mathcal{M}^\bullet \xrightarrow{w} \mathcal{K}^\bullet[1]$ . A morphism of triangles is defined in the obvious way, and we say that a triangle is *distinguished* or *exact* if it is isomorphic to the triangle defined by the cone of a morphism  $f: \mathcal{K}^\bullet \rightarrow \mathcal{L}^\bullet$ , which is the triangle  $\mathcal{K}^\bullet \xrightarrow{f} \mathcal{L}^\bullet \xrightarrow{\alpha} \text{Cone}(f) \xrightarrow{\beta} \mathcal{K}^\bullet[1]$ . From Proposition 1.2, an exact triangle in  $D(\mathfrak{A})$  induces a long exact sequence in cohomology

$$\begin{aligned} \dots \rightarrow \mathcal{H}^i(\mathcal{A}^\bullet) \xrightarrow{\mathcal{H}^i(u)} \mathcal{H}^i(\mathcal{B}^\bullet) \xrightarrow{\mathcal{H}^i(v)} \mathcal{H}^i(\mathcal{C}^\bullet) \xrightarrow{\mathcal{H}^i(w)} \\ \mathcal{H}^{i+1}(\mathcal{A}^\bullet) \xrightarrow{\mathcal{H}^{i+1}(u)} \mathcal{H}^{i+1}(\mathcal{B}^\bullet) \xrightarrow{\mathcal{H}^{i+1}(v)} \mathcal{H}^{i+1}(\mathcal{C}^\bullet) \xrightarrow{\mathcal{H}^{i+1}(w)} \dots \end{aligned}$$

**Definition 1.12.** If  $\mathfrak{B}$  is another abelian category, an additive functor  $F: D_{\mathfrak{A}}^*(\mathfrak{A}) \rightarrow D(\mathfrak{B})$  is said to be *exact* if it commutes with the shift functor,  $F(\mathcal{K}^\bullet[1]) \simeq F(\mathcal{K}^\bullet)[1]$ , and maps exact triangles to exact triangles.  $\triangle$

Then, for any exact triangle  $\mathcal{K}^\bullet \xrightarrow{u} \mathcal{L}^\bullet \xrightarrow{v} \mathcal{M}^\bullet \xrightarrow{w} \mathcal{K}^\bullet[1]$  we have a long exact sequence

$$\begin{aligned} \dots \rightarrow \mathcal{H}^i(F(\mathcal{K}^\bullet)) \rightarrow \mathcal{H}^i(F(\mathcal{L}^\bullet)) \rightarrow \mathcal{H}^i(F(\mathcal{M}^\bullet)) \rightarrow \\ \mathcal{H}^{i+1}(F(\mathcal{K}^\bullet)) \rightarrow \mathcal{H}^{i+1}(F(\mathcal{L}^\bullet)) \rightarrow \mathcal{H}^{i+1}(F(\mathcal{M}^\bullet)) \rightarrow \dots \end{aligned}$$

**1.3. Derived Functors.** The cohomology groups of a sheaf  $\mathcal{F}$  on an algebraic variety  $X$  are the cohomology objects of the complex of global sections  $\Gamma(X, \mathcal{I}^\bullet)$  of a resolution  $\mathcal{I}^\bullet$  of  $\mathcal{F}$  by injective sheaves; the resulting groups do not depend on the injective resolution, due to a result known as abstract de Rham theorem. In order to define derived functors on the derived category we shall mimic this construction. Let  $\mathfrak{A}$  be an abelian category with enough injectives. Thus, any object  $\mathcal{M}$  in  $\mathfrak{A}$  has an injective resolution  $\mathcal{M} \rightarrow I^0(\mathcal{M}) \rightarrow I^1(\mathcal{M}) \rightarrow \dots$  which can be chosen to be functorial in  $\mathcal{M}$ . One can prove (by using bicomplexes, a notion we have not introduced in these notes) that for any complex  $\mathcal{M}^\bullet$  there is a complex of injective objects  $I(\mathcal{M}^\bullet)$  and a quasi-isomorphism

$$\mathcal{M}^\bullet \rightarrow I(\mathcal{M}^\bullet),$$

which defines a functor  $I: K(\mathfrak{A}) \rightarrow K(\mathfrak{A})$ . Let  $\mathfrak{B}$  be another abelian category and  $F: \mathfrak{A} \rightarrow \mathfrak{B}$  a left-exact functor. Then  $F$  induces a functor  $\mathbf{R}F: K^+(\mathfrak{A}) \rightarrow D^+(\mathfrak{B})$  defined by  $\mathbf{R}F(\mathcal{M}^\bullet) = F(I(\mathcal{M}^\bullet))$ . Moreover, if  $\mathcal{J}^\bullet$  is an acyclic complex of injective objects then  $F(\mathcal{J}^\bullet)$  is acyclic, because  $\mathcal{J}^\bullet$  splits. This implies that  $\mathbf{R}F$  maps quasi-isomorphisms to isomorphisms and thus (cf. Proposition 1.8) yields a functor

$$\mathbf{R}F: D^+(\mathfrak{A}) \rightarrow D^+(\mathfrak{B}),$$

which is the *right derived functor* of  $F$ .

We can also derive on the right functors from  $K(\mathfrak{A})$  to  $K(\mathfrak{B})$  that are not induced by a left-exact functor. We shall give some examples in Subsection 1.3.3.

As it is customary for the “classical” right  $i$ -th derived functor of  $F$ , we use the notation  $\mathbf{R}^i F(\mathcal{M}^\bullet) = H^i(\mathbf{R}F(\mathcal{M}^\bullet))$ . The right derived functor  $\mathbf{R}F$  is exact. In particular, an exact triangle in  $K(\mathfrak{A})$ ,  $\mathcal{M}'^\bullet \rightarrow \mathcal{M}^\bullet \rightarrow \mathcal{M}''^\bullet \rightarrow \mathcal{M}'^\bullet[1]$  induces a long exact sequence

$$\begin{aligned} \cdots \rightarrow \mathbf{R}^i F(\mathcal{M}'^\bullet) \rightarrow \mathbf{R}^i F(\mathcal{M}^\bullet) \rightarrow \mathbf{R}^i F(\mathcal{M}''^\bullet) \rightarrow \\ \mathbf{R}^{i+1} F(\mathcal{M}'^\bullet) \rightarrow \mathbf{R}^{i+1} F(\mathcal{M}^\bullet) \rightarrow \mathbf{R}^{i+1} F(\mathcal{M}''^\bullet) \rightarrow \cdots \end{aligned}$$

For any bounded below complex  $\mathcal{M}^\bullet$  there is a natural morphism  $F(\mathcal{M}^\bullet) \rightarrow \mathbf{R}F(\mathcal{M}^\bullet)$  in the derived category.  $\mathcal{M}^\bullet$  is said to be *F-acyclic* if this morphism is an isomorphism in  $D^+(\mathfrak{B})$ .

The right derived functor  $\mathbf{R}F$  satisfies a version of the de Rham theorem, namely, if a complex  $\mathcal{M}^\bullet$  is isomorphic in the derived category  $D^+(\mathfrak{A})$  to an *F-acyclic* complex  $\mathcal{J}^\bullet$ , then  $\mathbf{R}F(\mathcal{M}^\bullet) \simeq F(\mathcal{J}^\bullet)$  in  $D^+(\mathfrak{B})$ .

Let  $\mathfrak{C}$  be a third abelian category and  $G: \mathfrak{B} \rightarrow \mathfrak{C}$  another left-exact functor.

**Proposition 1.13** (Composite functor theorem of Grothendieck). *If  $F$  transforms complexes of injective objects into  $G$ -acyclic complexes, one has a natural isomorphism  $\mathbf{R}(G \circ F) \xrightarrow{\sim} \mathbf{R}G \circ \mathbf{R}F$ .*  $\square$

The theory of right derived functors can be applied when  $\mathfrak{A}$  is one of the categories  $\mathfrak{Mod}(X)$  or  $\mathfrak{Qco}(X)$  because both have enough injectives.

One can develop a parallel theory of derived left exact functors if one assumes that  $\mathfrak{A}$  has enough projectives, so that any object  $\mathcal{M}$  has a functorial projective resolution  $\cdots \rightarrow P^1(\mathcal{M}) \rightarrow P^0(\mathcal{M}) \rightarrow \mathcal{M} \rightarrow 0$ . Then for every bounded above complex  $\mathcal{M}^\bullet$  there exists a bounded above complex  $P(\mathcal{M}^\bullet)$  of projective objects which defines a functor  $P: K^-(\mathfrak{A}) \rightarrow K^-(\mathfrak{A})$ . The functor  $\mathbf{L}F: K^-(\mathfrak{A}) \rightarrow K^-(\mathfrak{B})$  given by  $\mathbf{L}F(\mathcal{M}^\bullet) = F(P(\mathcal{M}^\bullet))$  defines as above a *left derived functor*

$$\mathbf{L}F: D^-(\mathfrak{A}) \rightarrow D^-(\mathfrak{B}).$$

Analogous properties to those stated for right derived functors hold for left derived functors.

One should note that the categories  $\mathfrak{Mod}(X)$ ,  $\mathfrak{Qco}(X)$  and  $\mathfrak{Coh}(X)$  do not have enough projectives. However if  $X$  is a (quasi-)projective, any quasi-coherent sheaf has a (possibly infinite) resolution by locally free sheaves which may have infinite rank, and the problem is circumvented by considering complexes  $\mathcal{P}^\bullet$  of locally free sheaves. We shall come again to this point in Subsection 1.3.2.

**1.3.1. Derived Direct Image.** Let  $f: X \rightarrow Y$  be a morphism of algebraic varieties. Since the direct image functor  $f_*: \mathfrak{Mod}(X) \rightarrow \mathfrak{Mod}(Y)$  is left-exact, it induces a right derived functor

$$\mathbf{R}f_*: D^+(X) \rightarrow D^+(Y)$$

described as  $\mathbf{R}f_* \mathcal{M}^\bullet \simeq f_*(\mathcal{I}^\bullet)$  where  $\mathcal{I}^\bullet$  is a complex of injective  $\mathcal{O}_X$ -modules quasi-isomorphic to  $\mathcal{M}^\bullet$ . When  $f$  is quasi-compact and locally of finite type (as it often happens), the direct image of a quasi-coherent sheaf is also quasi-coherent; in this case,  $\mathbf{R}f_*$  defines a functor  $\mathbf{R}f_*: D_{qc}^+(X) \rightarrow D_{qc}^+(Y)$ . When  $f$  is proper, so that the higher direct images of a coherent sheaf are coherent as well (cf. [25, Thm.3.2.1] or [26, Thm. 5.2] in the projective case), we also have a functor  $\mathbf{R}f_*: D_c^+(X) \rightarrow D_c^+(Y)$ .

Moreover, the dimension of  $X$  bounds the number of higher direct images of a sheaf of  $\mathcal{O}_X$ -modules, so that  $\mathbf{R}f_*$  induces also a functor  $\mathbf{R}f_*: D_{qc}^b(X) \rightarrow D_{qc}^b(Y)$ . In this case  $\mathbf{R}f_*$  extends to a functor  $\mathbf{R}f_*: D_{qc}(X) \rightarrow D_{qc}(Y)$  which maps  $D_c^b(X)$  to  $D_c^b(Y)$ .

If  $Y$  is a point,  $\mathfrak{A}_Y$  is the category of abelian groups and  $f_*$  is the functor of global sections  $\Gamma(X, \cdot)$ . In this case,  $\mathbf{R}f_*\mathcal{M}^\bullet = \mathbf{R}\Gamma(X, \mathcal{M}^\bullet)$  and  $\mathbf{R}^i f_*\mathcal{M}^\bullet$  is called the  $i$ -th hypercohomology group  $\mathbb{H}^i(X, \mathcal{M}^\bullet)$  of the complex  $\mathcal{M}^\bullet$ . It coincides with the cohomology group  $H^i(X, \mathcal{M})$  when the complex reduces to a single sheaf.

**1.3.2. The derived inverse image.** Let  $f: X \rightarrow Y$  be a morphism of algebraic varieties. One can prove that any sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{M}$  is a quotient of a *flat* sheaf of  $\mathcal{O}_X$ -modules  $P(\mathcal{M})$  and that one can choose  $P(\mathcal{M})$  depending functorially on  $\mathcal{M}$ . One then shows that for any bounded above complex  $\mathcal{M}^\bullet$  there is a complex  $P(\mathcal{M}^\bullet)$  of flat sheaves and a quasi-isomorphism  $P(\mathcal{M}^\bullet) \rightarrow \mathcal{M}^\bullet$  which defines a functor  $K^-(\mathfrak{Mod}(Y)) \rightarrow K^-(\mathfrak{Mod}(Y))$ . Moreover  $\mathbf{L}f^*(\mathcal{M}^\bullet) = f^*(P(\mathcal{M}^\bullet))$  gives a left derived functor

$$\mathbf{L}f^*: D^-(Y) \rightarrow D^-(X),$$

which induces functors  $\mathbf{L}f^*: D_{qc}^-(X) \rightarrow D_{qc}^-(Y)$  and  $\mathbf{L}f^*: D_c^-(X) \rightarrow D_c^-(Y)$ . In some cases it also induces a functor  $\mathbf{L}f^*: D_c(X) \rightarrow D_c(Y)$  which maps  $D_c^b(X)$  to  $D_c^b(Y)$ ; this happens for instance when  $Y$  is smooth. Another case is when  $f$  is of *finite homological dimension*, that is, when for every coherent sheaf  $\mathcal{G}$  on  $Y$  there are only a finite number of nonzero derived inverse images  $\mathbf{L}_j f^*(\mathcal{G}) = \mathcal{H}^{-j}(\mathbf{L}f^*(\mathcal{G}))$ ; in particular, flat morphisms are of finite homological dimension.

