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NUMERICAL PROPERTIES OF HIGGS BUNDLES

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Abstract. We introduce two notions of numerical effectiveness for a Higgs bundle $\mathfrak{E} = (E, \phi)$, in the framework of (compact) Kähler manifolds and complex projective manifolds, respectively. In the first case the definition is given in terms of fibre metrics on the bundle E, in the second in terms of the usual numerical effectiveness of a set of line bundles associated to $\mathfrak{E}$. In both cases we establish several properties of these Higgs bundles, including the existence of a special filtration for numerically flat Higgs bundles, which implies that the Chern classes of such bundles vanish. We study the relation between numerical effectiveness and semistability, and apply this to study the semistable Higgs bundles having vanishing discriminant (in the projective case, these satisfy the remarkable property that they are semistable after restriction to any curve in the base manifold). In the case of projective manifolds we compare the two definitions.

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1. Introduction

The Nakai-Moishezon criterion states that a line bundle L on an n-dimensional projective variety X is ample if and only if $c_1(L)^k \cdot [V] > 0$ for every $k = 0, \ldots, n$ and every irreducible k-dimensional subvariety of V of X. In a sense, a line bundle is *numerically effective* when it lies in the closure of the space of ample line bundles. This amounts to saying that $c_1(L) \cdot [C] \geq 0$ for every irreducible curve C in X. One can extend this notion to vector bundles E of any rank in terms of the universal rank one quotient $\mathcal{O}_{\mathbb{P}E}(1)$ on the projectivized bundle $\mathbb{P}E$: the bundle E is said to be numerically effective if $\mathcal{O}_{\mathbb{P}E}(1)$ is.

There is also a notion of numerical effectiveness tailored for vector bundles on Kähler manifolds; this is given in terms of (possibly singular) Hermitian fibre metrics [10, 8].

The notion of numerical effectiveness is tightly related to that of semistability. For instance, numerically flat bundles (i.e., bundles that are numerically effective together with their dual) are semistable. Moreover, numerical effectiveness may be used to characterize semistable bundles, along the lines of a result by Miyaoka which states that a vector bundle E on a smooth projective curve X is semistable if and only if the numerical class $\lambda = c_1(\mathcal{O}_{\mathbb{P}E}(1)) - \frac{1}{r}\pi^*(c_1(E))$ is nef, where $\pi : \mathbb{P}E \to X$ is the projection. A number of results generalizing Miyaoka's criterion have recently been proved [6, 2, 3] (these apply to Higgs or principal bundles on projective or Kähler manifolds). One should also cite results of Gieseker [14], which have been generalized in [6, 4].

In this paper we review these notions of numerical effectiveness for Higgs bundles and study the main properties of the class of bundles so identified. Thus we give two notions of numerical effectiveness for a Higgs bundle $\mathfrak{E} = (E, \phi)$, in the framework of (compact) Kähler manifolds and complex projective manifolds, respectively. In the first case the definition is given in terms of fibre metrics on the bundle E, in the second it is formulated in a way which is tantamount to the usual numerical effectiveness of a set of line bundles associated to $\mathfrak{E}$. In both cases we establish several properties of these Higgs bundles, including the existence of a special filtration for numerically flat Higgs bundles, which implies that the Chern classes of such bundles vanish. We study the relation between numerical effectiveness and semistability, and apply this to study the semistable Higgs bundles having vanishing discriminant (in the projective case, these satisfy the remarkable property that they are semistable after restriction to any curve in the base manifold). In the case of projective manifolds we compare the two definitions.

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2. Metrics and connections on semistable Higgs bundles

Our notion of numerical effectiveness for Higgs bundles on Kähler manifolds is quite closely related to the notion of (approximate) Hermitian-Yang-Mills structure, i.e., to the circle of ideas usually known as *Hitchin-Kobayashi correspondence.* It is quite easy to show that a vector bundle satisfying the Hermitian-Yang-Mills condition is polystable (i.e., it is a direct sum of stable sheaves having the same slope). The converse result is much deeper, and was proved first by Donaldson in the projective case [11, 12], and later by Uhlenbeck and Yau in the compact Kähler case [23]; they showed that a stable bundle admits a (unique up to homotheties) Hermitian metric which satisfies the Hermitian-Yang-Mills condition. Analogously, one can show that if Hermitian bundle satisfies the Hermitian-Yang-Mills condition in an approximate sense, then it is semistable, while the converse has been proved only in the case X is projective [18].

Simpson [20, 21] proved a Hitchin-Kobayashi correspondence for Higgs bundles: given a Higgs bundle with an Hermitian metric, one defines a natural connection whose specification involve all the data characterizing the Hermitian Higgs bundle (metric, complex structure, Higgs field). When this connection satisfies the Hermitian-Yang-Mills condition, the Higgs bundle is polystable, and *vice versa.* In this section we show that whenever an Hermitian Higgs bundle satisfies an approximate Hermitian-Yang-Mills condition, then it is semistable. To this end we shall need to prove a vanishing result.

2.1. **Main definitions.** Let X be an n-dimensional compact Kähler manifold, with Kähler form ω. The degree $\deg(F)$ of a coherent sheaf F on X is defined as

$$\deg(F) = \int_X c_1(F) \cdot \omega^{n-1},$$

and its slope as

$$\mu(F) = \frac{\deg(F)}{r}$$

provided that $r = \mathrm{rk}(F) > 0$.

Definition 2.1. *A Higgs sheaf $\mathfrak{E}$ on X is a pair $\mathfrak{E} = (E, \phi)$, where E is a coherent sheaf, and ϕ is a morphism $E \to E \otimes \Omega^1_X$ such that $\phi \wedge \phi : E \to E \otimes \Omega^2_X$ vanishes. A Higgs subsheaf F of a Higgs sheaf $\mathfrak{E} = (E, \phi)$ is a subsheaf of E such that $\phi(F) \subset F \otimes \Omega^1_X$. A Higgs bundle is a Higgs sheaf $\mathfrak{E}$ such that E is a locally-free $\mathcal{O}_X$-module.*

A Higgs sheaf $\mathfrak{E} = (E, \phi)$ on X is semistable (resp. stable) if E is torsion-free, and $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$) for every proper nontrivial Higgs subsheaf F of $\mathfrak{E}$.

Let h be an Hermitian fibre metric on a Higgs bundle $\mathfrak{E}$, and let $D_{(E,h)}$ be the unique connection on E which is compatible with both the metric h and the holomorphic structure of E (the *Chern connection* of the Hermitian bundle (E, h)). Moreover, let $\bar{\phi}$ be the adjoint of ϕ with respect to the metric h, i.e., the morphism $\bar{\phi} : E \to E \otimes \Omega^{0,1}_X$ such that

$$h(s, \phi(t)) = h(\bar{\phi}(s), t)$$

for all sections s, t of E. The operator

$$\mathcal{D}_{(\mathfrak{E},h)} = D_{(E,h)} + \phi + \bar{\phi} \tag{1}$$

defines a connection on the bundle E, which is neither compatible with the holomorphic structure of E, nor with the Hermitian metric h, and is called the *Hitchin-Simpson connection* of the Hermitian Higgs bundle $(\mathfrak{E}, h)$. Its curvature will be denoted by $\mathcal{R}_{(\mathfrak{E},h)} = \mathcal{D}_{(\mathfrak{E},h)} \circ \mathcal{D}_{(\mathfrak{E},h)}$; if this vanishes, we say that $(\mathfrak{E}, h)$ is *Hermitian flat*.

Let $\mathcal{K}_{(\mathfrak{E},h)} \in \operatorname{End}(E)$ be the mean curvature of the Hitchin-Simpson connection, i.e., $\mathcal{K}_{(\mathfrak{E},h)} = i\Lambda\mathcal{R}_{(\mathfrak{E},h)}$, where $\Lambda : \mathcal{A}^p \to \mathcal{A}^{p-2}$ is the adjoint of the operation of wedging by the Kähler 2-form (here $\mathcal{A}^p$ is the sheaf of $\mathbb{C}$-valued smooth p-forms on X). We may regard the mean curvature as a bilinear form on E by letting

$$\mathcal{K}_{(\mathfrak{E},h)}(s, t) = h(\mathcal{K}_{(\mathfrak{E},h)}(s), t)\,.$$

We recall the form that the Hitchin-Kobayashi correspondence acquires for Higgs bundles [21, Thm. 1].

