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Finite Quotients of Hyperbolic Orbifold Groups of Small Covolume

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SUPERVISOR
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Thesis submitted for the degree of "Doctor Philosophiæ"
Academic Year 1996/97

SISSA - SCUOLA
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Il presente lavoro costituisce la tesi presentata da Luisa Paoluzzi, sotto la direzione del Prof. Bruno Zimmermann, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministero della Pubblica Istruzione n. 419 del 24.04.1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*

Il est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde.

From Jacobi's letter to Legendre
July 2nd, 1830

At the end of my staying in SISSA, it is a pleasure to thank all those who have supported me in these last three years. First of all, my supervisor, prof. B. Zimmermann, who introduced me to the fascinating world of hyperbolic geometry and group theory. His willingness to mathematical discussion was invaluable. Next I am indebted to dr. D. Franco for suggesting me the main references concerning arithmetic geometry and for his patience in explaining me all the questions I had in this field.

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INDEX

INTRODUCTION

As a consequence of Mostow's rigidity theorem (see [7]), volumes of hyperbolic 3-manifolds play a similar role to that of Euler characteristic in dimension 2 because they are a topological invariant. By Thurston-Jørgsen's theorem ([47], compare section 1.5 and Theorem 1.5.6), volumes of hyperbolic 3-manifolds and 3-*orbifolds* (recall that a manifold is an orbifold -see paragraph 1.5) are well-ordered; in particular if we choose a class of finite-volume orbifolds satisfying certain properties (e.g. closed manifolds, *cusped manifolds*, closed orbifolds, *cusped orbifolds*, manifolds with *totally geodesic boundary*, etc.) there exists an orbifold belonging to the class having minimum volume. In the past few years many papers have been devoted to establish the element of minimal volume of particular given classes (see [1], [2], [3], [26], [33], [34], [51]).

The orbifold of minimal volume is not known yet. The orbifolds of smallest known volumes, however, are *tetrahedral orbifolds* (see section 1.5 and 1.4 for definitions) in analogy to the 2-dimensional case where the orbifold of minimal volume is uniformized by the *triangle group* $(2, 3, 7)$. More precisely the orbifold of minimal known volume is double covered by a tetrahedral orbifold (see paragraph 3.1): its volume is ≈ 0.039050 while the lower bound given by Meyerhoff [35] is ≈ 0.0000013 (see Theorem 1.5.7). On the other hand, the cusped (non-compact, of finite volume, without boundary) orientable hyperbolic 3-orbifold of minimal volume is a tetrahedral orbifold (see [33]).

Another example is the limit orbifold of minimal volume (i.e. its volume is a limit of other volumes of orbifolds) which is a *polyhedral orbifold* (compare section 1.5), uniformized by the *Picard group* (see [2], [52], [15] and [16]).

In this work we shall study finite *admissible* (i.e. with torsion-free kernel, see Definition 1.5.5) quotients of type $PSL(2, q)$ and $PGL(2, q)$ of the tetrahedral groups uniformizing some *Lannér tetrahedral orbifolds* and some cusped tetrahedral orbifolds of smallest volumes.

The problem of classifying admissible quotients of *linear fractional type* (see paragraph 1.3) has been considered in [30] (see also [10]) for the triangle group $(2, 3, 7)$ and in [20] for two Lannér tetrahedral groups (one of which uniformizes the closed tetrahedral orbifold of minimum volume). The classification of admissible quotients of type $PSL(2, q)$ has also been considered in a classical work by Macbeath [30] for the *modular group* and in a work by Singermann [43] for the *extended modular group* (see Example 1.3.3 iii)).

The hyperbolic tetrahedral groups are the subgroups of index 2 of orientation-preserving isometries in the Coxeter group generated by the reflection in the faces of *hyperbolic (Coxeter) Lannér, cusped or unbounded tetrahedra* (see paragraph 1.4). We exploit a generalization of the method used in [19] and [20]: we make computations using matrices (representing the elements of $PSL(2,q)$ -see paragraph 1.3) thus obtaining certain algebraic conditions (see [48] and paragraph 2.1) equivalent to the existence of an admissible homomorphism from the tetrahedral groups to the linear fractional groups. These conditions are easily verifiable for four of the Lannér tetrahedral groups which are not considered in [19] and will lead us to a complete classification of admissible quotients of linear fractional type for these four groups; surjectivity will be ensured by the maximality of some spherical *vertex-groups* (see 1.4) of the tetrahedral groups, considered as subgroups inside $PSL(2,q)$ (see Theorem 1.3.8). For the remaining three groups, the conditions cannot be discussed for all q's at the same time but must be checked case by case (see paragraph 2.2). In any case we see that we have admissible quotients of linear fractional type only for $q = p$, $q = p^2$ or $q = p^4$.