**1.3.3. Derived homomorphism functor and derived tensor product.** We wish to construct a “derived functor” of the functor of global homomorphisms for complexes. Although this functor is not induced by a left-exact functor between the original abelian categories, we still can derive the complex of homomorphisms by mimicking the procedure used so far. We are not detailing here the entire process (see [1, Appendix A]); let us just mention that one eventually obtains a bifunctor

$$\mathbf{R}\mathrm{Hom}_X^\bullet: D(X)^0 \times D^+(X) \rightarrow D(k)$$

described as  $\mathbf{R}\mathrm{Hom}_X^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet) \simeq \mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$ , where  $\mathcal{I}^\bullet$  is a complex of injective sheaves quasi-isomorphic to  $\mathcal{L}^\bullet$ . We use the notation

$$\mathrm{Ext}_X^i(\mathcal{K}^\bullet, \mathcal{L}^\bullet) = \mathbf{R}^i \mathrm{Hom}_X^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet) = H^i(\mathbf{R}\mathrm{Hom}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)).$$

**Proposition 1.14.** (Yoneda’s formula) *If  $\mathcal{M}^\bullet \in D(X)$  and  $\mathcal{N}^\bullet \in D^+(X)$ , one has*

$$\mathrm{Ext}_X^i(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq \mathrm{Hom}_{D(X)}^i(\mathcal{M}^\bullet, \mathcal{N}^\bullet) := \mathrm{Hom}_{D(X)}(\mathcal{M}^\bullet, \mathcal{N}^\bullet[i]).$$

□

Let  $Y$  be another algebraic variety and  $\Psi: D(X) \rightarrow D(Y)$  an exact functor which maps bounded complexes to bounded complexes.

**Corollary 1.15.** (Parseval’s formula) *If  $\Psi$  is fully faithful, there are isomorphisms*

$$\begin{array}{ccc} \mathrm{Ext}_X^i(\mathcal{M}^\bullet, \mathcal{N}^\bullet) & \xrightarrow{\sim} & \mathrm{Ext}_Y^i(\Psi(\mathcal{M}^\bullet), \Psi(\mathcal{N}^\bullet)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_{D(X)}^i(\mathcal{M}^\bullet, \mathcal{N}^\bullet) & \xrightarrow{\sim} & \mathrm{Hom}_{D(Y)}^i(\Psi(\mathcal{M}^\bullet), \Psi(\mathcal{N}^\bullet)) \end{array}$$

□

One can define the complex  $\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$  of sheaves of homomorphisms; this is given by  $\mathcal{H}om^n(\mathcal{M}^\bullet, \mathcal{N}^\bullet) = \prod_i \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^i, \mathcal{N}^{i+n})$  with the differential  $df = f \circ d_{\mathcal{M}} + (-1)^{n+1} d_{\mathcal{N}} \circ f$ . There is also a derived sheaf homomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}^\bullet : D(X)^0 \times D^+(X) \rightarrow D(X).$$

By applying Grothendieck's composite functor theorem we obtain for every open  $U \subseteq X$  an isomorphism in the derived category  $D(U)$ :

$$\mathbf{R}\Gamma(U, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{K}^\bullet, \mathcal{L}^\bullet)) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_U}^\bullet(\mathcal{K}^\bullet|_U, \mathcal{L}^\bullet|_U).$$

By following a similar procedure one can derive the “functor tensor product of complexes”: there exists a bifunctor, called derived tensor product

$$\overset{\mathbf{L}}{\otimes} : D(X) \times D(X) \rightarrow D(X),$$

whose description is  $\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{N}^\bullet = \mathcal{M}^\bullet \otimes P(\mathcal{N}^\bullet)$ , where  $P(\mathcal{N}^\bullet) \rightarrow \mathcal{N}^\bullet$  is a quasi-isomorphism and  $P(\mathcal{N}^\bullet)$  is a complex of flat sheaves.

**1.3.4. Base change in the derived category.** There are some derived category remarkable formulas that relate the various derived functors (cf. [26]). Here we describe only the following strengthened version of the *derived category base change formula*.

**Proposition 1.16.** [1, Prop.A.74] *Let us consider a cartesian diagram of morphisms of algebraic varieties*

$$\begin{array}{ccc} X \times_Y \tilde{Y} & \xrightarrow{\tilde{g}} & X \\ \downarrow \tilde{f} & & \downarrow f \\ \tilde{Y} & \xrightarrow{g} & Y \end{array}$$

*For any complex  $\mathcal{M}^\bullet$  of  $\mathcal{O}_X$ -modules there is a natural morphism*

$$\mathbf{L}g^* \mathbf{R}f_* \mathcal{M}^\bullet \rightarrow \mathbf{R}\tilde{f}_* \mathbf{L}\tilde{g}^* \mathcal{M}^\bullet.$$

*Moreover, if  $\mathcal{M}^\bullet$  has quasi-coherent cohomology and either  $f$  or  $g$  is flat, the above morphism is an isomorphism.*  $\square$

## 2. INTEGRAL FUNCTORS AND FOURIER-MUKAI TRANSFORMS

**2.1. Definitions.** We start this section with a general definition of integral functor; Fourier-Mukai transforms will provide interesting examples. The varieties involved can be quite general algebraic varieties, though in the applications they will be mainly smooth and projective. Most of the results described here are valid over an arbitrary algebraically closed field  $\mathbb{k}$ , possibly requiring that  $\text{ch } \mathbb{k} = 0$ . However, in order to simplify our treatment, we shall assume that  $\mathbb{k}$  is the field  $\mathbb{C}$  of the complex numbers.

Let us consider the diagram

$$\begin{array}{ccc} & X \times Y & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & & Y \end{array}$$

involving the projections of the cartesian product  $X \times Y$  onto its factors. Any object  $\mathcal{K}^\bullet$  in the derived category  $D_c^-(X \times Y)$  is the *kernel* of an *integral functor*:

$$\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet} : D_c^-(X) \rightarrow D_c^-(Y)$$

$$\mathcal{E}^\bullet \mapsto \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{E}^\bullet) = \mathbf{R}\pi_{Y*}(\pi_X^* \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet).$$

The integral functor  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  is said to be a *Fourier-Mukai functor* if it is an equivalence of categories, and a *Fourier-Mukai transform* is a Fourier-Mukai functor whose kernel  $\mathcal{K}^\bullet$  reduces to a single sheaf  $\mathcal{K}$ .

If  $\mathcal{K}^\bullet$  is of *finite Tor-dimension over  $X$* , that is, if it is isomorphic in the derived category to a bounded complex of sheaves that are flat over  $X$ , then the integral functor  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  maps  $D_c^b(X)$  to  $D_c^b(Y)$  and can be extended to a functor  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D(X) \rightarrow D(Y)$  between the entire derived categories. Note that the condition of finite Tor-dimensionality is always fulfilled if  $X$  and  $Y$  are *smooth* and  $\mathcal{K}^\bullet$  is an object of  $D_c^b(X \times Y)$ , because in this case  $\mathcal{K}^\bullet$  is isomorphic in the derived category to a bounded complex of locally free sheaves.

Since an integral functor is the composition of the functors  $\mathbf{L}\pi_X^*$  (which is isomorphic to  $\pi_X^*$  because  $\pi_X$  is a flat morphism),  $(\ ) \otimes^{\mathbf{L}} \mathcal{K}^\bullet$  and  $\mathbf{R}\pi_{Y*}$ , and these functors are exact, *any integral functor is exact* as well. In particular, for any exact sequence  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0$  of sheaves in  $X$  we obtain an exact sequence

$$\dots \rightarrow \Phi^{i-1}(\mathcal{G}) \rightarrow \Phi^i(\mathcal{F}) \rightarrow \Phi^i(\mathcal{E}) \rightarrow \Phi^i(\mathcal{G}) \rightarrow \Phi^{i+1}(\mathcal{F}) \rightarrow \dots$$

where we have written  $\Phi^i(\ ) = \mathcal{H}^i(\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\ ))$ .

**Definition 2.1.** Let  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  be an integral functor. A complex  $\mathcal{F}^\bullet$  in  $D_c^-(X)$  is  $\text{WIT}_i$  if  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{F}^\bullet) \simeq \mathcal{G}[-i]$  in  $D(Y)$ , for a coherent sheaf  $\mathcal{G}$  on  $Y$ . If in addition  $\mathcal{G}$  is locally free, we say that  $\mathcal{F}^\bullet$  is the  $\text{IT}_i$ .  $\triangle$

In Section 3 we shall see a connection between integral functors and index theory. This will make it clear that the “IT” condition is related with the “index theorem”, and that the “W” of “WIT” stands for “weak”.

Using the cohomology base change theorem [26, III.12.11] one proves the following criterion for the WIT condition to hold.

**Proposition 2.2.** *Assume that the kernel  $\mathcal{Q}$  is a locally free sheaf on the product  $X \times Y$ . A coherent sheaf  $\mathcal{F}$  on  $X$  is  $\text{IT}_i$  if and only if  $H^j(X, \mathcal{F} \otimes \mathcal{Q}_y) = 0$  for all  $y \in Y$  and for all  $j \neq i$ , where  $\mathcal{Q}_y$  denotes the restriction of  $\mathcal{Q}$  to  $X \times \{y\}$ . Furthermore,  $\mathcal{F}$  is  $\text{WIT}_0$  if and only if it is  $\text{IT}_0$ .  $\square$*

Let us list the simplest examples of integral functors.

*Example 2.3.* Let  $\delta: X \hookrightarrow X \times X$  denote the diagonal immersion, and write  $\Delta$  for its image. The structure sheaf  $\mathcal{O}_\Delta = \delta_* \mathcal{O}_X$  of  $\Delta$  defines an integral functor  $\Phi_{X \rightarrow X}^{\mathcal{O}_\Delta}: D_c^-(X) \rightarrow D_c^-(X)$ , which is isomorphic to the identity functor  $\Phi_{X \rightarrow X}^{\mathcal{O}_\Delta} \simeq \text{Id}$ . Thus, the identity functor is a Fourier-Mukai transform.  $\triangle$

*Example 2.4.* If  $\mathcal{L}$  is a line bundle on  $X$ , the functor  $\Phi_{X \rightarrow X}^{\delta_* \mathcal{L}}$  consists of the twist by  $\mathcal{L}$ . It is also a Fourier-Mukai transform, whose quasi-inverse is the twist by  $\mathcal{L}^{-1}$ , that is, the Fourier-Mukai transform with kernel  $\delta_* \mathcal{L}^{-1}$ .  $\triangle$

*Example 2.5.* If  $f: X \rightarrow Y$  is a proper morphism and  $\mathcal{K}^\bullet$  is the structure sheaf of the graph  $\Gamma_f \subset X \times Y$ , one has isomorphisms of functors  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet} \simeq \mathbf{R}f_*: D_c^-(X) \rightarrow D_c^-(Y)$  and  $\Phi_{Y \rightarrow X}^{\mathcal{K}^\bullet} \simeq \mathbf{L}f^*: D_c^-(Y) \rightarrow D_c^-(X)$ .  $\triangle$

**2.1.1. The abelian Fourier-Mukai transform.** Let  $X$  be an abelian variety; in the complex case, we can think of  $X$  as a torus. Let  $\hat{X}$  be its dual abelian variety; a closed point  $\xi \in \hat{X}$  corresponds to a zero degree line bundle  $\mathcal{P}_\xi$  over  $X$  (or a flat line bundle in the complex case).  $\hat{X}$  is a “fine moduli space”, in the sense that there exists a line

bundle  $\mathcal{P}$  over  $X \times \hat{X}$ , whose restriction to the fibre  $X \simeq \pi_{\hat{X}}^{-1}\xi$  of the projection  $\pi_{\hat{X}}$  is precisely the flat line bundle  $\mathcal{P}_\xi$  corresponding to  $\xi$ . This *universal Poincaré line bundle* is uniquely characterised by this property up to twisting by pull-backs  $\pi_{\hat{X}}^*\mathcal{N}$  of line bundles  $\mathcal{N}$  on  $\hat{X}$ . To avoid any ambiguity,  $\mathcal{P}$  is normalised so that its restriction to the fibre  $\hat{X} \simeq \pi_X^{-1}(0)$  of the origin  $0 \in X$  is trivial. This fixes  $\mathcal{P}$  uniquely. An explicit description of  $\mathcal{P}$  in differential-geometric terms will be given in Section 3.

The first example of a Fourier-Mukai transform was introduced by Mukai in this setting [43]. Mukai's seminal idea was to use the normalised Poincaré bundle  $\mathcal{P}$  to define an integral functor  $\mathbf{S} = \Phi_{X \rightarrow \hat{X}}^{\mathcal{P}}: D_c^b(X) \rightarrow D_c^b(\hat{X})$ , which turns out to be an equivalence of triangulated categories, that is, a *Fourier-Mukai transform*. We shall give a short proof of this fact in Theorem 2.12. Moreover, the functor  $\Phi_{\hat{X} \rightarrow X}^{\mathcal{P}^*[g]}$ , where  $g = \dim X$ , is quasi-inverse to  $\Phi_{X \rightarrow \hat{X}}^{\mathcal{P}}$ . We shall call  $\mathbf{S} = \Phi_{X \rightarrow \hat{X}}^{\mathcal{P}}$  the *abelian Fourier-Mukai transform* and  $\hat{\mathbf{S}} = \Phi_{\hat{X} \rightarrow X}^{\mathcal{P}^*}$  the *dual abelian Fourier-Mukai transform*. (Actually, instead of  $\hat{\mathbf{S}}$  Mukai considers the functor  $\tilde{\mathbf{S}} = \Phi_{\hat{X} \rightarrow X}^{\mathcal{P}}: D_c^b(\hat{X}) \rightarrow D_c^b(X)$ . One has  $\tilde{\mathbf{S}} \circ \mathbf{S} \simeq \iota_{\hat{X}}^* \circ [-g]$ , where  $\iota_{\hat{X}}: \hat{X} \rightarrow \hat{X}$  is the involution which maps a line bundle to its dual.)

**2.1.2. Orlov's representation theorem.** We have seen a few examples of integral functors. A natural problem is the characterisation of the exact functors  $D_c^b(X) \rightarrow D_c^b(Y)$  that are integral. The most important result in this direction is due to Orlov [48]:

**Theorem 2.6.** *Let  $X$  and  $Y$  be smooth projective varieties. Any fully faithful exact functor  $\Psi: D_c^b(X) \rightarrow D_c^b(Y)$  is an integral functor.*  $\square$

Orlov's original statement assumed that the exact functor has a right adjoint; however, Bondal and Van den Bergh proved that any exact functor  $D_c^b(X) \rightarrow D_c^b(Y)$  satisfies this property [8].

**2.2. General properties of integral functors.** The first property of integral functors we would like to describe is that the composition of two of them is again an integral functor.

