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Quasi-periodic Solutions for a PDE model arising in hydrodynamics

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Introduction

Main result

This thesis concerns the existence and the stability of small amplitude quasi-periodic solutions for the Hamiltonian PDEs

$$\begin{cases} \partial_t \eta &= -\partial_x u - \frac{1}{3}\varepsilon^2 \partial_x^3 u - \frac{2}{15}\varepsilon^4 \partial_x^5 u - \varepsilon^2 \partial_x(\eta u) - \varepsilon^4 \partial_x^2(\eta \partial_x u) \\ \partial_t u &= -\partial_x \eta - \varepsilon^2 \frac{1}{2}\partial_x(u^2) + \varepsilon^4 \frac{1}{2}\partial_x(\partial_x u)^2, \end{cases} \tag{1}$$

which are the equations of motion derived by the following Hamiltonian

$$H = \int_{\mathbb{T}} \frac{1}{2}\left(u^2 + \eta^2 + \varepsilon^2\left(-\frac{1}{3}(\partial_x u)^2 + \eta u^2\right) + \varepsilon^4\left(\frac{2}{15}(\partial_x^2 u)^2 - (\partial_x u)^2 \eta\right)\right) dx\,. \tag{2}$$

The equations (1) arise from an approximate model derived by the water waves equations of hydrodynamics, in a regime of small amplitude solutions with long wavelength. This model has been suggested to us by Walter Craig [26], and we present its derivation in Appendix A. There is a large literature regarding such approximate models, for which we refer to [27], [30], [29] and references therein.

Very recently the existence of small amplitude quasi-periodic solutions for the full water waves equations has been proved by M. Berti and R. Montalto in [19]. The goal of this thesis is to follow the same approach in order to construct quasi-periodic solutions for the system (1). Actually many of the techniques that we shall employ are very general and in principle can be adapted to other models in hydrodynamics.

We recall that a time quasi-periodic function with values in a phase space $\mathfrak{H}$, is a function defined $\forall t \in \mathbb{R}$ of the form

$$z(t) = Z(\omega t) \in \mathfrak{H}\,, \quad \mathbb{T}^N \ni \theta \to Z(\theta) \in \mathfrak{H}, \tag{3}$$

where the function Z is continuous, $\mathbb{T}^N := (\mathbb{R}/2\pi\mathbb{Z})^N$, and the frequency vector $\omega := (\omega_1, ..., \omega_N)$ is rationally independent, namely $\omega \cdot l \neq 0,\ \forall l \in \mathbb{Z}^N \setminus \{0\}$.

For the equations (1) we consider as phase space the space of 2π-periodic, real functions with zero average in the space variable, namely

$$(\eta, u) \in H_0^p(\mathbb{T}_x, \mathbb{R}) \times H_0^p(\mathbb{T}_x, \mathbb{R})\,, \tag{4}$$

where

$$H^p_0(\mathbb{T}_x,\mathbb{R}) := \left\{ g = \sum_{j\in\mathbb{Z}} g_j e^{ijx} \ : \ g_j = \bar g_{-j}\,,\ g_0 = 0\,,\ \|g\|^2_{H^p_0(\mathbb{T}_x,\mathbb{R})} = \sum_{j\in\mathbb{Z}} |g_j|^2 \langle j\rangle^{2p} \right\}.$$

Note that we are allowed to consider a phase space of functions with zero average since this is invariant under the evolution of (1). Moreover, the subspace consisting of functions (η, u) where η is even and u is odd in the spatial variable,

$$\eta(x) = \eta(-x)\,, \quad u(x) = -u(-x)\,, \tag{5}$$

is also invariant under the evolution of (1). Therefore for simplicity we shall consider functions in (4) that satisfy (5).

We endow the phase space introduced above with the symplectic form

$$\mathcal{W}\left(\begin{pmatrix} \eta(x) \\ u(x) \end{pmatrix}, \begin{pmatrix} \eta_1(x) \\ u_1(x) \end{pmatrix} \right) := \int_0^{2\pi} \left\langle J^{-1} \begin{pmatrix} \eta(x) \\ u(x) \end{pmatrix}, \begin{pmatrix} \eta_1(x) \\ u_1(x) \end{pmatrix} \right\rangle dx\,, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ is the standard $\mathbb{R}^2$ scalar product and J^{-1} is the symplectic matrix given by

$$J^{-1} = \begin{pmatrix} 0 & \partial_x^{-1} \\ \partial_x^{-1} & 0 \end{pmatrix}. \tag{7}$$

Notice that, given a function $g = \sum_{j\in\mathbb{Z}} g_j e^{ijx}$ such that $g_0 = 0$, i.e. g has zero average, then

$$\partial_x^{-1} g = \sum_{j\in\mathbb{Z}} \frac{1}{ij} g_j e^{ijx}\,,$$

namely ∂_x^{-1} is the periodic primitive of the function g. The symplectic form in (6) is explicitly given by

$$\mathcal{W}\left(\begin{pmatrix} \eta(x) \\ u(x) \end{pmatrix}, \begin{pmatrix} \eta_1(x) \\ u_1(x) \end{pmatrix} \right) := \int_0^{2\pi} \left[(\partial_x^{-1}\eta(x))u_1(x) + (\partial_x^{-1}u(x))\eta_1(x) \right] dx\,,$$

The system (1) can be rewritten in the form (see Appendix A.1)

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + J\nabla H(\eta,u) = 0\,, \qquad \text{where} \qquad J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{8}$$

and ∇ denotes the L^2-gradient, or equivalently

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = X_H \begin{pmatrix} \eta \\ u \end{pmatrix}. \tag{9}$$

Another symmetry of the equations (1) is the reversible structure. Indeed the equations (1) are reversible with respect to the involution

$$\rho : (\eta, u) \mapsto (\eta, -u), \tag{10}$$

in the sense that, the Hamiltonian vector field X_H in (9) satisfies

$$X_H \circ \rho = -\rho \circ X_H \,.$$

Equivalently, the Hamiltonian H in (2) is even in u, i.e.

$$H \circ \rho = H, \quad H(\eta, u) = H(\eta, -u). \tag{11}$$

This reversible property implies that if $(\eta(t), u(t))$ is a solution of (1), then $\rho(\eta(-t), u(-t))$ is also a solution. As a consequence it is natural to look for "reversible solutions" of (1) satisfying

$$(\eta(-t), u(-t)) = \rho(\eta(t), u(t)), \quad \text{i.e.} \quad \eta(x, -t) = \eta(x, t)\,, \quad u(x, -t) = -u(x, t)\,, \quad \forall x \in \mathbb{T} \tag{12}$$

namely η is even in time and u is odd in time.

Since we are looking for small amplitude solutions, the dynamics of the linearized system at $(\eta, u) = (0, 0)$ plays an important role. At least in a neighborhood of the origin, the Hamiltonian (2) can be seen as a perturbation of the quadratic Hamiltonian

$$\tilde{L}(\eta, u) = \int_{\mathbb{T}} \left(\frac{u^2}{2} + \frac{\eta^2}{2} - \frac{\varepsilon^2}{6} u_x^2 + \frac{\varepsilon^4}{15} u_{xx}^2 \right) dx\,. \tag{13}$$

The corresponding linear system at zero is

$$\begin{cases} \partial_t \eta &= -\partial_x u - \frac{1}{3}\varepsilon^2 \partial_x^3 u - \frac{2}{15}\varepsilon^4 \partial_x^5 u \\ \partial_t u &= -\partial_x \eta\,. \end{cases} \tag{14}$$

The solutions of the linear system (14), satisfying the conditions (5) and (12), are

$$\eta(x, t) = \sum_{j \geq 1} a_j \cos(\omega_j t) \cos(jx)\,, \quad u(x, t) = \sum_{j \geq 1} a_j \omega_j \sin(\omega_j t) \sin(jx) \tag{15}$$

for parameters $a_j \in \mathbb{R}$, where the linear frequencies of oscillations ω_j are

$$\omega_j := \omega_j(\varepsilon) := \sqrt{\frac{2}{15}\varepsilon^4 j^6 - \frac{1}{3}\varepsilon^2 j^4 + j^2}\,, \quad j \geq 1\,. \tag{16}$$

Notice that ω_j are real for all $j \in \mathbb{N}$ (see Remark 1.2). Hence all the solutions (15) of the system (14) are either periodic, quasi-periodic or almost perdiodic in time.

The main result of the thesis is that most of the quasi-periodic solutions (15) of the linear system (14) can be continued to quasi-periodic solutions of the nonlinear Hamiltonian system (2) for most values of the parameter $\varepsilon \in [\varepsilon_1, \varepsilon_2]$.

Let us state precisely our main result. We arbitrarily fix a finite subset $\mathbb{S} \subseteq \mathbb{N}_0 := \{1, 2, ...\}$ (where $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$), called *tangential sites*, and we consider the linear solutions of (14) whose Fourier modes are supported in $\mathbb{S}$, namely

$$\eta(t, x) = \sum_{j \in \mathbb{S}} \sqrt{r_j} \cos(\omega_j t) \cos(jx)\,, \quad u(t, x) = \sum_{j \in \mathbb{S}} \sqrt{r_j}\omega_j \sin(\omega_j t) \sin(jx)\,, \quad r_j > 0. \tag{17}$$

In Theorem 1 below we prove that for most values of the parameter $\varepsilon \in [\varepsilon_1, \varepsilon_2]$ and for μ sufficiently small there exist quasi-periodic solutions $\mathtt{g}(x, \omega^\infty t) = (\eta, u)(x, \omega^\infty t)$ of (1), with frequency vector $\omega^\infty := (\omega_j^\infty)_{j\in\mathbb{S}}$, which are μ-close to the solutions (17) of (14). Let $N := |\mathbb{S}|$ denote the cardinality of $\mathbb{S}$. The function $\mathtt{g}(x,\theta) = (\eta, u)(x,\theta)$ with $\theta \in \mathbb{T}^N$ belongs to the Sobolev spaces of $(2\pi)^{1+N}-$ periodic real functions

$$\mathbf{H}^p(\mathbb{T}^{1+N}, \mathbb{R}^2) := \{\mathtt{g} = (\eta, u) : \eta, u \in H^p\}$$

where

$$\begin{aligned} H^p &:= H^p(\mathbb{T}^{1+N}, \mathbb{R}) \\ &:= \left\{ g = \sum_{(l,j)\in\mathbb{Z}^{N+1}} g_{l,j} e^{i(l\cdot\varphi + jx)} \ : \ g_{l,j} = \overline{g}_{-l,-j}\,, \ \|g\|_p^2 := \sum_{(l,j)\in\mathbb{Z}^{N+1}} |g_{l,j}|^2 \langle l, j\rangle^{2p} < \infty \right\} \end{aligned} \tag{18}$$

and $\langle l, j\rangle := \max\{1, |l|, |j|\}$ and $|l| := \max_{i=1,\dots,N} |l_i|$. For

$$p \geq \mathfrak{p}_0 := \left[\frac{N+1}{2}\right] + 1 \in \mathbb{N} \tag{19}$$

the Sobolev spaces $H^p(\mathbb{T}^{N+1}) \subset L^\infty(\mathbb{T}^{N+1})$ are an algebra with respect to the product of functions. In the Thesis we shall consider $\mathfrak{p}_0$ fixed.

Theorem 1. *Fix finitely many tangential sites $\mathbb{S} := \{0 < j_1 < ... < j_N,\ j_k \in \mathbb{N}\}$. There exists $\bar{p} > \mathfrak{p}_0$, $\mu_0 \in (0,1)$ such that for every $|r| \leq \mu_0^2, r := (r_j)_{j\in\mathbb{S}}$ there exists a Cantor like set $\mathcal{G} \subset [\varepsilon_1, \varepsilon_2]$ with asymptotically full measure as $r \to 0$, i.e.*

$$\lim_{r\to 0} |\mathcal{G}| = \varepsilon_2 - \varepsilon_1$$

such that for all ε in $\mathcal{G}$ the system (1) has a reversible quasi-periodic solution

$$\mathtt{g}(x, \omega^\infty t) = (\eta(x, \omega^\infty t), u(x, \omega^\infty t))\,,$$

with Sobolev regularity $(\eta, u)(x,\theta) \in \mathbf{H}^{\bar{p}}(\mathbb{T} \times \mathbb{T}^N, \mathbb{R}^2)$ where η is even in the spatial variable, and u is odd, of the form

$$\begin{cases} \eta(x, \omega^\infty t) &= \sum_{j\in\mathbb{S}} \sqrt{r_j} \cos(\omega_j^\infty t) \cos(jx) + o(\sqrt{|r|}) \\ u(x, \omega^\infty t) &= \sum_{j\in\mathbb{S}} \sqrt{r_j} \sin(\omega_j^\infty t) \sin(jx) + o(\sqrt{|r|}) \end{cases} \tag{20}$$

with frequency vector $\omega^\infty := (\omega_j^\infty(\varepsilon))_{j\in\mathbb{S}} \in \mathbb{R}^N$ that is Diophantine and satisfies $\omega_j^\infty - \omega_j(\varepsilon) \to 0,\ \ \forall j \in \mathbb{S}$, as $r \to 0$. The terms $o(\sqrt{|r|})$ are small in $H^{\bar{p}}(\mathbb{T}^N \times \mathbb{T}, \mathbb{R}^2)$. In addition these quasi-periodic solutions are linearly stable.

Theorem 1 will be deduced by Theorem 4.1 and Lemma 4.8 below. In order to prove Theorem 4.1 we use a Nash-Moser scheme (see Chapter 10). The Nash-Moser iterative procedure selects many values of the parameter $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, giving rise to the quasi-periodic solutions (20) defined for all times. By a Fubini-type argument it also results that, for most values of $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, there exist quasi-periodic solutions of (1) for most values of the amplitudes $|r| \le \mu_0^2$. In order to prove Theorem 1 we will split the phase space into two different subspaces, a finite dimensional one, which we shall call $\mathtt{H}_{\mathbb{S}}$ and its orthogonal, called $\mathtt{H}_{\mathbb{S}}^{\perp}$ (see (1.32)). On the finite dimensional subspace $\mathtt{H}_{\mathbb{S}}$ we will describe the dynamics by introducing the action-angle variables (see Chapter 1).

The quasi-periodic solutions $\mathtt{g}(\omega^\infty t) = (\eta(\omega^\infty t), u(\omega^\infty t))$ found in Theorem 1 are linearly stable. More precisely this means that there exist symplectic coordinates around each invariant torus,

$$(\psi, y, z) \in \mathbb{T}^N \times \mathbb{R}^N \times \mathtt{H}_{\mathbb{S}}^{\perp} ,$$

see (5.27), in which the Hamiltonian reads

$$\omega \cdot y + (K_{11}(\psi)y, z)_{L^2(\mathbb{T}_x)} + \frac{1}{2}K_{20}(\psi)y \cdot y + \frac{1}{2}(K_{02}(\psi)z, z)_{L^2(\mathbb{T}_x)} + K_{\ge 3}(\psi, y, z) , \tag{21}$$

where $K_{\ge 3}$ collects all the terms of order at least 3 in (y, z). In these coordinates the quasi-periodic solutions $\mathtt{g}(\omega^\infty t)$ read $t \mapsto (\omega^\infty t, 0, 0)$, and the corresponding linearized equations are

$$\begin{cases} \dot\psi & = K_{20}(\omega^\infty t)[y] + K_{11}^T(\omega^\infty t)[z] \\ \dot y & = 0 \\ \dot z & = JK_{02}(\omega^\infty t)[z] + JK_{11}(\omega^\infty t)[y] . \end{cases}$$

The actions $y(t) = y_0$ do not evolve in time and the third equation reduces to the linear PDE

$$\dot z = JK_{02}(\omega^\infty t)[z] + JK_{11}(\omega^\infty t)[y]. \tag{22}$$

The operator K_{02} (explicitly given in (6.1)) is the restriction to the infinite dimensional subspace $\mathtt{H}_{\mathbb{S}}^{\perp}$ of the linearized system (1) (see (6.10)) up to a finite dimensional remainder (see Lemma 6.1).

In Chapters 7-9 we prove the existence of a bounded and invertible "symmetrizer" map $\mathbf{W}_\infty$ (see (9.101), (9.102)) such that for all $\theta \in \mathbb{T}^N$ and under the change of variable

$$z = \mathbf{W}_\infty z_\infty , \quad z_\infty := (\mathtt{z}_\infty^{(1)}, \mathtt{z}_\infty^{(2)})$$

the equation (22) transforms into the diagonal system

$$\partial_t z_\infty = -i\mathbf{D}_\infty z_\infty + f_\infty(\omega^\infty t) , \quad f_\infty(\omega^\infty t) = \mathbf{W}_\infty(\omega^\infty t)^{-1} JK_{11}(\omega^\infty t)[y_0] = \begin{pmatrix} \mathtt{f}_\infty^{(1)}(\omega^\infty t) \\ \mathtt{f}_\infty^{(2)}(\omega^\infty t) \end{pmatrix} , \tag{23}$$

where, if we define $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\} := \mathbb{S}^\pm \cup (\mathbb{S}^\pm)^c$ with $\mathbb{S}^\pm := \mathbb{S} \cup (-\mathbb{S})$ (see (1.31)), the operator $\mathbf{D}_\infty$ can be written as follows

$$\mathbf{D}_\infty := \begin{pmatrix} D_\infty & 0 \\ 0 & -D_\infty \end{pmatrix} , \quad D_\infty := diag_{j \in (\mathbb{S}^\pm)^c}\{\lambda_j^\infty\} , \ \lambda_j^\infty \in \mathbb{R} ,$$

with D_∞ a Fourier multiplier operator that can be written in terms of (see (10.43))

$$\lambda_j^\infty := j\sqrt{\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1} + m_1^\infty j + r_j^\infty , \quad j \in (\mathbb{S}^\pm)^c , \ r_j^\infty = -r_{-j}^\infty , \quad m_1, r_j \in \mathbb{R} , \tag{24}$$

and, for some $\mathfrak{a} > 0$,

$$m_1^\infty = O(\mu^{\mathfrak{a}}), \quad \sup_{j \in (\mathbb{S}^\pm)^c} |\partial_\varepsilon^k r_j^\infty| = O(\mu^{\mathfrak{a}}), \ \ \forall 0 < |k| \le k_0$$

(see (4.8), (4.9) and (4.13)), where $k_0 \in \mathbb{N}$ is a constant fixed once and for all in Chapter 3 (see Remark 3.6), depending only on the linear frequencies $\omega_j(\varepsilon)$ defined in (16).

The λ_j^∞ are the Floquet exponents of the quasi-periodic solution. As we shall prove in Chapters 7-9 the solutions $z_\infty := (\mathtt{z}_\infty^{(1)}, \mathtt{z}_\infty^{(2)})$ satisfy $\mathtt{z}_\infty^{(1)}(-x) = \mathtt{z}_\infty^{(2)}(x)$. This condition, in the Fourier basis, reads $z_{\infty,j} := (\mathtt{z}_{\infty,j}^{(1)}, \mathtt{z}_{\infty,-j}^{(1)})$. Hence it suffices to solve the first equation in (23). Furthermore the system (23) reduces to the infinitely many decoupled scalar equations

$$\partial_t \mathtt{z}_{\infty,j}^{(1)} = -i\lambda_j^\infty \mathtt{z}_{\infty,j}^{(1)} + \mathtt{f}_{\infty,j}^{(1)}(\omega^\infty t), \quad \forall j \in (\mathbb{S}^\pm)^c.$$

By variation of constants the solutions are

$$\mathtt{z}_{\infty,j}^{(1)}(t) = c_j e^{-i\lambda_j^\infty t} + \mathtt{q}_{\infty,j}^{(1)}(t),$$

where

$$\mathtt{q}_{\infty,j}^{(1)}(t) := \sum_{l \in \mathbb{Z}^N} \frac{\mathtt{f}_{\infty,j,l}^{(1)} e^{i\omega^\infty \cdot lt}}{i(\omega^\infty \cdot l + \lambda_j^\infty)}, \quad \forall j \in (\mathbb{S}^\pm)^c. \tag{25}$$

Since the first order Melnikov conditions (see (4.10)) hold, the denominators of $\mathtt{q}_{\infty,j}^{(1)}(t)$ in (25) are non zero, so the functions $\mathtt{q}_{\infty,j}^{(1)}(t)$ are well defined. By the property of $\mathbf{W}_\infty$ in (9.101), recalling (23) we get

$$\|f_\infty(\omega^\infty t)\|_{H^p(\mathbb{T}_x)\times H^p(\mathbb{T}_x)} \le C|y_0|.$$

As a consequence, using also the properties of $\mathbf{W}_\infty$ and $\mathbf{W}_\infty^{-1}$ in (9.101) and (9.102), the Sobolev norm of the solution of (23) with initial condition $z_\infty(0) \in H^{\tilde{p}}(\mathbb{T}_x)$, with $\mathfrak{p}_0 \le \tilde{p} \le p$, satisfies

$$\|z_\infty(t)\|_{H^{\tilde{p}}(\mathbb{T}_x)\times H^{\tilde{p}}(\mathbb{T}_x)} \le C(p)\left(|y_0| + \|z_\infty(0)\|_{H^{\tilde{p}}(\mathbb{T}_x)\times H^{\tilde{p}}(\mathbb{T}_x)}\right),$$

for all $t \in \mathbb{R}$, which proves the linear stability of the torus. The above inequality can be translated in the original coordinates (η, u), which are related to the coordinates z by the change of variables $\mathbf{\Lambda}$ in (1.22) and $\mathcal{Z}$ in (7.15), as

$$\|(\eta, u)(t)\|_{H^{\tilde{p}+1}(\mathbb{T}_x)\times H^{\tilde{p}-1}(\mathbb{T}_x)} \lesssim_{\tilde{p}} \|(\eta_0, u_0)\|_{H^{\tilde{p}}(\mathbb{T}_x)\times H^{\tilde{p}}(\mathbb{T}_x)}.$$

In conclusion, we are able to prove both the existence and the linear stability of the quasi-periodic solutions of equations (1).

Historical preface

Since the 50′s the so called KAM (Kolmogorov [46]- Arnold [2]- Moser [53], [54]) theory played a key rôle in the knowledge of the dynamical behavior of “non integrable” Hamiltonian systems. The first results proved that, in a finite dimensional integrable Hamiltonian system subject to a "small" perturbation, under some non degeneracy assumptions, the quasi-periodic orbits form an asymptotically full-measure set of the phase space. The quasi-periodic solutions of the perturbed system are close to the quasi-periodic solutions of the unperturbed one.

The KAM theory is an important extension of the simpler problem of the existence of periodic solutions, that dates back to Poincaré in his studies of celestial mechanics [56]. Bifurcation theory on periodic solutions relies on the implicit function theorem.

On the contrary, in the search of quasi-periodic solutions, a serious non trivial problem arises, which prevents the use of the implicit function theorem: in the Fourier series expansion of the approximate solutions appears at the denominators the quantities $\omega \cdot l\,,\ l \in \mathbb{Z}^N$. For periodic solutions $\omega \cdot l = \omega l\,,\ l \in \mathbb{Z}$ and, if $\omega \neq 0$, the set $\{\omega l : l \in \mathbb{Z} \setminus \{0\}\}$ is at a positive distance from zero. On the other hand if $\omega \in \mathbb{R}^N\,,\ N \geq 2$, is a rationally independent vector, the set

$$\{\omega \cdot l : l \in \mathbb{Z}^N\}$$

is dense in $\mathbb{R}$, in particular it accumulates to zero. This is the so called "small divisor problem". Nevertheless Kolmogrov proved the existence of quasi-periodic solutions requiring that ω satisfies the non-resonance Diophantine condition

$$|\omega \cdot l| \geq \gamma |l|^{-\tau}\,,\ \ \forall l \neq 0\,, \quad \gamma \in (0,1).$$

See also [57].

Starting from the $80's$ the ideas of dynamical systems started to be extended to PDEs. It is known that many PDEs on a manifold can be rewritten as an infinite dimensional dynamical system of the form

$$\dot{u} = Lu + f(u) \tag{26}$$

where u is a function in some Banach space, L is a linear operator and f is a non linear term. The search of quasi-periodic solutions of (26), namely functions of the form $u := u(\omega t)$ as in (3), amounts to solve the equation for $u(\theta)$

$$\omega \cdot \partial_\theta u = Lu + f(u). \tag{27}$$

If $f(0) = f'(0) = 0$, then $u = 0$ is an equilibrium solution of the system (26), therefore it is natural to look for quasi-periodic solutions in a neighborhood of zero.

The first existence results for quasi-perdiodic solutions have been obtained by Kuksin [47] for the $1-$d non-linear Schrödinger equation (NLS) with Dirichlet boundary conditions where f is a bounded nonlinearity and Wayne [64] for the $1-$d nonlinear wave equation (NLW), still with Dirichlet boundary conditions. Their method of proof is a generalization of KAM theory.

As already discussed, because of the small divisor problem equation (27) cannot be solved by the classical implicit function theorem. Indeed the linearized operator of (27) at the equilibrium $u = 0$, i.e. $\omega \cdot \partial_\theta - L$, can be diagonalized in a Fourier basis (both in space and time) as $i\omega \cdot l - i\lambda_j$, where $l \in \mathbb{Z}^N$, λ_j, $j \in \mathbb{Z}$ are the eigenvalues of the linear operator L, and $i\omega \cdot l - i\lambda_j$ accumulate to zero. Note that the eigenvalues of the linear operator L are considered pure imaginary, as they could correspond to the interesting case of some resonance phenomena. In order to overcome this problem one can impose the first Melnikov non-resonance conditions, namely

$$|\omega \cdot l - \lambda_j| > \gamma \langle l \rangle^{-\tau} . \tag{28}$$

The previous results do not apply to spatial periodic boundary conditions. In this setting Craig and Wayne in [34] (see also [28]) proved the existence of periodic solutions, for the NLW and NLS equations. In such a case the eigenvalues of the Sturm-Liouville linear operator are (asymptotically) double, and the non-resonance conditions on the eigenvalues required by the KAM scheme in [47] and [64] are violated. Using the Lyapunov-Schmidt reduction method Craig and Wayne solved the range equation with a Nash-Moser iteration which requires less stringent conditions on the eigenvalues than the previous KAM scheme. Their approach was then generalized by Bourgain in [22] for quasi-periodic solutions, and in [23] and [24] for PDEs in higher spatial dimension where the multiplicity of the eigenvalues may be unbounded. We also mention more recent work such as [35], [14], [13], [17], [59], [58].

Let us now briefly describe the differentiable Nash-Moser scheme and the KAM methods. See for instance [25], [20], [17], [15], [16], [21]. The Nash-Moser scheme is a generalization of the tangent Newton method, plus a regularization procedure that we shall apply, to search for zeros of a functional operator of the form

$$F(u) = \omega \cdot \partial_\theta u - Lu - f(u). \tag{29}$$

The approximate solutions are defined iteratively by

$$u_{n+1} := u_n + h_{n+1} , \quad h_{n+1} := -S_n[DF(u_n)]^{-1}F(u_n),$$

where S_n is a suitable smoothing operator. The main difficulty is to invert the linearized operator $DF(u_n) := \mathcal{L}$ obtained at any step of the iteration and to prove that the inverse satisfies tame estimates albeit with loss of derivatives, i.e. $\mathcal{L}^{-1} : H^p \to H^{p-\tau}$. Actually, according to PDEs applications, the operator F in (29) will depend on some suitable parameters and one shall prove the invertibility of $\mathcal{L}$ for most values of these parameters. We underline that the loss of derivatives of $\mathcal{L}^{-1}$ will be compensated by the smoothing procedure and the super-quadratic convergence of the iteration .

Notice that for the unperturbed operator, $\omega \cdot \partial_\theta - L$, it is easy to prove tame estimates for the inverse, since it is represented as a diagonal matrix in the Fourier basis, whereas for the linearized operators $\omega \cdot \partial_\theta - L - f'(u)$ at a general approximate solution u such estimates requires hard work. The strategy that could be used is a KAM reducibility scheme, as we actually shall do.

The inductive $n + 1$-step of the reducibility KAM scheme, is the following: consider the operator

$$\mathcal{L}_n = \omega \cdot \partial_\theta + \mathcal{D}_n + \mathcal{R}_n ,$$

where $\mathcal{D}_n$ is a diagonal operator that in Fourier basis, both in space (described by the indexes j, k) and time (described by the index l), reads $(\mathcal{D}_n)_j^k(l) = (\mathcal{D}_n)_j^j(0) := diag_{j\in\mathbb{Z}} i\lambda_j^n$ and $\mathcal{R}_n$ is a bounded perturbation that in Fourier basis is $(\mathcal{R}_n)_j^k(l)$. Then the goal is to look for a transformation $\Phi_n = \mathbb{1} + \Psi_n$, with Ψ_n small enough, that diagonalizes the operator $\mathcal{L}_n$, by decreasing quadratically the size of the perturbation. To this end one has to solve the so called "homological equation" given by

$$\omega \cdot \partial_\theta \Psi_n + [\mathcal{D}_n, \Psi_n] + \Pi_{N_n} \mathcal{R}_n = [\mathcal{R}_n] , \tag{30}$$

with $[\mathcal{R}_n] := diag_{j\in\mathbb{Z}}(\mathcal{R}_n)_j^j(0)$, and Π_{N_n} the time Fourier truncation operator. This equation can be written in a Fourier basis and it reads

$$(\Psi_n)_j^k(l)(i\omega \cdot l + i\lambda_j^n - i\lambda_k^n) = (\mathcal{R}_n)_j^k(l) , \quad j \neq k , \ \ |l| \leq N_n .$$

In order to solve the homological equation above one has to impose the so called "second order non-resonance Melnikov conditions"

$$|\omega \cdot l + \lambda_j^n - \lambda_k^n| \geq \gamma \langle l \rangle^{-\tau} , \quad \forall (l, j, k) \neq (0, j, j) . \tag{31}$$

If the eigenvalues λ_j^n are double, (31) is violated for $(l, j, k) = (0, j, \pm j)$. In this thesis we choose a suitable phase space such that the eigenvalues of the linear system (14) are simple, and the previous problem does not appear. Then, if Ψ satisfies (30) we can consider the conjugated operator $\mathcal{L}_{n+1}$ that is

$$\begin{aligned} \mathcal{L}_{n+1} &:= \Phi_n^{-1} \mathcal{L}_n \Phi_n \\ &= \omega \cdot \partial_\theta + (\mathcal{D}_n + [\mathcal{R}_n]) + \Phi_n^{-1}(\Pi_{N_n}^\perp \mathcal{R}_n + \mathcal{R}_n \Psi_n - \Psi_n [\mathcal{R}_n]) \\ &= \omega \cdot \partial_\theta + \mathcal{D}_{n+1} + \mathcal{R}_{n+1} , \end{aligned}$$

where $\mathcal{D}_{n+1} := \mathcal{D}_n + [\mathcal{R}_n]$ is a diagonal operator, and $\mathcal{R}_{n+1} := \Phi_n^{-1}(\Pi_{N_n}^\perp \mathcal{R}_n + \mathcal{R}_n \Psi_n - \Psi_n[\mathcal{R}_n])$ is the remainder. It turns out that the remainder $\mathcal{R}_{n+1}$ is a bounded operator, whose size is quadratically smaller than the size of $\mathcal{R}_n$.

The previous scheme requires at any iterative step that the non resonance conditions (31) hold. In PDEs applications, usually, the eigenvalues λ_j of the linear operator $\mathcal{L}$ depend on some parameter. Therefore in order to be satisfied, the conditions (31) impose restrictions on the frequency ω and on such parameters. If the non linearity f of the system (27) is quasi-periodic in time with frequency ω, one could use ω itself as parameter in order to verify the non resonance conditions (31). This prospective has been used for instance in [4], [37], [13], [14], [16] or [17]. In the more difficult case, when the equation does not contains parameters, one can use the "initial conditions" as the parameters proving that the frequencies of the expected solution depends on the amplitude. This prospective has been introduced in [50] and then used in several other papers [10], [9], [11], [12], [45], [48], [49], [58] and [19]. In [8] all those problems are studied. In the present thesis the linear frequency λ_j defined in (16) depends on the external parameter ε that we shall use it in order to verify all the non resonants conditions by using the degenerate KAM theory as in [7].

The existence of quasi-periodic solutions for systems with an unbounded perturbation, i.e. the non linearity contains derivatives, has been proved in more recent years. The main difficulty is that the previous reducibility KAM scheme does not work. The first existence results for quasi-periodic solutions of PDEs with unbounded perturbations have been proved by Kuksin [49], see also Kappeler-Pöschel [45], for the Korteweg-de Vries equation (KdV) with periodic boundary conditions. The strategy introduced by Kuksin was then improved by Liu-Yuan [51], Zhang-Gao-Yuan [66] for derivative NLS. Subsequently existence of quasi-periodic solutions for derivative NLW has been proved by Berti-Biasco-Procesi [10]-[11] where the non linearity contains first order spatial and time derivatives. All these previous results still refer to semilinear perturbations, i.e. the order of the derivatives in the nonlinearity f in (27) is strictly lower than the order of the linear differential operator L.

The first results concerning the existence of quasi-periodic solutions for quasi-linear PDEs where the perturbation and the linear operator have the same order like $\partial_t u = -u_{xxx} - f(u_{xxx}, u_{xx}, u_x, u)$ have been proved by Baldi-Berti-Montalto in [4], [5], [6] for perturbations of Airy, KdV and mKdV equations. The strategy used by the authors is the following: to look for suitable transformations such that all the coefficients of the linearized operator at an approximate solutions become constant up to a bounded remainder. After this procedure one is back to an operator where the KAM reducibility scheme described above can be applied. This approach was extended in [37] and [36] to prove the existence of quasi-periodic solutions for quasi-linear perturbation of Schrödinger equation. See also [18] where the authors proved that perturbations of the defocusing nonlinear Schrödinger (dNLS) equation on the circle have an abundance of invariant tori of any size and (finite) dimension which support quasi-periodic solutions. In [3] the author proved the existence of periodic solutions of fully nonlinear autonomous equations of Benjamin-Ono type.

In this Thesis the model equations (1) are an approximation of the water waves equations as we shall present in Appendix A. The first results concerning the existence of small amplitude time periodic standing (namely even in space) pure gravity water waves is due to Plotnikov-Toland in [55]. In this paper the authors proved the result by using a Nash-Moser iteration method. This result has been then extended in [44], [40], [41], [42] . For other references and an historical survey of the background of this problem one can also see [31] and [43]. More recently in [1] Alazard-Baldi proved existence of standing wave periodic solutions for water wave equations with capillarity. This work was been extended by Berti-Montalto in [19] proving the existence also of quasi-periodic solutions. This result is the starting point of the present thesis.

Ideas of the proof of Theorem 1

Here we present in detail the strategy of the proof of Theorem 1, that will be deduced by Theorem 4.1 and Lemma 4.8 below.

Since we look for small amplitude solutions of (1), we rescale the functions (η, u) using a small amplitude parameter μ, i.e. we consider $(\mu\eta, \mu u)$. Then the Hamiltonian (2) reads

$$H(\eta, u) = \int_{\mathbb{T}} \Big(\frac{u^2}{2} + \frac{\eta^2}{2} - \frac{\varepsilon^2}{6} u_x^2 + \frac{\varepsilon^4}{15} u_{xx}^2 + \mu \frac{\varepsilon^2}{2} u^2 \eta - \mu \frac{\varepsilon^4}{2} u_x^2 \eta \Big) \, dx \, , \tag{32}$$

and the equations (1) become

$$\begin{cases} \partial_t \eta & = -\partial_x u - \frac{1}{3}\varepsilon^2 \partial_x^3 u - \frac{2}{15}\varepsilon^4 \partial_x^5 u - \mu\varepsilon^2 \partial_x(\eta u) - \mu\varepsilon^4 \partial_x^2(\eta \partial_x u) \\ \partial_t u & = -\partial_x \eta - \mu\varepsilon^2 \frac{1}{2} \partial_x(u^2) + \mu\varepsilon^4 \frac{1}{2} \partial_x (\partial_x u)^2 . \end{cases} \tag{33}$$

In order to find quasi-periodic solutions of the system (33) we shall perform a Nash-Moser scheme. The first approximate solution in the iterative scheme is the solution defined in (17) of the linear system (14). Notice that this linear solution is supported on the finitely many Fourier indices $\mathbb{S}$. In Chapter 1 we divide the phase space into two subspaces, $\mathtt{H}_{\mathbb{S}}$, which is finite dimensional and its orthogonal $\mathtt{H}_{\mathbb{S}}^{\perp}$. On $\mathtt{H}_{\mathbb{S}}$ we shall introduce action-angle variables $(\theta, I) \in \mathbb{T}^N \times \mathbb{R}^N$. After the introduction of these new coordinates we obtain a new Hamiltonian denoted $H_\mu(\theta, I, w)$.

- **Functional setting**. We look for an embedded invariant torus $i : \mathbb{T}^N \to \mathbb{T}^N \times \mathbb{R}^N \times \mathtt{H}_{\mathbb{S}}^{\perp}$, $\theta \mapsto i(\theta) = (\vartheta(\theta), I(\theta), w(\theta))$ of the Hamiltonian vector field X_{H_μ}, filled by quasi-periodic solutions with frequency ω_μ to be found. For that we define the non linear operator $\mathcal{F}(i, \cdot) = (\omega \cdot \partial_\theta - X_{H_\mu})(i(\theta))$. In order to find a solution of $\mathcal{F}(i, \cdot) = 0$ we implement a Nash-Moser scheme. The key point is to find an approximate right inverse of the linearized operator $d_i \mathcal{F}(i, \cdot)$. As a first step in Chapter 5 we follow the Berti-Bolle's approach developed in [15] (and implemented in [5] and [19]). The idea is to introduce symplectic coordinates near the approximate torus in which the linearized system $d_i \mathcal{F}(i, \cdot)$ becomes approximately decoupled into the action-angle components (defined on $\mathtt{H}_{\mathbb{S}}$) and into the normal ones (defined on $\mathtt{H}_{\mathbb{S}}^{\perp}$). Actually it is sufficient to invert the linearized operator $\mathcal{L}$ that differ from the one defined on the normal component for a finite dimensional remainder.

- **Linerized operator and KAM scheme** The goal is to diagonalize up to a bounded remainder, the operator $\mathcal{L}$ given by

$$\mathcal{L} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \cdot \partial_\theta + \begin{pmatrix} 0 & -iT(D) \\ -iT(D) & 0 \end{pmatrix} + \begin{pmatrix} a_1(x, \theta, D) & a_2(x, \theta, D) \\ a_3(x, \theta, D) & a_4(x, \theta, D) \end{pmatrix}$$

where the first two matrices arise from the linear terms of the equations (33) (after a change of variables, see Chapter 1), the linear operator $iT(D)$ is

$$iT(D) := j \sqrt{\frac{2}{15} \varepsilon^4 j^4 - \frac{1}{3} \varepsilon^2 j^2 + 1} \, , \; j \in \mathbb{Z}$$

and $a_k(x,\theta,D) \in OPS^m\,,\ m \in \mathbb{Z}\,,\ k = 1,...,4$ are pseudo-differential operators, with $C^\infty-$ symbols $a_k(x,\theta,\xi)$ in S^m. The last matrix in $\mathcal{L}$ arises from the linearization of the non linear terms in (33), and note that this matrix is of order μ. In Chapter 2 we present some useful tools of pseudo-differential operators theory that we shall use.

We divide this diagonalization procedure in two steps. The goal of the first step is to make the coefficients of the linearized operator $\mathcal{L}$ diagonal and constant, in (x,θ), up to a bounded remainder. This means that the operator obtained after the conjugation of $\mathcal{L}$ can be written in Fourier basis as a diagonal operator $\mathcal{D}$ plus a bounded remainder $\mathcal{R}$ (see Chapters 7, 8).

In the second step we perform a KAM reducibility scheme on the operator $\mathcal{D}+\mathcal{R}$ obtained above (see Chapter 9). We now present in more details the key points in these steps.

1. We expand the linear operator $\mathcal{L}$ as a sum of homogeneous operators of decreasing order plus a regularizing remainder in OPS^{-M-1}, obtaining

$$\begin{aligned}\mathcal{L} = &\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\cdot\partial_\theta + \begin{pmatrix}0 & iT(D)\\ iT(D) & 0\end{pmatrix} + \begin{pmatrix}0 & b_2(x,\theta)\\ b_3(x,\theta) & 0\end{pmatrix}\partial_x^2\\ &+\begin{pmatrix}c_1(x,\theta) & c_2(x,\theta)\\ c_3(x,\theta) & c_4(x,\theta)\end{pmatrix}\partial_x + \sum_{k=0}^{M}\begin{pmatrix}a_1^{(k)}(x,\theta) & a_2^{(k)}(x,\theta)\\ a_3^{(k)}(x,\theta) & a_4^{(k)}(x,\theta)\end{pmatrix}\partial_x^{-k}\\ &+\begin{pmatrix}\sigma_1^{(k)}(x,\theta,D) & \sigma_2^{(k)}(x,\theta,D)\\ \sigma_3^{(k)}(x,\theta,D) & \sigma_4^{(k)}(x,\theta,D)\end{pmatrix}.\end{aligned} \tag{34}$$

   The constant $-M$ denotes the smallest order of the homogeneous terms (see Section 7.1), and it is fixed once and for all in Chapter 9.

2. We consider a change of variables such that the linear operator $\begin{pmatrix}0 & -iT(D)\\ -iT(D) & 0\end{pmatrix}$ transforms into the diagonal operator $\begin{pmatrix}iT(D) & 0\\ 0 & -iT(D)\end{pmatrix}$, see Section 7.2.

3. We consider a transformation, close to the identity, such that after conjugation we get rid of the second order matrix operator in (34), see Section 8.1. Then we make the homogeneous terms block symmetrized, namely we eliminate the off diagonal entries in these terms up to ∂_x^{-M}, see Chapter 8. After these conjugations we arrive to an operator of the form

$$\begin{aligned}&\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\cdot\partial_\theta + \begin{pmatrix}iT(D) & 0\\ 0 & -iT(D)\end{pmatrix} + \sum_{k=-1}^{M}\begin{pmatrix}\tilde a_1^{(k)}(x,\theta) & 0\\ 0 & \tilde a_4^{(k)}(x,\theta)\end{pmatrix}\partial_x^{-k}\\ &+\begin{pmatrix}\sigma_1(x,\theta,D) & \sigma_2(x,\theta,D)\\ \sigma_3(x,\theta,D) & \sigma_4(x,\theta,D)\end{pmatrix}\end{aligned} \tag{35}$$

   where $\sigma_m\,,\ m = 1,...,4$ are pseudo-differential operators in OPS^{-M-1} and $\tilde a_m^{(k)}\,,\ m = 1,4$ are functions of (x,θ) (see Chapter 8).

4. Finally, in Section 8.2.3, we conjugate the operator in (35) with two transformations in order to make the coefficients of the first order operator constant. The net result is an operator of the form
$$\mathcal{L} := \mathcal{D} + \mathcal{R}\,,$$
where
$$\mathcal{D} := \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \cdot \partial_\theta + \begin{pmatrix} iT(D) & 0 \\ 0 & -iT(D) \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \partial_x\,,$$
$m \in \mathbb{R}$ and $\mathcal{R}$ is a bounded remainder of size μ.

5. In Chapter 9 we perform the KAM reducibility scheme on the linear operator $\mathcal{D}+\mathcal{R}$ obtained in the previous step. We follow the strategy introduced in [19] in which $\mathcal{R}$ satisfies tame estimate. Actually we are able to prove (see Chapter 9) that the operators
$$\mathcal{R}\,, \quad [\mathcal{R}, \partial_x]\,, \quad \partial_{\theta_j}^{\mathtt{p}_0}\mathcal{R}\,, \quad \partial_{\theta_j}^{\mathtt{p}_0}[\mathcal{R}, \partial_x]\,, \quad \partial_{\theta_r}^{\mathtt{p}_0+\mathtt{b}}\mathcal{R}\,, \quad \partial_{\theta_r}^{\mathtt{p}_0+\mathtt{b}}[\mathcal{R}, \partial_x]\,, \quad r = 1, ..., N\,,$$
are $\mathcal{D}^{k_0}$-tame (see Definition 10). For the convergence of the iterative procedure we need these properties for a suitable $\mathtt{b} := \mathtt{b}(\tau)$ fixed, where τ is the diophantine exponent in (31). We need also to prove that the $\partial_{(\omega,\varepsilon)}-$derivatives of the operator $\mathcal{R}$ are $\mathcal{D}^{k_0}-$tame these informations are required in order to prove that the eigenvalues of the perturbed system $\mathcal{D}+\mathcal{R}$ are C^{k_0}-close to the unperturbed one.

- **Nash-Moser scheme.** After this diagonalization procedure we are able to prove the required invertibility of the linearized operator $\mathcal{L}$ and the tame estimates for its inverse. Using this, in Chapter 10 we implement a differentiable Nash-Moser iterative scheme which gives a zero of the operator $\mathcal{F}(i, \cdot)$, that is a quasi-periodic solution of the equations (33). This proves Theorem 4.1.

- **Measure estimates.** As already discussed, in order to apply both the previous KAM and Nash-Moser scheme the eigenvalues of the linearized operators, have to satisfy the first and the second Melnikov non resonance conditions defined in (28) and (31). The linear frequencies ω_j defined in (16) depend on the parameter ε, i.e. $\omega_j := \omega_j(\varepsilon)$, and, as we shall prove, are $C^{k_0}-$close to the frequencies of the perturbed system. Thanks to these informations, and also using the degenerate KAM theory (introduced by Rüssmann [62] in a finite dimensional setting and developed by Bambusi-Berti-Magistrelli in [7] for the infinite dimensional system) in Lemma 4.8 we prove that the perturbed frequencies satisfy the non resonance conditions for many ε. More precisely we prove that
$$|\omega_\mu(\varepsilon) \cdot l| \geq \gamma \langle l \rangle^{-\tau}\,,\ \forall l \in \mathbb{Z}^N \setminus \{0\}, \tag{36}$$
and
$$\begin{aligned} &|\omega_\mu(\varepsilon) \cdot l + \Omega_j(\varepsilon)| \geq \gamma j^3 \langle l \rangle^{-\tau}\,,\ \forall l \in \mathbb{Z}^N,\, j \in \mathbb{N}_0 \setminus \mathbb{S}, \\ &|\omega_\mu(\varepsilon) \cdot l + \Omega_j(\varepsilon) - \Omega_{j'}(\varepsilon)| \geq \gamma |j^3 - {j'}^3| \langle l \rangle^{-\tau}\,,\ \forall l \in \mathbb{Z}^N,\, j, j' \in \mathbb{N}_0 \setminus \mathbb{S}, \\ &|\omega_\mu(\varepsilon) \cdot l + \Omega_j(\varepsilon) + \Omega_{j'}(\varepsilon)| \geq \gamma |j^3 + {j'}^3| \langle l \rangle^{-\tau}\,,\ \forall l \in \mathbb{Z}^N,\, j, j' \in \mathbb{N}_0 \setminus \mathbb{S}\,, \end{aligned} \tag{37}$$

where $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. The first line is the first Melnikov condition, the second one and the third one are the second Melnikov condition. In conclusion, since the non resonance conditions are satisfied for many parameter it is possible to apply the KAM and the Nash-Moser scheme.

Notations

- $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$
- $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$
- $\mathbb{S} := \{0 < j_1 < ... < j_N \;\; j_k \in \mathbb{N}\}$
- $\mathbb{S}^{\pm} := \mathbb{S} \cap (-\mathbb{S}) \subset \mathbb{Z}$
- $\mathfrak{p}_0 := \left[\frac{N+1}{2}\right] + 1$
- $p > \mathfrak{p}_0$
- $\zeta := (\omega, \varepsilon) \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$
- $a \leq_{p,k,M} b$ means $a \leq C(p, k, M)b$
- $k_0 \in \mathbb{N}$ is a fixed constant
- $\gamma \in (0, 1)$
- $\mathcal{N}(A, z) := \{\zeta \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2] \; : \; \mathrm{dist}(A, \zeta) \leq z\}$

Chapter 1

Phase space

We consider the scale of Sobolev spaces of $(2\pi)-$periodic real functions in the space variable

$$\mathbf{H}^p_x(\mathbb{T}) := \mathbf{H}^p_x(\mathbb{T},\mathbb{R}) := H^p_x(\mathbb{T}) \times H^p_x(\mathbb{T})\,, \tag{1.1}$$

where

$$\begin{aligned} H^p_x(\mathbb{T}) :=& H^p_x(\mathbb{T},\mathbb{R}) := H^p(\mathbb{T}_x) \\ :=& \Big\{ g(x) = \sum_{j\in\mathbb{Z}} g_j e^{ijx}\ ,\ \ g_j = \overline{g}_{-j} : \|g\|^2_{H^p_x} := \sum_{j\in\mathbb{Z}} |g_j|^2 \langle j\rangle^{2p} < \infty \Big\} \end{aligned} \tag{1.2}$$

and $\langle j\rangle := \max\{1,|j|\}$. For $w = (w_1,w_2) \in \mathbf{H}^p_x(\mathbb{T})$ we define (with slight abuse of notation)

$$\|w\|_{\mathbf{H}^p_x} := \max\{\|w_1\|_{H^p_x}, \|w_2\|_{H^p_x}\}\,. \tag{1.3}$$

1.1 Spatial invariant subspace

In order to prove Theorem 1 we shall perform a KAM iteration on the system (33), which also rely on a control of the differences of the eigenvalues of the linearized system (see Chapter 9) . If the eigenvalues of the linear system at $\mu = 0$ are not simple, such control can be hard to achieve. In the phase space $\mathbf{H}^p_x(\mathbb{T})$ defined in (1.1) this is precisely the case. Indeed if we consider the unperturbed equations of motion (14), i.e. (33) at $\mu = 0$, that is

$$\begin{cases} \partial_t \eta &= -\partial_x u - \frac{1}{3}\varepsilon^2\partial_x^3 u - \frac{2}{15}\varepsilon^4\partial_x^5 u \\ \partial_t u &= -\partial_x \eta \end{cases} \tag{1.4}$$

and we expand (η, u) in Fourier series, namely

$$\begin{pmatrix}\eta\\ u\end{pmatrix} = \sum_{j\in\mathbb{N}} \begin{pmatrix} \eta^{(1)}_j \cos jx + \eta^{(2)}_j \sin jx \\ u^{(1)}_j \cos jx + u^{(2)}_j \sin jx \end{pmatrix}\ ,$$

substituting into (1.4) we get

$$\begin{aligned}
\dot\eta_j^{(1)}\cos jx+\dot\eta_j^{(2)}\sin jx &= \left(-\partial_x-\frac{1}{3}\varepsilon^2\partial_x^3-\frac{2}{15}\varepsilon^4\partial_x^5\right)(u_j^{(1)}\cos jx+u_j^{(2)}\sin jx)\\
&= u_j^{(1)}\left(+j\sin jx-\frac{1}{3}j^3\varepsilon^2\sin jx+\frac{2}{15}\varepsilon^4 j^5\sin jx\right)+\\
&\quad + u_j^{(2)}\left(-j\cos jx+\frac{1}{3}j^3\varepsilon^2\cos jx-\frac{2}{15}\varepsilon^4 j^5\cos jx\right)\\
\dot u_j^{(1)}\cos jx+\dot u_j^{(2)}\sin jx &= -\partial_x(\eta_j^{(1)}\cos jx+\eta_j^{(2)}\sin jx)\\
&= \eta_j^{(1)} j\sin jx-\eta_j^{(2)} j\cos jx\,.
\end{aligned}$$

Hence we obtain two decoupled systems of harmonic oscillators,

$$\begin{cases}\dot\eta_j^{(1)} &= u_j^{(2)}\left(-j+\frac{1}{3}j^3\varepsilon^2-\frac{2}{15}\varepsilon^4 j^5\right)\\ \dot u_j^{(2)} &= \eta_j^{(1)} j\end{cases},\qquad \begin{cases}\dot\eta_j^{(2)} &= u_j^{(1)}\left(j-\frac{1}{3}j^3\varepsilon^2+\frac{2}{15}\varepsilon^4 j^5\right)\\ \dot u_j^{(1)} &= -\eta_j^{(2)} j\,,\end{cases} \tag{1.5}$$

with the same frequencies $\omega_j := \sqrt{\frac{2}{15}\varepsilon^4 j^6-\frac{1}{3}\varepsilon^2 j^4+j^2}\,,\quad j\in\mathbb{N}\setminus\{0\}$.

To overcome these double resonances situation we shall confine the phase space to the invariant subspace of real functions (η,u) such that η is even in x and u is odd in x, that is

$$\eta(x)=\eta(-x),\quad u(x)=-u(-x). \tag{1.6}$$

This subspace is invariant under (33). We recall that also the set

$$\left\{(\eta,u)\in H_x^p(\mathbb{T},\mathbb{R})\times H_x^p(\mathbb{T},\mathbb{R})\ :\ \int_{\mathbb{T}}\eta dx=\int_{\mathbb{T}}u dx=0\right\} \tag{1.7}$$

is invariant under the evolution of (33).

Summarizing thanks to (1.7) and (1.6), we restrict (η,u) to the phase space

$$X_0^p := \left\{\begin{pmatrix}\eta\\ u\end{pmatrix}\in \mathbf{H}_x^p(\mathbb{T},\mathbb{R})\ :\ \begin{pmatrix}\eta(x)\\ u(x)\end{pmatrix}=\begin{pmatrix}\eta(-x)\\ -u(-x)\end{pmatrix},\ \int_{\mathbb{T}}\eta dx=0\right\}, \tag{1.8}$$

where $\mathbf{H}_x^p(\mathbb{T},\mathbb{R})$ is defined in (1.1).

Remark 1.1. *The space X_0^p can be represented as a sequence space via Fourier expansion in two different ways:*

- *The trigonometric representation*

$$\left\{\begin{pmatrix}\eta\\ u\end{pmatrix}\in \mathbf{H}_x^p\ :\ \begin{pmatrix}\eta\\ u\end{pmatrix}=\sum_{j\in\mathbb{N}\setminus\{0\}}\begin{pmatrix}\eta_j\cos jx\\ u_j\sin jx\end{pmatrix}\right\}. \tag{1.9}$$

- *The exponential representation*

$$\left\{\begin{pmatrix}\eta\\ u\end{pmatrix}\in \mathbf{H}_x^p\ :\ \begin{pmatrix}\eta\\ u\end{pmatrix}=\sum_{j\in\mathbb{Z}}\begin{pmatrix}\eta_j e^{ijx}\\ u_j e^{ijx}\end{pmatrix},\eta_j=\eta_{-j},\eta_0=0,\eta_j=\bar\eta_{-j}\,,\quad u_j=-u_{-j},u_j=\bar u_{-j}\right\}. \tag{1.10}$$

Using the trigonometric representation defined above the symplectic form $\mathcal{W}$ in (6) can be written

$$\mathcal{W} := \sum_{j\in\mathbb{N}} \frac{2\pi}{j} d\eta_j \wedge du_j . \tag{1.11}$$

1.2 Preliminary symmetrization of the linear part

The Hamiltonian H defined in (32) is the sum of the quadratic Hamiltonian $\tilde{L}$ defined in (13) and the cubic terms $\tilde{P}$, multiplied by μ, given by

$$\tilde{P}(\eta, u) = \int_{\mathbb{T}} \left(\frac{\varepsilon^2}{2} u^2 \eta - \frac{\varepsilon^4}{2} u_x^2 \eta \right) dx \, . \tag{1.12}$$

Therefore the Hamiltonian H can be written

$$H = \tilde{L} + \mu \tilde{P}. \tag{1.13}$$

The corresponding equations of motion (33), can be written as

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = X_{\tilde{L}}(\eta, u) + \mu X_{\tilde{P}}(\eta, u) \tag{1.14}$$

where

$$X_{\tilde{L}}(\eta, u) := \begin{pmatrix} 0 & -\partial_x - \frac{1}{3}\varepsilon^2 \partial_x^3 - \frac{2}{15}\varepsilon^4 \partial_x^5 \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix} \tag{1.15}$$

$$X_{\tilde{P}}(\eta, u) := \begin{pmatrix} -\varepsilon^2 \partial_x(\eta u) - \varepsilon^4 \partial_x(\eta_x u_x + \eta u_{xx}) \\ -\varepsilon^2 \frac{1}{2} \partial_x(u^2) + \varepsilon^4 \frac{1}{2} \partial_x (u_x)^2 \end{pmatrix} . \tag{1.16}$$

We look for a symplectic transformation that "balances" the order of the operators in the linear part $X_{\tilde{L}}$, namely we look for a change of variables that transforms the 2×2 matrix in (1.15) into a new matrix whose out-of-diagonal operators are the same.

Under a change of variables of the form

$$\begin{cases} \eta & = \Lambda q \\ u & = \Lambda^{-1} p \end{cases} \tag{1.17}$$

the linear system (14) (i.e. (1.14) with $\mu = 0$) becomes

$$\begin{aligned} q_t &= -\Lambda^{-1} \left(\frac{2}{15} \varepsilon^4 \partial_x^5 + \frac{1}{3} \varepsilon^2 \partial_x^3 + \partial_x \right) \Lambda^{-1} p \\ p_t &= -\Lambda \partial_x \Lambda q \, . \end{aligned} \tag{1.18}$$

Choosing

$$\Lambda := Op(g(j)), \quad \text{where} \quad g(j) = \sqrt[4]{\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1} \, , \qquad \forall j \in \mathbb{Z}, \tag{1.19}$$

the system (1.18) takes the form

$$\begin{aligned} q_t &= -iT(D)p \\ p_t &= -iT(D)q \end{aligned} \tag{1.20}$$

where

$$iT(D) = Op(iT(j,\varepsilon)), \qquad T(j,\varepsilon) := j\left(\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1\right)^{\frac{1}{2}}, \; j \in \mathbb{Z}. \tag{1.21}$$

Remark 1.2. *We have that* $\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1 > 0$ *for all* $j \in \mathbb{Z}$, *indeed* $\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1 = \left(\sqrt{\frac{2}{15}}\varepsilon^2 j^2 - 1\right)^2 + \left(\frac{2\sqrt{2}}{\sqrt{15}} - \frac{1}{3}\right)\varepsilon^2 j^2$ *and* $\left(\frac{2\sqrt{2}}{\sqrt{15}} - \frac{1}{3}\right) > 0$.

The change of variable (1.17), whose matrix is

$$\mathbf{\Lambda} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}, \tag{1.22}$$

is symplectic, i.e.

$$\mathbf{\Lambda} J \mathbf{\Lambda}^T = J\,, \tag{1.23}$$

where J is defined in (8). Hence the symplectic form $\mathcal{W}$ defined in (6) (i.e. (1.11)) remains the same:

$$\mathcal{W} = \sum_{j \in \mathbb{N}} \frac{2\pi}{j} dq_j \wedge dp_j. \tag{1.24}$$

Moreover, under the change of variable $\mathbf{\Lambda}$ in (1.22), also the involution ρ defined in (10), remain the same, indeed

$$\mathbf{\Lambda}^{-1}\rho\mathbf{\Lambda} = \rho. \tag{1.25}$$

Since $\mathbf{\Lambda}$ is symplectic the Hamiltonian system (33) (i.e. (1.14)) transforms into the new Hamiltonian system generated by the Hamiltonian (see also Lemma A.2)

$$\mathcal{H} := H \circ \mathbf{\Lambda},$$

that is explicitly given by (recall that H is the Hamiltonian in (32) i.e. (1.13))

$$\mathcal{H}(q,p) = L(q,p) + \mu\mathcal{P}(q,p) \tag{1.26}$$

where $L := \tilde{L} \circ \mathbf{\Lambda}$ is the quadratic part

$$L(q,p) = \int_{\mathbb{T}} \left(\frac{(\Lambda^{-1}p)^2}{2} + \frac{(\Lambda q)^2}{2} - \frac{\varepsilon^2}{6}(\Lambda^{-1}\partial_x p)^2 + \frac{\varepsilon^4}{15}(\Lambda^{-1}\partial_x^2 p)^2 \right) dx \tag{1.27}$$

and $\mathcal{P} := \tilde{P} \circ \mathbf{\Lambda}$ is

$$\mathcal{P}(q,p) = \int_{\mathbb{T}} \left(\frac{\varepsilon^2}{2}(\Lambda^{-1}p)^2 \Lambda q - \frac{\varepsilon^4}{2}(\Lambda^{-1}\partial_x p)^2 \Lambda q \right) dx\,. \tag{1.28}$$

The Hamiltonian system (33), i.e. (1.14), transforms in the new coordinates into

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = -J\nabla_{q,p}\mathcal{H}(q,p) = X_{\mathcal{H}}(q,p)\,,$$

that is explicitly given by

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & -iT(D) \\ -iT(D) & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \mu X_{\mathcal{P}}(q,p)\,, \tag{1.29}$$

where (see Lemma A.1)

$$X_{\mathcal{P}}(q,p) := \varepsilon^2 \begin{pmatrix} -\Lambda^{-1}\partial_x\left((\Lambda q)(\Lambda^{-1}p)\right) - \varepsilon^2\Lambda^{-1}\partial_x\left((\Lambda q)(\Lambda^{-1}p_{xx}) + (\Lambda q_x)(\Lambda^{-1}p_x)\right) \\ -\frac{1}{2}\Lambda\partial_x(\Lambda^{-1}p)^2 + \frac{\varepsilon^2}{2}\Lambda\partial_x(\Lambda^{-1}p_x)^2 \end{pmatrix}. \tag{1.30}$$

We remark the following properties.

Lemma 1.3. *Let $\Lambda := Op(g(j))$ with $g(j)$ defined in* (1.19)*. Then Λ and Λ^{-1} send real functions in real functions. In addition Λ and Λ^{-1} send even, respectively odd, functions in even, respectively odd, functions.*

Proof. Let $f = \sum_{j\in\mathbb{Z}} f_j e^{ijx}$. Then

$$\Lambda f := \sum_{j\in\mathbb{Z}} g(j) f_j e^{ijx}\,, \quad \Lambda^{-1} f := \sum_{j\in\mathbb{Z}} g(j)^{-1} f_j e^{ijx}\,.$$

Let f be a real function, that is, $f_j = \bar{f}_{-j}$, then

$$\overline{\Lambda f} = \sum_{j\in\mathbb{Z}} \overline{g(j)}\bar{f}_j e^{-ijx} = \sum_{j\in\mathbb{Z}} \overline{g(j)} f_{-j} e^{-ijx} = \sum_{j\in\mathbb{Z}} g(j) f_j e^{ijx} = \Lambda f,$$

where we have used that $\overline{g(j)} = g(j) = g(-j)$. Clearly we can repeat the same argument also for Λ^{-1}.

By $g(j) = g(-j)$, follows immediately that the operators Λ and Λ^{-1} send the set of even, respectively odd, functions into itself. Indeed let f be a even function, in the exponential representation this condition reads $f_j = f_{-j}$. Then, by $g(j) = g(-j)$, we get $g(j)f_j = g(-j)f_{-j}$ (similar for the other). □

Lemma 1.4. *The operator $iT(D)$ defined in* (1.21) *sends real functions in real functions. Moreover $iT(D)$ sends even, respectively odd, functions in odd, respectively even, functions.*

Proof. By the explicit definition of $iT(D)$ in (1.21) we have that $T(j,\varepsilon) = \overline{T(j,\varepsilon)}$. Let f be a real function, then

$$\begin{aligned} \overline{iT(D)f} &= \sum_{j\in\mathbb{Z}} -ij\left(\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1\right)^{\frac{1}{2}} f_{-j} e^{-ijx} \\ &= \sum_{j\in\mathbb{Z}} ij\left(\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1\right)^{\frac{1}{2}} f_j e^{ijx} \\ &= iT(D)f\,. \end{aligned}$$

By the explicit definition of $iT(D)$ we also have $T(j,\varepsilon) = -T(-j,\varepsilon)$, hence if q is a even function and p is a odd function, we obtain $T(j,\varepsilon)q_j = -T(-j,\varepsilon)q_{-j}$ and $T(j,\varepsilon)p_j = T(-j,\varepsilon)p_{-j}$. □

Remark 1.5. *By Lemma 1.3 we have that $\Lambda q = \eta$ is real even and with zero average and $\Lambda^{-1}p = u$ is real, odd and with zero average. Therefore under the change of coordinates $\mathbf{\Lambda}$ the phase space remains the same, i.e. X_0^p (defined in* (1.8)*).*

1.3 Action-angle variables

We rewrite the phase space X_0^p defined in (1.8) as the direct sum of two symplectic subspaces defined as follows. Fix

$$\mathbb{S} := \{j_1, \dots, j_N\} \in \mathbb{N} \setminus \{0\}, \quad 0 < j_1 < \dots < j_N\,,\ \ j_k \in \mathbb{N}\,. \tag{1.31}$$

Then we decompose

$$X_0^p = \mathtt{H}_{\mathbb{S}} \oplus \mathtt{H}_{\mathbb{S}}^{\perp}. \tag{1.32}$$

Remark 1.6. *Using the trigonometric representation, defined in* (1.9)*, the subspaces defined above read*

$$\begin{aligned}
\mathtt{H}_{\mathbb{S}} &:= \left\{ (q,p) \in X_0^p \ :\ q = \sum_{j\in\mathbb{S}} \tilde{q}_j \cos(jx)\,, p = \sum_{j\in\mathbb{S}} \tilde{p}_j \sin(jx) \right\} \\
\mathtt{H}_{\mathbb{S}}^{\perp} &:= \left\{ (q,p) \in X_0^p \ :\ q = \sum_{j\notin\mathbb{S}} \tilde{q}_j \cos(jx)\,, p = \sum_{j\notin\mathbb{S}} \tilde{p}_j \sin(jx) \right\}\,.
\end{aligned} \tag{1.33}$$

Using the exponential representation, defined in (1.10)*, we can set* $-\mathbb{S} := \{-j_1, \dots, -j_N\}$ *and the subspaces read*

$$\begin{aligned}
\mathtt{H}_{\mathbb{S}} &:= \Bigg\{ (q,p) \in X_0^p \ :\ q = \sum_{j\in\mathbb{S}\cup(-\mathbb{S})} q_j e^{ijx}\,,\ q_j = \bar{q}_{-j}\,,\ q_j = q_{-j} \\
&\qquad p = \sum_{j\in\mathbb{S}\cup(-\mathbb{S})} p_j e^{ijx}\,,\ p_j = \bar{p}_{-j}\,,\ p_j = -p_{-j} \Bigg\} \\
\mathtt{H}_{\mathbb{S}}^{\perp} &:= \Bigg\{ (q,p) \in X_0^p \ :\ q = \sum_{j\notin\mathbb{S}\cup(-\mathbb{S})} q_j e^{ijx}\ \ q_j = \bar{q}_{-j}\,,\ q_j = q_{-j} \\
&\qquad p = \sum_{j\notin\mathbb{S}\cup(-\mathbb{S})} p_j e^{ijx}\,,\ p_j = \bar{p}_{-j}\,,\ p_j = -p_{-j} \Bigg\}\,.
\end{aligned}$$

Any $z = (q,p) \in X_0^p$ can be written as $z = z_T + z^{\perp}$, where $z_T \in \mathtt{H}_{\mathbb{S}}$ is the so called "tangential variable" and $z^{\perp} \in \mathtt{H}_{\mathbb{S}}^{\perp}$ is the so called "normal variable". Therefore, if $(q,p) \in X_0^p$, then

$$q = \sum_{j\in\mathbb{S}} q_j \cos jx + q^{\perp}\,, \quad p = \sum_{j\in\mathbb{S}} p_j \sin jx + p^{\perp}\,.$$

The symplectic form $\mathcal{W}$ defined in (1.24) can be decomposed as follows

$$\mathcal{W} = \sum_{j\in\mathbb{S}} \frac{2\pi}{j} dq_j \wedge dp_j \oplus \mathcal{W}|_{\mathtt{H}_{\mathbb{S}}^{\perp}}\,, \tag{1.34}$$

where $\mathcal{W}|_{\mathtt{H}_{\mathbb{S}}^{\perp}}$ is given in (6). Now, in a r-neighborhood of the origin of $\mathtt{H}_{\mathbb{S}}$, we introduce the action-angle variables using the trigonometric representation of X_0^p, by setting

$$\begin{aligned}
q &= \sum_{j\in\mathbb{S}} \sqrt{I_j + r_j} \cos\vartheta_j \cos jx + w_1, &&\text{where}\quad w_1 := q^{\perp} = \sum_{j\in\mathbb{S}^{\perp}} q_j \cos jx, \\
p &= \sum_{j\in\mathbb{S}} \sqrt{I_j + r_j} \sin\vartheta_j \sin jx + w_2 &&\text{where}\quad w_2 := p^{\perp} = \sum_{j\in\mathbb{S}^{\perp}} p_j \sin jx,
\end{aligned} \tag{1.35}$$

and $0 < r_j < 1$ is a constant such that the variable $|I_j| \leq r_j$, $\forall j = 1, ..., N$. In conclusion, let $z := (q, p) \in X_0^p$, then the change of variables $\mathcal{A} : (\vartheta, I, w) \mapsto z$ is

$$\mathcal{A}(\vartheta, I, w) := A(\vartheta, I) + w := \sum_{j \in \mathbb{S}} \sqrt{\frac{j}{\pi}} \begin{pmatrix} \sqrt{I_j + r_j} \cos \vartheta_j \cos jx \\ \sqrt{I_j + r_j} \sin \vartheta_j \sin jx \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{1.36}$$

where

$$A(\vartheta, I) := \sum_{j \in \mathbb{S}} \sqrt{\frac{j}{\pi}} \begin{pmatrix} \sqrt{I_j + r_j} \cos \vartheta_j \cos jx \\ \sqrt{I_j + r_j} \sin \vartheta_j \sin jx \end{pmatrix}, \quad w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{1.37}$$

After the change of variables (1.36) the symplectic form $\mathcal{W}$ defined in (1.34) becomes

$$\mathcal{W}_{\text{new}} := \sum_{j \in \mathbb{S}} dI_j \wedge d\vartheta_j \oplus \mathcal{W}|_{H_{\mathbb{S}}^{\perp}}, \tag{1.38}$$

where $\mathcal{W}|_{H_{\mathbb{S}}^{\perp}}$ is the symplectic form defined in (6).

Note that $\mathcal{W}_{\text{new}} = -d\Xi$ where Ξ is the Liouville 1-form

$$\Xi_{(\vartheta, I, w)}[\hat{\vartheta}, \hat{I}, \hat{w}] := -\sum_{j \in \mathbb{S}} I_j \hat{\vartheta}_j - \frac{1}{2} \left(J^{-1} w, \hat{w} \right)_{L_x^2}. \tag{1.39}$$

The Hamiltonian system (1.29) is transformed into the new Hamiltonian system

$$\begin{cases} \dot{\vartheta} = \partial_I H_\mu \\ \dot{I} = -\partial_\vartheta H_\mu \\ \partial_t w = -J \nabla_w H_\mu \end{cases} \tag{1.40}$$

generated by the Hamiltonian

$$H_\mu = \mathcal{H} \circ \mathcal{A} \tag{1.41}$$

where $\mathcal{H}$ is defined in (1.26) and $\mathcal{A}$ is defined in (1.36).

After the introduction of the action-angle variables, the involution ρ defined in (10) and (1.25) becomes

$$\tilde{\rho} : (\vartheta, I, w) \mapsto (-\vartheta, I, \rho w). \tag{1.42}$$

This is our new reversible structure, hence

$$H_\mu \circ \tilde{\rho} = H_\mu,$$

where H_μ is defined in (1.41). Then it is natural to look for reversible solutions of (1.40) satisfying

$$\begin{aligned} \vartheta(-\theta) &= -\vartheta(\theta), \\ I(-\theta) &= I(\theta), \\ w(-\theta) &= (\rho w)(\theta). \end{aligned} \tag{1.43}$$

We denote by

$$X_{H_\mu} := (\partial_I H_\mu, -\partial_\vartheta H_\mu, -J \nabla_w H_\mu)$$

the Hamiltonian vector field in the variables $(\vartheta, I, w) \in \mathbb{T}^N \times \mathbb{R}^N \times \mathtt{H}_{\mathbb{S}}^{\perp}$, where $\mathtt{H}_{\mathbb{S}}^{\perp}$ is defined in (1.32). Hence the Hamiltonian H_μ in (1.41) reads

$$H_\mu = \mathcal{N} + \mu P, \quad \mathcal{N} = L \circ \mathcal{A} = \vec{\omega}(\varepsilon) \cdot I + \frac{1}{2}(w, \mathbf{D}w)_{L^2_x}, \qquad P = \mathcal{P} \circ \mathcal{A} \tag{1.44}$$

where L is defined in (1.27), $\mathcal{P}$ is defined in (1.28), $\mathcal{A}$ is defined in (1.36), $\vec{\omega}(\varepsilon) = \Big(j\sqrt{\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1}\Big)_{j\in\mathbb{S}}$ represents the unperturbed tangential frequency vector and (recall the definition of Λ given in (1.19))

$$\begin{aligned} \mathbf{D} &:= \begin{pmatrix} \Lambda \mathbb{1} \Lambda & 0 \\ 0 & \Lambda^{-1}(\frac{2}{15}\varepsilon^4\partial_x^4 + \frac{1}{3}\varepsilon^2\partial_x^2 + \mathbb{1})\Lambda^{-1} \end{pmatrix}\Bigg|_{\mathtt{H}_{\mathbb{S}}^{\perp}} \\ &= \begin{pmatrix} (\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1)^{\frac{1}{2}} & 0 \\ 0 & (\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1)^{\frac{1}{2}} \end{pmatrix}\Bigg|_{\mathtt{H}_{\mathbb{S}}^{\perp}} . \end{aligned} \tag{1.45}$$

In what follows since $\mathbf{D}$ acts on $w \in \mathtt{H}_{\mathbb{S}}^{\perp}$ we shall not write the restriction on the operator.

Chapter 2

Functional Analytic Setting

Since we are looking for quasi-periodic solutions, we consider the following Sobolev spaces of $(2\pi)-$ periodic real functions in space and "time", namely

$$\mathbf{H}^p(\mathbb{T}\times\mathbb{T}^N) := \mathbf{H}^p(\mathbb{T}\times\mathbb{T}^N,\mathbb{R}) := H^p(\mathbb{T}\times\mathbb{T}^N)\times H^p(\mathbb{T}\times\mathbb{T}^N)\,, \tag{2.1}$$

where

$$\begin{aligned} H^p(\mathbb{T}\times\mathbb{T}^N) :=& H^p(\mathbb{T}\times\mathbb{T}^N,\mathbb{R}) \\ :=& \Bigg\{ \mathtt{w}(x,\theta) = \sum_{j\in\mathbb{Z},l\in\mathbb{Z}^N} \mathtt{w}_{j,l}e^{ijx+il\cdot\theta} \ : \ \mathtt{w}_{l,j} = \overline{\mathtt{w}}_{-l,-j}, \\ & \|\mathtt{w}\|_p^2 := \|\mathtt{w}\|_{H^p_{\theta,x}} := \sum_{j\in\mathbb{Z},l\in\mathbb{Z}^N} |\mathtt{w}_{j,l}|^2 \langle j,l\rangle^{2p} < \infty \Bigg\} \end{aligned}$$

and $\langle j,l\rangle := \max\{1,|j|,|l|\}$.

Remark 2.1. *We use the space $H^p(\mathbb{T}\times\mathbb{T}^N)$ whose functions have the same regularity both in space and time, since in Chapter 8 we have to consider the composition transformation T_{M+4} (see Lemma 8.8) that mixes regularity of time and space.*

With slight abuse of notation we define the so called $p-$norm of a vector $w = (w_1,w_2)\in\mathbf{H}^p(\mathbb{T}\times\mathbb{T}^N)$ as

$$\|w\|_p := \|w\|_{\mathbf{H}^p_{\theta,x}} := \max\{\|w_1\|_p,\|w_2\|_p\}\,. \tag{2.2}$$

We shall consider a function $w(x,\theta)\in L^2(\mathbb{T}\times\mathbb{T}^N,\mathbb{C})\times L^2(\mathbb{T}\times\mathbb{T}^N,\mathbb{C})$ of the space-time also as a θ-dependent family of functions $w(\cdot,\theta)\in L^2(\mathbb{T}_x,\mathbb{C})\times L^2(\mathbb{T}_x,\mathbb{C})$. We shall also write $L^2 = L^2(\mathbb{T}\times\mathbb{T}^N) = L^2(\mathbb{T}_x) = L^2_x$. We can expand a function $w := (w_1,w_2)$ in Fourier series as follows:

$$w(x,\theta) = \begin{pmatrix} w_1(x,\theta) \\ w_2(x,\theta)\end{pmatrix} = \sum_{j\in\mathbb{Z}} \begin{pmatrix} (w_1)_j(\theta) \\ (w_2)_j(\theta)\end{pmatrix} e^{ijx} = \sum_{j\in\mathbb{Z},l\in\mathbb{Z}^N} \begin{pmatrix} (w_1)_{j,l} \\ (w_2)_{j,l}\end{pmatrix} e^{ijx+il\cdot\theta}\,. \tag{2.3}$$

For notational convenience we will write $\| \cdot \|_p$ both for functions and for vectors. Moreover we have the following equivalence of norms:

$$\| \cdot \|_p \simeq \| \cdot \|_{H^p_\theta L^2_x} + \| \cdot \|_{L^2_\theta H^p_x} \,,$$

where

$$\|w\|^2_{H^p_\theta L^2_x} := \sum_l \langle l \rangle^{2p} \|w_l(\cdot)\|^2_{L^2_x} = \sum_l \langle l \rangle^{2p} \sum_j |w_{lj}|^2 \,,$$

and

$$\|w\|^2_{L^2_\theta H^p_x} := \sum_l \|w_l(\cdot)\|^2_{H^p_x} = \sum_l \sum_j |w_{lj}|^2 \, \langle j \rangle^{2p} \,.$$

Furthermore, given $v(\theta) \in \mathbb{R}^N$ we define the following norm

$$\|v\|^2_{H^p_\theta} = \sum_{l \in \mathbb{Z}^N} |v_l|^2 \, \langle l \rangle^{2p} \,. \tag{2.4}$$

We recall that the $p-$norm $\| \cdot \|_p$ defined in (2.2) satisfies the tame estimate for the product of functions (see for istance [5]), i.e. for all $p \geq \mathfrak{p}_0$, for all $\mathtt{w}, \mathtt{v} \in H^p(\mathbb{T} \times \mathbb{T}^N)$ the following inequality holds

$$\|\mathtt{w}\mathtt{v}\|_p \leq C(p) \|\mathtt{w}\|_p \|\mathtt{v}\|_{\mathfrak{p}_0} + C(\mathfrak{p}_0) \|\mathtt{w}\|_{\mathfrak{p}_0} \|\mathtt{v}\|_p \,. \tag{2.5}$$

Definition 1. *Given a function* $\mathtt{w} \in L^2(\mathbb{T} \times \mathbb{T}^N)$,

$$\mathtt{w}(x, \theta) = \sum_{j \in \mathbb{Z}, l \in \mathbb{Z}^N} \mathtt{w}_{l,j} e^{ijx + il \cdot \theta}$$

we define the ***majorant function***

$$|\mathtt{w}|(x, \theta) = \sum_{l \in \mathbb{Z}^N, j \in \mathbb{Z}} |\mathtt{w}_{l,j}| e^{ijx + il \cdot \theta} \,. \tag{2.6}$$

Note that the Sobolev norm $\| \cdot \|_p$ in (2.2) of w and $|w|$ is the same, i.e. $\|w\|_p = \||w|\|_p$.

In this work we have that the functions also depend on the parameter $\zeta := (\omega, \varepsilon)$. For a scalar valued functions $\lambda : \Lambda_0 \in \mathbb{R}^{N+1} \to \mathbb{R}$ which are k_0-times differentiable with respect to a parameter

$$\zeta := (\omega, \varepsilon) \in \Lambda_0 \subset \mathbb{R}^{N+1} \,,$$

we define, for $\gamma \in (0, 1)$, the weighted norm

$$|\lambda|^{k_0, \gamma} := |\lambda|^{k_0, \gamma}_{\Lambda_0} := \sum_{|k| \leq k_0} \gamma^{|k|} \sup_{\zeta \in \Lambda_0} |\partial^k_\zeta \lambda(\zeta)|. \tag{2.7}$$

We shall also consider families of Sobolev functions $\zeta \mapsto \mathtt{w}(\zeta) \in H^p(\mathbb{T} \times \mathbb{T}^N)$ which are k_0-times differentiable with respect to the parameter ζ (defined above) and we define the weighted Sobolev norm

$$\|\mathtt{w}\|^{k_0, \gamma}_p := \sum_{|k| \leq k_0} \gamma^{|k|} \sup_{\zeta \in \Lambda_0} \|\partial^k_\zeta \mathtt{w}(\zeta)\|_p \,. \tag{2.8}$$

In this thesis we will also consider vector valued Sobolev functions $\zeta \mapsto w(\zeta) \in \mathbf{H}^p(\mathbb{T} \times \mathbb{T}^N)$ which are k_0-times differentiable with respect to a parameter ζ. For $\gamma \in (0,1)$ the weighted Sobolev norm of $w(\zeta) \in \mathbf{H}^p(\mathbb{T} \times \mathbb{T}^N)$ is given by

$$\|w\|_p^{k_0,\gamma} := \sum_{|k| \le k_0} \gamma^{|k|} \sup_{\zeta \in \Lambda_0} \|\partial_\zeta^k w(\zeta)\|_p \,, \tag{2.9}$$

where $\|\cdot\|_p$ is defined in (2.2).

Remark 2.2. *In Chapters 7, 8 and in Appendix B we shall also consider* 2×2 *matrices of functions*

$$\mathbf{F} := \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \,,$$

with the norm

$$\|\mathbf{F}\|_p^{k_0,\gamma} := \max_{m=1,\dots,4} \|f_m\|_p^{k_0,\gamma}.$$

We also introduce the smoothing operators

$$(\Pi_K w)(x,\theta) := \sum_{|(l,j)| \le K} w_{l,j} e^{ijx + il\cdot\theta}, \qquad \Pi_K^\perp = \mathbb{1} - \Pi_K \tag{2.10}$$

which satisfy the smoothing properties

$$\|\Pi_K w\|_{p+b}^{k_0,\gamma} \le K^b \|w\|_p^{k_0,\gamma}, \qquad \|\Pi_K^\perp w\|_p^{k_0,\gamma} \le K^{-b} \|w\|_{p+b}^{k_0,\gamma} \quad \forall p, b \ge 0\,. \tag{2.11}$$

Now we introduce the class of operators that we shall use later. We shall consider a class of θ-dependent families of linear operators $A : \mathbb{T}^N \mapsto \mathcal{L}(L^2(\mathbb{T}_x))\,,\ \theta \mapsto A(\theta)$ acting on $L^2(\mathbb{T}_x)$. We may consider also an operator $A \in L^2(\mathbb{T} \times \mathbb{T}^N)$ which acts on functions $\mathtt{w}(x,\theta)$ of space-time, as

$$(A\mathtt{w})(x,\theta) := (A(\theta)\mathtt{w}(\cdot,\theta))(x)\,.$$

If A maps the space of real valued functions into itself we say that A is a real operator.

We represent a real operator $\mathbf{A}$ acting on $w = (w_1, w_2) \in L^2(\mathbb{T}^1 \times \mathbb{T}^N, \mathbb{R}^2)$ as follows

$$\mathbf{A}w = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \,, \tag{2.12}$$

and each A_m, $m = 1, \dots, 4$ acts linearly.

We may identify an operator $A \in \mathcal{L}(L^2(\mathbb{T} \times \mathbb{T}^N))$ with, respect to the exponential representation, an infinite matrix $(A_j^{j'}(l - l'))_{j,j' \in \mathbb{Z}, l, l' \in \mathbb{Z}^N}$. Consequently given

$$\mathtt{w}(x,\theta) = \sum_{j' \in \mathbb{Z}} \mathtt{w}_{j'}(\theta) e^{ij'x} = \sum_{j' \in \mathbb{Z}, l' \in \mathbb{Z}^N} \mathtt{w}_{j',l'} e^{ij'x + il'\cdot\theta}$$

the action of A on $\mathtt{w} \in L^2(\mathbb{T} \times \mathbb{T}^N, \mathbb{R})$ is

$$A\mathtt{w}(x,\theta) = \sum_{j,j' \in \mathbb{Z}} A_j^{j'}(\theta) \mathtt{w}_{j'}(\theta) e^{ijx} = \sum_{j',j \in \mathbb{Z}, l', l \in \mathbb{Z}^N} A_j^{j'}(l - l') \mathtt{w}_{j',l'} e^{ijx + il\cdot\theta}\,. \tag{2.13}$$

Moreover the operator $\partial_\theta A(\theta)$ is identified with the matrix with elements $i(l-l')A_j^{j'}(l-l')$ and the commutator $[\partial_x, A]$ is identified with the matrix with entries $i(j-j')A_j^{j'}(l-l')$.

We now introduce the following operators that will be used in Chapter 9.

Definition 2. *Given a linear operator A as above we define the following operators*

1. *The **majorant operator** $|A|$ whose matrix elements are $|A_j^{j'}(l-l')|$.*

2. *The **differentiated operator** $\langle\partial_\theta\rangle^b A, b\in\mathbb{R}$, whose matrix elements are $\langle l-l'\rangle^b A_j^{j'}(l-l')$.*

3. *The **smoothed operator** $\Pi_K A, K\in\mathbb{N}$ whose matrix elements are*

$$(\Pi_K A)_j^{j'}(l-l') = \begin{cases} A_j^{j'}(l-l') & \text{if} \quad |l-l'|\le K\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

A simple property is given in the following Lemma.

Lemma 2.3. *Given linear operators A, B as above, for all $\mathtt{w}\in H^p(\mathbb{T}\times\mathbb{T}^N)$ we have*

$$\| |A+B|\mathtt{w}\|_p \le \| |A||\mathtt{w}| \|_p + \| |B||\mathtt{w}| \|_p, \tag{2.14}$$

$$\| |AB|\mathtt{w}\|_p \le \| |A||B||\mathtt{w}| \|_p\,. \tag{2.15}$$

Proof. See Lemma 2.2 in [19]. □

Definition 3. ***Even-Odd Operator.*** *A linear operator A as in* (2.13) *is **even**, if each $A(\theta)$, $\theta\in\mathbb{T}^N$ leaves invariant the space of functions even, respectively odd, in the spatial variable. A linear operator A as in* (2.13) *is **odd**, if each $A(\theta)$, $\theta\in\mathbb{T}^N$ sends the space of functions even in the spatial variable into the space of functions odd in the spatial variable and vice-versa.*

A linear operator $\mathbf{A}$ as in (2.12) sends X_0^p defined in (1.8) in itself if A_1, A_4 are even operators and A_2, A_3 are odd operators.

Since the Fourier coefficients (in the exponential representation) of an even, respectively odd, function satisfy $\mathtt{w}_{-j}=\mathtt{w}_j$, respectively $\mathtt{w}_{-j}=-\mathtt{w}_j$, $\forall j\in\mathbb{Z}$, we have that a linear operator A is even, respectively odd, if

$$\forall\theta\in\mathbb{T}^N, \quad A_j^{j'}(\theta) = A_{-j}^{-j'}(\theta), \text{ respectively } A_j^{j'}(\theta) = -A_{-j}^{-j'}(\theta)\,. \tag{2.16}$$

Definition 4. ***Reversibility.*** *A family of operators $\mathbf{A}(\theta)$ as in (2.12) is*

1. ***reversible** if $\mathbf{A}(-\theta)\circ\rho = -\rho\circ\mathbf{A}(\theta), \forall\theta\in\mathbb{T}^N$, where the involution ρ is defined in* (10),

2. ***reversibility preserving** if $\mathbf{A}(-\theta)\circ\rho = \rho\circ\mathbf{A}(\theta)$, $\forall\theta\in\mathbb{T}^N$.*

The conjugation of an even and reversible (respectively odd and reversible) operator with a map Φ which is even and reversibility preserving is even and reversible (respectively odd and reversible).

A family of operator $\mathbf{A}(\theta)$ as in (2.12) is

1. **reversible** if and only if the maps $\theta \mapsto A_1(\theta), A_4(\theta)$ are odd and $\theta \mapsto A_2(\theta), A_3(\theta)$ are even.
2. **reversibility preserving** if and only if the maps $\theta \mapsto A_1(\theta), A_4(\theta)$ are even and $\theta \mapsto A_2(\theta), A_3(\theta)$ are odd.

2.1 Pseudo-differential operators

The change of variables $\mathbf{\Lambda}$ (see (1.22) and (1.19)) is given in terms of Fourier multipliers, which are a particular case of pseudo-differential operators. In this section we present some known results (see [39], [60], [19]) about pseudo-differential operators. Since we are working in a periodic setting, we introduce pseudo-differential operators on the torus. Let $a : \mathbb{Z} \to \mathbb{C}$ be a function. Let $(\Delta_j a)(j) := a(j+1) - a(j)$ be the discrete derivative. For $\beta \in \mathbb{N}$ we denote by $\Delta_j^\beta := \Delta_j \circ ... \circ \Delta_j$ the composition of β-discrete derivatives.

Definition 5. *Let $a : \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$, $a(x, j)$ be a function which is C^∞ with respect to x. Let $m \in \mathbb{R}$. We shall say that a is a* ***symbol of order*** *m, if for all $\alpha, \beta \in \mathbb{N}$ there exists a constant $C = C_{\alpha,\beta} > 0$ such that*

$$|\partial_x^\alpha \Delta_j^\beta a(x, j)| \leq C(1 + |j|)^{m-\beta}, \quad \forall (x, j) \in \mathbb{T} \times \mathbb{Z}\,. \tag{2.17}$$

We denote S^m the class of all symbols of order m.

Definition 6. *Given a symbol $a \in S^m$ and a function $u(x) = \sum_{j\in\mathbb{Z}} \hat{u}(j)e^{ijx}$, we define the operator*

$$a(x, D)u(x) = \sum_{j\in\mathbb{Z}} a(x, j)\hat{u}(j)e^{ijx}$$

and we say that $a(x, D) := Op(a)$ is the ***pseudo-differential operator*** *associated to the symbol a.*

We introduce another equivalent definition of pseudo-differential symbols of order m, that we shall use along all the thesis.