Theorem 2.2. *A Higgs vector bundle $\mathfrak{E} = (E, \phi)$ over a compact Kähler manifold is polystable if and only if it admits an Hermitian metric h such that the curvature of the Hitchin-Simpson connection of $(\mathfrak{E}, h)$ satisfies the Hermitian-Yang-Mills condition*

$$\mathcal{K}_{(\mathfrak{E},h)} = c \cdot \mathrm{Id}_E$$

for some constant real number c.

The constant c is related to the topological invariants of E by the formula

$$c \int_X \omega^n = 2n\pi\, \mu(E) \tag{2}$$

where $n = \dim(X)$.

2.2. **Approximate Hermitian-Yang-Mills structure and semistability.** Given an Hermitian vector bundle (E, h), we introduce a norm on the space of Hermitian endomorphisms ψ of (E, h) by letting

$$|\psi| = \max_X \sqrt{\operatorname{tr}(\psi^2)}\,.$$

Definition 2.3. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ has an* approximate Hermitian-Yang-Mills structure *if for every positive real number ξ there is an Hermitian metric h_ξ on E such that*

$$|\mathcal{K}_{(\mathfrak{E},h)} - c \cdot \mathrm{Id}_E\,| < \xi\,. \tag{3}$$

The constant c is again given by equation (2).

The next result was proved in [18, VI.10.13]) in the case of vector bundles.

Theorem 2.4. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ on a compact Kähler manifold admitting an approximate Hermitian-Yang-Mills structure is semistable.*

As in the vector bundle case, we need a vanishing result. A section s of a Higgs bundle $\mathfrak{E} = (E, \phi)$ is *ϕ-invariant* if it is an eigenvector of ϕ, namely, there is a holomorphic 1-form λ on X such that $\phi(s) = \lambda \otimes s$.

Proposition 2.5. *If a Higgs bundle $\mathfrak{E} = (E, \phi)$ has an Hermitian metric h such that the mean curvature $\mathcal{K}_{(\mathfrak{E},h)}$ of the Hitchin-Simpson connection is a seminegative definite form, and s is a ϕ-invariant section of E, then $D_{(E,h)}(s) = 0$ and $\mathcal{K}_{(\mathfrak{E},h)}(s, s) = 0$.*

Proof. We start by writing the relation between the curvatures of the Chern and Hitchin-Simpson connections for $(\mathfrak{E}, h)$. One has

$$\mathcal{R}_{(\mathfrak{E},h)} = R_{(E,h)} + D'_{(E,h)}(\phi) + D''_{(E,h)}(\bar{\phi}) + [\phi, \bar{\phi}] \tag{4}$$

where we have split the Chern connection $D_{(E,h)} = D'_{(E,h)} + D''_{(E,h)}$ into its (1,0) and (0,1) parts, and $[\phi, \bar{\phi}] = \phi \circ \bar{\phi} + \bar{\phi} \circ \phi$ is an element in $\Omega^{1,1}_X(\mathrm{End}(E))$.

Let s be a ϕ-invariant section of E. We have

$$\mathcal{R}_{(\mathfrak{E},h)}(s) = R_{(E,h)}(s) + d(\lambda + \bar{\lambda}) \otimes s\,.$$

Moreover, from the Weitzenböck formula [1] one has the identity

$$\partial\bar{\partial}h(s,s)) = h\big(D'_{(E,h)}(s), D'_{(E,h)}(s)\big) - h\big(\mathcal{R}_{(\mathfrak{E},h)}(s), s\big) + h(s,s)\, d(\lambda + \bar{\lambda})\,.$$

Let us set $f = h(s,s)$ and $L(f) = \Lambda(\partial\bar{\partial} f)$. Due to the current hypotheses,

$$L(f) = \|D'_{(E,h)}(s)\|^2 - \mathcal{K}_{(\mathfrak{E},h)}(s,s) \geq 0$$

where $\|D'_{(E,h)}(s)\|^2$ is the scalar product of $D'_{(E,h)}(s)$ with itself using the fibre metric h and the Kähler metric on X. By Hopf's maximum principle (see e.g. [18]) this implies $L(f) = 0$, which in turn yields $D'_{(E,h)}(s) = 0$ and $\mathcal{K}_{(\mathfrak{E},h)}(s,s) = 0$. Since s is holomorphic, we also have $D_{(E,h)}(s) = 0$. $\square$

Corollary 2.6. *Let $(\mathfrak{E}, h)$ be an Hermitian Higgs bundle. If the mean curvature $\mathcal{K}_{(\mathfrak{E},h)}$ of the Hitchin-Simpson connection is seminegative definite everywhere, and negative definite at some point of X, then E has no nonzero ϕ-invariant sections.*

Proof. If s is a nonzero ϕ-invariant section of E, then it never vanishes on X since $D_{(E,h)}(s) = 0$ by Proposition 2.5. By the same Proposition $\mathcal{K}_{(\mathfrak{E},h)}(s,s) = 0$, and this contradicts the fact that $\mathcal{K}_{(\mathfrak{E},h)}$ is negative at some point. $\square$

Corollary 2.7. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over X which admits an approximate Hermitian-Yang-Mills structure. If $\deg(E) < 0$ then E has no nonzero ϕ-invariant sections.*

Proof. Since $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure, for every $\xi > 0$ there exists an Hermitian metric h_ξ on E such that $\mathcal{K}_{(\mathfrak{E},h_\xi)} - c \cdot h_\xi < \xi \cdot h_\xi$ with $c < 0$. For ξ small enough $\mathcal{K}_{(\mathfrak{E},h_\xi)}$ is negative definite, and the result follows from the previous corollary. $\square$

Proof of Theorem 2.4. Assume that $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure and let $\mathfrak{F}$ be a Higgs subsheaf of $\mathfrak{E}$, with $\operatorname{rk}(F) = p$. Let $\mathfrak{G}$ be the Higgs bundle (G, ψ), where $G = \wedge^p E \otimes \det(F)^{-1}$, and ψ is the Higgs field naturally induced on G by the Higgs fields of $\mathfrak{E}$ and $\mathfrak{F}$. The inclusion $\mathfrak{F} \hookrightarrow \mathfrak{E}$ induces a morphism $\det(\mathfrak{F}) \to \wedge^p \mathfrak{E}$, and, tensoring by $\det(\mathfrak{F})^{-1}$, we obtain a ψ-invariant section of G. On the other hand, the approximate Hermitian-Yang-Mills structure of $\mathfrak{E}$ induces a structure of the same kind on $\mathfrak{G}$, with constant

$$c_G = \frac{2np\pi}{n!\,\mathrm{vol}(X)}(\mu(E) - \mu(F))\,.$$

By Corollary 2.7 we have $c_G \geq 0$, so that $\mu(F) \leq \mu(E)$, and $\mathfrak{E}$ is semistable. □

3. Metric characterization of numerical effectiveness for Higgs bundles

Since a Kähler manifold may not contain embedded curves at all, the usual notion of numerical effectiveness, which works well for projective manifolds, is no longer viable. An alternative approach, pursued by Demailly, Peternell and Schneider [10] and by de Cataldo [8], may be given in terms of fibre metrics. In particular, de Cataldo's treatment in terms of metrics on the bundle E seems to be well suited to an extension to the case of Higgs bundles, again replacing the Chern connection with the Hitchin-Simpson connection.

3.1. **Numerically effective Higgs bundles.** In this section X is a compact Kähler manifold of dimension n and (E,h) is a rank r Hermitian vector bundle on X. We adapt the De Cataldo's terminology to Higgs bundles. For finite-dimensional complex vector bundles, V, W and Hermitian forms θ_1, θ_2 on $V \otimes W$, let t be any positive integer. Then $\theta_1 \geq_t \theta_2$ means that the Hermitian form $\theta_1 - \theta_2$ is semipositive definite on all tensors in $V \otimes W$ of rank $\rho \leq t$ (where the rank is that of the associated linear map $V^* \to W$). The relevant range for t is $1 \leq t \leq N = \min(\dim V, \dim W)$.

Definition 3.1. *If (E,h) is equipped with a connection D, we may associate with the curvature R of D an Hermitian form $\widetilde{R}$ on $T_X \otimes E$, defined by*

$$\widetilde{R}(u \otimes s, v \otimes t) = \frac{i}{2\pi}\langle h(R^{(1,1)}(s), t), u \otimes v\rangle\,. \tag{5}$$

where $R^{(1,1)}$ is the $(1,1)$ part of R.