If  $Z$  is another proper variety, we consider the diagram

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow \pi_{XY} & \downarrow \pi_{Y,Z} & \searrow \pi_{XZ} & \\ X \times Y & & Y \times Z & & X \times Z \end{array}$$

Given kernels  $\mathcal{K}^\bullet$  in  $D_c^-(X \times Y)$  and  $\mathcal{L}^\bullet$  in  $D_c^-(Y \times Z)$ , the composition of the integral functors defined by them is given by the following Proposition:

**Proposition 2.7.** *There is a natural isomorphism of functors  $\Phi_{Y \rightarrow Z}^{\mathcal{L}^\bullet} \circ \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet} \simeq \Phi_{X \rightarrow Z}^{\mathcal{L}^\bullet * \mathcal{K}^\bullet}$ , where  $\mathcal{L}^\bullet * \mathcal{K}^\bullet = \mathbf{R}\pi_{XZ*}(\pi_{XY}^* \mathcal{K}^\bullet \otimes^{\mathbf{L}} \pi_{YZ}^* \mathcal{L}^\bullet)$  in  $D_c^-(X \times Z)$ .*  $\square$

**2.2.1. Action of integral functors on cohomology.** Integral functors act on cohomology, and the study of this action allows one to determine the topological invariants of the transform of a complex in terms of the topological invariants of the complex. This is very useful in studying the effect of integral functors on moduli spaces of sheaves.

We need to recall the notion of Mukai vector and pairing. The Mukai vector of a complex  $\mathcal{E}^\bullet$  in  $D_c^b(X)$  for a smooth projective variety  $X$  is defined as

$$v(\mathcal{E}^\bullet) = \text{ch}(\mathcal{E}^\bullet) \cdot \sqrt{\text{td}(X)},$$

where  $\text{td}(X) \in A^\bullet(X) \otimes \mathbb{Q}$  is the Todd class of  $X$ . We can define a symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on the rational Chow group  $A^\bullet(X) \otimes \mathbb{Q}$  by setting

$$\langle v, w \rangle = - \int_X v^* \cdot w \cdot \exp(\tfrac{1}{2}c_1(X)).$$

The Mukai pairing naturally extends to the even rational cohomology  $\bigoplus_j H^{2j}(X, \mathbb{Q})$ .

If  $X$  and  $Y$  are smooth proper varieties, and  $\mathcal{K}^\bullet$  is a kernel in  $D_c^b(X \times Y)$ , by the Grothendieck-Riemann-Roch theorem the integral functor  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D_c^b(X) \rightarrow D_c^b(Y)$  gives rise to the commutative diagram

$$\begin{array}{ccccc} D_c^b(X) & \xrightarrow{v} & A^\bullet(X) \otimes \mathbb{Q} & \longrightarrow & H^\bullet(X, \mathbb{Q}) \\ \downarrow \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet} & & \downarrow f^{\mathcal{K}^\bullet} & & \downarrow f^{\mathcal{K}^\bullet} \\ D_c^b(Y) & \xrightarrow{v} & A^\bullet(Y) \otimes \mathbb{Q} & \longrightarrow & H^\bullet(Y, \mathbb{Q}) \end{array}$$

where both  $f^{\mathcal{K}^\bullet}$  are the  $\mathbb{Q}$ -vector space homomorphisms defined by

$$f^{\mathcal{K}^\bullet}(\alpha) = \pi_{Y*}(\pi_X^* \alpha \cdot v(\mathcal{K}^\bullet)).$$

The map  $f^{\mathcal{K}^\bullet}$  sends  $H^{2\bullet}(X, \mathbb{Q})$  to  $H^{2\bullet}(Y, \mathbb{Q})$  and depends functorially on the kernel, i.e.,  $f^{\mathcal{L}^\bullet * \mathcal{K}^\bullet} = f^{\mathcal{L}^\bullet} \circ f^{\mathcal{K}^\bullet}$ . This implies the following result.

**Corollary 2.8.** *Let  $X$  and  $Y$  be smooth proper varieties and  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D_c^b(X) \rightarrow D_c^b(Y)$  a Fourier-Mukai functor. Then the morphisms  $f^{\mathcal{K}^\bullet}: A^\bullet(X) \otimes \mathbb{Q} \rightarrow A^\bullet(Y) \otimes \mathbb{Q}$  and  $f^{\mathcal{K}^\bullet}: H^\bullet(X, \mathbb{Q}) \rightarrow H^\bullet(Y, \mathbb{Q})$  are isomorphisms. Moreover, the latter induces an isomorphism of vector spaces between the even cohomology rings.*  $\square$

**2.2.2. Fully faithful integral functors and Fourier-Mukai functors.** In this Subsection all varieties are smooth and projective unless otherwise stated. The first step in characterising the kernels  $\mathcal{K}^\bullet$  in  $D_c^b(X \times Y)$  that give rise to equivalences is to determine the kernels for which  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D_c^b(X) \rightarrow D_c^b(Y)$  is fully faithful. The idea is to study the effect of a fully faithful integral functor  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  on the skyscraper sheaves  $\mathcal{O}_x$  of the points. Due to the Parseval formula (Corollary 1.15), for any pair of points  $x_1, x_2$  of  $X$  one has

$$(2.1) \quad \text{Hom}_{D(X)}^i(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) \simeq \text{Hom}_{D(Y)}^i(\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{x_1}), \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{x_2})) \simeq \text{Hom}_{D(Y)}^i(\mathbf{L}j_{x_1}^* \mathcal{K}^\bullet, \mathbf{L}j_{x_2}^* \mathcal{K}^\bullet).$$

It follows that if  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  is fully faithful, the kernel  $\mathcal{K}^\bullet$  fulfils the following properties:

- (1)  $\text{Hom}_{D(Y)}^i(\mathbf{L}j_{x_1}^* \mathcal{K}^\bullet, \mathbf{L}j_{x_2}^* \mathcal{K}^\bullet) = 0$  unless  $x_1 = x_2$  and  $0 \leq i \leq \dim X$ ;
- (2)  $\text{Hom}_{D(Y)}^0(\mathbf{L}j_x^* \mathcal{K}^\bullet, \mathbf{L}j_x^* \mathcal{K}^\bullet) = \mathbb{C}$ .

Kernels  $\mathcal{K}^\bullet$  satisfying these properties were called *strongly simple over  $X$*  by Maciocia [39], but this notion had already been used implicitly in [9]. The following crucial result was originally proved by Bondal and Orlov [9].

**Theorem 2.9.** *Let  $X$  and  $Y$  be smooth projective varieties, and  $\mathcal{K}^\bullet$  a kernel in  $D^b(X \times Y)$ . The functor  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  is fully faithful if and only if  $\mathcal{K}^\bullet$  is strongly simple over  $X$ .*  $\square$

Building on Bondal and Orlov's work, Bridgeland and Maciocia [18, 39] determined the kernels that give rise to equivalences of categories. They called *special* an object  $\mathcal{F}^\bullet$  in  $D_c^b(X)$  such that  $\mathcal{F}^\bullet \otimes \omega_X \simeq \mathcal{F}^\bullet$  in  $D_c^b(X)$ , where  $\omega_X$  is the canonical line bundle.

**Proposition 2.10.** *Let  $X$  and  $Y$  be smooth projective varieties of the same dimension  $n$ , and let  $\mathcal{K}^\bullet$  be a kernel in  $D_c^b(X \times Y)$ . Then  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  is a Fourier-Mukai functor if and only if  $\mathcal{K}^\bullet$  is strongly simple over  $X$  and  $\mathbf{L}j_x^* \mathcal{K}^\bullet$  is special for all  $x \in X$ .  $\square$*

When the canonical bundles of  $X$  and  $Y$  are trivial this takes a simpler form.

**Proposition 2.11.** *Let  $X$  and  $Y$  be smooth projective varieties of the same dimension with trivial canonical bundles and  $\mathcal{K}^\bullet$  an object in  $D_c^b(X \times Y)$  strongly simple over  $X$ . Then  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$  is a Fourier-Mukai functor and  $\Phi_{Y \rightarrow X}^{\mathcal{K}^{\bullet \vee[n]}}$  is a quasi-inverse to  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ .  $\square$*

The characterisation of the kernels that induce fully faithful integral functors or equivalences has been generalised to the case of singular Cohen-Macaulay varieties in [28, 29].

**2.2.3. The abelian Fourier-Mukai transform revisited.** Here we apply the characterisation of Fourier-Mukai functors to give a simple proof of the fact that the “abelian Fourier-Mukai transform”  $\mathbf{S}$  and the “dual abelian Fourier-Mukai transform”  $\hat{\mathbf{S}}$  are truly Fourier-Mukai transforms, that is, they are equivalences of categories.

Let  $X$  be an abelian variety of dimension  $g$  and  $\hat{X}$  its dual abelian variety. If  $\mathcal{P}$  is the Poincaré line bundle on  $X \times \hat{X}$ , the restriction  $\mathbf{L}j_\xi^* \mathcal{P}$  is the line bundle  $\mathcal{P}_\xi$  on  $X$  corresponding to the point  $\xi \in \hat{X}$ . Since  $\mathrm{Hom}_{D(X)}^i(\mathcal{P}_{\xi_1}, \mathcal{P}_{\xi_2}) \simeq H^i(X, \mathcal{P}_{\xi_1}^* \otimes \mathcal{P}_{\xi_2})$  for any pair of points  $\xi_1$  and  $\xi_2$  of  $\hat{X}$ , one has:

- (1)  $\mathrm{Hom}_{D(X)}^i(\mathcal{P}_{\xi_1}, \mathcal{P}_{\xi_2}) = 0$  unless  $\xi_1 = \xi_2$  and  $0 \leq i \leq g$ ;
- (2)  $\mathrm{Hom}_{D(X)}^0(\mathcal{P}_{\xi_1}, \mathcal{P}_{\xi_1}) = \mathbb{C}$  for any  $\xi \in \hat{X}$ .

In other words,  $\mathcal{P}$  is strongly simple over  $\hat{X}$ . By Proposition 2.11, it is strongly simple over  $X$  as well. So we have:

**Theorem 2.12.** *The functors  $\mathbf{S} = \Phi_{X \rightarrow \hat{X}}^{\mathcal{P}}: D_c^b(X) \rightarrow D_c^b(\hat{X})$  and  $\hat{\mathbf{S}} = \Phi_{\hat{X} \rightarrow X}^{\mathcal{P}^*}: D_c^b(\hat{X}) \rightarrow D_c^b(X)$  are equivalences of categories.  $\square$*

**2.2.4. Fourier-Mukai functors on K3 and abelian surfaces.** Let  $Y$  be a smooth projective surface and  $X$  a fine moduli space of *special* stable sheaves on  $Y$  with fixed Mukai vector  $v$ . Let  $\mathcal{Q}$  be a universal sheaf on  $X \times Y$  for the corresponding moduli problem, so that  $\mathcal{Q}$  is flat over  $X$  and  $\mathcal{Q}_x$  is a stable special sheaf on  $Y$  with Mukai vector  $v$ . Given closed points  $x$  and  $z$  in  $X$ , one has  $\chi(\mathcal{Q}_x, \mathcal{Q}_z) = -\langle v, v \rangle = -v^2$ .

**Proposition 2.13.** [20] *Assume that  $X$  is a projective surface. Then  $X$  is smooth if and only if  $v^2 = 0$ . Moreover, in this case  $\Phi_{X \rightarrow Y}^{\mathcal{Q}}: D_c^b(X) \rightarrow D_c^b(Y)$  is a Fourier-Mukai functor.  $\square$*

With the help of Proposition 2.13 one can construct Fourier-Mukai transforms for K3 surfaces. Let  $Y$  be a K3 surface with a polarisation  $H$  and  $v$  a Mukai vector with  $v^2 = 0$ . Assume that  $v = (r, c, s)$  is primitive (i.e., not divisible by an integer) and that the greatest common divisor of the numbers  $(r, c_1 \cdot H, s)$  is 1. If the moduli space  $X = M_v(Y)$  of sheaves on  $Y$  with Mukai vector  $v$  that are (Gieseker) stable with respect to  $H$  is nonempty, it is a projective variety of dimension  $v^2 + 2 = 2$  (cf. [41, 42]). Moreover, there exists a universal sheaf  $\mathcal{Q}$  on  $X \times Y$ . Proposition 2.13 implies that  $X$  is smooth and that  $\Phi_{X \rightarrow Y}^{\mathcal{Q}}: D_c^b(X) \rightarrow D_c^b(Y)$  is a Fourier-Mukai transform. We shall also see (cf. Proposition 2.30) that  $X$  is again a K3 surface.



*Example 2.14.* A first instance of this situation, which has also been the first example of a nontrivial Fourier-Mukai transform on K3 surfaces, was given in [3], where a class of K3 surfaces called *strongly reflexive* was introduced. A K3 surface  $Y$  is strongly reflexive if it carries a polarization  $H$  and a divisor  $\ell$  such that  $H^2 = 2$ ,  $H \cdot \ell = 0$ ,  $\ell^2 = -12$ , and  $X$  has no nodal curves of degree 1 or 2. Strongly reflexive K3 surfaces do exist, There is indeed a nonempty coarse moduli space of strongly reflexive K3 surfaces, which is an irreducible quasi-projective scheme of dimension 18 [4]. On a strongly reflexive K3 surface  $Y$  one may take the Mukai vector  $v = (2, \ell, -3)$ , which fulfils all the above requirements. One proves that  $M_v(Y) \neq \emptyset$  [3], and then  $\Phi_{X \rightarrow Y}^Q: D_c^b(X) \rightarrow D_c^b(Y)$  is a Fourier-Mukai transform. This may be used to construct further examples [21].  $\triangle$

*Example 2.15.* Later Mukai provided another example [46]. He considered a K3 surface  $Y$  such that there exist coprime positive integers  $r, s$  and a polarization  $H$  in  $Y$  with  $H^2 = 2rs$ . He proved that  $X = M_v(Y)$  is nonempty (and fine) and that the universal family induces a Fourier-Mukai transform  $D_c^b(X) \simeq D_c^b(Y)$ .  $\triangle$

**2.2.5. Relative integral functors and base change.** In this Subsection we generalise the notion of integral functor to the relative setting, namely, we shall deal with morphisms (or families) instead of single varieties.

We consider two (proper) morphisms of algebraic schemes  $p: X \rightarrow B$  and  $q: Y \rightarrow B$  and denote by  $\tilde{\pi}_X, \tilde{\pi}_Y$  the projections of the fibre product  $X \times_B Y$  onto its factors. If we set  $\rho = p \circ \tilde{\pi}_X = q \circ \tilde{\pi}_Y$  we have a cartesian diagram

$$\begin{array}{ccc} & X \times_B Y & \\ \tilde{\pi}_X \swarrow & \downarrow \rho & \searrow \tilde{\pi}_Y \\ X & & Y \\ p \searrow & & \swarrow q \\ & B & \end{array}$$

An object  $\mathcal{K}^\bullet$  in the derived category  $D_c^-(X \times_B Y)$  (a “relative kernel”), induces a relative integral functor  $\Phi: D^-(X) \rightarrow D^-(Y)$  which is defined as

$$\Phi(\mathcal{E}^\bullet) = \mathbf{R}\tilde{\pi}_{Y*}(\mathbf{L}\tilde{\pi}_X^* \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet).$$

This functor is an (ordinary) integral functor whose kernel in the derived category  $D_c^-(X \times Y)$  is  $j_* \mathcal{K}^\bullet$ , where  $j: X \times_B Y \hookrightarrow X \times Y$  is natural closed immersion. Thus the results about integral functors can be applied to relative integral functors as well. In the remainder of this Subection we assume that  $\mathcal{K}^\bullet$  is of finite Tor-dimension over  $X$ ; then,  $\Phi$  can be extended to a functor  $\Phi: D(X) \rightarrow D(Y)$  which maps  $D_c^b(X)$  to  $D_c^b(Y)$ .