Definition 7. *A linear operator A is called* ***pseudo-differential of order*** *m if its symbol $a(x, j)$ is the restriction to $\mathbb{R} \times \mathbb{Z}$ of a function $a(x, \xi)$ which is C^∞-smooth on $\mathbb{R} \times \mathbb{R}$, 2π-periodic in x and satisfies the following inequality*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\beta}, \quad \forall \alpha, \beta \in \mathbb{N}\,, \tag{2.18}$$

we say that $a(x, \xi)$ is the symbol of the operator A. We denote by OPS^m the set of the pseudo-differential operators whose symbols are in S^m .

Definition 6 is equivalent to the Definition 7 because a discrete symbol $a : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$ satisfying (2.17) can be extended to a C^∞-symbol $\tilde{a} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfying (2.18), see [60].

Lemma 2.4. *A pseudo-differential operator with symbol $a(x, \xi)$ is*

1. ***even***, *respectively* ***odd***, *if and only if* $a(x,\xi)=a(-x,-\xi)$, *respectively* $a(x,\xi)=-a(-x,-\xi)$,

2. ***real***, *i.e. it sends the space of real functions into itself, if and only if* $\overline{a(x,\xi)}=a(x,-\xi)$.

Proof. Let $u(x)=\sum_{j\in\mathbb{Z}}\hat u(j)e^{ijx}$ be a even function, i.e. $\hat u(j)=\hat u(-j)$, the action of an even pseudo-differential operator $a(x,D)$ (see Definition 6) with symbol $a(x,j)$, on u is given by

$$a(x,D)u(x)=\sum_j a(x,j)\hat u(j)e^{ijx}\,.$$

We have that

$$a(-x,D)u(-x)=\sum_j a(-x,j)\hat u(j)e^{-ijx}=\sum_j a(-x,-j)\hat u(-j)e^{ijx}=\sum_j a(x,j)\hat u(j)e^{ijx}$$

where the last equality holds if and only if $a(-x,-j)=a(x,j)$. The proof for the other cases is similar, therefore it is omitted. □

We now recall some properties of pseudo-differential operators, see [39] for more details. From now on we shall consider operators with C^∞-symbols.

Definition 8. *Let $a\in S^m$, and $a_{m-k}\in S^{m-k},\forall k\geq 0$. We call $\sum_{k\geq 0}a_{m-k}$ the* ***asymptotic expansion*** *of the symbol a and we write*

$$a(x,\xi)\sim\sum_{k\geq 0}a_{m-k}(x,\xi),$$

if for all $M\in\mathbb{N}$ we have

$$a(x,\xi)-\sum_{k\leq M}a_{m-k}(x,\xi)\in S^{m-M-1}\,.$$

We provide a fundamental result concerning composition of pseudo-differential operators.

Theorem 2.5. ***Composition.*** *Let $A:=Op(a(x,\xi))$ and $B:=Op(b(x,\xi))$ be two pseudo-differential operators with symbols of order respectively m and n with $m,n\in\mathbb{R}$. Then the composition operator $A\circ B=C$ is a pseudo-differential operator of order $m+n$ with symbol*

$$c(x,\xi)=\sum_{j\in\mathbb{Z}}a(x,\xi+j)\hat b(j,\xi)e^{ijx}=\sum_{j,k}\hat a(k-j,\xi+j)\hat b(j,\xi)e^{ikx}$$

where $\hat{\cdot}$ denotes the Fourier coefficients of the symbols $a(x,\xi)$ and $b(x,\xi)$ with respect to x. Moreover the symbol c admits the asymptotic expansion

$$c(x,\xi)\sim\sum_{\beta\geq 0}\frac{(-i)^\beta}{\beta!}\partial_\xi^\beta a(x,\xi)\partial_x^\beta b(x,\xi)\,,$$

namely, $\forall M\geq 1$

$$c(x,\xi)=\sum_{\beta=0}^{M-1}\frac{(-i)^\beta}{\beta!}\partial_\xi^\beta a(x,\xi)\partial_x^\beta b(x,\xi)+r_M(x,\xi)\,, \tag{2.19}$$

where $r_M\in S^{m+n-M}$. The remainder r_M has the explicit formula

$$r_M(x,\xi):=\frac{1}{(M-1)!i^M}\int_0^1(1-\tau)^{M-1}\sum_{j\in\mathbb{Z}}(\partial_\xi^M a)(x,\xi+j\tau)\widehat{\partial_x^M b}(x,\xi)e^{ijx}d\tau\,. \tag{2.20}$$

In this thesis we consider θ-dependent families of pseudo-differential operators. We work with pseudo-differential operators with symbol $a(x,\theta,\xi)$ that are C^∞-smooth also in θ. We still denote $A := A(\theta) = Op(a(\theta,\cdot)) = Op(a)$. Therefore given a symbol $a(x,\theta,\xi) \in C^\infty(\mathbb{T}\times\mathbb{T}^N\times\mathbb{R})$ we define the action of the operator A on a function $\mathtt{w}$ as follows

$$A\mathtt{w}(x,\theta) = \sum_{j\in\mathbb{Z}} a(x,\theta,j)\mathtt{w}_j(\theta)e^{ijx}\,.$$

One can extend the previous results to θ-dependent pseudo-differential operators; for instance the symbol of the composition operator $A\circ B$ is

$$c(x,\theta,\xi) = \sum_{j\in\mathbb{Z}} a(x,\theta,\xi+j)\hat{b}(j,\theta,\xi)e^{ijx} = \sum_{j,j'\in\mathbb{Z},l,l'\in\mathbb{Z}^N} \hat{a}(j'-j,l-l',\xi+j)\hat{b}(j,l',\xi)e^{ij'x+il\cdot\theta}\,.$$

In this thesis we consider family of pseudo-differential operators which are k_0-times differentiable with respect to a parameter ζ. Note that, if $A(\zeta) = Op(a(\zeta,x,\theta,\xi))$ is a pseudo-differential operator, then also $\partial_\zeta^k A$ is a pseudo-differential operator, that is

$$\partial_\zeta^k A = Op(\partial_\zeta^k a)\,, \quad \forall k\in\mathbb{N}^{N+1}\,.$$

As in [19] we define a suitable norm (inspired to the norm in [52]) which, given a symbol $b(x,\theta,\xi)\in S^m$, controls its regularity in (x,θ) and the decay in ξ in the Sobolev norm $\|\cdot\|_p$.

Definition 9. *Let $B := B(\zeta) := b(\zeta,x,\theta,D) \in OPS^m$, $m\in\mathbb{R}$ be a family of pseudo-differential operators with symbol $b(\zeta,x,\theta,\xi)\in S^m$, which are k_0-times differentiable with respect to $\zeta\in\Lambda_0\subset\mathbb{R}^{N+1}$. For $\gamma\in(0,1)$, $\alpha\in\mathbb{N}$, $p\geq 0$, we define the* ***weighted norm***

$$|B|_{m,p,\alpha}^{k_0,\gamma} := \sum_{k\leq k_0} \gamma^{|k|}\sup_{\zeta\in\Lambda_0} |\partial_\zeta^k B(\zeta)|_{m,p,\alpha}\,, \tag{2.21}$$

where $k=(k_1,...,k_{N+1})\in\mathbb{N}^{N+1}$ with $|k| := |k_1|+...+|k_{N+1}|$ and

$$|B|_{m,p,\alpha} := \max_{0\leq\beta\leq\alpha}\sup_{\xi\in\mathbb{R}} \|\partial_\xi^\beta b(\zeta,\cdot,\cdot,\xi)\|_p\,\langle\xi\rangle^{-m+\beta}\,. \tag{2.22}$$

Remark 2.6. *In what follows we shall always use the norm (2.21) with $\alpha=0$, that is $|\cdot|_{m,p,0}^{k_0,\gamma}$. We can use this semplification since all the symbols that we have to estimate are classical symbols, namely admit an asyntotic expansion in homogeneous symbols (see Chapter 7, 8 and Appendix B) . We shall systematically expand the symbols in homogeneous components in all the transformations that we shall do.*

Remark 2.7. *In what follows we shall consider matrices of pseudo-differential operators and, with a slightly abuse of notation, we shall use the norm $|\cdot|_{m,p,0}^{k_0,\gamma}$ defined in (2.21) both for pseudo-differential operators and for matrices. In other words, for $\mathbf{B}$ as follows*

$$\mathbf{B} := \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}\,, \quad B_i := b_i(\zeta,x,\theta,D)\in OPS^m\,, \quad i=1,...,4\,,$$

we set

$$|\mathbf{B}|^{k_0,\gamma}_{m,p,0} := \max_{i=1,\dots,4} |B_i|^{k_0,\gamma}_{m,p,0} \,.$$

For completeness, in this Section we decide to present the results for the norm $|\cdot|^{k_0,\gamma}_{m,p,\alpha}$. For each k_0, γ, m fixed, the norm (2.21) is non-decreasing both in p and α, namely

$$\forall p \le p',\, \forall \alpha \le \alpha', \quad |\cdot|^{k_0,\gamma}_{m,p,\alpha} \le |\cdot|^{k_0,\gamma}_{m,p',\alpha}\,, \quad |\cdot|^{k_0,\gamma}_{m,p,\alpha} \le |\cdot|^{k_0,\gamma}_{m,p,\alpha'}\,. \tag{2.23}$$

We also have that the norm (2.21) is non-increasing in m, that is

$$m \le m' \quad \Rightarrow \quad |\cdot|^{k_0,\gamma}_{m',p,\alpha} \le |\cdot|^{k_0,\gamma}_{m,p,\alpha}. \tag{2.24}$$

Given a function $a(\zeta, x, \theta) \in C^\infty-$smooth on $\mathbb{R} \times \mathbb{T}^N$ which is k_0-times differentiable with respect to ζ, the weighted norm of the corresponding multiplication operator is

$$|Op(a)|^{k_0,\gamma}_{0,p,\alpha} = \|a\|^{k_0,\gamma}_p\,, \quad \forall \alpha \in \mathbb{N}\,, \tag{2.25}$$

where the weighted Sobolev norm $\|\cdot\|^{k_0,\gamma}_p$ is defined in (2.8).

For a Fourier multiplier $g(D)$ with symbol $g \in S^m$, we have

$$|g(D)|_{m,p,\alpha} \le C(m,\alpha,g)\,, \quad \forall p \ge 0\,. \tag{2.26}$$

Proposition 2.8. ***Composition.*** *Let $A := a(\zeta, x, \theta, D)$ and $B := b(\zeta, x, \theta, D)$ be two pseudo-differential operators whose symbols $a(\zeta, x, \theta, \xi) \in S^m$ and $b(\zeta, x, \theta, \xi) \in S^n$, with $m, n \in \mathbb{R}$. Then $A(\zeta) \circ B(\zeta)$ is a pseudo-differential operator (see* (2.19)*) with symbol in S^{m+n} satisfying, for all $\alpha \in \mathbb{N}, p \ge \mathfrak{p}_0$,*

$$|AB|^{k_0,\gamma}_{m+n,p,\alpha} \le C(p)|A|^{k_0,\gamma}_{m,p,\alpha}|B|^{k_0,\gamma}_{n,\mathfrak{p}_0+\alpha+|m|,\alpha} + C(\mathfrak{p}_0)|A|^{k_0,\gamma}_{m,\mathfrak{p}_0,\alpha}|B|^{k_0,\gamma}_{n,p+\alpha+|m|,\alpha}\,. \tag{2.27}$$

Moreover, for any integer $M \ge 1$, the remainder $R_M = Op(r_M)$ (see (2.20)*) satisfies*

$$\begin{aligned} |R_M|^{k_0,\gamma}_{m+n-M,p,\alpha} \le \frac{1}{M!}\Big(& C(p)|A|^{k_0,\gamma}_{m,p,\alpha+M}|B|^{k_0,\gamma}_{n,\mathfrak{p}_0+2M+\alpha+|m|,\alpha} \\ & + C(\mathfrak{p}_0)|A|^{k_0,\gamma}_{m,\mathfrak{p}_0,\alpha+M}|B|^{k_0,\gamma}_{n,p+2M+\alpha+|m|,\alpha}\Big)\,. \end{aligned} \tag{2.28}$$

Proof. A complete proof is in [19].

□

By (2.25) and (2.26) and Proposition 2.8 we have that $\forall m \in \mathbb{Z}$ and for all $p \ge \mathfrak{p}_0$

$$|a(x,\theta)\partial_x^m|^{k_0,\gamma}_{m,p,\alpha} \le C(m,\alpha,p)\|a\|^{k_0,\gamma}_p + C(m,\alpha,\mathfrak{p}_0)\|a\|^{k_0,\gamma}_{\mathfrak{p}_0} \le C_1(m,\alpha,p)\|a\|^{k_0,\gamma}_p\,. \tag{2.29}$$

By (2.19) the commutator between two pseudo-differential operators $A := a(\zeta, x, \theta, D) \in OPS^m$ and $B := b(\zeta, x, \theta, D) \in OPS^n$ is a pseudo-differential operator $[A, B] \in OPS^{m+n-1}$.

Lemma 2.9. ***Commutators.*** *Let $A := a(\zeta, x, \theta, D)$, $B := b(\zeta, x, \theta, D)$ be pseudo-differential operators with symbols $a(\zeta, x, \theta, \xi) \in S^m$, $b(\zeta, x, \theta, \xi) \in S^n$, $m, n \in \mathbb{R}$. Then the commutator $[A, B] := AB - BA \in OPS^{m+n-1}$ satisfies*

$$\begin{aligned} |[A,B]|^{k_0,\gamma}_{m+n-1,p,\alpha} \leq & C(p)|A|^{k_0,\gamma}_{m,p+2+|n|+\alpha,\alpha+1}|B|^{k_0,\gamma}_{n,\mathfrak{p}_0+2+|m|+\alpha,\alpha+1} \\ & + C(p)|A|^{k_0,\gamma}_{m,\mathfrak{p}_0+2+|n|+\alpha,\alpha+1}|B|^{k_0,\gamma}_{n,p+2+|m|+\alpha,\alpha+1}. \end{aligned} \tag{2.30}$$

Proof. The estimate follows by (2.19), (2.28) for $M = 1$, and (2.23).

□

We finally state an invertibility Lemma.

Lemma 2.10. ***Invertibility.*** *Let $T := T(\zeta)$ and $T(\zeta) = \mathbb{1} + \Phi(\zeta)$ where $\Phi(\zeta)$ is a pseudo-differential operator in OPS^0. There exist constants $C(\mathfrak{p}_0, \alpha, k_0), C(p, \alpha, k_0) \geq 1$, $p \geq \mathfrak{p}_0$, such that, if*

$$C(\mathfrak{p}_0, \alpha, k_0)|\Phi|^{k_0,\gamma}_{0,\mathfrak{p}_0+\alpha,\alpha} \leq \frac{1}{2}, \tag{2.31}$$

then, for all ζ, the operator T is invertible, $T^{-1} \in OPS^0$ and, for all $p \geq \mathfrak{p}_0$

$$|T^{-1} - \mathbb{1}|^{k_0,\gamma}_{0,p,\alpha} \leq C(p, \alpha, k_0)|\Phi|^{k_0,\gamma}_{0,p+\alpha,\alpha}.$$

Proof. See [19]. □

2.2 $\mathcal{D}^{k_0}$- tame and modulo-tame operators

We consider linear operators $A := A(\zeta)$, k_0-times differentiable with respect to a parameter $\zeta \in \Lambda_0 \in \mathbb{R}^{N+1}$. Recall the weighted norm $\| \cdot \|^{k_0,\gamma}_p$ defined in (2.8). We now present some results, given in [19], that we shall use in Chapter 9.

Definition 10. *A family of linear operators $A := A(\zeta)$ is $\mathcal{D}^{k_0} - \sigma-$**tame** if the following weighted tame estimate holds: there exists $\sigma \geq 0$ such that for all $p \geq \mathfrak{p}_0$, for all $\mathtt{w} \in H^{p+\sigma}(\mathbb{T} \times \mathbb{T}^N)$,*

$$\sup_{|k| \leq k_0} \sup_{\zeta \in \Lambda_0} \gamma^{|k|} \|(\partial^k_\zeta A)\mathtt{w}\|_p \leq \mathcal{M}_A(\mathfrak{p}_0)\|\mathtt{w}\|_{p+\sigma} + \mathcal{M}_A(p)\|\mathtt{w}\|_{\mathfrak{p}_0+\sigma} \tag{2.32}$$

where the functions $p \mapsto \mathcal{M}_A(p) \geq 0$ are non-decreasing in p. We call $\mathcal{M}_A$ the tame constant of the operator A. Note that the constant $\mathcal{M}_A(\mathfrak{p}_0) := \mathcal{M}_A(k_0, \sigma, \mathfrak{p}_0)$ depends also on k_0, σ, but since k_0, σ do not vary along the thesis we shall omit to write them.

Remark 2.11. *In Chapter 9 we shall work with $\mathcal{D}^{k_0}$-tame operators with a finite $P < \infty$, whose tame constant $\mathcal{M}_A(p)$ may depend also on $C(P)$, for istance $\mathcal{M}_A(p) \leq C(P)\mu\|\mathtt{v}\|^{k_0,\gamma}_{p+\nu}$, $\forall \mathfrak{p}_0 \leq p \leq P$. We shall fix the highest P in the Nash-Moser iteration, see* (10.13).

When the "loss of derivatives" $\sigma = 0$ we call a $\mathcal{D}^{k_0} - 0-$tame operator to be $\mathcal{D}^{k_0}-$ tame.

Remark 2.12. *By (2.32) (with $k = 0, p = \mathfrak{p}_0$) we have*

$$\|A\|_{\mathcal{L}(H^{\mathfrak{p}_0+\sigma}, H^{\mathfrak{p}_0})} \leq 2\mathcal{M}_A(\mathfrak{p}_0)\,. \tag{2.33}$$

Let A be a linear operator, that can be identified with the infinite matrix $A_j^{j'}(l-l')$ where $j, j' \in \mathbb{Z}$ and $l, l' \in \mathbb{Z}^N$, then, $\forall |k| \leq k_0$,

$$\gamma^{2|k|} \sum_{l,j} \langle l, j \rangle^{2p} |\partial_\zeta^k A_j^{j'}(l-l')|^2 \leq C(k_0) \left(\mathcal{M}_A(\mathfrak{p}_0)^2 \langle l', j' \rangle^{2p+2\sigma} + \mathcal{M}_A(p)^2 \langle l', j' \rangle^{2\mathfrak{p}_0+2\sigma} \right).$$

The class of $\mathcal{D}^{k_0} - \sigma-$tame operators is closed under composition.

Lemma 2.13. ***Composition.*** *Let A, B be linear operators $\mathcal{D}^{k_0}-$tame. Then the composed operator $A \circ B := AB$ is a $\mathcal{D}^{k_0}-$tame operator with tame constant*

$$\mathcal{M}_{AB}(p) \leq C(k_0) \left(\mathcal{M}_A(\mathfrak{p}_0)\mathcal{M}_B(p) + \mathcal{M}_A(p)\mathcal{M}_B(\mathfrak{p}_0) \right).$$

Let A, B be respectively $\mathcal{D}^{k_0} - \sigma_A-$tame and $\mathcal{D}^{k_0} - \sigma_B-$tame operators with tame constants respectively $\mathcal{M}_A(p)$ and $\mathcal{M}_B(p)$. Then the composed operator $A \circ B$ is $\mathcal{D}^{k_0} - (\sigma_A + \sigma_B)-$tame operator with tame constant

$$\mathcal{M}_{AB}(p) \leq C(k_0) \left(\mathcal{M}_A(\mathfrak{p}_0)\mathcal{M}_B(p+\sigma_A) + \mathcal{M}_A(p)\mathcal{M}_B(\mathfrak{p}_0+\sigma_A) \right). \tag{2.34}$$

Proof. See [19]. □

The following lemmas are meant to prove that the norm $|\cdot|_{0,p,0}^{k_0,\gamma}$ controls the action of a pseudo-differential operator on $H^p(\mathbb{T} \times \mathbb{T}^N)$.

Lemma 2.14. *Let $B = b(\zeta, x, \theta, D)$ be a family of pseudo-differential operators which are k_0-times differentiable with respect to ζ and with symbol b in S^0. If $|B|_{0,p,0}^{k_0,\gamma} < \infty$, then B is $\mathcal{D}^{k_0}-$tame (see Definition 10) with tame constant $\forall p \geq \mathfrak{p}_0$*

$$\mathcal{M}_B(p) \leq C(p)|B|_{0,p,0}^{k_0,\gamma}\,. \tag{2.35}$$

Proof. See [19]. □

The action of a $\mathcal{D}^{k_0} - \sigma-$tame operator $A(\zeta)$ on functions $\mathtt{w}(\zeta) \in H^p(\mathbb{T} \times \mathbb{T}^N)$ that are k_0-times differentiable with respect to $\zeta \in \Lambda_0 \subset \mathbb{R}^{N+1}$ is given by the following Lemma.

Lemma 2.15. *Let $A := A(\zeta)$ be a $\mathcal{D}^{k_0} - \sigma-$tame operator. Then $\forall p \geq \mathfrak{p}_0$ and for any family of Sobolev functions $\mathtt{w} := \mathtt{w}(\zeta) \in H^{p+\sigma}(\mathbb{T} \times \mathbb{T}^N)$ which is k_0-times differentiable with respect to ζ the following tame estimate holds*

$$\|A\mathtt{w}\|_p^{k_0,\gamma} \leq_{k_0} \mathcal{M}_A(\mathfrak{p}_0)\|\mathtt{w}\|_{p+\sigma}^{k_0,\gamma} + \mathcal{M}_A(p)\|\mathtt{w}\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\,.$$

Proof. See lemma 2.14 in [19]. □

By Lemma 2.15, (2.25), (2.35) the tame estimate for the product of two functions in weighted Sobolev norm may be estimated as in the following Lemma:

Lemma 2.16. *For all $p \geq \mathfrak{p}_0$, for all $\mathtt{w}, \mathtt{v} \in H^p(\mathbb{T} \times \mathbb{T}^N)$ the following inequalities hold*

$$\begin{aligned} \|\mathtt{w}\mathtt{v}\|_p^{k_0,\gamma} &\leq_{k_0} C(p)\|\mathtt{w}\|_p^{k_0,\gamma}\|\mathtt{v}\|_{\mathfrak{p}_0}^{k_0,\gamma} + C(\mathfrak{p}_0)\|\mathtt{w}\|_{\mathfrak{p}_0}^{k_0,\gamma}\|\mathtt{v}\|_p^{k_0,\gamma}\,, \\ \|\mathtt{w}\mathtt{v}\|_p^{k_0,\gamma} &\leq_{k_0} C(p)\|\mathtt{w}\|_p^{k_0,\gamma}\|\mathtt{v}\|_p^{k_0,\gamma}\,. \end{aligned} \tag{2.36}$$

In view of the KAM reducibility scheme of Chapter 9 we also consider the stronger notion of $\mathcal{D}^{k_0}-$modulo-tame operators, that we need only for operators with loss of derivatives $\sigma = 0$.

Definition 11. *A linear operator $A := A(\zeta)$ is $\mathcal{D}^{k_0}$**-modulo-tame** if for all $k \in \mathbb{N}^{N+1}$, $|k| \leq k_0$, the majorant operators $|\partial_\zeta^k A|$ (see Definition 2) satisfy the following weighted tame estimate: for all $p \geq \mathfrak{p}_0, \mathtt{w} \in H^{p+\sigma}(\mathbb{T} \times \mathbb{T}^N)$,*

$$\sup_{|k| \leq k_0} \gamma^{|k|} \| |\partial_\zeta^k A| \mathtt{w}\|_p \leq \mathcal{M}_A^\sharp(\mathfrak{p}_0)\|\mathtt{w}\|_p + \mathcal{M}_A^\sharp(p)\|\mathtt{w}\|_{\mathfrak{p}_0}\,, \tag{2.37}$$

where the functions $p \mapsto \mathcal{M}_A^\sharp(p) \geq 0$ are non-decreasing in p. The constant $\mathcal{M}_A^\sharp(p)$ is called the modulo tame constant of the operator A.

Lemma 2.17. *Let A be a $\mathcal{D}^{k_0}-$ modulo-tame operator, then*

$$\mathcal{M}_A(p) \leq \mathcal{M}_A^\sharp(p)\,.$$

Proof. See Lemma 2.15 in [19]. □

The class of $\mathcal{D}^{k_0}$-modulo-tame operators is closed under sum and composition, indeed we have the following Lemma.

Lemma 2.18. ***Sum and composition.*** *Let A, B be $\mathcal{D}^{k_0}$-modulo-tame operators with modulo-tame constants respectively $\mathcal{M}_A^\sharp(p)$ and $\mathcal{M}_B^\sharp(p)$. Then $A + B$ is $\mathcal{D}^{k_0}$-modulo-tame with modulo-tame constant*

$$\mathcal{M}_{A+B}^\sharp(p) \leq \mathcal{M}_A^\sharp(p) + \mathcal{M}_B^\sharp(p)\,. \tag{2.38}$$

The composed operator $A \circ B := AB$ is C^{k_0}-modulo-tame with modulo-tame constant

$$\mathcal{M}_{AB}^\sharp(p) \leq C(k_0)\left(\mathcal{M}_A^\sharp(p)\mathcal{M}_B^\sharp(\mathfrak{p}_0) + \mathcal{M}_B^\sharp(p)\mathcal{M}_A^\sharp(\mathfrak{p}_0)\right)\,. \tag{2.39}$$

Assume in addition that $\langle\partial_\theta\rangle^b A, \langle\partial_\theta\rangle^b B$ (see Definition 2) are $\mathcal{D}^{k_0}$- modulo-tame with modulo-tame constant respectively $\mathcal{M}_{\langle\partial_\theta\rangle^b A}^\sharp(p)$ and $\mathcal{M}_{\langle\partial_\theta\rangle^b A}^\sharp(p)$, then $\langle\partial_\theta\rangle^b (AB)$ is $\mathcal{D}^{k_0}$ -modulo-tame with modulo-tame constant satisfsying

$$\begin{aligned} \mathcal{M}_{\langle\partial_\theta\rangle^b (AB)}^\sharp(p) \leq C(b)C(k_0)\Big(&\mathcal{M}_{\langle\partial_\theta\rangle^b A}^\sharp(p)\mathcal{M}_B^\sharp(\mathfrak{p}_0) + \mathcal{M}_{\langle\partial_\theta\rangle^b A}^\sharp(\mathfrak{p}_0)\mathcal{M}_B^\sharp(p) \\ &+ \mathcal{M}_{\langle\partial_\theta\rangle^b B}^\sharp(p)\mathcal{M}_A^\sharp(\mathfrak{p}_0) + \mathcal{M}_{\langle\partial_\theta\rangle^b B}^\sharp(\mathfrak{p}_0)\mathcal{M}_A^\sharp(p)\Big)\,. \end{aligned} \tag{2.40}$$

Proof. See Lemma 2.16 in [19]. □

As a consequence of the composition rule (2.39), if A is $\mathcal{D}^{k_0}$-modulo-tame, then, for all $n \geq 1$, each A^n is $\mathcal{D}^{k_0}$-modulo-tame and

$$\mathcal{M}^\sharp_{A^n}(p) \leq \Big(2C(k_0)\mathcal{M}^\sharp_A(\mathfrak{p}_0)\Big)^{n-1}\mathcal{M}^\sharp_A(p)\,. \tag{2.41}$$

Moreover, by (2.40), if $\langle\partial_\theta\rangle^b A$ is $\mathcal{D}^{k_0}$-modulo-tame then for all $n \geq 2$ each $\langle\partial_\theta\rangle^b A^n$ is $\mathcal{D}^{k_0}$-modulo-tame with

$$\begin{aligned}\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^b A^n}(p) \leq (4C(k_0)C(b))^{n-1}\Big(&\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^b A^n}(p)\Big(\mathcal{M}^\sharp_A(\mathfrak{p}_0)\Big)^{n-1} \\ &+ \mathcal{M}^\sharp_A(p)\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^b A^n}(\mathfrak{p}_0)\Big(\mathcal{M}^\sharp_A(\mathfrak{p}_0)\Big)^{n-2}\Big)\,.\end{aligned} \tag{2.42}$$

Lemma 2.19. ***Invertibility.*** *Consider the operator* $\Phi(\zeta) = \mathbb{1} + A(\zeta)$ *where* $A(\zeta) := A$ *is* $\mathcal{D}^{k_0}-$*modulo-tame with modulo-tame constant* $\mathcal{M}^\sharp_A(p)$*. Assume the smallness condition*

$$4C(b)C(k_0)\mathcal{M}^\sharp_A(\mathfrak{p}_0) \leq \frac{1}{2}\,. \tag{2.43}$$

Then the operator $\Phi(\zeta) := \Phi$ *is invertible,* $B := \Phi^{-1} - \mathbb{1}$ *is a* $\mathcal{D}^{k_0}-$*modulo-tame operator with modulo-tame constant*

$$\mathcal{M}^\sharp_B(p) \leq 2\mathcal{M}^\sharp_A(p)\,. \tag{2.44}$$

Moreover $\langle\partial_\theta\rangle^b B$ *is* $\mathcal{D}^{k_0}$*-modulo-tame with tame-constant*

$$\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^b B}(p) \leq 2\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^b A}(p) + 8C(b)C(k_0)\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^b B}(\mathfrak{p}_0)\mathcal{M}^\sharp_A(p)\,. \tag{2.45}$$

Proof. Using (2.33) and (2.43) the operator norm $\|A\|_{\mathcal{L}(H^{\mathfrak{p}_0})} \leq 2\mathcal{M}^\sharp_A(\mathfrak{p}_0) \leq \frac{1}{2}$.

Then Φ in invertible and the inverse $\Phi^{-1} = \mathbb{1} + B$ where $B = \sum_j (-1)^j A^j$ satisfy the estimate (2.44) by (2.38), (2.41) and (2.43). Similarly (2.45) follows by (2.38), (2.42) and (2.43). □

We now present further lemmas that we shall use in Chapter 9.

Lemma 2.20. ***Smoothing.*** *Suppose that* $\langle\partial_\theta\rangle^{\mathtt{b}} A$, $\mathtt{b} \geq 0$, *is* $\mathcal{D}^{k_0}$*- modulo-tame. Then the operator* $\Pi_N^\perp A$ *is* $\mathcal{D}^{k_0}$*- modulo-tame with tame constant*

$$\mathcal{M}^\sharp_{\Pi_N^\perp A}(p) \leq N^{-\mathtt{b}}\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^b A}(p)\,, \qquad \mathcal{M}^\sharp_{\Pi_N^\perp A}(p) \leq \mathcal{M}^\sharp_A(p)\,. \tag{2.46}$$

Proof. See Lemma 2.18 in [19]. □

Lemma 2.21. *Let* A *and* B *be linear operators satisfying* $|A|, |B|, |\langle\partial_\theta\rangle^{\mathtt{b}} A|, |\langle\partial_\theta\rangle^{\mathtt{b}} B| \in \mathcal{L}(H^{\mathfrak{p}_0})$*. Then*

$$|||A+B|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \leq |||A|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + |||B|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \tag{2.47}$$

$$|||AB|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \leq |||A|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||B|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \tag{2.48}$$

$$|||\langle\partial_\theta\rangle^{\mathtt{b}}(AB)|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \le |||\langle\partial_\theta\rangle^{\mathtt{b}} A|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||B|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + |||\langle\partial_\theta\rangle^{\mathtt{b}} B|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||A|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \tag{2.49}$$

$$|||\Pi_N^{\perp} A|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \le N^{-\mathtt{b}} \| \langle\partial_\theta\rangle^{\mathtt{b}} |A|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \tag{2.50}$$

$$|||\Pi_N^{\perp} A|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \le |||A|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\,. \tag{2.51}$$

Proof. See [19]. □

Lemma 2.22. *Let $\Phi_i := \mathbb{1} + \Psi_i, i = 1, 2$ satisfy*

$$|||\Psi_i|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \le \frac{1}{2}, \quad i = 1, 2\,. \tag{2.52}$$

Then $\Phi_i^{-1} = \mathbb{1} + \hat{\Psi}_i\,,\ \ i = 1, 2$ satisfy $|||\hat{\Psi}_1 - \hat{\Psi}_2|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \le 4|||\Psi_1 - \Psi_2|||_{\mathcal{L}(H^{\mathfrak{p}_0})}$ and

$$\begin{aligned} \| \langle\partial_\theta\rangle^{\mathtt{b}} |\hat{\Psi}_1 - \hat{\Psi}_2|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\le C(\mathtt{b}) \| \langle\partial_\theta\rangle^{\mathtt{b}} |\Psi_1 - \Psi_2|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \\ &+ C(\mathtt{b}) \left(1 + |||\langle\partial_\theta\rangle^{\mathtt{b}} \hat{\Psi}_1|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + \langle\partial_\theta\rangle^{\mathtt{b}} \hat{\Psi}_2|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right) |||\Psi_1 - \Psi_2|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\,. \end{aligned}$$

Proof. See Lemma 2.20 in [19]. □

2.3 Composition operators

The composition operator $\mathtt{w}(y) \mapsto \mathtt{w}(y + p(y))$ induced by a diffeomorphism of the torus $\mathbb{T}^n$ is tame.

Lemma 2.23. *Let $q := q(\zeta, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ be a family of 2π-periodic functions which is k_0-times differentiable with respect to $\zeta \in \Lambda_0 \subset \mathbb{R}^{N+1}$, satisfying*

$$\|q\|_{\mathcal{C}^{\mathfrak{p}_0+1}} \le \frac{1}{2}\,, \quad \|q\|_{\mathfrak{p}_0}^{k_0,\gamma} \le 1\,.$$

Let $g(y) := y + q(y)$, $y \in \mathbb{T}^n$. Then the composition operator

$$A : \mathtt{w}(y) \to (\mathtt{w} \circ g)(y) = \mathtt{w}(y + q(y))$$

satisfies the tame estimates

$$\|A\mathtt{w}\|_{\mathfrak{p}_0} \le C(\mathfrak{p}_0)\|\mathtt{w}\|_{\mathfrak{p}_0}, \qquad \|A\mathtt{w}\|_p \le C(\mathfrak{p}_0)\|\mathtt{w}\|_{\mathfrak{p}_0+1}\|q\|_p + C(p)\|\mathtt{w}\|_p,\ \forall p \ge \mathfrak{p}_0 + 1\,,$$

and for any $|k| \le k_0$,

$$\begin{aligned} &\|(\partial_\zeta^k A)\mathtt{w}\|_{\mathfrak{p}_0} \le C(\mathfrak{p}_0, k)\gamma^{-|k|}\|\mathtt{w}\|_{\mathfrak{p}_0+|k|} \\ &\|(\partial_\zeta^k A)\mathtt{w}\|_p \le \gamma^{-|k|} C(p, k) \left(\|\mathtt{w}\|_{\mathfrak{p}_0+|k|+1}\|q\|_p^{|k|,\gamma} + \|\mathtt{w}\|_{p+|k|}\right)\,, \quad \forall p \ge \mathfrak{p}_0 + 1\,. \end{aligned}$$

The map g is invertible with inverse $g^{-1}(x) = x + s(x)$. Suppose $\partial_\zeta^k q(\zeta, \cdot) \in C^\infty(\mathbb{T}^{N+1})$ for all $|k| \le k_0$. There exists a constant $\delta := \delta(\mathfrak{p}_0, k_0) \in (0, 1)$ such that, if $\|q\|_{2\mathfrak{p}_0+k_0+1}^{k_0,\gamma} \le \delta$, then

$$\|s\|_p^{k_0,\gamma} \le C(p, k_0)\|s\|_{p+k_0}^{k_0,\gamma}\,, \quad \forall p \ge \mathfrak{p}_0\,. \tag{2.53}$$

The composition operators A and A^{-1} are $\mathcal{D}^{k_0} - \sigma$-tame with $\sigma = (k_0+1)$ and tame constants satisfying for any $P > \mathfrak{p}_0$,

$$\mathcal{M}_A(p) \leq C(P,k_0)(1+\|q\|_p^{k_0,\gamma}), \quad \mathcal{M}_{A^{-1}}(p) \leq C(P,k_0)(1+\|q\|_{p+k_0}^{k_0,\gamma}), \qquad \forall \mathfrak{p}_0 \leq p \leq P. \tag{2.54}$$

Proof. See Lemma 2.21 in [19]. □

Finally we have the generalized Moser tame estimate for the composition operator.

Lemma 2.24. ***Composition operator.*** *Let $f \in C^\infty(\mathbb{R} \times \mathbb{T}^N, \mathbb{R})$, and let*

$$\mathtt{w}(x,\theta) \mapsto \mathtt{f}(\mathtt{w})(x,\theta) := f(x,\theta,\mathtt{w}(x,\theta)),$$

the induced composition operator. If $\mathtt{w}(\zeta) \in H^p(\mathbb{T}^{1+N})$ is a family of Sobolev functions satisfying $\|\mathtt{w}\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq 1$ then, $\forall p > \mathfrak{p}_0 := \frac{d+1}{2}$,

$$\|\mathtt{f}(\mathtt{w})\|_p^{k_0,\gamma} \leq C(p,k_0,f)(1+\|\mathtt{w}\|_p^{k_0,\gamma}).$$

Tame estimates for the translation operators.

We now prove some results for the composition with a particular change of variable that we shall consider in Chapter 8. Let $\zeta \in \Lambda_0$. We consider

$$\Psi(\zeta,\theta) := \Psi : h(x,\theta) \to h(x+\psi(\theta),\theta).$$

In order to simplify the notation in what follows we shall write ∂_θ^β instead of $\partial_{\theta_r}^\beta$, $r = 1, ..., N$. Moreover, since this particular composition operator acts only on the spatial component, we omit the θ-component in the function, namely we write $h(x)$ instead of $h(x,\theta)$. Note that

$$(\partial_\theta \Psi)[h] = (\partial_x h)(x+\psi(\theta))(\partial_\theta \psi(\theta)). \tag{2.55}$$

We start with the following Lemma

Lemma 2.25. *Let Ψ be the translation given above, that is $\Psi(\zeta,\theta) := \Psi : h(x) \to h(x+\psi(\theta))$. Then*

$$\partial_\theta \Psi \langle \partial_x \rangle^{-1} = \langle \partial_x \rangle^{-1} \partial_\theta \Psi .$$

Proof. Let $h = \sum_{j\in\mathbb{Z}} h_j e^{ijx}$ be a function. Then

$$\begin{aligned}(\partial_\theta \Psi)\langle \partial_x \rangle^{-1} h &= \partial_\theta \Psi \left(\sum_{j\in\mathbb{Z}} \frac{1}{\langle j \rangle} h_j e^{ijx} \right) \\ &= \partial_\theta \left(\sum_{j\in\mathbb{Z}} \frac{1}{\langle j \rangle} h_j e^{ijx+ij\psi(\theta)} \right) \\ &= \sum_{j\in\mathbb{Z}} \frac{ij}{\langle j \rangle} \psi_\theta(\theta) h_j e^{ijx+ij\psi(\theta)} \\ &= \langle \partial_x \rangle^{-1} (\partial_\theta \Psi)[h]\end{aligned}$$

where $\psi_\theta = \partial_\theta \psi$, □

Thanks to Lemma 2.25 we have that $\partial_\zeta^k \partial_{\theta_r}^\beta \Psi \langle \partial_x \rangle^{-\beta-|k|} = \langle \partial_x \rangle^{-\beta-|k|} \partial_\zeta^k \partial_{\theta_r}^\beta \Psi$. Also in what follows we shall write ∂_θ^β instead of $\partial_{\theta_r}^\beta$.

Lemma 2.26. *Let $k_0, \beta_0 \in \mathbb{N}$ and $\Psi(\zeta, \theta) := \Psi$ as before. Assume that*

$$\|\psi\|^{k_0,\gamma}_{\mathfrak{p}_0+\beta_0} \leq 1 \, . \tag{2.56}$$

Then for all $k \in \mathbb{N}^N, \beta \in \mathbb{N}$ with $|k| \leq k_0$, $\beta \leq \beta_0$ and for all $\mathfrak{p}_0 \leq p \leq P$

$$\|\partial_\zeta^k \partial_\theta^\beta \Psi \langle \partial_x \rangle^{-|k|-\beta} h\|_p \, , \ \ \|\partial_x \partial_\zeta^k \partial_\theta^\beta \Psi \langle \partial_x \rangle^{-|k|-\beta-1} h\|_p \leq_P \gamma^{-|k|} \Big(\|h\|_{\mathfrak{p}_0} \|\psi\|^{k_0,\gamma}_{p+\beta} + \|h\|_p \Big) \tag{2.57}$$

$$\|\partial_\theta^\beta \partial_i \Psi[\hat{\imath}] \langle \partial_x \rangle^{-\beta-1} h\|_p \, , \ \ \|\partial_\theta^\beta \partial_x \partial_i \Psi[\hat{\imath}] \langle \partial_x \rangle^{-\beta-2} h\|_p \leq_P \|h\|_{\mathfrak{p}_0} \|\partial_i \psi[\hat{\imath}]\|^{k_0,\gamma}_{p+\beta} + \|h\|_p \tag{2.58}$$

Proof. We prove $\|\partial_\zeta^k \partial_\theta^\beta \Psi \langle \partial_x \rangle^{-|k|-\beta} h\|_p \leq_P \gamma^{-|k|} \Big(\|h\|_{\mathfrak{p}_0} \|\psi\|^{k_0,\gamma}_{p+\beta+|k|} + \|h\|_p \Big)$. Set

$$f := \langle \partial_x \rangle^{-|k|-\beta} h, \tag{2.59}$$

then

$$\partial_\theta^\beta f(x+\psi(\theta)) = \sum_{1 \leq n \leq \beta} \sum_{\substack{\beta_1+...+\beta_n=\beta \\ \beta_1,...,\beta_n \geq 1}} C_{n,\beta_1,...,\beta_n} (\partial_x^n f)(x+\psi(\theta)) (\partial_\theta^{\beta_1} \psi)...(\partial_\theta^{\beta_n} \psi) \, . \tag{2.60}$$

We differentiate also for ∂_ζ^k:

$$\begin{aligned} \partial_\zeta^k \partial_\theta^\beta f(x+\psi(\theta)) &= \sum_{k_1+k_2=k} \sum_{\substack{1 \leq n \leq \beta \\ \beta_1+...+\beta_n=\beta \\ \beta_1,...,\beta_n \geq 1}} C_{n,\beta_1,...,\beta_n} \Big(\partial_\zeta^{k_1} (\partial_x^n f)(x+\psi(\theta)) \Big) \partial_\zeta^{k_2} \Big[(\partial_\theta^{\beta_1} \psi)...(\partial_\theta^{\beta_n} \psi) \Big] \\ &= \sum_{k_1+k_2=k} \sum_{\substack{1 \leq n \leq \beta \\ \beta_1+...+\beta_n=\beta \\ \beta_1,...,\beta_n \geq 1}} C_{n,\beta_1,...,\beta_n} \Bigg(\sum_{\substack{1 \leq m \leq |k_1| \\ a_1+...+a_m=k_1 \\ |a_1|,...,|a_m| \geq 1}} C_{m,k_1,...,k_m} (\partial_x^{m+n} f)(x+\psi(\theta)) \\ &\quad \times (\partial_\zeta^{a_1} \psi)...(\partial_\zeta^{a_m} \psi) \Bigg) \partial_\zeta^{k_2} \Big[(\partial_\theta^{\beta_1} \psi)...(\partial_\theta^{\beta_n} \psi) \Big] \, . \end{aligned} \tag{2.61}$$

Therefore, according to the previous formula, we need to estimate for any $1 \leq n \leq \beta$, $\beta_1 + ... + \beta_n = \beta$, $k_1 + k_2 = k$, $1 \leq m \leq |k_1|$, $a_1 + ... + a_m = k_1$ the term

$$\Big\| (\partial_x^{m+n} f)(x+\psi(\theta)) \Big[(\partial_\zeta^{a_1} \psi)...(\partial_\zeta^{a_m} \psi) \Big] \partial_\zeta^{k_2} \Big[(\partial_\theta^{\beta_1} \psi)...(\partial_\theta^{\beta_n} \psi) \Big] \Big\|_p \, . \tag{2.62}$$

First notice that for any $p \geq 0$, by (2.59)

$$\|\partial_x^{m+n} f\|_p = \|\partial_x^{m+n} \langle \partial_x \rangle^{-|k|-\beta} h\|_p \overset{m+n \leq \beta+|k|}{\leq} \|h\|_p \, . \tag{2.63}$$

Using Lemma 2.23, one has

$$\|(\partial_x^{m+n} f)(x+\psi(\theta))\|_p \leq_p \|\partial_x^{m+n} f\|_p + \|\psi\|_p \|\partial_x^{m+n} f\|_{\mathfrak{p}_0} \overset{(2.63)}{\leq_p} \|h\|_p + \|\psi\|_p \|h\|_{\mathfrak{p}_0} \, . \tag{2.64}$$

Using the interpolation estimates (2.5), the condition (2.56) and $a_1 + ... + a_m = k_1$, $\beta_1 + ... + \beta_n = \beta$ one has

$$\|(\partial_\zeta^{a_1} \psi)...(\partial_\zeta^{a_m} \psi)\|_p \leq_p \gamma^{-|k_1|} \|\psi\|^{k_0,\gamma}_p \, , \tag{2.65}$$

$$\|\partial_\zeta^{k_2}\left[(\partial_\theta^{\beta_1}\psi)...(\partial_\theta^{\beta_n}\psi)\right]\|_p \leq_p \gamma^{-|k_2|}\|(\partial_\theta^{\beta_1}\psi)...(\partial_\theta^{\beta_n}\psi)\|_p^{k_0,\gamma} \leq_p \gamma^{-|k_2|}\|\psi\|_{p+|\beta|}^{k_0,\gamma} \tag{2.66}$$

Then, by (2.64)-(2.66), using that $k_1 + k_2 = k$, and recalling the interpolation estimate for product the (2.36) one has that

$$(2.62) \leq_p \gamma^{-|k|}\left(\|h\|_p + \|\psi\|_{p+|\beta|}^{k_0,\gamma}\|h\|_{\mathfrak{p}_0}\right) .$$

As a consequence,by recalling (2.61), the first claimed estimate in (2.57) is proved. The other estimates follow similarly.

□

Chapter 3

Degenerate KAM theory

In this Chapter we verify that it is possible to develop degenerate KAM theory as in [7] and in [19].

Definition 12. *A function* $f := (f_1, ..., f_N) : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}^N$ *is called* ***non degenerate*** *if,* $\forall c := (c_1, ..., c_N) \in \mathbb{R}^N \setminus \{0\}$ *the function* $f \cdot c = f_1 c_1 + ... + f_N c_N$ *is not identically zero in the whole interval* $[\varepsilon_1, \varepsilon_2]$.

For a smooth function f, differentiating $(N-1)$-times the identity $f(\varepsilon) \cdot c = 0$ we see that

$$f(\varepsilon) \text{ degenerate} \quad \Longrightarrow \quad f(\varepsilon), (\partial_\varepsilon f)(\varepsilon), ..., (\partial_\varepsilon^{N-1} f)(\varepsilon) \quad \text{are linearly dependent} \quad \forall \varepsilon \in [\varepsilon_1, \varepsilon_2].$$

Let us consider

$$\omega_j(\varepsilon) = j\sqrt{\frac{2}{15} j^4 \varepsilon^4 - \frac{1}{3} j^2 \varepsilon^2 + 1}, \quad j \in \mathbb{N} \setminus \{0\}. \tag{3.1}$$

We define $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. We denote the unperturbed tangential frequency vector by

$$\begin{aligned} \vec{\omega} : [\varepsilon_1, \varepsilon_2] &\longrightarrow \mathbb{R}^N \\ \varepsilon \mapsto & \vec{\omega}(\varepsilon) := (\omega_j(\varepsilon))_{j \in \mathbb{S}} = (\omega_{j_1}(\varepsilon), \omega_{j_2}(\varepsilon), ..., \omega_{j_N}(\varepsilon)) \end{aligned}, \tag{3.2}$$

where $\mathbb{S}$ is defined in (1.31). The unperturbed normal frequency vector is defined as

$$\begin{aligned} \vec{\Omega} : & [\varepsilon_1, \varepsilon_2] \longrightarrow \mathbb{R}^N \\ & \varepsilon \mapsto \vec{\Omega}(\varepsilon) := (\Omega_j(\varepsilon))_{j \in \mathbb{N}_0 \setminus \mathbb{S}} := (\omega_j(\varepsilon))_{j \in \mathbb{N}_0 \setminus \mathbb{S}} \end{aligned}. \tag{3.3}$$

We show that the function $\varepsilon \to \omega_j(\varepsilon)$ is analytic in $\left(-\frac{\delta}{j}, \infty\right)$. Indeed the function $\omega_1(\varepsilon)$ is analyitic in $(-\delta, \infty)$. Let $\varepsilon_0 = 0$, then in $(-\delta, \delta)$ we have that the function

$$\omega_1(\varepsilon) = \sum_{n \geq 0} g_{2n} \frac{\varepsilon^{2n}}{(2n)!}, \tag{3.4}$$

Then if we expand in Taylor series at $\varepsilon_0 = 0$ also the functions

$$\omega_j(\varepsilon) = j\omega_1(j\varepsilon) = j \sum_{n \geq 0} g_{2n} j^{2n} \frac{\varepsilon^{2n}}{(2n)!}, \quad \forall j \in \mathbb{N}_0 \tag{3.5}$$

we obtain that are analytic in $\left(-\frac{\delta}{j}, \infty\right)$.

Lemma 3.1. *The frequency vectors*

$$\begin{aligned} &\vec{\omega}(\varepsilon) \in \mathbb{R}^N, \quad (\vec{\omega}(\varepsilon), \Omega_k(\varepsilon)) \in \mathbb{R}^{N+1}, \quad \forall k \in \mathbb{N}_0 \setminus \mathbb{S}, \\ &(\vec{\omega}(\varepsilon), \Omega_j(\varepsilon), \Omega_k(\varepsilon)) \in \mathbb{R}^{N+2}, \quad \forall j,k \in \mathbb{N}_0 \setminus \mathbb{S}\,,\ j \neq k\,, \quad (\vec{\omega}(\varepsilon), \varepsilon^2) \in \mathbb{R}^{N+1}, \end{aligned} \tag{3.6}$$

where $\mathbb{S}$ *is defined in* (1.31)*, are non-degenerate.*

Proof. Let us consider $\vec{\omega}(\varepsilon)\,,\ (\vec{\omega}(\varepsilon), \Omega_j(\varepsilon))\,, j \in \mathbb{N}_0 \setminus \mathbb{S}\,,\ (\vec{\omega}(\varepsilon), \Omega_j(\varepsilon), \Omega_k(\varepsilon))\,, j,k \in \mathbb{N}_0 \setminus \mathbb{S}, j \neq k$. By (3.3) we have that $\Omega_j(\varepsilon) := \omega_j(\varepsilon)\,,\ j \in \mathbb{N}_0 \setminus \mathbb{S}$, hence we can rewrite the vector above as follows

$$\vec{\omega}(\varepsilon)\,,\ (\vec{\omega}(\varepsilon), \omega_j(\varepsilon))\,, j \in \mathbb{N}_0 \setminus \mathbb{S}\,,\ (\vec{\omega}(\varepsilon), \omega_j(\varepsilon), \omega_k(\varepsilon))\,, j,k \in \mathbb{N}_0 \setminus \mathbb{S}, j \neq k.$$

There exist s Taylor coefficients $g_{2n} \neq 0$ of the analytic function ω_1, say $g_{2n_1}, ..., g_{2n_s}$ with $2n_1 < ... < 2n_s$ and $s = N$, $N{+}1$, $N{+}2$. Suppose, by contradiction, that the function $[\varepsilon_1, \varepsilon_2] \ni \varepsilon \mapsto (\omega_{j_1}(\varepsilon), ..., \omega_{j_s}(\varepsilon))$ with $j_1, .., j_s \geq 0\,,\ j_i \neq j_{i'}$ for all $i \neq i'$ is degenerate (according to Definition 12). Hence there exists $c \in \mathbb{R}^s \setminus \{0\}$ such that

$$c_1\omega_{j_1}(\varepsilon) + ... + c_s\omega_{j_s}(\varepsilon) = 0, \quad \forall \varepsilon \in (-\delta/j_s, +\infty), \quad \text{with} \quad s = N, N+1, N+2$$

where the function $[\varepsilon_1, \varepsilon_2] \ni \varepsilon \to c_1\omega_{j_1}(\varepsilon) + ... + c_s\omega_{j_s}(\varepsilon)$ is analytic. Hence we differentiate with respect to ε the identity $c_1\omega_{j_1}(\varepsilon) + ... + c_s\omega_{j_s}(\varepsilon) = 0$ and we find

$$\begin{cases} c_1\left(D_\varepsilon^{(2n_1)}\omega_{j_1}\right)(\varepsilon) + ... + c_s\left(D_\varepsilon^{(2n_1)}\omega_{j_s}\right)(\varepsilon) &= 0 \\ ... \\ c_1\left(D_\varepsilon^{(2n_s)}\omega_{j_1}\right)(\varepsilon) + ... + c_s\left(D_\varepsilon^{(2n_s)}\omega_{j_s}\right)(\varepsilon) &= 0\,. \end{cases}$$

Hence the $s \times s$ matrix

$$\mathcal{A}(\varepsilon) := \begin{pmatrix} \left(D_\varepsilon^{(2n_1)}\omega_{j_1}\right)(\varepsilon) & \left(D_\varepsilon^{(2n_1)}\omega_{j_2}\right)(\varepsilon) & ... & \left(D_\varepsilon^{(2n_1)}\omega_{j_s}\right)(\varepsilon) \\ \left(D_\varepsilon^{(2n_2)}\omega_{j_1}\right)(\varepsilon) & \left(D_\varepsilon^{(2n_2)}\omega_{j_2}\right)(\varepsilon) & ... & \left(D_\varepsilon^{(2n_2)}\omega_{j_s}\right)(\varepsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \left(D_\varepsilon^{(2n_s)}\omega_{j_1}\right)(\varepsilon) & \left(D_\varepsilon^{(2n_s)}\omega_{j_2}\right)(\varepsilon) & ... & \left(D_\varepsilon^{(2n_s)}\omega_{j_s}\right)(\varepsilon) \end{pmatrix}$$

is singular for all $\varepsilon \in (-\delta/j_s, \infty)$ therefore the analytic function

$$\det \mathcal{A}(\varepsilon) = 0, \quad \forall \varepsilon \in (-\delta/j_s, \infty). \tag{3.7}$$

In particular at $\varepsilon = 0$ we have $\det \mathcal{A}(0) = 0$. By (3.5) we can compute such determinant as

$$\det \mathcal{A}(0) := \det \begin{pmatrix} g_{2n_1} j_1^{2n_1+1} & ... & g_{2n_1} j_s^{2n_1+1} \\ g_{2n_2} j_1^{2n_2+1} & ... & g_{2n_2} j_s^{2n_2+1} \\ \vdots & \ddots & \vdots \\ g_{2n_s} j_1^{2n_s+1} & ... & g_{2n_s} j_s^{2n_s+1} \end{pmatrix} = g_{2n_1} ... g_{2n_s} \det \begin{pmatrix} j_1^{2n_1+1} & ... & j_s^{2n_1+1} \\ j_1^{2n_2+1} & ... & j_s^{2n_2+1} \\ \vdots & \ddots & \vdots \\ j_1^{2n_s+1} & ... & j_s^{2n_s+1} \end{pmatrix}.$$

This is the generalized Vandermonde determinant, we have $1 \le j_1 < j_2 < ... < j_s$ and the exponents $\alpha_j := 2n_j + 1$ are increasing, then

$$\det \begin{pmatrix} j_1^{2n_1+1} & ... & j_s^{2n_1+1} \\ j_1^{2n_2+1} & ... & j_s^{2n_2+1} \\ \vdots & \ddots & \vdots \\ j_1^{2n_s+1} & ... & j_s^{2n_s+1} \end{pmatrix} = \det \begin{pmatrix} j_1^{\alpha_1} & ... & j_s^{\alpha_1} \\ j_1^{\alpha_2} & ... & j_s^{\alpha_2} \\ \vdots & \ddots & \vdots \\ j_1^{\alpha_s} & ... & j_s^{\alpha_s} \end{pmatrix} > 0$$

see [61]. Since the Taylor coefficients $g_{2n_1}, ..., g_{2n_s} \neq 0$, we obtain that $\det \mathcal{A}(0) \neq 0$. This is in contradiction with (3.7).

Now we prove that $(\vec{\omega}(\varepsilon), \varepsilon^2) \in \mathbb{R}^{N+1}$ is non degenerate.

As before, suppose, by contradiction, that there exists $c = (c_1, ..., c_N, 1) \in \mathbb{R}^{N+1} \setminus \{0\}$ such that

$$c_1\omega_{j_1}(\varepsilon) + ... + c_N\omega_{j_s}(\varepsilon) + \varepsilon^2 = 0, \quad \forall \varepsilon \in (-\delta/j_N, +\infty),$$

where the function $[\varepsilon_1, \varepsilon_2] \ni \varepsilon \to c_1\omega_{j_1}(\varepsilon)+...+c_N\omega_{j_N}(\varepsilon)+\varepsilon^2$ is analytic. There exist N Taylor coefficients $g_{2n} \neq 0$ of the analytic function ω_1, say $g_{2n_1}, ..., g_{2n_N}$ with $2n_1 < ... < 2n_N$. Hence we differentiate with respect to ε the identity above and we find the $(N+1) \times (N+1)$- matrix

$$\mathcal{B}(\varepsilon) := \begin{pmatrix} \left(D_\varepsilon^{(2)}\omega_{j_1}\right)(\varepsilon) & \left(D_\varepsilon^{(2)}\omega_{j_2}\right)(\varepsilon) & ... & 2 \\ \left(D_\varepsilon^{(2n_1)}\omega_{j_1}\right)(\varepsilon) & \left(D_\varepsilon^{(2n_1)}\omega_{j_2}\right)(\varepsilon) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(D_\varepsilon^{(2n_N)}\omega_{j_1}\right)(\varepsilon) & \left(D_\varepsilon^{(2n_N)}\omega_{j_2}\right)(\varepsilon) & ... & 0 \end{pmatrix}$$

is singular for all $\varepsilon \in (-\delta/j_N, \infty)$ and so the analytic function $\det \mathcal{B}(\varepsilon) = 0$ for all $\varepsilon \in (-\delta/j_N, \infty)$, hence in $\varepsilon = 0$ we obtain

$$\det \mathcal{B}(0) = 2 \det \mathcal{A}(0) = 0\,. \tag{3.8}$$

By (3.5) we can compute such determinant as

$$\det \mathcal{B}(0) = 2g_{2n_1}...g_{2n_s} \det \begin{pmatrix} j_1^{2n_1+1} & ... & j_{s-1}^{2n_1+1} & j_s^{2n_1+1} \\ j_1^{2n_2+1} & ... & j_{s-1}^{2n_2+1} & j_s^{2n_2+1} \\ \vdots & \ddots & \vdots & \\ j_1^{2n_s+1} & ... & j_{s-1}^{2n_s+1} & j_s^{2n_s+1} \end{pmatrix} = 2 \det \mathcal{A}(0)\,.$$

As before, this is the generalized Vandermonde determinant, therefore is different from zero, in contradiction with (3.8). □

By Lemma 3.1 we can prove Lemmas 3.2, 3.4 and 3.5 below that we shall use in Chapter 4.

Lemma 3.2. *Let $\vec{\omega}(\varepsilon)$ as in (3.2). Then $\exists \rho_0 > 0, k_0 \in \mathbb{N}$ such that $\forall \varepsilon \in [\varepsilon_1, \varepsilon_2]$*

$$\max_{k \le k_0} \left|\partial_\varepsilon^k (\vec{\omega}(\varepsilon) \cdot l)\right| \ge \rho_0 \langle l \rangle \qquad \forall l \in \mathbb{Z}^N \setminus \{0\}\,. \tag{3.9}$$

Proof. Let us prove it by contradiction, i.e. $\forall \rho_0 > 0$ and $\forall k_0 \in \mathbb{N}$ there exists $l \in \mathbb{Z}^N \setminus \{0\}$ and $\varepsilon \in [\varepsilon_1, \varepsilon_2]$ such that $\max_{0 \le k \le k_0} |\partial_\varepsilon^k(\vec{\omega}(\varepsilon) \cdot l)| < \rho_0 \langle l \rangle$.

Then $\forall \lambda \in \mathbb{N}$ let $\rho_0 = \frac{1}{1+\lambda}$ there exists $l_\lambda \in \mathbb{Z}^N \setminus \{0\}, \varepsilon_\lambda \in [\varepsilon_1, \varepsilon_2]$ such that

$$\max_{0 \le k \le \lambda} \left| \partial_\varepsilon^k (\vec{\omega}(\varepsilon_\lambda) \cdot l_\lambda) \right| < \frac{1}{1+\lambda} \langle l_\lambda \rangle$$

and hence for all $k \in \mathbb{N}$, $\lambda \ge k$

$$\left| \partial_\varepsilon^k \left(\vec{\omega}(\varepsilon_\lambda) \cdot \frac{l_\lambda}{\langle l_\lambda \rangle} \right) \right| < \frac{1}{1+\lambda} \,. \tag{3.10}$$

The sequences $(\varepsilon_\lambda)_\lambda \in [\varepsilon_1, \varepsilon_2]$ and $\frac{l_\lambda}{\langle l_\lambda \rangle} \in \mathbb{R}^N$ are bounded, and by compactness there exists a subsequence $\lambda_r \to \infty$ such that $\varepsilon_{\lambda_r} \to \bar{\varepsilon} \in [\varepsilon_1, \varepsilon_2]$ and $\frac{l_{\lambda_r}}{\langle l_{\lambda_r} \rangle} \to \bar{c} \in \mathbb{R}^N \setminus \{0\}$.

Passing to the limit in (3.10) we obtain that

$$\left| \partial_\varepsilon^k \left(\vec{\omega}(\varepsilon_{\lambda_r}) \cdot \frac{l_{\lambda_r}}{\langle l_{\lambda_r} \rangle} \right) \right| < \frac{1}{1+\lambda_r} \to \left| \partial_\varepsilon^k \left(\vec{\omega}(\bar{\varepsilon}) \cdot \bar{c} \right) \right| = 0 \,, \quad \forall k \in \mathbb{N} \,.$$

Hence the analytic function $\varepsilon \mapsto \vec{\omega}(\varepsilon) \cdot \bar{c}$ is identically zero. Since $\bar{c} \neq 0$ this is in contradiction with the non degeneracy condition (3.6). □

In the following Lemma we divide the normal frequency in a suitable way and we will use this result in Lemmas 3.4 and 3.5.

Lemma 3.3. *Let Ω_j as in* (3.3)*, with $j \in \mathbb{N}_0 \setminus \mathbb{S}$. We can expand Ω_j as*

$$\Omega_j(\varepsilon) = \sqrt{\frac{2}{15}} \varepsilon^2 j^3 + r_j(\varepsilon) \quad \textit{where} \quad r_j(\varepsilon) = \frac{1}{1 + \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}}} \left(\sqrt{\frac{15}{2}} \frac{1}{\varepsilon^2 j^2} - \sqrt{\frac{5}{6}} \right) j.$$

Then

$$|r_j(\varepsilon)| \le C|j| \,, \quad \textit{and} \quad \sup_{j \in \mathbb{N}_0 \setminus \mathbb{S}} |\partial_\varepsilon^k r_j(\varepsilon)| \le C(k) j^{-1} \,, \quad \forall k \in \mathbb{N}_0.$$

Proof. We prove that the decomposition above holds, $\forall j \in \mathbb{N}_0$

$$\begin{aligned}
\sqrt{\frac{2}{15}} \varepsilon^2 j^3 + r_j(\varepsilon) &= \frac{\sqrt{\frac{2}{15}}\varepsilon^2 j^3 + \sqrt{\frac{2}{15}}\varepsilon^2 j^3 \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}} + \sqrt{\frac{15}{2}}\frac{j}{\varepsilon^2 j^2} - \sqrt{\frac{5}{6}} j}{1 + \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}}} \\
&= \frac{\sqrt{\frac{2}{15}}\varepsilon^2 j^3 \left(1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}\right) + \sqrt{\frac{2}{15}}\varepsilon^2 j^3 \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}}}{1 + \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}}} \\
&= \frac{\sqrt{\frac{2}{15}}\varepsilon^2 j^3 \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}} \left(1 + \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}}\right)}{1 + \sqrt{1 - \frac{5}{2}\frac{1}{j^2\varepsilon^2} + \frac{15}{2}\frac{1}{j^4\varepsilon^4}}} \\
&= \omega_j(\varepsilon).
\end{aligned}$$

We now compute the derivative of the remainder r_j. Notice that

$$|r_j| \le C|j|,$$

and $\forall k \geq 1, \forall \varepsilon \in [\varepsilon_1, \varepsilon_2]$

$$\partial_\varepsilon^k r_j(\varepsilon) \sim C(k) j^{-1}.$$

Hence

$$|\partial_\varepsilon^k r_j(\varepsilon)| \leq C(k)$$

uniformly in $j \in \mathbb{N}_0 \setminus \mathbb{S}$, for all $k \geq 1$ and $\varepsilon \in [\varepsilon_1, \varepsilon_2]$.

□

Lemma 3.4. *Let $\vec{\omega}(\varepsilon)$ as in (3.2), and Ω_j as in (3.3). Then $\exists \rho_0 > 0$, and $k_0 \in \mathbb{N}$ such that $\forall \varepsilon \in [\varepsilon_1, \varepsilon_2]$*

$$\max_{0 \leq k \leq k_0} \left| \partial_\varepsilon^k (\vec{\omega}(\varepsilon) \cdot l + \Omega_j(\varepsilon)) \right| \geq \rho_0 \langle l \rangle \qquad \forall l \in \mathbb{Z}^N, \forall j \in \mathbb{N}_0 \setminus \mathbb{S}\,. \tag{3.11}$$

Proof. We prove this lemma by contradiction. Suppose that for all $\rho_0 > 0, k_0 \in \mathbb{N}$, there exist $l \in \mathbb{Z}^N, j \in \mathbb{N}_0 \setminus \mathbb{S}$ and $\varepsilon \in [\varepsilon_1, \varepsilon_2]$ such that

$$\max_{0 \leq k \leq k_0} \left| \partial_\varepsilon^k (\vec{\omega}(\varepsilon) \cdot l + \Omega_j(\varepsilon)) \right| < \rho_0 \langle l \rangle.$$

Note that if $j^3 > C|l|$ then there is no small divisor problem, indeed $\forall \varepsilon \in [\varepsilon_1, \varepsilon_2]$ we have $|\vec{\omega}(\varepsilon) \cdot l + \Omega_j(\varepsilon)| \geq \Omega_j(\varepsilon) - |\vec{\omega}(\varepsilon)||l| \geq \varepsilon_1^2 j^3 - C|l| \geq |l|$ if $j^3 \geq C_0 |l|$, for some constant $C_0 > 0$.

Therefore we can restrict our attention to the indices $(l, j) \in \mathbb{Z}^N \times (\mathbb{N}_0 \setminus \mathbb{S})$ such that

$$j^3 \leq C_0 |l|. \tag{3.12}$$

We can suppose that for all $\lambda \in \mathbb{N}$ there exist $\varepsilon_\lambda \in [\varepsilon_1, \varepsilon_2], l_\lambda \in \mathbb{Z}^N, j_\lambda \in \mathbb{N}_0 \setminus \mathbb{S}$ such that

$$\max_{0 \leq k \leq \lambda} \left| \partial_\varepsilon^k (\vec{\omega}(\varepsilon_\lambda) \cdot l_\lambda + \Omega_{j_\lambda}(\varepsilon_\lambda)) \right| < \frac{1}{1+\lambda} \langle l_\lambda \rangle\,.$$

Hence

$$\forall k \in \mathbb{N},\ \lambda \geq k\,, \qquad \left| \partial_\varepsilon^k \left(\vec{\omega}(\varepsilon_\lambda) \cdot \frac{l_\lambda}{\langle l_\lambda \rangle} + \frac{\Omega_{j_\lambda}(\varepsilon_\lambda)}{\langle l_\lambda \rangle} \right) \right| < \frac{1}{1+\lambda}\,. \tag{3.13}$$

The sequences $(\varepsilon_\lambda)_\lambda \in [\varepsilon_1, \varepsilon_2]$ and $\frac{l_\lambda}{\langle l_\lambda \rangle}_{\lambda \in \mathbb{N}} \in \mathbb{R}^N$ are bounded, and by compactness there exists a subsequence $\lambda_r \to \infty$ such that

$$\varepsilon_{\lambda_r} \to \bar{\varepsilon} \in [\varepsilon_1, \varepsilon_2]\,, \qquad \frac{l_{\lambda_r}}{\langle l_{\lambda_r} \rangle} \to \vec{c} \in \mathbb{R}^N\,. \tag{3.14}$$

We have to consider two different cases, if $|l_\lambda|$ is bounded or not.

Case 1: $|l_{\lambda_r}| < c$, then $l_{\lambda_r} \to \bar{l} \in \mathbb{Z}^N$ and have that $|j_\lambda|^3 \leq C|l_\lambda| \leq c$ (see (3.12)) for all λ, hence $j_{\lambda_r} \to \bar{j}$. We consider the limit with $\lambda_r \to \infty$ hence we have that

$$\max_{0 \leq k \leq \lambda} \left| \partial_\varepsilon^k \left(\vec{\omega}(\varepsilon_\lambda) \cdot \frac{l_\lambda}{\langle l_\lambda \rangle} + \frac{1}{\langle l_\lambda \rangle} \Omega_{j_\lambda}(\varepsilon_\lambda) \right) \right| < \frac{1}{1+\lambda} \to \max_{0 \leq k \leq \lambda} \left| \partial_\varepsilon^k \left(\vec{\omega}(\bar{\varepsilon}) \cdot \frac{\bar{l}}{\langle \bar{l} \rangle} + \frac{1}{\langle \bar{l} \rangle} \Omega_{\bar{j}}(\bar{\varepsilon}) \right) \right| = 0$$

therefore, if $d := \frac{1}{\langle \bar{l} \rangle}$,

$$\partial_\varepsilon^k \left(\vec{\omega}(\varepsilon) \cdot \vec{c} + \Omega_j(\varepsilon) d \right) = 0 \quad \forall k \in \mathbb{N} \quad \text{and} \quad d \in \mathbb{R} \setminus \{0\}\,.$$

Hence the function $[\varepsilon_1, \varepsilon_2] \ni \varepsilon \to \vec{\omega}(\varepsilon)\cdot\vec{c}+\Omega_j(\varepsilon)d$ is identically zero. Since $(\vec{c}, d) \neq 0$ this is in contradiction with the non degeneracy condition (3.6).

Case 2: $|l_{\lambda_r}|$ unbounded. By $|j|^3 \leq c\langle l\rangle$, if we consider the limit with $\lambda \to \infty$ we have that

$$\sqrt{\frac{2}{15}}\frac{j^3_{\lambda_r}}{\langle l_{\lambda_r}\rangle} \to \bar{d} \in \mathbb{R} \setminus \{0\} \quad \varepsilon_{\lambda_r} \to \bar{\varepsilon} \in [\varepsilon_1, \varepsilon_2], \quad \frac{l_{\lambda_r}}{\langle l_{\lambda_r}\rangle} \to \vec{c} \in \mathbb{R}^N.$$

By Lemma 3.3 and (3.12) we have

$$\frac{\Omega_{\lambda_r}(\varepsilon_{\lambda_r})}{\langle l_{\lambda_r}\rangle} = \sqrt{\frac{2}{15}}\varepsilon^2_{\lambda_r}\frac{j^3_{\lambda_r}}{\langle l_{\lambda_r}\rangle} + \frac{r_{j_{\lambda_r}}(\varepsilon_{\lambda_r})}{\langle l_{\lambda_r}\rangle} \to \bar{d}\bar{\varepsilon}^2, \qquad \partial^k_\varepsilon \frac{\Omega_{\lambda_r}(\varepsilon_{\lambda_r})}{\langle l_{\lambda_r}\rangle} \to \bar{d}\partial^k_\varepsilon \bar{\varepsilon}^2, \ \forall\ k \geq 0\,.$$

Then, passing to the limit in (3.13) we obtain

$$\left|\partial^k_\varepsilon\left(\vec{\omega}(\varepsilon_\lambda)\cdot\frac{l_\lambda}{\langle l_\lambda\rangle} + \frac{1}{\langle l_\lambda\rangle}\Omega_{j_\lambda}(\varepsilon_\lambda)\right)\right| < \frac{1}{1+\lambda} \to \left|\partial^k_\varepsilon(\vec{\omega}(\bar{\varepsilon})\cdot\vec{c} + \bar{d}\bar{\varepsilon}^2)\right| = 0\,.$$

Therefore the analytic function $\varepsilon \to \vec{\omega}(\varepsilon)\cdot\vec{c} + \bar{d}\varepsilon^2$ is identically zero, in contradiction with the non degeneracy condition (3.6). □

Lemma 3.5. *Let $\vec{\omega}(\varepsilon)$ as in (3.2), and $\Omega_{j'}(\varepsilon), \Omega_j(\varepsilon)$ as in (3.3). Then $\exists \rho_0 > 0$ and $k_0 \in \mathbb{N}$ such that $\forall \varepsilon \in [\varepsilon_1, \varepsilon_2]$,*

$$\max_{0\leq k\leq k_0}\left|\partial^k_\varepsilon[\vec{\omega}(\varepsilon)\cdot l + \Omega_j(\varepsilon) - \Omega_{j'}(\varepsilon)]\right| \geq \rho_0\langle l\rangle\,, \quad \forall (l, j', j) \neq (0, j, j),\ l \in \mathbb{Z}^N\,,\ j, j' \in \mathbb{N}_0 \setminus \mathbb{S} \tag{3.15}$$

$$\max_{0\leq k\leq k_0}\left|\partial^k_\varepsilon[\vec{\omega}(\varepsilon)\cdot l + \Omega_j(\varepsilon) + \Omega_{j'}(\varepsilon)]\right| \geq \rho_0\langle l\rangle\,, \quad \forall l \in \mathbb{Z}^N\,,\ j, j' \in \mathbb{N}_0 \setminus \mathbb{S}\,. \tag{3.16}$$

Proof. We prove the lemma for $\vec{\omega}(\varepsilon)\cdot l+\Omega_j(\varepsilon)-\Omega_{j'}(\varepsilon)$ since the proof of the other is similar. The Lemma is proved by contradiction. Note that if $|j^3-j'^3| \geq C\langle l\rangle$ the non resonant condition $|\vec{\omega}(\varepsilon)\cdot l+\Omega_j(\varepsilon)-\Omega_{j'}(\varepsilon)| \geq \rho_0\langle l\rangle$ is satisfied, indeed, $\forall \varepsilon \in [\varepsilon_1, \varepsilon_2]$,

$$\begin{aligned}|\vec{\omega(\varepsilon)}\cdot l + \Omega_j(\varepsilon) - \Omega_{j'}(\varepsilon)| &\geq |\Omega_j(\varepsilon) - \Omega_{j'}(\varepsilon)| - |\vec{\omega}(\varepsilon)||l| \\ &\geq \sqrt{\frac{2}{15}}\varepsilon^2|j^3 - j'^3| - C|j-j'| - C|l| \\ &\geq C_1\varepsilon^2|j^3 - j'^3| - C_1|l| \geq \langle l\rangle\,, \qquad \text{if} \quad |j^3 - j'^3| \geq \tilde{C}\langle l\rangle\,,\end{aligned}$$

for some $\tilde{C} > 0$, where the second inequality follows by (3.3). Therefore we can restrict to the indices such that

$$|j^3 - j'^3| < C\langle l\rangle. \tag{3.17}$$

We can also assume $j' \neq j$, otherwise (3.15) reduces to (3.9). Suppose that for all $\lambda \in \mathbb{N}$ there exists $l_\lambda \in \mathbb{Z}^N, j_\lambda, j'_\lambda \in \mathbb{N}_0 \setminus \mathbb{S}, j'_\lambda \neq j_\lambda, \varepsilon_\lambda \in [\varepsilon_1, \varepsilon_2]$ such that for all $k \in \mathbb{N}, \forall \lambda \geq k$

$$\left|\partial^k_\varepsilon\left(\vec{\omega}(\varepsilon_\lambda)\cdot\frac{l_\lambda}{\langle l_\lambda\rangle} + \frac{\Omega_{j_\lambda}(\varepsilon_\lambda)}{\langle l_\lambda\rangle} - \frac{\Omega_{j'_\lambda}(\varepsilon_\lambda)}{\langle l_\lambda\rangle}\right)\right| < \frac{1}{1+\lambda}\,. \tag{3.18}$$

Since the sequences $(\varepsilon_\lambda)_{\lambda\in\mathbb{N}}, \left(\frac{l_\lambda}{\langle l_\lambda\rangle}\right)_{\lambda\in\mathbb{N}}$ are bounded, there exists $\lambda_r \to \infty$ such that

$$\varepsilon_{\lambda_r} \mapsto \bar{\varepsilon} \in [\varepsilon_1, \varepsilon_2], \qquad \frac{l_{\lambda_r}}{\langle l_{\lambda_r}\rangle} \mapsto \vec{c} \in \mathbb{R}^N\,. \tag{3.19}$$

We have to consider two cases:

Case 1: (l_{λ_r}) is bounded. Then $(l_{\lambda_r}) \to \bar{l} \in \mathbb{Z}^N$, by $|j^3 - j'^3| < C\langle l \rangle$ we can say that also $j_{\lambda_r}, j'_{\lambda_r}$ are bounded, indeed

$$|j^3 - j'^3| \geq |j - j'|(j + j') \geq (j + j')\,, \quad \forall j \neq j'$$

therefore

$$j_{\lambda_r} \to \bar{j}\,, \qquad j'_{\lambda_r} \to \bar{j}'. \tag{3.20}$$

Hence, passing to the limit in (3.18) for $\lambda_r \to \infty$, by (3.19) and (3.20) we deduce that

$$\forall k \in \mathbb{N}\,,\ \lambda \geq k\,, \qquad \partial_\varepsilon^k \left(\vec{\omega}(\bar{\varepsilon}) \cdot \vec{c} + \frac{\Omega_{\bar{j}}(\bar{\varepsilon})}{\langle \bar{l} \rangle} - \frac{\Omega_{\bar{j}'}(\bar{\varepsilon})}{\langle \bar{l} \rangle} \right) = 0\,.$$

Therefore the analytic function $\varepsilon \mapsto \vec{\omega}(\varepsilon) \cdot \vec{c} + \frac{\Omega_{\bar{j}}(\varepsilon)}{\langle l \rangle} - \frac{\Omega_{\bar{j}'}(\varepsilon)}{\langle l \rangle}$ is identically zero, in contradiction with the non degeneracy condition (3.6).

Case 2: (l_{λ_r}) is unbounded. We have $\Omega_{j_{\lambda_r}} - \Omega_{j'_{\lambda_r}} \simeq \sqrt{\frac{2}{15}}\varepsilon^2_{\lambda_r}|j^3_{\lambda_r} - j'^3_{\lambda_r}|$ and $\frac{\sqrt{\frac{2}{15}}|j^3_{\lambda_r} - j'^3_{\lambda_r}|}{\langle l_{\lambda_r} \rangle} \to \bar{d} \in \mathbb{R}$ indeed, from Lemma 3.3, $\forall k \in \mathbb{N}$

$$\partial_\varepsilon^k \frac{\Omega_{j_{\lambda_r}}(\varepsilon_{\lambda_r}) - \Omega_{j'_{\lambda_r}}(\varepsilon_{\lambda_r})}{\langle l_{\lambda_r} \rangle} = \sqrt{\frac{2}{15}} \partial_\varepsilon^k \varepsilon^2_{\lambda_r} \frac{j^3_{\lambda_r} - j'^3_{\lambda_r}}{\langle l_{\lambda_r} \rangle} + \frac{1}{\langle l_{\lambda_r} \rangle} \partial_\varepsilon^k \left(r_{j_{\lambda_r}}(\varepsilon_{\lambda_r}) \right) - \frac{1}{\langle l_{\lambda_r} \rangle} \partial_\varepsilon^k \left(r_{j'_{\lambda_r}}(\varepsilon_{\lambda_r}) \right)$$

and $\forall k \geq 1$

$$\begin{aligned} \left| \frac{1}{\langle l_{\lambda_r} \rangle} \partial_\varepsilon^k \left(r_{j_{\lambda_r}}(\varepsilon_{\lambda_r}) j_{\lambda_r} \varepsilon_{\lambda_r} \right) - \frac{1}{\langle l_{\lambda_r} \rangle} \partial_\varepsilon^k \left(r_{j'_{\lambda_r}}(\varepsilon_{\lambda_r}) i_{\lambda_r} \varepsilon_{\lambda_r} \right) \right| &\leq \frac{C}{\langle l_{\lambda_r} \rangle} \sup_{\varepsilon \in [\varepsilon_1, \varepsilon_2], j \in \mathbb{N}} \left| \partial_\varepsilon^k r_j(\varepsilon) \right| \\ &\leq \frac{C(k)}{\langle l_{\lambda_r} \rangle} \xrightarrow{\lambda_r \to \infty} 0\,. \end{aligned}$$

If $k = 0$ we have that $\varepsilon^2 j^3 + j < C\varepsilon^2 j^3$, therefore this part is controlled by the principal term. Hence

$$\partial_\varepsilon^k \frac{\Omega_{j_{\lambda_r}} - \Omega_{j'_{\lambda_r}}}{\langle l_{\lambda_r} \rangle} \to \bar{d} \partial_\varepsilon^k \bar{\varepsilon}^2, \quad \bar{d} \neq 0.$$

Passing to the limit in (3.18) for $\lambda_r \to \infty$ we deduce that

$$\forall k \in \mathbb{N} \qquad \partial_\varepsilon^k \left(\vec{\omega}(\bar{\varepsilon}) \cdot \vec{c} + \bar{d}\bar{\varepsilon}^2 \right) = 0.$$

Hence the analytic function $\varepsilon \mapsto \vec{\omega}(\varepsilon) \cdot \vec{c} + \bar{d}\varepsilon^2$ is identically zero, in contradiction with the non degeneracy condition (3.6). □

Remark 3.6. *We take as ρ_0 the smallest ρ_0 provided by Lemmas 3.2, 3.4, 3.5 . Moreover we take as k_0 the largest among the k_0 provided by Lemmas 3.2, 3.4, 3.5 and it is the so called "index of non-degeneracy".*

Chapter 4

Nash-Moser theorem and Measure estimates

In Chapter 1, after the introduction of the action-angle variables we arrived to the Hamiltonian H_μ defined in (1.44), that admits the reversible structure defined in (1.42). We look for an embedded invariant torus

$$i : \mathbb{T}^N \to \mathbb{T}^N \times \mathbb{R}^N \times \mathtt{H}_{\mathbb{S}}^{\perp}, \qquad \theta \mapsto i(\theta) = (\vartheta(\theta), I(\theta), w(\theta)) \tag{4.1}$$

of the Hamiltonian vector field $(\partial_I H_\mu, -\partial_\vartheta H_\mu, -J\nabla_w H_\mu)$ defined in (1.40) filled by quasi periodic solutions with Diophantine frequency $\omega \in \mathbb{R}^N$ which satisfies also the Melnikov non resonance conditions defined in (4.10).

4.1 Nash-Moser theorem

The Hamiltonian H_μ in (1.44) is a perturbation of the Hamiltonian $\mathcal{N}$. The quasi-periodic solutions of the Hamiltonian system (1.40) will have a shifted frequency which depends on the non linear term P. As in [19] we embed H_μ into the family of Hamiltonians

$$H_\alpha = \mathcal{N}_\alpha + \mu P, \qquad \mathcal{N}_\alpha = \alpha \cdot I + \frac{1}{2}\,(w, \mathbf{D}w)_{L^2_x} \qquad \alpha \in \mathbb{R}^N\,, \tag{4.2}$$

where $\mathbf{D}$ is defined in (1.45). The family H_α depends on the parameter α and for the value $\alpha = \vec{\omega}(\varepsilon)$, defined in (3.2), we have $H_\alpha = H_\mu$.

Then we look for a zero (i, α) of the non linear operator

$$\mathcal{F}(i, \alpha) = \mathcal{F}(i, \alpha, \omega, \mu) = \omega \cdot \partial_\theta i(\theta) - X_{H_\alpha} = \omega \cdot \partial_\theta i(\theta) - (X_{\mathcal{N}_\alpha} + \mu X_P)(i(\theta)), \tag{4.3}$$

that is explicitely given by

$$\mathcal{F}(i,\alpha,\omega,\mu) = \begin{pmatrix} \omega\cdot\partial_\theta\vartheta(\theta) - \alpha - \mu\partial_I P(i(\theta)) \\ \omega\cdot\partial_\theta I(\theta) + \mu\partial_\theta P(i(\theta)) \\ \omega\cdot\partial_\theta w(\theta) + J(\mathbf{D}w + \mu\nabla_w P(i(\theta))) \end{pmatrix} \tag{4.4}$$

for some Diophantine vector $\omega \in \mathbb{R}^N$. Thus $\theta \mapsto i(\theta)$ is an embedded torus, invariant for the vector field generated by the Hamiltonian H_μ, filled by quasi-periodic solutions with Diophantine frequency ω. Note that each Hamiltonian in (4.2) is reversible, that is $H_\alpha \circ \tilde\rho = H_\alpha$ where $\tilde\rho$ is the involution defined in (1.42). Then it is natural to look for reversible solutions of $\mathcal{F}(i,\alpha)=0$, namely satisfying $\tilde\rho\circ i(\theta) = i(-\theta)$, that is exactly the condition given in (1.43).

The Sobolev norm of the periodic component of the embedded torus

$$\mathcal{V}(\theta) := i(\theta) - (\theta,0,0) = (\Theta(\theta), I(\theta), w(\theta)), \quad \Theta(\theta) = \vartheta(\theta) - \theta$$

is

$$\|\mathcal{V}\|_p := \|\Theta\|_{H^p_\theta} + \|I\|_{H^p_\theta} + \|w\|_p\,, \tag{4.5}$$

where $\|w\|_p := \|w\|_{\mathbf{H}^p_{\theta,x}} = \max\{\|q\|_p,\, \|p\|_p\}$ is defined in (2.2) and $\|\cdot\|_{H^p_\theta}$ is defined in (2.4).

We look for quasi periodic solutions with frequency ω belonging to a δ-neighborhood (independent of μ)

$$\Omega := \left\{\omega \in \mathbb{R}^N : \mathrm{dist}(\omega, \vec\omega([\varepsilon_1,\varepsilon_2])) \le \delta,\ \delta > 0\right\}$$

of the unperturbed linear frequencies $\vec\omega(\varepsilon)$ for $\varepsilon \in [\varepsilon_1,\varepsilon_2]$ defined in (3.2).

Let $\mathbb{N}_0 := \mathbb{N}\setminus\{0\}$. Recall that $\mathbb{S}$ is defined in (1.31), the norm $|\cdot|^{k_0,\gamma}$ is defined in (2.7) and the norm $\|\cdot\|_p^{k_0,\gamma}$ is defined in (2.9).

Theorem 4.1. *Fix finitely many tangential sites $\mathbb{S}\subset\mathbb{N}_0$, and let N be the cardinality of $\mathbb{S}$. Let $\tau \ge 1$. There exist constants $\mu_0 > 0, a_0 := a_0(N,\tau,k_0) > 0$, and $k_1 := k_1(N,\tau,k_0) > 0$ such that, for $a_0 < (1+k_1)^{-1}$ and for all $\gamma = \mu^a$, with $0 < a < a_0$ and $\mu \in (0,\mu_0)$ there exist*

- *a k_0-times differentiable function*

$$\alpha_\infty : \Omega\times[\varepsilon_1,\varepsilon_2] \to \mathbb{R}^N, \quad \alpha_\infty(\omega,\varepsilon) = \omega + r_\mu(\omega,\varepsilon), \quad \text{with} \quad |r_\mu|^{k_0,\gamma} \le C\mu\gamma^{-(1+k_1)}\,, \tag{4.6}$$

- *a family of embedded tori i_∞ defined for all $\omega\in\Omega$ and $\varepsilon\in[\varepsilon_1,\varepsilon_2]$ satisfying the reversibility property (1.43) and*

$$\|i_\infty(\theta) - (\theta,0,0)\|_{\mathfrak{p}_0}^{k_0,\gamma} \le C\mu\gamma^{-(1+k_1)}\,, \tag{4.7}$$

- *a sequence of k_0-times differentiable functions $\lambda_j^\infty : \Omega\times[\varepsilon_1,\varepsilon_2]\to\mathbb{R}$, $j\in\mathbb{N}_0\setminus\mathbb{S}$, of the form*

$$\lambda_j^\infty(\omega,\varepsilon) = j\left(\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1\right)^{\frac12} + m_1^\infty(\omega,\varepsilon)j + r_j^\infty(\omega,\varepsilon)\,, \tag{4.8}$$

where m_1^∞ , r_j are real and m_1^∞ and r_j^∞ satisfy

$$|m_1^\infty|^{k_0,\gamma} \le C\mu\,, \qquad \sup_{j\in\mathbb{N}_0\setminus\mathbb{S}} |r_j^\infty|^{k_0,\gamma} \le C\mu\gamma^{-k_1}\,, \tag{4.9}$$

such that for all (ω,ε) in the Cantor like set

$$\begin{aligned} \mathcal{C}_\infty^\gamma = \{(\omega,\varepsilon)\in\Omega\times[\varepsilon_1,\varepsilon_2] : |\omega\cdot l| \ge \gamma\langle l\rangle^{-\tau}, \quad \forall l\in\mathbb{Z}^N\setminus\{0\}, \\ |\omega\cdot l + \lambda_j^\infty(\omega,\varepsilon)| \ge 4\gamma|j|^3\langle l\rangle^{-\tau}, \quad \forall l\in\mathbb{Z}^N, \forall j\in\mathbb{N}_0\setminus\mathbb{S} \\ |\omega\cdot l + \lambda_j^\infty(\omega,\varepsilon) - \lambda_{j'}^\infty(\omega,\varepsilon)| \ge 4\gamma|j^3 - j'^3|\langle l\rangle^{-\tau}\,, \quad \forall l\in\mathbb{Z}^N\,,\ j',j\in\mathbb{N}_0\setminus\mathbb{S} \\ |\omega\cdot l + \lambda_j^\infty(\omega,\varepsilon) + \lambda_{j'}^\infty(\omega,\varepsilon)| \ge 4\gamma|j^3 + j'^3|\langle l\rangle^{-\tau}\,, \quad \forall l\in\mathbb{Z}^N\,,\ j',j\in\mathbb{N}_0\setminus\mathbb{S}\}\,, \end{aligned} \tag{4.10}$$

the function $i_\infty(\theta) = i_\infty(\omega,\varepsilon,\mu)(\theta)$ is a solution of $\mathcal{F}(i_\infty,\alpha_\infty(\omega,\varepsilon),\omega,\varepsilon,\mu) = 0$. As a consequence the embedded torus $\theta\mapsto i_\infty(\theta)$ is invariant for the Hamiltonian vector field $X_{H_{\alpha_\infty(\omega,\varepsilon)}}$, and it is filled by quasi-periodic solutions with frequency ω.