We consider now a Hermitian Higgs bundle $\mathfrak{E} = (E, \phi, h)$ on X.

Definition 3.2. *Let $1 \leq t \leq N$. We say that $\mathfrak{E}$ is*

(i) *t-H-nef if for every $\xi > 0$ there is an Hermitian metric h_ξ on E such that*

$$\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_t -\xi\omega \otimes h_\xi;$$

(ii) *t-H-nflat if both $\mathfrak{E}$ and $\mathfrak{E}^*$ are t-H-nef.*

In the following we establish some basic properties of t-H-nef Higgs bundles on a compact Kähler manifold X.

Proposition 3.3. (i) *Let $f : X \to Y$ be a holomorphic map between compact Kähler manifolds, and let $\mathfrak{E} = (E, \phi)$ be a t-H-nef Higgs vector bundle on Y. Then $f^*\mathfrak{E} = (f^*E, f^*\phi)$ is a 1-H-nef Higgs bundle over X.*

(ii) *Let $\mathfrak{E} = (E, \phi_E)$ and $\mathfrak{F} = (F, \phi_F)$ be Higgs bundles. If $\mathfrak{E}$ is t'-H-nef and $\mathfrak{F}$ is t''-H-nef, then $\mathfrak{E} \otimes \mathfrak{F} = (E \otimes F, \rho)$ is t-H-nef, where*

$$\begin{array}{rccc} \rho : & E \otimes F & \longrightarrow & E \otimes F \otimes \Omega^1_X \\ & \rho(e \otimes f) & \mapsto & \phi_{\mathfrak{E}}(e) \otimes f + e \otimes \phi_{\mathfrak{F}}(f) \end{array}$$

and $t = \min(t', t'')$.

(iii) *If $\mathfrak{E} = (E, \phi)$ is a t-H-nef Higgs bundle, then for all $p = 2, \ldots, r = \mathrm{rk}(E)$ the p-th exterior power $\wedge^p\mathfrak{E} = (\wedge^p E, \wedge^p \phi)$ is a t-H-nef Higgs bundle, and for all m, the m-th symmetric power $S^m\mathfrak{E} = (S^m E, S^m \phi)$ is a t-H-nef Higgs bundle.*

(iv) *Let $(\mathfrak{Q}, h_Q)$ be an Hermitian Higgs quotient of $(\mathfrak{E}, h)$. The respective Hitchin-Simpson curvatures verify the inequality $\widetilde{\mathcal{R}}_{(\mathfrak{Q},h_{\mathfrak{Q}})} \geq_1 \widetilde{\mathcal{R}}_{(\mathfrak{E},h)|\mathfrak{Q}}$.*

(v) *A Higgs quotient $\mathfrak{Q}$ of a 1-H-nef Higgs bundle $\mathfrak{E} = (E, \phi)$ is 1-H-nef.*

(vi) *If $0 \to \mathfrak{S} \to \mathfrak{E} \to \mathfrak{Q} \to 0$ is an exact sequence of Higgs bundles, with $\mathfrak{E}$ and $\det(\mathfrak{Q})^{-1}$ 1-H-nef, then $\mathfrak{S}$ is 1-H-nef.*

(vii) *An extension of 1-H-nef Higgs bundles is 1-H-nef.*

Proof. (i) This is proved as in [8, Proposition 3.2.1(1)].

(ii) Since $\mathfrak{E} = (E, \phi_E)$ (analog., $\mathfrak{F} = (F, \phi_F)$) is t'-H-nef, for all $\xi > 0$ there exists a metric $h_{(\mathfrak{E},\xi/2)}$ over $\mathfrak{E}$ (analog. $h_{(\mathfrak{F},\xi/2)}$ over $\mathfrak{F}$) such that the Hitchin-Simpson curvature $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})}$ satisfies $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})} \geq_{t'} -\frac{\xi}{2}\omega \otimes h_{(\mathfrak{E},\xi/2)}$ (analogously, $\widetilde{\mathcal{R}}_{(\mathfrak{F},h_{(\mathfrak{F},\xi/2)})} \geq_{t''} -\frac{\xi}{2}\omega \otimes h_{(\mathfrak{F},\xi/2)}$). Considering on $\mathfrak{E} \otimes \mathfrak{F}$ the metrics $h_\xi = h_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})} \otimes h_{(\mathfrak{F},h_{(\mathfrak{F},\xi/2)})}$ we have

$$\widetilde{\mathcal{R}}_{(\mathfrak{E}\otimes\mathfrak{F},h_\xi)} = \widetilde{\mathcal{R}}_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})} \otimes h_{(\mathfrak{F},\xi/2)} + h_{(\mathfrak{E},\xi/2)} \otimes \widetilde{\mathcal{R}}_{(\mathfrak{F},h_{(\mathfrak{F},\xi/2)})} \geq_t -\xi\omega \otimes h_\xi.$$

(iii) This is proved much in the same way as (ii).

(iv) In [5] we have written equations of the Gauss-Codazzi type, which relate the Hitchin-Simpson curvatures of three Hermitian Higgs bundles sitting in an exact sequence. These equations state that the Hitchin-Simpson curvature of $\mathfrak{Q}$ is given by the restriction of the Hitchin-Simpson curvature of $\mathfrak{E}$ to $\mathfrak{Q}$ (if we embed $\mathfrak{Q}$ into $\mathfrak{E}$ by orthogonally splitting the latter) plus a semipositive term.

(v) Let $\xi > 0$ and h_ξ be an Hermitian metric on $\mathfrak{E}$ with $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\omega \otimes h_\xi$. We can endow $\mathfrak{Q}$ with the quotient metric $h_{(\mathfrak{Q},\xi)}$ and embed it into $\mathfrak{E}$ as a C^∞ Higgs subbundle. The claim follows from Lemma (iv).

(vi) The proof is as in Proposition 1.15(iii) of [10]. Let $r = \mathrm{rk}(E)$ and $p = \mathrm{rk}(S)$. By taking the $(p-1)$-th exterior power of the morphism $\mathfrak{E}^* \to \mathfrak{S}^*$ obtained from the exact sequence in the statement, and using the isomorphism $\mathfrak{S} \simeq \wedge^{p-1}\mathfrak{S}^* \otimes \det(\mathfrak{E})$, we get a surjection $\wedge^{p-1}\mathfrak{E}^* \to \mathfrak{S} \otimes \det(\mathfrak{S})^{-1}$. Tensoring by $\det(\mathfrak{S}) \simeq \det(\mathfrak{E}) \otimes \det(\mathfrak{Q})^{-1}$ we eventually obtain a surjection $\wedge^{r-p+1}\mathfrak{E} \otimes \det(\mathfrak{Q})^{-1} \to \mathfrak{S}$. Propositions (ii) and (v) now imply the claim.

(vi) Let us consider an extension of Higgs bundles $0 \to \mathfrak{F} \to \mathfrak{E} \to \mathfrak{G} \to 0$ where $\mathfrak{F}$ and $\mathfrak{G}$ are 1-H-nef. Then for every $\xi > 0$ the latter bundles carry Hermitian metrics $h_{(\mathfrak{F},\xi)}$ and $h_{(\mathfrak{G},\xi)}$ such that

$$\widetilde{\mathcal{R}}_{(\mathfrak{F},h_{(\mathfrak{F},\xi/3)})} \geq_1 -\tfrac{\xi}{3}\omega \otimes h_{(\mathfrak{F},\xi/3)}, \qquad \widetilde{\mathcal{R}}_{(\mathfrak{G},h_{(\mathfrak{G},\xi/3)})} \geq_1 -\tfrac{\xi}{3}\omega \otimes h_{(\mathfrak{G},\xi/3)}.$$

Fixing a C^∞ isomorphism $\mathfrak{E} \simeq \mathfrak{F} \oplus \mathfrak{G}$, these metrics induce an Hermitian metric h_ξ on $\mathfrak{E}$. A simple calculation, which involves the second fundamental form of $\mathfrak{E}$, shows that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\omega \otimes h_\xi$, so that $\mathfrak{E}$ is 1-H-nef (details in the ordinary case are given in [8]). □

3.2. **Numerical effectiveness, semistability and a characterization of 1-H-nflat Higgs bundles.** We study some relations between semistability and numerical effectiveness of Higgs bundles. Together with the vanishing result proved in the next Proposition, this will be the basic tool for providing a characterization of 1-H-nflat Higgs bundles which is one of the main results in this paper (Theorem 3.7).