One of the most interesting features of relative integral functors interesting is their compatibility with base changes. Let  $f: S \rightarrow B$  be a morphism. For any morphism  $g: Z \rightarrow B$  we denote by  $g_S: Z_S = Z \times_B S \rightarrow S$  and  $f_Z: Z_S \rightarrow Z$  the induced morphisms. We can consider the kernel  $\mathcal{K}_S^\bullet = \mathbf{L}f_{X \times_B Y}^* \mathcal{K}^\bullet$  and the induced relative integral functor

$$\Phi_S: D_c^-(X_S) \rightarrow D_c^-(Y_S); \quad \Phi_S(\mathcal{E}^\bullet) = \mathbf{R}\tilde{\pi}_{Y_S*}(\mathbf{L}\tilde{\pi}_{X_S}^* \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}_S^\bullet).$$

Since  $\mathcal{K}_S^\bullet$  is of finite Tor-dimension over  $S$ ,  $\Phi_S$  maps  $D_c^b(X_S)$  to  $D_c^b(Y_S)$ . Base change compatibility is expressed by the following result, whose proof uses base change in the derived category (cf. Proposition 1.16).

**Proposition 2.16.** *Assume either that  $f: S \rightarrow B$  or  $p: X \rightarrow B$  is flat. Then for every object  $\mathcal{E}^\bullet$  in  $D^b(X)$  there is a functorial isomorphism  $\mathbf{L}f_Y^* \Phi(\mathcal{E}^\bullet) \simeq \Phi_S(\mathbf{L}f_X^* \mathcal{E}^\bullet)$  in the derived category of  $Y_S$ .  $\square$*

If the morphism  $p: X \rightarrow B$  is flat there is no need to assume that the base change morphism is flat, a fact which is often neglected. In this case, by denoting by  $j_t$  the immersions of both fibres  $X_t = p^{-1}(t) \hookrightarrow X$  and  $Y_t = q^{-1}(t) \hookrightarrow Y$  over a closed point  $t \in B$ , one has  $\mathbf{L}j_t^* \Phi(\mathcal{E}^\bullet) \simeq \Phi_t(\mathbf{L}j_t^* \mathcal{E}^\bullet)$ . One has the following result.

**Corollary 2.17.** *Assume that  $p: X \rightarrow B$  is flat, and let  $\mathcal{E}^\bullet$  be an object in  $D^b(X)$ . Then the derived restriction  $\mathbf{L}j_t^* \mathcal{E}^\bullet$  to the fibre  $X_t$  is  $\text{WIT}_i$  for every  $t$  if and only if  $\mathcal{E}^\bullet$  is  $\text{WIT}_i$  and  $\Phi^i(\mathcal{E})$  is flat over  $B$ .  $\square$*

The condition of being  $\text{WIT}_i$  is open on the base, as the following Proposition asserts.

**Proposition 2.18.** *Let  $p: X \rightarrow B$  be a flat morphism and  $\mathcal{E}$  be a sheaf on  $X$  flat over  $B$ . The set  $U$  of points in  $B$  such that the restriction  $\mathcal{E}_t$  of  $\mathcal{E}$  to the fibre  $X_t$  is  $\text{WIT}_i$  is a nonempty open subscheme of  $B$ .  $\square$*

**2.2.6. Fourier-Mukai functors between moduli spaces.** Integral transforms define in many cases algebraic morphisms between moduli spaces. Assume that  $X$  and  $Y$  are smooth projective varieties and let  $\Phi: D^b(X) \rightarrow D^b(Y)$  be an integral functor. We consider the functor  $\mathbf{M}_{X,P}$  which associates to any variety  $T$  the set of all coherent sheaves  $\mathcal{E}$  on  $T \times X$ , flat over  $T$  and whose restrictions  $\mathcal{E}_t = j_t^* \mathcal{E}$  to the fibres  $X_t \simeq X$  of  $\pi_T: T \times X \rightarrow T$  have Hilbert polynomial  $P$ . Let  $\mathbf{M}_X$  be a subfunctor of  $\mathbf{M}_{X,P}$  which parametrises  $\text{WIT}_i$  sheaves for a certain index  $i$ . By Corollary 2.17, if  $\mathcal{E}$  is in  $\mathbf{M}_X(T)$  the sheaf  $\widehat{\mathcal{E}} = \Phi_T^i(\mathcal{E})$  is flat over  $T$ , so that for a fixed  $i$  the fibres  $(\widehat{\mathcal{E}})_t \simeq \widehat{\mathcal{E}}_t$  have the same Hilbert polynomial  $\widehat{P}$ . Thus the transforms  $\widehat{\mathcal{E}}$  are in  $\mathbf{M}_{Y,\widehat{P}}(T)$ . Proposition 2.16 implies now that  $\Phi$  yields a morphism of functors  $\Phi_{\mathbf{M}}: \mathbf{M}_X \rightarrow \mathbf{M}_{Y,\widehat{P}}$ .

**Proposition 2.19.** *If  $\mathbf{M}_X$  has a coarse moduli scheme  $M_X$  and  $\Phi$  is a Fourier-Mukai functor, then  $\mathbf{M}_Y = \Phi(\mathbf{M}_X)$  is coarsely representable by a moduli scheme  $M_Y$ , and  $\Phi$  induces an isomorphism of schemes  $M_X \simeq M_Y$ . Moreover  $M_X$  is a fine moduli scheme if and only if  $M_Y$  is a fine moduli scheme.  $\square$*

When  $\mathbf{M}_X$  is the moduli functor of all the skyscraper sheaves  $\mathcal{O}_x$ , one has:

**Corollary 2.20.** *If  $\Phi$  is a Fourier-Mukai functor, then  $X$  is a fine moduli space for the moduli functor of the sheaves  $\Phi^i(\mathcal{O}_x)$  over  $Y$ . Moreover these sheaves are simple.  $\square$*

**2.3. Fourier-Mukai partners.** The fact that two smooth algebraic varieties have equivalent derived categories entails strong constraints on their geometry. We already know that this fact is equivalent to the existence of a Fourier-Mukai functor between their derived categories (cf. Theorem 2.6). In this Section we describe some important results in this direction. All the varieties are projective.

**Definition 2.21.** Two varieties  $X$  and  $Y$  are Fourier-Mukai partners if there is an exact equivalence of triangulated categories  $F: D_c^b(X) \xrightarrow{\sim} D_c^b(Y)$ .  $\triangle$

**Lemma 2.22.** *Let  $X$  be a smooth variety. Every Fourier-Mukai partner of  $X$  is smooth.  $\square$*

**Theorem 2.23.** *Let  $X, Y$  be smooth Fourier-Mukai partners, so that there is a Fourier-Mukai functor  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D^b(X) \rightarrow D^b(Y)$ .*

- (1)  $X$  and  $Y$  have the same dimension.
- (2) There is an isomorphism  $H^0(X, \omega_X^i) \simeq H^0(Y, \omega_Y^i)$  for every integer  $i$ , so that  $X$  and  $Y$  have the same Kodaira dimension.
- (3)  $\omega_X$  and  $\omega_Y$  have the same order, that is,  $\omega_X^k$  is trivial if and only if  $\omega_Y^k$  is trivial. Thus,  $\omega_X$  is trivial if and only if  $\omega_Y$  is trivial and in this case the functor  $\Phi_{Y \rightarrow X}^{\mathcal{K}^\bullet \vee [n]}$  is a quasi-inverse to  $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ . Moreover,  $\omega_X^r \simeq \mathcal{O}_X$  and  $\omega_Y^r \simeq \mathcal{O}_Y$  with  $r = \text{rk}(\mathcal{K}^\bullet)$ .

□

Theorem 2.23 implies that if the kernel  $\mathcal{K}^\bullet$  has positive rank a certain power  $\omega_X^r$  of the canonical bundle of  $X$  is trivial, with  $r \neq 0$ . If  $X$  is a curve, it has to be elliptic (and then  $\omega_X \simeq \mathcal{O}_X$ ); if  $X$  is a surface, it is either abelian, K3, Enriques or bielliptic (corresponding to the cases  $\omega_X \simeq \mathcal{O}_X$ ,  $\omega_X^2 \simeq \mathcal{O}_X$  and  $\omega_X^{12} \simeq \mathcal{O}_X$ , cf. [26, Thm. 6.3]). In dimension 3 the most important example is provided by Calabi-Yau varieties (for which, by definition,  $\omega_X \simeq \mathcal{O}_X$ ).

The following result will be useful later on.

**Proposition 2.24.** *Let  $X, Y$  be proper smooth algebraic varieties of dimension  $n$  and  $\Phi: D_c^b(X) \xrightarrow{\sim} D_c^b(Y)$  a Fourier-Mukai functor. For every (closed) point  $x \in X$  the inequality  $\sum_i \dim \text{Hom}_{D(Y)}^1(\Phi^i(\mathcal{O}_x), \Phi^i(\mathcal{O}_x)) \leq n$  holds true.*

*Proof.* There is a spectral sequence  $E_2^{p,q} = \bigoplus_i \text{Hom}_{D(Y)}^p(\Phi^i(\mathcal{O}_x), \Phi^{i+q}(\mathcal{O}_x))$  converging to  $E_\infty^{p+q} = \text{Hom}_{D(Y)}^{p+q}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x))$ . The exact sequence of lower terms of the spectral sequence gives  $0 \rightarrow E_2^{1,0} \rightarrow E_\infty^1$ . By the Parseval formula (Corollary 1.15), one has  $\text{Hom}_{D(Y)}^1(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \simeq \text{Hom}_{D(X)}^1(\mathcal{O}_x, \mathcal{O}_x) \simeq \mathbb{C}^n$ . □

**2.3.1.  $D$ -equivalence implies  $K$ -equivalence.** Two smooth algebraic varieties  $X$  and  $Y$  are  $K$ -equivalent if there are a normal variety  $\tilde{Z}$  and birational morphisms  $\tilde{p}_X: \tilde{Z} \rightarrow X$ ,  $\tilde{p}_Y: \tilde{Z} \rightarrow Y$  such that  $\tilde{p}_X^* K_X$  and  $\tilde{p}_Y^* K_Y$  are  $\mathbb{Q}$ -linearly equivalent.

The next result is due to Kawamata.

**Theorem 2.25.** (“ $D$ -equivalence implies  $K$ -equivalence”) [34, 35] *Let  $X, Y$  be smooth Fourier-Mukai partners.*

- (1) *The line bundle  $\omega_X$  (resp.  $\omega_X^*$ ) is nef if and only if  $\omega_Y$  (resp.  $\omega_Y^*$ ) is nef.*
- (2) *If the Kodaira dimension  $\kappa(X)$  is equal to  $\dim X$  (or if  $\kappa(X, \omega_X^*) = \dim X$ ), then  $X$  and  $Y$  are  $K$ -equivalent.*

□

A consequence of this result is Bondal and Orlov’s “reconstruction theorem” [10].

**Theorem 2.26.** *Let  $X, Y$  be smooth Fourier-Mukai partners. If either  $\omega_X$  or  $\omega_Y$  is ample or anti-ample, there is an isomorphism  $X \simeq Y$ .* □

Very few results are available for Fourier-Mukai partners of singular varieties. It is known that any Fourier-Mukai partner of a Cohen-Macaulay (resp. Gorenstein) variety is Cohen-Macaulay (resp. Gorenstein) as well [29].

**2.3.2. Fourier-Mukai partners of curves.** Here we prove that the only Fourier-Mukai partner of a smooth projective curve is the curve itself.

**Theorem 2.27.** *A smooth curve  $X$  has no Fourier-Mukai partners but itself.*

*Proof.* Let  $Y$  be a Fourier-Mukai partner of  $X$  of genus  $g$ . By Theorem 2.26, one has  $X \simeq Y$  if  $g > 1$  or  $g = 0$  because in these cases the canonical line bundle is ample or anti-ample, respectively. Assume then that  $X$  is elliptic and take a Fourier-Mukai functor  $\Phi: D_c^b(Y) \rightarrow D_c^b(X)$ . One has  $\sum_i \dim \operatorname{Hom}_{D(X)}^1(\Phi^i(\mathcal{O}_y), \Phi^i(\mathcal{O}_y)) \leq 1$  for any point  $y \in Y$  (cf. Proposition 2.24), and then there is a unique value of  $i$  for which  $\Phi^i(\mathcal{O}_y) \neq 0$ . By Proposition 2.18,  $i$  is actually independent of  $y$ , and then  $Y$  is a fine moduli space of simple sheaves over  $X$  by Corollary 2.20. If the sheaves  $\Phi^i(\mathcal{O}_y)$  are torsion-free, they are stable by Corollary 4.6 and thus  $Y \simeq X$  by Corollary 4.7. If they have torsion, they are skyscraper sheaves of length 1, so that  $Y \simeq X$ .  $\square$

**2.3.3. Fourier-Mukai partners of surfaces.** The main result about Fourier-Mukai partners of algebraic surfaces is the following Theorem.

**Theorem 2.28.** *A smooth surface has a finite number of Fourier-Mukai partners.*  $\square$

This was proved by Bridgeland and Maciocia [20] for minimal surfaces. Kawamata [34] completed the result by including the surfaces with  $(-1)$ -curves. His proof is actually simpler and more direct, and exploits the geometric properties of the support of the kernel of the corresponding Fourier-Mukai functor.

The case of minimal surfaces is treated with a case-by-case approach, essentially based on the Enriques-Kodaira classification, in view of the fact that Fourier-Mukai partners have the same Kodaira dimension (Theorem 2.23).

**Proposition 2.29.** *Two smooth surfaces  $X$  and  $Y$  that are Fourier-Mukai partners have the same Picard number, the same Betti numbers, and therefore the same topological Euler characteristic.*

The study of the Fourier-Mukai partners of K3 and abelian surfaces is particularly interesting.