Remark 4.2. *The k_0 index appearing in Theorem 4.1 is the "index of non-degeneracy" defined in Lemmas 3.2, 3.4, and 3.5 and it depends only on the linear unperturbed frequencies.*

Theorem 4.1 above is proved in Chapter 10 using the results about the linerearized operator presented in Chapters 5-9.

4.2 Measure estimates

In this Section we want to deduce Theorem 1 by Theorem 4.1. Since a_0 (in Theorem 4.1) satisfies $a_0 < (1+k_1)^{-1}$ one has $|r_\mu|^{k_0,\gamma} \to 0$ as $\mu\to 0$ (where $|\cdot|^{k_0,\gamma}$ is defined in (2.7)) and hence for μ_0 small enough the map $\alpha_\infty(\cdot,\varepsilon) : \Omega \to \alpha_\infty(\Omega\times\{\varepsilon\})$ is invertible and moreover one has

$$\begin{aligned} \beta = \alpha_\infty(\omega,\varepsilon) = \omega + r_\mu(\omega,\varepsilon) \Leftrightarrow \omega = \alpha_\infty^{-1}(\beta,\varepsilon) = \beta + \tilde r_\mu(\beta,\varepsilon) \\ \text{with} \quad |\tilde r_\mu|^{k_0,\gamma} \le C\mu\gamma^{-(1+k_1)}\,. \end{aligned} \tag{4.11}$$

Indeed the inverse map $\beta\mapsto\alpha_\infty^{-1}(\beta,\varepsilon) = \beta + \tilde r_\mu(\beta,\varepsilon)$ satisfies the identity

$$\beta = \omega + r_\mu(\omega,\varepsilon) \Rightarrow \beta = \beta + \tilde r_\mu(\beta,\varepsilon) + r_\mu(\beta + \tilde r_\mu(\beta,\varepsilon),\varepsilon) \Rightarrow 0 = \tilde r_\mu(\beta,\varepsilon) + r_\mu(\beta+\tilde r_\mu(\beta,\varepsilon),\varepsilon)\,.$$

Thanks to the implicit function theorem $\tilde r_\mu$ is C^1 with respect to (β,ε) and it satisfies the identities

$$\begin{aligned} D_\beta \tilde r_\mu(\beta,\varepsilon) = -\left[\mathbb{1} + D_\omega r_\mu(\beta+\tilde r_\mu(\beta,\varepsilon),\varepsilon)\right]^{-1} D_\omega r_\mu(\beta+\tilde r_\mu(\beta,\varepsilon),\varepsilon) \\ \partial_\varepsilon \tilde r_\mu(\beta,\varepsilon) = -\left[\mathbb{1} + D_\omega r_\mu(\beta+\tilde r_\mu(\beta,\varepsilon),\varepsilon)\right]^{-1} \partial_\varepsilon r_\mu(\beta+\tilde r_\mu(\beta,\varepsilon),\varepsilon)\,, \end{aligned}$$

where D_β, D_ω denote the Frechet derivatives with respect to β,ω. Arguing by induction on $|k|\le k_0$ we obtain that $\tilde r_\mu$ is k_0-times differentiable and the estimate (4.11) follows as in [19].

Thanks to Theorem 4.1 the existence of an embedded invariant torus filled by quasi periodic solutions with Diophantine frequency $\omega = \alpha_\infty^{-1}(\beta, \varepsilon)$ is ensured. Indeed in Theorem 4.1 we prove the existence of solutions with frequency $\omega = \alpha_\infty^{-1}(\beta, \varepsilon)$ for the system for the Hamiltonian

$$H_\beta = \beta \cdot I + \frac{1}{2}(w, \mathbf{D}w)_{L^2_x} + \mu P.$$

Consider the curve of the unperturbed linear frequencies (defined also in (3.2))

$$[\varepsilon_1, \varepsilon_2] \ni \varepsilon \mapsto \vec{\omega}(\varepsilon) := \left(j\sqrt{\frac{2}{15}j^4\varepsilon^4 - \frac{1}{3}j^2\varepsilon^2 + 1} \right)_{j \in \mathbb{S}} \in \mathbb{R}^N.$$

We now prove that for most $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, the vector $\beta = \vec{\omega}(\varepsilon) \in \alpha_\infty(\mathcal{C}^\gamma_\infty)$ (see Lemma 4.8). Hence for such values of ε, by Theorem 4.1, we have found an embedded invariant torus for the Hamiltonian H_μ in (1.44), filled by quasi-periodic motions with Diophantine frequency $\omega = \alpha_\infty^{-1}(\vec{\omega}(\varepsilon), \varepsilon)$. This implies Theorem 1.

In the proof of Theorem 1 we have to prove that there exists a Cantor like set $\mathcal{G}$ with asymptotically full Lebesgue measure, that is exactly the condition: for most $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, the vector $\beta = \vec{\omega}(\varepsilon) \in \alpha_\infty(\mathcal{C}^\gamma_\infty)$. In what follows we prove exactly this (see Lemma 4.8).

By (4.11) we get

$$\omega_\mu(\varepsilon) = \alpha_\infty^{-1}(\vec{\omega}(\varepsilon), \varepsilon) = \vec{\omega}(\varepsilon) + \mathtt{r}_\mu(\varepsilon), \quad \mathtt{r}_\mu(\varepsilon) = \tilde{r}_\mu(\vec{\omega}(\varepsilon), \varepsilon) \tag{4.12}$$

where

$$|\partial_\varepsilon^k \mathtt{r}_\mu(\varepsilon)| \le \mu C \gamma^{-(1+k_1+k)}, \quad 0 \le k \le k_0\,. \tag{4.13}$$

We also denote

$$\begin{aligned} \lambda_j^\infty(\varepsilon) := \lambda_j^\infty(\omega_\mu(\varepsilon), \varepsilon) &:= j\sqrt{\frac{2}{15}j^4\varepsilon^4 - \frac{1}{3}j^2\varepsilon^2 + 1} + m_1^\infty(\varepsilon) j + r_j^\infty(\varepsilon), \quad \forall j \in \mathbb{N}_0 \setminus \mathbb{S} \\ &= \Omega_j(\varepsilon) + m_1^\infty(\varepsilon) j + r_j^\infty(\varepsilon) \quad \forall j \in \mathbb{N}_0 \setminus \mathbb{S} \end{aligned} \tag{4.14}$$

where $\Omega_j(\varepsilon)$ is defined in (3.3), m_1^∞ , r_j^∞ are real, and

$$m_1^\infty(\varepsilon) := m_1^\infty(\omega_\mu(\varepsilon), \varepsilon)\,, \quad r_j^\infty(\varepsilon) := r_j^\infty(\omega_\mu(\varepsilon), \varepsilon)\,. \tag{4.15}$$

By (4.9), (4.15) and (4.12), using that $\mu\gamma^{-1-k_1-k_0} \le 1$ that is satisfied for μ "small enough" (see Lemma 4.8), we get

$$|\partial_\varepsilon^k m_1^\infty(\varepsilon)| \le C\mu\gamma^{-k}, \quad \sup_{j \in \mathbb{N}_0 \setminus \mathbb{S}} |\partial_\varepsilon^k r_j^\infty(\varepsilon)| \le C\mu\gamma^{-k-k_1}, \quad \forall 0 \le k \le k_0\,. \tag{4.16}$$

We define the Cantor like set $\mathcal{G}$ in Theorem 1 as $\mathcal{G} = \mathcal{G}_\mu$, where $\mathcal{G}_\mu$ is given by

$$\begin{aligned} \mathcal{G}_\mu &:= \{\varepsilon \in [\varepsilon_1, \varepsilon_2] : \vec{\omega}(\varepsilon) \in \alpha_\infty(\mathcal{C}^\gamma_\infty)\} \\ &:= \left\{\varepsilon \in [\varepsilon_1, \varepsilon_2] : \left(\alpha_\infty^{-1}(\vec{\omega}(\varepsilon), \varepsilon), \varepsilon\right) \in \mathcal{C}^\gamma_\infty\right\}\,. \end{aligned} \tag{4.17}$$

By (4.10), (4.12) and (4.14) the set $\mathcal{G}_\mu$ can be written as

$$\begin{aligned}\mathcal{G}_\mu =&\{\varepsilon \in [\varepsilon_1, \varepsilon_2] : |\omega_\mu(\varepsilon)\cdot l| \geq \gamma \langle l\rangle^{-\tau} \quad \forall l \in \mathbb{Z}^N \setminus \{0\},\\ &|\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon)| \geq 4\gamma |j^3| \langle l\rangle^{-\tau}, \quad \forall l \in \mathbb{Z}^N, \forall j \in \mathbb{N}_0 \setminus \mathbb{S}\,,\\ &|\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon) - \lambda_{j'}^\infty(\varepsilon)| \geq 4\gamma |j^3 - j'^3| \langle l\rangle^{-\tau} \quad \forall l \in \mathbb{Z}^N, j', j \in \mathbb{N}_0 \setminus \mathbb{S}\,,\\ &|\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon) + \lambda_{j'}^\infty(\varepsilon)| \geq 4\gamma |j^3 + j'^3| \langle l\rangle^{-\tau} \quad \forall l \in \mathbb{Z}^N, j', j \in \mathbb{N}_0 \setminus \mathbb{S}\}.\end{aligned}$$

Now we prove that $\mathcal{G}_\mu$ has asymptotically full measure. We define the so called "resonant sets" as

$$\begin{aligned}\tilde{R}_l^{(0)} &:= \{\varepsilon \in [\varepsilon_1, \varepsilon_2] : |\omega_\mu(\varepsilon)\cdot l| < \gamma\langle l\rangle^{-\tau}\}\\ \tilde{R}_{l,j}^{(1)} &:= \{\varepsilon \in [\varepsilon_1, \varepsilon_2] : |\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon)| < 4\gamma\langle l\rangle^{-\tau}\}\\ \tilde{R}_{l,j,j'}^{(2)} &:= \{\varepsilon \in [\varepsilon_1, \varepsilon_2] : |\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon) - \lambda_{j'}^\infty(\varepsilon)| < 4\gamma|j^3 - j'^3|\langle l\rangle^{-\tau}\}\\ \tilde{R}_{l,j,j'}^{(3)} &:= \{\varepsilon \in [\varepsilon_1, \varepsilon_2] : |\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon) + \lambda_{j'}^\infty(\varepsilon)| < 4\gamma|j^3 + j'^3|\langle l\rangle^{-\tau}\}\,.\end{aligned} \tag{4.18}$$

Lemma 4.3. *Let $\mu\gamma^{-k_1}$ small enough. The resonant sets defined in* (4.18) *satisfy*

$$\begin{aligned}&\textit{if} \quad \tilde{R}_{l,j}^{(1)} \neq \emptyset \quad \textit{then} \quad |j|^3 \leq C\,\langle l\rangle\\ &\textit{if} \quad \tilde{R}_{l,j,j'}^{(2)} \neq \emptyset \quad \textit{then} \quad |j^3 - j'^3| \leq C\,\langle l\rangle\\ &\textit{if} \quad \tilde{R}_{l,j,j'}^{(3)} \neq \emptyset \quad \textit{then} \quad |j^3 + j'^3| \leq C\,\langle l\rangle\ .\end{aligned} \tag{4.19}$$

Proof. If $\varepsilon \in \tilde{R}_{l,j}^{(1)}$, then

$$|\lambda_j^\infty(\varepsilon)| \leq 4\gamma|j|^3\,\langle l\rangle^{-\tau} + |\omega_\mu(\varepsilon)||l| \leq 4\gamma|j|^3 + C|l| \tag{4.20}$$

by (4.14) and (4.16) we get

$$|\lambda_j^\infty(\varepsilon)| \geq |j|^3 - |m_1^\infty(\varepsilon)||j| - \sup_{j\in\mathbb{N}_0\setminus\mathbb{S}} |r_j^\infty(\varepsilon)| \geq |j|^3 - C\mu|j| - C\mu\gamma^{-k_1} \geq C_1\frac{|j|^3}{2}$$

for $2C\mu\gamma^{-k_1} \leq \frac{C_1}{2}$. Therefore if $\frac{C_1}{4} \geq 4\gamma$ then $\tilde{R}_{l,j}^{(1)} \neq \emptyset$.

If $\varepsilon \in \tilde{R}_{l,j,j'}^{(2)}$, then

$$|\lambda_j^\infty(\varepsilon) - \lambda_{j'}^\infty(\varepsilon)| \leq 4\gamma|j^3 - j'^3|\,\langle l\rangle^{-\tau} + |\omega_\mu(\varepsilon)||l| \leq 4\gamma|j^3 - j'^3| + C|l|\,. \tag{4.21}$$

As before, by (4.14) and (4.16) we get

$$\begin{aligned}|\lambda_j^\infty(\varepsilon) - \lambda_{j'}^\infty(\varepsilon)| &\geq |j^3 - j'^3| - |m_1^\infty(\varepsilon)||j - j'| - \sup_{j\in\mathbb{N}_0\setminus\mathbb{S}} |r_j^\infty(\varepsilon)|\\ &\geq |j^3 - j'^3| - C\mu|j - j'| - C\mu\gamma^{-k_1}\\ &\geq C_2\frac{|j^3 - j'^3|}{2}\end{aligned}$$

for $2C\mu\gamma^{-k_1} \leq \frac{C_2}{2}$. Therefore if $\frac{C_2}{4} \geq 4\gamma$ then $\tilde{R}_{l,j,j'}^{(2)} \neq \emptyset$. The other case follows similarly. □

Corollary 4.4. *The set* $\tilde{R}^{(2)}_{l,j,j'}$ *defined in* (4.18)*, is not empty if*

$$|j|, |j'| \leq C|l|^{1/2}\,, \quad \forall j \neq j'\,,\ j, j' \in \mathbb{N} \setminus \{0\}\,. \tag{4.22}$$

Proof. The proof follows by the condition given in (4.19) and by

$$|j^3 - j'^3| = |j - j'||j^2 + j'^2 + jj'| \geq |j^2 + j'^2 + jj'| \geq \frac{|j + j'|^2}{2}\,.$$

□

For estimate the measure of the set $\mathcal{G}_\mu$ we have to prove some Lemmas.

Lemma 4.5. *Consider* $\omega_\mu(\varepsilon)$ *defined in* (4.12)*. There exist* $k_0 \in \mathbb{N}$ *and* $\rho_0 > 0$ *such that for* μ *small enough and for all* $\varepsilon \in [\varepsilon_1, \varepsilon_2]$*,*

$$\max_{k \leq k_0} |\partial_\varepsilon^k (\omega_\mu(\varepsilon) \cdot l)| \geq \frac{1}{2}\rho_0 \langle l \rangle, \quad \forall l \in \mathbb{Z}^N \setminus \{0\}\,, \tag{4.23}$$

$$\max_{k \leq k_0} |\partial_\varepsilon^k (\omega_\mu(\varepsilon) \cdot l + \lambda_j^\infty(\varepsilon)| \geq \frac{1}{2}\rho_0 \langle l \rangle \quad \forall l \in \mathbb{Z}^N, j \in \mathbb{N}_0 \setminus \mathbb{S}\,, \tag{4.24}$$

$$\max_{k \leq k_0} |\partial_\varepsilon^k (\omega_\mu(\varepsilon) \cdot l + \lambda_j^\infty(\varepsilon) - \lambda_{j'}^\infty(\varepsilon))| \geq \frac{1}{2}\rho_0 \langle l \rangle \quad \forall l \in \mathbb{Z}^N j, j' \in \mathbb{N}_0 \setminus \mathbb{S} \tag{4.25}$$

$$\max_{k \leq k_0} |\partial_\varepsilon^k (\omega_\mu(\varepsilon) \cdot l + \lambda_j^\infty(\varepsilon) + \lambda_{j'}^\infty(\varepsilon))| \geq \frac{1}{2}\rho_0 \langle l \rangle \quad \forall l \in \mathbb{Z}^N j, j' \in \mathbb{N}_0 \setminus \mathbb{S}\,. \tag{4.26}$$

Proof. We prove (4.25), the other estimates follow analogously. We can consider

$$|j^3 - j'^3| \leq C \langle l \rangle\ , \tag{4.27}$$

otherwise $\tilde{R}^{(2)}_{l,j,j'}$ is empty. We can split $\lambda_j^\infty(\varepsilon) = \Omega_j(\varepsilon) + \lambda_j^\infty(\varepsilon) - \Omega_j(\varepsilon)$, where $\Omega_j(\varepsilon)$ is defined in (3.3). By Lemma 3.3 we have that

$$|\partial_\varepsilon^k [\Omega_j(\varepsilon) - \Omega_{j'}(\varepsilon)]\,| \leq C(k)|j^3 - j'^3|\,, \quad \forall k \geq 0\,. \tag{4.28}$$

Then for all $0 \leq k \leq k_0$, by (4.14) and (4.16) we have that

$$\begin{aligned} |\partial_\varepsilon^k \left[(\lambda_j^\infty - \lambda_{j'}^\infty)(\varepsilon) - (\Omega_j - \Omega_{j'})\right]| &\leq |\partial_\varepsilon^k m_1^\infty(\varepsilon)||j - j'| + 2 \sup_{j \in \mathbb{N}_0 \setminus \mathbb{S}} |\partial_\varepsilon^k r_j^\infty(\varepsilon)| \\ &\leq C\mu\gamma^{-(k+k_1)}|j - j'|\,. \end{aligned} \tag{4.29}$$

Using the definition of λ_j^∞ in (4.14), by (4.16), (4.13), (4.12), (4.29) and (3.15) we get

$$\begin{aligned}\max_{k\le k_0}|\partial_\varepsilon^k[\omega_\mu(\varepsilon)\cdot l+\lambda_j^\infty(\varepsilon)-\lambda_{j'}^\infty(\varepsilon)]| &\ge \max_{k\le k_0}\big(\partial_\varepsilon^k[\vec\omega(\varepsilon)\cdot l+\Omega_j(\varepsilon)-\Omega_{j'}(\varepsilon)|-|\partial_\varepsilon^k[m_1^\infty](j-j')|\big)\\ &-\max_{k\le k_0}\big(|\partial_\varepsilon^k\tilde r_\mu(\varepsilon)\cdot l|+|\partial_\varepsilon^k(r_j^\infty(\varepsilon)-r_{j'}^\infty(\varepsilon))|\big)\\ &\ge \max_{k\le k_0}|\partial_\varepsilon^k[\vec\omega(\varepsilon)\cdot l+\Omega_j(\varepsilon)-\Omega_{j'}(\varepsilon)|-C\mu|l|\gamma^{-1-k_1-k_0}\\ &-C\mu|j-j'|\gamma^{-k_1-k_0}\\ &\ge \max_{k\le k_0}|\partial_\varepsilon^k[\vec\omega(\varepsilon)\cdot l+\Omega_j(\varepsilon)-\Omega_{j'}(\varepsilon)|-C\mu|l|\gamma^{-1-k_1-k_0}\\ &-C\mu|j^3-j'^3|\gamma^{-k_1-k_0}\\ &\ge \rho_0\langle l\rangle - C\mu|l|\gamma^{-1-k_1-k_0}\\ &\ge \frac{\rho_0\langle l\rangle}{2}\,.\end{aligned}$$

The last equation follows if $\mu\gamma^{-1-k_1-k_0}\le\frac{\rho_0}{2C}$. □

We want to prove that $\mathcal{G}_\mu$ in (4.17) has asymptotically full Lebesgue measure. In order to do that we shall prove that the measure of the complementary set goes to zero as $\mu\to 0$. For this purpose we now estimate the measure of the resonant sets. We use the following classical Rüssmann's Lemma.

Lemma 4.6. *If* $\min_{\varepsilon\in[\varepsilon_1,\varepsilon_2]}\max_{0\le k\le k_0}\left|\partial_\varepsilon^k f(\varepsilon)\right|\ge\beta>0$ *then, for* α *small enough,*

$$|\{\varepsilon\in[\varepsilon_1,\varepsilon_2]:|f(\varepsilon)|<\alpha\}|\le c\alpha^{\frac{1}{k_0}}\,.$$

Proof. See Theorem 17.1 in [62]. □

Lemma 4.7. ***Estimates of the resonant sets.*** *Let* $\tau>\frac{4}{3}k_0$, *and* $\gamma=\mu^a$ *with* $0<a<\min\{a_0,1/(1+k_0+k_1)\}<1/2$. *Then the measure of the resonant sets defined in (4.18) satisfy*

$$\left|\tilde R_l^{(0)}\right|\le C\left(\gamma\langle l\rangle^{-\tau-1}\right)^{\frac{1}{k_0}}\,,\qquad \left|\tilde R_{l,j}^{(1)}\right|\le C\left(\gamma|j|^3\langle l\rangle^{-\tau-1}\right)^{\frac{1}{k_0}}$$
$$\left|\tilde R_{l,j,j'}^{(2)}\right|\le C\left(\gamma|j^3-j'^3|\langle l\rangle^{-\tau-1}\right)^{\frac{1}{k_0}}\,,\quad \left|\tilde R_{l,j,j'}^{(3)}\right|\le C\left(\gamma|j^3+j'^3|\langle l\rangle^{-\tau-1}\right)^{\frac{1}{k_0}}\,.$$

Proof. We rewrite (4.18) as follows

$$\begin{aligned}\tilde R_l^{(0)}&=\{\varepsilon\in[\varepsilon_1,\varepsilon_2]:|\omega_\mu(\varepsilon)\cdot l|\langle l\rangle^{-1}<\gamma\langle l\rangle^{-\tau-1}\}\\ \tilde R_{l,j}^{(1)}&=\{\varepsilon\in[\varepsilon_1,\varepsilon_2]:|\omega_\mu(\varepsilon)\cdot l+\lambda_j^\infty(\varepsilon)|\langle l\rangle^{-1}<4\gamma|j^3|\langle l\rangle^{-\tau-1}\}\\ \tilde R_{l,j,j'}^{(2)}&=\{\varepsilon\in[\varepsilon_1,\varepsilon_2]:|\omega_\mu(\varepsilon)\cdot l+\lambda_j^\infty(\varepsilon)-\lambda_{j'}^\infty(\varepsilon)|\langle l\rangle^{-1}<4\gamma|j^3-j'^3|\langle l\rangle^{-\tau-1}\}\\ \tilde R_{l,j,j'}^{(3)}&=\{\varepsilon\in[\varepsilon_1,\varepsilon_2]:|\omega_\mu(\varepsilon)\cdot l+\lambda_j^\infty(\varepsilon)+\lambda_{j'}^\infty(\varepsilon)|\langle l\rangle^{-1}<4\gamma|j^3+j'^3|\langle l\rangle^{-\tau-1}\}\,.\end{aligned}$$

Note that we are considering the sets defined above, with the restrictions on j,j',l provided in Lemma 4.3.

Then by Lemma 4.5 we have that

$$\max_{k\leq k_0} |\partial_\varepsilon^k [\omega_\mu(\varepsilon)\cdot l\langle l\rangle^{-1}]| \geq \frac{\rho_0}{2} \quad \forall \varepsilon\in[\varepsilon_1,\varepsilon_2]$$
$$\max_{k\leq k_0} |\partial_\varepsilon^k [(\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon))\langle l\rangle^{-1}]| \geq \frac{\rho_0}{2} \quad \forall \varepsilon\in[\varepsilon_1,\varepsilon_2]$$
$$\max_{k\leq k_0} |\partial_\varepsilon^k [(\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon) - \lambda_{j'}^\infty(\varepsilon))\langle l\rangle^{-1}]| \geq \frac{\rho_0}{2} \quad \forall \varepsilon\in[\varepsilon_1,\varepsilon_2]$$
$$\max_{k\leq k_0} |\partial_\varepsilon^k [(\omega_\mu(\varepsilon)\cdot l + \lambda_j^\infty(\varepsilon) + \lambda_{j'}^\infty(\varepsilon))\langle l\rangle^{-1}]| \geq \frac{\rho_0}{2} \quad \forall \varepsilon\in[\varepsilon_1,\varepsilon_2]\,.$$

By Lemma 4.6 the conclusion follows. □

Lemma 4.8. ***Measure Estimates.*** *Let*

$$\gamma = \mu^a\,, \quad \textit{with} \quad 0 < a < \min\{a_0, 1/(1+k_0+k_1)\} < 1/2\,,\ \ \tau \geq k_0(N+1). \tag{4.30}$$

Then the measure of the set $\mathcal{G}_\mu$ defined in (4.17) satisfies $|\mathcal{G}_\mu| \geq (\varepsilon_2-\varepsilon_1) - C\mu^{\frac{a}{k_0}}$ as $\mu\to 0$.

Proof. We estimate the measure of the complementary set

$$\mathcal{B}_\mu = \mathcal{G}_\mu^C = [\varepsilon_1,\varepsilon_2]\setminus\mathcal{G}_\mu = \left(\bigcup_l \tilde R_l^{(0)}\right)\cup\left(\bigcup_{l,j}\tilde R_{l,j}^{(1)}\right)\cup\left(\bigcup_{l,j,j'}\tilde R_{l,j,j'}^{(2)}\right)\cup\left(\bigcup_{l,j,j'}\tilde R_{l,j,j'}^{(3)}\right)$$

where $\tilde R_l^{(0)}, \tilde R_{l,j}^{(1)}, \tilde R_{l,j,j'}^{(2)}, \tilde R_{l,j,j'}^{(3)}$ are defined in (4.18). The estimates on the resonant sets follows by Lemma 4.7. Then, using the condition on the indices proved in Lemma 4.3 and in Corollary 4.4, we have

$$\begin{aligned}
|\mathcal{B}_\mu| &\leq \sum_l |\tilde R_l^{(0)}| + \sum_{l,j}|\tilde R_{l,j}^{(1)}| + \sum_{l,j,j'}|\tilde R_{l,j,j'}^{(2)}| + \sum_{l,j,j'}|\tilde R_{l,j,j'}^{(3)}| \\
&\leq \sum_l |\tilde R_l^{(0)}| + \sum_{j\leq C|l|^{1/3}}|\tilde R_{l,j}^{(1)}| + \sum_{j,j'\leq C|l|^{1/2}}|\tilde R_{l,j,j'}^{(2)}| + \sum_{j,j'\leq C|l|^{1/3}}|\tilde R_{l,j,j'}^{(3)}| \\
&\leq C\sum_l \left(\gamma\,\langle l\rangle^{-(\tau+1)}\right)^{\frac{1}{k_0}} + C\sum_{j\leq C|l|^{1/3}}\left(\gamma|j|^3\,\langle l\rangle^{-(\tau+1)}\right)^{\frac{1}{k_0}} \\
&+ C\sum_{j,j'\leq C|l|^{1/2}}\left(\gamma|j^3-j'^3|\,\langle l\rangle^{-(\tau+1)}\right)^{\frac{1}{k_0}} + C\sum_{j,j'\leq C|l|^{1/3}}\left(\gamma|j^3+j'^3|\,\langle l\rangle^{-(\tau+1)}\right)^{\frac{1}{k_0}} \\
&\leq \gamma^{\frac{1}{k_0}}\sum_l\langle l\rangle^{-\frac{\tau+1}{k_0}} + C\gamma^{\frac{1}{k_0}}\sum_l\langle l\rangle^{-\frac{\tau}{k_0}+\frac13} + C\gamma^{\frac{1}{k_0}}\sum_l\langle l\rangle^{-\frac{\tau}{k_0}+1} + C\gamma^{\frac{1}{k_0}}\sum_l\langle l\rangle^{-\frac{\tau}{k_0}+\frac23} \\
&\leq C\gamma^{\frac{1}{k_0}}\sum_l\langle l\rangle^{1-\frac{\tau}{k_0}} \\
&\leq C'\mu^{\frac{a}{k_0}}\,.
\end{aligned}$$

Then $|\mathcal{G}_\mu| \geq (\varepsilon_2-\varepsilon_1) - C'\mu^{\frac{a}{k_0}}$. □

Theorem 4.1 and Lemma 4.8 prove Theorem 1 with the Cantor-like set $\mathcal{G} := \mathcal{G}_\mu$ defined in (4.17) and frequency vector $\omega^\infty = \omega_\mu(\varepsilon)$ defined in (4.12).

Actually Theorem 4.1 is given in terms of the variables (θ, I, q, p), Theorem 1 is given in terms of the variables (η, u). In Chapter 1 we have given the relation between these variables (see (1.17)) and (1.36), i.e.

$$\begin{pmatrix} \eta \\ u \end{pmatrix} = \sum_{j \in \mathbb{S}} \sqrt{\frac{j}{\pi}} \begin{pmatrix} \Lambda_j \sqrt{I_j + r_j} \cos \vartheta_j \cos jx \\ \Lambda_j^{-1} \sqrt{I_j + r_j} \sin \vartheta_j \sin jx \end{pmatrix} + \begin{pmatrix} \Lambda q \\ \Lambda^{-1} p \end{pmatrix}.$$

Chapter 5

Approximate Inverse

5.1 Estimates on the perturbation P

In this Section we show tame estimates for the composition operator induced by the Hamiltonian vector field X_P, in (4.3). Since the functions $I_j \mapsto \sqrt{I_j + r_j}, \theta \mapsto \cos\theta$ and $\theta \mapsto \sin\theta$ are analytic for $|I_j| \leq r_j$, the composition Lemma 2.24 implies that, for all $\Theta, I \in H^p(\mathbb{T}^N, \mathbb{R}^N)$, $\|\Theta\|_{\mathfrak{p}_0}, \|I\|_{\mathfrak{p}_0} \leq 1$, setting $\vartheta(\theta) = \theta + \Theta(\theta)$,

$$\|\partial_\theta^\alpha \partial_I^\beta A(\theta(\cdot), I(\cdot))\|_p^{k_0,\gamma} \leq_p 1 + \|\mathcal{V}\|_p^{k_0,\gamma} \quad \forall \alpha, \beta \in \mathbb{R}^N, |\alpha| + |\beta| \leq 3$$

where A is given in (1.37), and $\mathcal{V}(\theta) = i(\theta) - (\theta, 0, 0) = (\Theta(\theta), I(\theta), w(\theta))$.

Let us consider the Hamiltonian vector field $X_P = (\partial_I P, -\partial_\theta P, -J\nabla_w P)$, where P is defined in (1.44).

Lemma 5.1. *Let $\mathcal{V}(\theta)$ satisfy $\|\mathcal{V}\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \leq 1$, for some $\sigma > 0$. Then*

$$\|X_P(i)\|_p^{k_0,\gamma} \leq_p 1 + \|\mathcal{V}\|_{p+2}^{k_0,\gamma} \tag{5.1}$$

and for all $\hat{\imath} := (\hat{\theta}, \hat{I}, \hat{w})$

$$\|d_i X_P(i)[\hat{\imath}]\|_p^{k_0,\gamma} \leq_p \|\hat{\imath}\|_{p+3} + \|\mathcal{V}\|_{s+3}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+3} \tag{5.2}$$

$$\|d_i^2 X_P(i)[\hat{\imath}, \hat{\imath}]\|_p^{k_0,\gamma} \leq_p \|\hat{\imath}\|_{p+3}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+3}^{k_0,\gamma} + \|\mathcal{V}\|_{p+4}^{k_0,\gamma} (\|\hat{\imath}\|_{\mathfrak{p}_0+3}^{k_0,\gamma})^2 \tag{5.3}$$

$$\|\partial_I d_i X_P(i)[\hat{\imath}]\|_p^{k_0,\gamma} \leq_p \|\hat{\imath}\|_{p+3}^{k_0,\gamma} + \|\mathcal{V}\|_{p+4}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+3}^{k_0,\gamma}. \tag{5.4}$$

Proof. We can write X_P as follows

$$X_P = \left(\left(\frac{\partial A(\theta, I)}{\partial I} \right)^T \nabla \mathcal{P}(\mathcal{A}(\theta, I, w)\,,\ - \left(\frac{\partial A(\theta, I)}{\partial \theta} \right)^T \nabla \mathcal{P}(\mathcal{A}(\theta, I, w)\,,\ \Pi_{\mathbb{S}}^{\perp}(-J)\nabla \mathcal{P}(\mathcal{A}(\theta, I, w)) \right)$$

where $\Pi_{\mathbb{S}}^{\perp}$ is the L^2-projection on the space $\mathtt{H}_{\mathbb{S}}^{\perp}$ defined in (1.32), A is defined in (1.37) and $\mathcal{P}$ is defined in (1.28). Hence the estimate (5.1) for X_P follows by direct computation using Lemma 2.16, and the estimates (5.2), (5.3) and (5.4) follow by differentiating X_P. $\square$

5.2 Almost approximate inverse

In order to find a solution of $\mathcal{F}(i,\alpha)=0$, with $\mathcal{F}$ defined in (4.3), we use a Nash-Moser scheme. The key point is to construct an almost approximate right inverse of the linearized operator

$$d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)[\hat{i},\hat{\alpha}]=\omega\cdot\partial_\theta\hat{i}-d_iX_{H_\alpha}(i_0(\theta))[\hat{i}]-(\hat{\alpha},0,0)$$

where the perturbation does not depend on α, hence $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)=d_{i,\alpha}\mathcal{F}(i_0)$. Note that the almost approximate right inverse is constructed at an approximate torus $i_0(\theta)=(\vartheta_0(\theta),I_0(\theta),w_0(\theta))$, at a given value of α_0 (see Theorem 5.13).

We use the general strategy in [15], that was implemented in [19]. An invariant torus i_0 with diophantine flow, that is, $|\omega\cdot l|\geq\gamma\langle l\rangle^{-\tau}$, $\forall l\in\mathbb{Z}^N\setminus\{0\}$, is isotropic (see [15]), namely $i_0^\star\Xi$ is closed, where Ξ is the 1-form defined in (1.39). If we differentiate Ξ we get the (opposite in sign) symplectic 2-form, that is $\mathcal{W}_{new}$, defined in (1.38). Hence the pull-back 1-form is closed if and only if the 2-form $-i_0^\star\mathcal{W}_{new}=i_0^\star d\Xi=di_0^\star\Xi=0$.

For an "approximately invariant" torus i_0, which the flow is "diophantine" for finitely many $l\in\mathbb{Z}^N$, the 1-form $i_0^\star\Xi$ is "approximately closed". In order to be more precisely we have that ω is in $\mathtt{DC}^\gamma_{K_n}$, that is

$$\mathtt{DC}^\gamma_{K_n}:=\{\omega\in\Omega\subset\mathbb{R}^N:|\omega\cdot l|\geq\gamma\,\langle l\rangle^{-\tau}\ ,\ \forall|l|\leq K_n\}\,, \tag{5.5}$$

where $K_n:=K_0^{(\frac{3}{2})^n}$.

Then we consider

$$i_0^\star\Xi=\sum_{k=1}^N a_k(\theta)d\varphi_k,\qquad a_k(\theta):=-\big([\partial_\theta\vartheta_0(\theta)]^TI_0(\theta)\big)_k-\frac{1}{2}\left(\partial_{\theta_k}w_0(\theta),Jw_0(\theta)\right)_{L^2(\mathbb{T}_x)} \tag{5.6}$$

and we quantify how small is the pull back of the 2-form

$$-\,i_0^\star\mathcal{W}_{new}=di_0^\star\Xi=\sum_{1\leq k<j\leq N}A_{jk}(\theta)d\theta_k\wedge d\theta_j,\qquad A_{jk}(\theta):=\partial_{\theta_k}a_j(\theta)-\partial_{\theta_j}a_k(\theta)\,, \tag{5.7}$$

in terms of the "error function"

$$Z(\theta):=(Z_1,Z_2,Z_3)(\theta):=\mathcal{F}(i_0,\alpha_0)(\theta)=\omega\cdot\partial_\theta(i_0(\theta))-X_{H_\alpha}(i_0(\theta),\alpha_0)\,. \tag{5.8}$$

Remark 5.2. *The frequency vector ω in* (5.8) *is only "approximate" Diophantine, that is $\omega\in\mathtt{DC}^\gamma_{K_n}$, where $\mathtt{DC}^\gamma_{k_n}$ is defined in* (5.5) .

Ansatz . The map $(\omega,\varepsilon)\mapsto\mathcal{V}_0(\omega,\varepsilon)=i_0(\theta,\omega,\varepsilon)-(\theta,0,0)$ is k_0-times differentiable with respect to the parameters $(\omega,\varepsilon)\in\mathbb{R}^N\times[\varepsilon_1,\varepsilon_2]$, and for some $\nu:=\nu(\tau,N)>0,\gamma\in(0,1)$

$$\|\mathcal{V}_0\|^{k_0,\gamma}_{\mathtt{p}_0+\nu}+|\alpha_0-\omega|^{k_0,\gamma}\leq C\mu\gamma^{-(1+k_1)}\,, \tag{5.9}$$

where $k_1=k_1(N,k_0)>0$ is given in Theorem 4.1. Moreover we assume $\mu\gamma^{-(1+k_1)}$ small enough. Actually in Lemma 4.7 and 4.8 we have required a stronger condition: $\mu\gamma^{-(1+k_1+k_0)}<1$.

Since in the Nash-Moser iteration (see Chapter 10) we shall construct an extension of each approximate solution that is k_0-times differentiable in the whole $\mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$ we suppose that the torus i_0 is defined for all $(\omega, \varepsilon) \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$.

Lemma 5.3. *Let Z as in* (5.8)*.Then*

$$\begin{aligned}\|Z\|_p^{k_0,\gamma} &\leq_p |\omega - \alpha_0|^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+3}^{k_0,\gamma} \\ &\leq_p \mu\gamma^{-(1+k_1)} + \|\mathcal{V}_0\|_{p+3}^{k_0,\gamma}.\end{aligned}$$

Proof. By (4.4), the estimate (5.1) on X_P and (5.9) one gets the result. □

In the Nash-Moser iteration in Chapter 10 we have to introduce the "ultra-violet" cut-off K_n. Moreover we require that $\omega \in \mathbb{R}^N$ satisfies finitely many non-resonance Diophantine conditions. Hence at every n-step we require that ω is in $\mathtt{DC}^\gamma_{K_n}$. In addition we will require that the frequency vector ω satisfies also finitely many first and second Melnikov non-resonance condition.

Since we have introduced the ultra-violet cut off it is better to split the coefficients $A_{kj} = A_{kj}(\theta)$ in (5.7) as

$$A_{kj} = A_{kj}^{(n)} + A_{kj}^{(n),\perp}, \quad A_{kj}^{(n)} := \Pi_{K_n} A_{kj}, \quad A_{kj}^{(n),\perp} := \Pi_{K_n}^\perp A_{kj} \tag{5.10}$$

where Π_{K_n} is defined as the orthogonal projection on the finite Fourier modes $|(l,j)| \leq K_n$, and $\Pi_{K_n}^\perp$ is defined as $\Pi_{K_n}^\perp := \mathbb{1} - \Pi_{K_n}$ (see (2.10)).

Lemma 5.4. *Assume that $\omega \in \mathtt{DC}^\gamma_{K_n}$. Then the coefficients $A_{jk}^{(n)}$ and $A_{jk}^{(n),\perp}$ defined in* (5.10) *satisfy the following tame estimate*

$$\|A_{kj}^{(n)}\|_p^{k_0,\gamma} \leq_p \gamma^{-1}(\|Z\|_{p+1+\tau(k_0+1)+k_0}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+1}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+1+\tau(k_0+1)+k_0}^{k_0,\gamma}). \tag{5.11}$$

Moreover for any $b > 0$ and for any $c > 0$ such that (5.9) holds with $\nu \geq \tau(k_0+1) + k_0 + 1 + c$ we have

$$\|A_{kj}^{(n),\perp}\|_p^{k_0,\gamma} \leq_p \|\mathcal{V}_0\|_{p+2}^{k_0,\gamma}, \quad \|A_{kj}^{(n),\perp}\|_{\mathfrak{p}_0+c}^{k_0,\gamma} \leq_{\mathfrak{p}_0,b} K_n^{-b}\|\mathcal{V}_0\|_{\mathfrak{p}_0+b+c}^{k_0,\gamma}. \tag{5.12}$$

Proof. We prove (5.11)

$$-\mathcal{L}ie_\omega(i_0^*\mathcal{W}_{new}) = \sum \omega \cdot \partial_\theta A_{jk}(\theta) d\theta_k \wedge d\theta_j$$

let $e_k = (0, ..., 1, 0..)$ with 1 in the k-entry then

$$\omega \cdot \partial_\theta A_{jk} = -\mathcal{L}ie_\omega(i_0^*\mathcal{W}_{new})[e_k, e_j] = -\mathcal{W}_{new}(\partial_\theta Z e_k, \partial_\theta i_0(\theta) e_j) - \mathcal{W}_{new}(\partial_\theta i_0(\theta) e_k, \partial_\theta Z e_j).$$

If we apply the projection we obtain

$$\omega \cdot \partial_\theta A_{jk}^{(n)} = -\Pi_{K_n}\left[\mathcal{W}_{new}(\partial_\theta Z e_k, \partial_\theta i_0(\theta) e_j) - \mathcal{W}_{new}(\partial_\theta i_0(\theta) e_k, \partial_\theta Z e_j)\right],$$

hence, by (5.9) and (2.36) we have

$$\|\omega \cdot \partial_\theta A_{kj}^{(n)}\|_p^{k_0,\gamma} \leq_p \|Z\|_{p+1}^{k_0,\gamma}\|\mathcal{V}_0\|_{\mathfrak{p}_0+1}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+1}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+1}^{k_0,\gamma} \leq_p \|Z\|_{p+1}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+1}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+1}^{k_0,\gamma}. \tag{5.13}$$

Therefore

$$\|A_{kj}^{(n)}\|_p^{k_0,\gamma} \leq_p \gamma^{-1}\left(\|Z\|_{p+\tau(k_0+1)+k_0}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+1}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\tau(k_0+1)+k_0}^{k_0,\gamma}\right)$$

where we have used $\|(\omega\cdot\partial_\theta)^{-1}\Pi_{K_n} g\|_p^{k_0,\gamma} \leq_p \gamma^{-1}\|g\|_{p+\tau(k_0+1)+k_0}^{k_0,\gamma}$ (recall that $\omega \in \mathtt{DC}_{K_n}^{\gamma}$).

For prove (5.12) we use the smooth properties (2.11), (2.36), and (5.9). □

Remark 5.5. *The splitting* (5.10) *is due to the fact that* $\omega \in \mathtt{DC}_{K_n}^{\gamma}$.

As in [15] and [5] we modify the approximate torus i_0 to obtain an isotropic torus i_δ which is still approximately invariant. We denote the laplacian $\Delta_\theta := \sum_{k=1}^N \partial_{\theta_k}^2$.

Lemma 5.6. ***Isotropic torus.*** *Let $\gamma^{-1}\mu < 1$. The torus $i_\delta(\theta) = (\vartheta_0(\theta), I_\delta(\theta), w_0(\theta))$, with*

$$\begin{aligned} I_\delta :=& I_0 + [\partial_\theta\vartheta_0(\theta)]^{-T}\rho(\theta), \\ \rho_j :=& \triangle_\theta^{-1}\sum_{k=1}^N \partial_{\theta_j}\left(\partial_{\theta_k}a_j - \partial_{\theta_j}a_k\right) = \triangle_\theta^{-1}\sum_{k=1}^N \partial_{\theta_j}A_{kj}(\theta)\,, \quad j=1,...,N\,, \end{aligned} \tag{5.14}$$

is isotropic. Moreover I_δ admits the splitting $I_\delta = I_\delta^{(n)} + I_\delta^{(n),\perp}$ where

$$I_\delta^{(n)} := I_0 + [\partial_\theta\vartheta_0(\theta)]^{-T}\rho^{(n)}(\theta)\,, \tag{5.15}$$

$$\rho_j^{(n)}(\theta) := \triangle_\theta^{-1}\sum_{k=1}^N \partial_{\theta_j}\Pi_{K_n}\left(\partial_{\theta_k}a_j - \partial_{\theta_j}a_k\right)(\theta) = \triangle_\theta^{-1}\sum_{k=1}^N \partial_{\theta_j}A_{kj}^{(n)}(\theta)\,,$$

$$I_\delta^{(n),\perp} := I_0 + [\partial_\theta\vartheta_0(\theta)]^{-T}\rho^{(n),\perp}(\theta)\,, \tag{5.16}$$

$$\rho_j^{(n),\perp}(\theta) := \triangle_\theta^{-1}\sum_{k=1}^N \partial_{\theta_j}\Pi_{K_n}^{\perp}\left(\partial_{\theta_k}a_j - \partial_{\theta_j}a_k\right)(\theta) = \triangle_\theta^{-1}\sum_{k=1}^N \partial_{\theta_j}A_{kj}^{(n),\perp}(\theta)\,.$$

There is $\sigma := \sigma(N,\tau,k_0) > 1$ and $c \geq 0$ such that, if (5.9) holds with $\sigma + c \leq \nu$, then

$$\begin{aligned} \|I_\delta - I_0\|_p^{k_0,\gamma} &\leq_p \left(\|I_\delta^{(n)} - I_0\|_p^{k_0,\gamma} + \|I_\delta^{(n),\perp}\|_p^{k_0,\gamma}\right) \\ &\leq_p \|\mathcal{V}_0\|_{p+1}^{k_0,\gamma} \end{aligned} \tag{5.17}$$

$$\|I_\delta^{(n)} - I_0\|_p^{k_0,\gamma} \leq_p \gamma^{-1}\left(\|Z\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma} + \|Z\|_{p+\sigma}^{k_0,\gamma}\right) \tag{5.18}$$

$$\|I_\delta^{(n),\perp}\|_{\mathfrak{p}_0+c}^{k_0,\gamma} \leq_{\mathfrak{p}_0,b} K_n^{-b}\|\mathcal{V}_0\|_{\mathfrak{p}_0+c+b+\sigma}^{k_0,\gamma} \quad \forall b > 0 \tag{5.19}$$

$$\|\partial_i i_\delta[\hat{\imath}]\|_p^{k_0,\gamma} \leq_p \|\hat{\imath}\|_p^{k_0,\gamma} + \left(\|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma}\|\hat{\imath}\|_{\mathfrak{p}_0}^{k_0,\gamma}\right)\,. \tag{5.20}$$

Moreover the "error" function $Z_\delta = \mathcal{F}(i_\delta, \alpha_0)$ of the isotropic torus i_δ (defined analogously to (5.8)*) can be splitted as $Z_\delta = Z_\delta^{(n)} + Z_\delta^{(n),\perp}$ with*

$$\|Z_\delta^{(n)}\|_p^{k_0,\gamma} \leq_p \|Z\|_{p+\sigma}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma} \tag{5.21}$$

$$\|Z_\delta^{(n),\perp}\|_p^{k_0,\gamma} \leq_p \|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma}, \qquad \|Z_\delta^{(n),\perp}\|_{\mathfrak{p}_0+c}^{k_0,\gamma} \leq_{\mathfrak{p}_0,b} K_n^{-b}\|\mathcal{V}_0\|_{\mathfrak{p}_0+\sigma+c+b}^{k_0,\gamma} \quad \forall b > 0\,. \tag{5.22}$$

Note that we denote by $\sigma := \sigma(N,\tau,k_0)$ possibly (larger) "loss of derivatives" constant.

Proof. In [15] it is proved that the torus i_δ is isotropics, hence we focus on the inequalities. We have $\|(\partial_\theta \theta_0)^{-T}\|_p^{k_0,\gamma} \leq_p 1 + \|\mathcal{V}_0\|_{p+1}^{k_0,\gamma}$. Then by

$$\|I_\delta - I_0\|_p^{k_0,\gamma} = \|(\partial_\theta \vartheta_0(\theta))^{-T} \rho^{(n)}(\theta)\|_p^{k_0,\gamma} + \|(\partial_\theta \vartheta_0(\theta))^{-T} \rho^{(n,\perp)}(\theta)\|_p^{k_0,\gamma},$$

by (5.14), (5.6), (5.7), (2.36), (5.9), the estimate (5.17) follows.

We have $I_\delta^{(n)} - I_0 = [\partial_\theta \vartheta_0(\theta)]^{-T} \rho^{(n)}(\theta)$, hence the estimate (5.18) follows by (5.11). The estimate (5.19) follows by (5.16) and (5.12). The estimate (5.20) follows by (5.6), (5.7), (5.9) and by (5.14).

For prove (5.21) and (5.22) we consider the following split.

$$\begin{aligned}
\mathcal{F}(i_\delta, \alpha_0) &= \mathcal{F}(i_0, \alpha_0) + \begin{pmatrix} 0 \\ \omega \cdot \partial_\theta (I_\delta - I_0) \\ 0 \end{pmatrix} + \mu \left(X_P(i_\delta) - X_P(i_0) \right) \\
&= \mathcal{F}(i_0, \alpha_0) + \begin{pmatrix} 0 \\ \omega \cdot \partial_\theta (I_\delta - I_0) \\ 0 \end{pmatrix} + + \mu \int_0^1 \partial_I X_P \left(t i_\delta + (1-t) i_0 \right) \cdot (I_\delta - I_0) \, dt \\
&= Z_\delta^{(n)} + Z_\delta^{(n),\perp}
\end{aligned}$$

where

$$\begin{aligned}
Z_\delta^{(n)} &:= \mathcal{F}(i_0, \alpha_0) + \begin{pmatrix} 0 \\ \omega \cdot \partial_\theta (I_\delta^{(n)} - I_0) \\ 0 \end{pmatrix} \\
&\quad + \mu \int_0^1 \partial_I X_P \left(t i_\delta + (1-t) i_0 \right) \cdot \left(I_\delta^{(n)} - I_0 \right) dt \,, \qquad (5.23)
\end{aligned}$$

$$Z_\delta^{(n),\perp} := \begin{pmatrix} 0 \\ \omega \cdot \partial_\theta I_\delta^{(n),\perp} \\ 0 \end{pmatrix} + \mu \int_0^1 \partial_I X_P \left(t i_\delta + (1-t) i_0 \right) \cdot I_\delta^{(n),\perp} dt \,. \qquad (5.24)$$

Differentiating (5.15) we have

$$\begin{aligned}
\omega \cdot \partial_\theta \left(I_\delta^{(n)} - I_0 \right) =& [\partial_\theta \vartheta_0(\theta)]^{-T} \omega \cdot \partial_\theta \rho^{(n)}(\theta) \\
& - \left([\partial_\theta \vartheta_0(\theta)]^{-T} (\omega \cdot \partial_\theta [\partial_\theta \vartheta_0(\theta)]^T) [\partial_\theta \vartheta_0(\theta)]^{-T} \right) \rho^{(n)}(\theta) \qquad (5.25) \\
\omega \cdot \partial_\theta [\partial_\theta \vartheta_0(\theta)] =& \mu \partial_\theta (\partial_I P)(i_0(\theta)) + \partial_\theta Z_1(\theta) \,, \qquad (5.26)
\end{aligned}$$

where Z_1 is the first component of the error function. Then for prove (5.21) we use (5.23), (5.25), (5.26), (5.18), (5.11), (5.15), (5.9), (5.2), (5.13), (2.36) and Lemma 5.3. The inequalities (5.22) follows by (5.24), (5.12), (5.17), (5.19), (5.16), (2.36) (5.2) and (5.9). □

In order to find an approximate inverse of the linearized operator $d_{i,\alpha}\mathcal{F}(i_\delta)$ we consider the symplectic diffeomorphism $G_\delta : (\psi, y, z) \to (\vartheta, I, w)$ of the phase space $\mathbb{T}^N \times \mathbb{R}^N \times \mathtt{H}_{\mathbb{S}}^{\perp}$ defined by

$$\begin{pmatrix} \vartheta \\ I \\ w \end{pmatrix} := G_\delta \begin{pmatrix} \psi \\ y \\ z \end{pmatrix} := \begin{pmatrix} \vartheta_0(\psi) \\ I_\delta(\psi) + [\partial_\psi \vartheta_0(\psi)]^{-T} y - [(\partial_\theta \tilde{w}_0)(\vartheta_0(\psi))]^T(-J)z \\ w_0(\psi) + z \end{pmatrix} \tag{5.27}$$

where $\tilde{w}_0 = w_0(\theta_0^{-1}(\theta))$. In [15] is proved that G_δ is symplectic.

In this coordinates, i_δ is the trivial embedded torus $(\psi, y, w) = (\psi, 0, 0)$. Under this symplectic change of variables the Hamiltonian vector field X_{H_α} generated by the Hamiltonian H_α in (4.2) changes into

$$X_{K_\alpha} = (DG_\delta)^{-1} X_{H_\alpha} \circ G_\delta, \quad \text{where} \quad K_\alpha := H_\alpha \circ G_\delta\,. \tag{5.28}$$

By (1.43) we have that the transformation G_δ is reversibility preserving thus K_α is reversible, that is $K_\alpha \circ \tilde{\rho} = K_\alpha$, where $\tilde{\rho}$ is defined in (1.42). We compute the Taylor expansion of the new Hamiltonian K_α at the trivial torus $(\psi, 0, 0)$, that is

$$\begin{aligned} K_\alpha(\psi, y, z) = {} & K_{00}(\psi, \alpha) + K_{10}(\psi, \alpha) \cdot y + (K_{01}(\psi, \alpha), z)_{L^2(\mathbb{T}_x)} + (K_{11}(\psi) y, z)_{L^2(\mathbb{T}_x)} \\ & + \frac{1}{2} K_{20}(\psi) y \cdot y + \frac{1}{2} (K_{02}(\psi) z, z)_{L^2(\mathbb{T}_x)} + K_{\geq 3}(\psi, y, z) \end{aligned} \tag{5.29}$$

where $K_{\geq 3}$ collects the terms at least cubic in the variables (y, z). The Taylor coefficient $K_{00}(\psi, \alpha) \in \mathbb{R}$, $K_{10}(\psi, \alpha) \in \mathbb{R}^N$, $K_{01}(\psi, \alpha) \in \mathtt{H}_{\mathbb{S}}^{\perp}$, $K_{20}(\psi, \alpha) \in \mathbb{R}^{N \times N}$, $K_{02}(\psi)$ is a linear self-adjoint operator of $\mathtt{H}_{\mathbb{S}}^{\perp}$ and $K_{11}(\psi) \in \mathcal{L}(\mathbb{R}^N, \mathtt{H}_{\mathbb{S}}^{\perp})$, where $\mathtt{H}_{\mathbb{S}}^{\perp}$ is defined in (1.32) .

By (5.27) and (4.2) the Taylor coefficients which depend on α are K_{00}, K_{10} and K_{01} .

The equations of motion associated to the Hamiltonian K_α in (5.29) are (recall (1.38) and the definition of J in (8) i.e. (1.23))

$$\begin{cases} \dot{\psi} & = K_{10}(\psi, \alpha) + K_{20}(\psi) y + K_{11}^T(\psi) z + \partial_y K_{\geq 3}(\psi, y, z) \\ \dot{y} & = \partial_\psi K_{00}(\psi, \alpha) - [\partial_\psi K_{10}(\psi, \alpha)]^T y - [\partial_\psi K_{01}(\psi, \alpha)]^T z - \partial_\psi (K_{11}(\psi) y, z)_{L^2(\mathbb{T}_x)} \\ & \quad - \frac{1}{2} \partial_\psi (K_{20}(\psi) y \cdot y) - \frac{1}{2} \partial_\psi (K_{02}(\psi) z, z)_{L^2(\mathbb{T}_x)} - \partial_\psi K_{\geq 3}(\psi, y, z) \\ \dot{z} & = (-J)(K_{01}(\psi, \alpha) + K_{11}(\psi) y + K_{02}(\psi) z + \nabla_z K_{\geq 3}(\psi, y, z))\,, \end{cases} \tag{5.30}$$

where $\partial_\psi K_{10}^T \in \mathbb{R}^{N \times N}$ and $\partial_\psi K_{01}^T, K_{11}^T : \mathtt{H}_{\mathbb{S}}^{\perp} \to \mathbb{R}^N$ are defined by the duality relation

$$\left(\partial_\psi K_{01}^T [\hat{\psi}], z \right)_{L^2(\mathbb{T}_x)} = \hat{\psi} \cdot [\partial_\psi K_{01}]^T z, \quad \forall \hat{\psi} \in \mathbb{R}^N, z \in \mathtt{H}_{\mathbb{S}}^{\perp}$$

and

$$K_{11}^T(\psi) z = \sum_{k=1}^N \left(K_{11}^T z \cdot e_k \right) e_k = \sum_{k=1}^N (z, K_{11} e_k)_{L^2(\mathbb{T}_x)} e_k \in \mathbb{R}^N\,, \quad \forall z \in \mathtt{H}_{\mathbb{S}}^{\perp}\,. \tag{5.31}$$

Note that the coefficients K_{00}, K_{10} and K_{01} vanish when $Z = 0$, in other words these coefficients vanish on an exact solution.

We consider $K_\alpha = H_\alpha \circ G_\delta$ (see (5.28)), and we define $\mathcal{F}(i_\delta, \alpha) = Z_\delta := (Z_{1,\delta}, Z_{2,\delta}, Z_{3,\delta})$ then differentiating it (see [15], [5]) we get

$$\begin{aligned} K_{10}(\psi,\alpha_0) =& \omega - [\partial_\psi \vartheta_0(\psi)]^T Z_{1,\delta}(\psi) \\ \partial_\psi K_{00}(\psi,\alpha_0) =& - [\partial_\psi \vartheta_0(\psi)]^T \big(-Z_{2,\delta} - [\partial_\psi I_\delta(\psi)][\partial_\psi \vartheta_0(\psi)]^{-1} Z_{1,\delta} - [(\partial_\psi \tilde{w}_0)(\vartheta_0(\psi))]^T(-J)Z_{3,\delta} \\ & -[(\partial_\psi \tilde{w}_0)(\vartheta_0(\psi))]^T(-J)\partial_\psi w_0(\psi)[(\partial_\psi \vartheta_0(\psi))]^{-1} Z_{1,\delta}\big) \\ K_{01}(\psi,\alpha_0) =& - JZ_{3,\delta} + J\partial_\psi w_0(\psi)[\partial_\psi \vartheta_0(\psi)]^{-1} Z_{1,\delta}(\psi)\,. \end{aligned} \tag{5.32}$$

If we consider the splitting $Z_\delta = Z_\delta^{(n)} + Z_\delta^{(n),\perp}$, given in Lemma 5.6, setting

$$Z_\delta^{(n)} := (Z_{1,\delta}^{(n)}, Z_{2,\delta}^{(n)}, Z_{3,\delta}^{(n)}) \quad \text{and} \quad Z_\delta^{(n),\perp} := (Z_{1,\delta}^{(n),\perp}, Z_{2,\delta}^{(n),\perp}, Z_{3,\delta}^{(n),\perp})$$

we can decompose the coefficients K_{00}, K_{01}, K_{02} in the Taylor expansion (5.29) as

$$\partial_\psi K_{00} = \partial_\psi K_{00}^{(n)} + \partial_\psi K_{00}^{(n),\perp}, \quad K_{10} = K_{10}^{(n)} + K_{10}^{(n),\perp}, \quad K_{01} = K_{01}^{(n)} + K_{01}^{(n),\perp} \tag{5.33}$$

where

$$\begin{aligned} \partial_\psi K_{00}^{(n)}(\psi,\alpha_0) =& - [\partial_\psi \vartheta_0(\psi)]^T \Big(-Z_{2,\delta}^{(n)} - [\partial_\psi I_\delta(\psi)][\partial_\psi \vartheta_0(\psi)]^{-1} Z_{1,\delta}^{(n)} \\ & - [(\partial_\psi \tilde{w}_0)(\theta_0(\psi))]^T(-J)Z_{3,\delta}^{(n)} \\ & -[(\partial_\psi \tilde{w}_0)(\vartheta_0(\psi))]^T(-J)\partial_\psi w_0(\psi)[(\partial_\psi \vartheta_0(\psi))]^{-1} Z_{1,\delta}^{(n)}\Big) \end{aligned} \tag{5.34}$$

$$\begin{aligned} \partial_\psi K_{00}^{(n),\perp}(\psi,\alpha_0) =& - [\partial_\psi \vartheta_0(\psi)]^T \Big(-Z_{2,\delta}^{(n),\perp}(\psi) - [\partial_\psi I_\delta(\psi)][\partial_\psi \vartheta_0(\psi)]^{-1} Z_{1,\delta}^{(n),\perp} \\ & - [(\partial_\psi \tilde{w}_0)(\theta_0(\psi))]^T(-J)Z_{3,\delta}^{(n),\perp} \\ & -[(\partial_\psi \tilde{w}_0)(\vartheta_0(\psi))]^T(-J)\partial_\psi w_0(\psi)[(\partial_\psi \vartheta_0(\psi))]^{-1} Z_{1,\delta}^{(n),\perp}\Big) \end{aligned} \tag{5.35}$$

$$K_{10}^{(n)}(\psi,\alpha_0) = \omega - [\partial_\psi \vartheta_0(\psi)]^T Z_{1,\delta}^{(n)}(\psi) \tag{5.36}$$

$$K_{10}^{(n),\perp}(\psi,\alpha_0) = - [\partial_\psi \vartheta_0(\psi)]^T Z_{1,\delta}^{(n),\perp}(\psi) \tag{5.37}$$

$$K_{01}^{(n)}(\psi,\alpha_0) = - JZ_{3,\delta}^{(n)} + J\partial_\psi w_0(\psi)[\partial_\psi \vartheta_0(\psi)]^{-1} Z_{1,\delta}^{(n)} \tag{5.38}$$

$$K_{01}^{(n),\perp}(\psi,\alpha_0) = - JZ_{3,\delta}^{(n),\perp} + J\partial_\psi w_0(\psi)[\partial_\psi \vartheta_0(\psi)]^{-1} Z_{1,\delta}^{(n),\perp}\,. \tag{5.39}$$

In the following two Lemmas we first give some estimates on the coefficients that vanish when $Z = 0$, then we estimate the variation of these coefficients with respect to α.

Lemma 5.7. *There exists $\sigma := \sigma(N, \tau, k_0) > 0$ such that if if (5.9) holds with $\nu \geq \sigma + c,\ c > 0$, then the splitted coefficients (5.33) satisfy*

$$\begin{aligned} &\|\partial_\psi K_{00}^{(n)}(\cdot,\alpha_0)\|_p^{k_0,\gamma} + \|\partial_\psi K_{10}^{(n)}(\cdot,\alpha_0) - \omega\|_p^{k_0,\gamma} \\ &\qquad + \|\partial_\psi K_{01}^{(n)}(\cdot,\alpha_0)\|_p^{k_0,\gamma} \leq_p \|Z\|_{p+\sigma}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma} \end{aligned} \tag{5.40}$$

$$\|\partial_\psi K_{00}^{(n),\perp}(\cdot,\alpha_0)\|_p^{k_0,\gamma} + \|\partial_\psi K_{10}^{(n),\perp}(\cdot,\alpha_0)\|_p^{k_0,\gamma} + \|\partial_\psi K_{01}^{(n),\perp}(\cdot,\alpha_0)\|_p^{k_0,\gamma} \leq_p \|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma} \tag{5.41}$$

$$\|\partial_\psi K_{00}^{(n),\perp}(\cdot,\alpha_0)\|^{k_0,\gamma}_{\mathfrak{p}_0+c} + \|\partial_\psi K_{10}^{(n),\perp}(\cdot,\alpha_0)\|^{k_0,\gamma}_{\mathfrak{p}_0+c} + \|\partial_\psi K_{01}^{(n),\perp}(\cdot,\alpha_0)\|^{k_0,\gamma}_{\mathfrak{p}_0+c} \leq_{\mathfrak{p}_0,b} K_n^{-b}\|\mathcal{V}_0\|^{k_0,\gamma}_{p+\sigma+c+b} \quad (5.42)$$

for all $b > 0$.

Proof. The estimate (5.40), (5.41) and (5.42) follows by the explicit expressions given in (5.34)-(5.39), by (2.36), (5.17) (5.21) and (5.22) . □

Lemma 5.8. *There exists* $\sigma := \sigma(N,\tau,k_0) > 0$ *such that, if* $\|\mathcal{V}_0\|^{k_0,\gamma}_{\mathfrak{p}_0+\sigma} \leq 1$, *then*

$$\begin{aligned}
\|\partial_\alpha K_{00}\|^{k_0,\gamma}_p + \|\partial_\alpha K_{10} - \mathbb{1}\|^{k_0,\gamma}_p + \|\partial_\alpha K_{01}\|^{k_0,\gamma}_p &\leq_p \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\sigma} \\
\|K_{20}\|^{k_0,\gamma}_p &\leq_p \mu\left(1 + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\sigma}\right) \\
\|K_{11}y\|^{k_0,\gamma}_p &\leq_p \mu\left(\|y\|^{k_0,\gamma}_{p+2} + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\sigma}\|y\|^{k_0,\gamma}_{\mathfrak{p}_0+2}\right) \\
\|K_{11}^T z\|^{k_0,\gamma}_p &\leq_p \mu\left(\|z\|^{k_0,\gamma}_{p+2} + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\sigma}\|z\|^{k_0,\gamma}_{\mathfrak{p}_0+2}\right) .
\end{aligned}$$

Proof. As in [15] we have

$$\begin{aligned}
\partial_\alpha K_{00}(\psi) &= I_\delta(\psi) \\
\partial_\alpha K_{10}(\psi) &= [\partial_\psi \vartheta_0(\psi)]^{-1} \\
\partial_\alpha K_{01}(\psi) &= (-J)\partial_\theta \tilde{w}_0(\vartheta_0(\psi)) \\
K_{20}(\theta) &= \mu[\partial_\theta\vartheta_0(\theta)]^{-1}\partial_{II}P(i_\delta(\theta))[\partial_\theta\vartheta_0(\theta)]^{-T} \\
K_{11}(\theta) &= \mu\left(\partial_I\nabla_w P(i_\delta(\theta))[\partial_\theta\vartheta_0(\theta)]^{-T} + (-J)(\partial_\theta\tilde{w}_0)(\vartheta_0(\theta))(\partial_{II}P(i_\delta(\theta))[\partial_\psi\vartheta_0(\psi)]^{-T}\right)
\end{aligned}$$

Then (2.36), (5.1), (5.11), (5.31) and (5.3) imply the lemma. □

If we consider the change of variables

$$DG_\delta(\vartheta,0,0)\begin{pmatrix}\hat{\psi}\\ \hat{y}\\ \hat{z}\end{pmatrix} := \begin{pmatrix}\partial_\psi\vartheta_0(\theta) & 0 & 0\\ \partial_\psi I_\delta(\theta) & [\partial_\psi\vartheta_0(\theta)]^{-T} & -[(\partial_\psi\tilde{w}_0)(\vartheta_0(\theta))]^T(-J)\\ \partial_\psi w_0(\theta) & 0 & \mathbb{1}\end{pmatrix}\begin{pmatrix}\hat{\psi}\\ \hat{y}\\ \hat{z}\end{pmatrix} \quad (5.43)$$

we have that the induced composition operator satisfies the following Lemma.

Lemma 5.9. *For all* $\hat{\imath} = (\hat{\psi},\hat{y},\hat{z})$ *we have*

$$\|DG_\delta(\vartheta,0,0)[\hat{\imath}]\|^{k_0,\gamma}_p + \|DG_\delta(\vartheta,0,0)^{-1}[\hat{\imath}]\|^{k_0,\gamma}_s \leq_p \|\hat{\imath}\|^{k_0,\gamma}_p + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\sigma}\|\hat{\imath}\|^{k_0,\gamma}_{\mathfrak{p}_0} \quad (5.44)$$

$$\|D^2G_\delta(\vartheta,0,0)[\hat{\imath}_1\,,\ \hat{\imath}_2]\|^{k_0,\gamma}_p \leq_p \|\hat{\imath}_1\|^{k_0,\gamma}_p\|\hat{\imath}_2\|^{k_0,\gamma}_{\mathfrak{p}_0} + \|\hat{\imath}_2\|^{k_0,\gamma}_p\|\hat{\imath}_1\|^{k_0,\gamma}_{\mathfrak{p}_0} + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\sigma}\|\hat{\imath}_1\|^{k_0,\gamma}_{\mathfrak{p}_0}\|\hat{\imath}_2\|^{k_0,\gamma}_{\mathfrak{p}_0} . \quad (5.45)$$

Proof. Use (5.43), (5.9), (2.36) and (5.17). □

Under the change of variables (5.43) the linearized operator $d_{i,\alpha}\mathcal{F}(i_\delta)$ is transformed into a new operator obtained by linearizing the equations of motion in (5.30) at $(\psi,y,z) = (\theta,0,0)$, differentiating also in α at α_0 and changing $\partial_t \rightsquigarrow \omega\cdot\partial_\theta$. Actually the linearized operator $d_{i,\alpha}\mathcal{F}(i_\delta)$ is "approximately" transformed into the new one, see (5.82) for the precise expression of the error. The new linearized operator is given by

$$\mathbf{N}[\hat\psi,\hat y,\hat z,\hat\alpha] := \begin{pmatrix} \omega\cdot\partial_\theta\hat\psi - \partial_\psi K_{10}(\theta)[\hat\psi] - \partial_\alpha K_{10}(\theta)[\hat\alpha] - K_{20}(\theta)\hat y - K_{11}^T(\theta)\hat z \\ \omega\cdot\partial_\theta\hat y - \partial_{\psi\psi}K_{00}(\theta)[\hat\psi] - \partial_\psi\partial_\alpha K_{00}(\theta)[\hat\alpha] + [\partial_\psi K_{10}(\theta)]^T\hat y + [\partial_\psi K_{01}(\theta)]^T\hat z \\ \omega\cdot\partial_\theta\hat z + J\partial_\psi K_{01}(\theta)[\hat\psi] + J\partial_\alpha K_{01}(\theta)[\hat\alpha] + JK_{11}(\theta)\hat y + JK_{02}(\theta)\hat z \end{pmatrix}. \tag{5.46}$$

In order to construct an "almost-approximate" inverse of (5.46) we need to solve

$$\mathbf{N}[h] = g\,, \quad \text{where} \quad h=(h_1,h_2,h_3) \quad \text{and} \quad g=(g_1,g_2,g_3). \tag{5.47}$$

We start by considering the third equation in the system defined in (5.47), that is, $\mathcal{L}_\omega\hat z = g_3 - J\partial_\psi K_{01}(\theta)[\hat\psi] - JK_{11}(\theta)\hat y - J\partial_\alpha K_{01}(\theta)[\hat\alpha]$ where

$$\mathcal{L}_\omega := \Pi_{\mathbb{S}}^\perp\left(\omega\cdot\partial_\theta + JK_{02}(\theta)\right)|_{\mathtt{H}_{\mathbb{S}}^\perp}. \tag{5.48}$$

We need that $\mathcal{L}_\omega$ is "almost invertible" up to a scales $K_n := K_0^{(3/2)^n}$ that we shall use for the non-linear Nash-Moser iteration in Chapter 10. Hence we have to require that the operator $\mathcal{L}_\omega$ is "almost" invertible, therefore we need following assumption:

- **Almost-invertibility assumption**. There exists a subset $\Lambda_0 \subset \Omega\times[\varepsilon_1,\varepsilon_2]$, such that for all $(\omega,\varepsilon)\in\Lambda_0$ the operator $\mathcal{L}_\omega$ in (5.48) can be decomposed as

$$\mathcal{L}_\omega = \mathbf{L}_\omega + \mathbf{R}_\omega + \mathbf{R}_\omega^\perp \tag{5.49}$$

  where $\mathbf{L}_\omega$ is invertible and $\mathbf{R}_\omega,\mathbf{R}_\omega^\perp$ satisfy the estimate (9.97), (9.98) and (9.99). More precisely for every $g\in H^{p+\sigma}(\mathbb{T}^{1+N})\cap\mathtt{H}_{\mathbb{S}}^\perp$ and such that $g(-\theta)=-\rho g(\theta)$ (see (1.43)) there exists a solution $h:=\mathbf{L}_\omega^{-1}g\in H^p(\mathbb{T}^{1+N})\cap\mathtt{H}_{\mathbb{S}}^\perp$, with $h(-\theta)=\rho h(\theta)$ of the linear equation $\mathbf{L}_\omega h = g$ which satisfies for all $\mathfrak{p}_0\le p\le P$ the tame estimate

$$\begin{aligned}\|\mathbf{L}_\omega^{-1}g\|_p^{k_0,\gamma} &\le_P \gamma^{-1}\left(\|g\|_{p+\sigma}^{k_0,\gamma} + \mu\gamma^{-1}\|g\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\left[\|\mathcal{V}_0\|_{p+\sigma+\nu(\mathtt{b})}^{k_0,\gamma} + \gamma^{-1}\|\mathcal{V}_0\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\|Z\|_{p+3}^{k_0,\gamma}\right]\right)\\ &\le_P \gamma^{-1}\left(\|g\|_{p+\sigma}^{k_0,\gamma} + \|g\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\right)\end{aligned} \tag{5.50}$$

  for some $\sigma:=\sigma(\tau,N,k_0)\ge 0$ and $\nu(\mathtt{b})$ defined in (9.25).

Remark 5.10. *This inversion assumption must be verified at each n step of the Nash-Moser nonlinear iteration, as we shall do thanks to Theorem 9.18. Note that in Chapter 8 and 9 we almost diagonalize $\mathcal{L}_\omega$ up to remainders of size $O(\mu N_{n-1}^{\mathfrak{a}-1})$ where the scales N_n are given by*

$$N_n := K_n^r\,, \quad \textit{i,e.} \quad N_0 := K_0^r\,, \tag{5.51}$$

with $r>1$ large enough, it satisfies (10.6). *This process allows us to verify the inverse assumption. Moreover the set of the good parameters Λ_0 is contained in $\mathtt{DC}_{K_n}^\gamma\times[\varepsilon_1,\varepsilon_2]$, where $\mathtt{DC}_{K_n}^\gamma$ is defined in* (5.5). *Actually the parameters $(\omega,\varepsilon)\in\Lambda_0$ have to satisfy the first and the second Melnikov non-resonance conditions* (9.94).

If we consider the operator defined in (5.46) we have that $\partial_\psi K_{10}, \partial_{\psi\psi} K_{00}, \partial_\psi K_{00}(\theta)$ and $\partial_\psi K_{01}(\theta)$ vanish at an exact solution (see Lemma 5.7), and also the small remainders $\mathbf{R}_\omega$ and $\mathbf{R}_\omega^\perp$ are equal to zero on an exact solution, hence it is natural to look for an almost inverse of the operator

$$\mathbb{D}[\hat\psi, \hat y, \hat z, \hat\alpha] := \begin{pmatrix} \omega \cdot \partial_\theta \hat\psi - \partial_\alpha K_{10}(\theta)[\hat\alpha] - K_{20}(\theta)\hat y - K_{11}^T(\theta)\hat z \\ \omega \cdot \partial_\theta \hat y + \partial_\psi \partial_\alpha K_{00}(\theta)[\hat\alpha] \\ \mathbf{L}_\omega \hat z + J\partial_\alpha K_{01}(\theta)[\hat\alpha] + JK_{11}(\theta)\hat y \end{pmatrix}, \tag{5.52}$$

where $\mathbf{L}_\omega = \omega \cdot \partial_\theta - (-J)K_{02}(\theta)$. In addition since we require only finitely many non resonance condition, i.e. $|\omega \cdot l| \le \gamma^{-1} \langle l \rangle^\tau$, $|l| \le K_n$ we also decompose $\omega \cdot \partial_\theta$ as:

$$\omega \cdot \partial_\theta = \mathcal{D}_\omega^{(n)} + \mathcal{D}_\omega^{(n),\perp}, \quad \mathcal{D}_\omega^{(n)} := \Pi_{K_n} \omega \cdot \partial_\theta \Pi_{K_n} + \Pi_{K_n}^\perp \quad \mathcal{D}_\omega^{(n),\perp} := \Pi_{K_n}^\perp \omega \cdot \partial_\theta \Pi_{K_n}^\perp - \Pi_{K_n}^\perp \tag{5.53}$$

and we also split the operator $\mathbb{D}$ in (5.52) as

$$\mathbb{D} = \mathbb{D}_n + \mathbb{D}_n^\perp, \qquad \text{where} \quad \mathbb{D}_n^\perp[\hat\psi, \hat y, \hat z, \hat\alpha] := \begin{pmatrix} \mathcal{D}_\omega^{(n),\perp} \hat\psi \\ \mathcal{D}_\omega^{(n),\perp} \hat y \\ 0 \end{pmatrix}, \tag{5.54}$$

$$\mathbb{D}_n[\hat\psi, \hat y, \hat z, \hat\alpha] := \begin{pmatrix} \mathcal{D}_\omega^{(n)} \hat\psi - \partial_\alpha K_{10}(\theta)[\hat\alpha] - K_{20}(\theta)\hat y - K_{11}^T(\theta)\hat z \\ \mathcal{D}_\omega^{(n)} \hat y + \partial_\psi \partial_\alpha K_{00}(\theta)[\hat\alpha] \\ \mathbf{L}_\omega \hat z + J\partial_\alpha K_{01}(\theta)[\hat\alpha] + JK_{11}(\theta)\hat y \end{pmatrix}. \tag{5.55}$$

By the smoothing properties (2.11) the operator $\mathcal{D}_\omega^{(n),\perp}$ in (5.53) satisfies

$$\|\mathcal{D}_\omega^{(n),\perp} h\|_{\mathfrak{p}_0}^{k_0,\gamma} \le K_n^{-b} \|h\|_{\mathfrak{p}_0+b+1}^{k_0,\gamma}, \quad \forall b > 0, \quad \|\mathcal{D}_\omega^{(n),\perp} h\|_p^{k_0,\gamma} \le \|h\|_{p+1}^{k_0,\gamma}. \tag{5.56}$$

Lemma 5.11. *Assume that $\omega \in \mathtt{DC}_{K_n}^\gamma$, defined in (5.5). Then, for all $g \in H^p(\mathbb{T} \times \mathbb{T}^N)$ with zero average, the linear equation $\mathcal{D}_\omega^{(n)} h = g$ has the unique solution $h = \big(\mathcal{D}_\omega^{(n)}\big)^{-1} g$ with zero average, which satisfies*

$$\|\big(\mathcal{D}_\omega^{(n)}\big)^{-1} g\|_p^{k_0,\gamma} \le_{k_0} \gamma^{-1} \|g\|_{p+\tau_1}^{k_0,\gamma}, \quad \tau_1 = \tau + k_0(\tau+1). \tag{5.57}$$

We are looking for an exact inverse of $\mathbb{D}_n$ defined in (5.55). Therefore we have to solve the system

$$\mathbb{D}_n[\hat\psi, \hat y, \hat z, \hat\alpha] = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \tag{5.58}$$

where (g_1, g_2, g_3) satisfy the reversibility property (see the Almost-invertibility assumption before and the definition of $\tilde\rho$ given in (1.42) i.e. (1.43))

$$g_1(-\theta) = g_1(\theta), \quad g_2(-\theta) = -g_2(\theta), \quad g_3(-\theta) = -(\rho g_3)(\theta). \tag{5.59}$$

We consider the second equation in (5.58), that is

$$\mathcal{D}_\omega^{(n)} \hat y + \partial_\theta \partial_\alpha K_{00}(\theta)[\hat\alpha] = g_2. \tag{5.60}$$

By reversibility, we have that the θ average of the right hand side of the equation above vanishes, that is

$$\int_{\mathbb{T}^N} \partial_\theta K_{00}(\theta) d\theta = 0 \quad \text{and} \quad \int_{\mathbb{T}^N} g_2(\theta) d\theta = 0 \,. \tag{5.61}$$

By (5.61) and Lemma 5.11 we have that the solution of (5.60) is well defined and it is given by

$$\hat{y} := (\mathcal{D}_\omega^{(n)})^{-1}(-\partial_\psi \partial_\alpha K_{00}(\theta)[\hat{\alpha}] + g_2) \,. \tag{5.62}$$

Under the assumption (5.50) we can solve the equation

$$\hat{z} := \mathbf{L}_\omega^{-1}((-J)\partial_\alpha K_{01}(\theta)[\hat{\alpha}] + (-J)K_{11}(\theta)\hat{y} + g_3) \,. \tag{5.63}$$

We now substitute (5.62) and (5.63) in the first equation in (5.58) and we found that

$$\begin{aligned}
\mathcal{D}_\omega^{(n)} \hat{\psi} =& \partial_\alpha K_{10}(\theta)[\hat{\alpha}] + K_{20}(\theta) \Big((\mathcal{D}_\omega^{(n)})^{-1}(-\partial_\psi \partial_\alpha K_{00}(\theta)[\hat{\alpha}] + g_2) \Big) \\
& + K_{11}^T(\theta) \Big(\mathbf{L}_\omega^{-1} \Big[(-J)\partial_\alpha K_{01}(\theta)[\hat{\alpha}] + (-J)K_{11}(\theta)(\mathcal{D}_\omega^{(n)})^{-1}(-\partial_\psi \partial_\alpha K_{00}(\theta)[\hat{\alpha}] + g_2)) + g_3 \Big] \Big) \\
=& \Big(\partial_\alpha K_{10}(\theta) - K_{20}(\theta)(\mathcal{D}_\omega^{(n)})^{-1} \partial_\psi \partial_\alpha K_{00}(\theta) + K_{11}^T(\theta)\mathbf{L}_\omega^{-1}(-J)\partial_\alpha K_{01}(\theta) \\
& - K_{11}^T(\theta)\mathbf{L}_\omega^{-1}(-J)K_{11}(\theta)(\mathcal{D}_\omega^{(n)})^{-1}(\partial_\psi \partial_\alpha K_{00}(\theta) \Big) [\hat{\alpha}] \\
& + \Big(K_{20}(\theta)(\mathcal{D}_\omega^{(n)})^{-1} + K_{11}^T(\theta)\mathbf{L}_\omega^{-1}(-J)K_{11}(\theta)(\mathcal{D}_\omega^{(n)})^{-1} \Big) g_2 \\
& + K_{11}^T(\theta)\mathbf{L}^{-1} g_3 + g_1 \\
=& M_1(\theta)[\hat{\alpha}] + M_2(\theta) g_2 + M_3(\theta) g_3 + g_1 \,,
\end{aligned} \tag{5.64}$$

where

$$\begin{aligned}
M_1(\theta) :=& \, \partial_\alpha K_{10}(\theta) - K_{20}(\theta)(\mathcal{D}_\omega^{(n)})^{-1} \partial_\psi \partial_\alpha K_{00}(\theta) + K_{11}^T(\theta)\mathbf{L}_\omega^{-1}(-J)\partial_\alpha K_{01}(\theta) \\
& - K_{11}^T(\theta)\mathbf{L}_\omega^{-1}(-J)K_{11}(\theta)(\mathcal{D}_\omega^{(n)})^{-1} \partial_\psi \partial_\alpha K_{00}(\theta)
\end{aligned} \tag{5.65}$$

$$M_2(\theta) := K_{20}(\theta)(\mathcal{D}_\omega^{(n)})^{-1} + K_{11}^T(\theta)\mathbf{L}_\omega^{-1}(-J)K_{11}(\theta)(\mathcal{D}_\omega^{(n)})^{-1} \tag{5.66}$$

$$M_3(\theta) := K_{11}^T(\theta)\mathbf{L}_\omega^{-1} \,. \tag{5.67}$$

Therefore, in order to solve (5.64) we have to choose $\hat{\alpha}$ such that the right hand side of (5.64) has zero average, that is

$$\int_{\mathbb{T}^N} (M_1(\theta)[\hat{\alpha}] + M_2(\theta) g_2 + M_3(\theta) g_3 + g_1) d\theta = 0 \,.$$

By (5.9), (5.57) and Lemma 5.8 we have that the θ-averaged matrix

$$\langle M_1 \rangle = \mathbb{1} + O(\mu \gamma^{-1(1+k_1)}) \,. \tag{5.68}$$

Therefore, for $\mu\gamma^{-1(1+k_1)}$ is small enough, $\langle M_1 \rangle$ is invertible and $\langle M_1 \rangle^{-1} = \mathbb{1} + O(\mu\gamma^{-1(1+k_1)})$. Thus we can define

$$\hat{\alpha} := - \langle M_1 \rangle^{-1} \left(\langle g_1 \rangle + \langle M_2 g_2 \rangle + \langle M_3 g_3 \rangle \right) \,. \tag{5.69}$$

Then, with this choice of $\hat{\alpha}$, by Lemma 5.11 the equation (5.64) has the solution

$$\hat{\psi} := (\mathcal{D}_\omega^{(n)})^{-1}\left(M_1(\theta)[\hat{\alpha}] + M_2(\theta)g_2 + M_3(\theta)g_3 + g_1\right) . \tag{5.70}$$

In conclusion $(\hat{\psi}, \hat{y}, \hat{z}, \hat{\alpha})$, with $\hat{y}$ given in (5.62), $\hat{z}$ in (5.63), $\hat{\alpha}$ in (5.69) and $\hat{\psi}$ in (5.70) is a solution of (5.58).

Lemma 5.12. *Assume (5.9) with $\nu = \sigma + \nu(\mathtt{b})$ and* (5.50)*. Then, for all $(\omega, \varepsilon) \in \Lambda_0$, $\forall g := (g_1, g_2, g_3)$ satisfying* (5.59)*, the system* (5.58) *has solution $\mathbb{D}_n^{-1} g := (\hat{\psi}, \hat{y}, \hat{z}, \hat{\alpha})$ where $(\hat{\psi}, \hat{y}, \hat{z}, \hat{\alpha})$ are defined in* (5.70)*,* (5.62)*,*(5.63) *and* (5.69)*, which satisfies* (1.43) *and for any $\mathfrak{p}_0 \le p \le P$*

$$\begin{aligned} \|\mathbb{D}_n^{-1} g\|_p^{k_0,\gamma} &\le_P \gamma^{-1}\Big(\|g\|_{p+\sigma}^{k_0,\gamma} + \mu\gamma^{-1}\|g\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}[\|\mathcal{V}_0\|_{p+\sigma+\nu(\mathtt{b})}^{k_0,\gamma} + \gamma^{-1}\|\mathcal{V}_0\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\|\mathcal{F}(i_0,\alpha_0)\|_{p+\sigma}^{k_0,\gamma}]\Big) \\ &\le_P \gamma^{-1}\Big(\|g\|_{p+\sigma}^{k_0,\gamma} + \|g\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\Big) . \end{aligned} \tag{5.71}$$

Proof. By the explicit definition of M_2 and M_3 in (5.66) and (5.67), and by (5.50), (5.9), (5.57) and Lemma 5.8 we have

$$\|M_2 g\|_{\mathfrak{p}_0}^{k_0,\gamma} + \|M_3 g\|_{\mathfrak{p}_0}^{k_0,\gamma} \le C\|g\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} .$$

By the explicit definition of α in (5.69) and by (5.68), we arrive to

$$|\hat{\alpha}|^{k_0,\gamma} \le C\|g\|_{\mathfrak{p}_0}^{k_0,\gamma} .$$

The explicit definition of $\hat{y}$ in (5.62) and (5.57) imply

$$\|\hat{y}\|_p^{k_0,\gamma} \le_p \gamma^{-1}(\|g\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\|g\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}) .$$

For estimate $\hat{z}$ we use (5.50), hence $\hat{z}$ satisfies (5.71). Finally by the explicit definition of $\hat{\psi}$, given in (5.70),and by (5.66), (5.67), (5.50), (5.57) and Lemma 5.8 we have that $\hat{\psi}$ satisfies (5.71) . □

Now we are ready to give the expression of the almost approximate right inverse. The operator

$$\mathbf{T}_0 := \mathbf{T}_0(i_0) := (D\tilde{G}_\delta)(\theta, 0, 0) \circ \mathbb{D}_n^{-1} \circ (DG_\delta)(\theta, 0, 0)^{-1} \tag{5.72}$$

is an approximate right inverse for $d_{i,\alpha}\mathcal{F}(i_0)$ (as we shall prove in Lemma 5.13) where

$$\tilde{G}_\delta(\psi, y, z, \alpha) = (G_\delta(\psi, y, z), \alpha)$$

is the identity on the α-component.

We denote the norm $\|(\psi, y, z, \alpha)\|_p^{k_0,\gamma} := \max\{\|(\psi, y, z)\|_p^{k_0,\gamma}, |\alpha|^{k_0,\gamma}\}$, where $\|(\psi, y, z)\|_p^{k_0,\gamma}$ is defined in (4.5) and $|\cdot|^{k_0,\gamma}$ is defined in (2.7).