In paragraph 2.3 we classify the admissible quotients of linear fractional type for some cusped tetrahedral groups uniformizing orbifolds of small volumes, in particular the cusped orbifold of minimal volume (Theorems 2.3.1 and 2.3.2). The cusped tetrahedral groups we are going to study are interesting also from a number-theoretical point of view since some of them are *Bianchi groups* or *extended Bianchi groups* (see [15] and Example 1.3.3 iv)) which can be considered as generalizations of the classical modular group. Indeed the groups uniformizing the three smallest cusped hyperbolic 3-orbifolds are the (extended) Bianchi groups studied in Theorems 2.3.1 case $k = 3$ and 2.3.2, 2.3.3 and 2.3.4 case $n = 3$ (see [37]). These groups admit natural homomorphisms to $PSL(2,q)$ or $PGL(2,q)$ given by *reduction of matrix coefficients mod p* (see 1.3.3 iv)). We prove that all the admissible quotients of linear fractional type of these groups are obtained as reductions mod p (with one exception, see Theorem 2.3.4). It turns out that we can have admissible quotients only for $q = p$ or $q = p^2$.

From a geometric viewpoint, looking for admissible quotients of linear fractional type of a group uniformizing a certain orbifold is equivalent to looking for hyperbolic 3-manifolds which are finite regular coverings of the orbifold itself (for the notion of orbifold covering see section 1.5). Consequently, the manifolds associated to the admissible quotients we classify are hyperbolic 3-manifolds (of small volumes) admitting actions of some linear fractional group. In paragraph 4.3, some application to the construction of closed 3-manifolds admitting large group actions (see also paragraph 1.6) is also discussed.

Using the results of chapter 2 and generalizing further the techniques adopted there, in chapter 3 we study some natural extensions of the tetrahedral groups. The first extension we consider (section 3.1) is the fundamental group of the hyperbolic 3-orbifold of minimal known volume. Notice that the tetrahedral orbifold which is its double covering is uniformized by one of the groups studied in detail in paragraph 2.2 (Theorem 2.2.1, case $k = 2, n = 5$). Again we find necessary and sufficient conditions to have an admissible quotient of linear fractional type. In this case (like for the three Lannér tetrahedral groups for which we do not give a complete classification) the conditions have to be checked case by case with the help of a computer.

The other type of extension we consider are the *Coxeter groups* (see 1.4). Since they contain the groups of global symmetries (also orientation-reversing) of some of the platonic solids, they cannot be mapped admissibly to the linear fractional groups (see Theorem 1.3.8, for a classification of all subgroups of $PSL(2, q)$, and paragraph 3.2). In this case it seems natural to consider admissible quotients of type $PSL(2, q) \times \mathbb{Z}_2$. The main result here is the complete classification of admissible surjections of this type for the Coxeter groups studied in chapter 2. One can compare [25] where some examples of admissible quotients of this type are found by computational methods.

In the last chapter we classify all admissible quotients of type $PSL(2, q)$ for another remarkable Bianchi group: the Picard group which is a polyhedral group associated to a cusped Coxeter pyramid. Here we exploit the result and method of Macbeath [30] concerning the (2,3)-generation of the groups $PSL(2, q)$ (see Theorem 1.3.9). The same question is considered in [43] for the extended modular group, as we have already noted. The Picard group is interesting from a geometric viewpoint as well since, as we have already said, it is the fundamental group of the limit orbifold of minimal volume (see [2]). The situation for the Picard group is more similar to that of the modular group rather than to that of the small Bianchi groups of tetrahedral type. In fact we shall see that $PSL(2, q)$ is an admissible quotient of the Picard group for most values of q (and in particular for infinite powers of any prime p). The reason is that the Picard orbifold (i.e. the quotient of the hyperbolic space by the Picard group) has a non-rigid cusp on which *hyperbolic Dehn surgery* can be performed while the cusped tetrahedral orbifolds have rigid cusps (see paragraph 1.6). Moreover the tetrahedral groups are unsplittable as a free-product with amalgamation or HNN-extension (as are the bounded triangle groups in dimension 2), see [56], whereas the Bianchi groups which are not of tetrahedral type, including the Picard group, are splittable (as is the modular group which is a cusped triangle group uniformizing the cusped orientable hyperbolic 2-orbifold of minimal volume), see [15] or [16]. The same behavior of the Picard group is shown by another Bianchi group of polyhedral type for which we give again the complete classification of all admissible quotients of type $PSL(2, q)$ in Theorem 4.2.2. Indeed our method works for all amalgams of type $\mathrm{G}_{k_1} *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathrm{G}_{k_2}$ generalizing the Picard group (for the definition of the groups G_k see 1.6.3; for $k_1 = 2, k_2 = 3$ we obtain the Picard group and for $k_1 = 2, k_2 = 4$ the group $PGL(2, \mathbb{Z}[i\sqrt{2}])$ studied in 4.2.2). These results also answer a question by Alperin (see [5]) showing that there are *imaginary quadratic rings of integers* whose associated Bianchi groups admit quotients of linear fractional type not induced by reduction of coefficients mod p. Nevertheless for the Bianchi group considered by Alperin all quotients of type $PSL(2, q)$, $q > 5$, arise by reduction of matrix coefficients mod p (Theorem 2.3.4) as it always happens in the non imaginary quadratic case.