**Proposition 2.30.** *Let  $X$  be a K3 (resp. an abelian) surface and  $Y$  a Fourier-Mukai partner of  $X$ . Then  $Y$  is a K3 (resp. an abelian) surface as well.*

*Proof.* Since  $\omega_X$  is trivial,  $\omega_Y$  is trivial as well by Theorem 2.23. Then  $Y$  is either K3 or abelian. Moreover  $H^\bullet(X, \mathbb{Q}) \simeq H^\bullet(Y, \mathbb{Q})$  by Corollary 2.8, then if  $X$  is K3,  $Y$  is also K3 and if  $X$  is abelian,  $Y$  is abelian as well.  $\square$

By a result of Orlov, the Fourier-Mukai partners of a K3 or an abelian surface are completely characterised in terms of isometries of the transcendental lattice  $\mathbf{T}(X)$ . This is defined as the orthogonal complement to  $\operatorname{Pic}(X)$  in  $H^2(X, \mathbb{Z})$ .

**Proposition 2.31.** [48, Thm. 3.3] *Let  $X, Y$  be two K3 or abelian surfaces.  $X$  and  $Y$  are Fourier-Mukai partners if and only if the lattices  $\mathbf{T}(X)$  and  $\mathbf{T}(Y)$  are Hodge isometric.*  $\square$

**2.3.4. Fourier-Mukai partners for threefolds.** Minimal models of threefolds are in general not characterised; at present, one is only able to prove in some special cases that birational threefolds have equivalent derived categories. Two important results in this sense are the following, both due to Bridgeland.

**Theorem 2.32.** [19, Thm. 1.1] *Let  $X$  be a (complex) threefold with terminal singularities and  $f_1: Y_1 \rightarrow X$ ,  $f_2: Y_2 \rightarrow X$  crepant resolutions of singularities. Then there is an equivalence of triangulated categories  $D_c^b(Y_1) \simeq D_c^b(Y_2)$ .*  $\square$

Since any birational map between smooth Calabi-Yau threefolds is crepant, we deduce the following result.

**Theorem 2.33.** [19] *Let  $X$  and  $Y$  be two birational smooth Calabi-Yau threefolds. Then  $X$  and  $Y$  are Fourier-Mukai partners.*  $\square$

The proof of Theorem 2.32 relies on the fact that any crepant birational map between threefolds with only terminal singularities can be decomposed into a sequence of particularly simple birational transformations, called *flops*. This reduces the proof to the case of flops. There are two different proofs, one due to Bridgeland [19], who explicitly constructs the flop using a moduli spaces of point perverse sheaves, and another due to Van den Bergh [52] who uses noncommutative techniques. The description of all the Fourier-Mukai partners of a Calabi-Yau threefold is still unknown. Căldăraru [22] has found some explicit models of Fourier-Mukai partners for three-dimensional Calabi-Yau threefolds.

### 3. THE NAHM TRANSFORM

This construction was introduced by Nahm in 1983 [47] in 1983. Starting from an instanton on a 4-dimensional flat torus, it yields an instanton on the dual torus. Later this was formalised by Schenk [51] and Braam and van Baal [11]. According to their picture, the Nahm transform is an index-theoretic construction, where, given a vector bundle  $E$  on a flat torus  $X$  equipped with an anti-self-dual connection  $\nabla$ , the dual torus  $\hat{X}$  is regarded as the parameter space of a family of Dirac operators twisted by  $\nabla$ . The index of this family yields, under suitable conditions, an instanton  $\hat{\nabla}$  on  $\hat{X}$ . A survey of some properties of the standard version of the Nahm transform was given by M. Jardim in [33].

The connection between the Nahm and the Fourier-Mukai transform was seemingly first realised by Braam-van Baal and Schenk, and a first formalization is given by Donaldson and Kronheimer [23]. The link between the two constructions is a relation between index bundles and higher direct images, in accordance with Illusie’s definition of the “analytical index” of a relative elliptic complex [32].

Mainly following [2], we shall describe here the relation between the Fourier-Mukai and Nahm transforms by considering the second as a particular case of a more general class of transforms, that we call *Kähler Nahm transforms*. We shall also introduce a special case of such transforms when the manifolds involved have a hyperkähler structure, considering a generalization of the notion of instanton (the *quaternionic instantons*) and proving that the “hyperkähler Fourier-Mukai transform” preserves the quaternionic instanton condition. This will be mainly taken from [6]. However, the whole theory recalled in this section is described in more detail in [1].

**3.1. Line bundles on complex tori.** We provide here a few basic facts about the description of line bundles on complex tori, which will be useful to introduce the Nahm transform. If  $V$  is a  $g$ -dimensional complex vector space, and  $\Xi$  a nondegenerate lattice in it, the quotient  $T = V/\Xi$  comes with a natural structure of  $g$ -dimensional complex manifold, and is said to be a *complex torus* of dimension  $g$ . Any generator of the lattice  $\Xi$  corresponding to a loop in  $T$ , one has natural identifications  $\Xi \simeq \pi_1(T) \simeq H_1(T, \mathbb{Z})$ , and as a consequence, also  $H^k(T, \mathbb{Z}) \simeq \Lambda^k \Xi^*$ . Moreover, the space  $\mathcal{H}(T)$  of hermitian forms  $H: V \times V \rightarrow \mathbb{C}$  that satisfy the condition  $\text{Im}(H(\Xi, \Xi)) \subset \mathbb{Z}$  may be identified with the Néron-Severi group  $\text{NS}(T)$ .

**Definition 3.1.** A *semicharacter* associated with an element  $H \in \mathcal{H}(T)$  is a map  $\chi: \Xi \rightarrow U(1)$  such that  $\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{iH(\lambda, \mu)}$ . An element  $H \in \mathcal{H}(T)$  and an associated semicharacter  $\chi$  define an *automorphy factor*  $a: V \times \Xi \rightarrow U(1)$  by  $a(v, \lambda) = \chi(\lambda)e^{\pi H(v, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)}$ .

**Proposition 3.2.** [7] *The holomorphic functions  $s$  on  $V$  that satisfy the condition  $s(v + \lambda) = a(v, \lambda)s(v)$  for all  $v \in V$  and  $\lambda \in \Xi$ , where  $a$  is an automorphy factor associated with an element  $H \in \mathcal{H}(T)$ , are in a one-to-one correspondence with sections of a line bundle  $L$  on  $T$  such that  $c_1(L) = H$ .*

One may define the dual torus  $T^*$  as follows. Let  $\Omega$  be the conjugate dual space of  $V$ , and let  $\Xi^* = \{\ell \in \Omega \mid \ell(\Xi) \subset \mathbb{Z}\}$  be the lattice dual to  $\Xi$ . If we set  $T^* = \Omega/\Xi^*$ , by the natural isomorphism  $T^* \simeq \text{Hom}_{\mathbb{Z}}(\Xi, U(1))$  and the exact sequence

$$0 \rightarrow T^* \rightarrow \text{Pic}(T) \rightarrow \text{NS}(T) \rightarrow 0$$

we see that  $T^*$  parametrises flat  $U(1)$  bundles on  $T$ .

Proposition 3.2 may be used to construct the Poincaré bundle  $\mathcal{P}$  on  $T \times T^*$ . Let  $H \in \mathcal{H}(T \times T^*)$  be given by

$$(3.1) \quad H(v, w, \alpha, \beta) = \overline{\beta(v)} + \alpha(w)$$

where  $v, w \in V$ ,  $\alpha, \beta \in \Omega$ , and consider associated semicharacter

$$(3.2) \quad \chi(\lambda, \mu) = e^{i\pi \mu(\lambda)}.$$

The Poincaré bundle is the line bundle  $\mathcal{P}$  given by the hermitian form (3.1) and the semicharacter (3.2). This Poincaré bundle is automatically normalised as in Section 2. Moreover, it comes with a natural hermitian metric, which is expressed on the functions on the universal covering of  $T \times T^*$  corresponding (via the automorphy condition) to sections of  $\mathcal{P}$  in terms of the standard hermitian metric on  $\mathbb{C}^n$ . For any element  $\xi \in T^*$  we shall denote  $\mathcal{P}_\xi = \mathcal{P}|_{T \times \{\xi\}}$  the line bundle on  $T$  parametrised by  $\xi$ .

**3.2. Nahm transform.** Let us briefly recall Nahm's transform in its original version. Let  $T$  be a flat Riemannian 4-torus, equipped with a compatible complex structure,  $T^*$  its dual torus. As we have seen, the Poincaré bundle on  $T \times T^*$  comes with a natural hermitian metric. Let  $\nabla_{\mathcal{P}}$  be the corresponding Chern connection (the unique connection on  $\mathcal{P}$  compatible both with the hermitian metric and the complex structure of  $\mathcal{P}$ ). Furthermore, let  $E$  be an hermitian vector bundle on  $E$  whose Chern connection  $\nabla$  is anti-self-dual (ASD), i.e., its curvature  $F_{\nabla}$  satisfies the ASD condition  $F_{\nabla} = -*F_{\nabla}$ , where  $*$  is Hodge duality on forms on  $T$ . A survey of the Nahm transform and their most recent generalisations can be founded in [33].

If  $\xi \in T^*$  one has a coupled connection  $\nabla_\xi$  in  $E \otimes \mathcal{P}_\xi$ , and correspondingly, a family of *twisted Dirac operators*

$$(3.3) \quad D_\xi: \Gamma(E \otimes \mathcal{P}_\xi \otimes S_+) \rightarrow \Gamma(E \otimes \mathcal{P}_\xi \otimes S_-)$$

where  $S_\pm$  is the spin bundle of positive/negative helicity on  $T$ . If the pair  $(E, \nabla)$  satisfies an irreducibility condition (it is *without flat factors*, i.e., there is no  $\nabla$ -compatible splitting  $E = E' \oplus L$ , where  $L$  is a flat line bundle), then for every  $\xi \in T^*$  we have  $\ker D_\xi = 0$ , and then by Atiyah-Singer's index theory (minus) the index of the family of Dirac operators is a vector bundle  $\hat{E}$  on  $T^*$ , whose fibre at  $\xi \in T^*$  is the vector space  $\text{coker } D_\xi$ .

The bundle  $\hat{E}$  may be equipped with a metric and a compatible connection. Indeed, completing the spaces of sections appearing in Eq. (3.3) in the natural  $L^2$  norms, we have for every  $\xi \in T^*$  an exact sequence

$$0 \rightarrow \hat{E}_\xi \rightarrow L^2(E \otimes \mathcal{P}_\xi \otimes S_-) \xrightarrow{D_\xi^*} L^2(E \otimes \mathcal{P}_\xi \otimes S_+) \rightarrow 0$$

By restricting the scalar product in the space in the middle in this sequence one defines an hermitian metric in the bundle  $\hat{E}$ . Moreover, if we regard the spaces  $L^2(E \otimes \mathcal{P}_\xi \otimes S_\pm)$  as the fibres of trivial  $\infty$ -dimensional bundles on  $T^*$ , this exact sequence allows one to define a connection on  $\hat{E}$  by a projection formula: one takes a section of  $\hat{E}$ , regards it as section of the bundle in the middle, takes the covariant derivative with respect to the trivial connection, and then projects back to  $\hat{E}$  using the scalar product. One shows that the resulting connection  $\hat{\nabla}$  is compatible with the metric, and is ASD.

The pair  $(\hat{E}, \hat{\nabla})$  is the *Nahm transform* of  $(E, \nabla)$ . The Atiyah-Singer theorem for families allows one to compute the topological invariants of  $\hat{E}$ , getting

$$(\hat{r}, c_1(\hat{E}), \text{ch}_2(\hat{E})) = -(\text{ch}_2(E), c_1(E), r)$$

where to compare the first Chern classes we use the natural identification of the groups  $H^2(T, \mathbb{Z})$  and  $H^2(\hat{T}, \mathbb{Z})$ .

**3.3. Fourier-Mukai vs. Nahm.** In order to compare the Fourier-Mukai transform with Nahm's construction, it is convenient to recast the latter into a more general form. To this end, we consider a flat proper submersive holomorphic morphism of complex manifolds  $f: Z \rightarrow Y$ . We call the sheaf  $\mathcal{O}_{Z/Y} = f^{-1}\mathcal{C}_Y^\infty \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_Z$  the sheaf of “relatively holomorphic functions” (locally, if  $x$  and  $y$  are holomorphic coordinates on the fibres of  $f$  and on  $Y$  respectively, the sections of  $\mathcal{O}_{Z/Y}$  are functions of the variables  $x, y, \bar{y}$ ).

If  $F$  is a holomorphic vector bundle on  $Z$  (whose sheaf of sections we shall denote by  $\mathcal{F}$ ), then  $\mathcal{F}^r = \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z/Y}$  has a relative holomorphic structure i.e., its sheaf of sections has a structure of  $\mathcal{O}_{Z/Y}$ -module. The relative Dolbeault complex provides a (fine) resolution of this  $\mathcal{O}_{Z/Y}$ -module:

$$0 \rightarrow \mathcal{F}^r \rightarrow \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{C}_Z^\infty \xrightarrow{\bar{\partial}_{Z/Y}} \mathcal{F} \otimes_{\mathcal{O}_Z} \Omega_{Z/Y}^{0,1} \rightarrow \dots$$

Moreover, one has  $R^i f_* \mathcal{F}^r \simeq R^i f_* \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{C}_Y^\infty$ , i.e., the higher direct images of  $\mathcal{F}^r$  come with a natural holomorphic structure.

**Definition 3.3.** The bundle  $F$  satisfies the even (odd) IT condition if  $R^i f_* \mathcal{F}^r = 0$  for  $i$  odd (even), and the non-vanishing higher direct images are locally free.

Now, let us assume that the sheaf of relative differentials  $\Omega_{Z/Y}^1$  has a *relative Kähler structure*, i.e., it is equipped with an hermitian metric such that the corresponding 2-form is closed under the relative exterior differential. We define relative spin bundles  $\Sigma_\pm = \bigoplus_{k \text{ even, odd}} \bigwedge^k \Omega_{Z/Y}^{0,1}$ , and a relative Dirac operator  $D = \bar{\partial}_{Z/Y} + \bar{\partial}_{Z/Y}^*: f_*(\mathcal{F} \otimes_{\mathcal{O}_Z} \Sigma_+) \rightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_Z} \Sigma_-)$ . By restricting to the fibres of  $f$ , this provides a family of Dirac operators parametrised by  $Y$ .