Theorem 5.13. ***Almost-approximate inverse***. *Assume that the inversion assumptions* (5.49)-(5.50) *hold. Then there exists $\bar{\sigma} := \bar{\sigma}(\tau, N, k_0) > 0$ such that, if* (5.9) *holds with $\nu = \bar{\sigma} + \nu(\mathtt{b})$, then for all*

$(\omega,\varepsilon)\in\Lambda_0$, for all $g:=(g_1,g_2,g_3)$ satisfying (5.59), the operator $\mathbf{T}_0$ defined in (5.72) satisfies, for all $\mathfrak{p}_0\le p\le P$

$$
\begin{aligned}
\|\mathbf{T}_0 g\|_p^{k_0,\gamma} &\le_P \gamma^{-1}\Big(\|g\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}+\mu\gamma^{-1}\|g\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}[\|\mathcal{V}_0\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma} \\
&\quad +\gamma^{-1}\|\mathcal{V}_0\|_{\mathfrak{p}_0+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\|\mathcal{F}(i_0,\alpha_0)\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}]\Big) \\
&\le_P \gamma^{-1}\Big(\|g\|_{p+\bar\sigma}^{k_0,\gamma}+\|g\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\Big)\,.
\end{aligned}
\tag{5.73}
$$

Moreover $\mathbf{T}_0$ is an approximate inverse of $d_{i,\alpha}\mathcal{F}(i_0)$, namely we may decompose $d_{i,\alpha}\mathcal{F}(i_0)\circ\mathbf{T}_0-\mathbb{1}$ as follows

$$
d_{i,\alpha}\mathcal{F}(i_0)\circ\mathbf{T}_0-\mathbb{1}=\mathcal{P}(i_0)+\mathcal{P}_\omega(i_0)+\mathcal{P}_\omega^\perp(i_0) \tag{5.74}
$$

where the operators $\mathcal{P}$, $\mathcal{P}_\omega$, $\mathcal{P}_\omega^\perp$ satisfy, for all $\mathfrak{p}_0\le p\le P$

$$
\begin{aligned}
\|\mathcal{P}g\|_p^{k_0,\gamma} &\le_P \gamma^{-1}\Big(\|\mathcal{F}(i_0,\alpha_0)\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}\|g\|_{p+\bar\sigma}^{k_0,\gamma}+ \\
&\quad +\|g\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}\Big[\|\mathcal{F}(i_0,\alpha_0)\|_{p+\bar\sigma}^{k_0,\gamma}+\|\mathcal{F}(i_0,\alpha_0)\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\Big]\Big)
\end{aligned}
\tag{5.75}
$$

$$
\|\mathcal{P}_\omega g\|_p^{k_0,\gamma} \le_P \mu\gamma^{-2}N_{n-1}^{-\mathtt{a}}\Big(\|g\|_{p+\bar\sigma}^{k_0,\gamma}+\|g\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\Big) \tag{5.76}
$$

$$
\|\mathcal{P}_\omega^\perp g\|_p^{k_0,\gamma} \le_P \gamma^{-1}\Big(\|g\|_{p+\bar\sigma}^{k_0,\gamma}+\|g\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\Big) \tag{5.77}
$$

$$
\|\mathcal{P}_\omega^\perp g\|_{\mathfrak{p}_0}^{k_0,\gamma} \le_{P,b} \gamma^{-1}K_n^{-b}\Big(\|g\|_{p+\bar\sigma+b}^{k_0,\gamma}+\|g\|_{\mathfrak{p}_0+\bar\sigma}^{k_0,\gamma}\|\mathcal{V}_0\|_{p+\bar\sigma+\nu(\mathtt{b})}^{k_0,\gamma}\Big)\,,\quad \forall b>0\,. \tag{5.78}
$$

Proof. The bound (5.73) follows by (5.72), (5.71) and by (5.43). By (4.4), since $X_{\mathcal{N}}$ does not depend on I, and i_δ differs by i_0 only on the I component, see (5.14), we have

$$
\begin{aligned}
d_{i,\alpha}\mathcal{F}(i_\delta)-d_{i,\alpha}\mathcal{F}(i_0) &= \mu\Big(d_iX_P(i_\delta)[\hat\imath]-d_iX_P(i_0)[\hat\imath]\Big) \\
&= \mu\int_0^1 \partial_I d_iX_P(\theta_0,I_0+s(I_\delta-I_0),w_0)[I_\delta-I_0,\Pi[\cdot]]ds \\
&:= \mathcal{E}_0 \\
&:= \mathcal{E}_0^{(n)}+\mathcal{E}_0^{(n),\perp}\,,
\end{aligned}
\tag{5.79}
$$

where $\Pi:(\hat\imath,\hat\alpha)\mapsto\hat\imath$ and (recall (5.16) and (5.15))

$$
\mathcal{E}_0^{(n)} := \mu\int_0^1 \partial_I d_iX_P(\theta_0,I_0+s(I_\delta-I_0),w_0)[I_\delta^{(n)}-I_0,\Pi[\cdot]]ds \tag{5.80}
$$

$$
\mathcal{E}_0^{(n),\perp} := \mu\int_0^1 \partial_I d_iX_P(\theta_0,I_0+s(I_\delta-I_0),w_0)[I_\delta^{(n),\perp}-I_0,\Pi[\cdot]]ds\,. \tag{5.81}
$$

Let us define $(\psi,y,z)=:v$, hence v is the symplectic coordinates induced by G_δ in (5.27). Then (recall the definition of K_α in (5.28) and the corresponding equations of motion given in (5.30)) the non linear operator $\mathcal{F}$ in (4.3) reads

$$
\mathcal{F}(G_\delta(v(\theta)),\alpha)=DG_\delta(v(\theta))(\omega\cdot\partial_\theta v(\theta)-X_{K_\alpha}(v(\theta),\alpha))\,.
$$

Differentiating the equation above at the trivial torus $(\theta,0,0) = G_\delta^{-1}(i_\delta)(\theta) := v_\delta(\theta)$ and $\alpha=\alpha_0$ we get

$$\begin{aligned} d_{i,\alpha}\mathcal{F}(i_\delta) =& DG_\delta(v_\delta)(\omega\cdot\partial_\theta - d_{v,\alpha}X_{K_\alpha}(v_\delta,\alpha))D\tilde{G}_\delta(v_\delta)^{-1} + \\ &+ D^2G_\delta(v_\delta)\left[DG_\delta(v_\delta)^{-1}\mathcal{F}(i_\delta,\alpha_0)][DG_\delta(v_\delta)^{-1}[\hat{\imath}]\right] \\ =& DG_\delta(v_\delta)(\omega\cdot\partial_\theta - d_{v,\alpha}X_{K_\alpha}(v_\delta,\alpha_0))D\tilde{G}_\delta(v_\delta)^{-1} + \mathcal{E}_1\,, \end{aligned} \tag{5.82}$$

where, recalling the splitting $\mathcal{F}(i_\delta,\alpha_0) = Z_\delta = Z_\delta^{(n)} + Z_\delta^{(n),\perp}$ we have

$$\mathcal{E}_1 := D^2G_\delta(v_\delta)\left[DG_\delta(v_\delta)^{-1}\mathcal{F}(i_\delta,\alpha_0)][DG_\delta(v_\delta)^{-1}[\hat{\imath}]\right] = \mathcal{E}_1^{(n)} + \mathcal{E}_1^{(n),\perp} \tag{5.83}$$

with

$$\mathcal{E}_1^{(n)} := D^2G_\delta(v_\delta)\left[DG_\delta(v_\delta)^{-1}Z_\delta^{(n)},\ DG_\delta(v_\delta)^{-1}\Pi[\cdot]\right] \tag{5.84}$$

$$\mathcal{E}_1^{(n),\perp} := D^2G_\delta(v_\delta)\left[DG_\delta(v_\delta)^{-1}Z_\delta^{(n),\perp},\ DG_\delta(v_\delta)^{-1}\Pi[\cdot]\right]\,. \tag{5.85}$$

By the decomposition (5.54), (5.55), (5.49), and by Lemma 5.7, we obtain

$$(\omega\cdot\partial_\theta - d_{v,\alpha}X_{K_\alpha}(v_\delta,\alpha_0))[\hat{v},\hat{\alpha}] = \left(\mathbb{D}_n + \mathbb{D}_n^\perp + R_Z^{(n)} + R_Z^{(n),\perp} + \mathbb{R}_\omega + \mathbb{R}_\omega^\perp\right)[\hat{v},\hat{\alpha}] \tag{5.86}$$

where $R_Z^{(n)}$ and $R_Z^{(n),\perp}$ are defined (by splitting R) as follows

$$R_Z^{(n)}[\hat{\psi},\hat{y},\hat{z},\hat{\alpha}] := \begin{pmatrix} -\partial_\psi K_{10}^{(n)}(\theta,\alpha_0)[\hat{\psi}] \\ -\partial_{\psi\psi}K_{00}^{(n)}(\theta,\alpha_0)[\hat{\psi}] + [\partial_\psi K_{10}^{(n)}(\theta,\alpha_0)]^T\hat{y} + [\partial_\psi K_{01}^{(n)}(\theta,\alpha_0)]^T\hat{z} \\ +J\partial_\psi K_{01}^{(n)}(\theta,\alpha_0)[\hat{\psi}] \end{pmatrix}$$

$$R_Z^{(n),\perp}[\hat{\psi},\hat{y},\hat{z},\hat{\alpha}] := \begin{pmatrix} -\partial_\psi K_{10}^{(n),\perp}(\theta,\alpha_0)[\hat{\psi}] \\ -\partial_{\psi\psi}K_{00}^{(n),\perp}(\theta,\alpha_0)[\hat{\psi}] + [\partial_\psi K_{10}^{(n),\perp}(\theta,\alpha_0)]^T\hat{y} + [\partial_\psi K_{01}^{(n),\perp}(\theta,\alpha_0)]^T\hat{z} \\ +J\partial_\psi K_{01}^{(n),\perp}(\theta,\alpha_0)[\hat{\psi}] \end{pmatrix}$$

and

$$\mathbb{R}_\omega[\hat{\psi},\hat{y},\hat{z},\hat{\alpha}] := \begin{pmatrix} 0 \\ 0 \\ \mathbf{R}_\omega[\hat{z}] \end{pmatrix}, \quad \mathbb{R}_\omega^\perp[\hat{\psi},\hat{y},\hat{z},\hat{\alpha}] := \begin{pmatrix} 0 \\ 0 \\ \mathbf{R}_\omega^\perp[\hat{z}] \end{pmatrix}. \tag{5.87}$$

Hence by (5.79), (5.82), (5.83) and (5.86) we can write

$$\begin{aligned} d_{i,\alpha}\mathcal{F}(i_\delta) =& DG_\delta(v_\delta)(\omega\cdot\partial_\theta - d_{v,\alpha}X_{K_\alpha}(v_\delta,\alpha_0))D\tilde{G}_\delta(v_\delta)^{-1} + \mathcal{E}_1 \\ =& DG_\delta(v_\delta)\circ\mathbb{D}_n\circ D\tilde{G}_\delta(v_\delta)^{-1} + \mathcal{E}^{(n)} + \mathcal{E}_\omega + \mathcal{E}_\omega^\perp \end{aligned} \tag{5.88}$$

where

$$\mathcal{E}^{(n)} := \mathcal{E}_0^{(n)} + \mathcal{E}_1^{(n)} + DG_\delta(v_\delta)R_Z^{(n)}D\tilde{G}_\delta(v_\delta)^{-1}, \quad \mathcal{E}_\omega := DG_\delta(v_\delta)\mathbb{R}_\omega D\tilde{G}_\delta(v_\delta)^{-1}, \tag{5.89}$$

$$\mathcal{E}^{(n),\perp} := \mathcal{E}_0^{(n)\perp} + \mathcal{E}_1^{(n)\perp} + DG_\delta(v_\delta)\left[\mathbb{R}_\omega^\perp + \mathbb{D}_n^\perp + R_Z^{(n)\perp}\right]D\tilde{G}_\delta(v_\delta)^{-1}. \tag{5.90}$$

By the definition of $\mathbf{T}_0$ in (5.72), and by (5.88), since $\mathbb{D}_n \circ \mathbb{D}_n^{-1} = \mathbb{1}$ (see Lemma 5.12), we get

$$d_{i,\alpha}\mathcal{F}(i_0) \circ \mathbf{T}_0 - \mathbb{1} = \mathcal{P} + \mathcal{P}_\omega + \mathcal{P}_\omega^\perp$$

$$\mathcal{P} := \mathcal{E}^{(n)} \circ \mathbf{T}_0, \quad \mathcal{P}_\omega := \mathcal{E}_\omega \circ \mathbf{T}_0, \quad \mathcal{P}_\omega^\perp := \mathcal{E}_\omega^\perp \circ \mathbf{T}_0 \, .$$

Hence thanks to Lemma 5.1, by (2.36), (5.9), (5.17), (5.18), (5.21), (5.44) and (5.45) we obtain

$$\begin{aligned} \|\mathcal{E}^{(n)}[\hat{\imath}, \hat{\alpha}]\|_p^{k_0,\gamma} &\leq \|Z\|_{p+\sigma}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \|\hat{\imath}\|_{p+\sigma}^{k_0,\gamma} + \|Z\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma} \\ &= \|\mathcal{F}(i_0,\alpha_0)\|_{p+\sigma}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} + \|\mathcal{F}(i_0,\alpha_0)\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \|\hat{\imath}\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{F}(i_0,\alpha_0)\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \|\mathcal{V}_0\|_{p+\sigma}^{k_0,\gamma} \end{aligned} \tag{5.91}$$

where we have used $Z := \mathcal{F}(i_0, \alpha_0)$. The estimate (5.75) follows by (5.73), (5.91) and (5.9). The estimates (5.76), (5.77) and (5.78) follow by (9.97), (9.98), (9.99), (5.73), (5.44),(5.17), (5.19), (5.22), (5.41), (5.9) and (5.56). □

Chapter 6

Linearized operator in the normal directions

In order to write and explicit expression of the linearized operator $\mathcal{L}_\omega$ in (5.48), we have to compute $\frac{1}{2}(K_{02}(\psi)z, z)_{L^2(\mathbb{T}_x)}$ with $z \in \mathtt{H}_{\mathbb{S}}^{\perp}$, that is the quadratic term in z of $(H_\alpha \circ G_\delta)(\psi, 0, z)$ defined in (5.29).

Lemma 6.1. *The operator $K_{02}(\psi)$ is*

$$K_{02}(\psi) = \Pi_{\mathbb{S}}^{\perp} \partial_{\mathtt{v}} \nabla_{\mathtt{v}} \mathcal{H}(\mathcal{A}(i_\delta(\psi))) + \mu R(\psi) \tag{6.1}$$

where $\mathtt{v} = (q, p)$ and $\mathcal{H}$ is the Hamiltonian defined in (1.26) evaluated at the torus

$$\mathcal{A}(i_\delta(\psi)) = \mathcal{A}(\vartheta_0(\psi), I_\delta(\psi), w_0(\psi)) = A(\vartheta_0(\psi), I_\delta(\psi)) + w_0(\psi) \tag{6.2}$$

where $\mathcal{A}$ is defined in (1.36) and A is defined in (1.37). *The operator $K_{02}(\psi)$ is reversibility preserving. The remainder $R(\psi)$ has the finite dimensional form*

$$R(\psi)[h] = \sum_{j=1}^{N} (h, g_j(\psi))_{L^2(\mathbb{T}_x)} \chi_j, \qquad \forall h \in \mathtt{H}_{\mathbb{S}}^{\perp} \tag{6.3}$$

for functions $g_j, \chi_j \in \mathtt{H}_{\mathbb{S}}^{\perp}$ which satisfy the tame estimates: for some $\sigma := \sigma(\tau, N) > 0, \forall p \geq \mathfrak{p}_0$,

$$\begin{aligned} \|g_j\|_p^{k_0,\gamma} + \|\chi_j\|_p^{k_0,\gamma} &\leq 1 + \|\mathcal{V}_\delta\|_{p+\sigma}^{k_0,\gamma} \\ \|\partial_i g_j[\hat{\imath}]\|_p^{k_0,\gamma} + \|\partial_i \chi_j[\hat{\imath}]\|_p^{k_0,\gamma} &\leq \|\hat{\imath}\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_\delta\|_{p+\sigma}^{k_0,\gamma} \|\hat{\imath}\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} . \end{aligned} \tag{6.4}$$

Proof. We consider G_δ defined in (5.27) and $\mathcal{A}$ defined in (1.36), then

$$\begin{aligned}
\mathcal{A}\circ G_\delta(\psi,y,z) &= \mathcal{A}\begin{pmatrix} \vartheta_0(\psi) \\ I_\delta(\psi)+[\partial_\psi\vartheta_0(\psi)]^{-T}y-[(\partial_\theta\tilde{w}_0)(\vartheta_0(\psi))]^T(-J)z \\ w_0(\psi)+z \end{pmatrix} = \\
&= A(\vartheta_0(\psi), I_\delta(\psi)+[\partial_\psi\vartheta_0(\psi)]^{-T}y-[(\partial_\theta\tilde{w}_0)(\vartheta_0(\psi))]^T(-J)z)+(w_0(\psi)+z) \\
&= \sum_{j\in\mathbb{S}}\sqrt{\frac{j}{\pi}}\begin{pmatrix} \sqrt{r_j+I_\delta^j(\psi)+([\partial_\psi\vartheta_0(\psi)]^{-T}y)_j-([(\partial_\theta\tilde{w}_0)(\vartheta_0(\psi))]^T(-J)z)_j}\cos(\vartheta_0)_j\cos jx \\ \sqrt{r_j+I_\delta^j(\psi)+([\partial_\psi\vartheta_0(\psi)]^{-T}y)_j-([(\partial_\theta\tilde{w}_0)(\vartheta_0(\psi))]^T(-J)z)_j}\sin(\vartheta_0)_j\sin jx \end{pmatrix} \\
&+ w_0(\psi)+z \\
&= \sum_{j\in\mathbb{S}}\sqrt{\frac{j}{\pi}}\begin{pmatrix} \sqrt{r_j+I_\delta^j(\psi)+(L_1(\psi)y)_j+(L_2(\psi)z)_j}\cos(\vartheta_0)_j\cos jx \\ \sqrt{r_j+I_\delta^j(\psi)+(L_1(\psi)y)_j+(L_2(\psi)z)_j}\sin(\vartheta_0)_j\sin jx \end{pmatrix} + w_0(\psi)+z\,,
\end{aligned}$$

where

$$L_1(\psi) := [\partial_\psi\vartheta_0(\psi)]^{-T}, \qquad L_2(\psi) := -[(\partial_\theta\tilde{w}_0)(\vartheta_0(\psi))]^T(-J)\,.$$

Let $H_\alpha = \mathcal{N}_\alpha + \mu P$, as in (4.2), then the operator K_{02} is given by

$$K_{02}(\psi) = \partial_z\nabla_z K_\alpha(\psi,0,0) = \partial_z\nabla_z(H_\alpha\circ G_\delta)(\psi,0,0) = \mathbf{D}\,|_{\mathbb{H}_{\mathbb{S}}^\perp} + \mu\partial_z\nabla_z(P\circ G_\delta)(\psi,0,0) \tag{6.5}$$

where $\mathbf{D}$ is defined in (1.45). If we consider the perturbed part of the Hamiltonian H_α (defined in (4.2)) composed with the change of variable G_δ, we get

$$(P\circ G_\delta)(\psi,y,z) = P\left(\theta_0(\psi), I_\delta(\psi)+L_1(\psi)y-L_2(\psi)z, w_0(\psi)+z\right)\,. \tag{6.6}$$

We now differentiate (6.6) with respect to z, and we obtain

$$\nabla_z(P\circ G_\delta)(\psi,y,z) = L_2(\psi)^T\partial_I P(G_\delta(\psi,y,z)) + \nabla_w P(G_\delta(\psi,y,z))\,. \tag{6.7}$$

Therefore

$$\begin{aligned}
\partial_z\nabla_z(P\circ G_\delta)(\psi,0,0) =& \partial_w\nabla_w P(i_\delta(\psi)) + L_2(\psi)^T\partial_{II}P(i_\delta(\psi))L_2(\psi) + L_2(\psi)^T\partial_w\partial_I P(i_\delta(\psi)) \\
&+ \partial_I\nabla_w P(i_\delta(\psi))L_2(\psi) \\
=& \partial_w\nabla_w P(i_\delta(\psi)) + R_1(\psi)+R_2(\psi)+R_3(\psi) \\
=& \partial_w\nabla_w P(i_\delta(\psi)) + R(\psi)
\end{aligned} \tag{6.8}$$

where $R(\psi) := R_1(\psi)+R_2(\psi)+R_3(\psi)$ and

$$\begin{aligned}
R_1(\psi) &:= L_2(\psi)^T\partial_{II}P(i_\delta(\psi))L_2(\psi), \\
R_2(\psi) &:= L_2(\psi)^T\partial_w\partial_I P(i_\delta(\psi)), \\
R_3(\psi) &:= \partial_I\nabla_w P(i_\delta(\psi))L_2(\psi)\,.
\end{aligned}$$

Note that each $R_i\,, i=1,2,3$ has the finite dimensional form (6.3) because it is the composition of at least one operator with finite rank $\mathbb{R}^N$. Indeed if we write $L_2(\psi):\mathtt{H}_{\mathbb{S}}^{\perp}\to\mathbb{R}^N$ as follows

$$L_2(\psi)[h]=\sum_{j=1}^{N}\big(h, L_2(\psi)^T[e_j]\big)_{L^2(\mathbb{T}_x)}[e_j], \quad \forall h\in \mathtt{H}_{\mathbb{S}}^{\perp}$$

then

$$R_1(\psi)[h]=\sum_{j=1}^{N}\big(h, L_2(\psi)^T[e_j]\big)_{L^2(\mathbb{T}_x)}\big(L_2(\psi)^T\partial_{II}P(i_\delta(\psi))\big)[e_j]\,.$$

Similarly we can write R_2 and R_3 as

$$R_2(\psi)[h]=\sum_{j=1}^{N}\left(h, \partial_w\partial_I P(i_\delta(\psi))[e_j]\right)_{L^2(\mathbb{T}_x)} L_2(\psi)^T[e_j]$$
$$R_3(\psi)[h]=\sum_{j=1}^{N}\big(h, L_2(\psi)^T[e_j]\big)_{L^2(\mathbb{T}_x)}\left(\partial_w\partial_I P(i_\delta(\psi))\right)[e_j]\,.$$

Therefore (6.4) follows by Lemma 5.1. By (6.5), (6.7), (1.44), (1.36), (1.27) and (1.28) we get

$$\begin{aligned}K_{02}(\psi)=&\mathbf{D}\,|_{\mathtt{H}_{\mathbb{S}}^{\perp}}+\mu\Pi_{\mathbb{S}}^{\perp}\partial_{\mathtt{v}}\nabla_{\mathtt{v}}P(\mathcal{A}(i_\delta(\psi)))+\mu R(\psi)\\ =&\Pi_{\mathbb{S}}^{\perp}\partial_{\mathtt{v}}\nabla_{\mathtt{v}}\mathcal{H}(\mathcal{A}(i_\delta(\psi)))+\mu R(\psi)\end{aligned}$$

which proves (6.1). □

In conclusion, by Lemma 6.1 the linear operator $\mathcal{L}_\omega$ defined in (5.48) has the form

$$\mathcal{L}_\omega=\Pi_{\mathbb{S}}^{\perp}(\mathcal{L}+\mu R)\Pi_{\mathbb{S}}^{\perp} \quad \text{where} \quad \mathcal{L}=\Omega\cdot\partial_\theta+J\partial_{\mathtt{v}}\nabla_{\mathtt{v}}\mathcal{H}(\mathcal{A}(i_\delta)(\theta))\,. \tag{6.9}$$

It is obtained linearizing the system (1.29) at the torus $\mathcal{A}(i_\delta(\theta))$ defined in (6.2), changing $\partial_t \rightsquigarrow \omega\cdot\partial_\theta$, and denoting Ω the 2×2-matrix given by

$$\Omega:=\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}.$$

Hence the linearized operator $\mathcal{L}$ is

$$\begin{aligned}\mathcal{L}:=&\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\cdot\partial_\theta+\begin{pmatrix}0 & iT(D)\\ iT(D) & 0\end{pmatrix}\\ &+\mu\varepsilon^4\begin{pmatrix}\partial_x\Lambda^{-1}((\Lambda^{-1}p_{xx})\Lambda) & \partial_x\Lambda^{-1}((\Lambda q)\partial_x^2\Lambda^{-1})\\ 0 & 0\end{pmatrix}\\ &+\mu\varepsilon^4\begin{pmatrix}\partial_x\Lambda^{-1}((\Lambda^{-1}p_x)\partial_x\Lambda) & \partial_x\Lambda^{-1}((\Lambda q_x)\partial_x\Lambda^{-1})\\ 0 & -\partial_x\Lambda((\Lambda^{-1}p_x)\partial_x\Lambda^{-1})\end{pmatrix}\\ &+\mu\varepsilon^2\begin{pmatrix}\partial_x\Lambda^{-1}((\Lambda^{-1}p)\Lambda) & \partial_x\Lambda^{-1}((\Lambda q)\Lambda^{-1})\\ 0 & \partial_x\Lambda((\Lambda^{-1}p)\Lambda^{-1})\end{pmatrix},\end{aligned} \tag{6.10}$$

where $iT(D)$ is defined in (1.21) and $\mathtt{v} := (p,q) := (p(x,\theta), q(x,\theta)) = \mathcal{A}(i_\delta(\theta))$ are functions in x and θ. By (1.17), (6.2), (1.36) and (1.43) we have that the function q is even in x, while the function p is odd in x (see (1.6)). Moreover the function q is even in θ, while the function p is odd in θ (see (12)). The operators $\mathcal{L}_\omega$ and $\mathcal{L}$ are real, reversible and send X_0^p defined in (1.8) in itself.

In the next two Chapters we reduce the linear operator $\mathcal{L}$ in (6.10) to constant coefficients up to a bounded remainder. The finite dimensional remainder R transforms under conjugation into an operator of the same form (see Lemma 9.2) and therefore it will be dealt only once at the end of Chapter 9.

From now on we will assume that for some $\nu := \nu(\tau, N) > 0\,,\ \gamma \in (0,1)$

$$\|\mathcal{V}_0\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu} \leq 1 \quad \overset{5.17}{\Rightarrow} \quad \|\mathcal{V}_\delta\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu} \leq 2\,. \tag{6.11}$$

Note that this condition will be satisfied by the approximate solutions at every step of the Nash-Moser iteration. Actually $\nu := \nu(b) + \sigma_1$ where $\nu(b)$ is defined in (9.25) and σ_1 is defined in (10.3), is fixed in the Nash-Moser iteration of Chapter 10.

In order to estimate the variation of the eigenvalues with respect to the approximate invariant torus, we have to estimate the derivatives with respect to the torus $i(\theta)$ in a low norm $\|\cdot\|_{p_1}$. Note that for all the Sobolev indices p_1 such that

$$p_1 + \sigma \leq \mathfrak{p}_0 + \nu\,, \quad \text{for some} \quad \sigma := \sigma(\tau, N) > 0\,, \tag{6.12}$$

we have

$$\|\mathcal{V}_0\|^{k_0,\gamma}_{p_1+\sigma} \leq 1 \quad \overset{5.17}{\Rightarrow} \quad \|\mathcal{V}_\delta\|^{k_0,\gamma}_{p_1+\sigma} \leq 1\,.$$

The constants ν and σ represent losses of derivatives at any step of the reduction procedure in Chapters (8), (9). It (possibly) will increase along the finitely many steps of such a procedure. We shall fix the largest loss of derivatives $\sigma := \sigma(b)$ in Chapter 9.

Note that the Sobolev index p_1 is introduced since in the reducibility scheme (see Chapter 9) the remainder $\mathbf{Q}_0$ satisfy the estimates (9.23). In Lemma 9.5 we consider $\mathbf{Q}_0 = \mathbf{Q}$ defined in Proposition 9.3 and so we want that (9.14) holds with $p_1 = \mathfrak{p}_0$. For this reason we estimate (in Chapters 7, 8 and in Appendix B) the derivatives ∂_i of functions, operators, pseudo-differential operators, in the intermediate norm $\|\cdot\|_{p_1}$, where p_1 satisfies (6.12).

As a consequence of the Moser composition Lemma 2.24 the Sobolev norm of the function $\mathtt{v} = (q,p)(x,\theta) = \mathcal{A}(i_\delta(\theta))$ (see (6.2)) satisfies

$$\|\mathtt{v}\|^{k_0,\gamma}_p \leq C(p)\left(1 + \|\mathcal{V}_0\|^{k_0,\gamma}_p\right)\,, \quad \forall p \geq \mathfrak{p}_0. \tag{6.13}$$

Similarly for $p_1 + \sigma \leq \mathfrak{p}_0 + \nu$

$$\|\partial_i \mathtt{v}[\hat{\imath}]\|_{p_1} \leq_{p_1} \mu \|\hat{\imath}\|_{p_1}\,.$$

Note that in Chapters 9 and 10 we have to estimate the finite difference $\|\mathtt{v}(i_1) - \mathtt{v}(i_2)\|_{p_1}$ in terms of the difference $\|i_1 - i_2\|_{p_1+\sigma}$. In order to do that we consider the derivatives ∂_i. It is sufficient to estimate only this low norm since it gives enough informations required in order to control the variation of the eigenvalues of $\mathcal{L}$ with respect to the torus.

By the extension procedure of Chapter 10 we have that $\mathcal{V}_0 := \mathcal{V}_0(\omega, \varepsilon)$ is defined for all $(\omega, \varepsilon) \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$. Moreover all the functions appearing in $\mathcal{L}$ defined in (6.10) are $C^\infty-$functions both in x and θ as the approximate torus $\mathtt{v} = \mathcal{A}(i_\delta(\theta))$. This enables us to use directly pseudo-differential operators theory presented in Chapter 2.

Chapter 7

Symmetrization of the linear part

In order to prove the inversion assumptions (5.49), (5.50) we now perform a reduction of the linear operator $\mathcal{L}$, in (6.10), in decreasing symbols. In Section 7.1, we provide the asymptotic expansion of $\mathcal{L}$ in homogeneous symbols up to order $-M$ plus a suitable bounded remainder with symbol in S^{-M-1}. The constant M will be fixed in Chapter 9 and it depends only on the "absolute constants" $k_0, \mathfrak{p}_0, \mathtt{b}$ see (9.24). In Section 7.2 we block diagonalize the highest order of the linear part of $\mathcal{L}$.

7.1 Asymptotic expansion of the linearized operator

The linearized operator given in (6.10) is the composition of some pseudo-differential operators. We recall Definition 7 for pseudodifferential opeators with a $C^\infty-$symbol, and Theorem 2.5, that we shall use in the following Lemma in order to write the composition of pseudo-differential operators as a homogeneous terms plus a suitable remainder.

Lemma 7.1. *Let* $\Lambda := Op\left(\left(\frac{2}{15}\varepsilon^4\xi^4 - \frac{1}{3}\varepsilon^2\xi^2 + 1\right)^{1/4}\right)$ *be the pseudo-differential operator introduced in* (1.19). *Let* $a(x,\theta) \in H^p(\mathbb{T}\times\mathbb{T}^N)$. *Then* $\forall M \in \mathbb{N}$ *we have the following asymptotic expansion,*

$$
\begin{aligned}
\partial_x\Lambda^{-1}(a(x,\theta)\partial_x^2\Lambda^{-1}) &= -\left(\frac{2}{15}\right)^{-1/2}\varepsilon^{-2}a(x,\theta)\partial_x + \sum_{k=0}^{M} c_k^{(1)}\tilde{a}_k^{(1)}(x,\theta)\partial_x^{-k} + Op(\sigma_1(x,\theta,\xi))\,,\\
\partial_x\Lambda^{-1}(a(x,\theta)\partial_x\Lambda) &= a(x,\theta)\partial_x^2 + \sum_{k=0}^{M} c_k^{(2)}\tilde{a}_k^{(2)}(x,\theta)\partial_x^{-k} + Op(\sigma_2(x,\theta,\xi))\,,\\
\partial_x\Lambda^{-1}(a(x,\theta)\partial_x\Lambda^{-1}) &= \sum_{k=0}^{M} c_k^{(3)}\tilde{a}_k^{(3)}(x,\theta)\partial_x^{-k} + Op(\sigma_3(x,\theta,\xi))\,,\\
\partial_x\Lambda^{-1}(a(x,\theta)\Lambda) &= a(x,\theta)\partial_x + \sum_{k=0}^{M} c_k^{(4)}\tilde{a}_k^{(4)}(x,\theta)\partial_x^{-k} + Op(\sigma_4(x,\theta,\xi))\,,\\
\partial_x\Lambda^{-1}(a(x,\theta)\Lambda^{-1}) &= \sum_{k=0}^{M} c_k^{(5)}\tilde{a}_k^{(5)}(x,\theta)\partial_x^{-k} + Op(\sigma_5(x,\theta,\xi))\,,
\end{aligned}
\tag{7.1}
$$

$$\partial_x\Lambda(a(x,\theta)\partial_x\Lambda^{-1}) = a(x,\theta)\partial_x^2 + 2\partial_x a(x,\theta)\partial_x + \sum_{k=0}^{M} c_k^{(6)}\tilde{a}_k^{(6)}(x,\theta)\partial_x^{-k} + Op(\sigma_6(x,\theta,\xi)),$$

$$\partial_x\Lambda(a(x,\theta)\Lambda^{-1}) = a(x,\theta)\partial_x + \sum_{k=0}^{M} c_k^{(7)}\tilde{a}_k^{(7)}(x,\theta)\partial_x^{-k} + Op(\sigma_7(x,\theta,\xi)),$$

where $c_k^{(i)} \in \mathbb{R}$ *for* $i = 1, ..., 7,$ *and* $k \leq M$*, are some real constant coefficients,*

$$\tilde{a}_k^{(i)}(x,\theta) = \sum_{j=0}^{k+1} d_j\partial_x^j a(x,\theta)\,,\ i = 1,2,4\,,\ d_j \in \mathbb{R}\,, \textit{possibly equal to zero for some } j,\ k \leq M\,,$$

$$\tilde{a}_k^{(i)}(x,\theta) = \sum_{j=0}^{k} d_j\partial_x^j a(x,\theta)\,,\ i = 3,5\,,\ d_j \in \mathbb{R}\,, \textit{possibly equal to zero for some } j,\ k \leq M\,,$$

$$\tilde{a}_k^{(6)}(x,\theta) = \sum_{j=0}^{k+3} d_j\partial_x^j a(x,\theta)\,,\ d_j \in \mathbb{R}\,, \textit{possibly equal to zero for some } j,\ k \leq M\,,$$

$$\tilde{a}_k^{(7)}(x,\theta) = \sum_{j=0}^{k+2} d_j\partial_x^j a(x,\theta)\,,\ d_j \in \mathbb{R}\,, \textit{possibly equal to zero for some } j,\ k \leq M\,,$$

and $\Sigma_i(x,\theta,D) := Op(\sigma_i(x,\theta,\xi))$, $i = 1, ..., 7$ *is the remainder belonging to* OPS^{-M-1} *for all* $i = 1, ..., 7$. *Furthermore*

$$|\Sigma_i|_{-M-1,p,0}^{k_0,\gamma} \leq C(p,M)\|a\|_{p+M+3}^{k_0,\gamma}, \quad i = 1, ..., 7\,. \tag{7.2}$$

Proof. As previously discussed, since we are working with pseudo-differential operators, a good strategy is to consider their asymptotic expansion. Therefore, instead of Λ^d for $d \in \mathbb{Z}$ we can consider its asymptotic expansion (recall Definition 8)

$$\left(\frac{2}{15}\varepsilon^4\xi^4 - \frac{1}{3}\varepsilon^2\xi^2 + 1\right)^{d/4} = \left(\frac{2}{15}\right)^{d/4}\varepsilon^d|\xi|^d\left(1 - \frac{5}{2\varepsilon^2\xi^2} + \frac{15}{2\varepsilon^4\xi^4}\right)^{d/4}$$

$$= \begin{cases} \left(\frac{2}{15}\right)^{d/4}\varepsilon^d|\xi|^d\left[1 - \frac{5d}{8\varepsilon^2\xi^2} + \frac{15d}{8\varepsilon^4\xi^4} + ... + \binom{\frac{d}{4}}{\frac{M+d}{2}}\left(-\frac{5}{2\varepsilon^2\xi^2} + \frac{15}{2\varepsilon^4\xi^4}\right)^{\frac{M+d}{2}}\right] + \sigma_{M+1}(\xi) \\ \text{if } M+d \text{ is even}, \\ \left(\frac{2}{15}\right)^{d/4}\varepsilon^d|\xi|^d\left[1 - \frac{5d}{8\varepsilon^2\xi^2} + \frac{15d}{8\varepsilon^4\xi^4} + ... + \binom{\frac{d}{4}}{\frac{M+d+1}{2}}\left(-\frac{5}{2\varepsilon^2\xi^2} + \frac{15}{2\varepsilon^4\xi^4}\right)^{\frac{M+1+d}{2}}\right] + \sigma_{M+1}(\xi) \\ \text{if } M+d \text{ is odd}, \end{cases} \tag{7.3}$$

where $\sigma_{M+1}(\xi) \in S^{-M-1}$.

Note that, only for notational reasons, in what follows we are not writing the θ-component of the above operators and functions, since the pseudo-differential operators Λ and Λ^{-1} defined in (1.19) act only on the spatial component.

Remark 7.2. *Let* $A := Op(a(x,\xi))$ *and* $B := Op(b(x,\xi))$ *be two pseudo-differential operators and let* $\mathtt{n} \in \mathbb{N}$. *Then* $A \circ B := C$ *is a pseudo-differential operator (see Theorem 2.5) and it admits the following* *asymptotic expansion*

$$
\begin{aligned}
C =& Op(a(x,\xi)\circ b(x,\xi)) \\
=& Op\left(\sum_{k\geq 0}\frac{(-i)^k}{k!}(\partial_\xi^k a(x,\xi))(\partial_x^k b(x,\xi))\right) \\
=& Op\left(\sum_{k=0}^{\mathtt{n}}\frac{(-i)^k}{k!}(\partial_\xi^k a(x,\xi))(\partial_x^k b(x,\xi))\right)+\Sigma(x,\xi)\,.
\end{aligned}
\tag{7.4}
$$

Let

$$
(e,b,c):=\{(-1,2,-1),\,(-1,1,1),\,(-1,1,-1),\,(-1,0,1),\,(-1,0,-1),\,(1,1,-1),\,(1,0,-1)\}\,. \tag{7.5}
$$

Then

$$
\begin{aligned}
&\partial_x\Lambda^e(a(x)\partial_x^b\Lambda^c)= \\
&\quad Op\left(\sum_{k=0}^{\mathtt{n}}\frac{(-i)^k}{k!}i(i\xi)^b\left[\frac{2}{15}\varepsilon^4\xi^4-\frac{1}{3}\varepsilon^2\xi^2+1\right]^{c/4}\left(\partial_\xi^k\left[\xi\left[\frac{2}{15}\varepsilon^4\xi^4-\frac{1}{3}\varepsilon^2\xi^2+1\right]^{e/4}\right]\right)(\partial_x^k a(x))\right) \\
&+\Sigma(x,\xi) \\
&=a(x)Op\left(\left(\frac{2}{15}\varepsilon^4\xi^4-\frac{1}{3}\varepsilon^2\xi^2+1\right)^{\frac{c}{4}+\frac{e}{4}}(i\xi)^{(b+1)}\right) \\
&+Op\left(\sum_{k\neq 0,\,\text{odd}}^{\mathtt{n}}\frac{(-i)^k}{k!}i(i\xi)^b\left(\frac{2}{15}\varepsilon^4\xi^4-\frac{1}{3}\varepsilon^2\xi^2+1\right)^{\frac{c}{4}+\frac{e}{4}-k}(\partial_x^k a(x))\left(\sum_{s=1}^{k+\frac{k+1}{2}}\xi^{2s}\right)\right) \\
&+Op\left(\sum_{k\neq 0,\,\text{even}}^{\mathtt{n}}\frac{(-i)^k}{k!}\xi i(i\xi)^b\left(\frac{2}{15}\varepsilon^4\xi^4-\frac{1}{3}\varepsilon^2\xi^2+1\right)^{\frac{c}{4}+\frac{e}{4}-k}(\partial_x^k a(x))\left(\sum_{s=1}^{k+\frac{k}{2}}\xi^{2s}\right)\right)+\Sigma(x,\xi) \\
&\overset{(7.3)}{=}Op\left(a(x)\left(\frac{2}{15}\varepsilon^4\xi^4-\frac{1}{3}\varepsilon^2\xi^2+1\right)^{\frac{c}{4}+\frac{e}{4}}(i\xi)^{(b+1)}\right) \\
&+Op\left(\sum_{k\text{ odd}}^{\mathtt{n}}\frac{(-i)^k}{k!}i(i\xi)^bC(k,\varepsilon)|\xi|^{c+e-4k}\left[1+C_1(k,\varepsilon)\xi^{-2}+C_2(k,\varepsilon)\xi^{-4}\right.\right. \\
&\left.\left.+\ldots+C_{\tilde{M}_k}\xi^{-\tilde{M}_k}+\sigma_{M+1}(\xi,x)\right]\partial_x^k a(x)\left(\sum_{s=1}^{k+\frac{k+1}{2}}\xi^{2s}\right)\right) \\
&+Op\left(\sum_{k\text{ even}}^{\mathtt{n}}\frac{(-i)^k}{k!}i(i\xi)^bC(k,\varepsilon)|\xi|^{c+e-4k+1}\left[1+C_1(k,\varepsilon)\xi^{-2}+C_2(k,\varepsilon)\xi^{-4}\right.\right. \\
&\left.\left.+\ldots+C_{\tilde{M}_k}(k,\varepsilon)\xi^{-\tilde{M}_k}+\sigma_{M+1}(x,\xi)\right](\partial_x^k a(x))\left(\sum_{s=1}^{k+\frac{k}{2}}\xi^{2s}\right)\right)\,,
\end{aligned}
\tag{7.6}
$$

where $\tilde{M}_k$ is such that $-\tilde{M}_k+b+c+e-4k+2\geq -M-1$, and $\mathtt{n}$ is such that $4\mathtt{n}:=M-b-c-e$, this choose of $\mathtt{n}$ ensures that we are considering all the terms of order bigger than $-M$. Actually in what follows, we will consider $\mathtt{n}:=[\frac{M}{2}]+1>(M-b-c-e)/4$. Note that $e+c=-2,0$, so we can consider ξ^{e+c-4k} instead of $|\xi|^{e+c-4k}$.

Therefore if $\mathtt{n}$ is even (if $\mathtt{n}$ is odd we obtain a similar operator, but we have to consider as last term the opportune one) we obtain

$$\begin{aligned}\partial_x\Lambda^e(a(x)\partial_x^b\Lambda^c) = {} & a(x)\partial_x^{c+e+b+1} + ... + a(x)\partial_x^{-M}\\ & + a_x(x)(\partial_x^{c+e+b} + \partial_x^{c+e+b-2})(1 + c_2^{(1)}\partial_x^{-2} + ... + c_{\tilde{M}}^{(1)}\partial_x^{-\tilde{M}_1})\\ & + a_{x,x}(x)(\partial_x^{c+e+b-1} + \partial_x^{c+e+b-3} + \partial_x^{c+e+b-5})(1 + c_2^{(2)}\partial_x^{-2} + ... + c_{\tilde{M}}^{(2)}\partial_x^{-\tilde{M}_2}) + ... +\\ & + (\partial_x^{\mathtt{n}}a(x))(\partial_x^{c+e+b-4\mathtt{n}+1+2\mathtt{n}+\mathtt{n}} + ... + \partial_x^{c+e+b-4\mathtt{n}+1+2})(1 + c_2^{(\mathtt{n})}\partial_x^{-2} + ... + c_{\tilde{M}}^{(\mathtt{n})}\partial_x^{-\tilde{M}_{\mathtt{n}}})\\ & + \Sigma_M(x,D)\,,\end{aligned}$$

where $c_k^{(i)}$ in the equation above are some real constants derived by (7.3). Also note that $e + b + c \leq 1$ (see (7.5)). Collecting all the terms of the same order with respect to the derivative in x we can prove (7.1).

Now we prove (7.2). Let us consider $\partial_x\Lambda^e \circ a(x)\partial_x^b\Lambda^c$ with (e, b, c) as in (7.5), hence $e + b + c \leq 1$, then, by (2.26), (2.25) and (2.28) one has that the pseudo-differential operators $\Sigma_i,\ i = 1, ..., 7$ satisfy

$$\begin{aligned}|\Sigma_i|^{k_0,\gamma}_{1+e+b+c-\mathtt{n},p,0} \leq {} & C(p)|\partial_x\Lambda^e|^{k_0,\gamma}_{1+e,p,\mathtt{n}}|a(x)\partial_x^b\Lambda^c|^{k_0,\gamma}_{c+b,\mathtt{p}_0+2\mathtt{n}+1+e,0}\\ & + C(\mathtt{p}_0)|\partial_x\Lambda^e|^{k_0,\gamma}_{1+e,\mathtt{p}_0,\mathtt{n}}|a(x)\partial_x^b\Lambda^c|^{k_0,\gamma}_{c+b,p+2\mathtt{n}+1+e,0}\\ \leq {} & C(p)|\partial_x\Lambda^e|^{k_0,\gamma}_{1+e,p,\mathtt{n}}|a(x)|^{k_0,\gamma}_{0,\mathtt{p}_0+2\mathtt{n}+1+e,0}|\partial_x^b\Lambda^c|^{k_0,\gamma}_{c+b,\mathtt{p}_0+2\mathtt{n}+1+e,0}\\ & + C(p)|\partial_x\Lambda^e|^{k_0,\gamma}_{1+e,\mathtt{p}_0,\mathtt{n}}|a(x)|^{k_0,\gamma}_{0,\mathtt{p}_0+2\mathtt{n}+1+e,0}|\partial_x^b\Lambda^c|^{k_0,\gamma}_{b+c,p+2\mathtt{n}+1+e,0}\\ & + C(p)|\partial_x\Lambda^e|^{k_0,\gamma}_{1+e,\mathtt{p}_0,\mathtt{n}}|a(x)|^{k_0,\gamma}_{0,p+2\mathtt{n}+1+e,0}|\partial_x^b\Lambda^c|^{k_0,\gamma}_{b+c,\mathtt{p}_0+2\mathtt{n}+1+e,0}\\ \leq {} & C(p,\mathtt{n},e,b,c)\|a\|^{k_0,\gamma}_{\mathtt{p}_0+2\mathtt{n}+1+e} + C(\mathtt{p}_0,\mathtt{n})\|a\|^{k_0,\gamma}_{p+2\mathtt{n}+1+e}\\ \leq {} & C(p,\mathtt{n},e,b,c)\|a\|^{k_0,\gamma}_{p+2\mathtt{n}+1+e}\\ \leq {} & C(p,M)\|a\|^{k_0,\gamma}_{p+M+3}\,.\end{aligned} \tag{7.7}$$

where we have used $e \leq 1$ and $2\mathtt{n} \leq M + 1$. □

We now want to apply Lemma 7.1 to the linear operator $\mathcal{L}$ defined in (6.10). For this reason instead of a we shall consider, opportunely, $\Lambda q\,,\ \Lambda q_x\,,\ \Lambda^{-1}p\,,\ \Lambda^{-1}p_x\,,\ \Lambda^{-1}p_{xx}$. With this definition of a the following estimate holds

$$\|a\|_p^{k_0,\gamma} \leq \|\mathtt{v}\|^{k_0,\gamma}_{p+2}. \tag{7.8}$$

Then the operator $\mathcal{L}$ in (6.10) reads

$$\begin{aligned}\mathcal{L} = {} & \begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\cdot\partial_\theta + \begin{pmatrix}0 & iT(D)\\ iT(D) & 0\end{pmatrix} + \mu\varepsilon^4\begin{pmatrix}(\Lambda^{-1}p_x) & 0\\ 0 & -(\Lambda^{-1}p_x)\end{pmatrix}\partial_x^2\\ & + \mu\varepsilon^2\begin{pmatrix}(\Lambda^{-1}p) + \varepsilon^2(\Lambda^{-1}p_{xx}) & -(\frac{2}{15})^{-1/2}(\Lambda q)\\ 0 & (\Lambda^{-1}p) - 2\varepsilon^2(\Lambda^{-1}p_{xx})\end{pmatrix}\partial_x + \mu\begin{pmatrix}\tilde{R}_1 & \tilde{R}_2\\ \tilde{R}_3 & \tilde{R}_4\end{pmatrix}.\end{aligned} \tag{7.9}$$

We define

$$\tilde{\mathbf{R}} := \begin{pmatrix} \tilde{R}_1 & \tilde{R}_2 \\ \tilde{R}_3 & \tilde{R}_4 \end{pmatrix}, \text{ where } \tilde{R}_m := \sum_{k=0}^{M} c_k A_k^{(m)}(x,\theta)\partial_x^{-k} + \Sigma^{(s)}(x,\theta,D), \text{ for } m = 1,...,4. \tag{7.10}$$

The operators $\tilde{R}_m$, $m = 1,...,4$ are the sum of the homogeneous terms $A_k^{(m)}\partial_x^{-k}$ for $k = 0,...,M$ and $\Sigma^{(m)}(x,\theta,D)$ which is a pseudo-differential operator, whose symbol $\sigma^{(m)}$ belong to S^{-M-1}.

In addition if we define

$$\mathbf{A}_k\partial_x^{-k} := \begin{pmatrix} c_k A_k^{(1)}(x,\theta) & c_k A_k^{(2)}(x,\theta) \\ c_k A_k^{(3)}(x,\theta) & c_k A_k^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-k} \tag{7.11}$$

we obtain by Lemma 7.1, (2.27) and (2.29), recall also Remark 2.7, that, for all $0 \le k \le M$

$$\begin{aligned} |\mathbf{A}_k\partial_x^{-k}|_{-k,p,0}^{k_0,\gamma} &\le C(p)\|\mathbf{A}_k\|_p^{k_0,\gamma} \\ &\le C(p)\|\mathtt{v}\|_{p+k+5}^{k_0,\gamma}, \end{aligned} \tag{7.12}$$

where we have used that the functions a defined in Lemma 7.1 satisfy (7.8).

Furthermore, by (7.2) and (7.8) we get

$$\begin{aligned} |\Sigma^{(m)}|_{-M-1,p,0}^{k_0,\gamma} &\le C(p,M)\|\mathtt{v}\|_{p+2\mathtt{n}+5}^{k_0,\gamma} \\ &\le C(p,M)\|\mathtt{v}\|_{p+M+6}^{k_0,\gamma}, \end{aligned} \tag{7.13}$$

where $\mathtt{n}$ is given in Lemma 7.1, and we use $2\mathtt{n} \le M+1$. In addition, by (7.10) and (2.36), the following estimates hold

$$\begin{aligned} \|\partial_i\mathbf{A}_k[\hat{\imath}]\|_{p_1} &\le_{p_1} \|\hat{\imath}\|_{p_1+5+k}, \quad k = 0,...,M \\ |\partial_i\Sigma[\hat{\imath}]|_{-M,p_1,0} &\le_{p_1} \|\hat{\imath}\|_{p_1+M+6}. \end{aligned} \tag{7.14}$$

Remark 7.3. *Note that in the definition of $\tilde{\mathbf{R}}$ (see (7.10)) we are summing in k, with $k = 0,...,M$. It has no relation with the index of non-degeneracy k_0.*

7.2 Symmetrization of the highest order

In this Section we look for a transformation that makes the highest order of $\mathcal{L}$ defined in (7.9) diagonal. We consider the change of variables given in matrix form by

$$\mathcal{Z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{Z}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{7.15}$$

Hence the linear system defined in (7.9), becomes

$$\mathcal{Z}^{-1}\mathcal{L}\mathcal{Z} := \mathcal{L}_0 = \Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{B}^{(1)}(x,\theta)\partial_x^2 + \mathbf{C}^{(1)}(x,\theta)\partial_x + \mathbf{R} \tag{7.16}$$

where

$$\Omega = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \tag{7.17}$$

$$\mathbf{T(D)} = \begin{pmatrix} iT(D) & 0 \\ 0 & -iT(D) \end{pmatrix}, \tag{7.18}$$

$$\mathbf{B}^{(1)} = \mu \begin{pmatrix} 0 & \varepsilon^4(\Lambda^{-1}p_x) \\ \varepsilon^4(\Lambda^{-1}p_x) & 0 \end{pmatrix}, \tag{7.19}$$

$$\mathbf{C}^{(1)} = \mu\varepsilon^2 \begin{pmatrix} -\frac{1}{2}(\Lambda^{-1}p_{xx})\varepsilon^2 - \frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) + (\Lambda^{-1}p) & \frac{3}{2}(\Lambda^{-1}p_{xx})\varepsilon^2 + \frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) \\ \frac{3}{2}(\Lambda^{-1}p_{xx})\varepsilon^2 - \frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) & -\frac{1}{2}(\Lambda^{-1}p_{xx})\varepsilon^2 + \frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) + (\Lambda^{-1}p) \end{pmatrix}, \tag{7.20}$$

$$\mathbf{R} = \mu \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}, \tag{7.21}$$

where $R_1 = \frac{1}{2}(\tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 + \tilde{R}_4)$, $R_2 = \frac{1}{2}(\tilde{R}_1 - \tilde{R}_2 + \tilde{R}_3 - \tilde{R}_4)$, $R_3 = \frac{1}{2}(\tilde{R}_1 + \tilde{R}_2 - \tilde{R}_3 - \tilde{R}_4)$ and $R_4 = \frac{1}{2}(\tilde{R}_1 - \tilde{R}_2 - \tilde{R}_3 + \tilde{R}_4)$ and $\tilde{R}_m$, $m = 1, ..., 4$ are defined in (7.10).

From (7.10) the remainder $\mathbf{R}$ can be written, with an abuse of notation, as

$$\sum_{k=0}^{M} \mu \begin{pmatrix} A_k^{(1)}(x,\theta) & A_k^{(2)}(x,\theta) \\ A_k^{(3)}(x,\theta) & A_k^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-k} + \mu \begin{pmatrix} Op(\sigma_1(x,\theta,\xi)) & Op(\sigma_2(x,\theta,\xi)) \\ Op(\sigma_3(x,\theta,\xi)) & Op(\sigma_4(x,\theta,\xi)) \end{pmatrix}. \tag{7.22}$$

It is clear that $Op(\sigma_m(x,\theta,\xi)) := \Sigma_m(x,\theta,D)$, $m = 1, ...4$ are a linear combination of the remainder terms defined in (7.10), while $A_k^{(m)}(x,\theta)$ are linear conbination of the coefficient functions defined in (7.10). Moreover this new remainder satisfies the same estimates of the previous one, so , for all $k = 0, ..., M$, by (7.12), and Remark 2.2 we get

$$\|\mathbf{A}_k\|_p^{k_0,\gamma} \leq C(p)\|\mathrm{v}\|_{p+k+5}^{k_0,\gamma}, \tag{7.23}$$

and if we define

$$\Sigma := \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix}$$

by (7.13) and Remark 2.7 we obtain

$$|\Sigma|_{-M-1,p,0}^{k_0,\gamma} \leq C(p,M)\|\mathrm{v}\|_{p+M+6}^{k_0,\gamma}. \tag{7.24}$$

Moreover, by (7.14) we have

$$\begin{aligned} \|\partial_i \mathbf{A}_k[\hat{\imath}]\|_{p_1} &\leq_{p_1} \|\hat{\imath}\|_{p_1+5+k}, \quad k = 0, ..., M \\ |\partial_i \Sigma[\hat{\imath}]|_{-M,p_1,0} &\leq_{p_1} \|\hat{\imath}\|_{p_1+M+6}. \end{aligned} \tag{7.25}$$

Lemma 7.4. *Let E be as follows*

$$E := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \; f, g \in H^p(\mathbb{T} \times \mathbb{T}^N, \mathbb{R}) : \; f(-x,\theta) = g(x,\theta), \; \int_{\mathbb{T}} g(x)dx = \int_{\mathbb{T}} f(x)dx = 0 \right\}. \tag{7.26}$$

Then the linear operator $\mathcal{L}_0$ in (7.16) leaves E invariant.

Proof. After the rotation (7.15) the invariant (for $\mathcal{L}$ in (7.9)) subspace X_0^p defined in (1.8) reads

$$\begin{aligned} E :=& \mathcal{Z}^{-1} X_0^p \\ =& \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \ f, g, \in H^p(\mathbb{T}\times\mathbb{T}^N, \mathbb{R}) : \ f(-x,\theta) = g(x,\theta) \,, \ \int_{\mathbb{T}} g(x)dx = \int_{\mathbb{T}} f(x)dx = 0 \right\} . \end{aligned}$$

Indeed given $(\Lambda q, \Lambda^{-1}p) \in X_0^p$ we have

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \Lambda q \\ \Lambda^{-1} p \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda q + \Lambda^{-1}p \\ \Lambda q - \Lambda^{-1} p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} .$$

Since Λq is real and even in x, while $\Lambda^{-1}p$ is real and odd in the spatial variable we have that f, g are real and $f(-x) = g(x)$. Moreover both Λq and $\Lambda^{-1}p$ have zero average in the spatial varibale, hence f and g have zero average in the spatial variable. □

In Chapter 8 we will conjugate the operator $\mathcal{L}_0$ with other operators of the following form

$$T = \mathbb{1} + \begin{pmatrix} 0 & \varphi_2(x,\theta) \\ \varphi_3(x,\theta) & 0 \end{pmatrix} \quad \text{or} \quad T = \mathbb{1} + \begin{pmatrix} \varphi_1(x,\theta) & 0 \\ 0 & \varphi_4(x,\theta) \end{pmatrix} .$$

We want that every T leaves the space E invariant. For this reason in the following Lemma we give the general rules that a transformation has to satisfy in order to leave the space E invariant.

Lemma 7.5. *Let $k \in \mathbb{Z}$. Let $\mathbf{E}$ be*

$$\mathbf{E} := \begin{pmatrix} e_1(x,\theta) & e_2(x,\theta) \\ e_3(x,\theta) & e_4(x,\theta) \end{pmatrix} \partial_x^k$$

with

$$(-1)^k e_1(-x,\theta) = e_4(x,\theta) \,, \quad (-1)^k e_2(-x,\theta) = e_3(x,\theta) \,. \tag{7.27}$$

Then $\mathbf{E}$ leaves E invariant.

Proof. Let $(f,g) \in E$, then for every $k \in \mathbb{Z}$

$$\partial_x^k(g(x,\theta)) = \partial_x^k(f(-x,\theta)) = (-1)^k(\partial_x^k f)(-x,\theta) \,. \tag{7.28}$$

Hence, for every $k \in \mathbb{Z}$ given $\mathbf{E}$, by the formula above (7.27) we have that $\mathbf{E} : E \to E$. □

In Chapter 9 we shall use the matrix representation of operators. For this reason we present the following Lemma, that gives the conditions that the operators have to satisfy for sending E in itself.

Lemma 7.6. *Let* $\mathbf{B} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$. *Then $\mathbf{B} : E \to E$ if and only if*

$$\begin{gathered} (B_1)_k^j = \overline{(B_1)_{-k}^{-j}} \,, \ (B_2)_k^j = \overline{(B_2)_{-k}^{-j}} \,, \quad (B_3)_k^j = \overline{(B_3)_{-k}^{-j}} \,, \ (B_4)_k^j = \overline{(B_4)_{-k}^{-j}} \\ (B_1)_{-k}^j + (B_2)_{-k}^{-j} = (B_3)_k^j + (B_4)_k^{-j} \,. \end{gathered} \tag{7.29}$$

Proof. Let $w := (f, g) \in E$. Then

$$\mathbf{B}w = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} B_1 f + B_2 g \\ B_3 f + B_4 g \end{pmatrix} = \sum_{j,k\in\mathbb{Z}} \begin{pmatrix} (B_1)_k^j f_j + (B_2)_k^j g_j \\ (B_3)_k^j f_j + (B_4)_k^j g_j \end{pmatrix} e^{ikx}.$$

We have that $\mathbf{B}w$ is real if $B_1 f,\ B_2 g,\ B_3 f,\ B_4 g$ are real. Let us consider $B_1 f$, where f is a real function, then

$$\overline{(B_1)_k^j} \bar{f}_j e^{-ikx} = \overline{(B_1)_k^j} f_{-j} e^{-ikx} = \overline{(B_1)_{-k}^{-j}} f_j e^{ikx}.$$

Hence if

$$\overline{(B_1)_{-k}^{-j}} = (B_1)_k^j$$

then $B_1 f$ is real. Similar for the others. Now we want to find the conditions such that $(B_1 f + B_2 g)(-x) = (B_3 f + B_4 g)(x)$. Using the matrix rapresentation of the operators we have

$$\begin{aligned} (B_1 f + B_2 g)(-x) &= \sum_{j,k} \big((B_1)_k^j f_j + (B_2)_k^j g_j\big) e^{-ikx} \\ &= \sum_{j,k} \big((B_1)_{-k}^j f_j + (B_2)_{-k}^j g_j\big) e^{ikx} \\ &= \sum_{j,k} (B_3)_k^j f_j + (B_4)_k^j g_j \\ &= (B_3 f + B_4 g) x\,. \end{aligned}$$

Since

$$\sum_j (B_1)_{-k}^j f_j + \sum_j (B_2)_{-k}^j g_j = \sum_j ((B_1)_{-k}^j + (B_2)_{-k}^{-j}) f_j$$

and

$$\sum_j (B_3)_k^j f_j + \sum_j (B_4)_k^j g_j = \sum_j ((B_3)_k^j + (B_4)_k^{-j}) f_j$$

we arrive to

$$(B_1)_{-k}^j + (B_2)_{-k}^{-j} = (B_3)_k^j + (B_4)_k^{-j}\,.$$

□

The involution ρ, defined in (10), which is represented by the matrix

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

after the transformation $\mathcal{Z}$ defined in (7.15), becomes

$$\hat{\rho} := \mathcal{Z}^{-1} \rho \mathcal{Z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore (as in Definition 4) we have that a linear operator

$$\mathbf{B}(\theta) := \begin{pmatrix} B_1(\theta) & B_2(\theta) \\ B_3(\theta) & B_4(\theta) \end{pmatrix} \tag{7.30}$$

is

- is **reversible** if $\mathbf{B}(-\theta) \circ \hat{\rho} = -\hat{\rho} \circ \mathbf{B}(\theta)$,
- is **reversibility preserving** if $\mathbf{B}(-\theta) \circ \hat{\rho} = \hat{\rho} \circ \mathbf{B}(\theta)$.

Hence an operator $\mathbf{B}$ as in (7.30) is reversible if

$$B_1(-\theta) = -B_4(\theta)\,, \quad \text{and} \quad B_2(-\theta) = -B_3(\theta)\,,$$

and it is reversibility preserving if

$$B_1(-\theta) = B_4(\theta)\,, \quad \text{and} \quad B_2(-\theta) = B_3(\theta)\,.$$

In Chapter 9 we shall use these conditions in the Fourier exponential base. Hence an operator $\mathbf{B}$ as in (7.30) is reversible if

$$(B_1)_k^j(-l) = -(B_4)_k^j(l)\,, \quad \text{and} \quad (B_2)_k^j(-l) = -(B_3)_k^j(l) \;\; \forall j,k \in \mathbb{Z},\; l \in \mathbb{Z}^N\,, \tag{7.31}$$

and it is reversibility preserving if

$$(B_1)_k^j(-l) = (B_4)_k^j(l)\,, \quad \text{and} \quad (B_2)_k^j(-l) = (B_3)_k^j(l) \quad \forall j,k \in \mathbb{Z},\; l \in \mathbb{Z}^N. \tag{7.32}$$

The linear operator $\mathcal{L}_0$ defined in (7.16) is reversible with respect to $\tilde{\rho}$.

In the next Chapter we shall conjugate the operator $\mathcal{L}_0$ with operators T_j that are reversibility preserving in the sense presented above.

Chapter 8

Symmetrization at lower order

In this Chapter we conjugate $\mathcal{L}_0$ defined in (7.16) to a block diagonal constant coefficients up to a bounded remainder. We start by

$$\mathcal{L}_0 = \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{B}^{(1)}(x,\theta)\partial_x^2 + \mathbf{C}^{(1)}(x,\theta)\partial_x + \mathbf{R}\,, \tag{8.1}$$

where $\mathbf{T}(\mathbf{D})$ is defined in (7.18), Ω in (7.17), $\mathbf{B}^{(1)}$ in (7.19) , $\mathbf{C}^{(1)}$ is defined in (7.20) and $\mathbf{R}$ can be decomposed as in (7.22); then the next three steps are the following: in the first step we eliminate the off-diagonal coefficients up to order zero; In the second step (Section 8.2.2) we study the remainder, that can be written in a block diagonal form up to order $-M$ plus a pseudo-differential regularizing operator; Finally we make constant the first order coefficient.

8.1 Elimination of the second order operator

We want to eliminate the coefficient of the second order derivatives in (8.1).

Lemma 8.1. *There exists a real, reversibility preserving operator acting in E of the form*

$$T_1 = \mathbb{1} + \mu \begin{pmatrix} 0 & \varphi_2^{(1)}(x,\theta) \\ \varphi_3^{(1)}(x,\theta) & 0 \end{pmatrix} \partial_x^{-1} = \mathbb{1} + \Phi_1(x,\theta)\partial_x^{-1} \tag{8.2}$$

such that

$$\mathcal{L}_1 := (T_1)^{-1}\mathcal{L}_0 T_1 = \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(2)}(x,\theta)\partial_x + \mathbf{R}_1\,, \tag{8.3}$$

where

$$\begin{aligned}\mathbf{C}^{(2)} = \mu &\begin{pmatrix} -\frac{\sqrt{15}}{2\sqrt{2}}\mu\varepsilon^8(\Lambda^{-1}p_x)^2 - \frac{1}{2}(\Lambda^{-1}p_{xx})\varepsilon^4 & \frac{3}{2}(\Lambda^{-1}p_{xx})\varepsilon^4 + \varepsilon^2\frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) \\ \frac{3}{2}(\Lambda^{-1}p_{xx})\varepsilon^4 - \varepsilon^2\frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) & \frac{\sqrt{15}}{2\sqrt{2}}\mu\varepsilon^8(\Lambda^{-1}p_x)^2 - \frac{1}{2}(\Lambda^{-1}p_{xx})\varepsilon^4 \end{pmatrix} \\ + \mu &\begin{pmatrix} -\varepsilon^2\frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) + \varepsilon^2(\Lambda^{-1}p) & -\frac{3}{2\sqrt{15}}\varepsilon^4(\Lambda^{-1}p_{xx}) \\ -\frac{3}{2\sqrt{15}}\varepsilon^4(\Lambda^{-1}p_{xx}) & +\varepsilon^2\frac{\sqrt{15}}{2\sqrt{2}}(\Lambda q) + \varepsilon^2(\Lambda^{-1}p) \end{pmatrix},\end{aligned} \tag{8.4}$$

and $\mathbf{R}_1$ *is the matrix of symbols in* S^0 *defined in* (8.17) *that satisfy the estimates in Lemma 8.2. Moreover,*

$$\|\mathbf{C}^{(2)}\|_p^{k_0,\gamma} \leq \mu C(p)\|\mathrm{v}\|_{p+2}^{k_0,\gamma}\,. \tag{8.5}$$

The linear operator $\mathcal{L}_1$ *is real, reversible and acts in* E.

Proof. Note that with $\Phi_1(x,\theta)$ small enough (see (6.11)), the operator T_1 is invertible thanks to Neumann series, and $\partial_x^{-1}\cdot\partial_x = \partial_x\cdot\partial_x^{-1} = \pi_0$ where π_0 is the L^2-projector on the subspace of functions with zero average in the spatial variable; Furthermore $\partial_x\cdot\pi_0 = \pi_0\cdot\partial_x = \partial_x$, $\partial_x^{-1}\cdot\partial_{xxx} = \partial_{xx}$ and $\partial_x^{-1}\cdot\partial_{xx} = \partial_x$.

Then

$$\mathcal{L}T_1 - T_1(\Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D})) = [\mathbf{T}(\mathbf{D}),\Phi_1\partial_x^{-1}] + \mathbf{B}^{(1)}(x,\theta)\partial_x^2 + \mathbf{C}^{(1)}(x,\theta)\partial_x + \mathbf{B}^{(1)}\Phi_1\partial_x + \tilde{\mathbf{R}}_1\,. \tag{8.6}$$

We denote $(\cdot)_x\,,\,(\cdot)_{xx} := \partial_x(\cdot)\,,\,\partial_{xx}(\cdot)$; moreover we define

$$\tilde{\mathbf{R}}_1 := (\omega\cdot\partial_\theta\Phi_1)\partial_x^{-1} + \mathbf{C}^{(1)}(\Phi_1\pi_0) + \mathbf{C}^{(1)}(\Phi_1)_x\partial_x^{-1} + \mathbf{R} + \mathbf{R}\Phi\partial_x^{-1} + \mathbf{B}^{(1)}(\Phi_1)_{xx}\partial_x^{-1} + 2\mathbf{B}^{(1)}(\Phi_1)_x\pi_0\,, \tag{8.7}$$

that is the remainder that contains all the terms of order less or equal to zero in the space derivatives, and $\mathbf{R}$ is given in (7.21), $\mathbf{B}^{(1)}$ is given in (7.19) and $\mathbf{C}^{(1)}$ is given in (7.20).

We have to calculate

$$[\mathbf{T}(\mathbf{D}),\Phi_1(x,\theta)\partial_x^{-1}] = \\ =\mu\begin{pmatrix} 0 & iT(D)(\varphi_2^{(1)}(x,\theta)\partial_x^{-1}) + \varphi_2^{(1)}(x,\theta)\partial_x^{-1}iT(D) \\ -iT(D)(\varphi_3^{(1)}(x,\theta)\partial_x^{-1}) - \varphi_3^{(1)}(x,\theta)\partial_x^{-1}iT(D) & 0\end{pmatrix}.$$

For computing this commutator we use the asymptotic expansion of the operator $iT(D)$ defined in (1.21):

$$\begin{aligned} iT(D) :&= Op\left(i\xi\sqrt{\frac{2}{15}}\varepsilon^2\xi^2\left(1 - \frac{15}{2}\frac{1}{3\varepsilon^2\xi^2} + \frac{15}{2}\frac{1}{\varepsilon^4\xi^4}\right)^{1/2}\right) \\ &= Op\left(\sqrt{\frac{2}{15}}\varepsilon^2 i\xi^3\sum_{k=0}^{q}\binom{1/2}{k}\left(-\frac{15}{2}\frac{1}{3\varepsilon^2\xi^2} + \frac{15}{2}\frac{1}{\varepsilon^4\xi^4}\right)^k + r(\xi)\right) \\ &= -\sqrt{\frac{2}{15}}\varepsilon^2\partial_x^3 - \frac{\sqrt{5}}{2\sqrt{6}}\partial_x + \sum_{k=1}^{M-1}c_k\partial_x^{-k} + Op(r(\xi))\,, \end{aligned} \tag{8.8}$$

where $c_k\in\mathbb{R}$ are some constant, possibly equal to zero, $Op(r(\xi))$ is in OPS^{-M} and $q = \frac{M}{2}$ if M is even or $q = \frac{M+1}{2}$ if M is odd. Hence

$$\begin{aligned} [\mathbf{T}(\mathbf{D}),\Phi_1\partial_x^{-1}] &= \mu\varepsilon^2\begin{pmatrix} 0 & -\frac{2\sqrt{2}}{\sqrt{15}}\varphi_2^{(1)}(x,\theta)\partial_x^2 \\ \frac{2\sqrt{2}}{\sqrt{15}}\varphi_3^{(1)}(x,\theta)\partial_x^2 & 0\end{pmatrix} \\ &+ \mu\varepsilon^2\begin{pmatrix} 0 & -\frac{3\sqrt{2}}{\sqrt{15}}(\varphi_2^{(1)})_x\partial_x \\ \frac{3\sqrt{2}}{\sqrt{15}}(\varphi_3^{(1)})_x\partial_x & 0\end{pmatrix} + \begin{pmatrix} 0 & U_2 \\ U_3 & 0\end{pmatrix}\pi_0 + \begin{pmatrix} 0 & W_2 \\ W_3 & 0\end{pmatrix}\partial_x^{-1} \\ &+ \begin{pmatrix} 0 & P_2 \\ P_3 & 0\end{pmatrix} \\ &:= \mu\varepsilon^2\begin{pmatrix} 0 & -\frac{2\sqrt{2}}{\sqrt{15}}\varphi_2^{(1)}(x,\theta)\partial_x^2 \\ \frac{2\sqrt{2}}{\sqrt{15}}\varphi_3^{(1)}(x,\theta)\partial_x^2 & 0\end{pmatrix} + \mathbf{D}^{(1)}\partial_x + \mathbf{U}\pi_0 + \mathbf{W}\partial_x^{-1} + \mathbf{P}\,, \end{aligned} \tag{8.9}$$

where

$$\mathbf{U} := \mu \begin{pmatrix} 0 & U_2 \\ U_3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{cases} U_2 := -\sqrt{\frac{5}{6}}\varphi_2^{(1)} - 3\sqrt{\frac{2}{15}}\varepsilon^2(\varphi_2^{(1)})_{xx}\,, \\ U_3 := \sqrt{\frac{5}{6}}\varphi_3^{(1)} + 3\sqrt{\frac{2}{15}}\varepsilon^2(\varphi_3^{(1)})_{xx}\,, \end{cases} \tag{8.10}$$

while

$$\mathbf{W} := \mu \begin{pmatrix} 0 & W_2 \\ W_3 & 0 \end{pmatrix} \quad \text{and} \quad \begin{cases} W_2 := -\sqrt{\frac{2}{15}}\varepsilon^2(\varphi_2^{(1)})_{xxx} - \frac{1}{2}\sqrt{\frac{5}{6}}(\varphi_2^{(1)})_x \quad , \\ W_3 := \sqrt{\frac{2}{15}}\varepsilon^2(\varphi_3^{(1)})_{xxx} + \frac{1}{2}\sqrt{\frac{5}{6}}(\varphi_3^{(1)})_x \quad , \end{cases} \tag{8.11}$$

and

$$\mathbf{D}^{(1)} := \mu\varepsilon^2 \begin{pmatrix} 0 & -\frac{3\sqrt{2}}{\sqrt{15}}(\varphi_2^{(1)})_x \\ \frac{3\sqrt{2}}{\sqrt{15}}(\varphi_3^{(1)})_x & 0 \end{pmatrix}\,. \tag{8.12}$$

Finally $\mathbf{P} := \begin{pmatrix} 0 & P_2 \\ P_3 & 0 \end{pmatrix}$ and P_2, respectively P_3, are given by

$$\begin{aligned} P_2 &:= \mu\left(\sum_{k=1}^{M-1} c_k\partial_x^{-k} + Op(r(\xi))\right) \circ \varphi_2^{(1)}(x,\theta)\partial_x^{-1} + \mu\left(\varphi_2^{(1)}(x,\theta)\partial_x^{-1}\right) \circ \left(Op(r(\xi)) + \sum_{k=1}^{M-1} c_k\partial_x^{-k}\right) \\ P_3 &:= \mu\left(-\sum_{k=1}^{M-1} c_k\partial_x^{-k} - Op(r(\xi))\right) \circ \varphi_3^{(1)}(x,\theta)\partial_x^{-1} - \mu\left(\varphi_3^{(1)}(x,\theta)\partial_x^{-1}\right) \circ \left(\sum_{k=1}^{M-1} c_k\partial_x^{-k} + Op(r(\xi))\right)\,. \end{aligned} \tag{8.13}$$

We look for a transformation T_1 such that

$$\mathbf{B}^{(1)}(x,\theta) + \mu\varepsilon^2\frac{2\sqrt{2}}{\sqrt{15}} \begin{pmatrix} 0 & -\varphi_2^{(1)}(x,\theta) \\ \varphi_3^{(1)}(x,\theta) & 0 \end{pmatrix} = 0\,, \tag{8.14}$$

whose solution is, recalling (7.19),

$$\begin{aligned} \varphi_2^{(1)}(x,\theta) &:= \left(\frac{2\sqrt{2}}{\sqrt{15}}\right)^{-1}\varepsilon^2(\Lambda^{-1}p_x)(x,\theta)\,, \\ \varphi_3^{(1)}(x,\theta) &:= -\left(\frac{2\sqrt{2}}{\sqrt{15}}\right)^{-1}\varepsilon^2(\Lambda^{-1}p_x)(x,\theta)\,, \end{aligned} \tag{8.15}$$

then by (8.6), (8.9) and (8.15) we obtain

$$\begin{aligned} \mathcal{L}_1 :&= T_1^{-1}\mathcal{L}_0T_1 \\ &= \omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + T_1^{-1}\left(\mathbf{C}^{(1)}\partial_x + \mathbf{B}^{(1)}\Phi_1\partial_x + \mathbf{D}^{(1)}\partial_x\right) + T_1^{-1}\left(\mathbf{U}\pi_0 + \mathbf{W}\partial_x^{-1} + \mathbf{P} + \tilde{\mathbf{R}}_1\right) \\ &= \omega\cdot\partial_\varphi + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(2)}\partial_x + \mathbf{R}_1\,, \end{aligned} \tag{8.16}$$

where we define

$$\mathbf{C}^{(2)} := \mathbf{C}^{(1)} + \mathbf{B}^{(1)}\Phi_1 + \mathbf{D}^{(1)}\,,$$

and

$$\mathbf{R}_1 := T_1^{-1}(\mathbf{U}\pi_0 + \mathbf{W}\partial_x^{-1} + \mathbf{P} + \tilde{\mathbf{R}}_1) + (T_1^{-1} - \mathbb{1})\mathbf{C}^{(2)}\partial_x \tag{8.17}$$

where $\mathbf{P}$ is defined in (8.13), $\tilde{\mathbf{R}}_1$ is defined in (8.7), $\mathbf{U}$ is defined in (8.10) and $\mathbf{W}$ is defined in (8.11). We also have that $\mathbf{D}^{(1)}$ in (8.12), using (8.15), reads

$$\mathbf{D}^{(1)} = \begin{pmatrix} 0 & -\frac{3}{2}\mu\varepsilon^4\Lambda^{-1}p_{xx} \\ -\frac{3}{2}\mu\varepsilon^4\Lambda^{-1}p_{xx} & 0 \end{pmatrix} .$$

The inequality (8.5) follows by (2.36) and the definition of $\mathbf{C}^{(2)}$ in (8.4) (recall also Remark 2.2). Moreover let T_1 as in (8.2), then, by (2.25) and (2.2) (see also Remark 2.2) we have

$$|T_1|^{k_0,\gamma}_{0,p,0} + |T_1^{-1}|^{k_0,\gamma}_{0,p,0} \le C(p)(1+\mu\|\Phi_1\|_p^{k_0,\gamma}) \le C(p)(1+\mu\|\mathtt{v}\|^{k_0,\gamma}_{p+1}) . \tag{8.18}$$

In addition, by the explicit definition of $\varphi_2^{(1)}$ and $\varphi_3^{(1)}$ given in (8.15), using that Λq is even in θ while $\Lambda^{-1}p$ is odd in θ we have that the transformation T_1 defined in (8.2) is reversibility preserving (see (7.32)), hence $\mathcal{L}_1$ in (8.3) is reversible (see 7.31).

Moreover by the explicit definition of $\varphi_2^{(1)}$ and $\varphi_3^{(1)}$ given in (8.15) we have that T_1 is real (i.e. sends real values functions into real valued functions) and $-\varphi_2^{(1)}(-x,\theta) = \varphi_3^{(1)}(x,\theta)$ (see Lemma 7.5), hence $T_1 : E \to E$. This implies that the operator $\mathcal{L}_1$ sends E into itself.

Finally, since T_1 is reversibility preserving, and $\mathcal{L}_0$ is reversible, the operator $\mathcal{L}_1$ is reversible. □

Lemma 8.2. *The operator* $\mathbf{R}_1$ *defined in* (8.17) *admits an asymptotic expansion*

$$\mathbf{R}_1 = \mu\sum_{k=0}^{M}\begin{pmatrix} A_k^{(1)} & A_k^{(2)} \\ A_k^{(3)} & A_k^{(4)} \end{pmatrix}\partial_x^{-k} + \mu\begin{pmatrix} \Sigma_{R_1,1} & \Sigma_{R_1,2} \\ \Sigma_{R_1,3} & \Sigma_{R_1,4} \end{pmatrix}$$
$$= \mu\sum_{k=0}^{M}\mathbf{A}_k\partial_x^{-k} + \mu\Sigma_{R_1} ,$$

where ∂_x^0 *denotes one of the operator belonging to* $\{a\pi_0 + b\mathbb{1} ,\ a,b\in\{0,1\}\}$.

Moreover, for all $m = 1,...,4$ *,* $k = 0,...,M$ *and* $\sigma := \sigma(\tau,N,k_0) > 0$ *we have*

$$\begin{aligned} \|A_k^{(m)}\|_p^{k_0,\gamma} &\le \|\mathtt{v}\|^{k_0,\gamma}_{p+k+5+\sigma} \\ |\Sigma_{R_1,m}|^{k_0,\gamma}_{-M-1,p,0} &\le \|\mathtt{v}\|^{k_0,\gamma}_{p+3M+6+\sigma} \\ \|\partial_i\mathbf{A}_k^{(m)}[\hat{\imath}]\|_{p_1} &\le_{p_1} \|\hat{\imath}\|_{p_1+5+k+\sigma} \\ |\partial_i\Sigma_{R_1,m}[\hat{\imath}]|_{-M,p_1,0} &\le_{p_1} \|\hat{\imath}\|_{p_1+3M+6+\sigma} . \end{aligned} \tag{8.19}$$

Proof. This lemma follows by Lemmas B.8, B.9 . □

8.2 Diagonalization of the first-order operator

Now we want to make constant the first order coefficient, for that we have to compute three steps. First of all we eliminate the out of diagonal terms in $\mathbf{C}^{(2)}$ defined in (8.4). Then we block symmetrize the remainder up to order $-M$ (see Section 8.2.2). Finally with a change of the space variable and the composition with an operator close to the identity we are able to make the first order coefficient constant (see Section 8.2.3).

8.2.1 Symmetrization of the first order

Lemma 8.3. *There exists a real reversibility preserving operator, acting in E, of the form*

$$\begin{aligned} T_2 &= \mathbb{1} + \mu \begin{pmatrix} 0 & \varphi_2^{(2)}(x,\theta) \\ \varphi_3^{(2)}(x,\theta) & 0 \end{pmatrix} \partial_x^{-2} \\ &= \mathbb{1} + \Phi_2(x,\theta)\partial_x^{-2}\,, \end{aligned} \tag{8.20}$$

such that, given $\mathcal{L}_1$ defined in (8.3)*, we have*

$$\mathcal{L}_2 := T_2^{-1}\mathcal{L}_1 T_2 = \Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)}(x,\theta)\partial_x + \mathbf{R}_2 \tag{8.21}$$

where

$$\begin{aligned} \mathbf{C}^{(3)} =&\mu \begin{pmatrix} \frac{1}{2}\left(-\varepsilon^2\sqrt{\frac{15}{2}}(\Lambda q) + 2\varepsilon^2(\Lambda^{-1}p) - \varepsilon^4(\Lambda^{-1}p_{xx})\right) & 0 \\ 0 & \frac{1}{2}\left(+\varepsilon^2\sqrt{\frac{15}{2}}(\Lambda q) + 2\varepsilon^2(\Lambda^{-1}p) - \varepsilon^4(\Lambda^{-1}p_{xx})\right) \end{pmatrix} \\ &+\mu \begin{pmatrix} -\frac{\sqrt{15}}{2\sqrt{2}}\mu\varepsilon^8(\Lambda^{-1}p_x)^2 & 0 \\ 0 & +\frac{\sqrt{15}}{2\sqrt{2}}\mu\varepsilon^8(\Lambda^{-1}p_x)^2 \end{pmatrix} \\ :=&\mu \begin{pmatrix} c_3^{(1)} & 0 \\ 0 & c_3^{(4)} \end{pmatrix} \end{aligned} \tag{8.22}$$

and $\mathbf{R}_2$ is in OPS^0, and satisfy the estimates in Lemma 8.4. In addition

$$\|\mathbf{C}^{(3)}\|_p^{k_0,\gamma} \leq \mu C(p)\|\mathrm{v}\|_{p+2}^{k_0,\gamma}\,. \tag{8.23}$$

The linear operator $\mathcal{L}_1$ is real, reversible and acts in E.

Proof. We conjugate $\mathcal{L}_1$ in (8.3) with the operator T_2. As before for Φ_2 small enough (see (6.11)), T_2 is invertible. Then

$$\mathcal{L}_1 T_2 - T_2(\omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D})) = [\mathbf{T}(\mathbf{D}), \Phi_2\partial_x^{-2}] + \mathbf{C}^{(2)}\partial_x + \tilde{\mathbf{R}}_2\,, \tag{8.24}$$

where

$$\tilde{\mathbf{R}}_2 := \mathbf{R}_1 T_2 + \mathbf{C}^{(2)}(\Phi_2)_x\partial_x^{-2} + (\omega\cdot\partial_\theta\Phi_2)\partial_x^{-2} + \mathbf{C}^{(2)}\Phi_2\partial_x^{-1}\,, \tag{8.25}$$

$\mathbf{R}_1$ is defined in (8.17) and $\mathbf{C}^{(2)}$ is defined in (8.4). By (8.8) we have the following asymptotic expansion:

$$iT(D) := -\sqrt{\frac{2}{15}}\varepsilon^2\partial_x^3 + \sum_{k=-1}^{M-2} c_k\partial_x^{-k} + Op(r(\xi))\,. \tag{8.26}$$

Actually we are considering this expansion instead of (8.8) because $\partial_x\circ\partial_x^{-2} = \partial_x^{-2}\circ\partial_x = \partial_x^{-1}\in OPS^{-1}$, therefore we can consider only the highest order.

$$[\mathbf{T}(\mathbf{D}), \Phi_2(x,\theta)\partial_x^{-2}] =\mu \begin{pmatrix} 0 & iT(D)\circ\varphi_2^{(2)}\partial_x^{-2} + \varphi_2^{(2)}\partial_x^{-2}\circ iT(D) \\ -iT(D)\circ\varphi_3^{(2)}\partial_x^{-2} - \varphi_3^{(2)}\partial_x^{-2}\circ iT(D) & 0 \end{pmatrix}$$

$$= \mu\varepsilon^2 \begin{pmatrix} 0 & -\frac{2\sqrt{2}}{\sqrt{15}}\varphi_2^{(2)}(x,\theta)\partial_x \\ \frac{2\sqrt{2}}{\sqrt{15}}\varphi_3^{(2)}(x,\theta)\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & P_2^{(2)} \\ P_3^{(2)} & 0 \end{pmatrix}, \tag{8.27}$$

where we define $\mathbf{P}_2$ as

$$\mathbf{P}_2 := \begin{pmatrix} 0 & P_2^{(2)} \\ P_3^{(2)} & 0 \end{pmatrix} \tag{8.28}$$

and $P_i^{(2)} \in OPS^0$ for $i = 1, 2$ is given by

$$\begin{aligned} P_2^{(2)} :=& \mu \left(\sum_{k=-1}^{M-2} c_k \partial_x^{-k} + Op(r(\xi)) \right) \circ \left(\varphi_2^{(2)} \partial_x^{-2} \right) + \mu \left(\varphi_2^{(2)} \partial_x^{-2} \right) \circ \left(\sum_{k=-1}^{M-2} c_k \partial_x^{-k} + Op(r(\xi)) \right) \\ &- \frac{3\sqrt{2}}{\sqrt{15}} (\varphi_2^{(2)})_x \pi_0 - \frac{3\sqrt{2}}{\sqrt{15}} (\varphi_2^{(2)})_{xx} \partial_x^{-1} - \frac{\sqrt{2}}{\sqrt{15}} (\varphi_2^{(2)})_{xxx} \partial_x^{-2} \end{aligned} \tag{8.29}$$

$$\begin{aligned} P_3^{(2)} :=& -\mu \left(\sum_{k=-1}^{M-2} c_k \partial_x^{-k} + Op(r(\xi)) \right) \circ \left(\varphi_3^{(2)} \partial_x^{-2} \right) - \mu \left(\varphi_3^{(2)} \partial_x^{-2} \right) \circ \left(\sum_{k=-1}^{M-2} c_k \partial_x^{-k} + Op(r(\xi)) \right) \\ &+ \frac{3\sqrt{2}}{\sqrt{15}} (\varphi_3^{(2)})_x \pi_0 + \frac{3\sqrt{2}}{\sqrt{15}} (\varphi_3^{(2)})_{xx} \partial_x^{-1} + \frac{\sqrt{2}}{\sqrt{15}} (\varphi_3^{(2)})_{xxx} \partial_x^{-2}. \end{aligned} \tag{8.30}$$

We look for a transformation T_2 such that

$$\mathbf{C}^{(2)} + \mu\varepsilon^2 \begin{pmatrix} 0 & -\frac{2\sqrt{2}}{\sqrt{15}}\varphi_2^{(2)}(x,\theta)\partial_x \\ \frac{2\sqrt{2}}{\sqrt{15}}\varphi_3^{(2)}(x,\theta)\partial_x & 0 \end{pmatrix} = \text{diagonal matrix}.$$

Therefore, recalling the definition of $\mathbf{C}^{(2)}$ in (8.4), we define Φ_2 as

$$\begin{aligned} \frac{2\sqrt{2}}{\sqrt{15}} \varphi_2^{(2)}(x,\theta) &:= \left(\frac{\sqrt{15}}{2\sqrt{2}} (\Lambda q) + \frac{3}{2}\varepsilon^2 (\Lambda^{-1} p_{xx}) - \varepsilon^2 \frac{3}{2\sqrt{15}} (\Lambda^{-1} p_{xx}) \right)(x,\theta), \\ \frac{2\sqrt{2}}{\sqrt{15}} \varphi_3^{(2)}(x,\theta) &:= -\left(-\frac{\sqrt{15}}{2\sqrt{2}} (\Lambda q) + \frac{3}{2}\varepsilon^2 (\Lambda^{-1} p_{xx}) - \frac{3}{2\sqrt{15}} \varepsilon^2 (\Lambda^{-1} p_{xx}) \right)(x,\theta). \end{aligned} \tag{8.31}$$

Hence by (8.27) (8.31) and (8.4) we have

$$\mathbf{C}^{(2)} + [\mathbf{T}(\mathbf{D}), \Phi_2 \partial_x^{-2}] = \mathbf{C}^{(3)} \partial_x + \mathbf{P}_2$$

where $\mathbf{C}^{(3)}$ is the diagonal matrix defined in (8.22). Then, by (8.24)

$$\begin{aligned} \mathcal{L}_2 :&= T_2^{-1} \mathcal{L}_1 T_2 = \\ &= \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + T_2^{-1} \left(\mathbf{C}^{(3)} \partial_x + \mathbf{P}_2 + \tilde{\mathbf{R}}_2 \right) \\ &= \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)}(x,\theta)\partial_x + \mathbf{R}_2, \end{aligned} \tag{8.32}$$

where

$$\mathbf{R}_2 := T_2^{-1} \left(\mathbf{P}_2 + \tilde{\mathbf{R}}_2 \right) + (T_2^{-1} - \mathbb{1}) \mathbf{C}^{(3)} \partial_x, \tag{8.33}$$

and $\tilde{\mathbf{R}}_2$ is defined in (8.25).