In section 4.3 we exploit the fact that the Picard orbifold has a non-rigid cusp and construct closed hyperbolic 3-manifolds with $PSL(2, q)$-actions. By the above minimality property of the Picard orbifold it seems reasonable that for many of the groups $PSL(2, q)$ the minimal volume of a manifold with such an action is realized in this way. In the same section we also give a necessary and sufficient condition for an admissible surjection obtained by reduction of coefficients mod p of the Picard group to factorize through the fundamental group of a hyperbolic orbifold obtained by Dehn surgery with surgery coefficients (k, n) along the cusp of the Picard orbifold.

We remark that subgroups of small index in some Bianchi and tetrahedral groups have been classified in [22] and [12], by computational methods using the group-theory packages GAP [17] and Cayley, see also [8] and [15] for the Picard group.

NOTATION

Let us fix some notations which will be used throughout this work.

$\mathbb{Z}_n$ The cyclic group of order n.

$\mathbb{Z}$ The cyclic group of infinite order (i.e. the free group of rank 1). We shall use the same notation for the ring of integers.

$\mathbf{D}_n$ The dihedral group of order $2n$; it has the following presentation

$$\langle x, y \mid x^2, y^2, (xy)^n \rangle.$$

$\mathbf{S}_n$ The symmetric group acting on n elements (of order $n!$).

$\mathbf{A}_n$ The alternating group on n elements which is the index 2 subgroup of $\mathbf{S}_n$.

$\mathbb{Q}$ The field of rational numbers.

$\mathbb{R}$ The field of real numbers.

$\mathbb{C}$ The field of complex numbers.

$\mathbb{R}^h$ The Euclidean space of dimension h.

$\mathbf{S}^h$ The unit sphere of dimension h, i.e. the subset of points of $\mathbb{R}^{h+1}$ at distance 1 from the origin.

$\mathbf{B}^h$ The closed ball of dimension h centred at 0; observe that $\partial\mathbf{B}^h = \mathbf{S}^{h-1}$.

$\mathbf{H}^h$ The hyperbolic space of dimension h.

1. BACKGROUND

In this chapter we give some standard results and basic definitions which we shall use in the following. Some of the results are not strictly needed but we state them for the sake of completeness.

1.1. Useful properties of fields

In this paragraph we shall recall some basic facts about fields. Results are standard and their proofs, which we shall omit, can be found in any book of algebra (see, for instance, [27]). Throughout rings are understood to be commutative with unity. Let us start with finite (Galois) fields.

1.1.1 Theorem: *For each prime p and each positive integer m there exists exactly one (up to isomorphism) field $\mathbb{F}_q$ of order $q := p^m$.*

$\mathbb{F}_q$ is the splitting field of the polynomial $t^q - t$.

Remark that, if p is a prime, $\mathbb{F}_p$ is the ring of residues (mod p) $\mathbb{Z}_p$.

1.1.2 Corollary: *$\mathbb{F}_{q'}$ is a subfield of $\mathbb{F}_q$ if and only if q is a power of q'.*

1.1.3 Theorem: *The multiplicative group $\mathbb{F}_q^* := \mathbb{F}_q - \{0\}$ of $\mathbb{F}_q$ is cyclic.*

1.1.4 Definition: Let $q = p^m$ and define a field homomorphism in the following way

$$\mathcal{F} : \mathbb{F}_q \longrightarrow \mathbb{F}_q$$

$$\mathcal{F}(\alpha) = \alpha^p \,.$$

It is not difficult to prove that $\mathcal{F}$ is an automorphism of order m which leaves pointwise fixed exactly $\mathbb{F}_p$. It is called *Frobenius automorphism.*

Suppose that $m = 2m'$ and let $\mathcal{I}_{\mathcal{F}} := \mathcal{F}^{m'}$. $\mathcal{I}_{\mathcal{F}}$ is an automorphism of order 2 that we shall call *Frobenius involution* or *conjugation*; it leaves pointwise invariant the subfield $\mathbb{F}_{p^{m'}}$.