Proposition 3.4. *Let $\mathfrak{E} = (E, \phi)$ and $\mathfrak{E}^* = (E^*, \phi^*)$ be a 1-H-nef Higgs bundle and its dual Higgs bundle respectively. A ϕ^*-invariant section of E^* has no zeroes.*

Proof. Our proof is closely inspired by Proposition 1.16 in [10]. For a given $\xi > 0$, let h_ξ be the metric on E such that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\omega \otimes h_\xi$, and consider the closed $(1,1)$ current

$$T_\xi = \frac{i}{2\pi}\partial\bar{\partial}\log h^*_\xi(s,s);$$

this satisfies the inequality

$$T_\xi \geq -\frac{\widetilde{R}_{(E^*,h^*_\xi)}(s,s)}{h^*_\xi(s,s)},$$

where $\widetilde{R}_{(E^*,h_\xi^*)}(s,s)$ is regarded as a 2-form on X. If s is ϕ^*-invariant one has $[\phi^*,\overline{\phi}^*](s)=0$, so that $\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h_\xi^*)}(s,s) = \widetilde{R}_{(E^*,h_\xi^*)}(s,s)$. On the other hand, since $\mathfrak{E}$ is 1-H-nef, we have $-\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h_\xi^*)}(s,s) \geq -\xi\, h_\xi(s,s)\,\omega$. Thus, $T_\xi \geq -\xi\omega$.

Since $\partial\bar{\partial}\omega^{n-1}=0$, we have

$$\int_X (T_\xi + \xi\omega)\wedge\omega^{n-1} = \xi\int_X \omega^n.$$

For ξ ranging in the interval $(0,1]$ the masses of the currents $T_\xi+\xi\omega$ are uniformly bounded from above, so that the sequence $\{T_\xi+\xi\omega\}$ contains a subsequence which, by weak compactness, converges weakly to zero. (For details on this technique see e.g. [9]). Therefore, the Lelong number of T_ξ at each point $x\in X$ (which coincides with the vanishing order of s at that point) is zero [22], so that s never vanishes. □

Theorem 3.5. *A 1-H-nflat Higgs bundle $\mathfrak{E}=(E,\phi)$ is semistable.*

Proof. Since the mean curvature $\mathcal{K}_{(\mathfrak{E},h_\xi)}$ may be written in the form

$$\mathcal{K}_{(\mathfrak{E},h_\xi)}(s,s) = -2\pi\sum_{i=1}^n \widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)}(e_i\otimes s, e_i\otimes s),$$

where the e_i's are a unitary frame field on X, the fact that $\mathfrak{E}$ is 1-H-nef implies

$$\mathcal{K}_{(\mathfrak{E},h_\xi)}(s,s)\leq 2\pi\, n\,\xi\, h_\xi(s,s).$$

On the other hand, since $\det(\mathfrak{E})^{-1}$ is 1-H-nef, the Higgs bundle $\mathfrak{E}^*\simeq\wedge^{r-1}\mathfrak{E}\otimes\det(\mathfrak{E})^{-1}$ is 1-H-nef with the dual metric h_ξ^*, so that $\mathcal{K}_{(\mathfrak{E}^*,h_\xi^*)} = -\mathcal{K}^t_{(\mathfrak{E},h_\xi)}$, and

$$\mathcal{K}_{(\mathfrak{E},h_\xi)}(s,s)\geq -2\pi\, n\,\xi\, h_\xi(s,s).$$

As $c_1(E)=0$ because $\det(\mathfrak{E})$ is numerically flat [10, Corollary 1.5], after rescaling ξ these equations imply $|\mathcal{K}_{(\mathfrak{E},h_\xi)}|\leq\xi$, so that $\mathfrak{E}$ is semistable by Theorem 2.4. □

Let $\mathfrak{E}=(E,\phi)$ be a Higgs bundle on X.

Lemma 3.6. (i) *If $\mathfrak{E}$ is 1-H-nef with $c_1(E)=0$ then it is 1-H-nflat.*

(ii) *If rank$(\mathfrak{E})=1$, and moreover $\mathfrak{E}$ is 1-H-nef and has zero degree, then it is Hermitian flat.*

(iii) *If $\mathfrak{E}$ is 1-Hnflat, and $\{h_\xi\}$ is a family of metrics which makes $\mathfrak{E}$ 1-H-nef, then the family of dual metrics $\{h_\xi^*\}$ makes $\mathfrak{E}^*$ 1-H-nef.*

(iv) *If $\mathfrak{E}$ is stable and 1-H-nflat then it is Hermitian flat.*

Proof. (i) This follows again from the fact that $\mathfrak{E}^* \simeq \wedge^{r-1}\mathfrak{E} \otimes \det(\mathfrak{E})^{-1}$ is an isomorphism of Higgs bundles.

(ii) We proceed as in [10, Cor. 1.19]. For every $\xi > 0$ one has on $\mathfrak{L}$ an Hermitian metric k_ξ satisfying the inequality

$$0 \leq \int_X (\tfrac{i}{2\pi}\mathcal{R}_{(\mathfrak{L},k_\xi)} + \xi\omega) \cdot \omega^{n-1} = \deg(L) + \xi \int_X \omega^n.$$

By the same argument as in the proof of Proposition 3.4, if $\deg(L) = 0$ by taking the limit $\xi \to 0$ one shows that $c_1(L) = 0$, so that $\mathfrak{L}$ is Hermitian flat.

(iii) The determinant line bundle $\det(E)$ is 1-nef with respect to the family of determinant metrics $\{\det h_\xi\}$. The dual line bundle $\det^{-1}(E)$ is 1-nef as well, and it is such with respect to a family $\{a(\xi)\det^{-1} h_\xi\}$, where the homothety factor $a(\xi)$ only depends on ξ [10, Cor. 1.5]. From the isomorphism $\mathfrak{E}^* \simeq \wedge^{r-1}\mathfrak{E} \otimes \det^{-1}(\mathfrak{E})$ (where $r = \operatorname{rk}(E)$) we see that $\mathfrak{E}^*$ is made 1-H-nef by the family of metrics $\{h'_\xi = a(\xi)\, h^*_\xi\}$, so that for every $\xi > 0$ the condition $\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h'_\xi)} \geq_1 -\xi\,\omega \otimes h'_\xi$ holds. But this implies $\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h^*_\xi)} \geq_1 -\xi\,\omega \otimes h^*_\xi$.

(iv) As before, let us denote by $\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|^2$ the scalar product of the Hitchin-Simpson curvature with itself obtained by using the Hermitian metric of the bundle E and the Kähler form on X (thus, $\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|$ is a function on X). Note that in terms of a local orthonormal frame $\{e_\alpha\}$ on X and a local orthonormal basis of sections $\{s_a\}$ of E we may write

$$\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|^2 = 4\pi^2 \sum_{\alpha,a} \left(\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)}(e_\alpha \otimes s_a, e_\alpha \otimes s_a)\right)^2 .$$

Since $\mathfrak{E}$ is 1-H-nef, for every $\xi > 0$ there is an Hermitian metric h_ξ on E such that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\,\omega \otimes h_\xi$. Taking Lemma (iii) into account, for every ξ we have the inequalities

$$\xi \geq \widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)}(e_\alpha \otimes s_a, e_\alpha \otimes s_a) \geq -\xi\,.$$

So we have $\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\| \leq a_1\,\xi$ for some constant a_1. In the same way we have $\|\mathcal{K}_{(\mathfrak{E},h_\xi)}\| \leq a_2\,\xi$ for some constant a_2.

Assume that $n = \dim X > 1$. Since $c_1(E) = 0$, we have the representation formula [18, Chap. IV.4]

$$\int_X c_2(E) \cdot \omega^{n-2} = \frac{1}{8\pi^2\, n(n-1)} \int_X (\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|^2 - \|\mathcal{K}_{(\mathfrak{E},h_\xi)}\|^2)\,\omega^n$$

The previous inequalities imply $\int_X c_2(E) \cdot \omega^{n-2} = 0$. For every value of n, may apply Theorem 1 and Proposition 3.4 in [20] to show that $\mathfrak{E}$ admits an Hermitian metric whose corresponding Hitchin-Simpson curvature vanishes. $\square$

Theorem 3.7. *A Higgs bundle $\mathfrak{E}$ is 1-H-nflat if and only if it has a filtration in Higgs subbundles whose quotients are Hermitian flat Higgs bundles. As a consequence, all Chern classes of a 1-H-nflat Higgs bundle vanish.*

Proof. Assume that $\mathfrak{E}$ has such a filtration. Then any quotient of the filtration is 1-H-nflat, and the claim follows from Proposition 3.3(vii).