**Theorem 3.4.** Assume that the relative canonical bundle of  $f: Z \rightarrow Y$  is trivial. If  $F$  satisfies (e.g.) the even IT condition, then  $\ker D = 0$ , and

$$-\text{ind}(D) = \text{coker}(D) \simeq \bigoplus_{k \text{ odd}} R^k f_* \mathcal{F}^r.$$

If  $F$  has an hermitian metric, let  $\nabla$  be the corresponding Chern connection. By generalizing the constructions we have seen in the case of the original Nahm transform, one can use these data to induce on  $\text{coker}(D)$  an hermitian metric and a connection  $\hat{\nabla}$ , which turns out to be a Chern connection, and may be regarded as a “direct image” of  $\nabla$ .

Now we apply these constructions to the case when  $Z$  is a product  $X \times Y$  to define a generalised Nahm transform. We denote by  $\pi_X, \pi_Y$  the projections of  $X \times Y$  onto its factors. Let  $X$  be a compact Kähler manifold with trivial canonical bundle,  $Y$  a Kähler manifold,  $E$  an hermitian holomorphic vector bundle on  $X$ , and finally, let  $Q$  be an hermitian holomorphic vector bundle on  $X \times Y$ . Applying the previous construction to the (hermitian holomorphic) bundle  $\pi_X^* E \otimes Q$  on  $X \times Y$  we construct an hermitian bundle  $\hat{E}$  on  $Y$ . The pair  $(\hat{E}, \hat{\nabla})$ , where  $\hat{\nabla}$  is the Chern connection of the hermitian holomorphic bundle  $\hat{E}$ , is the *generalised Nahm transform* of  $(E, \nabla)$ .

The formalism we have so far developed provides a direct proof of the following compatibility condition between the Fourier-Mukai and generalised Nahm transforms.

**Theorem 3.5.** *Assume that  $F = \pi_X^* E \otimes Q$  satisfies an IT condition (say the even one). Then the sheaf of holomorphic sections of  $\hat{E} = \text{coker}(D)$  is isomorphic to the Fourier-Mukai transform  $\Phi_{X \rightarrow Y}^Q(\mathcal{E})$  of the sheaf  $\mathcal{E}$  of holomorphic sections of  $E$ .*

Here  $\mathcal{Q}$  is the sheaf of holomorphic sections of the bundle  $Q$ .

**3.4. Hyperkähler Fourier-Mukai transform.** In some situations the Fourier-Mukai transform preserves the stability of the sheaves it acts on. We have seen that original Nahm transform maps instantons to instantons. These two results are actually related by the so-called Hitchin-Kobayashi correspondence, according to which on a compact Kähler manifold, a holomorphic vector bundle is polystable (i.e., it is a direct sum of stable sheaves having the same slope) if and only if it carries an hermitian metric which satisfies a certain differential condition (it is an Hermitian-Yang-Mills metric). In the case of complex dimension 2, and for bundles of zero degree, the Hermitian-Yang-Mills condition is equivalent to saying that the Chern connection is ASD, and this establishes the link between the two “preservation” results. One might wonder if our generalised Nahm transform may provide further examples of preservation of some instanton-like condition. One such instance is provided by hyperkähler geometry.

Let  $X$  be a hyperkähler manifold, and let  $I_k$  be 3 basic complex structures. The automorphism  $\sum_k I_k \otimes I_k$  acting on  $\Lambda^2 T^* X$  has two eigenspaces, with eigenvalues 3 and -1 respectively:  $\Lambda^2 T^* X = \mathfrak{e}_1 \oplus \mathfrak{e}_2$ . One has  $(\mathfrak{e}_1)_x = \bigcap_{u \in Z_x} (\Lambda^2 T^* X)_u^{1,1}$ , where  $Z \rightarrow X$  is the twistor space of  $X$ .

**Definition 3.6.** A connection  $\nabla$  on a complex vector bundle  $E$  on  $X$  is a *quaternionic instanton* if its curvature  $F_\nabla$  takes values in  $\mathfrak{e}_1$ .

Let us say that a complex vector bundle  $E$  is *hyperstable* if it is stable with respect to any Kähler structure in the hyperkähler family of  $X$ .

**Theorem 3.7.** *Let  $X$  and  $Y$  be hyperkähler manifolds,  $(E, \nabla)$  a quaternionic instanton on  $X$ , and  $(Q, \tilde{\nabla})$  a quaternionic instanton on  $X \times Y$ . The Nahm transform of  $(E, \nabla)$  with kernel  $(Q, \tilde{\nabla})$  is a quaternionic instanton on  $Y$ .*

This may be regarded as a “stability preservation” result in view of the following extension of the Hitchin-Kobayashi correspondence, which is not difficult to prove.



**Theorem 3.8.** *Let  $E$  be a vector bundle on a hyperkähler manifold  $X$  which has zero degree with respect to all Kähler structures in the hyperkähler family of  $X$ . Then  $E$  is hyperstable if and only if it admits an hermitian metric such that the corresponding Chern connection is an irreducible quaternionic instanton.*

The proof of Theorem 3.7 exploits a generalization of the classical Atiyah-Ward correspondence. The latter states that there is a one-to-one correspondence between instantons on a compact, connected, orientable ASD Riemannian 4-manifold  $X$ , and holomorphic vector bundles on the twistor space  $Z$  of  $X$  that are holomorphically trivial along the fibres of the projection  $Z \rightarrow X$  (as stated, this correspondence holds true for instantons whose structure group is the general linear group. Unitary instantons on  $X$  correspond to bundles on  $Z$  that carry an additional structure, called a *real form*).

The generalization of this correspondence to quaternionic instantons on higher-dimensional hyperkähler manifolds reads as follows.

**Theorem 3.9.** *There is a one-to-one correspondence between the following objects:*

- (1) *gauge equivalence classes of (hermitian) quaternionic instantons on a hyperkähler manifold  $X$ ;*
- (2) *isomorphism classes of holomorphic vector bundles on the twistor space  $Z$  of  $X$ , holomorphically trivial along the fibres of  $Z$  (carrying a positive real form).*

The proof of Theorem 3.7 is based on the natural isomorphism  $Z_{X \times Y} \simeq Z_X \times_{\mathbb{P}^1} Z_Y$  and on the following commutative diagram

$$\begin{array}{ccccc} Z_X & \xleftarrow{t_1} & Z_{X \times Y} & \xrightarrow{t_2} & Z_Y \\ p_1 \downarrow & & q \downarrow & & p_2 \downarrow \\ X & \xleftarrow{\pi_1} & X \times Y & \xrightarrow{\pi_2} & Y \end{array} .$$

All data in the spaces in the bottom row are lifted to the first row by using the generalised Atiyah-Ward correspondence. Then one performs a (relative) Fourier-Mukai transform between  $Z_X$  and  $Z_Y$ , and descends from  $Z_Y$  to  $Y$  using the generalised Atiyah-Ward correspondence again, after several consistency checks.

#### 4. MODULI SPACES OF SHEAVES AND COHERENT SYSTEMS ON ELLIPTIC CURVES

Since its very first appearance the Fourier-Mukai transform has been an important tool in the study of moduli spaces of sheaves. A key feature is that, under suitable hypotheses, the Fourier-Mukai transform preserves the stability (or semistability) of sheaves and thus produces isomorphisms between different moduli spaces. Among such applications, one can list the original contributions by Mukai [45, 44] and the study of moduli spaces of stable sheaves on abelian or K3 surfaces [40, 24, 21, 5].

Since a complete account of all the applications of the Fourier-Mukai transform exceeds the scope of these notes (see [1] for a comprehensive treatment), we devote this Section to describing two particularly interesting examples. The first, which can be nowadays considered as classical, is the moduli spaces of sheaves on elliptic curves. The second is the theory of coherent systems on an elliptic curve.

In the first case, the Fourier-Mukai transform provides new and easier proofs of Atiyah's classical theorems. This approach was introduced in [17, 49, 27] but our treatment, taken directly from [1, Chap. 3], is somehow different.

The application of the Fourier-Mukai transform to the study of coherent systems on elliptic curves follows recent work by two of the authors [30]. Coherent systems on algebraic curves are “decorated objects” and their definition, notion of stability and moduli spaces have been introduced and studied by Le Potier [38], King and Newstead [36], García-Prada, Bradlow, Muñoz and Newstead [12, 13]. The specific case of elliptic curves has been studied by Lange and Newstead [37].

**4.1. Application of Fourier-Mukai transforms to the moduli spaces of sheaves on elliptic curves.** Let  $X$  be an elliptic curve, i.e., a smooth curve of genus 1 with a fixed point  $x^0$ . Then  $X$  can be regarded as an abelian variety of dimension one such that  $x^0$  is the identity of the group law in  $X$ . The morphism  $X \rightarrow \hat{X}$  mapping  $x$  to the line bundle  $\mathcal{O}_X(x - x^0)$  is an isomorphism, so that we can identify  $X$  with its dual variety  $\hat{X}$ . Using this identification we can write the Poincaré bundle described in Section 2 in the form

$$(4.1) \quad \mathcal{P} \simeq \mathcal{O}_{X \times X}(\Delta_\iota) \otimes \pi_1^* \mathcal{O}_X(-x^0) \otimes \pi_2^* \mathcal{O}_X(-x^0),$$

where  $\Delta_\iota$  is the graph of the isomorphism  $\iota: X \rightarrow X$  defined as  $\iota(x) = -x$ . Both the abelian Fourier-Mukai transform  $\mathbf{S}$  and the dual abelian Fourier-Mukai transform  $\hat{\mathbf{S}}$  are autoequivalences of  $D_c^b(X)$ .

Given an object  $\mathcal{E}^\bullet$  of  $D_c^b(X)$ , we can write its Chern character as  $\text{ch}(\mathcal{E}^\bullet) = (n, d)$ , where  $n = \text{ch}_0(\mathcal{E}^\bullet)$  is its rank and  $d = \text{ch}_1(\mathcal{E}^\bullet)$  its degree, thought as an integer number. Equation (4.1) and Grothendieck-Riemann-Roch give the following result.

**Proposition 4.1.** *If  $\text{ch}(\mathcal{E}^\bullet) = (n, d)$ , then  $\text{ch}(\mathbf{S}(\mathcal{E}^\bullet)) = (d, -n)$ .* □

When  $\mathcal{E}^\bullet$  is  $\text{WIT}_i$ , one has  $\mathbf{S}(\mathcal{E}) = \hat{\mathcal{E}}[-i]$  so that  $\text{ch}(\mathbf{S}(\mathcal{E}^\bullet)) = (-1)^i \text{ch}(\hat{\mathcal{E}})$ .

**4.1.1. (Semi)stable sheaves on an elliptic curve.**  $\mu$ -semistable sheaves on an elliptic curve  $X$  are characterized by the following result [49, Lemma 14.5].

**Proposition 4.2.** *Any indecomposable torsion-free sheaf on  $X$  is semistable.*

*Proof.* If  $\mathcal{E}$  is a torsion-free sheaf on  $X$  and  $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$  is its Harder-Narasimhan filtration, the quotient sheaves  $\mathcal{G}_i = \mathcal{E}_i / \mathcal{E}_{i-1}$  are  $\mu$ -semistable with  $\mu(\mathcal{G}_i) > \mu(\mathcal{G}_{i+1})$ . Thus  $\text{Hom}_X(\mathcal{G}_i, \mathcal{G}_{i+1}) = 0$  so that  $\text{Ext}_X^1(\mathcal{G}_{i+1}, \mathcal{G}_i) = 0$  by Serre duality. This implies that the Harder-Narasimhan filtration splits, then if  $\mathcal{E}$  is indecomposable it is also semistable. □

**Corollary 4.3.** *Let  $\mathcal{E}$  be a semistable sheaf of rank  $n$  and degree  $d$  on  $X$ .*

- (1) *If  $d < 0$ , then  $\mathcal{E}$  is  $\text{IT}_1$  for both  $\mathbf{S}$  and  $\hat{\mathbf{S}}$ , and both transforms are semistable.*
- (2) *If  $d > 0$ , then  $\mathcal{E}$  is  $\text{IT}_0$  for both  $\mathbf{S}$  and  $\hat{\mathbf{S}}$ , and both transforms are semistable.*
- (3) *If  $d \neq 0$ , then  $\mathcal{E}$  is locally free.*
- (4) *If  $d = 0$  and  $\mathcal{E}$  is stable, then  $\mathcal{E}$  is a line bundle. Thus, any semistable sheaf of degree 0 is  $\text{WIT}_1$  and the unique transform  $\hat{\mathcal{E}}$  is a skyscraper sheaf. Moreover a torsion-free sheaf of degree 0 is semistable if and only if it is  $S$ -equivalent to a direct sum of degree 0 line bundles:  $\mathcal{E} \sim \bigoplus_i \mathcal{L}_i^{\oplus n_i}$ , with  $\sum_i n_i = n$ .*

*Proof.* 1. One has  $H^0(X, \mathcal{E} \otimes \mathcal{P}_\xi) \simeq \text{Hom}_X(\mathcal{P}_\xi^*, \mathcal{E}) = 0$  for every point  $\xi \in \hat{X}$  since  $\mathcal{E}$  is semistable and  $d < 0$ . Then  $\mathcal{E}$  is  $\text{IT}_1$  by Proposition 2.2. Assume that  $\mathcal{E}$  is semistable. To prove that  $\hat{\mathcal{E}}$  is semistable we can assume that  $\mathcal{E}$  is indecomposable;

then  $\widehat{\mathcal{E}}$  is indecomposable as well, and then it is semistable by Proposition 4.2. A similar argument proves the semistability of  $\widehat{\mathbf{S}}(\mathcal{E})$ .

2. One has  $H^1(X, \mathcal{E} \otimes \mathcal{P}_\xi)^* \simeq \text{Hom}_X(\mathcal{E} \otimes \mathcal{P}_\xi, \mathcal{O}_X) \simeq \text{Hom}_X(\mathcal{E}, \mathcal{P}_\xi^*)$  by Serre duality. Since  $\mathcal{E}$  is semistable of positive degree, the second group is zero and then  $\mathcal{E}$  is  $\text{IT}_0$ . Proceeding as in the first part one proves the semistability of  $\widehat{\mathcal{E}}$ . The proof for  $\widehat{\mathbf{S}}$  is analogous.

3. If  $d \neq 0$ ,  $\mathcal{E}$  is either  $\text{IT}_0$  or  $\text{IT}_1$  with respect to  $\mathbf{S}$ , according to whether  $d > 0$  or  $d < 0$ , due to part 1 or 2. Moreover,  $\widehat{\mathcal{E}}$  is semistable of nonzero degree, so that it is  $\text{IT}_1$  or  $\text{IT}_0$  with respect to  $\widehat{\mathbf{S}}$ . It follows that  $\mathcal{E} \simeq \widehat{\mathbf{S}}^{1-i}(\mathbf{S}^i(\mathcal{E}))$  (with  $i = 0$  or  $1$ ) is locally free.