1.1.5 Proposition: *Assume that q^2 is odd and let $\mathcal{I}_{\mathcal{F}} : \mathbb{F}_{q^2} \longrightarrow \mathbb{F}_{q^2}$ be the Frobenius involution. Since $[\mathbb{F}_{q^2} : \mathbb{F}_q] = 2$ and by Theorem 1.1.3, we see that all elements of $\mathbb{F}_q$ admit square-roots in $\mathbb{F}_{q^2}$. Let $\iota \in \mathbb{F}_{q^2} - \mathbb{F}_q$ be the square-root of a non-square element of $\mathbb{F}_q$. Then the elements of the form $\iota\alpha$, $\alpha \in \mathbb{F}_q$ are exactly* *those on which $\mathcal{I}_{\mathcal{F}}$ acts as multiplication by -1. We shall denote the set of these elements by $\iota\mathbb{F}_q$ and by $\iota\mathbb{F}_q^*$ the set $\iota\mathbb{F}_q - \{0\}$.*

Proof:

If β is such that $\mathcal{I}_{\mathcal{F}}(\beta) = -\beta$, then squaring both sides, we get $\mathcal{I}_{\mathcal{F}}(\beta^2) = \beta^2$ so $\beta^2 \in \mathbb{F}_q$ but $\beta \notin \mathbb{F}_q^*$ else $\beta = \mathcal{I}_{\mathcal{F}}(\beta) = -\beta$ which is a contradiction. Viceversa $\mathcal{I}_{\mathcal{F}}(\iota\alpha)^2 = \mathcal{I}_{\mathcal{F}}((\iota\alpha)^2) = (\iota\alpha)^2$ and we deduce $\mathcal{I}_{\mathcal{F}}(\iota\alpha) = \pm\iota\alpha$ but we cannot have $\mathcal{I}_{\mathcal{F}}(\iota\alpha) = \iota\alpha$ unless $\alpha = 0$.

1.1.6 Remark: In order to simplify notations, in the following we shall always write α^q or $\bar{\alpha}$ instead of $\mathcal{I}_{\mathcal{F}}(\alpha)$ to denote the Frobenius involution.

Notice that for all $\alpha \in \mathbb{F}_{q^2}$ we have $\alpha + \alpha^q \in \mathbb{F}_q$ while $\alpha - \alpha^q \in \iota\mathbb{F}_q$; the Frobenius conjugation plays a similar role to that of conjugation of complex numbers.

The elements of $\mathbb{F}_q$ which admit square-roots in $\mathbb{F}_q$ are said to be *squares* in $\mathbb{F}_q$ while the remaining are said to be *non-squares* in $\mathbb{F}$. Remark that the product of two elements of $\mathbb{F}_q^*$ is a square if and only if both elements are either squares or non-squares. Notice that if q is even all elements of $\mathbb{F}_q$ are squares and admit exactly one square-root (belonging to $\mathbb{F}_q$) since squaring an element is exactly applying the Frobenius automorphism, instead if q is odd the number of squares in $\mathbb{F}_q$ is $(q+1)/2$ and they have two square-roots each, apart from 0.

The following Theorem is a classical result due to Gauss (see [41, page 315]).

1.1.7 Theorem (law of quadratic reciprocity): *For two distinct odd primes p and q we have:*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

where $\left(\frac{\alpha}{p}\right)$ denotes the Legendre symbol defined as

$$\left(\frac{\alpha}{p}\right) = \begin{cases} +1 & \textit{if } \alpha \textit{ is a square in } \mathbb{F}_p \\ -1 & \textit{otherwise.} \end{cases}$$

Mainly to fix notations, we give some other definitions.

1.1.8 Definition: Let R be a ring and R' a subring. We shall call *integral closure* of R' in R the set of elements of R which are roots of some monic polynomial with coefFicients in R'. It can be proved that the integral closure of R' is a subring of R containing R'.

1.1.9 Definition: Let F be an extension of $\mathbb{Q}$, the field of rational numbers. The *ring of integers* of F is the integral closure of $\mathbb{Z}$ in F; we shall denote it by $\mathcal{O}_F$. If F is a finite extension of $\mathbb{Q}$ we shall say that F is a *number field*.