To prove the converse, let $\mathfrak{F}$ be a Higgs subsheaf of $\mathfrak{E}$ of rank p. We have an exact sequence of Higgs sheaves

$$0 \to \det(\mathfrak{F}) \to \wedge^p \mathfrak{E} \to \mathfrak{G} \to 0,$$

where $\mathfrak{G}$ is not necessarily locally-free. Since $\det(\mathfrak{E})$ is 1-H-nflat we have $c_1(E) = 0$. By Theorem 3.5 $\wedge^p \mathfrak{E}$ is semistable, so that $\deg(F) \leq 0$. Let h_ξ be a family of Hermitian metrics which makes $\mathfrak{E}$ a 1-H-nef Higgs bundle, and let h_ξ^p be the induced metrics on $\wedge^p \mathfrak{E}$. After rescaling the dual metrics $(h_\xi^p)^*$ we obtain a family of metrics which makes $\wedge^p \mathfrak{E}^*$ a 1-H-nef Higgs bundle (cf. Lemma 3.6(iii)). Let U be the open dense subset of X where $\mathfrak{G}$ is locally free; then the metrics $(h_\xi^p)^*$ induce on $\det(\mathfrak{F})^{-1}_{|U}$ metrics making it 1-H-nef. These metrics extend to the whole of X, since they are homothetic by a constant factor to the duals of the metrics induced on $\det(\mathfrak{F})$ by the metrics on $\wedge^p \mathfrak{E}$. Thus, $\det(\mathfrak{F})^{-1}$ is 1-H-nef. If $\deg(F) = 0$ by Lemma 3.6(ii) $\det(\mathfrak{F})$ is Hermitian flat, so that $\wedge^p \mathfrak{E} \otimes \det(\mathfrak{F})^{-1}$ is 1-H-nflat. Then by Proposition 3.4 the morphism of Higgs bundles $\det(\mathfrak{F}) \to \wedge^p \mathfrak{E}$ has no zeroes, so that $\mathfrak{G}$ is locally-free.

In view of Lemma 3.6(iv) we may assume that $\mathfrak{E}$ is not stable. Let us then identify $\mathfrak{F}$ with a destabilizing Higgs subsheaf of minimal rank and zero degree. We need $\mathfrak{F}$ to be reflexive; we may achieve this by replacing $\mathfrak{F}$ with its double dual $\mathfrak{F}^{**}$. By Lemma 1.20 in [10], $\mathfrak{F}$ is locally-free and a Higgs subbundle of $\mathfrak{E}$. Now, $\mathfrak{F}^*$ is 1-H-nef because it is a Higgs quotient of $\mathfrak{E}^*$, while $\mathfrak{F}$ is 1-H-nef by Proposition 3.3(vi), so that $\mathfrak{F}$ is 1-H-nflat. Since $\mathfrak{F}$ is stable by construction, by Lemma 3.6(iv) it is Hermitian flat. The existence of the filtration follows by induction on the rank of $\mathfrak{E}$ since the quotient $\mathfrak{E}/\mathfrak{F}$ is locally-free and 1-H-nflat, hence we may apply to it the inductive hypothesis. □

3.3. **Projective curves.** The last part of this section is devoted to the case when X is a smooth projective curve. The first Proposition generalizes results given in [14, 6, 4]. Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on X.

Proposition 3.8. *If $\mathfrak{E}$ is semistable with* $\deg(E) \geq 0$, *then it is 1-H-nef.*

Proof. If $\mathfrak{E}$ is stable it admits an Hermitian-Yang-Mills metric h, so that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h)} = c\,h$ with $c \geq 0$ (note that we essentialy identify $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_{\mathfrak{E}})}$ with the mean curvature since we are on a curve). Then $\mathfrak{E}$ is 1-H-nef. If $\mathfrak{E}$ is properly semistable, we may filter it in such a way that the quotients of the filtration are stable Higgs bundles of nonnegative degree. By the previous argument, every quotient is 1-H-nef. One then concludes by Proposition 3.3(vii). □

The following result extends to 1-H-nef Higgs bundles of any rank a characterization of rank 2 ample vector bundles on a smooth projective curve given in [16].

Proposition 3.9. *If $\mathfrak{E}$ has nonnegative degree and all its locally-free Higgs quotients are 1-H-nef, then it is 1-H-nef.*

Proof. If $\mathfrak{E}$ is semistable, by Proposition 3.8 it is 1-H-nef. If $\mathfrak{E}$ is not semistable, let $\mathfrak{K}$ be a destabilizing semistable Higgs subbundle. Since $\deg(E) \geq 0$, then $\deg(K) > 0$, and again we have that $\mathfrak{K}$ is 1-H-nef. Thus $\mathfrak{E}$ is an extension of 1-H-nef Higgs bundles, and is 1-H-nef by Proposition 3.3(vii). □

4. The projective case

We give now a definition of numerical flatness for Higgs bundles on smooth projective varieties and compare it with the definition we have given here in the case of Kähler manifolds.

4.1. **Grassmannians of Higgs quotients.** Given a Higgs bundle $\mathfrak{E} = (E, \phi)$, we consider the Grassmann bundle $\mathrm{Gr}_s(E)$ of s-planes in E, which is a parametrizaton of the rank s locally-free quotients of E, and we construct closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \mathrm{Gr}_s(E)$ parametrizing rank s locally-free Higgs quotients (see again [5] for details). The scheme $\mathfrak{Gr}_s(\mathfrak{E})$ will be called the *Grassmannian scheme of Higgs quotients* of $\mathfrak{E}$.

We denote by p_s and ρ_s respectively the projection over X from $\mathrm{Gr}_s(E)$ and $\mathfrak{Gr}_s(\mathfrak{E})$. The restriction of the universal exact sequence on the Grassmann bundle gives the following exact universal sequence, which defines the universal Higgs quotient $Q_{s,\mathfrak{E}}$:

$$0 \to S_{r-s,\mathfrak{E}} \to \rho_s^*(\mathfrak{E}) \to Q_{s,\mathfrak{E}} \to 0\,. \tag{6}$$

We consider the numerical classes

$$\lambda_{s,\mathfrak{E}} = \left[c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1) - \frac{1}{r}\pi_s^*(c_1(E))\right] \in N^1(\mathbb{P}Q_{s,\mathfrak{E}}) \tag{7}$$

where $\pi_s \colon \mathbb{P}Q_{s,\mathfrak{E}} \to X$ is the natural epimorphism, and

$$\theta_{s,\mathfrak{E}} = \left[c_1(Q_{s,\mathfrak{E}}) - \frac{s}{r}\,\rho_s^*(c_1(E))\right] \in N^1(\mathfrak{Gr}_s(\mathfrak{E})), \tag{8}$$

where, for every projective scheme Z, we denote by $N^1(Z)$ the vector space of $\mathbb{R}$-divisors modulo numerical equivalence:

$$N^1(Z) = \frac{\operatorname{Pic}(Z)}{num.\,eq.} \otimes \mathbb{R}.$$

We recall from [5] our definition of ampleness and numerical effectiveness for Higgs bundles on projective varieties.

Definition 4.1. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ is a H-ample (resp. H-nef) if it is ample (resp. numerical effective) in the usual sense. If* $\operatorname{rk}\mathfrak{E} \geq 2$ *we require that:*

(i) *all bundles $Q_{s,\mathfrak{E}}$ are H-ample (resp. H-nef);*
(ii) *the line bundle* $\det(E)$ *is ample (resp. nef).*

If both $\mathfrak{E}$ and $\mathfrak{E}^*$ are H-nef, we say that $\mathfrak{E}$ is H-nflat.

Remark 4.2. (i) One should note that, since the schemes $\mathfrak{Gr}_s(\mathfrak{E})$ may be highly singular, our definition of numerical effectiveness for Higgs bundles on projective varieties requires to consider Higgs bundles on singular spaces. This may be done by using the theory of the De Rham complex on general schemes [15].

(ii) In Definition 4.1 we require that $\det(E)$ is ample, or nef, to avoid the existence of H-ample or H-nef Higgs bundles with zero or negative degree. Cf. [4] for a discussion of this point.