4. If  $\mathcal{E}$  is stable of degree 0, then  $H^0(X, \mathcal{E} \otimes \mathcal{P}_\xi) \simeq \text{Hom}_X(\mathcal{P}_\xi^*, \mathcal{E}) = 0$  unless  $\mathcal{E} \simeq \mathcal{P}_\xi^*$ . Thus if  $\mathcal{E}$  is not a line bundle, it is  $\text{IT}_1$ ; by Proposition 4.1  $\widehat{\mathcal{E}}$  is locally free of rank 0; thus  $\widehat{\mathcal{E}} = 0$  so that  $\mathcal{E} = 0$  by the invertibility of  $\mathbf{S}$ . For the second part, assume that  $\mathcal{E}$  is semistable of degree 0; then it has a Jordan-Holder filtration  $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$  whose quotients  $\mathcal{G}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$  are stable of degree 0, that is, they are line bundles of degree 0. Thus,  $\mathcal{G}_i \simeq \mathcal{P}_{\xi_i}$  for a point  $\xi_i \in \hat{X}$ . Since the sheaves  $\mathcal{P}_{\xi_i}$  are  $\text{WIT}_1$  and  $\widehat{\mathcal{P}}_{\xi_i} \simeq \mathcal{O}_{\iota(\xi_i)}$ , we see that  $\mathcal{E}$  is  $\text{WIT}_1$  and  $\widehat{\mathcal{E}}$  is a skyscraper sheaf. Analogous arguments prove that  $\mathcal{E}$  is  $\text{WIT}_1$  with respect to  $\widehat{\mathbf{S}}$  and that  $\widehat{\mathbf{S}}^1(\mathcal{E})$  is a skyscraper sheaf.  $\square$

4.1.2. *Geometry of the moduli spaces of stable sheaves on elliptic curves.* Let us consider the full subcategory  $\mathfrak{Coh}_{n,d}^{ss}(X)$  of the category  $\mathfrak{Coh}(X)$  of coherent sheaves on  $X$  whose objects are semistable sheaves of rank  $n$  and degree  $d$ . We also consider the category  $\mathfrak{Skh}_n(X)$  of skyscraper sheaves of length  $n$  on  $X$ . Corollary 4.3 implies the following result.

**Proposition 4.4.** *The abelian Fourier-Mukai transform induces equivalences of categories  $\mathfrak{Coh}_{n,d}^{ss}(X) \simeq \mathfrak{Coh}_{d,-n}^{ss}(X)$  if  $d > 0$ , and  $\mathfrak{Coh}_{n,0}^{ss}(X) \simeq \mathfrak{Skh}_n(X)$ .*  $\square$

The Fourier-Mukai transform  $\Psi = \Phi_{X \rightarrow X}^{\delta_* \mathcal{L}} : D^b(X) \xrightarrow{\sim} D^b(X)$ , which is nothing but the twist by  $\mathcal{L} = \mathcal{O}_X(x^0)$ , also induces an equivalence  $\mathfrak{Coh}_{n,d}^{ss}(X) \simeq \mathfrak{Coh}_{n,d+n}^{ss}(X)$ . By composing the Fourier-Mukai transforms  $\mathbf{S}$  and  $\Psi$  in an appropriate way, and using Euclid's algorithm we have:

**Proposition 4.5.** *For every pair  $(n, d)$  of integers ( $n > 0$ ), there is a Fourier-Mukai functor  $\tilde{\Phi} : D^b(X) \xrightarrow{\sim} D^b(X)$  which induces an equivalence of categories*

$$\mathfrak{Coh}_{n,d}^{ss}(X) \simeq \mathfrak{Coh}_{\bar{n},0}^{ss}(X) \xrightarrow{\mathbf{S}} \mathfrak{Skh}_{\bar{n}}(X),$$

where  $\bar{n} = \text{gcd}(n, d)$ .  $\square$

**Corollary 4.6.** *Let  $\mathcal{E}$  be a torsion-free sheaf of rank  $n$  and degree  $d$  on  $X$ . Then  $\mathcal{E}$  is stable if and only if it is simple, and if and only if it is semistable and  $\text{gcd}(n, d) = 1$ . Thus, the integral functors of Proposition 4.5 map stable sheaves to stable sheaves.*  $\square$

The structure of the coarse moduli space  $\mathcal{M}^{ss}(n, d)$  of semistable sheaves of rank  $n$  and degree  $d$  on  $X$  can be also obtained in a similar way. Let  $\text{Sym}^n X$  be the  $n$ -th symmetric product.

**Corollary 4.7.** *For every pair  $(n, d)$  of integers ( $n > 0$ ), there is a Fourier-Mukai functor which induces an isomorphism of moduli spaces*

$$\mathcal{M}^{ss}(n, d) \simeq \mathcal{M}^{ss}(\bar{n}, 0) \xrightarrow{\mathbf{S}} \text{Sym}^{\bar{n}} X,$$

where  $\bar{n} = \gcd(n, d)$ . Then, if  $Y$  is a nonempty moduli space of stable torsion-free sheaves on  $X$ , there is an isomorphism  $Y \simeq X$ .

*Proof.* Propositions 4.5 and 2.19 imply the first part, because  $\mathrm{Sym}^{\bar{n}} X$  is a coarse moduli space for the moduli functor of skyscraper sheaves of length  $\bar{n}$  on  $X$ . The last part follows now from Corollary 4.6.  $\square$

**4.1.3. Autoequivalences of the derived category of an elliptic curve.** Let  $X$  be an elliptic curve. Since  $H^{\mathrm{even}}(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ , if  $\Phi = \Phi_{X \rightarrow X}^{\mathcal{K}^\bullet} : D_c^b(X) \rightarrow D_c^b(X)$  is an integral functor ( $\mathcal{K}^\bullet \in D_c^b(X \times X)$ ), Equation (2.2.1) yields a diagram

$$\begin{array}{ccc} D_c^b(X) & \xrightarrow{\Phi} & D_c^b(X) \\ v \downarrow & & \downarrow v \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\Phi_*} & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

where  $\Phi_* = f^{\mathcal{K}^\bullet}$ . When  $\Phi$  is a Fourier-Mukai functor,  $\Phi_*$  is a matrix in  $SL(2, \mathbb{Z})$ . For instance, if  $\mathcal{P}$  is the Poincaré line bundle on  $X \times X$  and  $\mathbf{S} = \Phi_{X \rightarrow X}^{\mathcal{P}}$  is the abelian Fourier-Mukai transform, one has  $\mathbf{S}_* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  (cf. Proposition 4.1).

Due to Orlov's representation Theorem 2.6, we have a representation in  $SL(2, \mathbb{Z})$  of the group  $\mathrm{Aut}(D_c^b(X))$  of derived auto-equivalences of the derived category of  $X$ . The study of such representation is due to Bridgeland [16], who proved the following result.

**Proposition 4.8.** *Given a matrix  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$  such that  $\beta > 0$ , there exist vector bundles on  $X \times X$  that are strongly simple over both factors, and which restrict to give bundles of Chern character  $(\beta, \alpha)$  on the first factor and  $(\beta, \delta)$  on the second. For any such bundle  $\mathcal{Q}(A)$ , the associated integral functor  $\Phi^{\mathcal{Q}(A)}$  is a Fourier-Mukai transform, and moreover  $\Phi_*^{\mathcal{Q}(A)} = A$ .*  $\square$

Thus, Proposition 4.8 essentially describes all the Fourier-Mukai functors on an elliptic curve. A more precise result was proved by Hille and van den Bergh [31].

**Theorem 4.9.** *Let  $X$  be an elliptic curve. There is an exact sequence of groups*

$$0 \rightarrow 2\mathbb{Z} \times \mathrm{Aut}(X) \ltimes \mathrm{Pic}^0(X) \rightarrow \mathrm{Aut}(D_c^b(X)) \xrightarrow{\mathrm{ch}} SL(2, \mathbb{Z}) \rightarrow 0,$$

where  $n \in \mathbb{Z}$  acts as shift functor  $[n]$ , the transform corresponding to  $(\varphi, \mathcal{L}) \in \mathrm{Aut}(X) \ltimes \mathrm{Pic}^0(X)$  sends  $\mathcal{E}^\bullet$  to  $\varphi_*(\mathcal{L} \otimes \mathcal{E}^\bullet)$ .  $\square$

Here we have set  $\mathrm{ch}(\Phi) = \Phi_*$ . Given  $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$  with  $\beta > 0$ , by proceeding as in the proof of Corollary 4.3 one can determine the behaviour of the transforms  $\Phi^{\mathcal{Q}(A)}$  [30].

**Proposition 4.10.** *Let  $\mathcal{E}$  be a semistable (resp. stable) vector bundle of Chern character  $\mathrm{ch}(\mathcal{E}) = (r, d)$ .*

- (1) *If  $\alpha r + \beta d > 0$  then  $\mathcal{E}$  is  $IT_0$  with respect to  $\Phi^{\mathcal{Q}(A)}$  and the unique transform  $\widehat{\mathcal{E}}$  is also semistable (resp. stable).*
- (2) *If  $\alpha r + \beta d = 0$  then  $\mathcal{E}$  is  $WIT_1$  with respect to  $\Phi^{\mathcal{Q}(A)}$  and the unique transform  $\widehat{\mathcal{E}}$  is a torsion sheaf.*
- (3) *If  $\alpha r + \beta d < 0$  then  $\mathcal{E}$  is  $IT_1$  with respect to  $\Phi^{\mathcal{Q}(A)}$  and the unique transform  $\widehat{\mathcal{E}}$  is also semistable (resp. stable).*

$\square$

If we write  $\Psi^{\mathcal{Q}(A)} = \Phi^{\mathcal{Q}(A)*}$ , the functor  $\Psi^{\mathcal{Q}(A)}[1]$  is a quasi-inverse of  $\Phi^{\mathcal{Q}(A)}$ .

We are now interested in the Fourier-Mukai functors  $\Phi$  such that  $\Phi(\mathcal{O}_X) \simeq \mathcal{O}_X[i]$  for some integer  $i$ ; that is,  $\mathcal{O}_X$  is  $\text{WIT}_i$  with respect to  $\Phi$  and  $\Phi^i(\mathcal{O}_X) = \mathcal{O}_X$ . These Fourier-Mukai functors will be relevant in the study of coherent systems on an elliptic curve, as we will see in Section 4.2. Using Proposition 4.10 and the similar statement for  $\Psi$  one proves:

**Proposition 4.11.** *Let  $a$  be a positive integer. There exists a Fourier-Mukai transform  $\Phi_a: D_c^b(X) \rightarrow D_c^b(X)$ , unique up to composition with an automorphism of  $X$ , such that  $(\Phi_a)_* = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$  and  $\Phi_a(\mathcal{O}_X) \simeq \mathcal{O}_X$  (that is,  $\mathcal{O}_X$  is  $IT_0$  and  $\Phi_a^0(\mathcal{O}_X) = \mathcal{O}_X$ ).  $\square$*

**4.2. Coherent systems.** A coherent system of type  $(r, d, k)$  on a smooth projective curve  $X$  is defined as a pair  $(\mathcal{E}, V)$  consisting of a vector bundle  $\mathcal{E}$  (a locally free sheaf) of rank  $r$  and degree  $d$  over  $X$  and a vector subspace  $V \subset H^0(X, \mathcal{E})$  of dimension  $k$ . A morphism  $f: (\mathcal{E}', V') \rightarrow (\mathcal{E}, V)$  of coherent systems is a homomorphism of vector bundles  $f: \mathcal{E}' \rightarrow \mathcal{E}$  such that  $f(V') \subset V$ . If  $\mathcal{E}'$  is a subbundle of  $\mathcal{E}$  then we say that  $(\mathcal{E}', V')$  is a *coherent subsystem* of  $(\mathcal{E}, V)$ . Coherent systems on  $X$  form an additive category  $\mathfrak{S}(X)$  (see [38] §4.1).

As for many other decorated objects, the notion of stability (and semistability) for coherent systems depends on the choice of a real parameter. For any real number  $\alpha$ , the  $\alpha$ -slope of a coherent system  $(\mathcal{E}, V)$  of type  $(r, d, k)$  is defined by

$$\mu_\alpha(\mathcal{E}, V) = \frac{d}{r} + \alpha \frac{k}{r}.$$

A coherent system  $(\mathcal{E}, V)$  is called  $\alpha$ -stable ( $\alpha$ -semistable) if

$$\mu_\alpha(\mathcal{E}', V') < \mu_\alpha(\mathcal{E}, V) \quad (\mu_\alpha(\mathcal{E}', V') \leq \mu_\alpha(\mathcal{E}, V))$$

for every proper coherent subsystem  $(\mathcal{E}', V')$  of  $(\mathcal{E}, V)$ .

A coherent system  $(\mathcal{E}, V)$  gives rise to an evaluation map  $V \otimes \mathcal{O}_X \rightarrow \mathcal{E}$ . This enables us to consider coherent systems as objects of the abelian category  $\mathfrak{C}(X)$  whose objects are arbitrary sheaf maps  $\varphi: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}$ , where  $V$  is a finite dimensional vector space and  $\mathcal{E}$  is any coherent sheaf (cf. [36]). A morphism from  $\varphi_1: V_1 \otimes \mathcal{O}_X \rightarrow \mathcal{E}_1$  to  $\varphi_2: V_2 \otimes \mathcal{O}_X \rightarrow \mathcal{E}_2$  in  $\mathfrak{C}(X)$  is defined by a linear map  $f: V_1 \rightarrow V_2$  and a sheaf morphism  $g: \mathcal{E}_1 \rightarrow \mathcal{E}_2$  such that the obvious diagram commutes. With this definition, the category  $\mathfrak{S}(X)$  of coherent systems is a full subcategory of  $\mathfrak{C}(X)$ .

One can easily see that the condition for an object  $\varphi: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  of  $\mathfrak{C}(X)$  to represent a coherent system is that  $\mathcal{E}$  is a vector bundle and the induced map  $H^0(\varphi): V \rightarrow H^0(X, \mathcal{E})$  is injective. The latter condition is equivalent to  $H^0(X, \ker \varphi) = 0$ .

One can extend to  $\mathfrak{C}(X)$  the notion of  $\alpha$ -(semi)stability of coherent systems, and this extension does not introduce new semistable objects (cf. [36]). Moreover, the full subcategory  $\mathfrak{S}_{\alpha, \mu}(X)$  of  $\mathfrak{C}(X)$  consisting of  $\alpha$ -semistable coherent systems with fixed  $\alpha$ -slope  $\mu$  is a Noetherian and Artinian abelian category whose simple objects are precisely the  $\alpha$ -stable coherent systems [36, 50].