We shall consider algebraic quadratic extensions of the form $\mathbb{Q}(\sqrt{-d})$, where d is a positive square-free integer (*imaginary quadratic number fields*). In this case, the ring of integers will be denoted by $\mathcal{O}_d$. It can be shown that these rings of integers are all of the form $\mathbb{Z}[\zeta]$ for a suitable $\zeta \in \mathbb{C}$. Observe that the minimal polynomial of ζ $P_\zeta(t) \in \mathbb{Z}[t]$ must be monic of second degree.

For all prime numbers p which do not divide d, consider the unique unitary ring homomorphism $\mathbb{Z} \longrightarrow \mathbb{F}_p$ (i.e. reduction mod p) and the homomorphism it naturally induces on the polynomial rings $\mathbb{Z}[t] \longrightarrow \mathbb{F}_p[t]$. Let $P_{\zeta,p}$ the image of P_ζ in $\mathbb{F}_p[t]$. This polynomial either admits two (non-zero, since p does not divide d) roots $\zeta_1, \zeta_2 \in \mathbb{F}_p$ or it does not. In this latter case the polynomial has two roots in $\mathbb{F}_{p^2}$. Anyhow, one can define two unitary ring homomorphisms

$$\phi_j : \mathbb{Z}[\zeta] \longrightarrow \mathbb{F}_{p^2} \;\; j = 1, 2$$

defined by $\phi_j(\zeta) := \zeta_j$. Their images are $\mathbb{F}_p$ resp. $\mathbb{F}_{p^2}$ according as the ζ_j belong to $\mathbb{F}_p$ or not.

Since they will be needed later, we give the following

1.1.10 Examples: We have

$$\mathcal{O}_1 = \mathbb{Z}[i]$$

$$\mathcal{O}_2 = \mathbb{Z}[i\sqrt{2}]$$

$$\mathcal{O}_3 = \mathbb{Z}[\omega]$$

where i and ω denote a primitive forth-root of unity (i.e. the imaginary unit) and a primitive cubic-root of unity respectively. We see the number fields associated to $\mathcal{O}_1$ and $\mathcal{O}_3$ are in particular *cyclotomic*, i.e. generated by primitive roots of unity.

1.1.11 Remark: In what follows primitive forth- and cubic-roots of unity even when dealing with finite fields will be denoted by i and ω respectively. It will be clear from the context which field they belong to and no confusion should arise.

1.1.12 Definition: Let F be a field. The unique algebraic extension of F which is also algebraically closed is called *algebraic closure* of F and denoted by $\bar{F}$.

1.2. Algebraic and arithmetic geometry

In this paragraph we give a sketchy review of some definition and results of algebraic and arithmetic geometry. The approach will be the simplest possible.

We refer to [24] for basic definitions and results. We shall use the material of this section to solve some systems of equations over finite fields. In fact we shall need less, i.e. an estimate of the number of solutions of the system.

1.2.1 Definitions: Let F be a field and F^h the h-dimensional vector space over F. A set $\mathbb{A}^h_F$ is called *affine space* of dimension h over F if there exists an action of (the additive group of) F^h on the set which is transitive and free (for a definition of action see paragraph 1.5; here the group homomorphism maps to the group of bijections of the set). We can identify $\mathbb{A}^h_F$ with F^h forgetting the special role played by 0. Points in the affine space are determined by their *coordinates*, i.e. h-tuples of elements of F.

The set of all lines (i.e. vector subspaces of dimension 1) in F^{h+1} is called *projective space* of dimension h over F and denoted by $F\mathbb{P}^h$. Since a line of F^{h+1} is determined by any of its non-zero vectors, a point in the projective space is determined by its *homogeneous coordinates*, i.e. an $(h+1)$-tuple $[\lambda_0 : ... : \lambda_h]$ of elements of F not all 0 defined up to multiplication by an element μ in F^* (i.e. $[\lambda_0 : ... : \lambda_h] = [\mu\lambda_0 : ... : \mu\lambda_h]$).