(iii) Due to our iterative definition of H-nefness, a Higgs bundle $\mathfrak{E}$ is H-nef if and only if a finite number of line bundles L_i (each defined on a projective scheme Y_i for which a surjective morphism $Y_i \to X$ exists) are nef. For instance, if $\mathfrak{E}$ is a rank 3 Higgs bundle of X, one is requiring the usual nefness of the following line bundles:

- $\det(\mathfrak{E})$ on X
- $Q_{1,\mathfrak{E}}$ on $\mathfrak{Gr}_1(\mathfrak{E})$
- $\det(Q_{2,\mathfrak{E}})$ on $\mathfrak{Gr}_2(\mathfrak{E})$
- $Q_{1,Q_{2,\mathfrak{E}}}$ on $\mathfrak{Gr}_1(Q_{2,\mathfrak{E}})$.

△

Example 4.3. There are examples of Higgs bundles that are H-nflat but not numerically flat as ordinary bundles. Let $\mathfrak{E} = (E, \phi)$ be a semistable Higgs bundle which is not semistable as an ordinary bundle, and let $\mathfrak{F} = \mathfrak{E} \otimes \mathfrak{E}^*$ with its natural Higgs field. Then $\mathfrak{F}$ enjoys the stated properties. See [4] for details. △

We prove now some properties of H-nef Higgs bundles that will be useful in the sequel. These generalize properties given in [16, 7] for ordinary vector bundles.

Proposition 4.4. *Let X be a smooth projective variety.*

(i) *If $f\colon Y \to X$ is a finite surjective morphism of smooth projective varieties, and $\mathfrak{E}$ is a Higgs bundle on X, then $\mathfrak{E}$ is H-ample (resp. H-nef) if and only if $f^*\mathfrak{E}$ is H-ample (resp. H-nef).*

(ii) *Every quotient Higgs bundle of a H-nef Higgs bundle $\mathfrak{E}$ on X is H-nef.*

Proof. (i) This is standard in the rank one case [16]. In the higher rank case we first notice that $\det(f^*(E)) \simeq f^*(\det(E))$, so that the condition on the determinant is fulfilled. Moreover, by functoriality the morphism f induces a morphism $\bar{f}\colon \mathfrak{Gr}_s(f^*\mathfrak{E}) \to \mathfrak{Gr}_s(\mathfrak{E})$, and $Q_{s,f^*\mathfrak{E}} \simeq \bar{f}^*(Q_{s,\mathfrak{E}})$. One concludes by induction.

(ii) Let $\mathfrak{F} = (F, \phi_F)$ be a rank s Higgs quotient of $\mathfrak{E}$. This corresponds to a section $\sigma\colon X \to \mathfrak{Gr}_s(\mathfrak{E})$ such that $\mathfrak{F} \simeq \sigma^*(Q_{s,\mathfrak{E}})$. Since $Q_{s,\mathfrak{E}}$ is H-nef, $\mathfrak{F}$ is H-nef as well by the previous point. □

Miyaoka's criterion for semistability has been generalized in [6] and [5] to Higgs bundles on smooth projective varieties of any dimension (the same criterion has been generalized to principal bundles in [2]). Let $\Delta(E)$ be the characteristic class

$$\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = \frac{1}{2r} c_2(E \otimes E^*)\,.$$

This is called the *discriminant* of the bundle E. The following result was proved in [6] and [5].

Theorem 4.5. *Let $\mathfrak{E}$ be a Higgs bundle on a smooth polarized projective variety. The following conditions are equivalent.*

(i) *All classes $\lambda_{s,\mathfrak{E}}$ are nef, for $0 < s < r$.*

(ii) *$\mathfrak{E}$ is semistable and $\Delta(E) = 0$.*

(iii) *All classes* $\theta_{s,\mathfrak{E}}$ *are nef, for* $0 < s < r$.
(iv) *For any smooth projective curve* C *in* X, *the restriction* $\mathfrak{E}_{|C}$ *is semistable.*

Remark 4.6. Since condition (iv) is independent of the choice of the polarization, we obtain the interesting observation that a semistable Higgs bundle with vanishing discriminant is semistable with respect to *every* polarization. △

Corollary 4.7. [6] *A semistable Higgs bundle* $\mathfrak{E} = (E, \phi)$ *on an* n*-dimensional polarized smooth projective variety* (X, H) *such that* $c_1(E) \cdot H^{n-1} = \mathrm{ch}_2(E) \cdot H^{n-2} = 0$ *is H-nflat.*

Theorem 4.5 makes use of Theorem 2 in [21], which will also be further needed in the present paper. We recall it here in a simplified form which is enough for our purposes.

Theorem 4.8. *Let* $\mathfrak{E} = (E, \phi)$ *be a semistable Higgs bundle on an* n*-dimensional polarized smooth projective variety* (X, H), *and assume* $c_1(E) \cdot H^{n-1} = \mathrm{ch}_2(E) \cdot H^{n-2} = 0$. *Then* $\mathfrak{E}$ *admits a filtration whose quotients are stable and have vanishing Chern classes.*

4.2. **Numerically flat Higgs bundles and stability.** Analogously to what we did in the Kählerian case, we wish to stidu stability properties of H-nef and H-nflat Higgs bundles. The first result generalizes Corollary 3.6 in [6] and Theorem 1.2 in [14]. The proof does not differ much from the one given in [14] but we include it here for the reader's convenience.

Proposition 4.9. *Let* $\mathfrak{E} = (E, \phi)$ *be a Higgs bundle on a smooth projective variety* X *such that all classes* $\lambda_{s,\mathfrak{E}}$ *are nef.*

(i) *If the class* $c_1(E)$ *is nef, then all universal quotient bundles* $Q_{s,\mathfrak{E}}$ *are nef (so that* $\mathfrak{E}$ *is H-nef).*
(ii) *If* X *is a curve and* $c_1(E)$ *is ample, then all universal quotient bundles* $Q_{s,\mathfrak{E}}$ *are ample (so that* $\mathfrak{E}$ *is H-ample).*
(iii) *If* $c_1(E)$ *is positive (i.e.,* $c_1(E) \cdot [C] > 0$ *for all irreducible curves* $C \subset X$*), then the class* $c_1(Q_{s,\mathfrak{E}})$ *is positive for all* s.

Proof. (i). If $Q_{s,\mathfrak{E}}$ is not nef there is an irreducible curve $C \subset \mathbb{P}Q_{s,\mathfrak{E}}$ such that $c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)) \cdot [C] < 0$. Let $f \colon C' \to C$ be the normalization of C, and let $p \colon C' \to \mathfrak{Gr}_s(\mathfrak{E})$ be the induced map. If L is the pullback of $\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)$ to C', then L is a Higgs quotient of $p^* \circ \rho_s^*(E)$, and

$$\deg(L) = [f(C')] \cdot c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)) < 0 .$$

On the other hand, one has

$$\deg(p^* \circ \rho_s^*(E)) = [p(C')] \cdot c_1(\rho_s^*(E)) \geq 0$$

since $c_1(E)$ is nef, so that

$$\mu(L) < \mu(p^* \circ \rho_s^*(E)). \tag{9}$$

Now, in view of Theorem 4.5, the fact that all classes $\lambda_{s,\mathfrak{E}}$ are nef implies that $\mathfrak{E}$ is semistable, and also that the restriction of $\mathfrak{E}$ to any smooth projective curve in X is semistable. Combining this with Lemma 3.3 in [6], one shows that $p^* \circ \rho_s^*(E)$ is semistable. But then eq. (9) is a contradiction.

(ii). This proof is a slight variation of the previous one, due to the fact that Nakai's criterion for ampleness requires to check positive intersections with subvarieties of all dimensions. Let C be a smooth projective curve and $f\colon C \to X$ a morphism which is of degree larger than $r = \operatorname{rk} E$. Given a point $p \in C$ let F be the class of the fibre of $\mathbb{P}(f^*Q_{s,\mathfrak{E}})$ over p. The Higgs bundle $\mathfrak{E}' = f^*\mathfrak{E} \otimes \mathcal{O}_C(-p)$ is semistable by the same argument as in the previous proof. Moreover, $\deg(\mathfrak{E}') > 0$, so that $\mathfrak{E}'$ is H-nef by the previous point. If L is the pullback to C of the bundle $\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)$, then $L(-F)$ is nef since it is the hyperplane bundle in $\mathbb{P}Q_{s,\mathfrak{E}'}$. If V is any subvariety of $\mathbb{P}(f^*Q_{s,\mathfrak{E}})$ of dimension k, then $c_1(L)^k \cdot [V] > 0$, so that L is ample. Thus the pullback of $\mathfrak{E}$ to C is H-ample, and hence $\mathfrak{E}$ is H-ample as well by Proposition 4.4.