**4.3. Moduli spaces of coherent systems.** There exists a (coarse) moduli space for the  $\alpha$ -stable coherent systems of type  $(r, d, k)$  on  $X$ . It is a quasiprojective variety which we denote by  $G(\alpha; r, d, k)$ . The reader is referred to [13, Section 2.1] and [14] for further information about  $G(\alpha; r, d, k)$ .

Here, we only recall that  $\alpha$ -stable coherent systems exist only for  $\alpha > 0$  if  $k \geq 1$ . The range of the parameter  $\alpha$  is divided into open intervals determined by a finite number

of critical values

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L.$$

The moduli spaces for all values of  $\alpha$  in the interval  $(\alpha_i, \alpha_{i+1})$  are isomorphic; if  $k \geq r$  this is also true for the interval  $(\alpha_L, \infty)$  (see [13, Propositions 4.2 & 4.6]).

In this Section we describe some well known fact about the moduli spaces of coherent systems for the two limit cases of small and large values of  $\alpha$ . We will always assume  $d \neq 0$  and  $k > 0$ .

**4.3.1. Small values of the parameter.** We denote by  $G_0(r, d, k)$  the moduli space of  $\alpha$ -stable coherent systems of type  $(r, d, k)$  with  $0 < \alpha < \alpha_1$ , where  $\alpha_1$  is the first critical value.

**Proposition 4.12.** [50] *A coherent system  $(\mathcal{E}, V)$  of type  $(r, d, k)$  is  $\alpha$ -stable, with  $0 < \alpha < \alpha_1$ , if and only if  $\mathcal{E}$  is semistable and  $k'/r' < k/r$ , for all coherent subsystems  $(\mathcal{E}', V')$  of type  $(r', d', k')$  with  $0 \neq \mathcal{E}' \neq \mathcal{E}$  and  $\mu(\mathcal{E}') = \mu(\mathcal{E})$ .  $\square$*

**4.3.2. Large values of the parameter.** Let us denote by  $G_L(r, d, k)$  the moduli space of  $\alpha$ -stable coherent systems of type  $(r, d, k)$  with  $\alpha_L < \alpha < \frac{d}{r-k}$  (we are assuming that  $0 < k < r$ ).  $G_L(r, d, k)$  has been described by Bradlow and García-Prada [12] (see also [13]) in terms of the Brambilla-Grzegorzczuk-Newstead extensions, BGN extensions for short [15].

BGN extensions of type  $(r, d, k)$  are defined as extensions of vector bundles

$$0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0,$$

where  $\mathcal{E}$  has rank  $r > k$  and degree  $d > 0$ , which satisfy the following conditions:

- (1)  $H^0(X, \mathcal{F}^*) = 0$
- (2) If  $(e_1, \dots, e_k) \in \text{Ext}_X^1(\mathcal{F}, \mathcal{O}_X^{\oplus k}) \simeq H^1(X, \mathcal{F}^*)^{\oplus k}$  denotes the class of the extension, then  $e_1, \dots, e_k$  are linearly independent as vectors in  $H^1(X, \mathcal{F}^*)$ .

Coherent systems can be regarded as BGN-extensions due to the following result.

**Proposition 4.13.** [12, Proposition 4.1] *Let  $(\mathcal{E}, V)$  be an  $\alpha$ -semistable coherent system of type  $(r, d, k)$  with  $\alpha_L < \alpha < \frac{d}{r-k}$ . The evaluation map of  $(\mathcal{E}, V)$  defines a BGN extension  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$ , with  $\mathcal{F}$  semistable. Moreover, any BGN extension  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$  where the quotient  $\mathcal{F}$  is stable gives rise to an  $\alpha$ -stable coherent system, with  $\alpha_L < \alpha < \frac{d}{r-k}$ .  $\square$*

A complete characterisation of the BGN-extensions which give rise to  $\alpha$ -stable coherent systems has been given in [30].

**Proposition 4.14.** *A BGN extension of type  $(r, d, k)$ ,  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$  defines an  $\alpha$ -stable coherent system, with  $\alpha_L < \alpha < \frac{d}{r-k}$ , if and only if  $\mathcal{F}$  is semistable and one has  $k'/r' > k/r$  for all subextensions  $0 \rightarrow \mathcal{O}_X^{\oplus k'} \rightarrow \mathcal{E}' \rightarrow \mathcal{F}' \rightarrow 0$  of type  $(r', d', k')$  with  $\mu(\mathcal{F}') = \mu(\mathcal{F})$ .  $\square$*

We now denote by  $\mathcal{BGN}(r, d, k)$ ,  $\mathcal{BGN}^s(r, d, k)$  the families of BGN extension classes of type  $(r, d, k)$  on  $X$  in which the quotient is semistable or stable, respectively. Due to Proposition 4.13 we have inclusions  $\mathcal{BGN}^s(r, d, k) \hookrightarrow G_L(r, d, k) \hookrightarrow \mathcal{BGN}(r, d, k)$ .

**Proposition 4.15.** [37, Proposition 3.2 and Lemma 4.1] *Let  $(\mathcal{E}, V)$  be an  $\alpha$ -stable coherent system of type  $(r, d, k)$  on an elliptic curve  $X$  with  $d \neq 0$ ,  $k > 0$ . Then every indecomposable direct summand of  $\mathcal{E}$  has positive degree.  $\square$*

**4.4. Fourier-Mukai transforms of coherent systems on elliptic curves.** In this Subsection  $X$  is an elliptic curve. We also assume that  $d \neq 0$ ,  $k > 0$

Recall (Proposition 4.11) that for every integer number  $a > 0$  there is a Fourier-Mukai transform  $\Phi_a$  such that  $\mathcal{O}_X$  is  $\text{IT}_0$ ,  $\Phi_a^0(\mathcal{O}_X) = \mathcal{O}_X$  and  $(\Phi_a)_* = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ . If  $\Psi_a[1]: D_c^b(X) \rightarrow D_c^b(X)$  is the quasi-inverse of  $\Phi_a$ , the Fourier-Mukai transforms  $\Phi_a$ ,  $\Psi_a$  define functors  $\Phi_a^0: \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$ ,  $\Psi_a^1: \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$  that send the object  $\varphi: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  to  $\Phi_a^0(\varphi): V \otimes \mathcal{O}_X \rightarrow \Phi_a^0(\mathcal{E})$  and  $\Psi_a^1(\varphi): V \otimes \mathcal{O}_X \rightarrow \Psi_a^1(\mathcal{E})$ , respectively.

**Proposition 4.16.** *Let  $\varphi: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  be a coherent system.*

- (1) *The  $\Phi_a^0$ -transform  $\Phi_a^0(\varphi): V \otimes \mathcal{O}_X \rightarrow \Phi_a^0(\mathcal{E})$  is a coherent system.*
- (2) *If  $\mathcal{E}$  is  $\text{IT}_1$  with respect to  $\Psi_a$ ,  $\Psi_a^1(\varphi): V \otimes \mathcal{O}_X \rightarrow \Psi_a^1(\mathcal{E})$  is a coherent system.*

□

Therefore, the functor  $\Phi_a^0: \mathfrak{C}(X) \rightarrow \mathfrak{C}(X)$  preserves the subcategory  $\mathfrak{S}(X)$  of coherent systems and induces a functor  $\Phi_a^0: \mathfrak{S}(X) \rightarrow \mathfrak{S}(X)$ .

**4.4.1. Preservation of stability. Small  $\alpha$ .** In this subsection  $\alpha$ -stability refers to a positive  $\alpha$  which smaller than the first critical value.

As a consequence of Propositions 4.12 and 4.15, if  $\varphi: V \otimes \mathcal{O}_X \rightarrow \mathcal{E}$  is a coherent system in the moduli space  $G_0(r, d, k)$ , then  $\mathcal{E}$  is semistable of positive degree. Hence,  $r + ad > 0$  and Proposition 4.10 implies that  $\mathcal{E}$  is  $\text{IT}_0$  with respect to  $\Phi_a$  and that  $\Phi_a^0(\mathcal{E})$  is semistable with Chern character  $\text{ch}(\Phi_a^0(\mathcal{E})) = (r + ad, d)$ . This is not enough to prove that the transformed coherent system  $\Phi_a^0(\varphi): V \otimes \mathcal{O}_X \rightarrow \Phi_a^0(\mathcal{E})$  is stable, but one can prove that the remaining conditions required by Proposition 4.12 are also fulfilled. One then gets the following result [30].

**Theorem 4.17.** *The Fourier-Mukai transform  $\Phi_a$  induces an isomorphism of moduli spaces*

$$\Phi_a^0: G_0(r, d, k) \xrightarrow{\sim} G_0(r + ad, d, k),$$

*whose inverse is induced by  $\Psi_a$ . Therefore, the isomorphism type of  $G_0(r, d, k)$  depends only on the class  $[r] \in \mathbb{Z}/d\mathbb{Z}$ .*

□

**4.4.2. Preservation of stability. Large  $\alpha$ .** In this subsection we suppose that  $0 < k < r$ . Under this assumption, Lange and Newstead proved in [37, Theorem 5.2] that for an elliptic curve the moduli space  $G(\alpha; r, d, k)$  is non empty if and only if  $0 < \alpha < \frac{d}{r-k}$  and either  $k < d$  or  $k = d$  and  $\gcd(r, d) = 1$ . Moreover, in this case the largest critical value  $\alpha_L$  verifies  $\alpha_L < \frac{d}{r-k}$ .

Recall that for large  $\alpha$  the moduli spaces  $G(\alpha; r, d, k)$  are described in terms of BGN-extensions. If  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$  is a BGN-extension, the quotient  $\mathcal{F}$  is  $\text{IT}_0$  with respect to  $\Phi_a$  and  $\Phi_a^0(\mathcal{F})$  is semistable by Proposition 4.10. Since  $\mathcal{O}_X$  is also  $\text{IT}_0$ , it follows that we have an exact sequence  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \Phi_a^0(\mathcal{E}) \rightarrow \Phi_a^0(\mathcal{F}) \rightarrow 0$ . Using again Proposition 4.10, one can prove that this exact sequence is actually a BGN-extension and that it is stable if the original BGN-extension is so. One then has:

**Theorem 4.18.** ([30, Thm. 4.12]) *The Fourier-Mukai transform  $\Phi_a$  induces an isomorphism  $\Phi_a^0: \mathcal{BGN}(r, d, k) \xrightarrow{\sim} \mathcal{BGN}(r + ad, d, k)$  by sending a BGN extension  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$  to  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \Phi_a^0(\mathcal{E}) \rightarrow \Phi_a^0(\mathcal{F}) \rightarrow 0$ . This restricts to an isomorphism  $\Phi_a^0: \mathcal{BGN}^s(r, d, k) \xrightarrow{\sim} \mathcal{BGN}^s(r + ad, d, k)$ . The inverse isomorphism is induced by  $\Psi_a$ .*

□

**Theorem 4.19.** [30] *The Fourier-Mukai transform  $\Phi_a$  induces an isomorphism*

$$\Phi_a^0: G_L(r, d, k) \xrightarrow{\sim} G_L(r + ad, d, k).$$

*Therefore, the isomorphism type of  $G_L(r, d, k)$  depends only on the class  $[r] \in \mathbb{Z}/d\mathbb{Z}$ .*

*Proof.* We only sketch an idea of the proof. Given a coherent system in  $G_L(r, d, k)$ , we know by Proposition 4.13 that it defines an extension  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$  which belongs to  $\mathcal{BGN}(r, d, k)$ . By Proposition 4.18 the transformed extension  $0 \rightarrow \mathcal{O}_X^{\oplus k} \rightarrow \Phi_a^0(\mathcal{E}) \rightarrow \Phi_a^0(\mathcal{F}) \rightarrow 0$  belongs to  $\mathcal{BGN}(r + ad, d, k)$ . The importance of Proposition 4.14 becomes apparent here, because it tells us what conditions a BGN-extension has to fulfil in order to define an  $\alpha$ -stable coherent system. The proof then consists in checking that the transformed extension fulfils those additional conditions. This is a rather technical issue and we shall omit it here.  $\square$

One may draw a diagram which summarises all this information.

$$\begin{array}{ccccc} \mathcal{BGN}^s(r, d, k) & \hookrightarrow & G_L(r, d, k) & \hookrightarrow & \mathcal{BGN}(r, d, k) \\ \wr \downarrow \Phi_a^0 & & \wr \downarrow \Phi_a^0 & & \wr \downarrow \Phi_a^0 \\ \mathcal{BGN}^s(r + ad, d, k) & \hookrightarrow & G_L(r + ad, d, k) & \hookrightarrow & \mathcal{BGN}(r + ad, d, k) \end{array}$$

4.4.3. *Birational type of the moduli spaces  $G(\alpha; r, d, k)$ .* Let  $0 = \alpha_0 < \alpha_1 < \dots < \alpha_L$  be the critical values for coherent systems of type  $(r, d, k)$ , so that the moduli spaces  $G(\alpha; r, d, k)$  for any two values of  $\alpha \in (\alpha_i, \alpha_{i+1})$  coincide. Then, there is only a finite number of different moduli spaces. Moreover, one has:

**Theorem 4.20.** [37, Theorem 4.4] *The birational type of  $G(\alpha; r, d, k)$  is independent of  $\alpha \in (\alpha_0, \alpha_L)$ .*  $\square$

It follows that we can determine the birational type of any of the finitely many different moduli spaces simply by computing one of them. We then choose  $G_0(r, d, k)$ , which has been studied in full generality.

**Theorem 4.21.** ([30, Thm. 5.2]) *Let  $a$  be a positive integer. The birational types of  $G(\alpha; r, d, k)$  and  $G(\alpha; r + ad, d, k)$  are the same. Therefore, the birational type of  $G(\alpha; r, d, k)$  depends only on the class  $[r] \in \mathbb{Z}/d\mathbb{Z}$ .*

*Proof.* The birational types of  $G(\alpha; r, d, k)$  and  $G(\alpha; r + ad, d, k)$  are the same as those of  $G_0(r, d, k)$  and  $G_0(r + ad, d, k)$ , respectively, by Theorem 4.20. Moreover, the Fourier-Mukai transform  $\Phi_a$  induces an isomorphism  $\Phi_a^0: G_0(r, d, k) \rightarrow G_0(r + ad, d, k)$  by Theorem 4.17 and we finish by Theorem 4.20.  $\square$

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