The zero locus in $\mathbb{A}^h_F$ of a collection of polynomials in $F[t_1, ..., t_h]$ is called *affine variety.* Notice that the zero locus of a collection of polynomials coincides with the zero locus of the polynomials belonging to the ideal generated by the collection. In particular, since the ring of polynomials is Noetherian, it is enough to consider finite collections of polynomials, generators of the ideals of the ring. The set of all affine varieties in $\mathbb{A}^h_F$ are the closed sets for a topology on $\mathbb{A}^h_F$ called the *Zariski topology.*

The zero locus in $F\mathbb{P}^h$ of a collection of homogeneous polynomials belonging to $F[t_0, ..., t_h]$ is called *projective variety.* The set of all projective varieties in $F\mathbb{P}^h$ are the closed sets for a topology on $F\mathbb{P}^h$ called again *Zariski topology.*

1.2.2 Examples: A variety (affine or projective) defined by a single non-constant polynomial is called *hypersurface.*

A variety defined by $h - r$ linearly independent linear (of degree 1) polynomials is called *r-plane.* A projective r-plane is the image of a vector subspace of dimension $r + 1$ of F^{h+1} in the projective space. An $(h - 1)$-plane is called *hyperplane.* A 1-plane is usually called a *line.*

It can be proved that $\mathbb{A}^h_F$ can be immersed as an open subspace in $F\mathbb{P}^h$, e.g. as the set $\{t_j \neq 0\}$. This means that the projective space of dimension h is covered by $h+1$ open sets which are copies of the affine space of dimension h. The closure of an affine variety contained in $\mathbb{A}^h_F$ thought of as an open subset of the projective space (in the Zariski topology) is called *projective closure.*

Any linear isomorphism of the vector space F^{h+1} induces an automorphism of the projective space of dimension h since it preserves lines. Such automorphism (with respect to the Zariski topology) of the projective space is called *projective equivalence.*

Let Λ be a vector subspace of dimension $h-r$ in F^{h+1} and Π a complementary vector subspace of dimension $r+1$. Denote by $\mathbb{P}(\Lambda) \cong F\mathbb{P}^{h-r}$ resp. $\mathbb{P}(\Pi) \cong F\mathbb{P}^r$ their images in the projective space $F\mathbb{P}^h$. Consider the linear endomorphism of F^{h+1} which is the identity on Π and has kernel Λ. The induced map

$$F\mathbb{P}^h - \mathbb{P}(\Lambda) \longrightarrow \mathbb{P}(\Pi)$$

is called *projection from* $\mathbb{P}(\Lambda)$ *to* $\mathbb{P}(\Pi)$. Projections are continuous maps in the Zariski topology sending projective varieties to projective varieties.

1.2.3 Definition: A continuous map

$$f : \mathcal{V}_1 \subset F\mathbb{P}^r \longrightarrow \mathcal{V}_2 \subset F\mathbb{P}^h$$

is called *regular* if it can be locally expressed as a $(h+1)$-tuple of homogeneous polynomials of the same degree, i.e.

$$[\lambda_0 : ... : \lambda_r] \mapsto [P_0(\lambda_0, ..., \lambda_r) : ... : P_h(\lambda_0, ..., \lambda_r)]$$

with $P_0, ..., P_h \in F[t_0, ..., t_r]$.

An *isomorphism* between two projective varieties is a regular map which admits an inverse regular map; the two varieties are *isomorphic.*

Let $P \in \mathcal{V}_1$; if $f^{-1}(f(P))$ has finite cardinality the map is said to be $of\,finite$ *degree.* The cardinality of the "generical" fibre is called *degree* of the map.

Observe that projections and projective equivalences are regular maps.

Even if all the definitions above make sense for arbitrary fields, it is better to assume that F is algebraically closed. Else let $\bar{F}$ the algebraic closure of F. Consider $\bar{F}\mathbb{P}^h$; the points in the projective space which admit homogeneous coordinates belonging to F are called *F-rational points.* We shall say that a projective variety is $defined$ *over* F if it is the zero locus of a family of polynomials in $F[t_0, ..., t_h] (\subset \bar{F}[t_0, ..., t_h])$.

Analogous definitions can be given in the affine case.

From now on fields are understood to be algebraically closed (if not otherwise specified).

An affine variety is called *irreducible* if it is the zero locus of a prime ideal. It is not difficult to prove that any affine variety is the union of irreducible ones so it is not restrictive to consider only irreducible varieties (the same definition can be given in the case of projective varieties but we shall not insist on that). It can be proved that two non-empty open sets of an irreducible variety have non-empty intersection, in fact this property characterizes irreducibility (even in the projective space).

1.2.4 Definition: Consider an affine irreducible variety $\mathcal{V} \neq \varnothing$ defined by polynomials $P_1, ..., P_\ell \in F[t_1, ..., t_h]$. Consider the Jacobian matrix

$$M_{ij} = (\partial P_i / \partial t_j)$$

and let

$$r := h - \max_{p \in \mathcal{V}} \{\mathrm{rk}(M_{ij}(p))\}.$$

It can be proved that the set of *smooth points* (i.e. those for which the Jacobian matrix has rank $h - r$) is open. The value r is called *dimension of* $\mathcal{V}$. A variety is said to be *smooth* if all its points are. We call *curves* the varieties of dimension 1 and surfaces those of dimension 2.