Claim (iii) is proved as claim (ii). □

Corollary 4.10. *Given a Higgs bundle $\mathfrak{E} = (E, \phi)$, if all classes $\lambda_{s,\mathfrak{E}}$ are nef, and $c_1(E)$ is numerically equivalent to zero, then $\mathfrak{E}$ is H-nflat.*

Proposition 4.11. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on a smooth polarized projective variety X, such that all universal quotients $Q_{s,\mathfrak{E}}$ and $Q_{s,\mathfrak{E}^*}$ are nef. Then $\mathfrak{E}$ is semistable. If* $\deg(E) \neq 0$, *then $\mathfrak{E}$ is stable.*

Proof. Under the isomorphism $\mathfrak{Gr}_{r-s}(\mathfrak{E}^*) \simeq \mathfrak{Gr}_s(\mathfrak{E})$ the bundle $S_{r-s,\mathfrak{E}}$ is identified with $Q^*_{r-s,\mathfrak{E}^*}$. Therefore all the universal quotient bundles $Q_{s,\mathfrak{E}}$ and the bundles $S^*_{r-s,\mathfrak{E}}$ on $\mathfrak{Gr}_s(\mathfrak{E})$ are nef. Since

$$c_1(S_{r-s,E}) = -\theta_{s,E} + \frac{r-s}{r}\, p_s^*(c_1(E))$$

we have, after restricting to $\mathfrak{Gr}_s\mathfrak{E}$,

$$c_1(S^*_{r-s,\mathfrak{E}}) = \theta_{s,\mathfrak{E}} + \frac{s-r}{r}\rho_s{}^*(c_1(E))\,. \tag{10}$$

By [7, Prop. 1.2 (11)] this class is nef.

Let us assume at first that X is a curve, and let us suppose that $\deg(E) \geq 0$. By a slight generalization of [10, Prop. 1.8(i)] or [13, Prop. 2.2], the class $p_s^*(c_1(E))$ is positive, and as $\mathfrak{Gr}_s(\mathfrak{E})$ is a closed subscheme of $\mathrm{Gr}_s(E)$, the class $\rho_s^*(c_1(E))$ is positive as well. But since $c_1(S^*_{r-s,\mathfrak{E}})$ is nef this implies that all classes $\theta_{s,\mathfrak{E}}$ are nef and so from Proposition 4.5 it follows that $\mathfrak{E}$ is semistable.

If $\deg(E) \leq 0$, the same argument shows that $\mathfrak{E}^*$ is semistable, and then $\mathfrak{E}$ is semistable as well.

We now show that if $\deg(E) \neq 0$ then $\mathfrak{E}$ is stable. Assume for instance that $\deg(E) > 0$. Proposition 4.9 proves that in this case $c_1(Q_{s,\mathfrak{E}}) > 0$ for all s. Without loss of generality we may assume that $\mathfrak{Gr}_s(\mathfrak{E})$ has a section $\sigma : X \to \mathfrak{Gr}_s(\mathfrak{E})$. Then the bundle $Q_\sigma = \sigma^*(Q_{s,\mathfrak{E}})$ is an ample Higgs quotient of $\mathfrak{E}$. So one has the exact sequence

$$0 \to K \to E \to Q_\sigma \to 0$$

and $-c_1(K) = c_1(Q_\sigma \otimes \det^{-1} E) = \sigma^*(c_1(S^*_{r-s,\mathfrak{E}}))$ is nef as well. Thus $c_1(K) \leq 0$ and $\mu(K) < \mu(E)$. Hence $\mathfrak{E}$ is stable. If $\deg(E) < 0$ by applying the same argument to the dual of $\mathfrak{E}$ we obtain that $\mathfrak{E}^*$ is stable, and hence $\mathfrak{E}$ is stable again.

These results are then extended to an arbitrary dimension of X by the usual induction argument, considering a smooth divisor in the linear system $|mH|$ for m big enough. □

Corollary 4.12. *An H-nflat Higgs bundle is semistable.*

Proof. It is enough to check that if $\mathfrak{E}$ is H-nef and $c_1(E) \equiv 0$, then all universal quotient bundles $Q_{s,\mathfrak{E}}$ are nef. Indeed, in this case the classes $\theta_{s,\mathfrak{E}} = [c_1(Q_{s,\mathfrak{E}})]$ are nef, so that the classes $\lambda_{s,\mathfrak{E}}$ are nef by Theorem 4.5. But $\lambda_{s,\mathfrak{E}} = [c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1))]$, so that $Q_{s,\mathfrak{E}}$ is nef. □

Remark 4.13. Proposition 4.11 raises the question of the existence of stable H-nflat Higgs bundles. An example is provided by a Higgs bundle $\mathfrak{E}$ such that $E = L_1 \oplus L_2$ with $\phi(L_1) \subset L_2 \otimes \Omega^1_X$, $\phi(L_2) = 0$, if we choose $\deg(L_1) = 1$ and $\deg(L_2) = -1$, and assume that the genus of the curve X at least 2. Then $\mathfrak{E}$ is stable and H-nflat. For details see [4]. △

Proposition 4.11 admits as a simple consequence the characterization of H-nflat Higgs bundles in terms of filtrations.

Theorem 4.14. *A Higgs bundle $\mathfrak{E}$ on X is H-nflat if and only if it admits a filtration whose quotients are flat stable Higgs bundles.*

Proof. If $\mathfrak{E}$ is H-nflat by Corollary 4.12 it is semistable. Since all Chern classes of $\mathfrak{E}$ vanish, by Theorem 4.8 $\mathfrak{E}$ has a filtration whose quotients are stable and have vanishing Chern classes. We may assume that $\mathfrak{E}$ is an extension

$$0 \to \mathfrak{F} \to \mathfrak{E} \to \mathfrak{G} \to 0 \tag{11}$$

of stable Higgs bundles with vanishing Chern classes, otherwise one simply iterates the following argument. Let us consider the bundle $\mathfrak{F} = (F, \phi_F)$; the same will apply to $\mathfrak{G}$. By results given in [21], the bundle F admits a Hermitian-Yang-Mills metric. Let Θ be the curvature of the associated Chern connection. Since $c_1(F) = c_2(F) = 0$, we have

$$0 = \int_X \operatorname{tr}(\Theta \wedge \Theta) \cdot H^{n-2} = \gamma_1 \|\Theta\|^2 - \gamma_2 \|\Lambda \Theta_i\|^2 = \gamma_1 \|\Theta\|^2$$

for some positive constants γ_1, γ_2, so that the Chern connection of F is flat, i.e., F is flat.

Conversely, let assume that $\mathfrak{E}$ has a filtration as in the statement. Then $\mathfrak{E}$ is semistable with vanishing Chern classes, and by Corollary 4.7 it is numerically flat. □

4.3. **Comparison between the projective and Kählerian cases.** We study in this section the relation between the notions of 1-H-nefness and H-nefness on complex projective manifolds. One sees that 1-H-nefness impies H-nefness, and that the two notions are equivalent on curves. We do not know whether they are equivalent in any dimension.

Proposition 4.15. *A 1-H-nef Higgs bundle $\mathfrak{E} = (E, \phi)$ is H-nef.*

Proof. We proceed by induction on the rank r of $\mathfrak{E}$. If $r = 1$ there is nothing to prove. If $r > 1$, for every $s = 1, \dots, r-1$ let us consider the universal sequence (6) on the Higgs Grassmannian $\mathfrak{Gr}_s(\mathfrak{E})$. Since the Higgs Grassmannian is in general singular, we consider a resolution of singularities $\beta_s : B_s(\mathfrak{E}) \to \mathfrak{Gr}_s(\mathfrak{E})$, and pullback the universal sequence to $B_s(\mathfrak{E})$:

$$0 \to \beta_s^* S_{r-s,\mathfrak{E}} \to \gamma_s^* \mathfrak{E} \to \beta_s^* Q_{s,\mathfrak{E}} \to 0,$$

where $\gamma_s = \rho_s \circ \beta_s$. Since $\mathfrak{E}$ is 1-H-nef, the pullback $\gamma_s^*(\mathfrak{E})$ is 1-H-nef as well, and its Higgs quotient $\beta_s^* Q_{s,\mathfrak{E}}$ is 1-H-nef, hence H-nef by the inductive hypothesis.