The inequality (8.23) follows by (2.36) and the explicit definition of $\mathbf{C}^{(3)}$ given in (8.22). By the explicit definition of T_2 in (8.20) and (8.31) (recall Remark 2.2) we have

$$|T_2|_{0,p,0}^{k_0,\gamma} + |(T_2)^{-1}|_{0,p,0}^{k_0,\gamma} \leq C(p)(1 + \mu\|\mathtt{v}\|_{p+2}^{k_0,\gamma}) \,. \tag{8.34}$$

Moreover, by the explicit definition of T_2 in (8.20) and (8.31), since Λq is even in θ while $\Lambda^{-1}p$ is odd in θ, we have that the transformation T_2 is reversibility preserving. Since $\mathcal{L}_1$ is reversible (see Lemma 8.1), we have that $\mathcal{L}_2$ in (8.21) is reversible. In addition $T_2 : E \to E$ and it is real, indeed $\varphi_2^{(2)}(-x,\theta) = \varphi_3^{(2)}(x,\theta)$, see Lemma 7.5. Hence $\mathcal{L}_2 : E \to E$ and it is real. □

Lemma 8.4. *The operator* $\mathbf{R}_2$ *defined in* (8.33) *admits the asymptotic expansion*

$$\begin{aligned} \mathbf{R}_2 &= \mu \sum_{k=0}^{M} \begin{pmatrix} (A_k^0)^{(1)} & (A_k^0)^{(2)} \\ (A_k^0)^{(3)} & (A_k^0)^{(4)} \end{pmatrix} \partial_x^{-k} + \mu \begin{pmatrix} \Sigma_{R_2,1} & \Sigma_{R_2,2} \\ \Sigma_{R_2,3} & \Sigma_{R_2,4} \end{pmatrix} \\ &= \sum_{k=0}^{M} \mu \mathbf{A}_k^0 \partial_x^{-k} + \mu \Sigma_{R_2} \,, \end{aligned} \tag{8.35}$$

where ∂_x^0 *denotes one of the operators belonging to* $\{a\pi_0 + b\mathbb{1}\,,\ a,b \in \{0,1\}\}$.

Moreover, for all $m = 1, ..., 4\,,\ k = 0, ..., M$ *and* $\sigma := \sigma(\tau, N, k_0) > 0$ *we have*

$$\begin{aligned} \|(A_k^0)^{(m)}\|_p^{k_0,\gamma} &\leq_p \|\mathtt{v}\|_{p+k+5+\sigma}^{k_0,\gamma}\,, \quad k = 0,1 \\ \|(A_k^0)^{(m)}\|_p^{k_0,\gamma} &\leq_p \|\mathtt{v}\|_{p+2k+5+\sigma}^{k_0,\gamma}\,, \quad 2 \leq k \leq M \\ |\Sigma_{R_2,m}|_{-M-1,p,0}^{k_0,\gamma} &\leq_p \|\mathtt{v}\|_{p+4M+6+\sigma}^{k_0,\gamma} \\ \|\partial_i(\mathbf{A}_k^0)^{(m)}[\hat{\imath}]\|_{p_1} &\leq_{p_1} \|\hat{\imath}\|_{p_1+5+k+\sigma}\,, \quad k = 0,1 \\ \|\partial_i(A_k^0)^{(m)}[\hat{\imath}]\|_{p_1} &\leq_{p_1} \|\mathtt{v}\|_{p_1+2k+5+\sigma}\,, \quad 2 \leq k \leq M \\ |\partial_i \Sigma_{R_2,m}[\hat{\imath}]|_{-M,p_1,0} &\leq_{p_1} \|\hat{\imath}\|_{p_1+4M+6+\sigma}\,. \end{aligned} \tag{8.36}$$

Proof. This lemma follows by Lemmas B.11 and B.10 . □

8.2.2 Block symmetrization up to smoothing remainders

The change of variable that we will do in the next Section (i.e. T_{M+4} defined in (8.54)) acts differently on the out of diagonal entries of a matrix (see Lemma 8.13). For this reason we also have to take care of the remainder. The idea is to use the same procedure introduced in the previous Sections. Hence we conjugate the operator $\mathcal{L}_2$ in (8.21) with M transformations close to the identity, invertible, such that the matrices of the coefficients up to order M (fixed), can be written in a block diagonal form, i.e. the out of diagonal entries are equal to zero.

Let $\mathcal{L}_2$ as in (8.21), and $\mathbf{C}^{(3)}$ as in (8.22).

Lemma 8.5. *There exist M real, reversibility preserving operators T_j, $j = 3, ..., M+3$ acting on E of the form*

$$T_j = \mathbb{1} + \mu \begin{pmatrix} 0 & \varphi_2^{(j)}(x,\theta) \\ \varphi_3^{(j)}(x,\theta) & 0 \end{pmatrix} \partial_x^{-j} := \mathbb{1} + \Phi_j \partial_x^{-j}\,, \tag{8.37}$$

such that

$$\mathcal{L}_{M+3} := T_{M+3}^{-1} \circ ... \circ T_3^{-1} \mathcal{L}_2 T_3 \circ ... \circ T_{M+3}$$

has the form

$$\mathcal{L}_{M+3} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \cdot \partial_\theta + \begin{pmatrix} iT(D) & 0 \\ 0 & -iT(D) \end{pmatrix} + \mu \begin{pmatrix} c_3^{(1)}(x,\theta) & 0 \\ 0 & c_3^{(4)}(x,\theta) \end{pmatrix} \partial_x + \mathbf{R}_{M+3}\,, \tag{8.38}$$

where

$$\begin{aligned} \mathbf{R}_{M+3} &:= \sum_{k=0}^{M} \mu \begin{pmatrix} (A_k^k)^{(1)}(x,\theta) & 0 \\ 0 & (A_k^k)^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-k} + \mu \begin{pmatrix} \Sigma_1(x,\theta,D) & \Sigma_2(x,\theta,D) \\ \Sigma_3(x,\theta,D) & \Sigma_4(x,\theta,D) \end{pmatrix} \\ &:= \mu \sum_{k=0}^{M} (\mathbf{A}_k^k)^D \partial_x^{-k} + \mu \Sigma\,, \end{aligned} \tag{8.39}$$

with ∂_x^0 that denotes one of the operator belonging to $\{a\pi_0 + b\mathtt{1}\,,\ a,b \in \{0,1\}\}$ and

$$(\mathbf{A}_k^k)^D := \begin{pmatrix} (A_k^k)^{(1)}(x,\theta) & 0 \\ 0 & (A_k^k)^{(4)}(x,\theta) \end{pmatrix}\,, \quad k = 0, ..., M$$

and

$$\Sigma := \begin{pmatrix} \Sigma_1(x,\theta,D) & \Sigma_2(x,\theta,D) \\ \Sigma_3(x,\theta,D) & \Sigma_4(x,\theta,D) \end{pmatrix}$$

where Σ_m, $m = 1, ..., 4$ is a pseudo-differential operator in OPS^{-M-1}. In addition $(\mathbf{A}_k^k)^D$ and Σ satisfy the estimate in Lemma 8.6. The operator $\mathcal{L}_{M+3}$ is real, reversible and acts in E.

Proof. By Lemma 8.4 we can write the linear operator (8.21), as

$$\begin{aligned} \mathcal{L}_2 = & \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \cdot \partial_\theta + \begin{pmatrix} iT(D) & 0 \\ 0 & -iT(D) \end{pmatrix} + \mu \begin{pmatrix} c_3^{(1)}(x,\theta) & 0 \\ 0 & c_3^{(4)}(x,\theta) \end{pmatrix} \partial_x + \\ & + \mu \begin{pmatrix} (A_0^0)^{(1)}(x,\theta) & (A_0^0)^{(2)}(x,\theta) \\ (A_0^0)^{(3)}(x,\theta) & (A_0^0)^{(4)}(x,\theta) \end{pmatrix} \partial_x^0 + ... + \mu \begin{pmatrix} (A_M^0)^{(1)}(x,\theta) & (A_M^0)^{(2)}(x,\theta) \\ (A_M^0)^{(3)}(x,\theta) & (A_M^0)^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-M} \\ & + \mu \begin{pmatrix} \Sigma_{R_2,1}(x,\theta,D) & \Sigma_{R_2,2}(x,\theta,D) \\ \Sigma_{R_2,3}(x,\theta,D) & \Sigma_{R_2,4}(x,\theta,D) \end{pmatrix} \\ & := \Omega \cdot \partial_\theta + \mathbf{T(D)} + \mathbf{C}^{(3)} \partial_x + \mu \mathbf{A}_0^0 \partial_x^0 + \mu \mathbf{A}_1^0 \partial_x^{-1} + ... + \mu \mathbf{A}_M^0 \partial_x^{-M} + \mu \Sigma_{R_2}\,. \end{aligned} \tag{8.40}$$

We prove the lemma by induction. After $k-1$ transformations we obtain a new linear operator, that can be written in a block diagonal form up to order $-k+1$. The matrices of the coefficients change at every step, so we call them $\mathbf{A}_j^{k-1}$ where the index j represents the homogeneous degree, and $k-1$ represents the step of the block symmetrization.

At the first step we symmetrize $\mathbf{A}_0^0$, and we call it $(\mathbf{A}_0^0)^D$. After the block symmetrization of the zero order coefficient the other matrix coefficients change. For this reason we decide to call the new coefficients $\mathbf{A}_j^1$ with $j = 1, ..., M$. At the second step we symmetrize $\mathbf{A}_1^1$ and we call it $(\mathbf{A}_1^1)^D$, while for the other coefficients we use $\mathbf{A}_j^2$, with $j = 2, ..., M$. At the k step we arrive to a operator that can be written in a block diagonal form up to order $-k+1$ (see Appendix B.4 for more details). The coefficients that are written in a block diagonal form do not change during the block symmetrization of the other coefficients. In other words the block diagonal matrix coefficients remain the same during the iterative procedure. Let

$$\begin{aligned}
\mathcal{L}_{k+2} &:= T_{k+2}^{-1} \circ ... \circ T_3^{-1} \mathcal{L}_2 T_3 \circ ... \circ T_{k+2} \\
&= \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \cdot \partial_\theta + \begin{pmatrix} iT(D) & 0 \\ 0 & -iT(D) \end{pmatrix} + \mu \begin{pmatrix} c_3^{(1)}(x,\theta) & 0 \\ 0 & c_3^{(4)}(x,\theta) \end{pmatrix} \partial_x \\
&+ + \sum_{j=0}^{k-1} \mu \begin{pmatrix} (A_j^j)^{(1)}(x,\theta) & 0 \\ 0 & (A_j^j)^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-j} + \mu \begin{pmatrix} (A_k^k)^{(1)}(x,\theta) & (A_k^k)^{(2)}(x,\theta) \\ (A_k^k)^{(3)}(x,\theta) & (A_k^k)^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-k} \\
&+ ... + \mu \begin{pmatrix} (A_M^k)^{(1)}(x,\theta) & (A_M^k)^{(2)}(x,\theta) \\ (A_M^k)^{(3)}(x,\theta) & (A_M^k)^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-M} + \mu \begin{pmatrix} \Sigma_{R_{k+2},1}(x,\theta,D) & \Sigma_{R_{k+2},2}(x,\theta,D) \\ \Sigma_{R_{k+2},3}(x,\theta,D) & \Sigma_{R_{k+2},4}(x,\theta,D) \end{pmatrix} \\
&:= \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)} \partial_x + \mu \sum_{j=0}^{k-1} (\mathbf{A}_j^j)^D \partial_x^{-j} + \mu \mathbf{A}_k^k \partial_x^{-k} \\
&+ ... + \mu \mathbf{A}_M^k \partial_x^{-M} + \mu \Sigma_{R_{k+2}} .
\end{aligned} \tag{8.41}$$

Now we want to eliminate the out of diagonal terms of the $\mathbf{A}_k$ matrix. Hence we have to conjugate the operator $\mathcal{L}_{k-1}$ with T_{k+3}. We have that

$$\begin{aligned}
\mathcal{L}_{k+2} T_{k+3} &= \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)} \partial_x + \mu \sum_{s=0}^{k-1} (\mathbf{A}_s^s)^D \partial_x^{-s} + \mu \sum_{s=k}^{M} \mathbf{A}_s^k \partial_x^{-s} + \mu \Sigma_{R_{k+2}} T_{k+3} \\
&+ \mu(\omega \cdot \partial_\theta) \Phi_{k+3} \partial_x^{-k-3} + \mathbf{T}(\mathbf{D}) \circ \Phi_{k+3} \partial_x^{-k-3} + \mathbf{C}^{(3)} \partial_x \circ \Phi_{k+3} \partial_x^{-k-3} \\
&+ \mu \sum_{s=0}^{k-1} (\mathbf{A}_s^s)^D \partial_x^{-s} \circ \Phi_{k+3} \partial_x^{-k-3} + \mu \sum_{s=k}^{M} \mathbf{A}_s^k \partial_x^{-s} \circ \Phi_{k+3} \partial_x^{-k-3}
\end{aligned}$$

and

$$\begin{aligned}
T_{k+3} &\left(\Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)} \partial_x + \mu \sum_{s=0}^{k-1} (\mathbf{A}_s^s)^D \partial_x^{-s} \right) = \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)} \partial_x \\
&+ \mu \sum_{s=0}^{k-1} (\mathbf{A}_s^s)^D \partial_x^{-s} + \Phi_{k+3} \partial_x^{-k-3} \circ \Omega \cdot \partial_\theta + \Phi_{k+3} \partial_x^{-k-3} \circ \mathbf{T}(\mathbf{D}) + \Phi_{k+3} \partial_x^{-k-3} \circ \mathbf{C}^{(3)} \partial_x \\
&+ \mu \Phi_{k+3} \partial_x^{-k-3} \circ \sum_{s=0}^{k-1} (\mathbf{A}_s^s)^D \partial_x^{-s} ,
\end{aligned}$$

where, as usual, ∂_x^0 denotes one of the operator belonging to $\{a\pi_0 + b\mathtt{1}\,,\ a, b \in \{0,1\}\}$. Hence

$$\begin{aligned}
\mathcal{L}_{k+3} :&= T_{k+3}^{-1}\mathcal{L}_{k+2}T_{k+3} \\
&= \Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)}\partial_x + \mu(\mathbf{A}_0^0)^D\partial_x^0 + ... + \mu(\mathbf{A}_{k-1}^{k-1})^D\partial_x^{-k+1} + T_{k+3}^{-1}(\omega\partial_\theta\Phi_{k+3}) \\
&+ T_{k+3}^{-1}\left(\mu\mathbf{A}_k^k\partial_x^{-k} + ... + \mu\mathbf{A}_M^k\partial_x^{-M} + [\mathbf{T}(\mathbf{D}), \Phi_{k+3}\partial_x^{-k-3}] + [\mathbf{C}^{(3)}\partial_x\,,\ \Phi_{k+3}\partial_x^{-k-3}]\right) \\
&+ T_{k+3}^{-1}\left(\sum_{j=0}^{k+1}\left[\mu(\mathbf{A}_j^j)^D\partial_x^{-j}\,,\ \Phi_{k+3}\partial_x^{-k-3}\right]\right) \\
&+ T_{k+3}^{-1}\left(\mu\mathbf{A}_k^k(\partial_x^{-k}\Phi_{k+3}\partial_x^{-k-3}) + ... + \mu\mathbf{A}_M^k(\partial_x^{-M}\Phi_{k+3}\partial_x^{-k-3}) + \mu\Sigma_{R_{k+2}}T_{k+3}^{-1}\right)\,,
\end{aligned} \tag{8.42}$$

We develop the commutator $[\mathbf{T}(\mathbf{D}), \Phi_{k+3}]$ as in the previous case. Using (8.26) we have

$$iT(D) := -\sqrt{\frac{2}{15}}\varepsilon^2\partial_x^3 + \sum_{j=-1}^{M-k-3} c_j\partial_x^{-j} + Op(r(\xi))\,,$$

then

$$\begin{aligned}
&[\mathbf{T}(\mathbf{D}), \Phi_{k+3}\partial_x^{-k-3}] := \\
&\mu\begin{pmatrix} 0 & iT(D)\circ\varphi_2^{(k+3)}\partial_x^{-k-3} + \varphi_2^{(k+3)}\partial_x^{-k-3}\circ iT(D) \\ -iT(D)\circ\varphi_3^{(k+3)}\partial_x^{-k-3} - \varphi_3^{(k+3)}\partial_x^{-k-3}\circ iT(D) & 0 \end{pmatrix} \\
&= \mu\varepsilon^2\begin{pmatrix} 0 & -\frac{2\sqrt{2}}{\sqrt{15}}\varphi_2^{(k+3)}(x,\theta)\partial_x^{-k} \\ \frac{2\sqrt{2}}{\sqrt{15}}\varphi_3^{(k+3)}(x,\theta)\partial_x^{-k} & 0 \end{pmatrix} + \mathbf{P}_k\,,
\end{aligned} \tag{8.43}$$

where $\mathbf{P}_k := \begin{pmatrix} 0 & P_2^{(k)} \\ P_3^{(k)} & 0 \end{pmatrix}$ and $P_2^{(k)}$, respectively $P_3^{(k)}$, are given by

$$\begin{aligned}
P_2^{(k)} &:= \mu\Bigg[-\frac{\sqrt{2}}{\sqrt{15}}\varepsilon^2(\varphi_2^{(k+3)})_{xxx}\partial_x^{-k-3} - \frac{3\sqrt{2}}{\sqrt{15}}\varepsilon^2(\varphi_2^{(k+3)})_{xx}\partial_x^{-k-2} - \frac{3\sqrt{2}}{\sqrt{15}}\varepsilon^2(\varphi_2^{(k+3)})_x\partial_x^{-k-1} \\
&+ \left(\sum_{j=-1}^{M-k-3} c_j\partial_x^{-j} + Op(r(\xi))\right)\circ\varphi_2^{(k+3)}\partial_x^{-k-3} + \varphi_2^{(k+3)}\partial_x^{-k-3}\circ\left(\sum_{j=-1}^{M-k-3} c_j\partial_x^{-j} + Op(r(\xi))\right)\Bigg] \\
&:= \mu\left(\sum_{j=k+1}^{M} a_j^{(2)}(x,\theta)\partial_x^{-j} + Op(r(x,\theta,\xi))\right) \\
P_3^{(k)} &:= \mu\Bigg[\frac{\sqrt{2}}{\sqrt{15}}\varepsilon^2(\varphi_3^{(k+3)})_{xxx}\partial_x^{-k-3} + \frac{3\sqrt{2}}{\sqrt{15}}\varepsilon^2(\varphi_3^{(k+3)})_{xx}\partial_x^{-k-2} + \frac{3\sqrt{2}}{\sqrt{15}}\varepsilon^2(\varphi_3^{(k+3)})_x\partial_x^{-k-1} \\
&+ \left(\sum_{j=-1}^{M-k-3} c_j\partial_x^{-j} + Op(r(\xi))\right)\circ\varphi_3^{(k+3)}\partial_x^{-k-3} + \varphi_3^{(k+3)}\partial_x^{-k-3}\circ\left(\sum_{j=-1}^{M-k-3} c_j\partial_x^{-j} + Op(r(\xi))\right)\Bigg] \\
&:= \mu\left(\sum_{j=k+1}^{M} a_j^{(3)}(x,\theta)\partial_x^{-j} + Op(r(x,\theta,\xi))\right)\,,
\end{aligned} \tag{8.44}$$

where $a_j^{(2)}$ respectively $a_j^{(3)}$ are some functions depending on the derivative of $\varphi_2^{(k+3)}$, respectively $\varphi_3^{(k+3)}$.

Hence, if

$$\frac{2\sqrt{2}}{\sqrt{15}}\varepsilon^2\varphi_2^{(k+3)} - (A_k^k)^{(2)} = 0\,, \quad \frac{2\sqrt{2}}{\sqrt{15}}\varepsilon^2\varphi_3^{(k+3)} + (A_k^k)^{(3)} = 0\,, \tag{8.45}$$

we have

$$[\mathbf{T}(\mathbf{D}), \Phi_{k+3}\partial_x^{-k-3}] + \mu\mathbf{A}_k^k\partial_x^{-k} = \mu\begin{pmatrix}(A_k^k)^{(1)} & 0\\ 0 & (A_k^k)^{(4)}\end{pmatrix}\partial_x^{-k} + \mathbf{P}^{(k)}.$$

At every step we can define with $\tilde{\mathbf{R}}_k$ the sum of the pseudo-differential operators in OPS^{-k-1}, hence $\tilde{\mathbf{R}}_k$ is given by

$$\begin{aligned}\tilde{\mathbf{R}}_k =& T_{k+3}^{-1}\Big((\omega\partial_\theta\Phi_{k+3})\partial_x^{-k-3} + \mu\mathbf{A}_{k+1}^k\partial_x^{-k-1} + \ldots + \mu\mathbf{A}_M^k\partial_x^{-M} + \mathbf{P}_k + [\mathbf{C}^{(2)}\partial_x\,,\, \Phi_{k+3}\partial_x^{-k-3}]\Big)\\ &+ T_{k+3}^{-1}\left(\mu\sum_{j=0}^{k+1}\Big[(\mathbf{A}_j^j)^D\partial_x^{-j}\,,\, \Phi_{k+3}\partial_x^{-k-3}\Big]\right)\\ &+ T_{k+3}^{-1}\left(\mu\mathbf{A}_k^k(\partial_x^{-k}\Phi_{k+3}\partial_x^{-k-3}) + \ldots + \mu\mathbf{A}_M^k(\partial_x^{-M}\Phi_{k+3}\partial_x^{-k-3}) + \mu\Sigma_{R_{k+2}}T_{k+3}\right)\\ &:= \mu\sum_{j=k+1}^{M}\mathbf{A}_{j+1}^{k+1}\partial_x^{-j-1} + \mu\Sigma_{R_{k+3}}\,,\end{aligned} \tag{8.46}$$

where $\Sigma_{R_{k+3}} \in OPS^{-M-1}$. Note that, with an abuse of notation, we are now (and only here) calling $\tilde{\mathbf{R}}_k$ the sum of the homogeneous terms of order less than $-k$ plus the pseudo-differential operator $\Sigma_{R_{k+3}} \in OPS^{-M-1}$.

Therefore, by (8.42), (8.43) and (8.45), we arrive to

$$\begin{aligned}\mathcal{L}_{k+3} &:= T_{k+3}^{-1}\mathcal{L}_{k+2}T_{k+3}\\ &= \begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\cdot\partial_\theta + \begin{pmatrix}iT(D) & 0\\ 0 & -iT(D)\end{pmatrix} + \mu\begin{pmatrix}c_3^{(1)}(x,\theta) & 0\\ 0 & c_3^{(4)}(x,\theta)\end{pmatrix}\partial_x\\ &\quad+ \mu\sum_{j=0}^{k}\begin{pmatrix}(A_j^j)^{(1)}(x,\theta) & 0\\ 0 & (A_j^j)^{(4)}(x,\theta)\end{pmatrix}\partial_x^{-j} + \tilde{\mathbf{R}}_k\,,\end{aligned}$$

where $\tilde{\mathbf{R}}_k$ is defined in (8.46) and it contains all the remainder terms in OPS^{-k-1}, and ∂_x^0 denotes one of the operator belonging to $\{a\pi_0 + b\mathtt{1}\,,\ a,b\in\{0,1\}\}$. We point out that by Lemmas B.1, B.3 and B.4, $\tilde{\mathbf{R}}_k$ can be written as follows

$$\tilde{\mathbf{R}}_k := \mu\sum_{j=k}^{M}\begin{pmatrix}(A_j^k)^{(1)} & (A_j^k)^{(2)}\\ (A_j^k)^{(3)} & (A_j^k)^{(4)}\end{pmatrix}\partial_x^{-j} + \mu\begin{pmatrix}\Sigma_{R_{k+3}}^{(1)} & \Sigma_{R_{k+3}}^{(2)}\\ \Sigma_{R_{k+3}}^{(3)} & \Sigma_{R_{k+3}}^{(4)}\end{pmatrix}.$$

Hence, iterating the procedure above, by Lemmas B.1, B.3 and B.4, after M step we arrive to $\mathcal{L}_{M+3}$ defined in (8.38).

By the explicit definition of

$$\Phi_{k+3} = \mu\begin{pmatrix}0 & \varphi_2^{(k+3)}\\ \varphi_3^{(k+3)} & 0\end{pmatrix}$$

given in (8.45), that is

$$\Phi_{k+3} = \begin{pmatrix} 0 & \frac{\sqrt{15}}{2\sqrt{2}\varepsilon^2}(A_k^k)^{(2)} \\ -\frac{\sqrt{15}}{2\sqrt{2}\varepsilon^2}(A_k^k)^{(2)} & 0 \end{pmatrix}$$

we have that the transformation T_{k+3} defined in (8.37), with $j = k+3$, is reversibility preserving. Indeed, by the reversible structure of $\mathcal{L}_{k+2}$ we have that $(A_k^k)^{(2)}(-\theta) = -(A_k^k)^{(3)}(\theta)$.

By the iterative procedure we can prove that all the T_k , $k = 3, ..., M+3$ are reversibility preserving. Hence $\mathcal{L}_{M+3}$ is reversible.

Now we prove, by induction on k that all the T_k in (8.37) with $k = 3, ..., M+3$ map E into itself. Let $\mathcal{L}_{k-1}$ as in (8.41), and $\mathcal{L}_{k-1} : E \to E$. In particular

$$\mathbf{A}_k^k \partial_x^{-k} := \begin{pmatrix} (A_k^k)^{(1)} & (A_k^k)^{(2)} \\ (A_k^k)^{(3)} & (A_k^k)^{(4)} \end{pmatrix} \partial_x^{-k} ,$$

maps E in itself. This means that $(-1)^k (A_k^k)^{(1)}(-x,\theta) = (A_k^k)^{(4)}(x,\theta)$ and $(-1)^k (A_k^k)^{(2)}(-x,\theta) = (A_k^k)^{(3)}(x,\theta)$, see Lemma 7.5. We now consider T_{k+3} as in (8.37), using (8.45) we have that

$$T_{k+3} = \mathbb{1} + \left(\frac{2\sqrt{2}}{\sqrt{15}}\right)^{-1} \mu\varepsilon^{-2} \begin{pmatrix} 0 & (A_k^k)^{(2)} \\ -(A_k^k)^{(3)} & 0 \end{pmatrix} \partial_x^{-k-3} , \tag{8.47}$$

and then by the hypothesis on the $\mathbf{A}_k$ coefficient, we have that

$$(-1)^{k+3} A_k^{(2)}(-x,\theta) = -A_k^{(3)}(x,\theta) .$$

Finally, by the explicit definition of T_{k+3} in (8.47) we have that the transformation is real, therefore $\mathcal{L}_{M+3}$ is real. □

Lemma 8.6. *Let $\mathcal{L}_{M+3}$ as in* (8.38)*. Then*

$$\begin{aligned} \|(\mathbf{A}_k^k)^D\|_p^{k_0,\gamma} &\leq_p \|\mathtt{v}\|_{p+k^2+5+\sigma}^{k_0,\gamma} , \quad k = 0, ..., M \\ |\Sigma|_{-M-1,p,0}^{k_0,\gamma} &\leq_p \|\mathtt{v}\|_{p+(M+1)M+3M+6+\sigma}^{k_0,\gamma} \\ \|\partial_i (\mathbf{A}_k^k)^D[\hat{i}]\|_{p_1} &\leq \|\hat{i}\|_{p_1+k^2+5+\sigma} , \quad k = 0, ..., M \\ |\partial_i \Sigma[\hat{i}]|_{-M,p,0} &\leq_{p_1} \mu \|\hat{i}\|_{p+(M+1)M+3M+6+\sigma} . \end{aligned} \tag{8.48}$$

Proof. It follows by Lemma B.13. □

Lemma 8.7. *Let T_j , $j = 3, ..., M+3$ as in* (8.37)*. Let $\mathfrak{p}_1 \in \mathbb{R}$, such that $\|\mathtt{v}\|_{\mathfrak{p}_0+\mathfrak{p}_1} \leq 1$ where $\mathfrak{p}_1 := M^2 + 5 + \sigma$. Then for every $j = 3, ..., M+3$,*

$$\|\Phi_j\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq C(p,j)\mu \|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma} , \ \forall j = 3, ..., M+3 , \tag{8.49}$$

$$|T_j|_{0,p,0}^{k_0,\gamma} \leq C(p,j)\left(1 + \mu \|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma}\right) \tag{8.50}$$

$$|T_3 \circ ... \circ T_{M+3}|_{0,p,0}^{k_0,\gamma} \leq C(p,M)\left(1 + \mu \|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma}\right) . \tag{8.51}$$

Proof. The first inequality follows by Lemmas B.12, B.13. Actually, if $T_{3+j} := \mathbb{1} + \Phi_{3+j}\partial_x^{-j-3}$ we have that

$$|T_{3+j}|_{0,p,0}^{k_0,\gamma} \le C(p,j)(1+\mu\|\mathtt{v}\|_{p+j^2+5+\sigma}^{k_0,\gamma}), \quad j=0,...,M.$$

Therefore, for all $j = 0, ..., M$ we have

$$|T_{3+j}|_{0,p,0}^{k_0,\gamma} \le C(p,j)(1+\mu\|\mathtt{v}\|_{p+j^2+5+\sigma}^{k_0,\gamma}) \le C(p,M)(1+\mu\|\mathtt{v}\|_{p+M^2+5+\sigma}^{k_0,\gamma}) = C(p,M)(1+\mu\|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma}).$$

The second inequality (8.50) follows by the definition of T_j and (8.49). We now prove (8.51) by induction. By (8.50) and Lemma 2.8 we have

$$\begin{aligned}|T_3T_4|_{0,p,0}^{k_0,\gamma} &\le C(p)|T_3|_{0,p,0}^{k_0,\gamma}|T_4|_{0,\mathfrak{p}_0,0}^{k_0,\gamma} + C(\mathfrak{p}_0)|T_3|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|T_4|_{0,p,0}^{k_0,\gamma}\\ &\le C(p,M)\left(1+\mu\|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma}\right)\left(1+\mu\|\mathtt{v}\|_{\mathfrak{p}_0+\mathfrak{p}_1}^{k_0,\gamma}\right)\\ &\le C(p,M)\left(1+\mu\|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma}\right),\end{aligned}$$

where the initial and the last constant are different. Suppose that (8.51) is true for $T_3 \circ T_4 \circ ... \circ T_{k-1}$, then, using Lemma 2.8 we get

$$\begin{aligned}|T_3\circ...\circ T_{k-1}\circ T_k|_{0,p,0}^{k_0,\gamma} &\le C(p,k)|T_3T_3\circ...\circ T_{k-1}|_{0,p,0}^{k_0,\gamma}|T_k|_{0,\mathfrak{p}_0,0}^{k_0,\gamma} + C(\mathfrak{p}_0,k)|T_3T_3\circ...\circ T_{k-1}|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|T_k|_{0,p,0}^{k_0,\gamma}\\ &\le C(p,k)\left(1+\mu\|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma}\right)\left(1+\mu\|\mathtt{v}\|_{\mathfrak{p}_0+\mathfrak{p}_1}^{k_0,\gamma}\right)\\ &\le C(p,k)\left(1+\mu\|\mathtt{v}\|_{p+\mathfrak{p}_1}^{k_0,\gamma}\right).\end{aligned}$$

Note that the lemma follows without complication just because we are considering a finite number of compositions, where we shall define M in Section 8.2.4, independent from the Sobolev index p. □

8.2.3 Elimination of the (x,θ) dependence in the first order coefficient

In this section we shall make the first order coefficient constant up to a reminder supported on the high Fourier frequencies. Indeed we are working with frequencies $\omega \in \mathtt{DC}_{K_n}^{\gamma}$ where $\mathtt{DC}_{K_n}^{\gamma}$ is defined in (5.5). For this reason we can not invert $\omega \cdot l$ for all $l \in \mathbb{Z}^N$, but we can invert it only for finitely many l. Fortunately we can neglect the first order coefficient supported on high Fourier frequencies and we will study it in Chapter 9.

In order to be more precisely consider $\mathcal{L}_{M+3}$ in (8.38). We define

$$\tilde{\mathcal{L}}_{M+3} := \begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix}\partial_\theta + \begin{pmatrix}iT(D) & 0\\ 0 & -iT(D)\end{pmatrix} + \Pi_{K_n}\begin{pmatrix}c_3^{(1)} & 0\\ 0 & c_3^{(4)}\end{pmatrix}\partial_x + \mathbf{R}_{M+3}, \tag{8.52}$$

that is $\mathcal{L}_{M+3}$ in where we have negleted

$$\Pi_{K_n}^{\perp}\mathbf{C}^{(3)} := \Pi_{K_n}^{\perp}\begin{pmatrix}c_3^{(1)} & 0\\ 0 & c_3^{(4)}\end{pmatrix}\partial_x. \tag{8.53}$$

Lemma 8.8. *Let $(h,k)\in E$. There exist two real, reversibility preserving transformations acting in E, T_{M+4} of the form*

$$T_{M+4}:\begin{pmatrix}h(x,\theta)\\ k(x,\theta)\end{pmatrix}\to\begin{pmatrix}h(x+\psi_1(\theta),\theta)\\ k(x+\psi_2(\theta),\theta)\end{pmatrix}=\begin{pmatrix}\Psi_1 h(x,\theta)\\ \Psi_2 k(x,\theta)\end{pmatrix}\tag{8.54}$$

with

$$\psi_1(\theta)=-\psi_2(\theta)\tag{8.55}$$

and T_{M+5} of the form

$$T_{M+5}=\mathbb{1}+\Phi_{M+5}(y,\theta)\partial_y^{-1}=\mathbb{1}+\mu\begin{pmatrix}\varphi_1^{(M+5)}(y,\theta) & 0\\ 0 & \varphi_4^{(M+5)}(y,\theta)\end{pmatrix}\partial_y^{-1}\tag{8.56}$$

such that, given $\tilde{\mathcal{L}}_{M+3}$ defined in (8.52) *we have*

$$\tilde{\mathcal{L}}_{M+5}:=T_{M+5}^{-1}T_{M+4}^{-1}\tilde{\mathcal{L}}_{M+3}T_{M+4}T_{M+5}=\omega\cdot\partial_\theta+\mathbf{T}(\mathbf{D})+\mathbf{M}_{K_n}\partial_x+\mathbf{R}_{M+5}\tag{8.57}$$

where

$$\mathbf{M}_{K_n}=\begin{pmatrix}\mathtt{m}_{1,K_n} & 0\\ 0 & \mathtt{m}_{4,K_n}\end{pmatrix},\quad \mathtt{m}_{1,K_n}=-\mathtt{m}_{4,K_n},\quad \mathtt{m}_{1,K_n},\mathtt{m}_{4,K_n}\in\mathbb{R}\tag{8.58}$$

and $\mathbf{R}_{M+5}$ is a bounded remainder (see Section 8.2.4 for the estimates). The operator $\tilde{\mathcal{L}}_{M+5}$ is real, reversible and acts in E. In addition we have that T_{M+4} and T_{M+5} are tame, and $\forall f\in E$

$$\begin{aligned}\|T_{M+4}^{-1}f\|_p^{k_0,\gamma}+\|T_{M+4}f\|_p^{k_0,\gamma}&\le C(P)(\|f\|_{p+\sigma}^{k_0,\gamma}+\mu\gamma^{-1}\|\mathtt{v}\|_{p+2+\sigma}^{k_0,\gamma}\|f\|_{\mathtt{p}_0+\sigma}^{k_0,\gamma})\\ \|T_{M+5}^{-1}f\|_p^{k_0,\gamma}+\|T_{M+5}f\|_p^{k_0,\gamma}&\le C(p)(\|f\|_p^{k_0,\gamma}+\mu\|f\|_{\mathtt{p}_0}^{k_0,\gamma}\|\mathtt{v}\|_{p+2}^{k_0,\gamma}).\end{aligned}\tag{8.59}$$

Proof. The proof is divided in two steps. The goal of the first step is to apply the change of variables T_{M+4} because we want to remove the spatial average by the coefficient in front of ∂_y. The change of variables T_{M+4} is induced by the diffeomorphism

$$x+\psi_i(\theta)=y\ \Leftrightarrow\ x=y-\psi_i(\theta)\quad i=1,2\,.$$

Note that T_{M+4} is invertible and the inverse is given by

$$T_{M+4}^{-1}:\begin{pmatrix}v(y,\theta)\\ w(y,\theta)\end{pmatrix}\to\begin{pmatrix}v(y-\psi_1(\theta),\theta)\\ w(y-\psi_2(\theta),\theta)\end{pmatrix}=\begin{pmatrix}\Psi_1^{-1}v(y,\theta)\\ \Psi_2^{-1}w(y,\theta)\end{pmatrix},\quad\forall(v,w)\in E\,.$$

We have the following conjugation rules

$$T_{M+4}^{-1}(\Omega\cdot\partial_\theta)T_{M+4}=\Omega\cdot\partial_\theta+\begin{pmatrix}\omega\cdot\partial_\theta\psi_1(\theta) & 0\\ 0 & \omega\cdot\partial_\theta\psi_2(\theta)\end{pmatrix}\partial_y$$

and

$$\Psi_1^{-1}\partial_x\Psi_1=\partial_y\qquad\Psi_2^{-1}\partial_x\Psi_2=\partial_y\qquad T_{M+4}^{-1}\mathbf{T}(\mathbf{D})T_{M+4}=\mathbf{T}(\mathbf{D})\,.$$

Hence

$$
\begin{aligned}
\tilde{\mathcal{L}}_{M+4} &:= T_{M+4}^{-1}\tilde{\mathcal{L}}_{M+3}T_{M+4} \\
&= \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}\cdot\partial_\theta + \begin{pmatrix} iT(D) & 0 \\ 0 & -iT(D) \end{pmatrix} \\
&+ \left[\begin{pmatrix} \omega\cdot\partial_\theta\psi_1(\theta) & 0 \\ 0 & \omega\cdot\partial_\theta\psi_2(\theta) \end{pmatrix} + \left(T_{M+4}^{-1}\Pi_{K_n}\mu\begin{pmatrix} c_3^{(1)} & 0 \\ 0 & c_3^{(4)} \end{pmatrix}\right)(y,\theta)\right]\partial_y + \tilde{\mathbf{R}}_{M+4}\,,
\end{aligned}
\tag{8.60}
$$

where $\tilde{\mathcal{L}}_{M+3}$ is defined in (8.52) and

$$
\tilde{\mathbf{R}}_{M+4} = T_{M+4}^{-1}\mathbf{R}_{M+3}T_{M+4}\,, \tag{8.61}
$$

with $\mathbf{R}_{M+3}$ is defined in (8.39).We look for $\psi_i,\ i=1,2$ such that

$$
\begin{aligned}
\begin{pmatrix} \omega\cdot\partial_\theta\psi_1(\theta) & 0 \\ 0 & \omega\cdot\partial_\theta\psi_2(\theta) \end{pmatrix} + \Pi_{K_n}\mu\begin{pmatrix} \psi_1^{-1}c_3^{(1)} & 0 \\ 0 & \psi_2^{-1}c_3^{(4)} \end{pmatrix}(y,\theta) &= \Pi_{K_n}\mu\begin{pmatrix} c_4^{(1)} & 0 \\ 0 & c_4^{(4)} \end{pmatrix}(y,\theta) \\
&:= \Pi_{K_n}\mathbf{C}^{(4)}\,,
\end{aligned}
\tag{8.62}
$$

where $c_3^{(1)}$ and $c_3^{(4)}$ are defined in (8.22), and $c_4^{(1)}$, $c_4^{(4)}$ satisfy the equations:

$$
\mu\frac{1}{2\pi}\int_{\mathbb{T}}\Pi_{K_n}c_4^{(1)}(y,\theta)dy = \mathtt{m}_{1,K_n} \qquad \text{and} \qquad \mu\frac{1}{2\pi}\int_{\mathbb{T}}\Pi_{K_n}c_4^{(4)}(y,\theta)dy = \mathtt{m}_{4,K_n}\,, \qquad \forall\theta\in\mathbb{T}^N\,, \tag{8.63}
$$

for some $\mathtt{m}_{1,K_n}\,,\ \mathtt{m}_{4,K_n}\in\mathbb{R}$ independent of θ. The equations in (8.62) are explicitly given by

$$
\begin{cases}
\omega\cdot\partial_\theta\psi_1(\theta) + \mu\Pi_{K_n}c_3^{(1)}(y-\psi_1(\theta),\theta) = \mu\Pi_{K_n}c_4^{(1)}(y,\theta) \\
\omega\cdot\partial_\theta\psi_2\theta + \mu\Pi_{K_n}c_3^{(4)}(y-\psi_2(\theta),\theta) = \mu\Pi_{K_n}c_4^{(4)}(y,\theta)\,.
\end{cases}
\tag{8.64}
$$

Taking the spatial average of (8.64), the request (8.63) implies

$$
\begin{cases}
\omega\cdot\partial_\theta\psi_1(\theta) + \mu\frac{1}{2\pi}\int_{\mathbb{T}}\Pi_{K_n}c_3^{(1)}(x,\theta)dx = \mathtt{m}_{1,K_n} \\
\omega\cdot\partial_\theta\psi_2\theta - \mu\frac{1}{2\pi}\int_{\mathbb{T}}\Pi_{K_n}c_3^{(4)}(x,\theta)dx = \mathtt{m}_{4,K_n}\,.
\end{cases}
$$

Since we are looking for periodic solutions $\psi_1(\theta)$, $\psi_2(\theta)$, taking the average with respect to θ, using that $\omega\in\mathtt{DC}_{K_n}^\gamma$ (defined in (5.5)) we get

$$
\begin{cases}
\frac{1}{(2\pi)^{N+1}}\mu\int_{\mathbb{T}^{N+1}}\Pi_{K_n}c_3^{(1)}(x,\theta)dxd\theta = \mathtt{m}_{1,K_n} \\
-\mu\frac{1}{(2\pi)^{N+1}}\int_{\mathbb{T}^{N+1}}\Pi_{K_n}c_3^{(4)}(x,\theta)dxd\theta = \mathtt{m}_{4,K_n}\,.
\end{cases}
\tag{8.65}
$$

With this choice of $\mathtt{m}_{1,K_n}$ and $\mathtt{m}_{4,K_n}$ the equations

$$
\begin{cases}
\omega\cdot\partial_\theta\psi_1(\theta) &= \mathtt{m}_{1,K_n} - \mu\frac{1}{2\pi}\int_{\mathbb{T}}\Pi_{K_n}c_3^{(1)}(x,\theta)dx = \mu\Pi_{K_n}c_5^{(1)}(\theta) \\
\omega\cdot\partial_\theta\psi_2(\theta) &= \mathtt{m}_{4,K_n} + \mu\frac{1}{2\pi}\int_{\mathbb{T}}\Pi_{K_n}c_3^{(4)}(x,\theta)dx = \mu\Pi_{K_n}c_5^{(4)}(\theta)
\end{cases}
$$

are solved by

$$\psi_1(\theta) = \mu(\omega\cdot\partial_\theta)^{-1}\Pi_{K_n} c_1^{(5)}(\theta), \quad \psi_2(\theta) = \mu(\omega\cdot\partial_\theta)^{-1}\Pi_{K_n} c_4^{(5)}(\theta)\,. \tag{8.66}$$

Since Λq is even in the spatial variable, while $\Lambda^{-1}p$ is odd, by the explicit definition of $\Pi_{K_n} c_3^{(1)}$ and $\Pi_{K_n} c_3^{(4)}$ in (8.22), one arrive to (8.58), with $\mathtt{m}_{1,K_n} = -\mathtt{m}_{4,K_n}$. This also implies

$$\Pi_{K_n} c_5^{(1)} = -\Pi_{K_n} c_5^{(4)}\,,$$

therefore one get (8.55).

Then we have that $\tilde{\mathcal{L}}_{M+4}$ in (8.60) reads

$$\tilde{\mathcal{L}}_{M+4} := T_{M+4}^{-1}\tilde{\mathcal{L}}_{M+3}T_{M+4} = \Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + \Pi_{K_n}\mathbf{C}^{(4)}(y,\theta)\partial_y + \tilde{\mathbf{R}}_{M+4} \tag{8.67}$$

where $\Pi_{K_n}\mathbf{C}^{(4)}$ is defined in (8.62) and $\tilde{\mathbf{R}}_{M+4}$ is defined in (8.61).

Now we want to make constant the coefficient in front of ∂_y. We conjugate the operator $\tilde{\mathcal{L}}_{M+4}$ in (8.67) with a transformation T_{M+5} of the form (8.56). Then we have

$$\begin{aligned}\tilde{\mathcal{L}}_{M+4}T_{M+5} - T_{M+5}(\Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}_{K_n}\partial_y) &= [\mathbf{T}(\mathbf{D}), \Phi_{M+5}(y,\theta)\partial_y^{-1}] \\ &\quad + (\Pi_{K_n}\mathbf{C}^{(4)}(y,\theta) - \mathbf{M}_{K_n})\partial_y + \tilde{\mathbf{R}}_{M+5}\end{aligned} \tag{8.68}$$

where

$$\begin{aligned}\tilde{\mathbf{R}}_{M+5} &= \tilde{\mathbf{R}}_{M+4} + \tilde{\mathbf{R}}_{M+4}\Phi_{M+5}\partial_y^{-1} + (\omega\cdot\partial_\theta\Phi_{M+5})\partial_y^{-1} + \Pi_{K_n}\mathbf{C}^{(4)}\Phi_{M+5}\pi_0 \\ &\quad + \Pi_{K_n}\mathbf{C}^{(4)}(\Phi_{M+5})_y\partial_y^{-1} - \mathbf{M}_{K_n}\Phi_{M+5}\pi_0.\end{aligned} \tag{8.69}$$

Using the asymptotic expansion of $iT(D)$ defined in (8.8) we have

$$\begin{aligned}&[\mathbf{T}(\mathbf{D}),\Phi_{M+5}\partial_y^{-1}] \\ &\quad = \mu\begin{pmatrix} i(T(D)\varphi_1^{(M+5)}\partial_y^{-1} - \varphi_1^{(M+5)}\partial_y^{-1}T(D)) & 0 \\ 0 & i(-T(D)\varphi_4^{(M+5)}\partial_y^{-1} + \varphi_4^{(M+5)}\partial_y^{-1}T(D))\end{pmatrix} \\ &\quad = \mu\begin{pmatrix} -3\frac{\sqrt{2}}{\sqrt{15}}\varepsilon^2(\varphi_1)_y^{(M+5)}\partial_x & 0 \\ 0 & +3\frac{\sqrt{2}}{\sqrt{15}}\varepsilon^2\varepsilon^2(\varphi_4)_y^{(M+5)}\partial_x\end{pmatrix} + \mathbf{P}_{M+5},\end{aligned}$$

where

$$\mathbf{P}_{M+5} := \begin{pmatrix} P_1^{(M+5)} & 0 \\ 0 & P_4^{(M+5)}\end{pmatrix} \tag{8.70}$$

and

$$\begin{aligned}P_1^{(M+5)} &= -3\mu\sqrt{\frac{2}{15}}\varepsilon^2(\varphi_1^{(M+5)})_{yy}\pi_0 - \mu\left(\sqrt{\frac{2}{15}}\varepsilon^2(\varphi_1^{(M+5)})_{yyy} + \sqrt{\frac{5}{24}}(\varphi_1^{(M+5)})_y\right)\partial_y^{-1} \\ &\quad + \mu\left[\sum_{j=0}^{M-1} c_j\partial_y^{-j} + Op(r(\xi)), (\varphi_1^{(M+5)})\partial_y^{-1}\right] \\ P_4^{(M+5)} &= 3\mu\sqrt{\frac{2}{15}}\varepsilon^2(\varphi_4^{(M+5)})_{yy}\pi_0 + \mu\left(\sqrt{\frac{2}{15}}\varepsilon^2(\varphi_4^{(M+5)})_{yyy} + \sqrt{\frac{5}{24}}(\varphi_4^{(4)})_y\right)\partial_y^{-1} \\ &\quad + \mu\left[\sum_{j=0}^{M-1} c_j\partial_y^{-j} + Op(r(\xi)), (\varphi_4^{(M+5)})\partial_y^{-1}\right].\end{aligned} \tag{8.71}$$

We define the functions

$$\begin{aligned}\varphi_1^{(M+5)}(y,\theta) &= \frac{\sqrt{15}}{3\sqrt{2}}\varepsilon^{-2}\partial_y^{-1}[-\mathtt{m}_{1,K_n} + \mu\Pi_{K_n}c_4^{(1)}(y,\theta)]\\ \varphi_4^{(M+5)}(y,\theta) &= \frac{\sqrt{15}}{3\sqrt{2}}\varepsilon^{-2}\partial_y^{-1}[\mathtt{m}_{4,K_n} - \mu\Pi_{K_n}c_4^{(4)}(y,\theta)]\,,\end{aligned} \tag{8.72}$$

that, thanks to (8.63), are periodic and well defined. Then, by (8.68) and (8.72)

$$\begin{aligned}\tilde{\mathcal{L}}_{M+5} :&= T_{M+5}^{-1}\tilde{\mathcal{L}}_{M+4}T_{M+5} =\\ &= \Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}_{K_n}\partial_y + \mathbf{R}_{M+5}\end{aligned} \tag{8.73}$$

where

$$\mathbf{R}_{M+5} := T_{M+5}^{-1}\tilde{\mathbf{R}}_{M+5} + T_{M+5}^{-1}\mathbf{P}_{M+5}, \tag{8.74}$$

and $\tilde{\mathbf{R}}_{M+5}$ is defined in (8.69). The tame estimate for T_{M+4} and T_{M+5} in (8.59) follows by (2.36), (8.72) and (8.66) In addition, using the explicit definition of T_{M+4} and T_{M+5} in (8.66) and (8.72), using (8.55) and $\Lambda q(\theta) = \Lambda q(-\theta)\,,\ \Lambda^{-1}p(\theta) = -\Lambda^{-1}p(-\theta)$ we have that T_{M+4} and T_{M+5} are reversibility preserving (see (7.32)). Moreover both T_{M+4}, defined in (8.54), and T_{M+5}, defined in (8.56), are real operators. By Lemma 7.5 the operator T_{M+5} maps E in E. We now prove that also the operator T_{M+4} maps E in E. Let $(h,k)\in E$, then

$$T_{M+4}\begin{pmatrix} h\\ k\end{pmatrix}(x,\theta) = \begin{pmatrix} h(x+\psi_1(\theta),\theta)\\ k(x+\psi_2(\theta),\theta)\end{pmatrix}$$

acts in E if and only if $h(-(x-\psi_1(\theta)),\theta) = k(x+\psi_2(\theta),\theta)$. By (8.55) the claim is proved. Finally, since the composition of the real reversible operator $\mathcal{L}_{M+3}$ acting on E (see Lemma 8.5) with the real and reversible preserving operators T_{M+4} and T_{M+5} acting on E, is real, reversible and acts on E, we have that $\tilde{\mathcal{L}}_{M+5} : E \to E$ is real and reversible. □

We can rewrite $\tilde{\mathcal{L}}_{M+5}$ defined in (8.57) as follows

$$\tilde{\mathcal{L}}_{M+5} := \Omega\cdot\partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}\partial_x + \mathbf{R}_{M+5} + \mathbf{R}^{\perp}_{M_{K_n}}\,, \tag{8.75}$$

where

$$\mathbf{R}^{\perp}_{M_{K_n}} = (-\mathbf{M} + \mathbf{M}_{K_n})\partial_x \tag{8.76}$$

is a remainder supported only on the high Fourier frequencies and

$$\mathbf{M} = \begin{pmatrix} m_1 & 0\\ 0 & m_4\end{pmatrix}, \tag{8.77}$$

with m_1, m_4 given by

$$\begin{aligned}\mu\frac{1}{(2\pi)^{N+1}}\int_{\mathbb{T}^{N+1}} c_3^{(1)}(x,\theta)dxd\theta &= m_1\\ -\mu\frac{1}{(2\pi)^{N+1}}\int_{\mathbb{T}^{N+1}} c_3^{(4)}(x,\theta)dxd\theta &= m_4\,.\end{aligned} \tag{8.78}$$

By the explicit definiton of $c_3^{(1)}$ and $c_3^{(4)}$ in (8.22) we have that $m_1 = -m_4$, hence $\mathbf{M}$ in (8.77) reads

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & -m_1 \end{pmatrix}. \tag{8.79}$$

In conclusion we have the following Lemma.

Lemma 8.9. *Let $\mathcal{L}_{M+3}$ be the operator defined in* (8.38)*, and let T_{M+4} and T_{M+5} be the transformations* (8.54) *and* (8.56) *given in Lemma 8.8. Then*

$$\begin{aligned}\hat{\mathcal{L}}_{M+5} &:= T_{M+5}^{-1} T_{M+4}^{-1} \mathcal{L}_{M+3} T_{M+4} T_{M+5} \\ &= \mathcal{L}_{M+5} + \mathbf{C}^{\perp} + \mathbf{R}_{M_{K_n}}^{\perp},\end{aligned} \tag{8.80}$$

where

$$\mathcal{L}_{M+5} := \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}\partial_x + \mathbf{R}_{M+5}, \tag{8.81}$$

and $\mathbf{T}(\mathbf{D})$ is defined in (7.18)*, $\mathbf{M}$ is defined in* (8.79)*, $\mathbf{R}_{M+5}$ is defined in* (8.74)*. The remainders $\mathbf{C}^{\perp}$ defined in* (8.83) *and $\mathbf{R}_{M_{K_n}}^{\perp}$ defined in* (8.76) *satisfy the tame estimates in* (9.8)*.*

Proof. We write $\mathcal{L}_{M+3}$ defined in (8.38) as

$$\mathcal{L}_{M+3} = \tilde{\mathcal{L}}_{M+3} + \Pi_{K_n}^{\perp} \mathbf{C}^{(3)} \partial_x$$

where $\tilde{\mathcal{L}}_{M+3}$ is defined in (8.52) and $\Pi_{K_n}^{\perp} \mathbf{C}^{(3)}$ is defined in (8.53). We conjugate $\mathcal{L}_{M+3}$ with T_{M+4} defined in (8.54) and we get

$$\mathcal{L}_{M+4} := T_{M+4}^{-1} \mathcal{L}_{M+3} T_{M+4} = \tilde{\mathcal{L}}_{M+4} + T_{M+4}^{-1} \Pi_{K_n}^{\perp} \mathbf{C}^{(3)} \partial_x T_{M+4}\,.$$

Now we conjugate $\mathcal{L}_{M+4}$ with T_{M+5} defined in (8.56) and we obtain

$$\begin{aligned}\hat{\mathcal{L}}_{M+5} &:= T_{M+5}^{-1} \mathcal{L}_{M+4} T_{M+5} \\ &= \tilde{\mathcal{L}}_{M+5} + T_{M+5}^{-1} T_{M+4}^{-1} \Pi_{K_n}^{\perp} \mathbf{C}^{(3)} \partial_x T_{M+4} T_{M+5} \\ &= \Omega \cdot \partial_\theta + \mathbf{M}_{K_n} \partial_y + \mathbf{R}_{M+5} + T_{M+5}^{-1} T_{M+4}^{-1} \Pi_{K_n}^{\perp} \mathbf{C}^{(3)} \partial_x T_{M+4} T_{M+5} \\ &\overset{(8.75)}{=} \Omega \cdot \partial_\theta + \mathbf{M} \partial_y + \mathbf{R}_{M+5} + T_{M+5}^{-1} T_{M+4}^{-1} \Pi_{K_n}^{\perp} \mathbf{C}^{(3)} \partial_x T_{M+4} T_{M+5} + (\mathbf{M}_{K_n} - \mathbf{M}) \partial_y\,.\end{aligned} \tag{8.82}$$

Finally we define

$$\mathbf{C}^{\perp} := T_{M+5}^{-1} T_{M+4}^{-1} \Pi_{K_n}^{\perp} \mathbf{C}^{(3)} \partial_x T_{M+4} T_{M+5} \tag{8.83}$$

hence, by (8.83), (8.75) and (8.76), $\hat{\mathcal{L}}_{M+5}$ in (8.82) reads

$$\begin{aligned}\hat{\mathcal{L}}_{M+5} &:= \tilde{\mathcal{L}}_{M+5} + \mathbf{C}^{\perp} \\ &= \mathcal{L}_{M+5} + \mathbf{C}^{\perp} + \mathbf{R}_{M_{K_n}}^{\perp} \\ &= \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}\partial_x + \mathbf{R}_{M+5} + \mathbf{C}^{\perp} + \mathbf{R}_{M_{K_n}}^{\perp},\end{aligned} \tag{8.84}$$

where

$$\mathcal{L}_{M+5} := \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}\partial_x + \mathbf{R}_{M+5}. \tag{8.85}$$

□

Lemma 8.10. *Let* $\mathbf{M}_{K_n}$ *as in* (8.58). *The following estimates hold*

$$|\mathtt{m}_{1,K_n}|^{k_0,\gamma} \le C\mu \tag{8.86}$$

$$|\partial_i \mathtt{m}_{1,K_n}[\hat{\imath}]| \le C\mu\|\hat{\imath}\|_\sigma \tag{8.87}$$

$$|m_1 - \mathtt{m}_{1,K_n}|^{k_0,\gamma} \le C\mu K_n^{-b}\,, \ \ \forall b > 0. \tag{8.88}$$

$$|m_1|^{k_0,\gamma} \le C\mu \tag{8.89}$$

$$|\partial_i m_1[\hat{\imath}]| \le C\mu\|\hat{\imath}\|_\sigma\,. \tag{8.90}$$

Proof. The estimates (8.86) and (8.87) follows by the explicit definiton of $\mathtt{m}_{1,K_n}$ and $\mathtt{m}_{2,K_n}$ in (8.63). The estimate (8.88) follows by (8.63), and the smoothing property (2.11). The estimates (8.89) and (8.90) follows by (8.78), (8.22). ☐

8.2.4 Tame estimates of the remainder $\mathbf{R}_{M+5}$

The goal of this Section is to prove that the operators $\partial_{\theta_r}^\beta \mathbf{R}_{M+5}\,,\ \partial_{\theta_r}^\beta[\partial_x, \mathbf{R}_{M+5}]$ for $r = 1, ..., N\,,\ \beta \in \mathbb{N}\,,\ \beta \le \beta_0$ are $\mathcal{D}^{k_0}-$tame (see Definition 10).

We want to prove the following Lemma.

Lemma 8.11. *Let* $\mathbf{R}_{M+5}$ *be the operator defined in* (8.74). *Then the operators* $\partial_{\theta_r}^\beta \mathbf{R}_{M+5}\,,\ \partial_{\theta_r}^\beta[\partial_x, \mathbf{R}_{M+5}]$ *are* $\mathcal{D}^{k_0}$*-tame for* $r = 1, ..., N\,,\ \beta \in \mathbb{N}\,,\ \beta \le \beta_0\,,\ \beta_0 + k_0 + 1 \le M$*, with tame constants for all* $\mathfrak{p}_0 \le p \le P$

$$\mathcal{M}_{\partial_{\theta_r}^\beta \mathbf{R}_{M+5}}(p)\,,\ \mathcal{M}_{\partial_{\theta_r}^\beta[\partial_x, \mathbf{R}_{M+5}]}(p) \le_{P,M} \mu\|\mathtt{v}\|_{p+(M+1)M+3M+6+\sigma+\beta}^{k_0,\gamma}\,. \tag{8.91}$$

Moreover if the constant ν *in* (6.11) *satisfies* $p_1 + (M+1)M + 3M + 6 + \beta + \sigma \le \mathfrak{p}_0 + \nu$ *then*

$$\|\partial_{\theta_r}^\beta \partial_i \mathbf{R}_{M+5}[\hat{\imath}]\|_{\mathcal{L}(H^{p_1})}\,,\ \|\partial_{\theta_r}^\beta[\partial_i \mathbf{R}_{M+5}[\hat{\imath}], \partial_x]\|_{\mathcal{L}(H^{p_1})} \le_{P,M} \mu\|\hat{\imath}\|_{p_1+(M+1)M+3M+6+\beta+\sigma}\,. \tag{8.92}$$

The rest of this Section is devoted to the proof of the Lemma above. We recall the definition of the remainder $\mathbf{R}_{M+5}$ given in (8.74), that is $\mathbf{R}_{M+5} = T_{M+5}^{-1}\tilde{\mathbf{R}}_{M+5} + T_{M+5}^{-1}\mathbf{P}_{M+5}$ where $\tilde{\mathbf{R}}_{M+5}$ is defined in (8.69) and $\mathbf{P}_{M+5}$ is defined in (8.70). Using the explicit expression of $\tilde{\mathbf{R}}_{M+5}$ the remainder $\mathbf{R}_{M+5}$ can be written as follows

$$\begin{aligned} \mathbf{R}_{M+5} &= T_{M+5}^{-1}\tilde{\mathbf{R}}_{M+5} + T_{M+5}^{-1}\mathbf{P}_{M+5} \\ &= \mathbf{W}_1 + T_{M+5}^{-1}\tilde{\mathbf{R}}_{M+4}T_{M+5}, \end{aligned}$$

where

$$\mathbf{W}_1 := T_{M+5}^{-1}\Big((\omega\cdot\partial_\theta\Phi_{M+5})\partial_y^{-1} + \mathbf{C}^{(4)}\Phi_{M+5}\pi_0 + \mathbf{C}^{(4)}(\Phi_{M+5})_y\partial_y^{-1} - \mathbf{M}_{K_n}\Phi_{M+5}\pi_0 + \mathbf{P}_{M+5}\Big)\,, \tag{8.93}$$

and Φ_{M+5} is given in (8.56) (see also (8.66)), $\mathbf{C}^{(4)}$ in (8.62), $\mathbf{M}_{K_n}$ in (8.58) and $\mathbf{P}_{M+5}$ in (8.70).

In the next Lemma we shall prove that $\partial_{\theta_r}^\beta \mathbf{W}_1\,,\ r = 1, ..., N\,,\ \beta \in \mathbb{N}$ is a $\mathcal{D}^{k_0}-$tame operator. Then in Lemmas 8.14 and 8.15 we will focus on $T_{M+5}^{-1}\tilde{\mathbf{R}}_{M+4}T_{M+5}$.

Lemma 8.12. *Let* $\mathbf{W}_1$ *be the operator defined in* (8.93). *Then* $\partial_{\theta_r}^{\beta}\mathbf{W}_1$, $\partial_{\theta_r}^{\beta}[\mathbf{W}_1,\partial_x]$ *for* $r=1,...,N$, $\beta\in\mathbb{N}$, $\beta\le\beta_0$ *are* $\mathcal{D}^{k_0}-$*tame with tame constants, for all* $p\ge\mathfrak{p}_0$

$$\mathcal{M}_{\partial_{\theta_r}^{\beta}\mathbf{W}_1}(p)\,,\mathcal{M}_{[\partial_{\theta_r}^{\beta}\mathbf{W}_1,\partial_x]}(p)\le\mu\|\mathtt{v}\|_{p+3+M+\sigma+\beta}. \tag{8.94}$$

Moreover if the constant ν *in* (6.11) *satisfies* $p_1+3+M+\sigma+\beta\le\mathfrak{p}_0+\nu$, *then*

$$\|\partial_{\theta_r}^{\beta}\partial_i\mathbf{W}_1[\hat{\imath}]\|_{\mathcal{L}(H^{p_1})}\,,\ \|\partial_{\theta_r}^{\beta}[\partial_i\mathbf{W}_1[\hat{\imath}],\partial_x]\|_{\mathcal{L}(H^{p_1})}\le\mu\|\hat{\imath}\|_{p_1+3+M+\sigma+\beta}\,. \tag{8.95}$$

Proof. We claim that the transformation T_{M+5} defined in (8.56), and the operators $(\omega\cdot\partial_\theta\Phi_{M+5})\partial_y^{-1}$, $\mathbf{C}^{(4)}\Phi_{M+5}\pi_0$, $\mathbf{C}^{(4)}(\Phi_{M+5})_y\partial_y^{-1}$, $\mathbf{M}_{K_n}\Phi_{M+5}\pi_0$ are $\mathcal{D}^{k_0}$-tame operators since they are pseudo-differential operators. We prove it for $T_{M+5}^{-1}\mathbf{C}^{(4)}\Phi_{M+5}\pi_0$ since for the other terms it is similar. We consider first the operator T_{M+5}^{-1}. By (2.29) we have

$$|T_{M+5}^{-1}|_{0,p,0}^{k_0,\gamma}=|\mathbb{1}+\mu\Phi_{M+5}\partial_y^{-1}|_{0,p,0}^{k_0\gamma}\le C(p)(1+\mu\|\mathtt{v}\|_{p+2+\sigma}).$$

By Lemma 2.8

$$\begin{aligned}|T_{M+5}^{-1}\mathbf{C}^{(4)}\Phi_{M+5}\pi_0|_{0,p,0}^{k_0,\gamma}&\le C(p)|T_{M+5}^{-1}|_{0,p,0}^{k_0,\gamma}|\mathbf{C}^{(4)}\Phi_{M+5}\pi_0|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}+C(\mathfrak{p}_0)|T_{M+5}^{-1}|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|\mathbf{C}^{(4)}\Phi_{M+5}\pi_0|_{0,p,0}^{k_0,\gamma}\\&\le C(p)C(\mathfrak{p}_0)|T_{M+5}^{-1}|_{0,p,0}^{k_0,\gamma}|\mathbf{C}^{(4)}|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|\Phi_{M+5}\pi_0|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}\\&+C(\mathfrak{p}_0)C(P)|T_{M+5}^{-1}|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|\mathbf{C}^{(4)}|_{0,p,0}^{k_0,\gamma}|\Phi_{M+5}\pi_0|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}\\&+C(\mathfrak{p}_0)C(P)|T_{M+5}^{-1}|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|\mathbf{C}^{(4)}|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|\Phi_{M+5}\pi_0|_{0,p,0}^{k_0,\gamma}\,.\end{aligned}$$

Then, by Lemma 2.14, the tame estimate for these operators follows. By Lemmas 2.8 and 2.14 and by (2.29) the pseudo-differential operator $\mathbf{P}_{M+5}$ defined in (8.70), see also (8.71), is $\mathcal{D}^{k_0}$-tame. Indeed consider for instance $Op(r(\xi))\circ\varphi_1^{M+5}\partial_y^{-1}$, then by the explicit definition of $\varphi_1^{(M+5)}$ in (8.72) by (2.26) and (2.29) we have

$$\begin{aligned}|\mu Op(r(\xi))\circ\varphi_1^{M+5}\partial_y^{-1}|_{-M-1,p,0}^{k_0,\gamma}&\le C(p)\mu|Op(r(\xi))|_{0,p,0}^{k_0,\gamma}|\varphi_1^{(M+5)}\partial_y^{-1}|_{0,\mathfrak{p}_0+M+1,0}^{k_0,\gamma}\\&+\mu C(\mathfrak{p}_0)|Op(r(\xi))|_{0,\mathfrak{p}_0,0}^{k_0,\gamma}|\varphi_1^{(M+5)}\partial_y^{-1}|_{0,p+M+1,0}^{k_0,\gamma}\\&\le_p\mu\|\mathtt{v}\|_{p+M+3+\sigma}\,.\end{aligned}$$

We have to estimate

$$\partial_{\theta_r}^{\beta}(T_{M+5}^{-1}\mathbf{C}^{(4)}\Phi_{M+5}\pi_0)=\sum_{\beta_1+\beta_2+\beta_3=\beta}(\partial_{\theta_r}^{\beta_1}T_{M+5}^{-1})(\partial_{\theta_r}^{\beta_2}\mathbf{C}^{(4)})(\partial_{\theta_r}^{\beta_3}\Phi_{M+5}\pi_0)\,.$$

By Lemma 2.8 we have

$$\begin{aligned}|\partial_{\theta_r}^{\beta_1}T_{M+5}^{-1}|_{0,p,0}^{k_0,\gamma}&\le C(\beta_1,p)|T_{M+5}^{-1}|_{0,p+\beta_1,0}^{k_0,\gamma}\\|\partial_{\theta_r}^{\beta_2}\mathbf{C}^{(4)}|_{0,p,0}^{k_0,\gamma}&\le C(\beta_2,p)|\mathbf{C}^{(4)}|_{0,p+\beta_2,0}^{k_0,\gamma}\\|\partial_{\theta_r}^{\beta_3}\Phi_{M+5}\pi_0|_{0,p,0}^{k_0,\gamma}&\le C(\beta_3,p)|\Phi_{M+5}|_{0,p+\beta_3,0}^{k_0,\gamma}\,.\end{aligned}$$

Hence by Lemma 2.14, (2.26) and (2.29) the estimates (8.94) follow. The proof of (8.95) follows analogously. □

Now we focus on $T_{M+5}^{-1}\tilde{\mathbf{R}}_{M+4}T_{M+5}$, where $\tilde{\mathbf{R}}_{M+4}$ is defined in (8.61). Since the operators T_{M+5} and T_{M+5}^{-1} are $\mathcal{D}^{k_0}-$tame (see Lemma above), and since the compositions of $\mathcal{D}^{k_0}-$tame operators is $\mathcal{D}^{k_0}-$tame (see Lemma 2.13), instead of studying $T_{M+5}^{-1}\tilde{\mathbf{R}}_{M+4}T_{M+5}$ it is sufficient to prove that the operators $\partial_{\theta_r}^b\tilde{\mathbf{R}}_{M+4}$ and $[\partial_{\theta_r}^b\tilde{\mathbf{R}}_{M+4},\partial_x]$ are $\mathcal{D}^{k_0}-$tame.

The operator $\tilde{\mathbf{R}}_{M+4}$ is explicitly given by $\tilde{\mathbf{R}}_{M+4}=T_{M+4}^{-1}\mathbf{R}_{M+3}T_{M+4}$ where $\mathbf{R}_{M+3}$ is defined in (8.39).

First of all note that the conjugation of $\mathbf{R}_{M+3}$ with the transformation T_{M+4} defined in (8.54) can not be represented as a pseudo-differential operator on the out of diagonal elements. Indeed

$$\begin{aligned}\tilde{\mathbf{R}}_{M+4}=\sum_{k=0}^{M}&\begin{pmatrix}\Psi_1^{-1}\circ A_k^{(1)}(x,\theta)\partial_x^{-k}\circ\Psi_1 & 0\\ 0 & \Psi_2^{-1}\circ A_k^{(4)}(x,\theta)\partial_x^{-k}\circ\Psi_2\end{pmatrix}\\ &+\begin{pmatrix}\Psi_1^{-1}\circ\Sigma_1\circ\Psi_1 & \Psi_1^{-1}\circ\Sigma_2\circ\Psi_2\\ \Psi_2^{-1}\circ\Sigma_3\circ\Psi_1 & \Psi_2^{-1}\circ\Sigma_4\circ\Psi_2\end{pmatrix},\end{aligned}\tag{8.96}$$

and, as we shall prove in the following Lemma, the diagonal elements still remain pseudo-differential operators after the conjugation with T_{M+4}, but the out of diagonal elements lose this structure.

In order to simplify the notation, since $\Psi_2(\theta)=\Psi_1^{-1}(\theta)$, $\forall\theta\in\mathbb{T}^N$, see (8.54) and (8.55), we shall write Ψ instead of Ψ_1 and Ψ^{-1} instead of Ψ_2, correspondingly for ψ_1, ψ_2.

Lemma 8.13. *Let $A=Op(a(x,\theta,j))$ be a family of pseudo-differential operators. Let $(\Psi h)(x,\theta)=h(x+\psi(\theta),\theta)$ whose inverse is given by $(\Psi^{-1}h)(x,\theta)=h(x-\psi(\theta),\theta)$. Then*

$$\Psi^{-1}\circ A\circ\Psi=Op(\tilde{a}(x,\theta,j))\tag{8.97}$$

$$\Psi^{-1}\circ A\circ\Psi^{-1}=Op(\tilde{a}(x,\theta,j))\psi^{-2}\,,\tag{8.98}$$

where $Op(\tilde{a}(x,\theta,j))=Op(a(x-\psi(\theta),\theta,j))$.

Proof. We prove (8.97) Let $h=\sum_j h_j(\theta)e^{ijx}$, then $\Psi h=\sum_j h_j(\theta)e^{i\psi(\theta)j}e^{ijx}$. So

$$Op(a(x,\theta,j))[\Psi h]=\sum_{j\in\mathbb{Z}}a(x,\theta,j)h_j(\theta)e^{i\psi(\theta)j}e^{ijx}\,.$$

Hence the final operator is

$$\begin{aligned}\Psi^{-1}Op(a(x,\theta,j))[\Psi h]&=\sum_{j\in\mathbb{Z}}a(x-\psi(\theta),\theta,j)h_j(\theta)e^{i\psi(\theta)j}e^{ijx}e^{-i\psi(\theta)j}\\ &=\sum_{j\in\mathbb{Z}}a(x-\psi(\theta),\theta,j)h_j(\theta)e^{ijx}\\ &=\sum_{j\in\mathbb{Z}}\tilde{a}(x,\theta,j)h_j(\theta)e^{ijx}\\ &=Op(\tilde{a}(x,\theta,j))h\,.\end{aligned}$$

Now we prove (8.98). We have that $\Psi^{-1}h=\sum_j h_j(\theta)e^{-i\psi(\theta)j}e^{ijx}$. So

$$Op(a(x,\theta,j))[\Psi^{-1}h]=\sum_{j}a(x,\theta,j)h_j(\theta)e^{-i\psi(\theta)j}e^{ijx}\,.$$

Hence

$$\begin{aligned}\Psi^{-1}Op(a(x,\theta,j))[\Psi^{-1}h] &= \sum_{j\in\mathbb{Z}} a(x-\psi(\theta),\theta,j)h_j(\theta)e^{-i\psi(\theta)j}e^{ijx}e^{-i\psi(\theta)j}\\ &= \sum_{j\in\mathbb{Z}} a(x-\psi(\theta),\theta,j)h_j(\theta)e^{ijx}e^{-2i\psi(\theta)j}\\ &= \sum_{j\in\mathbb{Z}} \tilde a(x,\theta,j)h_j(\theta)e^{-2i\psi(\theta)j}e^{ijx}\\ &= Op(\tilde a(x,\theta,j))\psi^{-2}h\,.\end{aligned}$$

Therefore $\Psi^{-1}Op(a(x,\theta,j))\Psi^{-1}$ is not a pseudo-differential operator. □

We recall that in order to simplify the notation, since $\Psi_2(\theta)=\Psi_1^{-1}(\theta)\,,\ \forall\theta\in\mathbb{T}^N$, see (8.54) and (8.55), we shall write Ψ instead of Ψ_1 and Ψ^{-1} instead of Ψ_2, correspondingly for $\psi_1\,,\,\psi_2$. Thanks to the Lemma above we can prove the following Lemma on the diagonal entries of the operator $\tilde{\mathbf{R}}_{M+4}$.

Lemma 8.14. *For $k=0,...,M$, $m=1,4$ $r=1,...,N\,,\ \beta\in\mathbb{N}\,,\ |\beta|\le\beta_0\,,$ and the operators $\partial_{\theta_r}^{\beta}\Psi^{-1}A_k^{(m)}\partial_x^{-k}\Psi\,,\ \partial_{\theta_r}^{\beta}\Psi^{-1}\Sigma_m\Psi\,,\ \partial_{\theta_r}^{\beta}[\Psi^{-1}A_k^{(m)}\partial_x^{-k}\Psi\,,\,\partial_x]\,,\ \partial_{\theta_r}^{\beta}[\Psi^{-1}\Sigma_m\Psi\,,\,\partial_x]\,,$ are $\mathcal{D}^{k_0}$-tame with tame constants satisfying for all $\mathfrak{p}_0\le p\le P$*

$$\mathcal{M}_{\partial_{\theta_r}^{\beta}(\Psi^{-1}\Sigma_m\Psi)}(p)\,,\mathcal{M}_{\partial_{\theta_r}^{\beta}[\Psi^{-1}\Sigma_m\Psi\,,\,\partial_x]}(p)\le_{P,M}\mu\|\mathtt{v}\|^{k_0,\gamma}_{p+\sigma+(M+1)M+3M+6+\beta}\,. \tag{8.99}$$

Moreover is the constant ν in (6.11) satisfies $p_1+(M+1)M+3M+6+\beta+\sigma\le\mathfrak{p}_0+\nu$, then

$$\|\partial_{\theta_r}^{\beta}\partial_i(\Psi^{-1}\Sigma_m\Psi)[\hat\imath]\|_{\mathcal{L}(H^{p_1})}\,,\ \|\partial_{\theta_r}^{\beta}[\partial_i(\Psi^{-1}\Sigma_m\Psi)[\hat\imath],\partial_x]\|_{\mathcal{L}(H^{p_1})}\le_{P,M}\mu\|\hat\imath\|_{p_1+(M+1)M+3M+6+\beta+\sigma}\,. \tag{8.100}$$

Proof. By Lemma 8.13 we have that for $m=1,4$ and $k=0,...,M$ the operators $\Psi^{-1}A_k^m(x,\theta)\partial_x^{-k}\Psi=A_k^m(x+\psi(\theta),\theta)\partial_x^{-k}$, remain pseudo-differential operators, similar for $\Psi^{-1}\Sigma_m\Psi$. Then, by (2.29) and Lemma 2.23

$$|A_k^m(x+\psi(\theta),\theta)\partial_x^{-k}|^{k_0,\gamma}_{0,p,0}\le_p\|A_k^m(x+\psi(\theta),\theta)\|_p^{k_0,\gamma}\le_P\|A_k^m\|_p^{k_0\gamma}\|\psi\|_{\mathfrak{p}_0}^{k_0,\gamma}+\|\psi\|_p^{k_0,\gamma}\|A\|_{\mathfrak{p}_0}^{k_0\gamma}$$

by the estimates (8.48) the Lemma follows. □

Now it remains to prove that for $\beta\in\mathbb{N}$ the operators $\partial_{\theta_r}^{\beta}(\Psi^{-1}\circ\Sigma_2\circ\Psi^{-1})\,,\ \partial_{\theta_r}^{\beta}(\Psi\circ\Sigma_3\circ\Psi)$ and $\partial_{\theta_r}^{\beta}[\Psi^{-1}\circ\Sigma_2\circ\Psi^{-1}\,,\,\partial_x]\,,\ \partial_{\theta_r}^{\beta}[\Psi\circ\Sigma_3\circ\Psi\,,\,\partial_x]$ are $\mathcal{D}^{k_0}$-tame operators.

It is clear that we can study only one case, e.g. $\Psi\circ\Sigma_3\circ\Psi$ instead of study both $\Psi\circ\Sigma_3\circ\Psi$ and $\Psi^{-1}\circ\Sigma_2\circ\Psi^{-1}$, since $\|\Psi h\|_p=\|\Psi^{-1}h\|_p$ for every h, similarly for the other operators above. For this reason, and also for simplify the notation, in what follows we shall write and study $\Psi\Sigma\Psi$.

Lemma 8.15. *For all $\beta\in\mathbb{N}\,,\ |\beta|\le\beta_0\,,\ |k|\le k_0$ with $\beta_0+k_0+1\le M$ the operators $\partial_{\theta_r}^{\beta}(\Psi\Sigma\Psi)$ and $\partial_{\theta_r}^{\beta}[\Psi\Sigma\Psi,\partial_x]$ for all $r=1,...,N$ are $\mathcal{D}^{k_0}$-tame with tame constants satisfying for all $\mathfrak{p}_0\le p\le P$*

$$\mathcal{M}_{\partial_{\theta_r}^{\beta}(\Psi\Sigma\Psi)}(p)\,,\ \mathcal{M}_{\partial_{\theta_r}^{\beta}[\Psi\Sigma\Psi,\partial_x]}(p)\le_P\mu\|\mathtt{v}\|^{k_0,\gamma}_{p+(M+1)M+3M+6+\beta+\sigma}\,. \tag{8.101}$$

Moreover is the constant ν in (6.11) *satisfies* $p_1+(M+1)M+3M+6+\beta+\sigma\le\mathfrak{p}_0+\nu$, *then*

$$\|\partial_{\theta_r}^{\beta}\partial_i(\Psi\Sigma\Psi)[\hat{\imath}]\|_{\mathcal{L}(H^{p_1})}\,,\ \|\partial_{\theta_r}^{\beta}[\partial_i(\Psi\Sigma\Psi)[\hat{\imath}],\partial_x]\|_{\mathcal{L}(H^{p_1})}\le_P \mu\|\hat{\imath}\|_{p_1+(M+1)M+3M+6+\beta+\sigma}\,. \tag{8.102}$$

Proof. We prove that $\partial_{\theta_r}^{\beta}(\Psi\Sigma\Psi)$ is $\mathcal{D}^{k_0}$-tame. We have that

$$\begin{aligned}\partial_\zeta^k\left(\partial_{\theta_r}^{\beta}\Psi\Sigma\Psi\right)&=\sum_{\beta_1+\beta_2+\beta_3=\beta}\ \sum_{k_1+k_2+k_3=k}C(\beta_1,\beta_2,\beta_3,k_1,k_2,k_3)\left(\partial_{\theta_r}^{\beta_1}\partial_\zeta^{k_1}\Psi\right)\left(\partial_{\theta_r}^{\beta_2}\partial_\zeta^{k_2}\Sigma\right)\\&\quad\times\left(\partial_{\theta_r}^{\beta_3}\partial_\zeta^{k_3}\Psi\right)\\&=\sum_{\beta_1+\beta_2+\beta_3=\beta}\ \sum_{k_1+k_2+k_3=k}C(\beta_1,\beta_2,\beta_3,k_1,k_2,k_3)\left(\partial_{\theta_r}^{\beta_1}\partial_\zeta^{k_1}\Psi\,\langle\partial_x\rangle^{-\beta_1-|k_1|}\right)\\&\quad\times\left(\langle\partial_x\rangle^{\beta_1+|k_1|}\,\partial_{\theta_r}^{\beta_2}\partial_\zeta^{k_2}\Sigma\,\langle\partial_x\rangle^{\beta_3+|k_3|}\right)\left(\langle\partial_x\rangle^{-\beta_3-|k_3|}\,\partial_{\theta_r}^{\beta_3}\partial_\zeta^{k_3}\Psi\right)\end{aligned}$$

where $\beta_1,\beta_2,\beta_3\in\mathbb{N}$ and $k_1,k_2,k_3\in\mathbb{N}^{1+N}$. Let $|k|\le k_0$ and $M\ge\beta+k_0+1$. Then we claim that

$$\begin{aligned}|\,\langle\partial_x\rangle^{|k_1|+\beta_1}\,\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma\,\langle\partial_x\rangle^{|k_3|+\beta_3}\,|_{-M-1+|k_1|+\beta_1+|k_3|+\beta_3,p,0}&\le_p\gamma^{-|k_2|}\|\mathtt{v}\|^{k_0,\gamma}_{p+\beta+k_0+(M+1)M+3M+6}\\|[\langle\partial_x\rangle^{|k_1|+\beta_1}\,\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma\,\langle\partial_x\rangle^{|k_3|+\beta_3}\,,\partial_x]|_{-M-1+|k_1|+\beta_1+|k_3|+\beta_3,p,0}&\le_p\gamma^{-|k_2|}\|\mathtt{v}\|^{k_0,\gamma}_{p+\beta+k_0+(M+1)M+3M+6}\,.\end{aligned}$$

Indeed by (2.25), (2.26) and Lemma 2.8 we have

$$\begin{aligned}&|\,\langle\partial_x\rangle^{|k_1|+\beta_1}\,\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma\,\langle\partial_x\rangle^{|k_3|+\beta_3}\,|_{-M-1+|k_1|+\beta_1+|k_3|+\beta_3,p,0}\le\\&\le C(p,k,M,\beta)\Big(|\,\langle\partial_x\rangle^{|k_1|+\beta_1}\,|_{|k_1|+\beta_1,p,0}|\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma\,\langle\partial_x\rangle^{|k_3|+\beta_3}\,|_{-M-1+|k_3|+\beta_3,\mathfrak{p}_0+|k_1|+\beta_1,0}\\&+|\,\langle\partial_x\rangle^{|k_1|+\beta_1}\,|_{|k_1|+\beta_1,\mathfrak{p}_0,0}|\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma\,\langle\partial_x\rangle^{|k_3|+\beta_3}\,|_{-M-1+|k_3|+\beta_3,p+|k_1|+\beta_1,0}\Big)\\&\le C(p,k,M,\beta)\Big(|\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma|_{-M-1,\mathfrak{p}_0+|k_1|+\beta_1,0}|\,\langle\partial_x\rangle^{|k_3|+\beta_3}\,|_{|k_3|+\beta_3,p+|k_1|+\beta_1+M+1,0}\\&+|\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma|_{-M-1,p+|k_1|+\beta_1,0}|\,\langle\partial_x\rangle^{|k_3|+\beta_3}\,|_{|k_3|+\beta_3,\mathfrak{p}_0+|k_1|+\beta_1+M+1,0}\Big)\\&\le C(p,k,M,\beta)|\partial_\zeta^{k_2}\partial_{\theta_r}^{\beta_2}\Sigma|_{-M-1,p+|k_1|+\beta_1,0}\\&\le C(p,k,M,\beta)\gamma^{-|k_2|}|\Sigma|^{k_0,\gamma}_{-M-1,p+|k_1|+\beta_1+\beta_2+\sigma,0}\\&\le C(p,k_0,M,\beta)\gamma^{-|k_2|}\|\mathtt{v}\|^{k_0,\gamma}_{p+k_0+\beta_1+\beta_2+(M+1)M+3M+6+\sigma}\\&\le C(p,k_0,M,\beta)\gamma^{-|k_2|}\|\mathtt{v}\|^{k_0,\gamma}_{p+k_0+\beta+(M+1)M+3M+6+\sigma}\,.\end{aligned}$$

By Lemmas 2.26 and 2.25 we have that

$$\mathcal{M}_{\partial_{\theta_r}^{\beta}\Psi\langle\partial_x\rangle^{-\beta}}(p)\,,\ \mathcal{M}_{\partial_{\theta_r}^{\beta}[\Psi\langle\partial_x\rangle^{-\beta},\partial_x]}(p)\le\mu\|\mathtt{v}\|^{k_0,\gamma}_{p+\sigma+\beta+2}.$$

Similarly for $\langle\partial_x\rangle^{-\beta-|k|}\,\partial_{\theta_r}^{\beta}\Psi$. Then the tame estimates for the operators $\langle\partial_x\rangle^{-\beta_3-|k_3|}\,\partial_{\theta_r}^{\beta_3}\partial_\zeta^{k_3}\Psi$ and $\partial_{\theta_r}^{\beta_1}\partial_\zeta^{k_1}\Psi\,\langle\partial_x\rangle^{-\beta_1-|k_1|}$ in (8.101) and (8.102) holds. ☐

Thanks to this Lemma the proof of Lemma 8.11 is completed. Indeed by Lemmas 8.12, 8.14 and 8.15 we can prove the Lemma presented at the beginning of the Section.

8.2.5 Structure of the remainder $\mathbf{R}_{M+5}$

In this Section we write the remainder $\mathbf{R}_{M+5}$ in a block diagonal form.

Lemma 8.16. *Every operator* $\mathbf{B} := \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} : E \to E$, *where* E *is defined in* (7.26), *can be written in a block diagonal form i.e.* $\begin{pmatrix} \tilde{B}_1 & 0 \\ 0 & \tilde{B}_4 \end{pmatrix}$.

Proof. Let $\mathcal{S} : u(x) \to u(-x)$, and let $(f, g) \in E$, then

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} B_1 f + B_2 g \\ B_3 f + B_4 g \end{pmatrix} = \begin{pmatrix} B_1 + B_2\mathcal{S} & 0 \\ 0 & B_3\mathcal{S} + B_4 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

□

Lemma 8.17. *Let* $\mathbf{B}$ *be a real operator acting on* E *as in Lemma 8.16. Then the Fourier coefficients (in the exponential basis) of the operators* B_1, B_2, B_3, B_4 *satisfy the following equalities*

$$\begin{aligned} (B_1)^j_{-k} + (B_2)^{-j}_{-k} &= (B_3)^j_k + (B_4)^{-j}_k \\ \overline{(B_1)^{-j}_{-k}} + \overline{(B_2)^j_{-k}} = (B_1)^j_k + (B_2)^{-j}_k, &\quad \overline{(B_4)^{-j}_{-k}} + \overline{(B_3)^j_{-k}} = (B_4)^j_k + (B_3)^{-j}_k. \end{aligned} \tag{8.103}$$

Proof. By Lemma (8.16) we have

$$\begin{pmatrix} B_1 + B_2\mathcal{S} & 0 \\ 0 & B_3\mathcal{S} + B_4 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \sum_{j,k\in\mathbb{Z}} \begin{pmatrix} (B_1)^j_k + (B_2)^{-j}_k & 0 \\ 0 & (B_3)^{-j}_k + (B_4)^j_k \end{pmatrix} \begin{pmatrix} f_j \\ g_j \end{pmatrix} e^{ikx}.$$

Since $\mathbf{B} : E \to E$ we have that $(B_1 f + B_2\mathcal{S}f)(-x) = (B_3\mathcal{S}g + B_4 g)(x)$, in Fourier basis this condition correspond to the first in (8.103). The second two conditions in (8.103) ensure that the operator $\mathbf{B}$ maps real valued functions into real valued functions. □

Remark 8.18. *By* (8.103) *we have that a block diagonal form operator*

$$\mathbf{A} := \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix},$$

is real and sends E *in itself if for all* $j, k \in \mathbb{Z}$,

$$(A_1)^{-j}_{-k} = (A_4)^j_k, \quad (A_m)^j_k = \overline{(A_m)^{-j}_{-k}}, \quad m = 1, 4, \tag{8.104}$$

Now we shall write the remainder

$$\mathbf{R}_{M+5} := \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} \tag{8.105}$$

defined in (8.74) in a block diagonal form. Moreover we also shall give some important properties on the coefficients that we shall use in Chapter 9. We recall that $\mathbf{R}_{M+5}$ is a real, reversible operator that acts in E defined in (7.26) (see Lemma 8.8).