By definition we impose the dimension of the empty variety to be -1.

For irreducible projective varieties, dimension can be computed by studying a non-empty intersection of the variety with the affine spaces covering $F\mathbb{P}^h$. This is an affine irreducible variety and dimension is well-defined because of irreducibility.

1.2.5 Examples: A hypersurface in $\mathbb{A}^h_F$ has dimension $h-1$ while an r-plane has dimension r. The projective space $F\mathbb{P}^h$ and the affine space $\mathbb{A}^h_F$ have dimension h.

1.2.6 Definition: Let $\mathcal{V}$ be a projective variety of dimension r and let $\mathcal{P}$ a "generic" $(h-r)$-plane (i.e. $\mathcal{V} \cap \mathcal{P}$ consists of a finite number of points). Then the number of points, counted with their multiplicity, of $\mathcal{V} \cap \mathcal{P}$ is called the *degree of* $\mathcal{V}$.

1.2.7 Examples: The projective space and any r-plane have degree 1. A hypersurface has degree equal to the minimal degree of a polynomial defining it.

1.2.8 Definitions: A projective variety of dimension 1 in $F\mathbb{P}^2$ is called *plane curve.* Any plane curve is a hypersurface. A plane curve is called a *conic* (*cubic, quartic* resp.) if it has degree 2 (3, 4 resp.).

In projective spaces of higher dimension a hypersurface of degree 2 is called *quadric.*

The analogous definitions hold in the affine space.

1.2.9 Definition: Let $\mathcal{V}$ be a smooth plane curve of degree d. The quantity

$$g := \frac{(d-1)(d-2)}{2}$$

is called *arithmetic genus* of the curve. A curve of genus 0 (e.g. a line, a smooth conic) is called *rational.* A curve of genus 1 (e.g. a smooth cubic) is called *elliptic.* It is actually possible to define the genus for every curve. We shall not insist on that here; we just remark that the genus is defined up to isomorphism.

1.2.10 Remarks: If $F = \mathbb{C}$ and $\mathbb{CP}^h$ is endowed with the standard topology, a smooth curve is a Riemann surface. In this case, the arithmetic genus and the topological genus coincide. It can be proved in purely algebraic terms (see [43, page 88]), that the Riemann-Hurwitz formula holds for the arithmetic genus over arbitrary fields. This fact allows us to compute the genus of non-plane curves in many cases, once we consider their projection onto a plane. We state it here

1.2.11 Theorem (Riemann-Hurwitz formula): *Let $f : \mathcal{V} \longrightarrow \mathcal{V}'$ a surjective regular map of finite degree d between two smooth projective curves of genera g and g' respectively. Then the following equality holds:*

$$(2-2g) = d(2-2g') - \sum_{P \in \mathcal{V}'} (d - |f^{-1}(P)|)$$

where $|f^{-1}(P)|$ denotes the cardinality of the fibre $f^{-1}(P)$.

In the case of Riemann surfaces, the result is basely established by computing the Euler characteristics of both manifolds. For a topological approach see [54, page 155]. Recall that all 2-dimensional manifolds are triangulable (the result is due to Rado [39]).

now we give some results concerning the number of points of curves defined over finite fields.

1.2.12 Theorem (Dickson): *Let $\alpha_1 t_1^2 + \alpha_2 t_2^2 - \beta$, $\alpha_j \neq 0$ $j = 1, 2$, be the polynomial associated to an affine conic def/ined over $\mathbb{F}_q$, q odd. Then the number of $\mathbb{F}_q$-rational points on the conic is:*

$\beta \neq 0$ $\quad$ *$q-1$ or $q+1$ according as $-\alpha_1\alpha_2$ is a square or a non-square in $\mathbb{F}_q$;*

$\beta = 0$ $\quad$ *$2q-1$ or 1 according as $-\alpha_1\alpha_2$ is a square or a non-square in $\mathbb{F}_q$.*

Proof:

Since $\alpha_1 \neq 0$ the equation $\alpha_1 t_1^2 + \alpha_2 t_2^2 - \beta = 0$ is equivalent to $\alpha_1^2 t_1^2 - (-\alpha_1\alpha_2) t_2^2 = \alpha_1\beta$. Now if $-\alpha_1\alpha_2 = \delta^2$, $\delta \in \mathbb{F}_q$ we can write

i)
$$(\alpha_1 t_1 - \delta t_2)(\alpha_1 t_1 + \delta t_2) = \alpha_1 \beta$$

else $-\alpha_1\alpha_2 = (\iota\delta)^2$ and we can write

ii)
$$(\alpha_1 t_1 - \iota\delta t_2)(\alpha_1 t_1 + \iota\delta t_2) = (\alpha_1 t_1 - \iota\delta t_2)^{q+1} = \alpha_1\beta$$

since $\iota^q = -\iota$.