We need to show that $Q_{s,\mathfrak{E}}$ is H-nef; in view of Remark 4.2, by base change this reduces to proving the following fact: if $f_i : Z_i \to Y_i$ are surjective morphisms of projective schemes, and L_i are line bundles on Y_i such that the pullbacks $f_i^* L_i$ are nef, then the line bundles L_i are nef. This follows from [13, Prop. 2.3]. □

Proposition 4.16. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ over a smooth projective curve X is 1-H-nef if and only if it is H-nef.*

Proof. We have just proved the necessary condition. We prove the sufficiency again by induction on the rank r of $\mathfrak{E}$. If $r = 1$ there is nothing to prove. If $r > 1$, note that since $\mathfrak{E}$ is H-nef, then $\deg(E) \geq 0$, and all its quotients $\mathfrak{Q}$ are H-nef. By the inductive hypothesis, all $\mathfrak{Q}$ are 1-H-nef; one concludes by Proposition 3.9. □

This strongly simplifies the proof of Theorem 3.3.1 of [8], which gives the same result in the case of ordinary bundles.

By using the fact that H-nefness may be checked on embedded curves, and the fact that on curves 1-H-nefness and H-nefness coincide, we may prove some additional properties of H-nef Higgs bundles.

Lemma 4.17. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ over a smooth projective variety X is H-nef if and only if $\mathfrak{E}_{|C} = (E_{|C}, \phi_{|C})$ is H-nef for all irreducible curves C in X.*

Proof. By Remark 4.2 the Higgs bundle $\mathfrak{E}$ is H-nef if and only if a finite number of line bundles L_i (each defined on a projective scheme Y_i for which a surjective morphism $Y_i \to X$ exists) are nef. The claim then follows. □

Proposition 4.18. *An extension of H-nef Higgs bundles is H-nef.*

Proof. In view of Lemma 4.17 we may assume that X is a curve. The result then follows from Propositions (vii) and 4.16. □

In the same way, by using Lemma 4.17 one can prove that the tensor, exterior and symmetric products of H-nef Higgs bundles are H-nef. Moreover we have:

Proposition 4.19. *Let $\mathfrak{E}$ be a Higgs bundle. If $S^m(\mathfrak{E})$ is H-nef for some m, then $\mathfrak{E}$ is H-nef.*

Proof. Since a rank s Higgs quotient of $\mathfrak{E}$ yields a Higgs quotient of $S^m(\mathfrak{E})$ of rank

$$N_{(m,s)} = \binom{m+s-1}{s-1},$$

one has a morphism $g : \mathfrak{Gr}_s(\mathfrak{E}) \to \mathfrak{Gr}_{N_{(m,s)}}(S^m(\mathfrak{E}))$ such that $g^*(Q_{N_{(m,s)},S^m(\mathfrak{E})}) \simeq S^m(Q_{s,\mathfrak{E}})$. Since $S^m(\mathfrak{E})$ is H-nef, the symmetric product $S^m(Q_{s,\mathfrak{E}})$ is H-nef. The claim follows by induction on the rank of $\mathfrak{E}$. □

4.4. **More semistability criteria.** Some semistabilty criteria in addition to those listed in Theorem 4.5 may be given in terms of the notion of H-nefness. One of these has the advantage that is expressed in terms of a bundle on the base manifold. Another is stated in terms of the Higgs bundles $T_{s,\mathfrak{E}} = S^*_{r-s,\mathfrak{E}} \otimes Q_{s,\mathfrak{E}}$ on the Higgs Grassmannians $\mathfrak{Gr}_s(\mathfrak{E})$. For an ordinary vector bundle E, the bundle $T_{s,E}$ is the vertical tangent bundle to $p_s : \mathrm{Gr}_s(E) \to X$.

Theorem 4.20. *Let $\mathfrak{E} = (E, \phi)$ be a rank r Higgs bundle on a complex projective manifold X. The following three conditions are equivalent:*

(i) *the Higgs bundle $\mathfrak{F} = S^r(\mathfrak{E}) \otimes (\det \mathfrak{E})^{-1}$ is H-nflat;*
(ii) *$\mathfrak{E}$ is semistable and $\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = 0$;*
(iii) *the Higgs bundles $T_{s,\mathfrak{E}}$ are all H-nef.*

Proof. We first prove that (i) implies (ii). Since $\det(\mathfrak{F})$ is trivial, the dual Higgs bundle $\mathfrak{F}^*$ is H-nef as well, i.e., $\mathfrak{F}$ is H-nflat, hence semistable by Theorem 3.1 of [4]. Then, the Higgs bundle $\mathfrak{F} \otimes \mathfrak{F}^* \simeq S^r(\mathfrak{E}) \otimes S^r(\mathfrak{E}^*)$ is semistable. This implies that $\mathfrak{E}$ is semistable.

One also has that $S^r(\mathfrak{E}) \otimes S^r(\mathfrak{E}^*)$ is H-nflat so that its Chern classes vanish by Theorem 4.14. But since

$$c_2(S^r(E) \otimes S^r(E^*)) = 4r(\mathrm{rk}\, S^r(E))^2 \Delta(E)$$

we conclude.

(ii) implies (i): we have that $\mathfrak{F}$ is semistable and

$$c_1(F) = 0, \qquad c_2(F) = 2r(\mathrm{rk}\, S^r(E))^2 \Delta(E) = 0.$$

By Theorem 4.8, $\mathfrak{F}$ has a filtration whose quotients are stable Higgs bundles with vanishing Chern classes. Proceeding as in the proof of Lemma 3.6(iv), these quotients are shown to be Hermitian flat, hence they are H-nflat. Then $\mathfrak{F}$ is H-nflat as well.

We prove now that (i) implies (iii). If $\mathfrak{F}$ is H-nef, then the $\mathbb{Q}$-Higgs bundle $\mathfrak{E}\otimes(\det(\mathfrak{E}))^{-1/r}$ is H-nef by Proposition 4.19, so that the Higgs bundle $\mathfrak{E}^*\otimes(\det(\mathfrak{E}))^{1/r}$ is H-nef (since $c_1(\mathfrak{E}\otimes(\det(\mathfrak{E}))^{-1/r})=0$), and $Q_{s,\mathfrak{E}}\otimes\rho_s^*(\det(\mathfrak{E}))^{-1/r}$ is H-nef as well, since it is a universal quotient of $\mathfrak{E}\otimes(\det(\mathfrak{E}))^{-1/r}$. From the exact sequence

$$0\to Q_{s,\mathfrak{E}}\otimes Q_{s,\mathfrak{E}}^*\to\rho_s^*(\mathfrak{E}^*)\otimes Q_{s,\mathfrak{E}}\to T_{s,\mathfrak{E}}\to 0,$$

one obtains the claim.

Finally, we prove that (iii) implies (ii). Note that the class $\theta_{s,\mathfrak{E}}$ defined in equation (8) equals $[c_1(T_{s,\mathfrak{E}})]$, so that if $T_{s,\mathfrak{E}}$ is H-nef, the class $\theta_{s,\mathfrak{E}}$ is nef. This holds true for every $s=1,\dots,r-1$. It was proved in [6] that this is equivalent to condition (ii) in the statement. □

Example 4.21. We give an example of an H-nef Higgs bundle which is not nef as an ordinary bundle. Let X be a projective surface of general type that saturates Miyaoka-Yau's inequality, i.e., $3c_2(X)=c_1(X)^2$ (surfaces of general type satisfying this condition are exactly those that are uniformized by the unit ball in $\mathbb{C}^2$ [20]). The Higgs bundle $\mathfrak{E}$ whose underlying vector bundle is $E=\Omega_X^1\oplus\mathcal{O}_X$ with the Higgs morphism $\phi(\omega,f)=(0,\omega)$ is semistable and satisfies $\Delta(E)=0$, so that the Higgs bundle $\mathfrak{F}=S^3(\mathfrak{E})\otimes(\det\mathfrak{E})^{-1}$ is 1-H-nef. On the other hand, the underlying vector bundle $F=S^3(\Omega_X^1\oplus\mathcal{O}_X)\otimes K_X^*$ contains K_X^* as a direct summand and therefore is not nef (note that we exclude that K_X is numerically flat). △

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