Lemma 8.19. *Let* $\mathbf{R}_{M+5}$ *as in* (8.105). *Then it can be written as*

$$\mathbf{Q} := \begin{pmatrix} Q_1 & 0 \\ 0 & Q_4 \end{pmatrix} . \tag{8.106}$$

where

$$Q_1 := R_1 + R_2\mathcal{S} , \quad Q_4 := R_4 + R_3\mathcal{S} , \ \ and \ \ \mathcal{S} : u(x) \to u(-x) . \tag{8.107}$$

In addition, $\mathbf{Q} : E \to E$ *and is real, therefore its coefficients, written in the Fourier exponential basis, satisfy*

$$(Q_4)_k^j = (Q_1)_{-k}^{-j} , \quad (Q_1)_k^j = \overline{(Q_1)_{-k}^{-j}} , , \quad (Q_4)_k^j = \overline{(Q_4)_{-k}^{-j}} . \tag{8.108}$$

Moreover since $\mathbf{Q}$ *is reversible the following equality holds*

$$(Q_1)_k^j(-l) = -(Q_4)_k^j(l). \tag{8.109}$$

Proof. By Lemma 8.103 we can write $\mathbf{R}_{M+5}$ as a block diagonal form operator. The equalities 8.108 follow by the equality (8.103) for a block diagonal form operator (see (8.104)). Since $\mathbf{R}_{M+5}$ is reversible (see (7.31)) we have that also the operator $\mathbf{Q}$ is reversible, that is $Q_1(-\theta) = -Q_4(\theta)$, in the Fourier exponential representation this condition reads (8.109). □

The remainder $\mathbf{Q}$ satisfies the same tame estimate of the remainder $\mathbf{R}_{M+5}$ as we prove in the following Lemma.

Lemma 8.20. *Let* $\mathbf{Q}$ *be the operator defined in Lemma 8.19. Then for* $j = 1, ..., N$, $\beta \in \mathbb{N}$, $|\beta| \le \beta_0$, $|k| \le k_0$, $\beta_0 + k_0 + 1 \le M$ *the following estimates hold*

$$\mathcal{M}_{\partial_{\theta_r}^{\beta}\mathbf{Q}}(p) \, , \ \mathcal{M}_{\partial_{\theta_r}^{\beta}[\partial_x, \mathbf{Q}]}(p) \le_P \mu \|\mathrm{v}\|_{p+(M+1)M+3M+6+\sigma+\beta}^{k_0,\gamma} . \tag{8.110}$$

Moreover is the constant ν *in* (6.11) *satisfies* $p_1 + (M+1)M + 3M + 6 + \beta + \sigma \le \mathfrak{p}_0 + \nu$ *then*

$$\|\partial_{\theta_r}^{\beta}\partial_i\mathbf{Q}[\hat{\imath}]\|_{\mathcal{L}(H^{p_1})} \, , \ \|\partial_{\theta_r}^{\beta}[\partial_i\mathbf{Q}[\hat{\imath}], \partial_x]\|_{\mathcal{L}(H^{p_1})} \le_P \mu \|\hat{\imath}\|_{p_1+(M+1)M+3M+6+\beta+\sigma} . \tag{8.111}$$

Proof. This Lemma follows by Lemma 8.11, by (8.107) and by $\|u\|_p^{k_0,\gamma} = \|\mathcal{S}u\|_p^{k_0,\gamma}$. □

In conclusion the operator $\hat{\mathcal{L}}_{M+5}$ defined in (8.84) reads

$$\begin{aligned} \hat{\mathcal{L}}_{M+5} &= \mathcal{L}_{M+5} + \mathbf{C}^{\perp} + \mathbf{R}_{M_{K_n}}^{\perp} \\ &= \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}\partial_x + \mathbf{Q} + \mathbf{C}^{\perp} + \mathbf{R}_{M_{K_n}}^{\perp} \end{aligned}$$

where $\mathbf{C}^{\perp}$ is defined in (8.83), $\mathbf{R}_{M_{K_n}}^{\perp}$ is defined in (8.76), the operator $\mathcal{L}_{M+5} = \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}\partial_x + \mathbf{Q}$ with $\mathbf{T}(\mathbf{D})$ defined in (7.18), $\mathbf{M}$ defined in (8.77) and $\mathbf{Q}$ is given in the Lemma 8.19 and satisfies the estimates in Lemma 8.20.

Chapter 9

Partial reduction of $\mathcal{L}_\omega$

By the study in Chapters 7 and 8 the operator $\mathcal{L}$ in (6.10) is conjugated to the operator $\hat{\mathcal{L}}_{M+5}$ defined in (8.84)

$$\hat{\mathcal{L}}_{M+5} = \mathcal{W}^{-1}\mathcal{L}\mathcal{W} \,. \tag{9.1}$$

Therefore, by (8.81), the operator $\mathcal{L}$ defined in (6.10) is semi-conjugated to the real operator $\mathcal{L}_{M+5}$, up to operators which are supported only on high Fourier frequencies, that is

$$\begin{aligned} \mathcal{L}_{M+5} &= \mathcal{W}^{-1}\mathcal{L}\mathcal{W} - \mathbf{C}^{\perp} - \mathbf{R}^{\perp}_{M_{K_n}} \,, \\ \mathcal{W} &= \mathcal{Z}T_1T_2T_3 \circ \ldots \circ T_{M+3}T_{M+4}T_{M+5} \,, \end{aligned} \tag{9.2}$$

where $\mathbf{C}^{\perp}$ and $\mathbf{R}^{\perp}_{M_{K_n}}$ are defined in (8.83) and (8.76). The map $\mathcal{W}^{-1}$ sends the subspace E defined in (7.26) into itself, moreover it is real and reversibility preserving. We denote by $\Pi_{\mathbb{S}}$ the L^2-orthogonal projection on $\mathbb{S}$ (defined in (1.31)) and $\Pi_{\mathbb{S}}^{\perp} := \mathbb{1} - \Pi_{\mathbb{S}}$.

Lemma 9.1. *For $\mu\gamma^{-1}$ small enough, the operator*

$$\mathcal{W}^{\perp} = \Pi_{\mathbb{S}}^{\perp}\mathcal{W}\Pi_{\mathbb{S}}^{\perp}$$

is invertible and for all $\mathfrak{p}_0 \le p \le P$ it satisfies the tame estimate

$$\|\mathcal{W}^{\perp}h\|_p^{k_0,\gamma} + \|(\mathcal{W}^{\perp})^{-1}h\|_p^{k_0,\gamma} \le_{P,M} \|h\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+5+M^2+\sigma}^{k_0,\gamma}\|h\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \,. \tag{9.3}$$

Moreover if ν in (6.11) *satisfies $p_1 + 5 + M^2 + \sigma \le \mathfrak{p}_0 + \nu$, then*

$$\|\partial_i\mathcal{W}^{\pm1}[\hat{\imath}]h\|_{p_1}, \|\partial_i(\mathcal{W}^{\perp})^{\pm1}[\hat{\imath}]h\|_{p_1} \le_{P,M} \|\hat{\imath}\|_{p_1+5+M^2+\sigma}\|h\|_{\mathfrak{p}_0+\sigma} \tag{9.4}$$

Proof. By Lemmas 2.13, 2.14, 2.15 and 8.7 and by (6.11) we have that the operator $\mathcal{W}$ is invertible and satisfies

$$\|(\mathcal{W})^{\pm1}h\|_{p+\sigma}^{k_0,\gamma} \le_P \|h\|_p^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+5+M^2+\sigma}^{k_0,\gamma}\|h\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}.$$

By the definition of $\Pi_{\mathbb{S}}^{\perp}$, in order to prove that $\mathcal{W}^{\perp}$ is invertible, it is sufficient to prove that $\Pi_{\mathbb{S}}\mathcal{W}\Pi_{\mathbb{S}}$ is invertible. This follows by a perturbative argument, for $\mu\gamma^{-1}$ small enough, using that $\Pi_{\mathbb{S}}$ is a finite dimensional projector. □

The operator $\mathcal{L}_\omega$ defined in (5.48) or in (6.9) is semi-conjugate to

$$(\mathcal{W}^{\perp})^{-1}\mathcal{L}_\omega\mathcal{W}^{\perp} = \Pi_{\mathbb{S}}^{\perp}\mathcal{L}_{M+5}\Pi_{\mathbb{S}}^{\perp} + R_F - \Pi_{\mathbb{S}}^{\perp}\mathbf{R}^{\perp}\Pi_{\mathbb{S}}^{\perp} - \Pi_{\mathbb{S}}^{\perp}\mathbf{R}^{\perp}_{M_{K_n}}\Pi_{\mathbb{S}}^{\perp}$$

where

$$R_F := (\mathcal{W}^{\perp})^{-1}\Pi_{\mathbb{S}}^{\perp}\mu R\mathcal{W}^{\perp} \tag{9.5}$$

and R is the finite dimensional remainder defined in (6.1).

Lemma 9.2. *The operator R_F has the finite dimensional form (6.3)-(6.4).*

Proof. We have that R has the form (6.3), hence we have to prove that, given $\mathcal{R} : h \to (h, g)_{L^2(\mathbb{T}_x)}\chi$, the operator $(\mathcal{W}^{\perp})^{-1}\mathcal{R}\mathcal{W}$ has the form (6.3) as well. We will use the following property: given a scalar function $a : \mathbb{T}^N \to \mathbb{C}$ and $\chi = \chi(\theta, \cdot) \in \mathtt{H}_{\mathbb{S}}^{\perp}$, we have

$$(\mathcal{W}^{\perp})^{\pm 1}[a(\theta)\chi] = a(\theta)(\mathcal{W}^{\perp})^{\pm 1}[\chi]\,.$$

Indeed $\Pi_{\mathbb{S}}^{\perp}a(\theta) = a(\theta)\Pi_{\mathbb{S}}^{\perp}$ and for operator of the following form $1+\varphi_k\partial_x^{-k}$ we have that $(1+\varphi_k\partial_x^{-k})a(\theta) = a(\theta) + \varphi_k a(\theta)\partial_x^{-k} = a(\theta)(1 + \varphi_k\partial_x^{-k})$.

For any $h(\theta, \cdot) \in \mathtt{H}_{\mathbb{S}}^{\perp}$ we have

$$\begin{aligned}(\mathcal{W}^{\perp})^{-1}\mathcal{R}\mathcal{W}^{\perp}[h] :=&(\mathcal{W}^{\perp})^{-1}\left[\left(\mathcal{W}^{\perp}[h], g\right)_{L^2(\mathbb{T}_x)}\right]\chi\\ =&\left[\left(\mathcal{W}^{\perp}[h], g\right)_{L^2(\mathbb{T}_x)}\right](\mathcal{W}^{\perp})^{-1}[\chi]\\ =&\left(h, (\mathcal{W}^{\perp})^{-1}g\right)_{L^2(\mathbb{T}_x)}(\mathcal{W}^{\perp})^{-1}[\chi]\\ =&\,(h, g_\star)_{L^2(\mathbb{T}_x)}[\chi_\star]\,,\end{aligned}$$

where , $g_\star := (\mathcal{W}^{\perp})^{-1}g$ and $\chi_\star := (\mathcal{W}^{\perp})^{-1}[\chi]$. Therefore $(\mathcal{W}^{\perp})^{-1}R\mathcal{W}^{\perp}[h]$ has exactly the form (6.3).

□

In conclusion we write $\mathcal{L}_\omega$, defined in (5.48) (i.e. (6.9)), as follows:

$$\mathcal{L}_\omega = (\mathcal{W}^{\perp})\mathcal{L}_{M+6}(\mathcal{W}^{\perp})^{-1} + \mathbf{G}^{\perp} \tag{9.6}$$

where

$$\mathcal{L}_{M+6} := \mathcal{L}_{M+5} + R_F \quad \text{and} \quad \mathbf{G}^{\perp} := -\mathcal{W}^{\perp}(\mathbf{C}^{\perp} + \mathbf{R}^{\perp}_{M_{K_n}})(\mathcal{W}^{\perp})^{-1}. \tag{9.7}$$

The operator $\mathbf{G}^{\perp}$ satisfies, for all $\mathfrak{p}_0 \le p \le P$ and $\sigma := \sigma(\tau, N, k_0) > 0$

$$\begin{aligned}\|\mathbf{G}^{\perp}h\|_{\mathfrak{p}_0}^{k_0,\gamma} &\le_P \mu K_n^{-b}\left(\|h\|_{\mathfrak{p}_0+\sigma+b}^{k_0,\gamma} + \|\mathcal{V}_0\|_{\mathfrak{p}_0+M^2+5+\sigma+b}^{k_0,\gamma}\|h\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma}\right),\forall b > 0\\ \|\mathbf{G}^{\perp}h\|_{p}^{k_0,\gamma} &\le_P \mu\left(\|h\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+2}^{k_0,\gamma}\|h\|_{\mathfrak{p}_0+M^2+5+\sigma}^{k_0,\gamma}\right).\end{aligned} \tag{9.8}$$

The estimates (9.8) follows by (8.83), (8.76), (8.23), (9.3), (2.36) and (6.13).

Proposition 9.3. *Assume* (6.11)*. For all* $(\omega, \varepsilon) \in \mathtt{DC}^{\gamma}_{K_n} \times [\varepsilon_1, \varepsilon_2]$ *(see* (5.5)*) the operator* $\mathcal{L}_\omega$ *defined in* (6.9) *is semiconjugated to the real, reversible operator* $\mathcal{L}_{M+6} : E \to E$ *up to the remainder* $\mathbf{G}^\perp$ *which satisfy* (9.8)*. The operator*

$$\mathcal{L}_{M+6} = \Pi^\perp_{\mathbb{S}}(\Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{M}\partial_x + \mathbf{Q})\Pi^\perp_{\mathbb{S}} \tag{9.9}$$

where $\mathbf{T}(\mathbf{D})$ *is defined in* (7.18)*, the diagonal constant coefficient* $\mathbf{M}$ *with entries* $m_1 := m_1(\omega, \varepsilon)$*, (see* (8.79)*) is defined for all* $(\omega, \varepsilon) \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$*, and satisfy*

$$|m_1|^{k_0,\gamma} \le C\mu\,, \qquad |\partial_i m_1[\hat{\imath}]| \le C\mu\|\hat{\imath}\|_\sigma. \tag{9.10}$$

The remainder (defined in Lemma 8.19)

$$\mathbf{Q} := \begin{pmatrix} Q_1 & 0 \\ 0 & Q_4 \end{pmatrix} \tag{9.11}$$

satisfy the following tame properties: $\forall \beta \in \mathbb{N}$*,* $\beta + k_0 + 1 \le M$ *the operators* $\partial^\beta_{\theta_r} Q_m\,, \partial^\beta_{\theta_r}[Q_m, \partial_x]$*, for* $m = 1, 4$ *and* $r = 1, ..., N$ *are* $\mathcal{D}^{k_0}$*-tame and their tame constant satisfy, for all* $\mathfrak{p}_0 \le p \le P$*,*

$$\max_{m=1,4} \Big(\mathcal{M}_{\partial^\beta_{\theta_r} Q_m}(p)\,,\ \ \mathcal{M}_{\partial^\beta_{\theta_r}[Q_m,\partial_x]}(p)\Big) \le_{M,P} \mu\gamma^{-1}\Big(1 + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+(M+1)M+3M+6+\sigma+\beta}\Big)\,, \tag{9.12}$$

for some $\sigma := \sigma(\tau, N, k_0) > 0$.

Moreover if the constant ν in (6.11) satisfies

$$p_1 + (M+1)M + 3M + 6 + \sigma + M - k_0 + 1 \le \mathfrak{p}_0 + \nu\,, \tag{9.13}$$

then, for all $\beta \in \mathbb{N}\,,\ \beta + k_0 + 1 \le M$ we have

$$\|\partial^\beta_{\theta_r}\partial_i Q_m[\hat{\imath}]\|_{\mathcal{L}(H^{p_1})}\,,\ \ \|\partial^\beta_{\theta_r}[\partial_i Q_m[\hat{\imath}], \partial_x]\|_{\mathcal{L}(H^{p_1})} \le_{M,P} \mu\gamma^{-1}\|\hat{\imath}\|_{p_1+(M+1)M+3M+6+\sigma+\beta}\,. \tag{9.14}$$

Proof. We have that the approximate solution (q, p) is defined for all $(\omega, \varepsilon) \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$ at each step of the Nash-Moser iteration in Chapter 10, as it is proved in the extension Lemma 10.5. For this reason m_1 in (8.78), and hence $\mathbf{M}$ in (8.79), is defined for all the parameters $(\omega, \varepsilon) \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$. By Section 8.2.4 and Lemma 8.20 we have that the estimate (9.14) and (9.12) holds for $\mathbf{Q}$. We have to prove that the estimates are satisfied for R_F defined in (9.5). We have that $\forall h \in E \cap \mathtt{H}^\perp_{\mathbb{S}}$

$$R_F[h] := (\mathcal{W}^\perp)^{-1} R \mathcal{W}^\perp[h] = (h, g_\star)_{L^2(\mathbb{T}_x)}\,[\chi_\star]\,,$$

where $g_\star := (\mathcal{W}^\perp)^{-1} g \in E \cap \mathtt{H}^\perp_{\mathbb{S}}$ and $\chi_\star := (\mathcal{W}^\perp)^{-1}[\chi] \in E \cap \mathtt{H}^\perp_{\mathbb{S}}$. Hence by (9.4) we have for $\mathfrak{p}_0 \le p \le P$

$$\begin{aligned} &\|\chi_\star\|^{k_0,\gamma}_p\,,\ \ \|g_\star\|^{k_0,\gamma}_p \le_{P,M} \mu\gamma^{-1}(1 + \|\mathcal{V}_0\|_{p+M^2+5+\sigma}) \\ &\|\partial_i\chi_\star[\hat{\imath}]\|_{p_1}\,,\ \ \|\partial_i g_\star[\hat{\imath}]\|_p \le_{P,M} \mu\gamma^{-1}(1 + \|\hat{\imath}\|_{p_1+M^2+5+\sigma})\,. \end{aligned} \tag{9.15}$$

Therefore, using

$$\partial_{\theta_r}^{\beta}\partial_{\zeta}^{k} R_F h = \sum_{\substack{\beta_1+\beta_2=\beta\\ k_1+k_2=k}} C(\beta_1,\beta_2,k_1,k_2)\Big(\partial_{\theta_r}^{\beta_1}\partial_{\zeta}^{k_1}\chi_\star(h,\partial_{\theta_j}^{\beta_2}\partial_{\zeta}^{k_2}g)_{L^2_x}+\partial_{\theta_r}^{\beta_1}\partial_{\zeta}^{k_1}\chi_\star(h,\partial_{\theta_r}^{\beta_2}\partial_{\zeta}^{k_2}g)_{L^2_x}\Big)$$

$$\partial_{\theta_r}^{\beta}\partial_{\zeta}^{k}[R_F,\partial_x] h = \sum_{\substack{\beta_1+\beta_2=\beta\\ k_1+k_2=k}} C(\beta_1,\beta_2,k_1,k_2)\Big(\partial_{\theta_r}^{\beta_1}\partial_{\zeta}^{k_1}\chi_\star(h,\partial_{\theta_j}^{\beta_2}\partial_{\zeta}^{k_2}(\partial_x g))_{L^2_x}+\partial_{\theta_r}^{\beta_1}\partial_{\zeta}^{k_1}(\partial_x\chi_\star)(h,\partial_{\theta_r}^{\beta_2}\partial_{\zeta}^{k_2}g)_{L^2_x}\Big)$$

we have that the estimates (9.12) follow. For $\partial_i\partial_{\theta_r}^{\beta}R_F[\hat{\imath}]$ and $\partial_{\theta_r}^{\beta}[\partial_i R_F[\hat{\imath}],\partial_x]$ we have similar expressions.

□

9.1 Almost diagonalization and invertibility of $\mathcal{L}_\omega$

The goal of this section is to diagonalize the operator $\mathcal{L}_{M+6}$. We neglect the remainder $\mathbf{G}^\perp$ supported on the high fourier modes, which will contribute to the remainder in (9.98) and (9.99). We shall apply an iterative reducibility scheme. Let $\mathbf{L}_0$ be an operator acting on $E\cap \mathtt{H}_{\mathbb{S}}^{\perp}$, where E is defined in (7.26) and $\mathtt{H}_{\mathbb{S}}^{\perp}$ is defined in (1.32). The operator can be written as

$$\mathbf{L}_0=\mathbf{L}_0(i):=\Omega\cdot\partial_\theta\mathbb{1}^\perp+i\mathbf{D}_0+\mathbf{Q}_0,\quad \mathbb{1}^\perp:=\mathbb{1}\Pi_{\mathbb{S}^\perp}:=\begin{pmatrix}\mathbb{1}&0\\0&\mathbb{1}\end{pmatrix}\Pi_{\mathbb{S}^\perp}\,. \tag{9.16}$$

Note that $\mathbf{L}_0$ is defined for all $(\omega,\varepsilon)\in\mathtt{DC}^{\gamma}_{K_n}\times[\varepsilon_1,\varepsilon_2]$, where $\mathtt{DC}^{\gamma}_{K_n}$ is defined in (5.5). Let $\mathbb{Z}_0:=\mathbb{Z}\setminus\{0\}$ and $\mathbb{S}^\pm:=\mathbb{S}\cup(-\mathbb{S})$ where $\mathbb{S}$ is defined in (1.31). The diagonal part (with respect to the exponential representation) is given by

$$\mathbf{D}_0=\begin{pmatrix}D_0&0\\0&-D_0\end{pmatrix},\quad D_0=diag_{j\in\mathbb{Z}_0\setminus\mathbb{S}^\pm}\lambda_j^{(0)}\quad \lambda_j^{(0)}=j\sqrt{\frac{2}{15}j^4\varepsilon^4+\frac{1}{3}j^2\varepsilon^2+1}+m_1 j \tag{9.17}$$

where $m_1=m_1(\omega,\varepsilon)\in\mathbb{R}$ is defined for all $(\omega,\varepsilon)\in\mathbb{R}^N\times[\varepsilon_1,\varepsilon_2]$. The remainder

$$\mathbf{Q}_0:E\cap\mathtt{H}_{\mathbb{S}}^{\perp}\to E\cap\mathtt{H}_{\mathbb{S}}^{\perp}\quad \mathbf{Q}_0=\begin{pmatrix}Q_1&0\\0&Q_4\end{pmatrix}, \tag{9.18}$$

is real and reversible. The operators Q_1, Q_4 satisfy (8.108) and (8.109). Moreover the operator $\mathbf{Q}_0$ satisfies the following tame estimates:

- **Smallness assumptions on $\mathbf{Q}_0$.** The operators

$$Q_m\,,\ [Q_m,\partial_x]\,,\ \partial_{\theta_r}^{\mathfrak{p}_0}Q_m\,,\ \partial_{\theta_r}^{\mathfrak{p}_0}[Q_m,\partial_x],\quad r=1,...,N\,,\quad m=1,4$$

are $\mathcal{D}^{k_0}$-tame with tame constants defined for all $\mathfrak{p}_0\le p\le P$,

$$\mathbb{M}_0(p)=\max_{m=1,4,r=1,\dots,N}\Big\{\mathcal{M}_{Q_m}(p),\mathcal{M}_{[Q_m,\partial_x]}(p),\mathcal{M}_{\partial_{\theta_r}^{\mathfrak{p}_0}Q_m}(p)\mathcal{M}_{\partial_{\theta_r}^{\mathfrak{p}_0}[Q_m,\partial_x]}(p)\Big\}\,. \tag{9.19}$$

In addition the operators

$$\partial_{\theta_r}^{\mathfrak{p}_0+\mathtt{b}}Q_m\,,\ \partial_{\theta_r}^{\mathfrak{p}_0+\mathtt{b}}[Q_m,\partial_x],\quad r=1,...N\quad m=1,4$$

are $\mathcal{D}^{k_0}$-tame with tame constant defined for all $\mathfrak{p}_0 \le p \le P$,

$$\mathbb{M}_0(p, \mathtt{b}) = \max_{m=1,4,r=1,\dots,N} \left\{ \mathcal{M}_{\partial_{\theta_r}^{\mathfrak{p}_0+\mathtt{b}} Q_m}(p)\,, \mathcal{M}_{\partial_{\theta_r}^{\mathfrak{p}_0+\mathtt{b}}[Q_m,\partial_x]}(p) \right\}\,, \tag{9.20}$$

where $\mathtt{b} \in \mathbb{N}$ satisfies

$$\mathtt{b} := [\mathfrak{a}] + 2 \in \mathbb{N}\,, \quad \mathfrak{a} := 3\tau_1\,, \quad \tau_1 := \tau(1+k_0) + k_0\,. \tag{9.21}$$

We assume that the tame constant satisfy

$$\mathcal{M}_0(\mathfrak{p}_0, \mathtt{b}) := \max\{\mathbb{M}_0(\mathfrak{p}_0), \mathbb{M}_0(\mathfrak{p}_0, \mathtt{b})\} \le C(P)\gamma^{-1}\mu \tag{9.22}$$

and that there is $\sigma(\mathtt{b}) > 0$, $(\sigma(\mathtt{b}) = \nu(\mathtt{b}) + \sigma)$ such that, for all $r = 1, ..., N$, $\beta \in \mathbb{N}$, $\beta \le \mathtt{b} + \mathfrak{p}_0$ we have

$$\max_{m=1,4}\{\|\partial_{\theta_r}^{\beta}\partial_i Q_m[\hat{\imath}]\|_{\mathcal{L}(H^{\mathfrak{p}_0})}, \|\partial_{\theta_r}^{\beta}[\partial_i Q_m[\hat{\imath}], \partial_x]\|_{\mathcal{L}(H^{\mathfrak{p}_0})}\} \le C(P)\gamma^{-1}\mu\|\hat{\imath}\|_{\mathfrak{p}_0+\sigma(\mathtt{b})}\,. \tag{9.23}$$

Remark 9.4. *The conditions $\mathtt{b} > \mathfrak{a} + \frac{2}{3}$ and $\mathfrak{a} > 3\tau_1$ arise for the convergence of the iterative scheme* (9.78), (9.79) *in Lemma 9.16. We take an integer $\mathtt{b} := [\mathfrak{a}] + 2 \in \mathbb{N}$ so that $\partial_{\theta_m}^{\mathfrak{p}_0+\mathtt{b}}$ are differential operators (since $\mathfrak{p}_0 \in \mathbb{N}$). Note also that $\mathfrak{a} \ge \frac{3}{2}k_0(\tau+2) + 1$ (as $\tau \ge 1$) which is used in the extension procedure in $(S2)_\nu$ (see* (9.43)*). Moreover $\mathfrak{a} > \frac{3}{2}[\tau + k_0(\tau+2)]$ which is used in Lemma 10.7.*

We have to choose $M \ge \mathtt{b} + \mathfrak{p}_0 + k_0 + 1$ and for definiteness we fix

$$M = \mathtt{b} + \mathfrak{p}_0 + k_0 + 1. \tag{9.24}$$

We also define

$$\begin{aligned} c(\mathtt{b}) &:= (M+1)M + 3M + 6 \\ &:= (\mathtt{b} + \mathfrak{p}_0 + k_0 + 2)(\mathtt{b} + \mathfrak{p}_0 + k_0 + 1) + 3(\mathtt{b} + \mathfrak{p}_0 + k_0 + 1) + 6\,, \\ \nu(\mathtt{b}) &:= c(\mathtt{b}) + \mathtt{b} \end{aligned} \tag{9.25}$$

where M is the regularization order that we require on the off-diagonal terms of the remainder, and $c(\mathtt{b}), \nu(\mathtt{b})$ represent the loss of derivatives on the coefficient and from the next Lemma and so on we shall use those constant. The operator $\mathbf{L}_0 := \mathcal{L}_{M+6}$, where $\mathcal{L}_{M+6}$ is defined in (9.9) satisfies the previous assumptions.

Lemma 9.5. ***Tame estimate for Q.*** *Assume* (6.11) *. Then the operator $\mathbf{Q} := \mathbf{Q}_0$ defined in* (9.11) *satisfies , for all $\mathfrak{p}_0 \le p \le P$ the tame estimates* (9.19) *and* (9.20) *with*

$$\mathbb{M}_0(p) \le_P \mu\gamma^{-1}\left(1 + \|\mathcal{V}_0\|_{p+\mathfrak{p}_0+c(\mathtt{b})+\sigma}\right) \quad \mathbb{M}_0(p, \mathtt{b}) \le_P \mu\gamma^{-1}\left(1 + \|\mathcal{V}_0\|_{p+\nu(\mathtt{b})+\sigma}\right) \tag{9.26}$$

and (9.22) *holds. Moreover for all $r = 1, ..., N$ and $\beta \in \mathbb{N}$, $\beta \le \mathtt{b} + \mathfrak{p}_0$ the operators*

$$\partial_{\theta_r}^{\beta}\partial_i Q_m[\hat{\imath}]\,,\ \partial_{\theta_r}^{\beta}[\partial_i Q_m[\hat{\imath}], \partial_x]\,, \quad m = 1, 4$$

satisfy the bounds (9.23) *with $\sigma(\mathtt{b}) = \nu(\mathtt{b}) + \sigma$.*

Proof. For prove (9.26) we use (9.12) and (9.25). If $\nu := \nu(\mathtt{b})+\sigma$ the condition (9.13) holds (with $p_1 = \mathfrak{p}_0$) and so the bounds (9.23) holds by (9.14) with $\nu := \nu(\mathtt{b}) + \sigma$. □

By this lemma follows that for all $\mathfrak{p}_0 \leq p \leq P$

$$\mathcal{M}_0(p, \mathtt{b}) = \max\{\mathbb{M}_0(p), \mathbb{M}_0(p, \mathtt{b})\} \leq C(P)\mu\gamma^{-1}\left(1 + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\nu(\mathtt{b})+\sigma}\right) . \tag{9.27}$$

Let

$$N_{-1} := 1, \quad N_\nu := N_0^{(\frac{3}{2})^\nu}, \quad \forall\nu \geq 1, \tag{9.28}$$

this is the scale that we will use when we shall perform the almost reducibility of $\mathbf{L}_0$. Given a set A we define $\mathcal{N}(A, \delta) \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$ as

$$\mathcal{N}(A, \delta) := \left\{\zeta \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2] : dist(A, \zeta) \leq \delta\right\} . \tag{9.29}$$

Let $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$ and $\mathbb{S}^\pm := \mathbb{S} \cup (-\mathbb{S})$ where $\mathbb{S}$ is defined in (1.31). Now we can enunciate the almost reducibility theorem, that is

Theorem 9.6. ***Almost Reducibility.*** *There exists $\tau_0 = \tau_0(\tau, N) > 0$ such that, for all $P > \mathfrak{p}_0$ there is $N_0 := N_0(P, \mathtt{b}) \in \mathbb{N}$ such that, if*

$$N_0^{\tau_0}\mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})\gamma^{-1} \leq 1 \tag{9.30}$$

then, for all $n \in \mathbb{N}, \nu = 0, ..., n$:

$(S1)_\nu$ *There exists a real, reversible operator*

$$\mathbf{L}_\nu = \Omega \cdot \partial_\theta + i\mathbf{D}_\nu + \mathbf{Q}_\nu \quad where \quad \mathbf{D}_\nu = \begin{pmatrix} \mathcal{D}_\nu & 0 \\ 0 & -\mathcal{D}_\nu \end{pmatrix} \qquad \mathcal{D}_\nu = diag_{j\in\mathbb{Z}_0\setminus\mathbb{S}^\pm}\lambda_j^\nu \tag{9.31}$$

which acts on E, defined in (7.26) for $(\omega, \varepsilon) \in \mathtt{DC}^\gamma_{K_n} \times [\varepsilon_1, \varepsilon_2]$ (where $\mathtt{DC}^\gamma_{K_n}$ is defined in (5.5)) for $\nu = 0$, and for all (ω, ε) in

$$\mathcal{N}(\Lambda^\gamma_\nu, \gamma N_{\nu-1}^{-\tau-2}) \subset \Lambda^{\gamma/2}_\nu \quad \forall\nu \geq 1 \tag{9.32}$$

where λ^ν_j are k_0-times differentiable functions of the form

$$\lambda^\nu_j(\omega, \varepsilon) = \lambda_j(\omega, \varepsilon)^0 + r^\nu_j(\omega, \varepsilon) \quad \lambda_j(\omega, \varepsilon)^0 = j\left(\frac{2}{15}\varepsilon^4 j^4 - \frac{1}{3}\varepsilon^2 j^2 + 1\right)^{1/2} + m_1 j\,, \tag{9.33}$$

satifying

$$\lambda^\nu_j = -\lambda^\nu_{-j} \quad i.e \quad r^\nu_j = -r^\nu_{-j} \quad and \quad |r^\nu_j|^{k_0,\gamma} \leq C(P)\mu\gamma^{-1} \quad \forall j \in \mathbb{Z} \setminus \mathbb{S}^\pm\,. \tag{9.34}$$

The sets Λ^γ_ν, are defined by $\Lambda^\gamma_0 := \Omega \times [\varepsilon_1, \varepsilon_2]$, and for all $\nu \geq 1$

$$\begin{aligned}\Lambda^\gamma_\nu := \Lambda^\gamma_\nu(i) := \Big\{&\zeta = (\omega, \varepsilon) \in \Lambda^\gamma_{\nu-1} \cap \left([\mathtt{DC}^\gamma_{K_n} \cap \mathtt{DC}^\gamma_{N_{\nu-1}}] \times [\varepsilon_1, \varepsilon_2]\right) : \\ &|\omega \cdot l + \lambda^{\nu-1}_j - \lambda^{\nu-1}_{j'}| \geq \gamma|j^3 - j'^3|\,\langle l\rangle^{-\tau}\,, \forall|l| \leq N_{\nu-1}, j, j' \in \mathbb{Z} \setminus \mathbb{S}^\pm\Big\}\,.\end{aligned} \tag{9.35}$$

The remainder $\mathbf{Q}_\nu$ given by

$$\mathbf{Q}_\nu := \begin{pmatrix} (Q_1^\nu) & 0 \\ 0 & (Q_4^\nu) \end{pmatrix} \tag{9.36}$$

satisfy (8.108), (8.109) *and it is $\mathcal{D}^{k_0}$-modulo-tame, more precisely the operators Q_1^ν, Q_4^ν and $\langle\partial_\theta\rangle^{\mathtt{b}}(Q_1^\nu), \langle\partial_\theta\rangle^{\mathtt{b}}(Q_4^\nu)$, are $\mathcal{D}^{k_0}$-modulo-tame with modulo-tame constants respectively*

$$\mathcal{M}_\nu^\sharp(p) := \max_{m=1,4} \mathcal{M}^\sharp_{Q_m^\nu}(p)\,, \qquad \mathcal{M}_\nu^\sharp(p,\mathtt{b}) := \max_{m=1,4} \mathcal{M}^\sharp_{\langle\partial_\theta\rangle^{\mathtt{b}} Q_m^\nu}(p,\mathtt{b})\,, \tag{9.37}$$

satisfying for all $\mathfrak{p}_0 \le p \le P$

$$\mathcal{M}_\nu^\sharp(p) \le \mathcal{M}_0(p,\mathtt{b}) N_{\nu-1}^{-\mathfrak{a}} \quad \mathcal{M}_\nu^\sharp(p,\mathtt{b}) \le \mathcal{M}_0(p,\mathtt{b}) N_{\nu-1}\,. \tag{9.38}$$

Moreover, for $\nu \ge 1$ there exists a real, reversibility preserving map (see (7.32)*), from E to E (see* (7.26)*)*

$$\mathbf{\Phi}_{\nu-1} = \mathbb{1}^\perp + \mathbf{\Psi}_{\nu-1} \quad \textit{where} \quad \mathbf{\Psi}_{\nu-1} = \begin{pmatrix} \psi_1^{\nu-1} & 0 \\ 0 & \psi_4^{\nu-1} \end{pmatrix}, \tag{9.39}$$

such that

$$\mathbf{L}_\nu := \mathbf{\Phi}_{\nu-1}^{-1} \mathbf{L}_{\nu-1} \mathbf{\Phi}_{\nu-1}\,, \tag{9.40}$$

the operators $\psi_m^{\nu-1}$ and $\langle\partial_\theta\rangle^{\mathtt{b}}\psi_m^{\nu-1}$, $m = 1,4$ are $\mathcal{D}^{k_0}$-modulo tame with modulo tame constants satisfying, for all $\mathfrak{p}_0 \le p \le P$

$$\begin{aligned} \mathcal{M}^\sharp_{\psi_m^{\nu-1}}(p) &\le \frac{C(k_0)}{\gamma} N_{\nu-1}^{\tau_1} N_{\nu-2}^{-\mathfrak{a}} \mathcal{M}_0(p,\mathtt{b}) \\ \mathcal{M}^\sharp_{\langle\partial_\theta\rangle^{\mathtt{b}}\psi_m^{\nu-1}}(p) &\le \frac{C(k_0)}{\gamma} N_{\nu-1}^{\tau_1} N_{\nu-2} \mathcal{M}_0(p,\mathtt{b}) \end{aligned} \tag{9.41}$$

where $\tau_1 := \tau(k_0+1) + k_0$, $\mathfrak{a} := 3\tau_1$ (see (9.21)*).*

$(S2)_\nu$ *For all $j \in \mathbb{Z} \setminus \mathbb{S}^\pm$ there exists a k_0-times differentiable extension $\tilde\lambda_j^\nu : \Omega \times [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ such that $\tilde\lambda_j^\nu = \lambda_j^\nu$ on Λ_ν^γ and*

$$\tilde\lambda_j^\nu(\omega,\varepsilon) = \lambda_j^0(\omega,\varepsilon) + \tilde r_j^\nu(\omega,\varepsilon) \in \mathbb{R}, \quad \tilde r_j^\nu = -\tilde r_{-j}^\nu, \quad |\tilde r_j^\nu|^{k_0,\gamma} \le_P \mu\gamma^{-1} N_0^{k_0(\tau+2)}, \quad \forall j \in \mathbb{Z}\setminus\mathbb{S}^\pm \tag{9.42}$$

and for all $\nu \ge 1$

$$|\tilde\lambda_j^\nu - \tilde\lambda_j^{\nu-1}|^{k_0,\gamma} \le C(k_0) N_{\nu-1}^{k_0(\tau+2)} \mathcal{M}^\sharp_{\nu-1}(\mathfrak{p}_0) \le C(k_0,P)\gamma^{-1}\mu N_{\nu-1}^{k_0(2+\tau)} N_{\nu-2}^{-\mathfrak{a}}\,. \tag{9.43}$$

$(S3)_\nu$ *Let $i_1(\omega,\varepsilon), i_2(\omega,\varepsilon)$ such that $\mathbf{Q}_0(i_1)$ and $\mathbf{Q}_0(i_2)$ satisfy* (9.22)*. Assume that also* (9.23) *holds. Then for all $\nu = 0,\ldots n$ and for all $(\omega,\varepsilon) \in \Lambda_\nu^{\gamma_1}(i_1) \cup \Lambda_\nu^{\gamma_2}(i_2)$ with $\gamma_1,\gamma_2 \in [\gamma/2, 2\gamma]$ there exists $\sigma := \sigma(\tau,\nu,k_0) > 0$ such that, for $m = 1,4$*

$$\begin{aligned} |||Q_m^\nu(i_1) - Q_m^\nu(i_2)|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\le_{P,\mathtt{b}} \gamma^{-1}\mu N_{\nu-1}^{-\mathfrak{a}} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} \\ |||\langle\partial_\theta\rangle^{\mathtt{b}}(Q_m^\nu(i_1) - Q_m^\nu(i_2))|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\le_{P,\mathtt{b}} \mu\gamma^{-1} N_{\nu-1} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\,. \end{aligned} \tag{9.44}$$

Moreover, for all $\nu = 1, ..., n$ and for all $j \in \mathbb{Z} \setminus \mathbb{S}^{\pm}$

$$\begin{aligned} |(r_j^\nu(i_1) - r_j^\nu(i_2)) - (r_j^{\nu-1}(i_1) - r_j^{\nu-1}(i_2))| &\le C|||\mathbf{Q}_\nu(i_1) - \mathbf{Q}_\nu(i_2)|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \\ |r_j^\nu(i_1) - r_j^\nu(i_2)| &\le_P \gamma^{-1}\mu\|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} \end{aligned} \tag{9.45}$$

where $\nu(\mathtt{b})$ is defined in (9.25), and we recall that $\|\mathbf{Q}^\nu h\|_p := \max_{m=1,4}\|Q_m^\nu h\|_p$.

$(S4)_\nu$ *Let i_1, i_2 be like in $(S3)_\nu$ and $0 < \rho < \gamma/2$. Then*

$$\mu\gamma^{-1}C(P)N_{n-1}^{\tau}\|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} \le \rho \quad \Rightarrow \quad \Lambda_\nu^\gamma(i_1) \subseteq \Lambda_\nu^{\gamma-\rho}(i_2)\,. \tag{9.46}$$

Remark 9.7. *Note that* (9.45) *are sufficient to prove $(S4)_\nu$ about the inclusion of the Cantor sets $\Lambda_\nu^\gamma(i_1)$, $\Lambda_\nu^{\gamma-\rho}(i_2)$ corresponding to two nearby approximate solutions. These bounds follow by* (9.44)*, which is in terms of the Sobolev index $\mathfrak{p}_0$ and not in terms of the derivatives with respect to (ω, ε).*

Remark 9.8. *In order to prove* (9.37) *for $\nu = 0$ we shall use $(|l_1| + ... + |l_N|)^\beta \le C_\beta(|l_1|^\beta + ... + |l_N|^\beta)$ for this reason in Section 8.2.4 we have studied $\partial_{\theta_r}^\beta$, $r = 1, ..., N$ instead of $\langle\partial_\theta\rangle^\beta$.*

Remark 9.9. *Note that we have to look for $\mathcal{D}^{k_0}-$modulo-tame operators (see* (9.37)*) because the second estimate in Lemma 2.20 does not hold for $\mathcal{D}^{k_0}-$tame operators.*

It is important to note that in Theorem 9.6 we require only the bound (9.30) for $\mathcal{M}_0^\sharp(\mathfrak{p}_0, \mathtt{b})$ in low norm. But it is also proved that both $\mathcal{M}_\nu^\sharp(p)$ and $\mathcal{M}_\nu^\sharp(p, \mathtt{b})$ for all $\nu \ge 0$ do not diverge too much (see (9.38)).

In addition Theorem 9.6 implies that there exist a transformation $\mathbf{U}_n$ such that the conjugation of $\mathbf{L}_0$ with $\mathbf{U}_n$ is a diagonal operator (up to a small remainder) as we shall prove in the Theorem below.

Theorem 9.10. ***KAM almost-reducibility**. Assume* (6.11) *with $\nu \ge \nu(\mathtt{b})$. Let τ_0 as in Theorem 9.6, For all $P > \mathfrak{p}_0$ there exists $N_0 = N_0(P, \mathtt{b}) > 0, \delta_0 = \delta_0(P) > 0$ such that, if the smallness condition:*

$$N_0^{\tau_0}\mu\gamma^{-2} \le \delta_0 \tag{9.47}$$

holds, then for all $n \in \mathbb{N}$, for all $\zeta = (\omega, \varepsilon)$ in

$$\Lambda_{n+1}^\gamma := \Lambda_{n+1}^\gamma(i) = \bigcap_{\nu=0}^{n+1} \Lambda_\nu^\gamma \tag{9.48}$$

where Λ_{n+1}^γ is defined in (9.35)*, the operator*

$$\mathbf{U}_n := \mathbf{\Phi}_0 \circ ... \circ \mathbf{\Phi}_n \tag{9.49}$$

is well defined and

$$\mathbf{L}_n := \mathbf{U}_n^{-1}\mathbf{L}_0\mathbf{U}_n = \Omega \cdot \partial_\theta \mathbb{1}^\perp + i\mathbf{D}_n + \mathbf{Q}_n \tag{9.50}$$

where $\mathbf{D}_n$ is defined in (9.31) and $\mathbf{Q}_n$ in (9.36) (with $\nu = n$). The operators Q_1^n, Q_4^n are $\mathcal{D}^{k_0}-$ modulo-tame with modulo-tame constants

$$\mathcal{M}^\sharp_{Q^n_m}(p) \leq_P \mu\gamma^{-1} N_{n-1}^{-\mathfrak{a}} (1 + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\nu(\mathtt{b})+\sigma}) \quad \forall \mathfrak{p}_0 \leq p \leq P, \quad m = 1,4 \,. \tag{9.51}$$

Moreover the operators $\mathbf{U}_n^\pm - \mathbb{1}^\perp$ are $\mathcal{D}^{k_0}-$ modulo-tame with modulo-tame constants

$$\mathcal{M}^\sharp_{\mathbf{U}_n^\pm - \mathbb{1}^\perp}(p) \leq_P \mu\gamma^{-2} N_0^{\tau_1} (1 + \|\mathcal{V}_0\|^{k_0,\gamma}_{p+\nu(\mathtt{b})+\sigma}) \quad \forall \mathfrak{p}_0 \leq p \leq P \tag{9.52}$$

where $\tau_1 := \tau(k_0+1) + k_0$. In addition the operators $\mathbf{U}_n$, $\mathbf{U}_n^{-1} : E \to E$ are real and reversibility preserving (see (7.32)). The operator $\mathbf{L}_n : E \to E$ is real and reversible (see (7.31)).

Proof. We consider

$$\mathbf{U}_{n+1} = \mathbf{U}_n \circ \mathbf{\Phi}_{n+1} = \mathbf{U}_n \circ \left(\mathbb{1}^\perp + \mathbf{\Psi}_{n+1}\right) \,.$$

Hence, by (9.41)

$$\begin{aligned}\mathcal{M}^\sharp_{\mathbf{U}_{n+1}}(p) &\leq \mathcal{M}^\sharp_{\mathbf{U}_n}(p)\left(1 + \mathcal{M}^\sharp_{\mathbf{\Phi}_{n+1}}(\mathfrak{p}_0)\right) + \mathcal{M}^\sharp_{\mathbf{U}_n}(\mathfrak{p}_0)\left(1 + \mathcal{M}^\sharp_{\mathbf{\Phi}_{n+1}}(p)\right) \\ &\leq_{k_0} \mathcal{M}^\sharp_{\mathbf{U}_n}(p)\left(1 + \gamma^{-1} N_{n+1}^{\tau_1} N_n^{-\mathfrak{a}} \mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})\right) + \mathcal{M}^\sharp_{\mathbf{U}_n}(\mathfrak{p}_0)\left(1 + \gamma^{-1} N_{n+1}^{\tau_1} N_n^{-\mathfrak{a}} \mathcal{M}_0(p, \mathtt{b})\right)\end{aligned} \tag{9.53}$$

and by (9.41), (9.22) and (9.47) we have

$$\begin{aligned}\mathcal{M}^\sharp_{\mathbf{U}_{n+1}}(\mathfrak{p}_0) &\leq_{k_0} \mathcal{M}^\sharp_{\mathbf{U}_n}(\mathfrak{p}_0)\left(1 + \gamma^{-1} N_{n+1}^{\tau_1} N_n^{-\mathfrak{a}} \mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})\right) \\ &\leq_{k_0} \mathcal{M}^\sharp_{\mathbf{U}_{n-1}}(\mathfrak{p}_0)\left(1 + \gamma^{-1} N_n^{\tau_1} N_{n-1}^{-\mathfrak{a}} \mathcal{M}_0(p, \mathtt{b})\right)\left(1 + \gamma^{-1} N_{n+1}^{\tau_1} N_n^{-\mathfrak{a}} \mathcal{M}_0(p, \mathtt{b})\right) \\ &\leq_{k_0} \mathcal{M}^\sharp_{\mathbf{U}_0}(\mathfrak{p}_0) \Pi_{\nu=0}^n \left(1 + \gamma^{-1} N_{\nu+1}^{\tau_1} N_\nu^{-\mathfrak{a}} \mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})\right) \\ &\leq_{k_0} \mathcal{M}^\sharp_{\mathbf{U}_0}(\mathfrak{p}_0) \Pi_{\nu=0}^n \left(1 + \alpha_n(\mathfrak{p}_0)\right)\end{aligned} \tag{9.54}$$

where $\alpha_n(\mathfrak{p}_0) = \gamma^{-1} N_{\nu+1}^{\tau_1} N_\nu^{-\mathfrak{a}} \mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})$. By $\Pi_{\nu \geq 0}(1 + \alpha_\nu) \leq \exp\left(C(P)\gamma^{-1}\mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})\right) \leq 2$, (9.41), (9.22) and (9.47) we have

$$\begin{aligned}\mathcal{M}^\sharp_{\mathbf{U}_{n+1}}(\mathfrak{p}_0) &\leq \mathcal{M}^\sharp_{\mathbf{U}_0}(\mathfrak{p}_0) \Pi_{\nu \geq 0}\left(1 + \alpha_\nu\right) \\ &\leq \mathcal{M}^\sharp_{\mathbf{U}_0}(\mathfrak{p}_0) \exp\left(C(p)\gamma^{-1}\mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})\right) \\ &\leq \mathcal{M}^\sharp_{\mathbf{U}_0}(\mathfrak{p}_0) \exp\left(C(p)\gamma^{-2}\mu\right) \\ &\leq 2.\end{aligned} \tag{9.55}$$

Iterating (9.53), using (9.55) and $\Pi_\nu\left(1 + \alpha_\nu(\mathfrak{p}_0)\right) \leq 2$ we get

$$\begin{aligned}\mathcal{M}^\sharp_{\mathbf{U}_{n+1}}(p) &\leq_{k_0} \mathcal{M}^\sharp_{\mathbf{U}_n}(p)\left(1 + \gamma^{-1} N_{n+1}^{\tau_1} N_n^{-\mathfrak{a}} \mathcal{M}_0(\mathfrak{p}_0, \mathtt{b})\right) + \mathcal{M}^\sharp_{\mathbf{U}_n}(\mathfrak{p}_0)\left(1 + \gamma^{-1} N_{n+1}^{\tau_1} N_n^{-\mathfrak{a}} \mathcal{M}_0(p, \mathtt{b})\right) \\ &\leq_{k_0} \sum_{\nu \geq 0} \alpha_\nu(p) + \mathcal{M}^\sharp_{\mathbf{U}_0}(p) \\ &\leq C(k_0)\left(1 + N_0^{\tau_1} \mathcal{M}_0(p, \mathtt{b})\gamma^{-1}\right)\end{aligned} \tag{9.56}$$

since $\mathbf{U}_0 = \mathbf{\Phi}_0 = \mathbb{1}^\perp + \mathbf{\Psi}_0$ and $\mathcal{M}_{\mathbf{U}_0}(p) \leq 1 + C(k_0) N_0^{\tau_1} \mathcal{M}_0(p, \mathtt{b})\gamma^{-1}$ by (9.41). Finally

$$\mathbf{U}_n - \mathbb{1}^\perp = (\mathbf{U}_n - \mathbf{\Phi}_0) + \left(\mathbf{\Phi}_0 - \mathbb{1}^\perp\right) = \sum_{\nu=0}^{n-1}\left(\mathbf{U}_{\nu+1} - \mathbf{U}_\nu\right) + \mathbf{\Psi}_0 = \sum_{\nu=0}^{n-1} \mathbf{U}_\nu \mathbf{\Psi}_{\nu+1} + \mathbf{\Psi}_0 \,.$$

Hence (9.52) for $\mathbf{U}_n - \mathbb{1}^\perp$ follows by Lemma 2.18, (9.55), (9.56), (9.47), (9.27), (6.11). The estimate for $\mathbf{U}_n^{-1} - \mathbb{1}^\perp$ follows by Lemma 2.19. □

9.2 Initialization

Proof of $(S1)_0$. For $\nu = 0$ we have that that (9.16), (9.17) and (9.18), are satisfied and imply (9.31), (9.32), (9.33) (9.34) and (9.35) with $r_j^0(\omega, \varepsilon) = 0$. Now we have to prove that (9.38) for $\nu = 0$ holds. Therefore we have to prove the following lemma:

Lemma 9.11. *Proof of (9.38) when $\nu = 0$, i.e.*

$$\mathcal{M}_0^\sharp(p) \leq_{\mathfrak{p}_0, \mathtt{b}} \mathcal{M}_0(p, \mathtt{b}), \quad \mathcal{M}_0^\sharp(p, \mathtt{b}) \leq_{\mathfrak{p}_0, \mathtt{b}} \mathcal{M}_0(p, \mathtt{b}) .$$

Proof. In what follows we shall write Q instead of Q_1 and Q_4.

The matrix element of the commutator $[Q, \partial_x]$ are $i(j'-j)Q_j^{j'}(l-l')$, of $\partial_{\theta_r}^b Q$ are $i^b(l_r - l_r')^b Q_j^{j'}(l-l')$ and of $\partial_{\theta_r}^b [Q, \partial_x]$ are $i^{b+1}(l_r - l_r')^b (j'-j) Q_j^{j'}(l-l')$. Hence

$$\begin{aligned}
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_C \mathbb{M}_0^2(\mathfrak{p}_0) \langle l', j'\rangle^{2p} + \mathbb{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} \\
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} \langle j'-j\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_C \mathbb{M}_0^2(\mathfrak{p}_0) \langle l', j'\rangle^{2p} + \mathbb{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} \\
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} \langle l-l'\rangle^{2\mathfrak{p}_0} |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_C \mathbb{M}_0^2(\mathfrak{p}_0) \langle l', j'\rangle^{2p} + \mathbb{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} \\
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} \langle l-l'\rangle^{2\mathfrak{p}_0} \langle j'-j\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_C \mathbb{M}_0^2(\mathfrak{p}_0) \langle l', p\rangle^{2p} + \mathbb{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} \\
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} \langle l-l'\rangle^{2(\mathfrak{p}_0+\mathtt{b})} |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_C \mathbb{M}_0^2(\mathfrak{p}_0) \langle l', j'\rangle^{2p} + \mathbb{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} \\
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} \langle l-l'\rangle^{2(\mathfrak{p}_0+\mathtt{b})} \langle j'-j\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_C \mathbb{M}_0^2(\mathfrak{p}_0) \langle l', j'\rangle^{2p} + \mathbb{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} .
\end{aligned}$$

Using the inequality

$$\langle l-l'\rangle^{2a} \langle j'-j\rangle^2 \leq 1 + |j'-j|^2 + \max_{r=1,\dots,N} |l_r - l_r'|^{2a} + |j-j'|^2 \max_{r=1,\dots,N} |l_r - l_r'|^{2a} \tag{9.57}$$

we obtain, for $p_1 = \mathfrak{p}_0,\ p = \mathfrak{p}_0 + \mathtt{b}$, by (9.57) and (9.22),

$$\begin{aligned}
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} \langle l-l'\rangle^{2\mathfrak{p}_0} \langle j'-j\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_{\mathtt{b}} \mathcal{M}_0^2(\mathfrak{p}_0) \langle l', j'\rangle^{2p} + \mathcal{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} \\
&\gamma^{2|k|} \sum_{j,l} \langle l, j\rangle^{2p} \langle l-l'\rangle^{2(\mathfrak{p}_0+\mathtt{b})} | \langle j'-j\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \leq_{\mathtt{b}} \mathcal{M}_0^2(\mathfrak{p}_0) \langle l', j'\rangle^{2p} + \mathcal{M}_0^2(p) \langle l', j'\rangle^{2\mathfrak{p}_0} .
\end{aligned}$$

Let us prove that if $Q, [\partial_x, Q] : H^p \to H^p$ then $|Q| : H^p \to H^p$ is a $\mathcal{D}^{k_0}-$tame, by (9.57) and the

Cauchy-Schwartz inequality, for all $|k| \le k_0$ we get

$$\begin{aligned}
|||\partial_\zeta^k Q|h\|_p^2 &\le \sum_{l,j} \langle l,j\rangle^{2p} \Big(\sum_{l',j'} |\partial_\zeta^k Q_j^{j'}(l-l')||h_{l',j'}| \Big)^2 \\
&\le \sum_{l,j} \langle l,j\rangle^{2p} \Big(\sum_{l',j'} \langle j-j'\rangle |\partial_\zeta^k Q_j^{j'}(l-l')||h_{l',j'}| \frac{1}{\langle j-j'\rangle} \Big)^2 \\
&\le C \sum_{l,j} \langle l,j\rangle^{2p} \sum_{l',j'} \langle j-j'\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 |h_{l',j'}|^2 \\
&\le C \sum_{l',j'} |h_{l',j'}|^2 \sum_{l,j} \langle l,j\rangle^{2p} \langle j-j'\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \\
&\le C\gamma^{-2|k|} \sum_{l',j'} |h_{l',j'}|^2 \Big(\mathcal{M}_0^2(p) \langle l',j'\rangle^{2\mathfrak{p}_0} + \mathcal{M}_0^2(\mathfrak{p}_0) \langle l',j'\rangle^{2p} \Big) \\
&\le C\gamma^{-2|k|} \left(\mathcal{M}_0^2(p) \|h\|_{\mathfrak{p}_0}^2 + \mathcal{M}_0^2(\mathfrak{p}_0) \|h\|_p^2 \right).
\end{aligned}$$

We now prove that $\forall |k| \le k_0$ also $| \langle \partial_\theta \rangle^{\mathtt{b}} Q| : H^p \to H^p$ is $\mathcal{D}^{k_0}$-tame. By (9.57) and the Cauchy-Schwartz inequality we have

$$\begin{aligned}
||| \langle \partial_\theta \rangle^{\mathtt{b}} \partial_\zeta^k Q|h\|_p^2 &\le \sum_{l,j} \langle l,j\rangle^{2p} \Big(\sum_{l',j'} | \langle l-l'\rangle^{\mathtt{b}} \partial_\zeta^k Q_j^{j'}(l-l')||h_{l',j'}| \Big)^2 \\
&= \sum_{l,j} \langle l,j\rangle^{2p} \Big(\sum_{l',j'} | \langle l-l'\rangle^{\mathtt{b}+\mathfrak{p}_0} \langle j'-j\rangle |\partial_\zeta^k Q_j^{j'}(l-l')||h_{l',j'}| \frac{1}{\langle l-l'\rangle^{\mathfrak{p}_0} \langle j'-j\rangle} \Big)^2 \\
&\le C \sum_{l,j} \langle l,j\rangle^{2p} \sum_{l',j'} | \langle l-l'\rangle^{2(\mathtt{b}+\mathfrak{p}_0)} \langle j'-j\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 |h_{l',j'}|^2 \\
&= C \sum_{l',j'} |h_{l',j'}|^2 \sum_{l,j} | \langle l,j\rangle^{2p} \langle l-l'\rangle^{2(\mathtt{b}+\mathfrak{p}_0)} \langle j'-j\rangle^2 |\partial_\zeta^k Q_j^{j'}(l-l')|^2 \\
&\le C\gamma^{-2|k|} \sum_{l',j'} |h_{l',j'}|^2 \Big(\mathcal{M}_0^2(\mathfrak{p}_0,\mathtt{b}) \langle l',j'\rangle^{2p} + \mathcal{M}_0^2(p,\mathtt{b}) \langle l',j'\rangle^{2\mathfrak{p}_0} \Big) \\
&\le C\gamma^{-2|k|} \left(\mathcal{M}_0^2(\mathfrak{p}_0,\mathtt{b}) \|h\|_p^2 + \mathcal{M}_0^2(p,\mathtt{b}) \|h\|_{\mathfrak{p}_0}^2 \right) . \qquad (9.58)
\end{aligned}$$

Therefore the Lemma is proved. □

Proof of $(S2)_0$. The function $m_1(\omega,\varepsilon)$ is k_0-times differentiable on $\Omega \times [\varepsilon_1,\varepsilon_2]$ because it depends on the torus $i_\delta(\omega,\varepsilon)$ that is k_0-times differentiable with respect to (ω,ε) on all $\Omega \times [\varepsilon_1,\varepsilon_2]$.

Proof of $(S3)_0$. This condition follows by the Lemma below.

Lemma 9.12.

$$|||\Delta_{12} Q_m|h\|_{\mathfrak{p}_0}^2 \le C(P)\gamma^{-2}\mu^2 \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}^2 \|h\|_{\mathfrak{p}_0}^2 , \quad m = 1,4 \qquad (9.59)$$

$$||| \langle \partial_\theta \rangle^{\mathtt{b}} \Delta_{12} Q_m|h\|_{\mathfrak{p}_0}^2 \le C(P,\mathtt{b})\gamma^{-2}\mu^2 \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}^2 \|h\|_{\mathfrak{p}_0}^2 , \quad m = 1,4 \qquad (9.60)$$

where $\Delta_{12} Q_m := Q_m(i_1) - Q_m(i_2)$.

Proof. We prove (9.60). Also in this case we shall write Q instead of Q_1 and Q_4. By the mean value teorem and the estimate (9.23) we have

$$\|\Delta_{12}Q\|_{\mathcal{L}(H^{\mathfrak{p}_0})}, \|[\Delta_{12}Q, \partial_x]\|_{\mathcal{L}(H^{\mathfrak{p}_0})}, \|\partial_{\theta_r}^{\mathfrak{p}_0+\mathtt{b}}\Delta_{12}Q\|_{\mathcal{L}(H^{\mathfrak{p}_0})},$$
$$\|\partial_{\theta_r}^{\mathfrak{p}_0+\mathtt{b}}[\Delta_{12}Q, \partial_x]\|_{\mathcal{L}(H^{\mathfrak{p}_0})} \le C(p,\mathtt{b})\gamma^{-1}\mu\|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}, \ \ \forall r = 1, ..., N\,.$$

Hence, for all $l' \in \mathbb{Z}^N$, $j' \in \mathbb{Z}\setminus\mathbb{S}^{\pm}$ we have

$$\sum_{l,j} \langle l,j\rangle^{2\mathfrak{p}_0}\langle j-j'\rangle^2\langle l-l'\rangle^{2(\mathfrak{p}_0+\mathtt{b})}|(\Delta_{12}Q)_j^{j'}(l-l')|^2 \le C(P,\mathtt{b})\mu^2\gamma^{-2}\|i_1-i_2\|^2_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\langle l',j'\rangle^{2\mathfrak{p}_0}$$

which arguing as in (9.58), proves (9.60). The proof of (9.59) is similar. □

Proof of $(S4)_0$. It follows by definition, indeed $\Omega = \Omega_0^{\gamma}(i_1)$ and $\Omega = \Omega_0^{\gamma-\rho}(i_2)$.

9.3 Reducibility step

The goal of this section is to describe the generic inductive step. We show how to define, $\mathbf{L}_{\nu+1}$, $\Psi_{\nu+1}$ and $\Phi_{\nu+1}$ from $\mathbf{L}_\nu$. We conjugate $\mathbf{L}_\nu = \Omega\cdot\partial_\theta + \mathbf{D}_\nu + \mathbf{Q}_\nu$ by a transformation close to the identity, of the form

$$\mathbf{\Phi}_\nu = \mathbb{1}^{\perp} + \mathbf{\Psi}_\nu, \quad \mathbf{\Psi}_\nu = \begin{pmatrix} \psi_1^{(\nu)} & 0 \\ 0 & \psi_4^{(\nu)} \end{pmatrix}, \tag{9.61}$$

see (9.39), where $(\psi_1^{(\nu)})_j^{j'}(l) = (\psi_4^{(\nu)})_{-j}^{-j'}(l)$, $(\psi_1^{(\nu)})_j^{j'}(l) = \overline{(\psi_1^{(\nu)})_{-j}^{-j'}(-l)}$, $\forall j, j' \in \mathbb{Z}$ and $l \in \mathbb{Z}^N$ (see (8.104)).

We have

$$\mathbf{L}_\nu\mathbf{\Phi}_\nu - \mathbf{\Phi}_\nu(\Omega\cdot\partial_\theta + \mathbf{D}_\nu + [\mathbf{Q}_\nu]) = \omega\cdot\partial_\theta\mathbf{\Psi}_\nu + [\mathbf{D}_\nu, \mathbf{\Psi}_\nu] + \mathbf{Q}_\nu + \mathbf{Q}_\nu\mathbf{\Psi}_\nu - \mathbf{\Psi}_\nu[\mathbf{Q}_\nu] - [\mathbf{Q}_\nu]. \tag{9.62}$$

We want to solve the homological equation

$$\omega\cdot\partial_\theta\mathbf{\Psi}_\nu + [\mathbf{D}_\nu, \mathbf{\Psi}_\nu] + \Pi_N\mathbf{Q}_\nu - [\mathbf{Q}_\nu] = 0, \tag{9.63}$$

where, $\forall j \in \mathbb{Z}$

$$\begin{aligned}[\mathbf{Q}_\nu] &:= \begin{pmatrix} [Q_1^{(\nu)}] & 0 \\ 0 & [Q_4^{(\nu)}] \end{pmatrix} = \\ &= \begin{pmatrix} \mathrm{diag}_j(Q_1^{(\nu)})_j^j(0) & 0 \\ 0 & \mathrm{diag}_j(Q_4^{(\nu)})_j^j(0) \end{pmatrix}\end{aligned} \tag{9.64}$$

(see Lemma 8.16) The equation (9.63) is equivalent to the two scalar homological equations

$$\omega\cdot\partial_\theta\psi_1^{(\nu)} + i[\mathcal{D}^{(\nu)}, \psi_1^{(\nu)}] + \Pi_N Q_1^{(\nu)} = [Q_1^{(\nu)}],$$
$$\omega\cdot\partial_\theta\psi_4^{(\nu)} - i[\mathcal{D}^{(\nu)}, \psi_4^{(\nu)}] + \Pi_N Q_4^{(\nu)} = [Q_4^{(\nu)}]\,.$$

The solutions of this equations are

$$(\psi_m^{(\nu)})_j^{j'}(l) := \begin{cases} -\frac{(Q_m^{(\nu)})_j^{j'}(l)}{i(\omega\cdot l+\sigma_m(\lambda_j-\lambda_{j'}))} & \forall (l,j,j')\neq(0,j,j), \quad |l|\leq N, j',j\in\mathbb{Z}_0\setminus\mathbb{S}^\pm \\ 0 & \text{otherwise} \end{cases} \tag{9.65}$$

where $m=1,4$, $\sigma_1:=1$ and $\sigma_4:=-1$; $\mathbb{Z}_0:=\mathbb{Z}\setminus\{0\}$ and $\mathbb{S}^\pm:=\mathbb{S}\cup(-\mathbb{S})$ with $\mathbb{S}$ defined in (1.31). Note that if $(\omega,\varepsilon)\in\Lambda_{\nu+1}^\gamma$, using (9.34) we have that $\omega\cdot l+\lambda_j-\lambda_{j'}$ is different from zero, therefore the maps ψ_m, $m=1,4$ are well defined.

Hence, by (9.62) we have

$$\begin{aligned} \mathbf{L}_{\nu+1} &:= \mathbf{\Phi}_\nu^{-1}\mathbf{L}_\nu\mathbf{\Phi}_\nu \\ &= \Omega\cdot\partial_\theta+\mathbf{D}_\nu+[\mathbf{Q}_\nu]+\mathbf{\Phi}_\nu^{-1}\left(\Pi_N^\perp\mathbf{Q}_\nu+\mathbf{Q}_\nu\mathbf{\Psi}_\nu-\mathbf{\Psi}_\nu[\mathbf{Q}_\nu]\right) \\ &= \Omega\cdot\partial_\theta+\mathbf{D}_{\nu+1}+\mathbf{Q}_{\nu+1}, \end{aligned} \tag{9.66}$$

where $\mathbf{D}_{\nu+1}=\mathbf{D}_\nu+[\mathbf{Q}_\nu]$ and $\mathbf{Q}_{\nu+1}=\mathbf{\Phi}_\nu^{-1}\left(\Pi_N^\perp\mathbf{Q}_\nu+\mathbf{Q}_\nu\Psi_\nu-\mathbf{\Psi}_\nu[\mathbf{Q}_\nu]\right)$.

To simplify the notation we drop the index ν.

Lemma 9.13. ***Homological equation.*** *For all $(\omega,\varepsilon)\in\Lambda_{\nu+1}^{\gamma/2}$ there exists a unique solution $\mathbf{\Psi}=\begin{pmatrix}\psi_1 & 0\\ 0 & \psi_4\end{pmatrix}$ of the homological equation (9.63). The map ψ_m, with $m=1,4$ satisfies*

$$\begin{aligned} \mathcal{M}^\sharp_{\psi_m}(p) &\leq_{k_0} N^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_Q(p) \\ \mathcal{M}^\sharp_{\langle\partial_\theta\rangle^{\mathtt{b}}\psi_m}(p) &\leq_{k_0} N^{\tau_1}\gamma^{-1-|k|}\mathcal{M}^\sharp_Q(p,\mathtt{b}), \end{aligned} \tag{9.67}$$

where τ_1 is defined in (9.21). Given i_1,i_2, denote

$$\Delta_{12}\psi_m=\psi_m(i_2)-\psi_m(i_1), \quad m=1,4.$$

If $\frac{\gamma}{2}\leq\gamma_1$, $\gamma_2\leq 2\gamma$, then, for all $(\omega,\varepsilon)\in\Lambda_{\nu+1}^{\gamma_1}(i_1)\cap\Lambda_{\nu+1}^{\gamma_2}(i_2)$, $m=1,4$,

$$\begin{aligned} |||\Delta_{12}\psi_m|||_{\mathcal{L}(H^{\mathtt{p}_0})} &\leq CN^{2\tau}\gamma^{-1}\left(|||Q_m(i_2)|||_{\mathcal{L}(H^{\mathtt{p}_0})}\|i_1-i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma}+|||\Delta_{12}Q_m|||_{\mathcal{L}(H^{\mathtt{p}_0})}\right) \\ |||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}\psi_m|||_{\mathcal{L}(H^{\mathtt{p}_0})} &\leq CN^{2\tau}\gamma^{-1}\Big(|||\langle\partial_\theta\rangle^{\mathtt{b}}Q_m(i_2)|||_{\mathcal{L}(H^{\mathtt{p}_0})}\|i_1-i_2\|_{2\mathtt{p}_+\nu(\mathtt{b})+\sigma} \\ &\qquad+|||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}Q_m|||_{\mathcal{L}(H^{\mathtt{p}_0})}\Big). \end{aligned} \tag{9.68}$$

Moreover $\mathbf{\Psi}:E\to E$ is real and reversibility preserving (see (7.32)*).*

Proof. In what follows we shall omit $m=1,4$. Let $(\omega,\varepsilon)\in\Lambda_{\nu+1}^{\gamma/2}$ defined in (9.35) with $\nu+1$ instead of ν. The inequalities (9.67) follows from the definition of ψ in (9.65), indeed for all $(l,j,j')\in\mathbb{Z}^N\times(\mathbb{Z}\setminus\mathbb{S}^\pm)\times(\mathbb{Z}\setminus\mathbb{S}^\pm)$ with $|l|\leq N$, $(l,j,j')\neq(0,j,j)$ we have

$$|\psi_j^{j'}(l)|\leq CN^\tau\gamma^{-1}|Q_j^{j'}(l)|.$$

Moreover differentiating (9.65) with respect to $\zeta = (\omega, \varepsilon)$, we get

$$\partial_\zeta^k \psi_j^{j'}(l) = \sum_{k_1+k_2=k} C(k_1,k_2) \left[\partial_\zeta^{k_1} \frac{1}{(\omega \cdot l + \lambda_j - \lambda_{j'})} \right] \partial_\zeta^{k_2} Q_j^{j'}(l)$$

and, by (9.33), (9.34), (9.35),

$$\sup_{|k_1| \le k_0} |\partial_\zeta^{k_1} \frac{1}{(\omega \cdot l + \lambda_j - \lambda_{j'})}| \le C(k_0) \langle l \rangle^{\tau(k_0+1)+k_0} \gamma^{-1-|k_1|}$$

hence, for all $0 < |k| \le k_0$

$$|\partial_\zeta^k \psi_j^{j'}(l)| \le C \langle l \rangle^{\tau(k_0+1)+k_0} \gamma^{-1-|k|} \sum_{|k_1| \le |k|} |\partial_\zeta^{k_2} Q_j^{j'}(l)|. \tag{9.69}$$

We have that for all $0 \le |k| \le k_0$, using (9.69), (9.37), (2.37), $\|h\|_p = |||h|||_p$ and (9.25)

$$\begin{aligned}
||| \langle \partial_\theta \rangle^{\mathtt{b}} \partial_\zeta^k \psi |h|||_p^2 &\le \sum_{j,l} \langle l,j \rangle^{2p} \left(\sum_{|l-l'| \le N, j'} | \langle l-l' \rangle^{\mathtt{b}} \partial_\zeta^k \psi_j^{j'}(l-l') || h_{l',j'}| \right)^2 \\
&\le C(k_0) N^{2\tau_1} \gamma^{-2(1+|k|)} \sum_{|k_2| \le |k|} \gamma^{2|k_2|} \sum_{l,j} \langle l,j \rangle^{2p} \left(\sum_{l',j'} | \langle l-l' \rangle^{\mathtt{b}} \partial_\zeta^{k_2} Q_j^{j'}(l-l') || h_{l',j'}| \right)^2 \\
&= C(k_0) N^{2\tau_1} \gamma^{-2(1+|k|)} \sum_{|k_2| \le |k|} \gamma^{2|k_2|} ||| \langle \partial_\theta \rangle^{\mathtt{b}} \partial_\zeta^{k_2} Q || h_{l',j'} |||_p^2 \\
&\le C(k_0) N^{2\tau_1} \gamma^{-2(1+|k|)} \left(\mathcal{M}^\sharp(p,\mathtt{b})^2 |||h|||_{\mathfrak{p}_0}^2 + \mathcal{M}^\sharp(\mathfrak{p}_0,\mathtt{b})^2 |||h|||_p^2 \right) \\
&\le C(k_0) N^{2\tau_1} \gamma^{-2(1+|k|)} \left(\mathcal{M}^\sharp(p,\mathtt{b})^2 \|h\|_{\mathfrak{p}_0}^2 + \mathcal{M}^\sharp(\mathfrak{p}_0,\mathtt{b})^2 \|h\|_p^2 \right) .
\end{aligned} \tag{9.70}$$

Similary one gets

$$\begin{aligned}
\|\partial_\zeta^k \psi h\|_p^2 &\le \sum_{l,j} \langle l,j \rangle^{2p} \left(\sum_{l',j'} |\partial_\zeta^k (\psi_j^{j'}(l-l') h_{l',j'}| \right)^2 \\
&\le_{k_0} N^{2\tau_1} \gamma^{-2(1+k_0)} \left(\mathcal{M}^\sharp(p)^2 \|h\|_{\mathfrak{p}_0}^2 + \mathcal{M}^\sharp(\mathfrak{p}_0)^2 \|h\|_p^2 \right) .
\end{aligned}$$

We now prove (9.68), by (9.65), for all $(\omega,\varepsilon) \in \Lambda_{\nu+1}^{\gamma_1}(i_1) \cap \Lambda_{\nu+1}^{\gamma_2}(i_2)$ we have

$$\Delta_{12} \psi_j^{j'}(l) := \psi_j^{j'}(l)(i_1) - \psi_j^{j'}(l)(i_2) = \frac{Q_j^{j'}(l)(i_1) - Q_j^{j'}(l)(i_2)}{\delta_{ljj'}(\lambda_1)} - Q_j^{j'}(l)(i_2) \frac{\delta_{ljj'}(i_1) - \delta_{ljj'}(i_2)}{\delta_{ljj'}(i_1)\delta_{ljj'}(i_2)}$$

where $\delta_{ljj'} = i(\omega \cdot l - \lambda_j + \lambda_{j'})$. Hence we have to estimate $\delta_{ljj'}(i_1) - \delta_{ljj'}(i_2)$. From (9.42) and (9.45), we get

$$\begin{aligned}
|\delta_{ljj'}(i_1) - \delta_{ljj'}(i_2)| &= |\Delta_{12}(\lambda_j - \lambda_{j'})| \\
&= |(\lambda_j - \lambda_{j'})(i_1 - i_2)| \\
&\le C|m_1(i_1) - m_1(i_2)||j - j'| + |r_j(i_1) - r_j(i_2)| + |r_{j'}(i_1) - r_{j'}(i_2)| \\
&\le \mu \gamma^{-1} C |j - j'| \|i_1 - i_2\|_{2\mathfrak{p}_0 + \nu(\mathtt{b}) + \sigma} ,
\end{aligned}$$

therefore, using $|\delta_{ljj'}| \geq \gamma \langle l \rangle^{-\tau} |j^3 - j'^3|$, where γ_1^{-1} , $\gamma_2^{-1} \leq \gamma^{-1}$, $\mu\gamma^{-1}$ small enough, we have

$$\begin{aligned} \frac{|\delta_{ljj'}(i_1) - \delta_{ljj'}(i_2)|}{|\delta_{ljj'}(i_1)||\delta_{ljj'}(i_2)|} &\leq C \frac{l^{2\tau}\mu|j-j'|\|i_1 - i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma}}{\gamma^2|j^3 - j'^3|^2} \\ &\leq C \frac{l^{2\tau}\mu|j-j'|\|i_1 - i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma}}{\gamma^2|j-j'|^2|j^2+j'^2+jj'|^2} \\ &\leq C \frac{l^{2\tau}\mu\|i_1 - i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma}}{\gamma^2|j^2+j'^2+jj'|^2} \\ &\leq C N^{2\tau}\mu\gamma^{-2}\|i_1 - i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma} \,. \end{aligned}$$

Hence

$$|\psi_j^{j'}(l)(i_1) - \psi_j^{j'}(l)(i_2)| \leq_C \frac{|Q_j^{j'}(l)(i_1) - Q_j^{j'}(l)(i_2)|}{|\delta_{ljj'}(\lambda_1)|} + |Q_j^{j'}(l)(i_2)| \frac{|\delta_{ljj'}(i_1) - \delta_{ljj'}(i_2)|}{|\delta_{ljj'}(i_1)||\delta_{ljj'}(i_2)|} \,.$$

Therefore

$$\begin{aligned} |\Delta_{12}\psi_j^{j'}(l)| &\leq |\Delta_{12}Q_j^{j'}(l)|\gamma^{-1}\langle l\rangle^{\tau} + |Q_j^{j'}(l)(i_2)|N^{2\tau}\mu\gamma^{-2}C\|i_1 - i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma} \\ &\leq N^{2\tau}C\gamma^{-1}\Big(|\Delta_{12}Q_j^{j'}(l)| + \mu\gamma^{-1}|Q_j^{j'}(i_2)|\|i_1 - i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma}\Big) \,, \end{aligned}$$

then, with $\gamma^{-1}\mu$ small enough

$$|\Delta_{12}\psi_j^{j'}(l)| \leq CN^{2\tau}\gamma^{-1}\Big(|Q_j^{j'}(l)(i_2)|\|i_1 - i_2\|_{2\mathtt{p}_0+\nu(\mathtt{b})+\sigma} + |\Delta_{12}Q_j^{j'}(l)|\Big)$$

and the other estimate follows as in (9.70). In addition we have that $\mathbf{Q}$ is real and leaves E invariant, (see (8.108)), hence also $\boldsymbol{\Psi} : E \to E$ and it is real, hence $\boldsymbol{\Psi}$ satisfy (8.104). Indeed

$$\begin{aligned} -(\psi_1)_j^{j'} &= \frac{(Q_1)_j^{j'}}{i(\omega \cdot l + \lambda_j - \lambda_{j'})} \\ &= \frac{(Q_4)_{-j}^{-j'}}{i(\omega \cdot l + \lambda_j - \lambda_{j'})} \\ &= (\psi_4)_{-j}^{-j'} \,, \end{aligned}$$

moreover

$$\begin{aligned} -\overline{(\psi_1)_j^{j'}(l)} &= \frac{\overline{(Q_1)_j^{j'}(l)}}{-i(\omega \cdot l + \lambda_j - \lambda_{j'})} \\ &= \frac{(Q_1)_{-j}^{-j'}(-l)}{i(\omega \cdot (-l) - \lambda_j + \lambda_{j'})} \\ &= (\psi_1)_{-j}^{-j'}(-l), \end{aligned}$$

similarly for ψ_4. Finally, by (8.109) we have $(Q_1)_j^{j'}(-l) = -(Q_4)_j^{j'}(l)$, hence by the definition of ψ in (9.65) we have that

$$(\psi_1)_j^{j'}(-l) = (\psi_4)_j^{j'}(l)$$

and so Ψ is reversibility preserving (see (7.32)). $\square$

By (9.66) we prove that at the step $\nu+1$ (9.40) and (9.31) are satisfied. By the explicit definition of (9.66) we have that the operator $\mathbf{L}_{\nu+1}$ has the same form of $\mathbf{L}_\nu$ with $\mathbf{Q}_{\nu+1}$ instead of $\mathbf{Q}_\nu$. Note that the new remainder $\mathbf{Q}_{\nu+1}$ is the sum of quadratic function of $\mathbf{\Phi}_\nu$ and $\mathbf{Q}_\nu$. Now we want to prove that the new normal form $\mathbf{D}_{\nu+1}$ is diagonal.

Lemma 9.14. ***New diagonal part.*** *The new normal form is*

$$i\mathbf{D}_{\nu+1} = i\mathbf{D}_\nu + [\mathbf{Q}_\nu] = i\begin{pmatrix} D_{\nu+1} & 0 \\ 0 & -D_{\nu+1}\end{pmatrix}, \quad D_{\nu+1} = diag_{j\in\mathbb{Z}\setminus\mathbb{S}^\pm}\lambda_j^{\nu+1}(\omega,\varepsilon), \quad \lambda_j^{\nu+1} = \lambda_j^\nu + \mathtt{r}_j^\nu \in \mathbb{R}$$

with $\mathtt{r}_j^\nu = -\mathtt{r}_{-j}^\nu$, $\lambda_j^{\nu+1} = -\lambda_{-j}^{\nu+1}$, $\forall j \in \mathbb{Z}\setminus\mathbb{S}^\pm$, *and*

$$|\lambda_j^{\nu+1}(\omega,\varepsilon) - \lambda_j^\nu(\omega,\varepsilon)|^{k_0,\gamma} = |\mathtt{r}_j^{\nu+1}(\omega) - \mathtt{r}_j^\nu(\omega)|^{k_0,\gamma} \le C\mathbb{M}^\sharp(\mathfrak{p}_0)\,. \tag{9.71}$$

Moreover given $i_1(\omega,\varepsilon)$, $i_2(\omega,\varepsilon)$ *then, for all* $(\omega,\varepsilon)\in\Lambda_\nu^{\gamma_1}(i_1)\cap\Lambda_\nu^{\gamma_2}(i_2)$ *we have*

$$|\Delta_{12}\mathtt{r}_j(\omega,\varepsilon)| \le C|||\Delta_{12}Q|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\,. \tag{9.72}$$

Proof. For simplicity in the first part of the proof we will drop the index ν. We have,

$$[\mathbf{Q}] = \begin{pmatrix} [Q_1] & 0 \\ 0 & [Q_4]\end{pmatrix},$$

is defined in (9.64). Due to $\mathbf{Q}$ is real and acts in E, defined in (7.26) , the operator $\mathbf{Q}$ satisfy (8.108). Since $\mathbf{Q}$ is reversible (see (8.109)) we also have $(Q_1)_j^j(0) = -(Q_4)_{-j}^{-j}(0)$. Hence if we define $(Q_m)_j^j(0) := \alpha_j^m + i\mathtt{r}_j^m$, with $m = 1,4$ we get

$$\begin{aligned}(Q_1)_j^j(0) &= \alpha_j^1 + i\mathtt{r}_j^1 \\ &= \alpha_{-j}^4 + i\mathtt{r}_{-j}^4 \\ &= \alpha_j^4 - i\mathtt{r}_j^4 \\ &= -\alpha_j^1 + i\mathtt{r}_j^1\end{aligned}$$

where the first equality follows by the $(Q_1)_{-j}^{-j}(0) = (Q_4)_j^j(0)$, for the second we used the reality condition, that in Fourier is $(Q_1)_{-j}^{-j}(0) = \overline{(Q_1)_j^j(0)}$ and for the third we use the reversible condition $(Q_1)_j^j(0) = -(Q_4)_j^j(0)$. Therefore we obtain that $\alpha_j^1 = 0$ and

$$(Q_1)_j^j(0) = i\mathtt{r}_j^1 \in i\mathbb{R},$$

similarly for $(Q_4)_j^j(0)$. Moreover we also obtain that $\mathtt{r}_j^1 = -\mathtt{r}_{-j}^1 = -\mathtt{r}_j^4$. Hence

$$\mathtt{r}_j^1,\ \mathtt{r}_{-j}^1 = \mathtt{r}_j^4 \in \mathbb{R}\,.$$

The statement follows with $\mathtt{r}_j^1 = \mathtt{r}_j$.

By the definition of $\mathcal{M}^\sharp(\mathfrak{p}_0)$ given in (9.37), and by Definition 11 we have that

$$|||\partial_\zeta^k Q_m^\nu|h\|_{\mathfrak{p}_0} \le 2\gamma^{-|k|}\mathcal{M}^\sharp(\mathfrak{p}_0)\|h\|_{\mathfrak{p}_0} \quad \text{for} \quad m = 1,4.$$

which implies that

$$|\partial_\zeta^k (Q_m^\nu)_j^j(0)| \le C\gamma^{-|k|}\mathcal{M}^\sharp(\mathfrak{p}_0), \quad m = 1,4$$

hence

$$|\lambda_{\nu+1}^{(j)} - \lambda_\nu^{(j)}|^{k_0,\gamma} = |\mathtt{r}_j^{\nu+1}(\omega,\varepsilon) - \mathtt{r}_j^\nu(\omega,\varepsilon)|^{k_0,\gamma} \le C \max_{m=1,4}\left(|(Q_m^\nu)_j^j(0)|^{k_0,\gamma}\right) \le C\gamma^{-|k|}\mathcal{M}^\sharp(\mathfrak{p}_0).$$

In a similar way we obtain

$$|\Delta_{12}(Q_m^\nu)_j^j(0)| \le C \max_{m=1,4} |||\Delta_{12}Q_m^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}.$$

By this we obtain (9.72). □

9.4 The iteration

Now we prove iteratively that $(S1)_\nu$, $(S2)_\nu$, $(S3)_\nu$, $(S4)_\nu$ in Theorem 9.6 are satisfied for every $\nu \ge 0$. To this end we suppose that the Theorem is true for $(S1)_\nu$, $(S2)_\nu$, $(S3)_\nu$, $(S4)_\nu$, and we shall prove it for $(S1)_{\nu+1}$, $(S2)_{\nu+1}$, $(S3)_{\nu+1}$, $(S4)_{\nu+1}$.

Proof of $(S1)_{\nu+1}$. Since the eigenvalues λ_j^ν are defined on $\mathcal{N}(\Lambda_\nu^\gamma, \gamma N_{\nu-1}^{-\tau-2})$, (see 9.32), the set $\Lambda_{\nu+1}^\gamma$ in (9.35) is well defined. Moreover λ_j^ν are well defined also on the set $\mathcal{N}(\Lambda_{\nu+1}^\gamma, \gamma N_{\nu-1}^{-\tau-2}) \subseteq \mathcal{N}(\Lambda_\nu^\gamma, \gamma N_{\nu-1}^{-\tau-2})$ because $\Lambda_{\nu+1}^\gamma \subseteq \Lambda_\nu^\gamma$. Let us prove $\mathcal{N}(\Lambda_{\nu+1}^\gamma, \gamma N_\nu^{-\tau-2}) \subset \Lambda_{\nu+1}^{\gamma/2}$, that is (9.32) at the step $\nu+1$. Let $\zeta_0 = (\omega_0, \varepsilon_0) \in \Lambda_{\nu+1}^\gamma$ and (ω,ε), with $|\zeta - \zeta_0| \le \gamma N_\nu^{-\tau-2}$. Then for all $|l| \le N_\nu$ and for all $j \ne k$ we have, by (9.34), with $\mu\gamma^{-2} \le 1$,

$$\begin{aligned}
|\omega\cdot l + \lambda_j^\nu(\zeta) - \lambda_{j'}^\nu(\zeta)| &= |\omega\cdot l + \omega_0\cdot l - \omega_0\cdot l + \lambda_j^\nu(\zeta_0) - \lambda_j^\nu(\zeta_0) + \lambda_j^\nu(\zeta) - \lambda_{j'}^\nu(\zeta) + \lambda_{j'}^\nu(\zeta_0) - \lambda_{j'}^\nu(\zeta_0)| \\
&\ge |\omega_0\cdot l + \lambda_j^\nu(\zeta_0) - \lambda_{j'}^\nu(\zeta_0)| - |l||\omega-\omega_0| - |(\lambda_j^\nu - \lambda_{j'}^\nu)(\zeta) - (\lambda_j^\nu - \lambda_{j'}^\nu)(\zeta_0)| \\
&\ge |\omega_0\cdot l + \lambda_j^\nu(\omega_0) - \lambda_{j'}^\nu(\omega_0)| - |\zeta-\zeta_0|\left(|l| + \mu C|j^3 - j'^3|\right) \\
&\ge \gamma|j^3 - j'^3|\,\langle l\rangle^{-\tau} - \gamma N_\nu^{-\tau-1} - \mu C|j^3 - j'^3|\gamma N_\nu^{-\tau-2} \\
&\ge |j^3 - j'^3|\frac{\gamma}{2}\,\langle l\rangle^{-\tau}
\end{aligned}$$

with $N_0 > 4C$ large enough. Hence $\zeta = (\omega,\varepsilon) \in \Lambda_{\nu+1}^{\gamma/2}$, defined in (9.35) with $\nu \rightsquigarrow \nu+1$ and $\gamma \rightsquigarrow \gamma/2$. By (9.32) at the step $\nu+1$ and by Lemma 9.13, for all $(\omega,\varepsilon) \in \mathcal{N}(\Lambda_{\nu+1}^\gamma, \gamma N_\nu^{-\tau-2})$ the solution Ψ_ν of the homological equation (9.63), defined componedwised in (9.65), is well defined, and by (9.67) and (9.38) satisfy for all $0 \le |k| \le k_0$ the estimate (9.41) at $\nu+1$, that is, at $\nu+1$ with $k=0, p=\mathfrak{p}_0$ that

$$\mathcal{M}^\sharp_{\psi_i^\nu}(\mathfrak{p}_0) \le C N_\nu^{\tau(k_0+1)+k_0} N_{\nu-1}^{-\mathfrak{a}}\mathcal{M}_0(p,\mathtt{b}), \quad i = 1,4. \tag{9.73}$$

We have $\mathbf{\Phi}_\nu = \mathbb{1}^\perp + \mathbf{\Psi}_\nu$ is invertible, indeed by (9.21), (9.30) the smallness condition (2.43) in Lemma 2.19 is satisfied for $N_0 := N_0(p,b)$ large enough. Hence we define $\mathbf{\Phi}_\nu^{-1}$ as

$$\mathbf{\Phi}_\nu^{-1} := \mathbb{1}^\perp + \hat{\mathbf{\Psi}}_\nu = \mathbb{1}^\perp + \begin{pmatrix} \hat{\psi}_1^\nu & 0 \\ 0 & \hat{\psi}_4^\nu \end{pmatrix} \tag{9.74}$$

in addition, by Lemma 2.19, we have that $\hat{\psi}_m^\nu\,,\ m = 1,4$ are $\mathcal{D}^{k_0}$-modulo tame with the same constants of $\psi_m^\nu\,,\ m = 1,4$, therefore (we drop the index 1 and 4) we obtain

$$\begin{aligned} \mathcal{M}_{\hat{\psi}^\nu}(p) &\leq C(k_0)\gamma^{-1} N_{\nu-1}^{-\mathfrak{a}} N_\nu^{\tau(k_0+1)+k_0} \mathcal{M}_0(p,\mathtt{b}) \\ \mathcal{M}_{\langle\partial_\theta\rangle^{\mathtt{b}}\hat{\psi}^\nu}(p) &\leq C(k_0,\mathtt{b})\gamma^{-1} N_{\nu-1} N_\nu^{\tau(k_0+1)+k_0} \mathcal{M}_0(p,\mathtt{b})\,. \end{aligned} \tag{9.75}$$

Note that this is (9.41) for $\nu+1$. Moreover since $\mathbf{\Psi}_\nu : E \to E$ and is reversibility preserving (see (7.32)), also $\hat{\mathbf{\Psi}}_\nu : E \to E$ and is reversibility preserving.

By Lemma 9.14 the operator $\mathbf{D}_{\nu+1}$ is diagonal and its eigenvalues $\lambda_j^{\nu+1} : \mathcal{N}(\Lambda_{\nu+1}^\gamma, \gamma N_\nu^{-\tau-2}) \to \mathbb{R}$ satisfy (9.34) at $\nu+1$. Now we shall estimate the remainder, $\mathbf{Q}_{\nu+1}$ defined in (9.66), that is

$$\mathbf{Q}_{\nu+1} = \mathbf{\Phi}_\nu^{-1}\mathbf{H}_\nu\,, \quad \mathbf{H}_\nu = \Pi_{N_\nu}^\perp \mathbf{Q}_\nu + \mathbf{Q}_\nu\mathbf{\Psi}_\nu - \mathbf{\Psi}_\nu[\mathbf{Q}_\nu]\,,$$

so

$$\mathbf{Q}_{\nu+1} = \mathbf{\Phi}_{\nu+1}^{-1}\left(\Pi_{N_\nu}^\perp \mathbf{Q}_\nu + \mathbf{Q}_\nu\mathbf{\Psi}_\nu - \mathbf{\Psi}_\nu[\mathbf{Q}_\nu]\right)\,. \tag{9.76}$$

By (9.74), (9.36) and (9.61) we have

$$\mathbf{Q}_{\nu+1} = \begin{pmatrix} Q_1^{\nu+1} & 0 \\ 0 & Q_4^{\nu+1} \end{pmatrix}\,. \tag{9.77}$$

Since $(Q_1^{\nu+1})_k^j = (Q_4^{\nu+1})_{-k}^{-j}$ we shall write $Q^{\nu+1}$ instead of $Q_m^{\nu+1}\,,\quad m = 1,4$.

Lemma 9.15. ***Nash-Moser Iterative scheme****. The operator $\mathbf{Q}_{\nu+1}$, respectively $\langle\partial_\theta\rangle^{\mathtt{b}}\,\mathbf{Q}_{\nu+1}$, is $\mathcal{D}^{k_0}$-modulo-tame with modulo tame constant satisfying, respectively*

$$\mathcal{M}_{\nu+1}^\sharp(p) \leq_{k_0} N_\nu^{-\mathtt{b}}\mathcal{M}_\nu^\sharp(p,\mathtt{b}) + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}_\nu^\sharp(p)\mathcal{M}_\nu^\sharp(\mathfrak{p}_0) \tag{9.78}$$

$$\mathcal{M}_{\nu+1}^\sharp(p,\mathtt{b}) \leq_{k_0,\mathtt{b}} \mathcal{M}_\nu^\sharp(p,\mathtt{b}) + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}_\nu^\sharp(p,\mathtt{b})\mathcal{M}_\nu^\sharp(\mathfrak{p}_0) + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}_\nu^\sharp(\mathfrak{p}_0,\mathtt{b})\mathcal{M}_\nu^\sharp(p)\,. \tag{9.79}$$

Proof. By Lemmas 2.20, 2.18, and by (9.67), (9.75) we can estimate each term in (9.76). We will write $Q^{\nu+1}$ and ψ^ν instead of $Q_1^{\nu+1}, Q_4^{\nu+1}, \psi_1^\nu, \psi_4^\nu$.

$$\begin{aligned} \|Q^{\nu+1}h\|_p &\leq_{k_0} N_\nu^{-\mathtt{b}}\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^{\mathtt{b}}Q^\nu}(\mathfrak{p}_0)\|h\|_p + N_\nu^{-\mathtt{b}}\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^{\mathtt{b}}Q^\nu}(p)\|h\|_{\mathfrak{p}_0} + \mathcal{M}^\sharp_{Q^\nu}(p)\|\bar{\psi}^\nu h\|_{\mathfrak{p}_0} \\ &+ \mathcal{M}^\sharp_{Q^\nu}(\mathfrak{p}_0)\|\bar{\psi}^\nu h\|_p + N^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_{Q^\nu}(p)\|[Q^\nu]h\|_{\mathfrak{p}_0} + N^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_{Q^\nu}(\mathfrak{p}_0)\|[Q^\nu]h\|_p \\ &+ N^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_{Q^\nu}(p)\|\Pi_{N_\nu}^\perp Q^\nu h\|_{\mathfrak{p}_0} + N^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_{Q^\nu}(\mathfrak{p}_0)\|\Pi_{N_\nu}^\perp Q^\nu h\|_p \\ &+ N^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_{Q^\nu}(p)\|Q^\nu\psi^\nu h\|_{\mathfrak{p}_0} + N^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_{Q^\nu}(\mathfrak{p}_0)\|Q^\nu\psi^\nu h\|_p \\ &\leq_{k_0} N_\nu^{-\mathtt{b}}\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^{\mathtt{b}}Q^\nu}(\mathfrak{p}_0)\|h\|_p + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}_\nu^\sharp(p)\mathcal{M}_\nu^\sharp(\mathfrak{p}_0)\|h\|_p \\ &+ N_\nu^{-\mathtt{b}}\mathcal{M}^\sharp_{\langle\partial_\theta\rangle^{\mathtt{b}}Q^\nu}(p)\|h\|_{\mathfrak{p}_0} + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}_\nu^\sharp(p)\mathcal{M}_\nu^\sharp(p)\|h\|_{\mathfrak{p}_0}\,. \end{aligned}$$

The proof of (9.79) follows by Lemmas 2.18, 2.20 and by (9.67), (9.75), (9.38). □

Thanks to the estimate (9.78) and (9.79) and using (9.21) we can prove that (9.38) holds at the step $\nu+1$.

Lemma 9.16.

$$\mathcal{M}^\sharp_{\nu+1}(p) \le N_\nu^{-\mathtt{a}} \mathcal{M}_0(p,\mathtt{b}), \quad \mathcal{M}^\sharp_{\nu+1}(p,\mathtt{b}) \le \mathcal{M}_0(p,\mathtt{b}) N_\nu.$$

Proof. We prove by induction. By (9.78), (9.30), (9.21) and (9.38), for $N_0 := N_0(P,\mathtt{b}) > 0$ large enough, we get

$$\begin{aligned}
\mathcal{M}^\sharp_{\nu+1}(p) &\le_{k_0} N_\nu^{-\mathtt{b}} \mathcal{M}^\sharp_\nu(p,\mathtt{b}) + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_\nu(p)\mathcal{M}^\sharp_\nu(\mathfrak{p}_0)\\
&\le_{k_0} N_\nu^{-\mathtt{b}} N_{\nu-2}\mathcal{M}_0(p,\mathtt{b}) + N_\nu^{\tau_1}\gamma^{-1} N_{\nu-1}^{-\mathtt{a}}\mathcal{M}_0(p,\mathtt{b}) N_{\nu-1}^{-\mathtt{a}}\mathcal{M}_0(\mathfrak{p}_0,\mathtt{b})\\
&\le_{k_0} \left[N_\nu^{-\mathtt{b}} N_{\nu-2} + N_\nu^{\tau_1}\gamma^{-1} N_{\nu-1}^{-2\mathtt{a}}\mathcal{M}_0(\mathfrak{p}_0,\mathtt{b})\right]\mathcal{M}_0(p,\mathtt{b})\\
&\le_{k_0} N_\nu^{-\mathtt{a}}\mathcal{M}^\sharp_0(p,\mathtt{b})\,.
\end{aligned}$$

This is true for $\mathtt{a}, b$ as in (9.21). Similarly by (9.79), (9.38), (9.21) and (9.30), with $N_0 := N_0(P,\mathtt{b}) > 0$ large enough, we get

$$\begin{aligned}
\mathcal{M}^\sharp_{\nu+1}(p,\mathtt{b}) &\le_{k_0,\mathtt{b}} \mathcal{M}^\sharp_\nu(p,\mathtt{b}) + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_\nu(p,\mathtt{b})\mathcal{M}^\sharp_\nu(\mathfrak{p}_0) + N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}^\sharp_\nu(\mathfrak{p}_0,\mathtt{b})\mathcal{M}^\sharp_\nu(p)\\
&\le_{k_0,\mathtt{b}} N_{\nu-1}\mathcal{M}_0(p,\mathtt{b}) + 2N_\nu^{\tau_1}\gamma^{-1}\mathcal{M}_0(p,\mathtt{b})N_{\nu-1}N_{\nu-1}^{-\mathtt{a}}\mathcal{M}_0(\mathfrak{p}_0,\mathtt{b})\\
&\le \mathcal{M}_0(p,\mathtt{b})N_\nu\,.
\end{aligned}$$

□

Since $\mathbf{\Phi}_\nu : E \to E$ is reversibility preserving we have that $\mathbf{Q}_{\nu+1} : E \to E$ is reversible (see (7.32)).

Proof of $(S2)_{\nu+1}$. We have to construct a smooth extension $\tilde\lambda_j^{\nu+1}$ on $\Omega\times[\varepsilon_1,\varepsilon_2]$. Thanks to the inductive hyphotesis, we have that there exists an extension $\tilde\lambda_j^\nu : \Omega\times[\varepsilon_1,\varepsilon_2]$, that is C^{k_0}-times differentiable, moreover $\lambda_j^\nu = \tilde\lambda_j^\nu$ on Λ_ν^γ and $\tilde\lambda_j^\nu = 0$ outside $\mathcal{N}(\Lambda_\nu^\gamma, \gamma N_{\nu+1}^{-\tau-2})$, where Λ_ν^γ is defined in (9.35). Note that all the sets Λ_ν^γ are defined by only finitely many non-resonance conditions, that is

$$\Lambda_\nu^\gamma = \bigcap_{\substack{|l|\le N_{\nu-1}\\ |j|,|j'|\le CN_{\nu-1}^{1/2}}} \Big\{\zeta = (\omega,\varepsilon) \in \Lambda_{\nu-1}^\gamma \cap \Big([\mathtt{DC}^\gamma_{K_n} \cap \mathtt{DC}^\gamma_{N_{\nu-1}}] \times [\varepsilon_1,\varepsilon_2]\Big) :$$
$$|\omega\cdot l + \lambda_j^{\nu-1} - \lambda_{j'}^{\nu-1}| \ge \gamma|j^3 - j'^3|\,\langle l\rangle^{-\tau}, \forall |l| \le N_{\nu-1}, j, j' \in \mathbb{Z}\setminus\mathbb{S}^\pm\Big\}\,.$$

Actually, if $|j^2 + j'^2| \ge CN_{\nu-1}$, $j \ne j'$, for all $(\omega,\varepsilon) \in \Lambda_{\nu-1}^\gamma$ then the functions

$$\begin{aligned}
|\omega\cdot l + \lambda_j^{\nu-1} - \lambda_{j'}^{\nu-1}| &\ge |\lambda_j^{\nu-1} - \lambda_{j'}^{\nu-1}| - |\omega||l|\\
&\ge |j^3 - j'^3| - C|l|\\
&\ge C|j^2 + j'^2| - CN_{\nu-1}\\
&\ge \frac12\,.
\end{aligned}$$

We have that $\lambda_j^{\nu+1} = \lambda_j^\nu + \mathtt{r}_j^\nu$, defined on $\mathcal{N}(\Lambda_{\nu+1}^\gamma, \gamma N_\nu^{-\tau-2})$ hence the idea is to extend just the function $\mathtt{r}_j^\nu$. For this reason we consider $\chi_\nu : \mathbb{R}^{1+N} \to \mathbb{R}$, let $\chi_\nu \in C^\infty$ be a cut off function, where

$$
\begin{aligned}
&0 \le \chi_\nu \le 1\,, \qquad \chi_\nu(\zeta) = 1\,,\ \ \forall \zeta \in \Lambda_{\nu+1}^\gamma\,, \quad supp(\chi_\nu) \subseteq \mathcal{N}(\Lambda_{\nu+1}^\gamma, \gamma N_\nu^{-\tau-2})\,,\\
&|\partial_\zeta^k \chi_\nu(\zeta)| \le C(k) \left(N_\nu^{\tau+2}\gamma^{-1}\right)^{|k|}\,, \quad \forall k \in \mathbb{N}^N, \quad \Rightarrow \quad |\chi_\nu(\zeta)|^{k_0,\gamma} \le C(k_0) N_\nu^{(\tau+2)k_0}\,, \quad \forall k\,.
\end{aligned}
$$

Hence, we define

$$
\begin{aligned}
\tilde{\mathtt{r}}_j^\nu :=& \chi_\nu \mathtt{r}_j^\nu\,,\\
\tilde\lambda_j^{\nu+1} :=& \tilde\lambda_j^\nu + \tilde{\mathtt{r}}_j^\nu\,.
\end{aligned}
\tag{9.80}
$$

By (9.80), Lemma 9.14 and (9.38), (9.27) we have the following estimate

$$
\begin{aligned}
|\tilde\lambda_j^{\nu+1} - \tilde\lambda_j^\nu|^{k_0,\gamma} &\le |\chi_\nu|^{k_0,\gamma} |\mathtt{r}_j^\nu|^{k_0,\gamma}\\
&\le C(k_0) N_\nu^{(\tau+2)k_0} \mathcal{M}_\nu^\sharp(\mathfrak{p}_0)\\
&\le \mu\gamma^{-1} C(k_0, P, \mathtt{b}) N_\nu^{(\tau+2)k_0} N_{\nu-1}^{-\mathfrak{a}}\,,
\end{aligned}
$$

this is (9.43) at $\nu+1$.

Proof of $(S3)_{\nu+1}$. Let $Q_m^\nu(i_s), \psi_m^{\nu-1}(i_s)$ with $m = 1,4$ and $s = 1,2$ be the operators constructed at the ν-step, defined on $\Lambda_\nu^{\gamma_1}(i_1) \cap \Lambda_\nu^{\gamma_2}(i_2)$.

$Q_m^\nu(i_m), \psi_m^{\nu-1}(i_s)$ satisfy (9.38) and (9.41). Since $(Q_1)_k^j = (Q_4)_{-k}^{-j}$ we can drop the $1,4$-index.

We now want to estimate the operator $\Delta_{12} Q_{\nu+1}$. By Lemma 9.63 we have constructed the operators ψ_m^ν, $m = 1,4$ defined for all $\omega \in \Lambda_{\nu+1}^{\gamma_1}(i_1) \cap \Lambda_{\nu+1}^{\gamma_2}(i_2)$. From now on we shall drop the index m. We estimate the operator $\Delta_{12}\psi^\nu$, by (9.68), (2.33), (9.38) and (9.44) we have

$$
\begin{aligned}
|||\Delta_{12}\psi^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\le_{\mathtt{b}} N_\nu^{2\tau}\gamma^{-1}\left(|||Q(i_2)|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} + |||\Delta_{12} Q|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right)\\
&\le_{\mathtt{b}} N_\nu^{2\tau}\gamma^{-1}\left(|||Q_0|||_{\mathcal{L}(H^{\mathfrak{p}_0})} N_{\nu-1}^{-\mathfrak{a}} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} + |||\Delta_{12} Q|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right)\\
&\le_{P,\mathtt{b}} N_\nu^{2\tau}\gamma^{-1}\left(|||Q_0|||_{\mathcal{L}(H^{\mathfrak{p}_0})} N_{\nu-1}^{-\mathfrak{a}} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} + \mu N_{\nu-1}^{-\alpha} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\right)\\
&\le C(P,\mathtt{b}) N_\nu^{2\tau}\gamma^{-2}\mu N_{\nu-1}^{-\mathfrak{a}} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}
\end{aligned}
\tag{9.81}
$$

we also have,by (9.68), (2.33), (9.38), (9.27) and (9.44)

$$
\begin{aligned}
||| \langle\partial_\theta\rangle^{\mathtt{b}} \Delta_{12}\psi^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\le_{\mathtt{b}} N_\nu^{2\tau}\gamma^{-1}\left(||| \langle\partial_\theta\rangle^{\mathtt{b}} Q(i_2)|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} + ||| \langle\partial_\theta\rangle^{\mathtt{b}} \Delta_{12} Q|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right)\\
&\le_{P,\mathtt{b}} N_\nu^{2\tau}\gamma^{-1}\left(N_{\nu-1}|||Q_0|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} + N_{\nu-1}|||Q_0|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right)\\
&\le_{P,\mathtt{b}} N_\nu^{2\tau}\gamma^{-2} N_{\nu-1}\mu \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\,.
\end{aligned}
\tag{9.82}
$$

By (9.73), for $\gamma^{-2}\nu$ small enough, the smallness condition (2.52) is verified. Therefore if we define $\hat{\mathbf{\Phi}}_\nu^{-1}$ as in (9.74), by (9.81), (9.82), (9.75) (2.33) and Lemma 2.22, we get (by dropping the $1,4$ index).

$$
\begin{aligned}
|||\Delta_{12}\hat\psi^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\le_{P,\mathtt{b}} N_\nu^{2\tau}\gamma^{-2} N_{\nu-1}^{-\mathfrak{a}}\mu \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\\
||| \langle\partial_\theta\rangle^{\mathtt{b}} \Delta_{12}\hat\psi^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\le_{P,\mathtt{b}} N_\nu^{2\tau}\gamma^{-2} N_{\nu-1}\mu \|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\,.
\end{aligned}
\tag{9.83}
$$

Now we estimate $\Delta_{12}Q^{\nu+1}$ where $Q^{\nu+1} = (\mathbb{1}+\psi^\nu)^{-1}\left(\Pi^\perp_{N_\nu}Q^\nu + Q^\nu\psi^\nu - \psi^\nu[Q^\nu]\right)$ because $(\mathbb{1}+\psi^\nu)^{-1}\psi[Q^\nu]$ satisfies the estimate we have to study the norm of $\Delta_{12}(Q^\star)^\nu$ where $(Q^\star)^\nu = (\mathbb{1}+\psi^\nu)^{-1}\left(\Pi^\perp_{N_\nu}Q^\nu + Q^\nu\psi^\nu\right)$. We have that

$$\begin{aligned}\Delta_{12}(Q^\star)^\nu =& \Delta_{12}\hat{\psi}^\nu\left(\Pi^\perp_{N_\nu}Q^\nu(i_1) + Q^\nu(i_1)\psi^\nu(i_1)\right)\\ &+ (\mathbb{1}+\hat{\psi}^\nu(i_2))\left(\Pi^\perp_{N_\nu}\Delta_{12}Q^\nu + \Delta_{12}Q^\nu(i_1)\psi^\nu(i_1) + \Delta_{12}Q^\nu(i_1)\psi^\nu(i_2)\right)\end{aligned}$$

then, by Lemma 2.21, (9.83), (2.33), (9.75) (9.68) and (9.67), taking $\gamma^{-2}\mu$ small enough, the following inequality holds

$$\begin{aligned}|||\Delta_{12}(Q^\star)^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \leq& |||\Delta_{12}\hat{\psi}^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\left(|||\Pi^\perp_{N_\nu}Q^\nu(i_1)|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + |||Q^\nu(i_1)|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||\psi^\nu(i_1)|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right) +\\ &+ |||(\mathbb{1}+\hat{\psi}^\nu)|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\left(|||\Pi^\perp_{N_\nu}\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + |||\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||\psi^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} +\right.\\ &\left.+ |||\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||\psi^\nu(i_2)|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right)\\ \leq_{\mathtt{b}}& |||\Delta_{12}\hat{\psi}^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\left(N_\nu^{-\mathtt{b}}|||\langle\partial_\theta\rangle^{\mathtt{b}}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + |||Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||\psi^\nu(i_1)|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right) +\\ &+ |||(\mathbb{1}+\hat{\psi}^\nu)|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\left(N_\nu^{-\mathtt{b}}|||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} +\right.\\ &\left.+ |||\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||\psi^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + |||\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||\psi^\nu(i_2)|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right)\\ \leq_{\mathtt{b}}& CN_\nu^{2\tau}\gamma^{-1}\mu N_{\nu-1}^{-\mathfrak{a}}\|i_1-i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\left(N_\nu^{-\mathtt{b}}|||\langle\partial_\theta\rangle^{\mathtt{b}}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} +\right.\\ &\left.+ N_\nu^{2\tau}N_{\nu-1}^{\mathfrak{a}}\gamma^{-1}|||Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||Q_0|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\right) +\\ &+ N_\nu^{-\mathtt{b}}|||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + N_\nu^{2\tau}N_{\nu-1}^{\mathfrak{a}}\gamma^{-1}|||\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}|||Q^0|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\\ \leq_{\mathtt{b}}& \left(N_\nu^{-\mathtt{b}}\mathcal{M}^\sharp_\nu(\mathfrak{p}_0,\mathtt{b}) + N_\nu^{\tau+(k_0+1)\tau}\gamma^{-1}\mathcal{M}^\sharp_\nu(\mathfrak{p}_0)^2\right)\|i_1-i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} +\\ &+ N_\nu^{-\mathtt{b}}|||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + N_\nu^{\tau+(k_0+1)\tau}\gamma^{-1}\mathcal{M}^\sharp_\nu(\mathfrak{p}_0)|||\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\end{aligned} \tag{9.84}$$

and, using (9.83), (9.68), since (9.38) and (9.30) imply $N_\nu^{\tau+(k_0+1)\tau}\gamma^{-1}\mathcal{M}^\sharp_\nu(\mathfrak{p}_0) \leq 1$ we obtain

$$\begin{aligned}|||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}(Q^\star)^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \leq_{P,\mathtt{b}}& \left(\gamma^{-1}\mu N_{\nu-1} + \mathcal{M}^\sharp_\nu(\mathfrak{p}_0,\mathtt{b})\right)\|i_1-i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} +\\ &+ |||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})} + N_\nu^{\tau+(k_0+1)\tau}\gamma^{-1}|||\Delta_{12}Q^\nu|||_{\mathcal{L}(H^{\mathfrak{p}_0})}\mathcal{M}^\sharp_\nu(\mathfrak{p}_0,\mathtt{b})\,.\end{aligned} \tag{9.85}$$

The other terms in (9.76) can be estimated in the same way, therefore $\Delta_{12}Q^{\nu+1}$ satisfies (9.84) and (9.85). We now have to prove (9.44) at the step $\nu+1$. By (9.84), (9.38), (9.22), (9.44) and (9.21), if $\gamma^{-2}\mu \leq 1$ and $N_0(P,\mathtt{b}) > 0$, we get

$$\begin{aligned}|||\Delta_{12}Q^{\nu+1}|||_{\mathcal{L}(H^{\mathfrak{p}_0})} &\leq_{P,\mathtt{b}} \|i_1-i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\left(N_\nu^{-\mathtt{b}}N_{\nu-1}\gamma^{-1}\mu + N_{\nu-1}^{-2\mathfrak{a}}N_\nu^{2\tau}\gamma^{-1}N_\nu^{\tau(k_0+1)+k_0}(\gamma^{-1}\mu)^2\right)\\ &\leq_{P,\mathtt{b}} \mu\gamma^{-1}N_\nu^{-\mathfrak{a}}\|i_1-i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\,.\end{aligned}$$

Similary, by (9.85), (9.38), (9.22) and (9.44) we get

$$|||\langle\partial_\theta\rangle^{\mathtt{b}}\Delta_{12}Q^{\nu+1}|||_{\mathcal{L}(H^{\mathfrak{p}_0})} \leq_{P,\mathtt{b}} \gamma^{-1}\mu N_\nu\|i_1-i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}$$

by $\mu\gamma^{-2} \leq 1$, (9.21) and taking $N_0 := N(P, \mathtt{b}) > 0$ large. Hence we have proved (9.44) at the step $\nu + 1$. The first inequality in (9.72) follows from Lemma 9.14, the second follows by a telescopic argument using the first inequality in (9.72) and (9.44).

Proof of $(S4)_{\nu+1}$. We have to prove that, if $\mu\gamma^{-1}C(P)N_{n-1}^{\tau}\|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} \leq \rho$, then

$$\zeta \in \Lambda_\nu^\gamma(i_1) \quad \Rightarrow \quad \zeta \in \Lambda_\nu^{\gamma-\rho}(i_2)\,.$$

Let $\zeta \in \Lambda_\nu^\gamma(i_1)$. By (9.35) and $(S4)_\nu$ we have that $\Lambda_{\nu+1}^\gamma(i_1) \subseteq \Lambda_\nu^\gamma(i_1) \subseteq \Lambda_\nu^{\gamma-\rho}(i_2)$. Therefore $\zeta \in \Lambda_\nu^{\gamma-\rho}(i_2) \subset \Lambda_\nu^\gamma(i_2)$. Using $(S1)_\nu$, we have that the eigenvalues $\lambda_j^\nu(\zeta, i_2)$ are well defined. Thanks to $\zeta \in \Lambda_\nu^\gamma(i_1) \cap \Lambda_\nu^{\gamma/2}(i_2)$ we got (9.45), then by (9.33) (9.45), and $|\partial_i m_1(i)[\hat{i}]| \leq \mu C\|\hat{i}\|_\sigma$

$$\begin{aligned}
|(\lambda_j^\nu - \lambda_{j'}^\nu)(\zeta, i_2) - (\lambda_j^\nu - \lambda_{j'}^\nu)(\zeta, i_1)| &\leq |(\lambda_j^0 - \lambda_{j'}^0)(\zeta, i_2) - (\lambda_j^0 - \lambda_{j'}^0)(\zeta, i_1)| \\
&\quad + 2\sup_j |\mathtt{r}_j^\nu(\zeta, i_2) - \mathtt{r}_j^\nu(\zeta, i_1)| \\
&\leq \mu\gamma^{-1}C(P)|j^3 - j'^3|\|i_2 - i_1\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma}\,. \qquad (9.86)
\end{aligned}$$

Using the definition of $\Lambda_{\nu+1}^\gamma(i_1)$ in (9.35) with $\nu + 1$ instead of ν, (9.86) we can conclude, for all $|l| \leq N_\nu$ that

$$\begin{aligned}
|\omega \cdot l + \lambda_j^\nu(i_2) - \lambda_{j'}^\nu(i_1)| &\geq |\omega \cdot l + \lambda_j^\nu(i_1) - \lambda_{j'}^\nu(i_i)| - |(\lambda_j^\nu - \lambda_{j'}^\nu)(i_2) - (\lambda_j^\nu - \lambda_{j'}^\nu)(i_1)| \\
&\geq \gamma|j^3 - j'^3|\,\langle l\rangle^{-\tau} - C\mu\gamma^{-1}|j^3 - j'^3|\|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} \\
&\geq (\gamma - \rho)|j^3 - j'^3|\,\langle l\rangle^{-\tau}
\end{aligned}$$

provided $C(P)\mu\gamma^{-1}N_\nu^\tau\|i_1 - i_2\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma} \leq \rho$. Hence $\zeta \in \Lambda_{\nu+1}^{\gamma-\rho}(i_2)$, and this proves (9.46) at the step $\nu + 1$.

9.5 Almost invertibility of $\mathcal{L}_\omega$

Let $\mathbf{L}_0 = \mathcal{L}_{M+6}$, where $\mathcal{L}_{M+6}$ is defined in (9.9). Then by (9.6) and Theorem 9.10 we obtain

$$\mathcal{L}_\omega = \mathbf{W}_n\mathbf{L}_n\mathbf{W}_n^{-1} + \mathbf{G}^\perp \quad \mathbf{W}_n = \mathcal{W}^\perp\mathbf{U}_n \qquad (9.87)$$

where $\mathbf{L}_n$ is the operator defined in (9.50) and $\mathbf{G}^\perp$ is defined in (9.7) and satisfy the estimate (9.8). Then (9.3), (9.52), (9.25), imply that for all $\mathfrak{p}_0 \leq p \leq P$,

$$\|\mathbf{W}^{\pm 1}h\|_p^{k_0,\gamma} \leq_P \|h\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+\nu(\mathtt{b})+\sigma}^{k_0,\gamma}\|h\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \qquad (9.88)$$

for some $\sigma := \sigma(\tau, N, k_0) > 0$. Since we want to use a Nash-Moser scheme we have to construct at each step an approximate inverse, that allows us to define the successive approximate solution of the Nash-Moser iteration. For construct the approximate inverse we have to verify that the inversion assumption given in (5.49) and (5.50) are satisfied. For this reason we decompose the linear operator $\mathbf{L}_n$ in (9.50) as

$$\mathbf{L}_n = \mathbf{D}_n^< + \mathbf{Q}_n^\perp + \mathbf{Q}_n \qquad (9.89)$$

where

$$\mathbf{D}_n^< := \Pi_{K_n}(\Omega \cdot \partial_\theta \mathbb{1}^\perp + i\mathbf{D}_n)\Pi_{K_n} + \Pi_{K_n}^\perp, \quad \mathbf{Q}_n^\perp = \Pi_{K_n}^\perp(\Omega \cdot \partial_\theta \mathbb{1}^\perp + i\mathbf{D}_n)\Pi_{K_n}^\perp - \Pi_{K_n}^\perp \tag{9.90}$$

the diagonal operator $\mathbf{D}_n$ is defined in (9.31) (with $\nu = n$), and the constant K_n is given in (5.51).

Lemma 9.17. ***First order Melinikov non-resonance conditions****. For all $\zeta = (\omega, \varepsilon)$ in*

$$\Lambda_{n+1}^{\gamma,I} := \Lambda_{n+1}^{\gamma,I}(i) := \left\{ \zeta \in \Lambda_{n+1}^\gamma : |\omega \cdot l + \lambda_j^n| \geq 2\frac{\gamma j^3}{\langle l \rangle^\tau}, \quad \forall |l| \leq K_n, \ j \in \mathbb{Z} \setminus \mathbb{S}^\pm \right\} \tag{9.91}$$

the operator $\mathbf{D}_n^<$ in (9.90) is invertible and

$$\|(\mathbf{D}_n^<)^{-1} g\|_s^{k_0,\gamma} \leq \gamma^{-1} \|g\|_{s+\tau(k_0+1)+k_0}^{k_0,\gamma}. \tag{9.92}$$

Proof. The estimate (9.92) follows by $|\partial_{(\omega,\varepsilon)}^k (\omega \cdot l + \lambda_j^n)^{-1}| \leq \langle l \rangle^{\tau(|k|+1)+|k|} \gamma^{-(|k|+1)}$ for all $|k| \leq k_0$. □

The smoothing properties defined in (2.11)imply that the operator $\mathbf{Q}_n^\perp$ defined in (9.90) satisfies, for all $b > 0$

$$\|\mathbf{Q}_n^\perp h\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq K_n^{-b} \|h\|_{\mathfrak{p}_0+b+3}^{k_0,\gamma} \quad \|\mathbf{Q}_n^\perp h\|_p^{k_0,\gamma} \leq \|h\|_{p+3}^{k_0,\gamma}. \tag{9.93}$$

Thanks to the decompositions (9.87) , (9.89), Theorem 9.10, Proposition 9.3 and (9.92), (9.93), (9.88) we can prove that $\mathcal{L}_\omega$ is almost invertible, indeed we have the following theorem:

Theorem 9.18. ***Almost invertibility of** $\mathcal{L}_\omega$. Assume (5.9), and that for all $P > \mathfrak{p}_0$ the smallness condition (9.47) holds. Let $\mathtt{a}, \mathtt{b}$ as in (9.21). Then for all*

$$(\omega, \varepsilon) \in \mathbf{\Lambda}_{n+1}^\gamma := \mathbf{\Lambda}_{n+1}^\gamma(i) := \Lambda_{n+1}^\gamma \cap \Lambda_{n+1}^{\gamma,I} \tag{9.94}$$

(see (9.48), (9.91)) the operator $\mathcal{L}_\omega$ defined in (5.48) can be decomposed as

$$\mathcal{L}_\omega = \mathbf{L}_\omega + \mathbf{Q}_\omega + \mathbf{Q}_\omega^\perp, \ \ \mathbf{L}_n = \mathbf{W}_n \mathbf{D}_n^< \mathbf{W}_n^{-1}, \quad \mathbf{Q}_\omega := \mathbf{W}_n \mathbf{Q}_n W_n^{-1} \quad \mathbf{Q}_\omega^\perp = \mathbf{W}_n \mathbf{Q}_n^\perp \mathbf{W}^{-1} + \mathbf{G}^\perp \tag{9.95}$$

where $\mathbf{Q}_n$ is defined in (9.36) (with n instead of ν), $\mathbf{Q}_n^\perp$ is defined in (9.90), and $\mathbf{G}^\perp$ is defined in (9.7). Moreover $\mathbf{L}_\omega$ is invertible and for some $\sigma := \sigma(N, \tau, k_0) \geq 0$, and for all $\mathfrak{p}_0 \leq p \leq P$, $g \in H^{p+\sigma}$ we have

$$\|\mathbf{L}_\omega^{-1} g\|_p^{k_0,\gamma} \leq_P \gamma^{-1} \left(\|g\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+\nu(\mathtt{b})+\sigma}^{k_0,\gamma} \|g\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \right) \tag{9.96}$$

where $\nu(\mathtt{b})$ is defined in (9.25), and

$$\|\mathbf{Q}_\omega h\|_p^{k_0,\gamma} \leq_P \gamma^{-1} \mu N_{n-1}^{-\mathtt{a}} \left(\|h\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+\nu(\mathtt{b})+\sigma}^{k_0,\gamma} \|h\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \right) \tag{9.97}$$

$$\|\mathbf{Q}_\omega^\perp h\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq_P K^{-b} \left(\|h\|_{p+\sigma+b}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+\nu(\mathtt{b})+\sigma+b}^{k_0,\gamma} \|h\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \right) \quad \forall b > 0 \tag{9.98}$$

$$\|\mathbf{Q}_\omega^\perp h\|_p^{k_0,\gamma} \leq_P \left(\|h\|_{p+\sigma}^{k_0,\gamma} + \|\mathcal{V}_0\|_{p+\nu(\mathtt{b})+\sigma}^{k_0,\gamma} \|h\|_{\mathfrak{p}_0+\sigma}^{k_0,\gamma} \right). \tag{9.99}$$

Proof. Use the decomposition (9.87), (9.89) and Theorem 9.10, and the estimates (9.92), (9.93) and (9.88). □

We finally define the operator $\mathbf{W}_\infty(\theta)$, as follows

$$\mathbf{W}_\infty = \mathcal{W}^\perp \mathbf{U}_\infty \,, \quad \text{where} \quad \mathbf{U}_\infty := \lim_{n\to\infty} \mathbf{U}_n \,, \tag{9.100}$$

where $\mathcal{W}$ is defined in (9.2) and $\mathbf{U}_n$ in (9.49). It completely diagonalizes the operator $\mathcal{L}_\omega$ defined in (5.48).

By the arguments of as in Chapter 7 and 8 one can prove that the operator $\mathbf{W}_\infty(\theta)$ satisfies the following tame estimates

$$\mathbf{W}_\infty(\theta) : (H^p(\mathbb{T}_x, \mathbb{C}) \times H^p(\mathbb{T}_x, \mathbb{C})) \cap \mathtt{H}_{\mathbb{S}}^\perp \to (H^p(\mathbb{T}_x, \mathbb{C}) \times H^p(\mathbb{T}_x, \mathbb{C})) \cap \mathtt{H}_{\mathbb{S}}^\perp \tag{9.101}$$

$$\mathbf{W}_\infty^{-1}(\theta) : (H^p(\mathbb{T}_x, \mathbb{C}) \times H^p(\mathbb{T}_x, \mathbb{C})) \cap \mathtt{H}_{\mathbb{S}}^\perp \to (H^p(\mathbb{T}_x, \mathbb{C}) \times H^p(\mathbb{T}_x, \mathbb{C})) \cap \mathtt{H}_{\mathbb{S}}^\perp \,. \tag{9.102}$$

Chapter 10

Nash-Moser Iteration

We define the finite-dimensional subspaces of trigonometric polynomials

$$E_n = \{\mathcal{V}(\theta) = (\Theta, I, w)(\theta),\ \Theta = \Pi_n \Theta\,,\ \ I = \Pi_n I\,,\ \ w = \Pi_n w\}$$

where Π_n is the projector

$$\Pi_n := \Pi_{K_n} : \mathtt{w}(\theta, x) := \sum_{l \in \mathbb{Z}^N, j \in \mathbb{Z}_0 \setminus \mathbb{S}^\pm} \mathtt{w}_{lj} e^{il\cdot\theta + ijx} \mapsto \Pi_n \mathtt{w}(\theta, x) := \sum_{|(l,j)| \le K_n} \mathtt{w}_{lj} e^{il\cdot\theta + ijx} \tag{10.1}$$

with $K_n = K_0^{(3/2)^n}$ (see (5.51) and (5.5)). With an abuse of notation we shall denote

$$\Pi_n \mathtt{q}(\theta) := \sum_{|l| \le K_n} \mathtt{q}_l e^{il\cdot\theta}.$$

In addition we define

$$\Pi_n^\perp := \mathbb{1} - \Pi_n\,.$$

We recall the smoothing properties (2.11) for $\mathcal{V} \in H^p$ that are

$$\|\Pi_n \mathcal{V}\|_{p+b}^{k_0,\gamma} \le K_n^b \|\mathcal{V}\|_p^{k_0,\gamma}\,, \qquad \|\Pi_n^\perp \mathcal{V}\|_p^{k_0,\gamma} \le K_n^{-b} \|\mathcal{V}\|_{p+b}^{k_0,\gamma} \qquad \forall b, p \ge 0 \tag{10.2}$$

where $\|\cdot\|_p^{k_0,\gamma}$ is defined in (2.9) . In view of the Nash-Moser Theorem 10.2 we introduce some constants

$$\sigma_1 := \max\{\bar\sigma, \sigma, 4\} \tag{10.3}$$

$$\mathtt{a}_1 := \max\{3(2\sigma_1 + 6) + 1, \frac{3}{2}[rk_0(\tau+2) + r\tau + \nu(\mathtt{b}) + 2\sigma_1] + 1\}\,,\ \mathtt{a}_2 := \ \frac{2}{3}\mathtt{a}_1 - rk_0(\tau+2) - \nu(\mathtt{b}) - 2\sigma_1 \tag{10.4}$$

$$b_1 := \nu(\mathtt{b}) + 3\sigma_1 + 3 + \mathtt{a}_1 + \frac{2}{3}\nu_1\,, \quad \nu_1 := 3(\nu(\mathtt{b}) + 2\sigma_1) + 1 \tag{10.5}$$

where $\bar\sigma := \bar\sigma(\tau, N, k_0) > 0$ is defined in Theorem 5.13, $\sigma := \sigma(\tau, N, k_0) > 0$ is the constant which appears in Theorem 9.10, 4 is the largest loss of regularity in the estimate of the Hamiltonian vector field X_P in

Lemma 5.1, $\nu(\mathtt{b})$ is defined in (9.25), the constant $b := [\mathfrak{a}] + 2 \in \mathbb{N}$ is defined in (9.21) and the exponent r in (5.51) satisfies

$$r\mathfrak{a} > \frac{1}{2}\mathtt{a}_1 + \frac{3}{2}\sigma_1 \,. \tag{10.6}$$

By Remark 9.4 the constant $\mathfrak{a} \geq \frac{3}{2}k_0(\tau+2)+1$. Hence, by the definition of $\mathtt{a}_1$ in (10.4), there exists $r := r(\tau, N, k_0)$ such that (10.6) holds. Indeed we can fix

$$r := \max\left\{ \frac{9\sigma_1 + 19}{3k_0(\tau+2)+2} \,,\ \frac{\frac{3}{2}(\nu(\mathtt{b}) + rk_0(\tau+2) + r\tau) + 6\sigma_1 + 1}{3k_0(\tau+2)+2} \right\}. \tag{10.7}$$

Remark 10.1. *The constant* $\mathtt{a}_1$ *is the exponent in* (10.12). *The constant* $\mathtt{a}_2$ *is the exponent in* (10.10). *The constant* ν_1 *is the exponent in* $(P3)_n$. *The conditions* $\mathtt{a}_1 > 3(2\sigma_1+6)$, $b_1 > \nu(\mathtt{b}) + 3\sigma_1 + 3 + \mathtt{a}_1 + \frac{2}{3}\nu_1$ *and* $r\mathfrak{a} > \frac{3}{2}(\sigma_1 + \frac{1}{3}\mathtt{a}_1)$ *arise for the convergence of the iterative scheme* (10.35), (10.36) *in Lemma 10.4. In addition we require that* $\mathtt{a}_1 \geq \frac{3}{2}[rk_0(\tau+2) + \nu(\mathtt{b}) + 2\sigma_1] + \frac{3}{2}r\tau + 1$ *so that* $\mathtt{a}_2 > r\tau$*, actually in Lemma 10.6 we need* $\mathtt{a}_2 \geq r\tau + \frac{2}{3}$.

Theorem 10.2 (Nash-Moser)**.** *There exist* $\delta_0, C_\star > 0$*, such that, if*

$$K_0^{\tau_2}\mu\gamma^{-2} \leq \delta_0 \,, \quad \tau_2 := \max(r\tau_0, 2\sigma_1 + \mathtt{a}_1 + 6) \,, \quad K_0 := \gamma^{-1} \,, \quad \gamma := \mu^a, \quad 0 \leq a \leq \frac{1}{3+\tau_2} \,, \tag{10.8}$$

where $\tau_0 := \tau_0(\tau, N)$ *is defined in Theorem 9.6. Then, for all* $n \geq 0$*:*

- $(P_1)_n$ *There exists a* k_0*-times differentiable function*

$$\tilde{W}_n : \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2] \ \to E_{n-1} \times \mathbb{R}^N \,, \ \zeta = (\omega, \varepsilon) \mapsto \tilde{W}_n(\zeta) := (\tilde{\mathcal{V}}_n, \tilde{\alpha}_n - \omega),$$

for $n \geq 1$ *and* $\tilde{W}_0 = 0$*, satisfying*

$$\|\tilde{W}_n\|^{k_0,\gamma}_{\mathfrak{p}_0 + \nu(\mathtt{b}) + \sigma_1} \leq C_\star K_0^{rk_0(\tau+2)}\gamma^{-1}\mu \,. \tag{10.9}$$

Let $\tilde{U}_n = U_0 + \tilde{W}_n$ *where* $U_0 := (\theta, 0, 0, \omega)$*. The difference* $\tilde{H}_n = \tilde{U}_n - \tilde{U}_{n-1}$*, for* $n \geq 1$ *satisfies*

$$\|\tilde{H}_1\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} \leq C_\star\mu\gamma^{-1}K_0^{rk_0(\tau+2)} \,, \quad \|\tilde{H}_n\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} \leq C_\star\mu\gamma^{-1}K_{n-1}^{-\mathtt{a}_2}, \quad \forall n > 1 \,. \tag{10.10}$$

- $(P_2)_n$ *Setting* $\tilde{\imath}_n := (\theta, 0, 0) + \tilde{\mathcal{V}}_n$ *we define*

$$\mathcal{G}_0 := \Omega \times [\varepsilon_1, \varepsilon_2], \quad \mathcal{G}_{n+1} = \mathcal{G}_n \bigcap \mathbf{\Lambda}^\gamma_{n+1}(\tilde{\imath}), \quad n \geq 0 \tag{10.11}$$

where $\mathbf{\Lambda}^\gamma_{n+1}(\tilde{\imath})$ *is defined in* (9.94). *Then for all* ζ *in* $\mathcal{N}(\mathcal{G}_n, \gamma K_{n-1}^{-r(\tau+2)})$ *setting* $\gamma_{-1} = \gamma$ *and* $K_{-1} = 1$ *we have*

$$\|\mathcal{F}(\tilde{U}_n)\|^{k_0,\gamma}_{\mathfrak{p}_0} \leq C_\star\mu K_{n-1}^{-\mathtt{a}_1} \tag{10.12}$$

- $(P_3)_n$

$$\|\tilde{W}_n\|^{k_0,\gamma}_p \leq C_\star\mu\gamma^{-1}K_{n-1}^{\nu_1} \,, \quad \forall\omega \in \mathcal{N}(\mathcal{G}_n, \gamma K_{n-1}^{-r(\tau+2)}) \,.$$

We prove this theorem by iteration. We have that $(P_1)_0, (P_2)_0, (P_3)_0$ follow by $\|\mathcal{F}(U_0)\|_p^{k_0,\gamma} = O(\mu)$ and taking $C_\star$ large enough. Let us assume that $(P_1)_n, (P_2)_n, (P_3)_n$ hold for some $n \geq 0$, we prove $(P_1)_{n+1}, (P_2)_{n+1}, (P_3)_{n+1}$. We shall define the successive approximation $\tilde{U}_{n+1}$ by the Nash-Moser scheme. Note that in order to define $\tilde{U}_{n+1}$ we need to prove the almost-approximate invertibility of the linear operator

$$L_n := L_n(\zeta) := d_{i,\alpha}\mathcal{F}(i_n(\zeta)) .$$

Theorem 5.13 allows us to prove that L_n is almost-approximate invertible, so we have to verify that the inverse assumptions (5.49) and (5.50) (of Theorem 5.13) are satisfied. For this reason we have to use Theorem 9.18, with $i = i_n$. By (10.8), with μ small enough, we have that the the smallness condition (9.47) holds. Hence we can apply Theorem 9.18, therefore we can prove that (5.49) and (5.50) are satisfied for all

$$\zeta \in \mathcal{N}\left(\mathbf{\Lambda}_{n+1}^{\gamma}(\tilde{i}_n, 2\gamma K_n^{-r(\tau+2)})\right),$$

where $\mathbf{\Lambda}_{n+1}^{\gamma}$ is defined in (9.94). Indeed by (9.32) and recalling the definition of $\Lambda_{n+1}^{\gamma,I}(\tilde{i}_n)$ in (9.91) with $\tilde{i}_n$ instead of i, that gives

$$\mathcal{N}\left(\Lambda_{n+1}^{\gamma,I}(\tilde{i}_n), 2\gamma K_n^{-r(\tau+2)}\right) \subseteq \Lambda_{n+1}^{\gamma/2,I}(\tilde{i}_n)$$

we have

$$\mathcal{N}\left(\mathbf{\Lambda}_{n+1}^{\gamma}(\tilde{i}_n), 2\gamma K_n^{-r(\tau+2)}\right) \subseteq \mathbf{\Lambda}_{n+1}^{\gamma/2}(\tilde{i}_n), \quad \forall n \geq 0 .$$

Now we can apply Theorem 5.13 to the linear operator $L_n(\zeta)$ with $\Lambda_0 = \mathcal{N}(\mathbf{\Lambda}_{n+1}^{\gamma}(\tilde{i}_n), 2\gamma K_n^{r(\tau+2)})$ and

$$P := \mathfrak{p}_0 + b_1 , \quad \text{where} \quad b_1 \quad \text{is defined in} \quad (10.5), \tag{10.13}$$

and P is the larger scale used in the Nash-Moser theorem. Finally we have the existence of an almost-approximate inverse $\mathbf{T}_n := \mathbf{T}_n(\zeta, \tilde{i}_n(\zeta))$ which satisfies

$$\|\mathbf{T}_n g\|_p^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \gamma^{-1}\left(\|g\|_{p+\sigma_1}^{k_0,\gamma} + \|g\|_{\mathfrak{p}_0+\sigma_1}^{k_0,\gamma}\|\tilde{\mathcal{V}}_n\|_{p+\sigma_1+\nu(\mathtt{b})}^{k_0,\gamma}\right), \quad \forall \mathfrak{p}_0 \leq p \leq \mathfrak{p}_0 + b_1 \tag{10.14}$$

$$\|\mathbf{T}_n g\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \gamma^{-1}\|g\|_{\mathfrak{p}_0+\sigma_1}^{k_0,\gamma} . \tag{10.15}$$

For all

$$\zeta \in \mathcal{N}(\mathcal{G}_{n+1}, 2\gamma K_n^{-r(\tau+2)}) \subset \mathcal{N}(\mathcal{G}_{n+1}, \gamma K_{n-1}^{-r(\tau+2)}), \quad n \geq 0 \tag{10.16}$$

we can define the successive approximation as follows

$$U_{n+1} := \tilde{U}_n + H_{n+1} , \quad H_{n+1} := (\hat{\mathcal{V}}_{n+1}, \hat{\alpha}_{n+1}) := -\mathbf{\Pi}_n \mathbf{T}_n \Pi_n \mathcal{F}(\tilde{U}_n) \in E_n \times \mathbb{R}^N \tag{10.17}$$

where $\mathbf{\Pi}_n$ is defined by (see (10.1))

$$\mathbf{\Pi}_n(\mathcal{V}, \alpha) := (\Pi_n \mathcal{V}, \alpha) , \quad \mathbf{\Pi}_n^{\perp}(\mathcal{V}, \alpha) := (\Pi_n^{\perp}\mathcal{V}, 0) , \quad \forall (\mathcal{V}, \alpha) . \tag{10.18}$$

At this point we have to prove that the iterative scheme defined in (10.17) is rapidly converging. By definition we have that $L_n = d_{i,\alpha}\mathcal{F}(\tilde{i}_n)$, then we can write

$$\mathcal{F}(U_{n+1}) = \mathcal{F}(\tilde{U}_n) + L_n H_{n+1} + \mathcal{Q}(\tilde{U}_n, H_{n+1}) \tag{10.19}$$

where

$$Q(\tilde{U}_n, H_{n+1}) := \mathcal{F}(\tilde{U}_n + H_{n+1}) - \mathcal{F}(\tilde{U}_n) - L_n H_{n+1}\,. \tag{10.20}$$

By the definition of H_{n+1} in (10.17) and using the definition of $\mathbf{\Pi}_n$ given in (10.18) we have

$$\begin{aligned}\mathcal{F}(U_{n+1}) &= \mathcal{F}(\tilde{U}_n) - L_n \mathbf{\Pi}_n \mathbf{T}_n \Pi_n F(\tilde{U}_n) + \mathcal{Q}(\tilde{U}_n, H_{n+1})\\ &= F(\tilde{U}_n) - L_n \mathbf{T}_n \Pi_n F(\tilde{U}_n) + L_n \mathbf{\Pi}_n^{\perp} \mathbf{T}_n \Pi_n F(\tilde{U}_n) + \mathcal{Q}(\tilde{U}_n, H_{n+1})\\ &= F(\tilde{U}_n) - \Pi_n L_n \mathbf{T}_n \Pi_n F(\tilde{U}_n) + (L_n \mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp} L_n)\mathbf{T}_n \Pi_n F(\tilde{U}_n) + \mathcal{Q}(\tilde{U}_n, H_{n+1})\\ &= \Pi_n^{\perp}\mathcal{F}(\tilde{U}_n) + \mathcal{R}_n + \mathcal{S}_n + \mathcal{Q}_n(\tilde{U}_n, H_{n+1})\end{aligned} \tag{10.21}$$

where

$$\mathcal{R}_n := (L_n \mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp} L_n)\mathbf{T}_n \Pi_n \mathcal{F}(\tilde{U}_n), \qquad \mathcal{S}_n := -\Pi_n (L_n \mathbf{T}_n - \mathbb{1})\Pi_n \mathcal{F}(\tilde{U}_n)\,. \tag{10.22}$$

Thanks to (5.2), (4.4), (10.3), (10.9) we have $\forall \zeta \in \Omega \times [\varepsilon_1, \varepsilon_2], \quad p \geq \mathfrak{p}_0$

$$\|\mathcal{F}(\tilde{U}_n)\|_p^{k_0,\gamma} \leq_p \|\mathcal{F}(U_0)\|_p^{k_0,\gamma} + \|\mathcal{F}(\tilde{U}_n) - \mathcal{F}(U_0)\|_p^{k_0,\gamma} \leq_p \mu + \|\tilde{W}\|_{p+\sigma_1}^{k_0,\gamma}\,, \tag{10.23}$$

and, by (10.9) and (10.8)

$$\gamma^{-1}\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq 1\,. \tag{10.24}$$

In order to prove that the scheme is rapidly convergent we have to prove that $\mathcal{F}(U_{n+1})$ and $W_{n+1} := \tilde{W}_n + H_{n+1}$ decrease fastly, for this reason (recalling Definition (10.19), and (10.17)) we start by proving some estimate for $H_{n+1}, \mathcal{Q}(\tilde{U}_n, H_{n+1})$, $\mathcal{S}_n$ and $\mathcal{R}_n$.

- **Estimates of** H_{n+1}. By (10.17), (2.11), (10.14), (10.15) and (10.9) we have

$$\begin{aligned}\|H_{n+1}\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} &\leq \|\mathbf{\Pi}_n \mathbf{T}_n \Pi_n \mathcal{F}(\tilde{U}_n)\|_p^{k_0,\gamma}\\ &\leq_{\mathfrak{p}_0+b_1} \gamma^{-1} K_n^{\sigma_1}\Big(\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} + K_n^{\nu(\mathtt{b})+2\sigma_1}\|\tilde{\mathcal{V}}_n\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma}\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma}\Big)\\ &\leq_{\mathfrak{p}_0+b_1} \gamma^{-1} K_n^{2\sigma_1+\nu(\mathtt{b})}(\mu + \|\tilde{W}\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma})\,,\end{aligned} \tag{10.25}$$

  where the last estimate follows by (10.23). Using (10.17), (2.11), (10.14), (10.15) and (10.9) and (10.24), we get

$$\|H_{n+1}\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq \|\mathbf{\Pi}_n \mathbf{T}_n \Pi_n \mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+\sigma_1}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \gamma^{-1} K_n^{\sigma_1}\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma}\,. \tag{10.26}$$

- **Estimate of** $\mathcal{Q}(\tilde{U}_n, H_{n+1})$. Using (10.20), (4.4), (5.2), (10.9) and (2.11) we obtain the quadratic estimate, $\forall H \in E_n \times \mathbb{R}^N$

$$\|\mathcal{Q}(\tilde{U}_n, H)\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq_{\mathfrak{p}_0} \mu K_n^6 (\|\hat{\mathcal{V}}\|_{\mathfrak{p}_0}^{k_0,\gamma})^2\,, \quad \hat{\mathcal{V}} \in E_n\,. \tag{10.27}$$

Then the term $\mathcal{Q}(\tilde{U}_n, H_{n+1})$ defined in (10.20) satisfies, by (10.26), (10.27) and $\mu\gamma^{-1} \leq 1$

$$\|Q(\tilde{U}_n, H_{n+1})\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq_{\mathfrak{p}_0} \gamma^{-1} K_n^{2\sigma_1+6} \left(\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma}\right)^2 . \tag{10.28}$$

- **Estimate of $\mathcal{S}_n$.** According to (5.74), we revrite the term $\mathcal{S}_n$ in 10.22 as

$$\mathcal{S}_n = -\Pi_n(L_n\mathbf{T}_n - \mathbb{1})\Pi_n\mathcal{F}(\tilde{U}_n) = -\mathcal{S}_n^{(1)} - \mathcal{S}_{n,\omega} - \mathcal{S}_{n,\omega}^{\perp}$$

where

$$\mathcal{S}_n^{(1)} = \Pi_n\mathcal{P}(\tilde{i}_n)\Pi_n\mathcal{F}(\tilde{U}_n), \quad \mathcal{S}_{n,\omega} = \Pi_n\mathcal{P}_\omega(\tilde{i}_n)\Pi_n\mathcal{F}(\tilde{U}_n), \quad \mathcal{S}_{n,\omega}^{\perp} = \Pi_n\mathcal{P}_\omega^{\perp}(\tilde{i}_n)\Pi_n\mathcal{F}(\tilde{U}_n) .$$

Using (10.2), (10.9), (10.8) and (10.24) we have

$$\begin{aligned}\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+\sigma_1}^{k_0,\gamma} &\leq \|\Pi_n\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+\sigma_1}^{k_0,\gamma} + \|\Pi_n^{\perp}\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+\sigma_1}^{k_0,\gamma} \\ &\leq K_n^{\sigma_1}\left(\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} + K_n^{-b_1}\|\|\mathcal{F}(\tilde{U}_n)\|_{p}^{k_0,\gamma}\right) .\end{aligned} \tag{10.29}$$

Hence by (10.29), using the bounds (5.75), (5.76), (5.77) , (5.78), and by (10.23), (10.2) we obtain

$$\begin{aligned}\|\mathcal{S}_n^{(1)}\|_{\mathfrak{p}_0}^{k_0,\gamma} &\leq_{\mathfrak{p}_0+b_1} \gamma^{-1}K_n^{2\sigma_1}\left(\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} + K_n^{-b_1}\|\mathcal{F}(\tilde{U}_n)\|_{p}^{k_0,\gamma}\right)\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} \\ &\leq_{\mathfrak{p}_0+b_1} \gamma^{-1}K_n^{2\sigma_1}\left[\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} + K_n^{\sigma_1-b_1}(\mu + \|\tilde{W}_n\|_{p}^{k_0,\gamma})\right]\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} \end{aligned} \tag{10.30}$$

$$\|\mathcal{S}_{n,\omega}\|_{\mathfrak{p}_0} \leq_{\mathfrak{p}_0+b_1} \mu\gamma^{-2}N_{n-1}^{-a}K_n^{\sigma_1}\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} \tag{10.31}$$

$$\begin{aligned}\|\mathcal{S}_{n,\omega}^{\perp}\|_{\mathfrak{p}_0}^{k_0,\gamma} &\leq_{\mathfrak{p}_0+b_1} \gamma^{-1}K_n^{2\sigma_1+\nu(\mathtt{b})-b_1}\left(\|\mathcal{F}(\tilde{U}_n)\|_{p}^{k_0,\gamma} + \mu\|\tilde{\mathcal{V}}_n\|_{p}^{k_0,\gamma}\right) \\ &\leq_{\mathfrak{p}_0+b_1} \gamma^{-1}K_n^{3\sigma_1+\nu(\mathtt{b})-b_1}\left(\mu + \|\tilde{W}_n\|_{p}^{k_0,\gamma}\right) .\end{aligned} \tag{10.32}$$

- **Estimate of $\mathcal{R}_n$** For $H := (\hat{\mathcal{V}}, \hat{\alpha})$ we have

$$\left(L_n\mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp}L_n\right)H = \mu[d_iX_P(\tilde{i}_n), \Pi_n^{\perp}]\hat{\mathcal{V}} = \mu[\Pi_n, d_iX_P(\tilde{i}_n)]\hat{\mathcal{V}} \quad H \in E_n \times \mathbb{R}^N$$

where X_P is the Hamiltonian vector field of the perturbation P defined in (1.44) (see (4.4)). Hence from (5.1) and (10.2), recalling the definition of σ_1 in (10.3) we obtain

$$\|\left(L_n\mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp}L_n\right)H\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \mu K_n^{-b_1+\sigma_1+3}\left(\|\hat{\mathcal{V}}\|_{p}^{k_0,\gamma} + \|\tilde{\mathcal{V}}_n\|_{p}^{k_0,\gamma}\|\hat{\mathcal{V}}\|_{\mathfrak{p}_0+3}^{k_0,\gamma}\right) . \tag{10.33}$$

Hence, $\mathcal{R}_n$ defined in (10.22), using (10.33),(10.14), (10.2), (10.8), (10.9), and (10.24) we have

$$\begin{aligned}\|\mathcal{R}_n\|_{\mathfrak{p}_0}^{k_0,\gamma} &\leq \|(L_n\mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp}L_n)(\mathbf{T}_n\Pi_n\mathcal{F}(\tilde{U}_n))\|_{\mathfrak{p}_0}^{k_0,\gamma} \\ &\leq_{\mathfrak{p}_0+b_1} \mu K_n^{-b_1+\sigma_1+3}(\|\mathbf{T}_n\Pi_n\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} + \|\mathcal{V}_n\|_{\mathfrak{p}_0}^{k_0,\gamma}\|\mathbf{T}_n\Pi_n\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+3}^{k_0,\gamma}) \\ &\leq_{\mathfrak{p}_0+b_1} K_n^{-b_1+2\sigma_1+3+\nu(\mathtt{b})}(\mu\gamma^{-1}\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} + \mu\|\tilde{\mathcal{V}}_n\|_{\mathfrak{p}_0}^{k_0,\gamma}) \\ &\leq_{\mathfrak{p}_0+b_1} K_n^{-b_1+3\sigma_1+3+\nu(\mathtt{b})}(\mu + \|\tilde{W}_n\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma})\end{aligned} \tag{10.34}$$

where the last inequation follows by (10.23).

Using the estimates above we can estimate $\mathcal{F}(U_{n+1})$ as proved in the following Lemma.

Lemma 10.3. *For all* $\zeta \in \mathcal{N}(\mathcal{G}_{n+1}, 2\gamma K_n^{-r(\tau+2)})$ *we have, setting* $\nu_2 := \nu(\mathtt{b}) + 3\sigma_1 + 3$

$$\|\mathcal{F}(U_{n+1})\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \frac{1}{\gamma} K^{\nu_2 - b_1}\left(\mu + \|\tilde{W}_n\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma}\right) + 2\frac{K_n^{2\sigma_1+6}}{\gamma}(\|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma})^2$$
$$+ K_{n-1}^{-ra} K_n^{\sigma_1} \frac{\mu}{\gamma^2} \|\mathcal{F}(\tilde{U}_n)\|_{\mathfrak{p}_0}^{k_0,\gamma} \tag{10.35}$$

$$\|W_1\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \gamma^{-1}\mu, \quad \|W_{n+1}\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} K_n^{\nu(\mathtt{b})+2\sigma_1}\gamma^{-1}(\mu + \|\tilde{W}_n\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma}), \quad n \geq 1\,. \tag{10.36}$$

Proof. The estimate (10.35) on $\mathcal{F}(U_{n+1})$ follows by (10.21), (10.30), (10.31), (10.32), (10.34), (10.8), (10.9). By (10.17) and (10.14) we have

$$\|W_1\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} = \|H_1\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \gamma^{-1}\|\mathcal{F}(U_0)\|_{p+\sigma_1}^{k_0,\gamma} \leq_{\mathfrak{p}_0+b_1} \mu\gamma^{-1}\,.$$

Finally the stimate (10.36) follows by $W_{n+1} := \tilde{W}_n + H_{n+1}$ and (10.25). □

Lemma 10.4. *For all* $\zeta \in \mathcal{N}(\mathcal{G}_{n+1}, 2\gamma K_n^{-r(\tau+2)})$ *we have*

$$\|\mathcal{F}(U_{n+1})\|_{\mathfrak{p}_0}^{k_0,\gamma} \leq C_\star \mu K_n^{-\mathtt{a}_1}\,, \qquad \|W_{n+1}\|_{\mathfrak{p}_0+b_1}^{k_0,\gamma} \leq C_\star \mu\gamma^{-1}K_n^{\nu_1} \tag{10.37}$$

$$\|H_1\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1}^{k_0,\gamma} \leq C\mu\gamma^{-1}\,, \qquad \|H_{n+1}\|_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1}^{k_0,\gamma} \leq_{\mathfrak{p}_0} \mu\gamma^{-1}K_n^{\nu(\mathtt{b})+2\sigma_1}K_{n-1}^{-\mathtt{a}_1}\,, \quad n \geq 1\,. \tag{10.38}$$

Proof. Note that, by (10.16), if $\zeta \in \mathcal{N}(\mathcal{G}_{n+1}, 2\gamma K_n^{-r(\tau+2)})$ then $\zeta \in \mathcal{N}(\mathcal{G}_{n+1}, \gamma K_{n-1}^{-r(\tau+2)})$. Hence (10.12) and $(P3)_n$ hold. The first inequality in (10.37) follows by (10.35), $(P2)_n$, $(P3)_n$, $\gamma^{-1} = K_0 \leq K_n$, $\mu\gamma^{-2} \leq c$ small, and by (10.4), (10.5), (10.6), (10.7). For $n = 0$ we use also (10.8).

The second inequality in (10.37) follows by (10.36), $(P3)_n$, (10.5), K_0 large enough.

Since $H_1 = W_1$ the first inequality in (10.38) follows by (10.36). For $n \geq 1$, the estimate (10.38) follows by (10.26), (10.12) and (10.2). □

Lemma 10.5. ***Extension***. *There is a* C^{k_0}*-smooth function* $\tilde{H}_{n+1}$ *defined on the whole* $\mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$ *such that*

$$\tilde{H}_{n+1} = H_{n+1}, \quad \forall \zeta \in \mathcal{N}(\mathcal{G}_{n+1}, \gamma K_n^{-r(\tau+2)})\,, \tag{10.39}$$

and (10.10) *holds also at the step* $n+1$.

Proof. Since the function H_{n+1} is defined for all $\zeta \in \mathcal{N}(\mathcal{G}_{n+1}, \gamma K_n^{-r(\tau+2)})$ and it is the extension of $\tilde{H}_{n+1}$ a good strategy is to consider the cut-off functions. Hence let ψ_{n+1} be a C^∞ cut-off functions satisfying

$$0 \leq \psi_{n+1} \leq 1, \quad \psi_{n+1}(\zeta) = 1, \quad \forall \zeta \in \mathcal{N}(\mathcal{G}_{n+1}, \gamma K_n^{-r(\tau+2)})\,,$$
$$supp(\psi_{n+1}) \subseteq \mathcal{N}(\mathcal{G}_{n+1}, 2\gamma K_n^{-r(\tau+2)})\,,$$
$$|\partial_\zeta^k \psi_{n+1}(\zeta)| \leq C(k)\left(\gamma^{-1}K_n^{r(\tau+2)}\right)^{|k|}\,, \quad \forall k \in \mathbb{N}^{N+1}\,.$$

Then we define

$$\tilde{H}_{n+1}(\zeta) := \begin{cases} \psi_{n+1}(\zeta) H_{n+1}(\zeta) & \forall \zeta \in \mathcal{N}(\mathcal{G}_{n+1}, 2\gamma K_n^{-r(\tau+2)}) \\ 0 & \forall \zeta \notin \mathcal{N}(\mathcal{G}_{n+1}, 2\gamma K_n^{-r(\tau+2)}) \,. \end{cases}$$

So (10.39) holds and we have the estimate

$$\|\tilde{H}_{n+1}\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} \leq_{\mathfrak{p}_0} K_n^{r(\tau+2)k_0} \|H_{n+1}\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} .$$

The first inequality in (10.10) for $n = 0$ follows by (10.38), while for $n \geq 1$, and also at the step $n+1$, we deduce the estimate (10.10) by the definition of $\mathtt{a}_2$ in (10.4) and by (10.38). □

We now define

$$\tilde{W}_{n+1} := \tilde{W}_n + \tilde{H}_{n+1} \,, \quad \tilde{U}_{n+1} := \tilde{U}_n + \tilde{H}_{n+1} = U_0 + \tilde{W}_n + \tilde{H}_{n+1} := U_0 + \tilde{W}_{n+1}$$

which are defined for all $\zeta \in \mathbb{R}^N \times [\varepsilon_1, \varepsilon_2]$ and satisfy

$$\tilde{W}_{n+1} := W_{n+1} \,, \quad \tilde{U}_{n+1} := U_{n+1} \,, \qquad \forall \zeta \in \mathcal{N}(\mathcal{G}_{n+1}, \gamma K_n^{-r(\tau+2)}).$$

Therefore $(P2)_{n+1}, (P3)_{n+1}$ are proved by Lemma 10.4. In addition by (10.10), which has been proved up to the step $n+1$ in Lemma 10.5, we have

$$\|\tilde{W}_{n+1}\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} \leq \sum_{k=1}^{n+1} \|\tilde{H}_k\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} \leq C_\star K_0^{pk_0(\tau+2)} \mu \gamma^{-1}$$

and thus (10.9) holds also at the step $n+1$. So the proof of Theorem 10.2 is completed.

10.1 Proof of Theorem 4.1

We now have to prove that the scheme in Theorem 10.2 converges when $n \to \infty$. Let $\gamma = \mu^a$ with $a \in (0, a_0)$ and $a_0 := 1/(2+\tau_2)$. Note that the smallness condition defined in (10.8) is satisfied for $0 < \mu < \mu_0$ small enough and also Theorem 10.2 holds. Thanks to (10.10) we have that the sequence of functions $\tilde{U}_n := (\tilde{i}_n, \tilde{\alpha}_n)$ is a Cauchy sequence in $\|\cdot\|^{k_0,\gamma}_{\mathfrak{p}_0}$, (see (2.9)) hence we can define its limit function as follows

$$U_\infty := (i_\infty, \alpha_\infty) = (\theta, 0, 0, \omega) + W_\infty \,, \quad W_\infty : \Omega \times [\varepsilon_1, \varepsilon_2] \to H^{\mathfrak{p}_0}_\theta \times H^{\mathfrak{p}_0}_\theta \times H^{\mathfrak{p}_0}_{x,\theta} \times \mathbb{R}^N \,, \quad W_\infty := \lim_{n\to\infty} \tilde{W}_n \,.$$

Then, using (10.9) and (10.10) we obtain that

$$\|U_\infty - U_0\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} \leq C_\star \mu \gamma^{-1} K_0^{pk_0(\tau+2)} \,, \quad \|U_\infty - \tilde{U}_n\|^{k_0,\gamma}_{\mathfrak{p}_0+\nu(\mathtt{b})+\sigma_1} \leq C\mu\gamma^{-1} K_n^{-\mathtt{a}_2} \quad \forall n \geq 1 \,. \tag{10.40}$$

In addition by Theorem 10.2, recalling the Definitions (9.94), (9.48) and (9.91), we deduce that $\mathcal{F}(\zeta, U_\infty(\zeta)) = 0$ for all ζ belonging to

$$\bigcap_{n\geq 0} \mathcal{G}_n = \Lambda \cap \bigcap_{n\geq 1} \boldsymbol{\Lambda}^\gamma_n(\tilde{i}_{n-1}) = \Lambda \cap \left[\bigcap_{n\geq 1} \Lambda^\gamma_n(\tilde{i}_{n-1}) \right] \bigcap \left[\bigcap_{n\geq 1} \Lambda^{\gamma,I}_n(\tilde{i}_{n-1}) \right] , \tag{10.41}$$

where $\mathtt{\Lambda} := \Omega \times [\varepsilon_1, \varepsilon_2]$. Therefore, by (10.40), for $n = 0$ and since $K_0 = \gamma^{-1}$ (see (10.8)) we deduce the estimates (4.6) and (4.7) with $k_1 := rk_0(\tau + 2)$.

We now have to provide the caractherization of $\mathcal{C}^\gamma_\infty$ in (4.10), in order to do that we firstly consider the following set

$$\mathcal{G}_\infty := \mathtt{\Lambda} \cap \left[\bigcap_{n\geq 1} \mathtt{\Lambda}_n^{2\gamma}(i_\infty) \right] \bigcap \left[\bigcap_{n\geq 1} \mathtt{\Lambda}_n^{2\gamma, I}(i_\infty) \right] . \tag{10.42}$$

Lemma 10.6. *Let* $\mathcal{G}_\infty$ *as in* (10.42) *and* $\mathcal{G}_n$ *as in* (10.11). *Then*

$$\mathcal{G}_\infty \subseteq \bigcap_{n\geq 0} \mathcal{G}_n .$$

Proof. By (10.40) and (10.8) we have

$$\mu\gamma^{-1} C(p) N_0^\tau \|i_\infty - i_0\|^{k_0,\gamma}_{\mathtt{p}_0+\nu(\mathtt{b})+\sigma_1} \leq \mu^2 \gamma^{-2} C(p) C_\star K_0^{r\tau} K_0^{rk_0(\tau+2)} \leq \gamma$$
$$\mu\gamma^{-1} C(p) N_{n-1}^\tau \|i_\infty - \tilde{i}_{n-1}\|^{k_0,\gamma}_{\mathtt{p}_0+\nu(\mathtt{b})+\sigma_1} \leq \mu^2 \gamma^{-2} C(p) C K_{n-1}^{r\tau} K_n^{-\mathtt{a}_2} \leq \gamma , \quad \forall n \geq 2$$

where τ_2 is defined in (10.8) and by (10.4) and that $\mathtt{a}_2 \geq r\tau + 2/3$ (defined in (10.4)) we have that $\tau_2 > \mathtt{a}_1 > 3(rk_0(\tau+2) + r\tau)/2$. Therefore Theorem 9.6 implies

$$\mathtt{\Lambda}_n^{2\gamma}(i_\infty) \subset \mathtt{\Lambda}_n^{\gamma}(\tilde{i}_{n-1}) , \quad \forall n \geq 1,$$

where $\mathtt{\Lambda}_n^\gamma$ is defined in (9.48). Using the definition of $\mathtt{\Lambda}_n^{\gamma,I}$ in (9.91) and similar arguments we have

$$\mathtt{\Lambda}_n^{2\gamma,I}(i_\infty) \subset \mathtt{\Lambda}_n^{\gamma,I}(\tilde{i}_{n-1}) , \quad \forall n \geq 1.$$

So the lemma is proved. □

We now can define the final eigenvalues as follows

$$\lambda_j^\infty := j\left(\frac{2}{15} j^4 \varepsilon^4 - \frac{1}{3} j^2 \varepsilon^2 + 1\right) + m_1^\infty j + r_j^\infty , \quad j \in \mathbb{N} \setminus \mathbb{S}, j \neq 0 , \tag{10.43}$$

where

$$m_1^\infty := m_1(i_\infty) , \quad r_j^\infty := \lim_{n\to 0} \tilde{r}_j^n(i_\infty) , \quad \forall j \in \mathbb{N} \setminus \mathbb{S} , \ j \neq 0 \tag{10.44}$$

where m_1 is defined in (8.78) and $\tilde{r}_j^n$ are defined in (9.42). Note that by (9.43) the sequence $(\tilde{r}_j^n(i_\infty))_{n\in\mathbb{N}}$ is a Cauchy sequence in $|\cdot|^{k_0,\gamma}$ defined in (2.7). As a consequence its limit function $r_j^\infty(\omega, \varepsilon)$ is well defined, it is k_0-times differentiable and satisfies

$$|r_j^\infty - \tilde{r}_j^n(i_\infty)|^{k_0,\gamma} \leq C\mu\gamma^{-1} N_n^{k_0(\tau+2)} N_{n-1}^{-\mathtt{a}} , \quad n \geq 0 . \tag{10.45}$$

Note that, since $\tilde{r}_j^0(i_\infty) = 0$ and $K_0 = \gamma^{-1}$, one has

$$|r_j^\infty|^{k_0,\gamma} \leq C\mu\gamma^{-1} K_0^{rk_0(\tau+2)+1}$$

and (4.9) holds with $k_1 = rk_0(\tau + 2) + 1$.

We are now ready to consider the set $\mathcal{C}^\gamma_\infty$ defined in (4.10).

Lemma 10.7. *Let $\mathcal{C}^\gamma_\infty$ be the set defined in* (4.10) *and $\mathcal{G}_\infty$ be the set defined in* (10.42)*. Then*

$$\mathcal{C}^\gamma_\infty \subseteq \mathcal{G}_\infty\,.$$

Proof. Thanks to (10.42) we have only to prove that $\mathcal{C}^\gamma_\infty \subseteq \Lambda^{2\gamma}_n(i_\infty)\,,\ \ \forall n \in \mathbb{N}$. We prove it by induction. For $n = 0$ the inclusion is verified because $\Lambda^{2\gamma}_0(i_\infty) = \Omega \times [\varepsilon_1, \varepsilon_2] = \Lambda$. Assume that $\mathcal{C}^\gamma_\infty \subseteq \Lambda^{2\gamma}_n(i_\infty)$ We shall prove that $\mathcal{C}^\gamma_\infty \subseteq \Lambda^{2\gamma}_{n+1}(i_\infty)$. By Theorem 9.6 we have that $\tilde{\lambda}^n_j(i_\infty)(\zeta) = \lambda^n_j(i_\infty)(\zeta)\,,\ \ \forall \zeta \in \Lambda^{2\gamma}_n(i_\infty)$. Hence $\forall \zeta \in \mathcal{C}^\gamma_\infty \subseteq \Lambda^{2\gamma}_n(i_\infty)$, by (9.33), (10.43) and (10.45) we obtain

$$|(\lambda^n_j - \lambda^n_{j'})(i_\infty) - (\lambda^\infty_j - \lambda^\infty_{j'})| \leq C\mu\gamma^{-1} N_n^{k_0(\tau+2)} N_{n-1}^{-\mathfrak{a}}\,,$$

and therefore, using (4.10) with $j \neq j'$ we have

$$\begin{aligned}
|\omega \cdot l + \lambda^n_j(i_\infty) - \lambda^n_{j'}(i_\infty)| &\geq |\omega \cdot l + \lambda^\infty_j - \lambda^\infty_{j'}| - |(\lambda^n_j - \lambda^n_{j'})(i_\infty) - (\lambda^\infty_j - \lambda^\infty_{j'})| \\
&\geq |\omega \cdot l + \lambda^\infty_j - \lambda^\infty_{j'}| - C\mu\gamma^{-1} N_n^{k_0(\tau+2)} N_{n-1}^{-\mathfrak{a}} \\
&\geq 4\gamma |j^3 - {j'}^3|\, \langle l \rangle^{-\tau} - C\mu\gamma^{-1} N_n^{k_0(\tau+2)} N_{n-1}^{-\mathfrak{a}} |j^3 - {j'}^3| \\
&\geq 2\gamma |j^3 - {j'}^3|\, \langle l \rangle^{-\tau}\,, \quad \forall |l| \leq N_n\,,
\end{aligned}$$

provided $\mu\gamma^{-2} \leq C N_n^{-k_0(\tau+2)} N_{n-1}^{\mathfrak{a}}\,,\quad \forall n \geq 0$, which holds true by (9.21) and (10.8). Hence we have proved that $\mathcal{C}^\gamma_\infty \subseteq \Lambda^{2\gamma}_{n+1}(i_\infty)$. One can prove similarly that $\mathcal{C}^\gamma_\infty \subseteq \Lambda^{2\gamma, I}_n(i_\infty)\,,\ \ \forall n \in \mathbb{N}$ which proves the lemma. □

By Lemmas 10.6 and 10.7 we have the following result

Lemma 10.8. *Let $\mathcal{C}^\gamma_\infty$ as in* (4.10) *and $\mathcal{G}_n$ as in* (10.11)*. Then*

$$\mathcal{C}^\gamma_\infty \subseteq \bigcap_{n \geq 0} \mathcal{G}_n\,.$$

Appendix A

Approximate model PDEs of water waves

A.1 Transformation laws of Hamiltonian systems

We now recall some well known properties (that can be found e.g. in [38]) of Hamiltonian systems. Let Y be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, $\mathcal{W}$ be a non degenerate symplectic two-form and H be an Hamiltonian. Then the associated Hamiltonian vector field X_H is defined by

$$\mathcal{W}\left(X_H(v), \cdot\right) = -dH(v)(\cdot)\,. \tag{A.1}$$

If $\mathcal{W}(a, b) = \left\langle J^{-1}a, b\right\rangle$, where J^{-1} is a non degenerate and anti-symmetric linear operator, then the condition (A.1) is equivalent to

$$\left\langle J^{-1}X_H(v), \cdot\right\rangle = -\left\langle \nabla_v H(v), \cdot\right\rangle\,,$$

that is $J^{-1}X_H(v) = -\nabla_v H(v)$, and

$$X_H(v) = -J\nabla_v H(v). \tag{A.2}$$

Therefore the associated Hamiltonian system can be written as

$$v_t + J\nabla_v H(v) = 0\,.$$

In the next Lemma we discuss how a vector field transforms under a linear change of variables.

Lemma A.1. *Let $X(v)$ be a vector field. Consider a linear change of variables $w = \Phi v$. Then the differential equation $v_t = X(v)$ transforms in*

$$w_t = \Phi \circ X \circ \Phi^{-1} w.$$

If the vector field $X(v)$ is Hamiltonian we have the following Lemma.

Lemma A.2. *Let $-J\nabla_v H(v)$ be an Hamiltonian vector field, whose Hamiltonian is H. Under the linear change of variables $w = \Phi v$, the differential equation $v_t = -J\nabla_v H(v)$ transforms*

$$w_t = -J_1 \nabla_w K(w)\,, \quad \textit{where} \quad J_1 := \Phi J \Phi^T\,,$$

and K is the Hamiltonian given by

$$K(w) = H(\Phi^{-1} w)\,.$$

A.2 Craig-Sulem-Zakharov's Hamiltonian formulation

In this Section we present the computations that W. Craig gave to us (in a private communication [26]), in order to arrive at the system (1).

This system is derived from the Hamiltonian formulation of the water waves equations introduced by Zakharov in [65] and Craig-Sulem in [33]. Let us precisely describe this system. We consider the evolution of a perfect, incompressible, irrotational fluid under the action of gravity which occupies the free boundary region

$$S_\eta := \{(x, y) \in \mathbb{R} \times \mathbb{R} : -h \le y \le \eta(x)\}\ .$$

We refer to the classical book of Stoker [63]. The unknowns of the problems are the free surface $y = \eta(x)$, and the velocity potential $\Phi : S_\eta \to \mathbb{R}$, i.e. the irrotational velocity field $\nabla\Phi$. The gravity water-waves problem can be written as follows

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = 0\,, & \text{at } y = \eta(x)\,, \quad \text{Bernoulli condition} \\ \triangle\Phi = 0 & \text{in } S_\eta\,, \quad \text{incompressibility} \\ \partial_y \Phi = 0 & \text{at } y = -h\,, \quad \text{impermeability} \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } y = \eta(x)\,, \quad \text{kinematic condition}\,, \end{cases} \tag{A.3}$$

where $g > 0$ is the acceleration of gravity. In addition we consider periodic boundary conditions:

$$\eta(x + 2\pi) = \eta(x)\,,\ \ \Phi(x + 2\pi, y) = \Phi(x, y)\,, \quad \forall x \in \mathbb{R}\,. \tag{A.4}$$

It was observed by Zakharov in [65] that the system (A.3), is an infinite dimensional Hamiltonian system in the variables η (the profile of the fluid) and $\xi(x) := \Phi(x, \eta(x))$, that is the value of the velocity potential Φ restricted to the free boundary. The first observation is that $\eta(x)$ and $\xi(x)$ *uniquely* determine the velocity potential Φ in the whole fluid domain S_η, solving the elliptic problem

$$\begin{cases} \triangle\Phi = 0 & \text{in } \ S_\eta \\ \partial_y \Phi = 0 & \text{at } \ y = -h \\ \Phi = \xi & \text{at } \ y = -\eta(x)\,, \end{cases} \tag{A.5}$$

with the periodicity conditions (A.4).

In [65] and [32] it is proved that the system (A.3) can be written in the variables (η, ξ) as the following Hamiltonian system

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = -J \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.6}$$

where δ_ξ , δ_η denote the L^2_x gradient, the Hamiltonian H is

$$H(\eta, \xi) = \frac{1}{2} \int_{\mathbb{T}} \left(\xi \cdot G(\eta)\xi + g\eta^2 \right) dx \tag{A.7}$$

and

$$G(\eta) := (\partial_y \Phi)(x, \eta(x)) - (\partial_x \Phi)(x, \eta(x)) \cdot \partial_x \eta(x)$$

is the so called the Dirichlet-Neumann operator. The first term in the Hamiltonian (A.7) represents the kinetic energy of the fluid, and the second term the potential energy.

A.3 Derivation of system (1)

We now present the derivation of system (1) from the Hamiltonian system (A.6).

In [32] it has been proved that the Dirichlet-Neumann operator admits the following Taylor expansion

$$\begin{aligned} G(\eta)\xi = D\tanh(hD)\xi + (D\eta D - G^{(0)}\eta G^{(0)})\xi + \\ + \frac{1}{2}(G^{(0)}\eta^2 D^2 + D^2\eta^2 G^{(0)} - 2G^{(0)}\eta G^{(0)}\eta G^{(0)})\xi + R^{(3)} \end{aligned}$$

where $D := -i\partial_x$, $G^{(0)}(D) := D\tanh(hD)$ and $R^{(3)}$ collects all the terms of order at least four in the variables η and ξ. Using this expansion the Hamiltonian (A.7) reads

$$\begin{aligned} H(\eta, \xi) =& \frac{1}{2} \int_{\mathbb{T}} \Big[\xi \cdot \Big(D\tanh(hD)\xi + (D\eta D - G^{(0)}\eta G^{(0)})\xi \; + \\ &+ \frac{1}{2}(G^{(0)}\eta^2 D^2 + D^2\eta^2 G^{(0)} - 2G^{(0)}\eta G^{(0)}\eta G^{(0)})\xi \Big) + R^{(3)} + g\eta^2 \Big] dx \\ =& \frac{1}{2} \int_{\mathbb{T}} \Big[\xi \cdot D\tanh(hD)\xi + \xi \cdot (D\eta D - G^{(0)}\eta G^{(0)})\xi \; + \\ &+ \frac{1}{2}\xi \cdot G^{(0)}\eta^2 D^2 \xi + \frac{1}{2}\xi \cdot D^2 \eta^2 G^{(0)}\xi - \xi \cdot G^{(0)}\eta G^{(0)}\eta G^{(0)}\xi + R^{(3)} + g\eta^2 \Big] dx \, . \end{aligned} \tag{A.8}$$

We now introduce the long wave and the small amplitude regime scaling taking

$$\eta = \varepsilon^2 \eta', \qquad \xi = \varepsilon \xi', \qquad \varepsilon x = X, \qquad D_x = \varepsilon D_X \, . \tag{A.9}$$

After this rescaling the Fourier multiplier $\tanh(hD_X)$ reads

$$\tanh(\varepsilon h D_X) = \varepsilon h D_X - \varepsilon^3 \frac{h^3}{3} D_X^3 + \varepsilon^5 \frac{2h^5}{15} D_X^5 + O(\varepsilon^7) \, . \tag{A.10}$$

The transformation (A.9) changes the matrix J in (A.6) only for a scalar factor. Denoting by A the matrix that represents the change of variables (A.9)

$$A := \begin{pmatrix} \varepsilon^{-2} & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$$

then (see Lemma A.2)

$$J_1 := AJA^T = \varepsilon^{-3} J.$$

Introducing the transformation (A.9) into the Hamiltonian (A.8), thanks to (A.10) and Lemma A.2, we get

$$\begin{aligned} H = \frac{1}{2}\int_{\mathbb{T}} \Bigg(& \varepsilon^3 \xi' D_X \tanh(h\varepsilon D_X)\xi' + \varepsilon^4 {\eta'}^2 g + \varepsilon^6 \xi' D_X \eta' D_X \xi' \\ & - \varepsilon^6 \xi' D_X \tanh(h\varepsilon D_X)\eta' D_X \tanh(h\varepsilon D_X)\xi' \\ & + O(\varepsilon^9)\Bigg)\frac{dX}{\varepsilon} \\ = \frac{1}{2}\int_{\mathbb{T}} \Bigg[& \varepsilon^3 \xi' D_X \Big(\varepsilon h D_X - \varepsilon^3 \frac{h^3}{3} D_X^3 + \varepsilon^5 \frac{2h^5}{15} D_X^5\Big)\xi' + \varepsilon^6 \xi' D_X \eta' D_X \xi' \\ & - \varepsilon^6 \xi' D_X \Big(\varepsilon h D_X - \varepsilon^3 \frac{h^3}{3} D_X^3 + \varepsilon^5 \frac{2h^5}{15} D_X^5\Big)\eta' D_X \Big(\varepsilon h D_X \\ & \qquad - \varepsilon^3 \frac{h^3}{3} D_X^3 + \varepsilon^5 \frac{2h^5}{15} D_X^5\Big)\xi' + \varepsilon^4 {\eta'}^2 g + O(\varepsilon^9)\Bigg]\frac{dX}{\varepsilon} \\ = \frac{1}{2}\int_{\mathbb{T}} \Bigg[& \varepsilon^3 \Big(h\xi' D_X^2 \xi' + \eta'^2 g\Big) - \varepsilon^5 \Big(\frac{h^3}{3}\xi' D_X^4 \xi' - \xi' D_X \left(\eta' D_X \xi'\right)\Big) \\ & - \varepsilon^7 \Big(h^2 \xi' D_X^2 \left(\eta' D_X^2 \xi'\right) - \frac{2h^5}{15}\xi' D_X^6 \xi'\Big) + O(\varepsilon^9)\Bigg] dX. \end{aligned}$$

From now on we drop the primes in our notation and we shall omit both $O(\varepsilon^9)$ and $R^{(3)}$ terms. Then, by $D_X^2 = -\partial_X^2\,,\ D_X^4 = \partial_X^4\,,\ D_X^6 = -\partial_X^6$ we rewrite the above Hamiltonian as follows

$$
\begin{aligned}
2H &= \varepsilon^3 \int_{\mathbb{T}} \Big[-h\partial_X\big((\partial_X\xi)\xi\big) + h(\partial_X\xi)^2 + g\eta^2\Big] dX + \\
&+ \varepsilon^5 \int_{\mathbb{T}} \Bigg[\frac{h^3}{3}\Big(-\partial_X(\xi(\partial_X^3\xi)) - (\partial_X^2\xi)^2 + \partial_X((\partial_X\xi)(\partial_X^2\xi))\Big) \\
&\qquad - \partial_X(\xi\eta(\partial_X\xi)) + \eta(\partial_X\xi)^2\Bigg] dX \\
&+ \varepsilon^7 \int_{\mathbb{T}} \Bigg[\frac{2h^5}{15}\Big(-\partial_X(\xi(\partial_X^5\xi)) + \partial_X((\partial_X\xi)(\partial_X^4\xi)) - \partial_X((\partial_X^2\xi)(\partial_X^3\xi)) + (\partial_X^3\xi)^2\Big) \\
&\qquad + h^2\bigg(-\partial_X\big(\xi\partial_X(\eta(\partial_X^2\xi))\big) + \partial_X\big((\partial_X\xi)\eta(\partial_X^2\xi)\big) - (\partial_X^2\xi)^2\eta\bigg)\Bigg] dX \\
&= \varepsilon^3 \int_{\mathbb{T}} h(\partial_X\xi)^2 + g\eta^2 dX + \varepsilon^5 \int_{\mathbb{T}} \left(-\frac{h^3}{3}(\partial_X^2\xi)^2 + \eta(\partial_X\xi)^2\right) dX + \\
&+ \varepsilon^7 \int_{\mathbb{T}} \left(\frac{2h^5}{15}(\partial_X^3\xi)^2 - h^2(\partial_X^2\xi)^2\eta\right) dX\,.
\end{aligned}
\tag{A.11}
$$

We now introduce the surface elevation-velocity coordinates $(\eta,\xi) \mapsto (\eta, u := \partial_X\xi)$. If we call B the matrix that represents this change of variable, that is

$$
B := \begin{pmatrix} 1 & 0 \\ 0 & \partial_X \end{pmatrix}
$$

then (see Lemma A.2)

$$
J_2 := BJ_1B^T = \varepsilon^{-3}\begin{pmatrix} 0 & \partial_X \\ \partial_X & 0 \end{pmatrix}\,. \tag{A.12}
$$

After the change of variable B the Hamiltonian in (A.11) reads (see Lemma A.2)

$$
\begin{aligned}
2H &= \varepsilon^3 \int_{\mathbb{T}} \left(hu^2 + g\eta^2\right) dX + \varepsilon^5 \int_{\mathbb{T}} \left(-\frac{h^3}{3}(\partial_X u)^2 + \eta u^2\right) dX + \\
&+ \varepsilon^7 \int_{\mathbb{T}} \left(\frac{2h^5}{15}(\partial_X^2 u)^2 - h^2(\partial_X u)^2\eta\right) dX\,.
\end{aligned}
\tag{A.13}
$$

The Hamiltonian equations corresponding to this Hamiltonian are

$$
\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = -J_2 \begin{pmatrix} \delta_\eta H \\ \delta_u H \end{pmatrix}
$$

where J_2 in defined in (A.12), i.e.

$$
\begin{aligned}
\partial_t\eta &= -\varepsilon^{-3}\partial_X\delta_u H(\eta,u) \\
\partial_t u &= -\varepsilon^{-3}\partial_X\delta_\eta H(\eta,u)\,.
\end{aligned}
$$

The Hamiltonian equations defied above are explicitly given by

$$
\begin{aligned}
\partial_t\eta &= -\partial_X\left(hu + \varepsilon^2\left(\frac{h^3}{3}\partial_X^2 u + \eta u\right) + \varepsilon^4\left(\frac{2h^5}{15}\partial_X^4 u + h^2\partial_X(\eta\partial_X u)\right)\right) \\
\partial_t u &= -\partial_X\left(g\eta + \frac{\varepsilon^2}{2}u^2 - \varepsilon^4\frac{h^2}{2}(\partial_X u)^2\right)\,.
\end{aligned}
\tag{A.14}
$$

For simplicity we assume $h, g = 1$. Moreover we can also assume, instead of the Hamiltonian in (A.13) the Hamiltonian divided by a factor ε^{-3}, and we can redefine

$$J_2 := J := \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}.$$

With these assumptions the Hamiltonian (A.13) (divided by ε^3) become the Hamiltonian (2), whose the corresponding equations of motion (1) are equal to (A.14) (with $h, g = 1$).

Appendix B

Asymptotic expansions

In this Section we will prove that each remainder, obtained along the descent method in Chapter 8 has always the same structure (7.22), in homogeneous components up to smoothing operator in S^{-M-1}. Moreover we provide some explicit estimates on the coefficients and the symbols in this expansion. For that we use systematically the asymptotic expansion for the composition operators (see (2.19)) in homogeneous symbols and the estimates given in Proposition 2.8 (with $\alpha = 0$). This method is slightly different by [19]. We decide to use this strategy because the homogeneous structure allows us to eliminate the out of diagonal terms up to order $-M$ by means some easy transformations. We underline that the order $-M$, at which we arrest the expansion, is a fixed constant provided by the KAM iteration in Chapter 9, see (9.24).

In what follows we shall use the norm $\|\cdot\|_p^{k_0,\gamma}$, defined in (2.8), for functions and for 2×2 matrices of functions (see Remarks 2.2), similarly we shall use the norm $|\cdot|_{m,p,0}^{k_0,\gamma}$, defined in (2.21), both for operators and for 2×2 matrices of operators (see Remarks 2.7).

B.1 Inverse of T_k

In this Section we invert operators $T_k = \mathbb{1} + \Phi_k \partial_x^{-k}$, $k = 1, ..., M+3$, where $\Phi_k := \Phi_k(x,\theta)$ are functions. By the composition formula in (2.19) we obtain

$$\Phi\partial_x^{-k}\Phi\partial_x^{-k} = \Phi\sum_{\beta=0}^{p} C(\beta)(\partial_x^{\beta}\Phi)(\partial_x^{-2k-\beta}) + r \tag{B.1}$$

where $-2k-p=-M$, $C(\beta)$ is a constant and $r\in OPS^{-M-1}$. Hence iterating this formula we have arrive to

$$(\Phi\partial_x^{-k})^{\delta} = \Phi\sum_{\beta_1=0}^{p_1}\sum_{\beta_2=0}^{p_2}...\sum_{\beta_{\delta-1}=0}^{p_{\delta-1}} C(\beta_1,...,\beta_{\delta-1})\partial_x^{\beta_{\delta-1}}(\Phi\partial_x^{\beta_{\delta-2}}(...\partial_x^{\beta_2}\Phi(\partial_x^{\beta_1}\Phi)...))\partial_x^{-\delta k-\beta_1-\beta_2-...-\beta_{\delta-1}} + r$$

$$= \sum_{j=0}^{M-k\delta} C(j)\tilde{\Phi}_{j+k\delta}\partial_x^{-k\delta-j} + r_\delta\,, \tag{B.2}$$

for some suitable constant $C(j)$ and where $r_\delta \in OPS^{-M-1}$ is the sum of r and all the pseudo-differential operators in OPS^{-M-1} generated by the composition.

Actually we are interested in $T_k = \mathbb{1} + \Phi_k\partial_x^{-k}$ where Φ_k for $k = 1, ..., M+3$ is an out of diagonal matrix, small enough, that is

$$\Phi_k := \mu \begin{pmatrix} 0 & \varphi_2^{(k)} \\ \varphi_3^{(k)} & 0 \end{pmatrix} \tag{B.3}$$

see (8.2), (8.20) and (8.37). Moreover we shall require that the functions Φ_k satisfy

$$\|\Phi_k\|^{k_0,\gamma}_{\mathfrak{p}_0+\chi(M)+\sigma} \le 1, \tag{B.4}$$

where $\chi(M)$ is a constant depending on M and $\sigma := \sigma(\tau, N, k_0)$.

Lemma B.1. *Let $T_k = \mathbb{1} + \Phi_k\partial_x^{-k}$. Then*

$$(\Phi_k\partial_x^{-k})^\delta = \sum_{j=0}^{M-k\delta} C(j)(\tilde{\Phi}_k)_{j+k\delta}\partial_x^{-k\delta-j} + \nu_{k,\delta}\,,$$

for some suitable functions $\tilde{\Phi}_k$ and constants $C(j)$. The operator $\nu_{k,\delta} \in OPS^{-M-1}$.

Moreover for $j = 0, ..., M - k\delta$,

$$\|(\tilde{\Phi}_k)_{j+k\delta}\|^{k_0,\gamma}_p \le C(p)\|\Phi_k\|^{k_0,\gamma}_{p+j}\,, \quad |\nu_{k,\delta}|^{k_0,\gamma}_{-M-1,p,0} \le C(p,M)\|\Phi_k\|^{k_0,\gamma}_{p+(\delta-1)k+2M-4k}\,. \tag{B.5}$$

Proof. We prove (B.5) by induction on k and δ. Let $k = 1, \delta = 2$ then using (B.2) we have

$$\Phi_1\partial_x^{-1}\Phi_1\partial_x^{-1} = \sum_{s=0}^{M-2} C(s)(\tilde{\Phi}_1)_{s+2}\partial_x^{-s-2} + \nu_2 \tag{B.6}$$

where the functions $(\tilde{\Phi}_1)_{s+2}$ are defined as follows

$$(\tilde{\Phi}_1)_{s+2} := \Phi_1\partial_x^s\Phi_1\,. \tag{B.7}$$

Therefore, by the definition above and (2.36) immediately follows that

$$\|(\tilde{\Phi}_1)_{s+2}\|^{k_0,\gamma}_p \le C(p)\|\Phi_1\|^{k_0,\gamma}_{p+s}\,. \tag{B.8}$$

For the pseudo-differential operator $\nu_{1,2}$, by (2.25), (2.26) and (2.28) we have

$$|\nu_{1,2}|^{k_0,\gamma}_{-M-1,p,0} \le C(p,M)\|\Phi_1\|^{k_0,\gamma}_{p+1+2M-4}\,. \tag{B.9}$$

Now we suppose that it is true for $k = 1$ and $\delta = m$, that is (see (B.2))

$$(\Phi_1\partial_x^{-1})^m := \sum_{k=0}^{M-m} C(k)(\hat{\Phi}_1)_{k+m}\partial_x^{-m-k} + \nu_{1,m}\,, \tag{B.10}$$

where $\hat{\Phi}_1$ are some suitable functions. Then

$$\|(\hat{\Phi}_1)_{k+m}\|_p^{k_0,\gamma} \leq C(p)\|(\Phi_1)\|_{p+k}^{k_0,\gamma} . \tag{B.11}$$

The pseudo-differential operator $\nu_{1,m}$ is defined as follows

$$\nu_{1,m} := \tilde{\nu}_{1,m} + \Phi_1 \partial_x^{-1} \nu_{1,m-1} , \tag{B.12}$$

hence by (2.28), we have

$$|\nu_{1,m}|_{-M-1,p,0}^{k_0,\gamma} \leq C(p,M)\|\Phi_1\|_{p+2M+m-1-4}^{k_0,\gamma} . \tag{B.13}$$

Remark B.2. *Note that the formula* (B.10) *follows by iteration, indeed*

$$\begin{aligned}
(\Phi_1\partial_x^{-1})^m &= \Phi_1\partial_x^{-1} \circ (\Phi_1\partial_x^{-1})^{m-1} \\
&= \Phi_1\partial_x^{-1} \circ \left(\sum_{k=0}^{M-m+1} C(k)(\hat{\hat{\Phi}}_1)_{k+m-1}\partial_x^{-m+1-k} + \nu_{1,m-1} \right) \\
&= \Phi_1\partial_x^{-1} \left(\sum_{k=0}^{M-m+1} C(k)(\hat{\hat{\Phi}}_1)_{k+m-1}\partial_x^{-m+1-k} \right) + \Phi_1\partial_x^{-1}\nu_{1,m-1} \\
&= \sum_{k=0}^{M-m} C(k)(\hat{\Phi}_1)_{k+m}\partial_x^{-m-k} + \tilde{\nu}_{1,m} + \Phi_1\partial_x^{-1}\nu_{1,m-1} ,
\end{aligned}$$

where we define $\nu_{1,m} := \tilde{\nu}_{1,m} + \Phi_1\partial_x^{-1}\nu_{1,m-1}$.

Now we prove the formula for $k = 1$ and $\delta = m + 1$. By (B.2) we have

$$\begin{aligned}
\Phi_1\partial_x^{-1}(\Phi_1\partial_x^{-1})^m &= \Phi_1\partial_x^{-1} \left(\sum_{k=0}^{M-m-1} C(k)(\hat{\Phi}_1)_{k+m}\partial_x^{-m-k} + \nu_{1,m} \right) \\
&= \Phi_1 \sum_{k=0}^{M-m-1-j} \sum_{j=0}^{M-m-1-k} C(k,j)\partial_x^j(\hat{\Phi}_1)_{k+m}\partial_x^{-m-k-1-j} + \tilde{\nu}_{m+1} + \Phi_1\partial_x^{-1}\nu_{1,m} \\
&= \sum_{s=0}^{M-m-1} C(s)(\check{\Phi}_1)_{s+m+1}\partial_x^{-m-s-1} + \nu_{1,m+1} ,
\end{aligned}$$

where $\nu_{1,m+1}$ collects all the terms in OPS^{-M-1}, and the functions $\check{\Phi}_1$ are defined

$$(\check{\Phi}_1)_{s+m+1} := \sum_{j=0}^{s} \Phi_1\partial_x^j(\hat{\Phi}_1)_{m+s-j} .$$

Hence, by (2.36), (B.11), (B.12) and (B.13) we have the following estimates

$$\begin{aligned}
\|(\check{\Phi}_1)_{s+m+1}\|_p^{k_0,\gamma} \leq& C(p) \left(\|(\hat{\Phi}_1)_{m+s}\|_p^{k_0,\gamma} + \|(\hat{\Phi}_1)_{m+s-1}\|_{p+1}^{k_0,\gamma} + ... + \|(\hat{\Phi}_1)_m\|_{p+s}^{k_0,\gamma} \right) \\
\leq& C(p)\|\Phi_1\|_{p+s}^{k_0,\gamma} , \ s = 0, ..., M-m-1 , \\
|\nu_{1,m+1}|_{-M-1,p,0}^{k_0,\gamma} \leq& C(p,M)\|\Phi_1\|_{p+2M+m-4}^{k_0,\gamma} .
\end{aligned}$$

The Lemma is obviously true for $k = s+1$ and $\delta = 1$. It is also true for $k = s+1\,, \delta = 2$, indeed, by (B.2) we have

$$\Phi_{s+1}\partial_x^{-s-1}\Phi_{s+1}\partial_x^{-s-1} = \sum_{j=0}^{M-2s-2} C_j \hat{\Phi}_{2s+2+j}\partial_x^{-2s-2-j} + \nu_{(s+1,2)}$$

where the functions $\hat{\Phi}_{s+1}$ are defined as follows

$$(\hat{\Phi}_{s+1})_{2s+2+j} := \Phi_{s+1}\partial_x^j \Phi_{s+1}$$

hence by (2.36), (2.27), (2.28), (2.29) we have

$$\begin{aligned} \|(\hat{\Phi}_{s+1})_{2s+2+j}\|_p^{k_0,\gamma} &\leq C(p)\|\Phi_{s+1}\|_{p+j}^{k_0,\gamma} \\ |\nu_{(s+1,2)}|_{-M-1,p,0}^{k_0,\gamma} &\leq C(p,M)\|\Phi_{s+1}\|_{p+s+2M-4s-4}^{k_0,\gamma}\,. \end{aligned}$$

Suppose that the lemma is true for $k = s+1$ and $\delta = m$, that is

$$(\Phi_{s+1}\partial_x^{-s-1})^m = \sum_{k=0}^{M-ms-m} C(k)(\bar{\Phi}_{s+1})_{k+m+ms}\partial_x^{-ms-m-k} + \nu_{s+1,m} \tag{B.14}$$

for some suitable $\bar{\Phi}_{s+1}$, with

$$\begin{aligned} \|(\bar{\Phi}_{s+1})_{k+m+ms}\|_p^{k_0,\gamma} &\leq C(p)\|\Phi_{s+1}\|_{p+k}^{k_0,\gamma}\,,\ k = 0, ..., M-ms-m\,, \\ |\nu_{s+1,m}|_{-M-1,p,0}^{k_0,\gamma} &\leq C(p,M)\|\Phi_{s+1}\|_{p+(m-1)(s+1)+2M-4-4s}^{k_0,\gamma}\,. \end{aligned} \tag{B.15}$$

By (B.14), (2.19) and (B.2) with $k = s+1$ and $\delta = m+1$, we have

$$\begin{aligned} \Phi_{s+1}\partial_x^{-s-1}(\Phi_{s+1}\partial_x^{-s-1})^m &= \Phi_{s+1}\partial_x^{-s-1}\sum_{k=0}^{M-ms-m} C(k)(\bar{\Phi}_{s+1})_{k+sm+m}\partial_x^{-sm-m-k} + \Phi_{s+1}\partial_x^{-s-1}\nu_{s+1,m} \\ &= \Phi_{s+1}\sum_{k=0}^{M-(m+1)(s+1)-j}\ \sum_{j=0}^{M-(m+1)(s+1)-k} C(k)\partial_x^j(\bar{\Phi}_{s+1})_{k+sm+m} \\ &\quad \times \partial_x^{-(m+1)(s+1)-k-j} + \tilde{\nu}_{s+1,m+1} + \Phi_{s+1}\partial_x^{-s-1}\nu_{s+1,m} \\ &= \sum_{j=0}^{M-(m+1)(s+1)} C(j)(\check{\Phi}_{s+1})_{(s+1)(m+1)+j}\partial_x^{-(m+1)(s+1)-j} + \nu_{s+1,m+1} \end{aligned}$$

where the functions $\check{\Phi}_{s+1}$ are defined as follows

$$(\check{\Phi}_{s+1})_{(s+1)(m+1)+j} := \sum_{k=0}^{j} \Phi_{s+1}\partial_x^k(\bar{\Phi}_{s+1})_{(s+1)m+j-k}\,,$$

and the pseudo-differential operator $\nu_{s+1,m+1}$ collects all the terms in OPS^{-M-1}, that is

$$\nu_{s+1,m+1} := \tilde{\nu}_{s+1,m+1} + \Phi_{s+1}\partial_x^{-s-1}\nu_{s+1,m}\,.$$

Hence, by the explicit definition of $(\check{\Phi}_{s+1})_{(s+1)(m+1)+j}$, $\nu_{s+1,m+1}$ given above, by (B.15), (2.29) and (2.36), (2.27), (2.28) we have

$$\begin{aligned} \|(\check{\Phi}_{s+1})_{(s+1)(m+1)+j}\|_p^{k_0,\gamma} &\leq C(p)\left(\|(\bar{\Phi}_{s+1})_{m(1+s)+j}\|_p^{k_0,\gamma} + ... + \|(\bar{\Phi}_{s+1})_{m(s+1)}\|_{p+j}^{k_0,\gamma}\right) \\ &\leq C(p)\|\Phi_{s+1}\|_{p+j}^{k_0,\gamma}\,,\ j = 0, ..., M-(m+1)(s+1) \\ |\nu_{s+1,m+1}|_{-M-1,p,0}^{k_0,\gamma} &\leq C(p,M)\|\Phi_{s+1}\|_{p+m(s+1)+2M-4s-4}^{k_0,\gamma}\,. \end{aligned}$$

So the Lemma is proved. □

Thanks to this Lemma we can prove that, if an operator admits the asymptotic expansion in homogeneous components up to order $-M$, then, under some suitable assumption, by the Neumann series also the inverse operator admits the asymptotic expansion up to the same order.

Lemma B.3. *Let $T_k = \mathbb{1} + \Phi_k \partial_x^{-k}$, with Φ_k as in* (B.3) *for $k = 1, ...M+3$. Let Φ_k satisfies* (2.31) *(with $\alpha = 0$). Then T_k^{-1} can be expanded as follows*

$$T_k^{-1} = \mathbb{1} + \Phi_k \partial_x^{-k} + \sum_{j=0}^{M-2k} C(j)(\hat{\Phi}_k)_{j+2k} \partial_x^{-j-2k} + \nu_k \,,$$

for some suitable functions $\hat{\Phi}_k$ and constants $C(j)$. The operator $\nu_k \in OPS^{-M-1}$. Moreover we have that

$$\|(\hat{\Phi}_k)_{j+2k}\|_p^{k_0,\gamma} \le C(p)\|\Phi_k\|_{p+j}^{k_0,\gamma} \,, \quad |\nu_k|_{-M-1,p,0}^{k_0,\gamma} \le C(p,M)\|\Phi_k\|_{p+(\mathtt{n}-1)k+2M-4k}^{k_0,\gamma} \,, \tag{B.16}$$

where $\mathtt{n} = \left[\frac{M}{k}\right] \ge 2$.

Proof. By the Neumann series and Lemma B.1 we have

$$\begin{aligned} T_k^{-1} &= \mathbb{1} + \sum_{\delta \ge 1} (-1)^\delta (\Phi_k \partial_x^{-k})^\delta \\ &= \mathbb{1} + \Phi_k \partial_x^{-k} + (\Phi_k \partial_x^{-k})^2 + ... + (\Phi_k \partial_x^{-k})^{\mathtt{n}} + r \\ &= \mathbb{1} + \Phi_k \partial_x^{-k} + \left(\sum_{j=0}^{M-2k} C''(j)(\tilde{\Phi}_k)_{j+2k} \partial_x^{-2k-j} + \nu_{k,2} \right) + ... \\ &+ \left(\sum_{j=0}^{M-k\mathtt{n}} C'(j)(\tilde{\Phi}_k)_{j+\mathtt{n}k} \partial_x^{-k\mathtt{n}-j} \right) + \nu_{k,\mathtt{n}} + r \,, \end{aligned} \tag{B.17}$$

where $\mathtt{n} = \left[\frac{M}{k}\right]$, $C'(j), C''(j)$ are some constants, $\nu_{k,\delta}$ for $\delta = 1, ..., M-2k$ is the pseudo-differential operator in OPS^{-M-1} given in Lemma B.1 and r is the remainder of the convergent series. Then, we redefine the homogeneous terms as follows

$$\sum_{j=0}^{M-2k} C''(j)(\tilde{\Phi}_k)_{j+2k} \partial_x^{-2k-j} + ... + \sum_{j=0}^{M-k\mathtt{n}} C'(j)(\tilde{\Phi}_k)_{j+\mathtt{n}k} \partial_x^{-k\mathtt{n}-j} := \sum_{j=0}^{M-2k} C(j)\hat{\Phi}_{j+2k} \partial_x^{-j-2k} \,.$$

We define the new pseudo-differential operator of order $-M-1$ as the sum of the other pseudo-differential operators, that is

$$\nu_{k,2} + ... + \nu_{k,\mathtt{n}} + r := \nu_k .$$

By Lemma 2.10 and (2.27) the remainder r can be estimate by induction. Let $s = M - 2k$, then

$$
\begin{aligned}
|(\Phi_k\partial_x^{-k})^{s+1}|^{k_0,\gamma}_{-k(s+1),\mathfrak{p}_0,0} =& |(\Phi_k\partial_x^{-k})^{s}\circ(\Phi_k\partial_x^{-k})|^{k_0,\gamma}_{-k(s+1),\mathfrak{p}_0,0} \\
\leq& 2C(\mathfrak{p}_0,k_0)|(\Phi_k\partial_x^{-k})^{s}|^{k_0,\gamma}_{0,\mathfrak{p}_0,0}|\Phi_k\partial_x^{-k}|^{k_0,\gamma}_{-k,\mathfrak{p}_0,0} \\
\leq& C(\mathfrak{p}_0,k_0)^s\left(|\Phi_k\partial_x^{-k}|^{k_0,\gamma}_{0,\mathfrak{p}_0,0}\right)^{s+1} \\
\overset{(2.31)}{\leq}& \frac{1}{2^s}\|\Phi\|^{k_0,\gamma}_{\mathfrak{p}_0} \\
\leq& \frac{1}{2^s}\,.
\end{aligned}
\tag{B.18}
$$

Moreover by (2.27) and (2.29) we also have

$$
\begin{aligned}
|(\Phi_k\partial_x^{-k})^{s+1}|^{k_0,\gamma}_{-k(s+1),p,0} &= |(\Phi_k\partial_x^{-k})^{s}\circ(\Phi_k\partial_x^{-k})|^{k_0,\gamma}_{-k(s+1),p,0} \\
&\leq C(p)|(\Phi_k\partial_x^{-k})^{s}|^{k_0,\gamma}_{0,p,0}|\Phi_k\partial_x^{-k}|^{k_0,\gamma}_{-k,\mathfrak{p}_0,0} + C(\mathfrak{p}_0)|(\Phi_k\partial_x^{-k})^{s}|^{k_0,\gamma}_{0,\mathfrak{p}_0,0}|\Phi_k\partial_x^{-k}|^{k_0,\gamma}_{-k,p,0} \\
&\leq (s+1)C(p,k_0)\left(C(\mathfrak{p}_0,k_0)\|\Phi_k\|^{k_0,\gamma}_{\mathfrak{p}_0}\right)^s\|\Phi_k\|^{k_0,\gamma}_{p}\,.
\end{aligned}
\tag{B.19}
$$

Then (recall that $s = M - 2k$) by (B.19) and (2.31)

$$
\begin{aligned}
|\sum_{s>0}(\Phi_k\partial_x^{-k})^{s+1}|^{k_0,\gamma}_{0,p,0} &\leq \sum_{s>0}|(\Phi_k\partial_x^{-k})^{s+1}|^{k_0,\gamma}_{0,p,0} \\
&\leq \left(\sum_{s>0}(s+1)\left(C(\mathfrak{p}_0,k_0)\|\Phi_k\|^{k_0,\gamma}_{\mathfrak{p}_0}\right)^s\right)C(p,k_0)\|\Phi_k\|^{k_0,\gamma}_{p} \\
&\leq C'(p,k_0)\|\Phi_k\|^{k_0,\gamma}_{p}\,.
\end{aligned}
\tag{B.20}
$$

Therefore we can decompose the inverse operator as the sum of a homogeneous terms plus a bounded regularizing remainder. The estimates (B.16) follow by Lemma B.1. □

Now we want to prove that the composition of two operator that can be written as the sum of homogeneous terms plus a bounded regularizing remainder has the same structure. Note that given two operator $\mathbf{B} = \sum_{j=0}^{M}\mathbf{B}_j\partial_x^{-k} + \nu_B$ and $\mathbf{A} = \sum_{k=0}^{M}\mathbf{A}_k\partial_x^{-k} + \nu_A$ such that $\|\mathbf{A}\|^{k_0,\gamma}_{\mathfrak{p}_0+\chi(M)+\sigma}, \|\mathbf{B}\|^{k_0,\gamma}_{\mathfrak{p}_0+\chi(M)+\sigma} \leq 1$ then the composition operator it is given by

$$
\begin{aligned}
\mathbf{B}\circ\mathbf{A} &= \left(\sum_{j=0}^{M}\mathbf{B}_j\partial_x^{-k} + \nu_B\right)\left(\sum_{k=0}^{M}\mathbf{A}_k\partial_x^{-k} + \nu_A\right) \\
&= \left(\sum_{j=0}^{M}\mathbf{B}_j\partial_x^{-k} + \nu_B\right)\circ\left(\sum_{k=0}^{M}\mathbf{A}_k\partial_x^{-k}\right) + \nu_{AB}
\end{aligned}
$$

where

$$
\nu_{AB} := \left(\sum_{j=0}^{M}\mathbf{B}_j\partial_x^{-k}\right)\circ\nu_A + \nu_B\circ\nu_A\,.
$$

Then, by (2.27) we have immediately that ν_{AB} is a pseudo-differential operator in OPS^{-M-1} and

$$|\nu_{AB}|^{k_0,\gamma}_{-M-1,p,0} \leq \sum_{j=0}^{M} C(p,j)\|\mathbf{B}_j\|^{k_0,\gamma}_p |\nu_A|^{k_0,\gamma}_{-M-1,\mathfrak{p}_0+j,0} + \sum_{j=0}^{M} C(\mathfrak{p}_0,j)|\nu_A|^{k_0,\gamma}_{-M-1,p+j,0}$$
$$+ C(p)|\nu_B|^{k_0,\gamma}_{-M-1,p,0}|\nu_A|^{k_0,\gamma}_{-M-1,\mathfrak{p}_0,0} + C(\mathfrak{p}_0)|\nu_B|^{k_0,\gamma}_{-M-1,\mathfrak{p}_0,0}|\nu_A|^{k_0,\gamma}_{-M-1,p,0}\,.$$

Therefore we now want to prove the following Lemma.

Lemma B.4. *Let* $\mathbf{A}\partial_x^{-m}$, $m = 0, ..., M$ *and* $T_k = \mathbb{1} + \Phi_k \partial_x^{-k}$, $k = 1, ..., M+3$ *such that* $\|\mathbf{A}\|^{k_0,\gamma}_{\mathfrak{p}_0+\chi(M)+\sigma}, \leq 1$*, where* $\chi(M)$ *is a constant and* $\sigma := \sigma(\tau, N, k_0)$*. Then the following asymptotic expansion holds*

$$T_k^{-1}\mathbf{A}\partial_x^{-m} = \mathbf{A}\partial_x^{-m} + \sum_{s=0}^{M-k} C(s)\tilde{\mathbf{A}}_{k+s}\partial_x^{-k-s} + \sigma_{\mathbf{A}}\,, \tag{B.21}$$

for some suitable functions $\tilde{\mathbf{A}}_{k+s}$ *and constants* $C(s)$*.The operator* $\sigma_{\mathbf{A}} \in OPS^{-M-1}$*.*

Moreover we have

$$\begin{aligned} \|\tilde{\mathbf{A}}_{k+s}\|^{k_0,\gamma}_p &\leq C(p)\left(\|\Phi_k\|^{k_0,\gamma}_{p+s} + \|\mathbf{A}\|^{k_0,\gamma}_{p+s}\right), \quad \forall k+s \geq m\,, \\ |\sigma_{\mathbf{A}}|^{k_0,\gamma}_{-M-1,p,0} &\leq C(p,M)\left(\|A\|^{k_0,\gamma}_{p+3M-k-2m+1} + \|\Phi_k\|^{k_0,\gamma}_{p+(\mathtt{n}-1)k+2M-4k}\right), \end{aligned} \tag{B.22}$$

where $\mathtt{n} = \left[\frac{M}{k}\right] \geq 2$ *(see* (B.17)*).*

Proof. We shall write $C(\cdot)$ for the constants. The proof follows by Lemmas B.3 and B.1, indeed

$$\begin{aligned} T_k^{-1}\mathbf{A}\partial_x^{-m} &= \mathbf{A}\partial_x^{-m} + \Phi_k\partial_x^{-k}\mathbf{A}\partial_x^{-m} + \sum_{j=0}^{M-2k-m} C(j)\hat{\Phi}_{j+2k}\partial_x^{-j-2k}\mathbf{A}\partial_x^{-m} + \nu_k\mathbf{A}\partial_x^{-m} \\ &= \mathbf{A}\partial_x^{-m} + \sum_{s=0}^{M-k-m} C(s)\tilde{\mathbf{A}}_{m+k+s}\partial_x^{-m-k-s} + \sigma_{\mathbf{A}}\,, \end{aligned}$$

where $\tilde{\mathbf{A}}_{m+k+s}$ is a suitable matrix whose entries are some suitable functions. After reordering the terms of the series, using (2.19) one arrive to the expansion defined in the Lemma. The first estimate follows by Lemma B.3. Now we prove the second inequality in (B.22). Let $\nu_{2,A}$ be the remainder of $\Phi_k\partial_x^{-k}\mathbf{A}\partial_x^{-m}$, then, by (2.28), (2.25) and (2.26)

$$|\nu_{2,\mathbf{A}}|^{k_0,\gamma}_{-M-1,p,0} \leq C(p,M)\|\mathbf{A}\|^{k_0,\gamma}_{p+2M-k-2m}\,.$$

Let $\nu_{j,\mathbf{A}}$ be the remainder of $\hat{\Phi}_{j+2k}\partial_x^{-j-2k}\mathbf{A}\partial_x^{-m}$, for $j = 0, ..., M-2k-m$, then by (2.28), (2.25) and (2.26)

$$|\nu_{j,\mathbf{A}}|^{k_0,\gamma}_{-M-1,p,0} \leq C(p,M)\|\mathbf{A}\|^{k_0,\gamma}_{p+2M+j-2k-2m} \leq C(p,M)\|\mathbf{A}\|^{k_0,\gamma}_{p+3M-4k-3m}\,, \quad \forall j = 0, ..., M-2k-m\,.$$

Using (2.27) , (2.25) and (2.26) we can estimate $\nu_k\mathbf{A}\partial_x^{-m}$ as follows:

$$|\nu_k\mathbf{A}\partial_x^{-m}|^{k_0,\gamma}_{-M-1-m,p,0} \leq C(p,M)\left(\|\Phi_k\|^{k_0,\gamma}_{p+(\mathtt{n}-1)k+2M-4k} + \|\mathbf{A}\|^{k_0,\gamma}_{p+M+1}\right).$$

Finally we define $\sigma_{\mathbf{A}} := \nu_{2,\mathbf{A}} + \nu_{j,\mathbf{A}} + \nu_k \mathbf{A}\partial_x^{-m}$. Then, by (2.27) and (2.29), the following estimate follows

$$
\begin{aligned}
|\sigma_{\mathbf{A}}|_{-M-1,p,0}^{k_0,\gamma} \leq & C(p,M)\left(\|\mathbf{A}\|_{p+2M-k-2m}^{k_0,\gamma} + \|\mathbf{A}\|_{p+3M-4k-3m}^{k_0,\gamma} + \|\Phi_k\|_{p+(\mathtt{n}-1)k+2M-4k}^{k_0,\gamma} + \|\mathbf{A}\|_{p+M+1}^{k_0,\gamma}\right) \\
\leq & C(p,M)\left(\|\mathbf{A}\|_{p+3M-k-2m+1}^{k_0,\gamma} + \|\Phi_k\|_{p+(\mathtt{n}-1)k+2M-4k}^{k_0,\gamma}\right).
\end{aligned}
$$

This complete the proof of the Lemma. □

Lemma B.5. *Let $R = \sum_{k=0}^{M} \mathbf{A}_k \partial_x^{-k} + \Sigma_R$, and $T_s = (\mathbb{1} + \Phi_s \partial_x^{-s})$, $s = 0, ..., M+3$, such that $\|\mathbf{A}_k\|_{\mathfrak{p}_0+\chi(M)+\sigma}, \leq 1$, where $\chi(M)$ is a constant and $\sigma := \sigma(\tau, N, k_0)$. Then the following asymptotic expansion holds*

$$
RT_s = \sum_{k=0}^{M} C(k)\tilde{\mathbf{A}}_k \partial_x^{-k} + \Sigma\,,
$$

for some suitable functions $\tilde{\mathbf{A}}_k$ and constants $C(k)$. The operator $\Sigma \in OPS^{-M-1}$.

Moreover

$$
\|\tilde{\mathbf{A}}_k\|_p^{k_0,\gamma} \leq C(p)\left(\|\mathbf{A}_k\|_p^{k_0,\gamma} + \|\Phi_s\|_{p+k}^{k_0,\gamma}\right), \quad |\Sigma|_{-M-1,p,0}^{k_0,\gamma} \leq C(p,M)\left(|\Sigma_R|_{-M-1,p,0}^{k_0,\gamma} + \|\Phi_s\|_{p+2M+1}^{k_0,\gamma}\right).
$$

Proof. By Lemma B.3, we have

$$
\begin{aligned}
RT_s :=& \sum_{k=0}^{M} \mathbf{A}_k \partial_x^{-k} + \sum_{k=0}^{M-s-m} \mathbf{A}_k \left(\sum_{m=0}^{M-s-k} C(m)(\partial_x^m \Phi)\partial_x^{-s-k-m} + \sigma_k\right) + \Sigma_R T_s \\
=& \sum_{k=0}^{M} \mathbf{A}_k \partial_x^{-k} + \sum_{k=0}^{M-s-m} C(k)\hat{\mathbf{A}}_{k+s}\partial_x^{-s-k} + \sigma_{k,A} + \Sigma_R T_s
\end{aligned}
$$

where the functions $\hat{\mathbf{A}}_{s+k}$ are defined as follows

$$
\hat{\mathbf{A}}_{s+k} := \sum_{j=0}^{s} C(j)\partial_x^j \Phi\,.
$$

Then the estimate on the coefficient follows immediately, for the estimate on Σ we have to use (2.28) (2.27), (2.25) and (2.26).

$$
\begin{aligned}
|\sigma_{k,\mathbf{A}}|_{-M-1,p,0}^{k_0,\gamma} \leq & C(p,M)\left(\|\mathbf{A}\|_p^{k_0,\gamma} + \|\Phi_s\|_{p+2M-2s-k}^{k_0,\gamma}\right) \\
\leq & C(p,M)\left(\|\mathbf{A}\|_p^{k_0,\gamma} + \|\Phi_s\|_{p+2M-2s}^{k_0,\gamma}\right), \quad \forall k = 0, ..., M-s-m. \\
|\Sigma_R T_s|_{-M-1,p,0}^{k_0,\gamma} \leq & C(p,M)\left(|\Sigma_R|_{-M-1,p,0}^{k_0,\gamma} + \|\Phi_s\|_{p+M+1}^{k_0,\gamma}\right).
\end{aligned}
$$

Then the Lemma is proved. □

B.2 The remainder $\mathbf{R}_1$

We can apply the general tools proved in Section B.1, to the operators defined in Chapter 7 and 8. In this Section we prove that the operator $\mathbf{R}_1$ in (8.17) admits an asymptotic expansion and the estimates given in Lemma 8.2.

As before we shall write $C(\cdot)$ for the constants. Moreover we shall assume that $\|\mathtt{v}\|_{\mathfrak{p}_0+\chi(M)+\sigma} \leq 1$, for some constant $\chi(M)$ and for $\sigma := \sigma(\tau, N, k_0)$.

Lemma B.6. *Let* $\tilde{\mathbf{R}}_1$ *defined in* (8.7)*,* $\mathbf{U}\pi_0$ *in* (8.10)*,* $\mathbf{W}\partial_x^{-1}$ *in* (8.11)*,* $\mathbf{P}$ *in* (8.13) *and* T_1 *in* (8.2)*. Then we have an asymptotic expansion of the form*

$$\tilde{\mathbf{R}}_1 + \mathbf{U}\pi_0 + \mathbf{W}\partial_x^{-1} + \mathbf{P} = \sum_{j=0}^{M} \mu \begin{pmatrix} F_j^{(1)}(x,\theta) & F_j^{(2)}(x,\theta) \\ F_j^{(3)}(x,\theta) & F_j^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-j} + \mu \begin{pmatrix} \beta_1(x,\theta,D) & \beta_2(x,\theta,D) \\ \beta_3(x,\theta,D) & \beta_4(x,\theta,D) \end{pmatrix} \tag{B.23}$$

where ∂_x^0 *denotes one of the operators belonging to* $\{a\pi_0 + b\mathbb{1}\,,\ a,b \in \{0,1\}\}$*.* F_j^m*,* $m = 1,...,4$ *and* $j = 1,...,M$ *are some suitable functions and* $\beta_k \in OPS^{-M-1}$ *for* $k = 1,...,4$*.*

Moreover for all $j = 0,...,M$ *and for all* $k = 1,...,4$ *we have*

$$\|F_j^{(k)}\|_p^{k_0,\gamma} \leq_{p,j} \|\mathtt{v}\|_{p+j+5+\sigma}^{k_0,\gamma} \tag{B.24}$$

$$|\beta_k|_{-M-1,p,0}^{k_0,\gamma} \leq_{p,M} \|\mathtt{v}\|_{p+2M+6+\sigma}^{k_0,\gamma} \tag{B.25}$$

$$\|\partial_i F_j^{(k)}[\hat{\imath}]\|_{p_1} \leq_{p_1,j} \|\hat{\imath}\|_{p_1+j+5+\sigma} \tag{B.26}$$

$$|\partial_i \beta_k[\hat{\imath}]|_{-M,p_1,0} \leq_{p_1,M} \|\hat{\imath}\|_{p_1+2M+6+\sigma}\,. \tag{B.27}$$

Proof. Now we prove that we can decompose $\tilde{\mathbf{R}}_1 + \mathbf{U}\pi_0 + \mathbf{W}\partial_x^{-1} + \mathbf{P}$ as in (B.23). First of all we consider the remainder $\tilde{\mathbf{R}}_1$ defined in (8.7). It is clear that it can be written as

$$\begin{aligned} &\tilde{\mathbf{R}}_1 = \mathbf{V}_0\pi_0 + \mathbf{V}_1\partial_x^{-1} + \mathbf{R}T_1\,, \quad \mathbf{V}_0 = \mathbf{C}^{(1)}\Phi_1 + 2\mathbf{B}^{(1)}(\Phi)_x\,, \\ &\mathbf{V}_1 = (\omega\cdot\partial_\theta\Phi_1) + \mathbf{C}^{(1)}(\Phi_1)_x + \mathbf{B}^{(1)}(\Phi)_{xx}\,. \end{aligned} \tag{B.28}$$

Therefore we have the following estimate (recall that, by (8.2) and (8.15), we have $\|\Phi_1\|_p^{k_0,\gamma} \leq \|\mathtt{v}\|_{p+1}^{k_0,\gamma}$)

$$\|\mathbf{V}_0\|_p^{k_0,\gamma} \leq_p \|\mathtt{v}\|_{p+2}^{k_0,\gamma} \tag{B.29}$$

$$\|\mathbf{V}_1\|_p^{k_0,\gamma} \leq_p \|\mathtt{v}\|_{p+3+\sigma}^{k_0,\gamma}\,. \tag{B.30}$$

The linear operator $\mathcal{L}$ in (6.10) can be written in homogeneous component plus a regularizing remainder (see (7.9) in Chapter 7). The remainder $\mathbf{R}$ can be written as in (7.22), therefore

$$\begin{aligned} \mathbf{R}T_1 &= \sum_{j=0}^{M} \mu \begin{pmatrix} A_j^{(1)}(x,\theta) & A_j^{(2)}(x,\theta) \\ A_j^{(3)}(x,\theta) & A_j^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-j} + \mu \begin{pmatrix} Op(\sigma_1(x,\theta,\xi)) & Op(\sigma_2(x,\theta,\xi)) \\ Op(\sigma_3(x,\theta,\xi)) & Op(\sigma_4(x,\theta,\xi)) \end{pmatrix} T_1 \\ &+ \mu \sum_{j=0}^{M} \begin{pmatrix} A_j^{(1)}(x,\theta) & A_j^{(2)}(x,\theta) \\ A_j^{(3)}(x,\theta) & A_j^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-j} \circ \Phi_1\partial_x^{-1} \\ &:= \sum_{j=0}^{M} \mu\mathbf{A}_j\partial_x^{-j} + \mu\Sigma T_1 + \sum_{j=0}^{M} \mu\mathbf{A}_j\partial_x^{-j} \circ \Phi_1\partial_x^{-1}\,, \end{aligned} \tag{B.31}$$

where $\mathbf{A}_j$ is the matrix that represents the j-th coefficient, and Σ is in OPS^{-M-1} and represents the matrix whose entries are $\sigma_i\,, i = 1,...,4$. Note that we can estimate the first term as in (7.23). We now consider only the third term, that is

$$\begin{aligned}
\sum_{j=0}^{M-1} \mathbf{A}_j \partial_x^{-j} \circ \Phi_1 \partial_x^{-1} &= \sum_{j=0}^{M-1-k} \mathbf{A}_j \left(\sum_{k=0}^{M-j-1} C(k)(\partial_x^k \Phi_1) \partial_x^{-k-j-1} + \tilde{\nu}_{\tilde{\mathbf{A}}_s} \right) \\
&= \sum_{s=0}^{M-1} C(s) \tilde{\mathbf{A}}_{s+1} \partial_x^{-s-1} + \sum_{j=0}^{M} \mathbf{A}_j \tilde{\nu}_{\tilde{\mathbf{A}}_s} \\
&= \sum_{s=0}^{M-1} C(s) \tilde{\mathbf{A}}_{s+1} \partial_x^{-s-1} + \nu_{\tilde{\mathbf{A}}} \,.
\end{aligned} \tag{B.32}$$

where

$$\nu_{\tilde{\mathbf{A}}} := \sum_{j=0}^{M} \mathbf{A}_j \tilde{\nu}_{\tilde{\mathbf{A}}_s} = \sum_{j=0}^{M} \begin{pmatrix} A_j^{(1)}(x,\theta) & A_j^{(2)}(x,\theta) \\ A_j^{(3)}(x,\theta) & A_j^{(4)}(x,\theta) \end{pmatrix} \tilde{\nu}_{\tilde{\mathbf{A}}_s} \,,$$

and

$$\tilde{\mathbf{A}}_{s+1} := \begin{pmatrix} \tilde{A}_{s+1}^{(1)}(x,\theta) & \tilde{A}_{s+1}^{(2)}(x,\theta) \\ \tilde{A}_{s+1}^{(3)}(x,\theta) & \tilde{A}_{s+1}^{(4)}(x,\theta) \end{pmatrix}$$

with

$$\sum_{s=0}^{M-1} \begin{pmatrix} \tilde{A}_{s+1}^{(1)}(x,\theta) & \tilde{A}_{s+1}^{(2)}(x,\theta) \\ \tilde{A}_{s+1}^{(3)}(x,\theta) & \tilde{A}_{s+1}^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-s-1} = \mu \sum_{j=0}^{M-1-k} \sum_{k=0}^{M-j-1} C(j,k) \begin{pmatrix} A_j^{(2)} \partial_x^k \varphi_3^{(1)} & A_j^{(1)} \partial_x^k \varphi_2^{(1)} \\ A_j^{(4)} \partial_x^k \varphi_3^{(1)} & A_j^{(3)} \partial_x^k \varphi_2^{(1)} \end{pmatrix} \partial_x^{-k-j-1} \,.$$

Note that

$$\mu \sum_{j=0}^{M-1} \sum_{k=0}^{M-j-1} C(j,k) A_j^{(\cdot)} \partial_x^k \varphi_{(\cdot)}^{(1)} = \sum_{s=0}^{M-1} \tilde{A}_{s+1}^{(\cdot)} \quad \text{with} \quad \tilde{A}_{s+1}^{(\cdot)} = \mu \sum_{k=0}^{s} C'(k) A_k^{(\cdot)} (\partial_x^{s-k} \varphi_{(\cdot)}^{(1)}) \,. \tag{B.33}$$

Then, using (2.36), (2.28) , (2.25) and (2.26) and (7.23), by the explicit definition of $\tilde{\mathbf{A}}_{s+1}$ given in (B.33), we obtain that

$$\|\tilde{\mathbf{A}}_{s+1}\|_p^{k_0,\gamma} \leq_{p,s} \mu \|\mathtt{v}\|_{p+s+5+\sigma}^{k_0,\gamma} \,, \quad \forall s = 0, ..., M-1 \tag{B.34}$$

$$\begin{aligned}
|\tilde{\nu}_{\tilde{\mathbf{A}}_s}|_{-M-1,p,0}^{k_0,\gamma} &\leq_{p,M} \|\Phi_1\|_{p+j+2M-2j-2}^{k_0,\gamma} \,, \quad j = 0, ..., M-1 \\
&\leq_{p,M} \|\Phi_1\|_{p+2M-j-2}^{k_0,\gamma} \,, \quad j = 0, ..., M-1 \\
&\leq_{p,M} \mu \|\mathtt{v}\|_{p+2M-1+\sigma}^{k_0,\gamma}
\end{aligned}$$

$$|\nu_{\tilde{\mathbf{A}}}|_{-M-1,p,0}^{k_0,\gamma} \leq_{p,M} \sum_{j=0}^{M} \left(\mu |\mathbf{A}_j|_{0,p,0}^{k_0,\gamma} |\tilde{\nu}_{\tilde{\mathbf{A}}_s}|_{-M-1,\mathfrak{p}_0,0}^{k_0,\gamma} + \mu |\mathbf{A}_j|_{0,\mathfrak{p}_0,0}^{k_0,\gamma} |\tilde{\nu}_{\tilde{\mathbf{A}}_s}|_{-M-1,p,0}^{k_0,\gamma} \right) \tag{B.35}$$

$$\leq_{p,M} \mu \|\mathtt{v}\|_{p+2M-1+\sigma}^{k_0,\gamma} \,. \tag{B.36}$$

By (B.28), (B.31) and (B.32) we can rewrite $\tilde{\mathbf{R}}_1$ as

$$\begin{aligned}
\tilde{\mathbf{R}}_1 &:= \sum_{j=0}^{M} \mu \begin{pmatrix} Y_j^{(1)}(x,\theta) & Y_j^{(2)}(x,\theta) \\ Y_j^{(3)}(x,\theta) & Y_j^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-j} + \mu \begin{pmatrix} Op(\nu_1(x,\theta,\xi)) & Op(\nu_2(x,\theta,\xi)) \\ Op(\nu_3(x,\theta,\xi)) & Op(\nu_4(x,\theta,\xi)) \end{pmatrix} \\
&:= \sum_{j=0}^{M} \mu \mathbf{Y}_j \partial_x^{-j} + \mu \nu_Y
\end{aligned} \tag{B.37}$$

where we define,

$$\begin{aligned}
\nu_Y :=& \Sigma T_1 + \nu_{\tilde{\mathbf{A}}} \\
\mathbf{Y}_j :=& \begin{pmatrix} Y_j^{(1)}(x,\theta) & Y_j^{(2)}(x,\theta) \\ Y_j^{(3)}(x,\theta) & Y_j^{(4)}(x,\theta) \end{pmatrix}, \\
\mathbf{Y}_0 :=& \mathbf{V}_0\pi_0 + C\mathbf{A}_0\,, \\
\mathbf{Y}_1 :=& \mathbf{V}_1 + C\mathbf{A}_1 + C'\mu\tilde{\mathbf{A}}_1\,, \\
\mathbf{Y}_s :=& \mu C'(s)\tilde{\mathbf{A}}_s + C(s)\mathbf{A}_s\,, \quad \text{for} \quad s = 2, ..., M\,.
\end{aligned}$$

By (2.28), (7.13), (B.36) , (2.25) and (2.26) we have $i = 1, ..., 4$

$$\begin{aligned}
|\nu_i|^{k_0,\gamma}_{-M-1,p,0} \le& |\sigma_i|^{k_0,\gamma}_{-M-1,p,0} + \mu|\sigma_i \circ \Phi_1 \partial_x^{-1}|^{k_0,\gamma}_{-M-1,p,0} + \mu|\nu_{\tilde{\mathbf{A}}}|^{k_0,\gamma}_{-M-1,p,0} \\
\le_{p,M}& \|\mathrm{v}\|^{k_0,\gamma}_{p+M+6} + \mu\|\mathrm{v}\|^{k_0,\gamma}_{p+2M-1} + \mu\|\mathrm{v}\|^{k_0,\gamma}_{p+2M-1+\sigma} \\
\le_{p,M}& \|\mathrm{v}\|^{k_0,\gamma}_{p+2M+6+\sigma}\,.
\end{aligned} \tag{B.38}$$

Moreover, by (7.12) and (B.34) we have that, for evey $j = 0, ..., M$

$$\|\mathbf{Y}_j\|^{k_0,\gamma}_p \le_{p,j} \|\mathrm{v}\|^{k_0,\gamma}_{p+j+5+\sigma}\,. \tag{B.39}$$

We have to study the commutator (8.9), we start by $\mathbf{P}$ defined in (8.13). Let $i = 2, 3$, then

$$\begin{aligned}
P_i =& \left(\sum_{k=1}^{M-1} c_k \partial_x^{-k}\right) \circ \varphi_i^{(1)}\partial_x^{-1} + Op(r(\xi)) \circ \varphi_i^{(1)}\partial_x^{-1} + \varphi_i^{(1)}\partial_x^{-1} \circ Op(r(\xi)) \\
&+ \varphi_i^{(1)}\partial_x^{-1} \circ \left(\sum_{k=1}^{M-1} c_k\partial_x^{-k}\right)\,.
\end{aligned} \tag{B.40}$$

We study the first term

$$\begin{aligned}
\left(\sum_{k=1}^{M-1} c_k\partial_x^{-k}\right) \circ \varphi_i^{(1)}\partial_x^{-1} =& \sum_{k=1}^{M-1-\beta} c_k \left(\sum_{\beta=0}^{M-k-1} (\partial_x^\beta \varphi_i^{(1)})\partial_x^{-1-\beta-k} + \sigma_i^{(k)}(\xi,x,D)\right) \\
=& \sum_{s=0}^{M-2} c_s \tilde{\varphi}^{(1)}_{i,s+2}\partial_x^{-2-s} + \sigma_i^{(k)}(\xi,x,D)
\end{aligned} \tag{B.41}$$

where $i = 2, 3$ and $\tilde{\varphi}^{(1)}_{i,s+2} := \sum_{k=0}^s c_k\partial_x^k\varphi_i^{(1)}$, with $c_k \in \mathbb{R}$. We recall that c_k are the real constants generated by the asympotic expansion of Λ (see (7.3)) . Using (2.28), , (2.25) and (2.26) for $i = 2, 3$ and by (8.15), we have

$$|\sigma_i^{(k)}|^{k_0,\gamma}_{-M-1,p,0} \le_{p,M} \|\varphi_i^{(1)}\|^{k_0,\gamma}_{p+2M-k-2} \le_{p,M} \|\mathrm{v}\|^{k_0,\gamma}_{p+2M-k-1}$$

and

$$\|\tilde{\varphi}^{(1)}_{i,s+2}\|^{k_0,\gamma}_p \le_{p,s} \|\varphi_i^{(1)}\|^{k_0,\gamma}_{p+s} \quad s = 0, ..., M-2\,. \tag{B.42}$$

Hence for all $k = 0, ..., M-1$,$i = 2, 3$ we have

$$|\sigma_i^{(k)}(\xi,x,\theta)|^{k_0,\gamma}_{-M-1,p,0} \le_{p,M} \|\varphi_i^{(1)}\|^{k_0,\gamma}_{p+2M-2} \le_{p,M} \|\mathrm{v}\|^{k_0,\gamma}_{p+2M-1}\,. \tag{B.43}$$

Note that the last term in (B.40) can be written as $\varphi_i^{(1)}\sum_{k=0}^{M-2} c_{k+2}\partial_x^{-k-2}$, hence we do not lose derivatives on the coefficients. The other two terms in (B.40) are in OPS^{-M-1}, then, by (2.28) , (2.25) and (2.26), we can estimate they

$$\begin{aligned} &|Op(r(\xi))\circ\varphi_i^{(1)}\partial_x^{-1}|_{-M-2,p,0}^{k_0,\gamma} \leq_{p,M} \|\varphi_i^{(1)}\|_{p+M+1}^{k_0,\gamma} \leq_{p,M} \|\mathtt{v}\|_{p+M+2}^{k_0,\gamma} \\ &|\varphi_i^{(1)}\partial_x^{-1}\circ Op(r(\xi))|_{-M-2,p,0}^{k_0,\gamma} \leq_{p,M} \|\varphi_i^{(1)}\|_p^{k_0,\gamma} \leq_{p,M} \|\mathtt{v}\|_{p+1}^{k_0,\gamma} . \end{aligned} \tag{B.44}$$

We now consider $\mathbf{U}$, and $\mathbf{W}$ defined in (8.10) and (8.11). By the explicit definition we have that $\mathbf{U}$, and $\mathbf{W}$ satisfy

$$\|\mathbf{U}\|_p^{k_0,\gamma} \leq_p \mu\|\mathtt{v}\|_{p+3}^{k_0,\gamma} , \quad \|\mathbf{W}\|_p^{k_0,\gamma} \leq_p \mu\|\mathtt{v}\|_{p+4}^{k_0,\gamma}. \tag{B.45}$$

In addition, using (B.41) we can define

$$\begin{aligned} \mathbf{E} :=& \mathbf{W}\partial_x^{-1} + \mathbf{U}\pi_0 + \mathbf{P} \\ =& \sum_{j=0}^{M} \mu \begin{pmatrix} 0 & X_j^{(2)}(x,\theta) \\ X_j^{(3)}(x,\theta) & 0 \end{pmatrix} \partial_x^{-j} + \mu \begin{pmatrix} 0 & Op(\tilde\nu_2(x,\theta,\xi)) \\ Op(\tilde\nu_3(x,\theta,\xi)) & 0 \end{pmatrix} \\ :=& \sum_{j=0}^{M} \mu\mathbf{X}_j\partial_x^{-j} + \mu\nu_X , \end{aligned} \tag{B.46}$$

where

$$\begin{aligned} &\mathbf{X}_0 := \mathbf{U}_0 , \quad \partial_x^0 := \pi_0 \\ &\mathbf{X}_1 := \mathbf{W} \\ &\mathbf{X}_j := \begin{pmatrix} 0 & c_j(\tilde\varphi_2^{(1)})_j + \tilde c_j(\varphi_2^{(1)})_j \\ -c_j(\tilde\varphi_3^{(1)})_j - \tilde c_j(\varphi_3^{(1)})_j & 0 \end{pmatrix} , \quad \forall j = 2, ...M-1 \\ &Op(\tilde\nu_i) = \sigma_i^{(k)} + Op(r(\xi))\circ\varphi_i^{(1)}\partial_x^{-1} + \varphi_i^{(1)}\partial_x^{-1}\circ Op(r(\xi)) , \quad i = 2,3 \end{aligned}$$

then, by (B.45), for $i = 2, 3$ we have

$$\|X_0^{(i)}\|_p^{k_0,\gamma} \leq_p \|\mathtt{v}\|_{p+3}^{k_0,\gamma} , \quad \|X_1^{(i)}\|_p^{k_0,\gamma} \leq_p \|\mathtt{v}\|_{p+4}^{k_0,\gamma}$$

and for $j = 2, ...M$ by (B.42) we have

$$\|X_j^{(i)}\|_p^{k_0,\gamma} \leq_{p,j} \|\mathtt{v}\|_{p+j-1}^{k_0,\gamma} , \quad |\tilde\nu_i|_{-M-1,p,0}^{k_0,\gamma} \leq_{p,M} \|\mathtt{v}\|_{p+2M+1}^{k_0,\gamma} . \tag{B.47}$$

Finally by (B.37) and (B.46) we define

$$\begin{aligned} \tilde{\mathbf{R}}_1 + \mathbf{E} :=& \sum_{j=0}^{M} \mu \begin{pmatrix} F_j^{(1)}(x,\theta) & F_j^{(2)}(x,\theta) \\ F_j^{(3)}(x,\theta) & F_j^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-j} + \mu \begin{pmatrix} Op(\beta_1(x,\theta,\xi)) & Op(\beta_2(x,\theta,\xi)) \\ Op(\beta_3(x,\theta,\xi)) & Op(\beta_4(x,\theta,\xi)) \end{pmatrix} \\ :=& \sum_{j=0}^{M} \mu\mathbf{F}_j\partial_x^{-j} + \mu\beta , \end{aligned} \tag{B.48}$$

where $\mathbf{F}_j = \mathbf{Y}_j + \mathbf{X}_j$ for all $j = 0, ..., M$, and $\beta = \nu_Y + \nu_X := \begin{pmatrix} Op(\beta_1(x,\theta,\xi)) & Op(\beta_2(x,\theta,\xi)) \\ Op(\beta_3(x,\theta,\xi)) & Op(\beta_4(x,\theta,\xi)) \end{pmatrix} \in OPS^{-M-1}$. By (B.47), (B.45), (B.47), (B.39) we can prove (B.24).

In order to prove (B.25) we have to use (B.44), (B.43) (B.47) and (B.38). The estimates (B.26) and (B.27) follows by the definition of $\mathbf{F}_j$ and β in (B.48) (recall also the estimates (7.25)). □

Thanks to Lemma B.3 we can write the inverse of the operator T_1 as follows

Lemma B.7. *The inverse of the operator T_1 defined in* (8.2) *admits the following asymptotic expansion*

$$T_1^{-1} = \mathbb{1} + \Phi_1 \partial_x^{-1} + \sum_{k=0}^{M-2} C(k)(\tilde{\Phi}_1)_{k+2}\partial_x^{-k-2} + \nu_1 \,, \tag{B.49}$$

where

$$(\tilde{\Phi}_1)_{k+2} = \begin{pmatrix} (\tilde{\Phi}_1)^{(1)}_{k+2} & (\tilde{\Phi}_1)^{(2)}_{k+2} \\ (\tilde{\Phi}_1)^{(3)}_{k+2} & (\tilde{\Phi}_1)^{(4)}_{k+2} \end{pmatrix} \qquad \nu_1 = \begin{pmatrix} \nu_1^{(1)} & \nu_1^{(2)} \\ \nu_1^{(3)} & \nu_1^{(4)} \end{pmatrix}$$

are some suitable matrices and pseudo-differential operators in OPS^{-M-1}.

Moreover, for $k = 0, .., M-2$ and for all $i = 1, ..., 4$ we have the following estimates

$$\|(\tilde{\Phi}_i)_{k+2}\|_p^{k_0,\gamma} \leq_{p,k} \mu \|\mathtt{v}\|^{k_0,\gamma}_{p+k+1+\sigma} \tag{B.50}$$

$$|\nu_1|^{k_0,\gamma}_{-M-1,p,0} \leq_{p,M} \mu \|\mathtt{v}\|^{k_0,\gamma}_{p+3M-6+\sigma} \,. \tag{B.51}$$

Proof. By Lemma B.3 and the explicit definition of Φ_1 in (8.2) and (8.15) the Lemma follows. □

Lemma B.8. *Let T_1 in* (8.2)*, and consider its decomposition defined in* (B.49)*. Let $\mathbf{E}$ be the operator defined in* (B.46) *and $\tilde{\mathbf{R}}_1$ defined in* (B.37)*, so that $\mathbf{E}+\tilde{\mathbf{R}}_1$ can be written as in* (B.48)*. Then the following asymptotic expansion holds*

$$\begin{aligned} T_1^{-1}(\mathbf{E} + \tilde{\mathbf{R}}_1) &= \sum_{j=0}^{M} \mu \begin{pmatrix} H_j^{(1)}(x,\theta) & H_j^{(2)}(x,\theta) \\ H_j^{(3)}(x,\theta) & H_j^{(4)}(x,\theta) \end{pmatrix} \partial_x^{-j} + \mu \begin{pmatrix} \delta_1(x,\theta,D) & \delta_2(x,\theta,D) \\ \delta_3(x,\theta,D) & \delta_4(x,\theta,D) \end{pmatrix} \\ &:= \sum_{j=0}^{M} \mu \mathbf{H}_j \partial_x^{-j} + \mu\delta \,, \end{aligned} \tag{B.52}$$

for some suitable functions $\mathbf{H}_j$ and $\delta \in OPS^{-M-1}$.

Moreover for $s = 1, ..., 4$ and $\forall j = 0, ..., M$

$$\|H_j^{(s)}\|_p^{k_0,\gamma} \leq_{p,j} \|\mathtt{v}\|^{k_0,\gamma}_{p+j+5+\sigma} \tag{B.53}$$

$$|\delta_1|^{k_0,\gamma}_{-M-1,p,0} \leq_{p,M} \|\mathtt{v}\|^{k_0,\gamma}_{p+3M+6+\sigma} \tag{B.54}$$

$$\|\partial_i H_j^{(s)}[\hat{\imath}]\|_{p_1} \leq_{p_1,j} \|\hat{\imath}\|_{p_1+j+5+\sigma} \tag{B.55}$$

$$|\partial_i \delta_s[\hat{\imath}]|_{-M,p_1,0} \leq_{p_1,M} \|\hat{\imath}\|_{p_1+3M+6+\sigma} \,. \tag{B.56}$$

Proof. By (B.49), (B.48) and Lemma B.4 we can write

$$\begin{aligned} T_1^{-1}(\mathbf{E} + \tilde{\mathbf{R}}_1) &= \left(\mathbb{1} + \Phi_1\partial_x^{-1} + \sum_{k=0}^{M-2} C(k)(\tilde{\Phi}_1)_{k+2}\partial_x^{-k-2} + \nu_1 \right) \left(\mu \sum_{j=0}^{M} C(j)\mathbf{F}_j \partial_x^{-j} \right) + \mu T_1^{-1}\beta \\ &= \sum_{j=0}^{M} C(j)\mu\mathbf{F}_j\partial_x^{-j} + \sum_{j=0}^{M-1} C(j)\mu\tilde{\mathbf{F}}_{j+1}\partial_x^{-j-1} + \sum_{k=0}^{M-2-j}\sum_{j=0}^{M-2-k} \mu C(j)\hat{\mathbf{F}}_{k+2+j}\partial_x^{-k-2-j} \\ &\quad + \mu\sigma_{j,F} + \mu T_1^{-1}\beta, \end{aligned}$$

where

$$\tilde{\mathbf{F}}_{j+1} = \Phi_1 \sum_{s=0}^{j} C(s)\partial_x^s \mathbf{F}_{j-s}\,, \quad \hat{\mathbf{F}}_{k+2+j} = (\tilde{\Phi}_1)_{j+2} \sum_{m=0}^{j} C(m)\partial_x^m \mathbf{F}_{k-m}$$

and $\sigma_{j,F}$ collects all the terms in OPS^{-M-1} generated by the composition (see Theorem 2.5).

We define $\delta := \sigma_{j,F} + T_1^{-1}\beta$. By riorganizing the series above we arrive to (B.52). Hence, by Lemma B.4 and (B.24) we have

$$\|\tilde{\mathbf{F}}_{j+1}\|_p^{k_0,\gamma}\,,\ \|\hat{\mathbf{F}}_{k+2+j}\|_p^{k_0,\gamma} \leq_{p,j} \mu \|\mathtt{v}\|_{p+j+5+\sigma}^{k_0,\gamma}\,,$$

that, also by (B.24) proved (B.53). The estimate B.54 follows by (B.25) and Lemma B.4. The estimates (B.55) and (B.56) follows by Lemma B.6. □

Lemma B.9. *Let T_1 be th operator in* (8.2) *and let $\mathbf{C}^{(2)}$ the matrix of functions defined in* (8.4)*. Then the following asymptotic expansion holds:*

$$(T_1^{-1} - \mathbb{1})\mathbf{C}^{(2)}\partial_x^1 = \mu^2 \sum_{j=0}^{M} \begin{pmatrix} \check{F}_j^{(1)} & \check{F}_j^{(2)} \\ \check{F}_j^{(3)} & \check{F}_j^{(4)} \end{pmatrix} \partial_x^{-j} + \mu^2 \begin{pmatrix} \tilde{\beta}_1 & \tilde{\beta}_2 \\ \tilde{\beta}_3 & \tilde{\beta}_4 \end{pmatrix}\,,$$

for some suitable functions and pseudo-differential operators in OPS^{-M-1}. Moreover, for all $j = 0, ..., M$ and $s = 1, ..., 4$

$$\|\check{F}_j^{(s)}\|_p^{k_0,\gamma} \leq_{p,j} \|\mathtt{v}\|_{p+j+1+\sigma}^{k_0,\gamma} \tag{B.57}$$

$$|\tilde{\beta}_s|_{-M-1,p,0}^{k_0,\gamma} \leq_{p,M} \|\mathtt{v}\|_{p+2M-2+\sigma}^{k_0,\gamma} \tag{B.58}$$

$$\|\partial_i \check{F}_j^{(s)}[\hat{\imath}]\|_p \leq_{p,j} \|\hat{\imath}\|_{p+j+1+\sigma} \tag{B.59}$$

$$|\partial_i \tilde{\beta}_s[\hat{\imath}]|_{-M,p,0} \leq_{p,M} \|\hat{\imath}\|_{p+2M-2+\sigma}\,. \tag{B.60}$$

Proof. By Lemma B.7, we have

$$\begin{aligned}(T_1^{-1} - \mathbb{1})\mathbf{C}^{(2)}\partial_x^1 &= \Phi_1 \partial_x^{-1}\mathbf{C}^{(2)}\partial_x + \sum_{j=0}^{M-1} C(j)\tilde{\Phi}_{j+2}\partial_x^{-j-2}\mathbf{C}^{(2)}\partial_x^{-j-1+s} + \tilde{\beta} \\ &= \sum_{k=0}^{M} C(k)\acute{\mathbf{F}}_k \partial_x^{-k} + \sum_{j=0}^{M-1} C(j)\bar{\mathbf{F}}_{j+1}\partial_x^{-j-1}\tilde{\beta}\end{aligned}$$

where

$$\acute{\mathbf{F}}_k = \Phi_1 \partial_x^k \mathbf{C}^{(2)}\,, \quad \bar{\mathbf{F}}_{j+1} = \tilde{\Phi}_{j+2} \sum_{s=0}^{k} C(s)\partial_x^s \mathbf{C}^{(2)}\,,$$

and $\tilde{\beta}$ collects all the terms in OPS^{-M-1} generated by the composition (see Theorem 2.5). Therefore we can define

$$\check{\mathbf{F}}_0 = \acute{\mathbf{F}}_0\,, \quad \check{\mathbf{F}}_k = \acute{\mathbf{F}}_k + \bar{\mathbf{F}}_{k-1}\,, \quad \forall k > 0.$$

The estimate follows by (2.27), (2.28) and (8.4) □

In conclusion, by Lemmas B.8 and B.9 we can expand the remainder $\mathbf{R}_1$ in (8.17) as follows

$$\mathbf{R}_1 := \mu \sum_{k=0}^{M} \begin{pmatrix} A_k^{(1)} & A_k^{(2)} \\ A_k^{(3)} & A_k^{(4)} \end{pmatrix} \partial_x^{-k} + \mu \begin{pmatrix} \Sigma_{R_1,1} & \Sigma_{R_1,2} \\ \Sigma_{R_1,3} & \Sigma_{R_1,4} \end{pmatrix} \tag{B.61}$$

where ∂_x^0 that denotes one of the operator belonging to $\{a\pi_0 + b\mathbb{1}\,,\ a, b \in \{0,1\}\}$,

$$\begin{pmatrix} A_k^{(1)} & A_k^{(2)} \\ A_k^{(3)} & A_k^{(4)} \end{pmatrix} := \begin{pmatrix} \check{F}_k^{(1)} & \check{F}_k^{(2)} \\ \check{F}_k^{(3)} & \check{F}_k^{(4)} \end{pmatrix} + \begin{pmatrix} H_k^{(1)} & H_k^{(2)} \\ H_k^{(3)} & H_k^{(4)} \end{pmatrix}, \quad k = 0, ..., M$$

and

$$\begin{pmatrix} \Sigma_{R_1,1} & \Sigma_{R_1,2} \\ \Sigma_{R_1,3} & \Sigma_{R_1,4} \end{pmatrix} := \begin{pmatrix} \tilde{\beta}_1 & \tilde{\beta}_2 \\ \tilde{\beta}_3 & \tilde{\beta}_4 \end{pmatrix} + \begin{pmatrix} \delta_1(x,\theta,D) & \delta_2(x,\theta,D) \\ \delta_3(x,\theta,D) & \delta_4(x,\theta,D) \end{pmatrix} \in OPS^{-M-1}.$$

Moreover the estimates in Lemma 8.2 holds.

B.3 The remainder $\mathbf{R}_2$

We now want to prove the expansion and the estimates given in Lemma 8.4 for the remainder $\mathbf{R}_2$.

For all the section we shall assume that $\|\mathtt{v}\|^{k_0,\gamma}_{\mathfrak{p}_0+\chi(M)+\sigma} \leq 1$, where $\chi(M) \in \mathbb{R}$ is a constant and $\sigma := \sigma(\tau, N, k_0)$.

Lemma B.10. *Let T_2^{-1} be the inverse of the operator T_2 defined in* (8.20)*. Let $\tilde{\mathbf{R}}_2$ in* (8.25) *and $\mathbf{P}_2$ in* (8.28) *(see also* (8.29) *and* (8.31)*).Then the following asymptotic expansion holds:*

$$\begin{aligned} T_2^{-1}(\tilde{\mathbf{R}}_2 + \mathbf{P}_2) &= \sum_{j=0}^{M} \mu \begin{pmatrix} H_k^{(1)} & H_k^{(2)} \\ H_k^{(3)} & H_k^{(4)} \end{pmatrix} \partial_x^{-k} + \mu \begin{pmatrix} \delta_1(x,\theta,D) & \delta_2(x,\theta,D) \\ \delta_3(x,\theta,D) & \delta_4(x,\theta,D) \end{pmatrix} \\ &:= \sum_{k=1}^{M} \mu \mathbf{H}_k \partial_x^{-k} + \mu\delta\,, \end{aligned}$$

for some suitable functions and pseudo-differential operators in OPS^{-M-1}. Moreover, for all $s = 1, ..., 4$ and for all $k = 1, ..., M$

$$\|H_k^{(s)}\|_p^{k_0,\gamma} \leq_{p,k} \|\mathtt{v}\|^{k_0,\gamma}_{p+2k+5+\sigma}\,, \quad k = 0, ..., M \tag{B.62}$$

$$|\delta|^{k_0,\gamma}_{-M-1,p,0} \leq_{p,M} \|\mathtt{v}\|^{k_0,\gamma}_{p+4M+6+\sigma} \tag{B.63}$$

$$\|\partial_i H_k^{(s)}[\hat{\imath}]\|_{p_1} \leq_{p_1,k} \|\hat{\imath}\|_{p_1+2k+5+\sigma}\,, \quad k = 0, ..., M \tag{B.64}$$

$$|\partial_i \delta[\hat{\imath}]|_{-M,p_1,0} \leq_{p_1,M} \|\hat{\imath}\|_{p_1+4M+6+\sigma}\,. \tag{B.65}$$

Proof. The proof is similar to the one in the previous Section and it is omitted. It follows by Lemmas B.3 and B.4. □

Lemma B.11. *Let T_2 in* (8.20)*, and $\mathbf{C}^{(3)}$ in* (8.22)*. Then the following asymptotic expansion holds*

$$(T_2^{-1} - \mathbb{1})\mathbf{C}^{(3)}\partial_x = \mu^2 \sum_{j=0}^{M} \begin{pmatrix} \tilde{F}_j^{(1)} & \tilde{F}_j^{(2)} \\ \tilde{F}_j^{(3)} & \tilde{F}_j^{(4)} \end{pmatrix} \partial_x^{-j} + \mu^2 \begin{pmatrix} \tilde{\beta}_1 & \tilde{\beta}_2 \\ \tilde{\beta}_3 & \tilde{\beta}_4 \end{pmatrix} = \mu^2 \sum_{j=0}^{M} \mathbf{F}_j \partial_x^{-j} + \mu^2 \beta\,,$$

for some suitable functions and pseudo-differential operators in OPS^{-M-1}. *Moreover, for* $s = 1, ..., 4$ *and for* $j = 0, ..., M$

$$
\begin{aligned}
\|F_j^{(s)}\|_p^{k_0,\gamma} &\leq_{p,j} \|\mathtt{v}\|_{p+j+2+\sigma}^{k_0,\gamma} \\
|\tilde{\beta}|_{-M-1,p,0}^{k_0,\gamma} &\leq_{p,M} \|\mathtt{v}\|_{p+M+1+\sigma}^{k_0,\gamma} \\
\|\partial_i F_j^{(s)}[\hat{\imath}]\|_{p_1} &\leq_{p,j} \|\hat{\imath}\|_{p_1+j+2+\sigma}^{k_0,\gamma} \\
|\partial_i \tilde{\beta}[\hat{\imath}]|_{-M-1,p_1,0} &\leq_{p,M} \|\hat{\imath}\|_{p_1+M+1+\sigma}^{k_0,\gamma} .
\end{aligned}
$$

Proof. The proof follows by Lemma B.5, and by the explicit definition of $\mathbf{C}^{(3)}$ and Φ_3 given in (8.22) and (8.31) . □

In conclusion by Lemmas B.11 and B.10 the remainder $\mathbf{R}_2$ in (8.33) as the following asymptotic expansion

$$
\mathbf{R}_2 := \mu \sum_{k=0}^{M} \begin{pmatrix} (A_k^0)^{(1)} & (A_k^0)^{(2)} \\ (A_k^0)^{(3)} & (A_k^0)^{(4)} \end{pmatrix} \partial_x^{-k} + \mu \begin{pmatrix} \Sigma_{R_2,1} & \Sigma_{R_2,2} \\ \Sigma_{R_2,3} & \Sigma_{R_2,4} \end{pmatrix} \tag{B.66}
$$

where ∂_x^0 that denotes one of the operator belonging to $\{a\pi_0 + b\mathtt{1}\ ,\ a, b \in \{0, 1\}\}$ and

$$
\begin{aligned}
\begin{pmatrix} (A_j^0)^{(1)} & (A_j^0)^{(2)} \\ (A_j^0)^{(3)} & (A_j^0)^{(4)} \end{pmatrix} &:= \begin{pmatrix} \tilde{F}_j^{(1)} & \tilde{F}_j^{(2)} \\ \tilde{F}_j^{(3)} & \tilde{F}_j^{(4)} \end{pmatrix} + \begin{pmatrix} H_j^{(1)} & H_j^{(2)} \\ H_j^{(3)} & H_j^{(4)} \end{pmatrix} \ , \ j = 0, ..., M \\
\begin{pmatrix} \Sigma_{R_2,1} & \Sigma_{R_2,2} \\ \Sigma_{R_2,3} & \Sigma_{R_2,4} \end{pmatrix} &:= \begin{pmatrix} \tilde{\beta}_1 & \tilde{\beta}_2 \\ \tilde{\beta}_3 & \tilde{\beta}_4 \end{pmatrix} + \begin{pmatrix} \delta_1(x,\theta,D) & \delta_2(x,\theta,D) \\ \delta_3(x,\theta,D) & \delta_4(x,\theta,D) \end{pmatrix} .
\end{aligned} \tag{B.67}
$$

Moreover the estimates in Lemma 8.4 holds.

B.4 Smoothing remainders along the block symmetrization

We now want to study the loss of derivatives that we have on the coefficients obtained in Section 8.2.2 during the block symmetrization. In order to give an explicit estimate of the coefficients we want to iterate Lemmas 8.2 and 8.4. The coefficients of the remainder at the $n-$th step (of the block symmetrization), depend on the coefficients of the $(n-1)$-th step. Hence for convenience we provide different numeration of the coefficients, e.g. we define $\mathbf{A}_k^0$ the matrix coefficient of the homogeneous terms ∂_x^{-k} at the "step 0", for $k = 0, ..., M$ (see also $\mathbf{R}_2$ in (8.35) or above).

By Section B.1 we have that every time that we are considering the remainder, we are allowed to write it as the sum of homogeneous terms plus a pseudo-differential operator in OPS^{-M-1}. In addition, ∂_x^0 shell denote one of the operators belonging to $\{a\pi_0 + b\mathbb{1}\, , a, b \in \{0, 1\}\}$.

We assume that $\|\mathtt{v}\|_{\mathtt{p}_0+\chi(M)+\sigma}^{k_0,\gamma} \leq 1$ where $\chi(M) \in \mathbb{R}$ is a constant and $\sigma := \sigma(\tau, N, k_0)$.

We start with $\mathcal{L}_2$ defined in (8.21), where the remainder $\mathbf{R}_2$ is written as in Lemma (8.4). Therefore the operator $\mathcal{L}_2$ is given by,

$$\mathcal{L}_2 = \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)}\partial_x + \mu \sum_{j=0}^{M} \mathbf{A}_j^0 \partial_x^{-j} + \mu \Sigma_{R_2}. \tag{B.68}$$

We consider the first transformation given in Lemma 8.5, that is $T_3 = \mathbb{1} + \Phi_3 \partial_x^{-3}$. By Lemmas B.5 and B.4, the conjugation of $\mathcal{L}_2$ with T_3 can be written as an homogeneous part plus a remainder in OPS^{-M-1}.

Moreover for every matrix $\mathbf{A} = \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix}$ we define

$$(\mathbf{A})^D := \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(4)} \end{pmatrix}.$$

Lemma B.12. *Let $\mathcal{L}_2$ and T_3 as above, with*

$$\varphi_3^{(2)} = \frac{2\sqrt{2}}{\sqrt{15}} \varepsilon^{-2} (A_0^0)^{(2)}, \quad \varphi_3^{(3)} = -\frac{2\sqrt{2}}{\sqrt{15}} \varepsilon^{-2} (A_0^0)^{(3)}.$$

Then

$$\mathcal{L}_3 := T_3^{-1} \mathcal{L}_2 T_3 := \Omega \cdot \partial_\theta + \mathbf{T}(\mathbf{D}) + \mathbf{C}^{(3)}\partial_x + \mu (\mathbf{A}_0^0)^D + \mu \sum_{j=0}^{M-1} \mathbf{A}_{j+1}^{(1)} \partial_x^{-j-1} + \mu \Sigma_{R_3}. \tag{B.69}$$

Moreover

$$\|\Phi_3\|_p^{k_0,\gamma} \le_p \mu \|\mathbf{A}_0^0\|_p^{k_0,\gamma} \le_p \mu \|\mathrm{v}\|_{p+5+\sigma}^{k_0,\gamma}, \qquad \|(\mathbf{A}_0^0)^D\|_p^{k_0,\gamma} \le_p \|\mathrm{v}\|_{p+5+\sigma}^{k_0,\gamma} \tag{B.70}$$

$$\|\mathbf{A}_1^1\|_p^{k_0,\gamma} \le_p \|\mathbf{A}_1^0\|_p^{k_0,\gamma} \le_p \|\mathrm{v}\|_{p+6+\sigma}^{k_0,\gamma}, \qquad \|\mathbf{A}_2^1\|_p^{k_0,\gamma} \le_p \|\mathbf{A}_2^0\|_p^{k_0,\gamma} \le_p \|\mathrm{v}\|_{p+9+\sigma}^{k_0,\gamma},$$

$$\|\partial_i \mathbf{A}_1^1[\hat{\imath}]\|_{p_1} \le_{p_1} \|\hat{\imath}\|_{p_1+6+\sigma}, \qquad \|\partial_i \mathbf{A}_2^1[\hat{\imath}]\|_{p_1} \le_{p_1} \|\hat{\imath}\|_{p_1+9+\sigma} \tag{B.71}$$

$$\|\mathbf{A}_j^1\|_p^{k_0,\gamma} \le_{p,j} \|\mathrm{v}\|_{p+3j+5+\sigma}^{k_0,\gamma}, \qquad \|\partial_i \mathbf{A}_j^1[\hat{\imath}]\|_{p_1} \le_{p_1,j} \|\mathrm{v}\|_{p_1+3j+5+\sigma} \quad j = 3, ..., M \tag{B.72}$$

$$|\sigma_3|_{-M-1,p,0}^{k_0,\gamma} \le_{p,M} \|\mathrm{v}\|_{p+5M+6+\sigma}^{k_0,\gamma}, \qquad |\partial_i \sigma_3[\hat{\imath}]|_{-M,p_1,0} \le_{p_1,M} \|\hat{\imath}\|_{p_1+5M+6+\sigma}. \tag{B.73}$$

Proof. The estimates (B.70) follow immediately by (8.36), with $k = 0$. Moreover, by the explicit definition of the remainder $\tilde{\mathbf{R}}_k$ in (8.46) (that is the collection of all the homogeneous terms and symbols of order higher then $-k$), with $k = 0$, and by (8.44) we have that

$$\mathbf{A}_1^1 = \mathbf{A}_1^0 + c_1(\Phi_3)_x, \qquad \mathbf{A}_2^1 = A_2^0 + c_2(\Phi_3)_{xx}, \qquad c_1, c_2 \in \mathbb{R}. \tag{B.74}$$

Therefore the estimate (B.71) follows by (8.36) and (B.74).

Note that $\mathbf{A}_j^1$, $j = 3, ..., M$ are linear combination of the derivatives of $\mathbf{C}^{(3)}, \mathbf{A}_j^0$, $j = 0, ..., M$, (see (8.46)), hence, iterating (8.19) and (8.36) we can prove the estimate (B.72) and (B.73). $\square$

We now argue inductively. Suppose that after k transformations we have

$$\begin{aligned}\mathcal{L}_{k+2} :=&T_{k+2}^{-1}\mathcal{L}_{k+1}T_{k+2}\\ :=&\Omega\cdot\partial_\theta+\mathbf{T}(\mathbf{D})+\mathbf{C}^{(3)}\partial_x+\mu(\mathbf{A}_0^0)^D+\mu(\mathbf{A}_1^1)^D\partial_x^{-1}+\mu(\mathbf{A}_2^2)^D\partial_x^{-2}\\ &+...+\mu(\mathbf{A}_{k-1}^{k-1})^D\partial_x^{-k-1}+\mu\sum_{j=0}^{M-k}(\mathbf{A}_{j+k}^k)\partial_x^{-k-j}+\mu\Sigma_{R_{k+2}}.\end{aligned}\tag{B.75}$$

In addition suppose that the following estimates hold:

$$\begin{aligned}&\|(\mathbf{A}_s^s)^D\|_p^{k_0,\gamma}\le_{p,s}\|\mathtt{v}\|_{p+s^2+5+\sigma}^{k_0,\gamma}, \qquad \|\partial_i(\mathbf{A}_s^s)^D[\hat{\imath}]\|_{p_1}\le_{p_1,s}\|\hat{\imath}\|_{p_1+s^2+5+\sigma}, \quad 0\le s\le k-1\\ &\|\mathbf{A}_k^k\|_p^{k_0,\gamma}\le_{p,k}\|\mathbf{A}_k^{k-1}\|_p^{k_0,\gamma}\le_{p,k}\|\mathtt{v}\|_{p+k^2+5+\sigma}^{k_0,\gamma}, \qquad \|\partial_i\mathbf{A}_k^k[\hat{\imath}]\|_{p_1}\le_{p_1,k}\|\hat{\imath}\|_{p_1+k^2+5+\sigma}\\ &\|\mathbf{A}_j^k\|_p^{k_0,\gamma}\le_{p,j}\|\mathtt{v}\|_{p+(k+1)j+5+\sigma}^{k_0,\gamma}, \qquad \|\partial_i\mathbf{A}_j^k[\hat{\imath}]\|_{p_1}\le_{p_1,j}\|\hat{\imath}\|_{p_1+(k+1)j+5+\sigma}, \quad j=k+1,...,M\\ &|\Sigma_{R_{k+2}}|_{-M-1,p,0}^{k_0,\gamma}\le_{p,M}\|\mathtt{v}\|_{p+(k+1)M+3M+6+\sigma}^{k_0,\gamma}, \qquad |\partial_i\Sigma_{R_{k+2}}[\hat{\imath}]|_{-M,p_1,0}\le_{p_1,M}\|\hat{\imath}\|_{p_1+(k+1)M+3M+6+\sigma}.\end{aligned}\tag{B.76}$$

Now we want to prove that the same estimate holds for $\mathcal{L}_{k+3}$.

Lemma B.13. *Let $\mathcal{L}_{k+2}$ in* (B.75), *and $T_{k+3}=\mathbb{1}+\Phi_{k+3}\partial_x^{-k-3}$, as in* (8.37) *with*

$$\varphi_{k+3}^{(2)}=\frac{2\sqrt{2}}{\sqrt{15}}\varepsilon^{-2}(A_k^k)^{(2)}, \quad \varphi_{k+3}^{(3)}=-\frac{2\sqrt{2}}{\sqrt{15}}\varepsilon^{-2}(A_k^k)^{(3)}.$$

Then

$$\begin{aligned}\mathcal{L}_{k+3} :=&T_{k+3}^{-1}\mathcal{L}_{k+2}T_{k+2}\\ =&\Omega\cdot\partial_\theta+\mathbf{T}(\mathbf{D})+\mathbf{C}^{(3)}\partial_x+\mu(\mathbf{A}_0^0)^D+\mu(\mathbf{A}_1^1)^D\partial_x^{-1}+\mu(\mathbf{A}_2^2)^D\partial_x^{-2}+...+\mu(\mathbf{A}_{k-1}^{k-1})^D\partial_x^{-k-1}\\ &+\mu(\mathbf{A}_k^k)^D\partial_x^{-k}+\mu\sum_{j=0}^{M-k-1}\mathbf{A}_{j+k+1}^{k+1}\partial_x^{-k-j-1}+\mu\Sigma_{R_{k+3}}.\end{aligned}\tag{B.77}$$

Moreover the following estimates hold

$$\begin{aligned}&\|\Phi_{k+3}\|_p^{k_0,\gamma}\le_p\mu\|\mathtt{v}\|_{p+k^2+5+\sigma}^{k_0,\gamma}\\ &\|\mathbf{A}_{k+1}^{k+1}\|_p^{k_0,\gamma}\le_{p,k}\|\mathbf{A}_{k+1}^k\|_p^{k_0,\gamma}\le_{p,k}\|\mathtt{v}\|_{p+(k+1)k+4+\sigma}^{k_0,\gamma},\\ &\|\partial_i\mathbf{A}_{k+1}^{k+1}[\hat{\imath}]\|_{p_1}\le_{p_1,k}\|\hat{\imath}\|_{p_1+(k+1)k+4+\sigma}\\ &\|\mathbf{A}_j^k\|_p^{k_0,\gamma}\le_{p,j}\|\mathtt{v}\|_{p+(k+2)j+4+\sigma}^{k_0,\gamma},\\ &\|\partial_i\mathbf{A}_j^k[\hat{\imath}]\|_{p_1}\le_{p_1,j}\|\hat{\imath}\|_{p_1+(k+2)j+4+\sigma}, \quad j=k+2,...,M\\ &|\Sigma_{R_{k+3}}|_{-M-1,p,0}^{k_0,\gamma}\le_{p,M}\|\mathtt{v}\|_{p+(k+2)M+3M+4+\sigma}^{k_0,\gamma},\\ &|\partial_i\Sigma_{R_{k+3}}[\hat{\imath}]|_{-M,p_1,0}\le_{p_1,M}\|\hat{\imath}\|_{p_1+(k+2)M+3M+4+\sigma}.\end{aligned}\tag{B.78}$$

Proof. The Lemma follows by Lemmas B.1, B.4, B.5 and by (B.76). □

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