In case i) if $\beta = 0$ the solutions are found imposing that at least one of the two factors is 0. For a fixed $t_2 \in \mathbb{F}_q$ one finds two values t_1 satisfying the equation except for the case $t_2 = 0 = t_1$. So there are $2q - 1$ possible couples (t_1, t_2).

If $\beta \neq 0$ the equation is equivalent to the system

$$\begin{cases} \alpha_1 t_1 - \delta t_2 = \gamma \ \in \mathbb{F}_q^* \\ \alpha_1 t_1 + \delta t_2 = \alpha_1 \beta \gamma^{-1} \,. \end{cases}$$

Summing and subtracting one can express t_1 and t_2 and see that they depend on the parameter $\gamma \in \mathbb{F}_q^*$, so there are $q - 1$ couples (t_1, t_2) on the conic.

In case ii) if $\beta = 0$ we must have $\alpha_1 t_1 - \iota\delta t_2 = 0$ and for t_1, t_2 to belong to $\mathbb{F}_q$ the only possibility is $t_1 = t_2 = 0$. If $\beta \neq 0$ instead, exploiting the fact that the multiplicative group of the field is cyclic (Theorem 1.1.3) and $\alpha_1\beta \in \mathbb{F}_q$ we can find exactly $q + 1$ elements γ of $\mathbb{F}_{q^2}^*$ whose $(q + 1)$-power is $\alpha_1\beta$. We again have to solve a system

$$\begin{cases} \alpha_1 t_1 - \iota\delta t_2 = \gamma \\ \alpha_1 t_1 + \iota\delta t_2 = \alpha_1 \beta \gamma^q \end{cases}$$

and it is easy to see that the solutions (t_1, t_2) are in $\mathbb{F}_q$ (e.g. check that they are fixed by the Frobenius involution).

Remark that when q is even the equation is equivalent to a linear one because of the Frobenius automorphism.

1.2.13 Theorem (Hasse-Weil bound): *Let N be the number of $\mathbb{F}_q$-rational points of a projective curve of genus g defined over $\mathbb{F}_q$. Then the following inequality holds:*

$$|N - (q + 1)| \leq 2gq^{1/2}i\,.$$

The proof of the above result can be found in [44, page 170], while the following Theorem is proved in [28].

1.2.14 Theorem (Lang-Weil): *Let N be the number of $\mathbb{F}_q$-rational points of a projective variety of dimension r and degree d contained in $\overline{\mathbb{F}_q}\mathbb{P}^h$ defined over $\mathbb{F}_q$. Then there exists a constant A depending only on h, d and r (but not on the variety) such that the following estimate holds:*

$$|N - q^r| \leq (d - 1)(d - 2)q^{r-1/2} + Aq^{r-1}\,.$$

Note that for $h = 2$ and $r = 1$ we obtain the previous result with $A = 1$.

1.3. Review on linear fractional groups

We collect some definitions and properties of general linear groups and projective linear groups. We omit proofs which can be found in [45], mainly paragraph 3.6, or in [13].

1.3.1 Definitions: Let R be a ring (commutative, with unity) and $\mathcal{M}(h,R)$ the unitary ring of $h\times h$ matrices with entries in R (actually it can be also considered as an algebra over R). The multiplicative subgroup of invertible elements in $\mathcal{M}(h,R)$ is called *general linear group* of degree h over R and denoted by $GL(h,R)$. Observe that we have a surjective homomorphism of multiplicative semigroups given by the determinant:

$$\det : \mathcal{M}(h,R) \longrightarrow R$$

and $GL(h,R) = \det^{-1}(R^*)$, where R^* is the multiplicative subgroup of R (remark that $GL(h,R) = (\mathcal{M}(h,R))^*$). We shall call *special linear group* the group $SL(h,R) :=$Ker(det).

If $R = \mathbb{F}_q$ is a finite field, we shall write $GL(h,q)$ and $SL(h,q)$ instead of $GL(h,\mathbb{F}_q)$ and $SL(h,\mathbb{F}_q)$ respectively.

The group R^* can be identified with the central subgroup of $GL(h,R)$ consisting of diagonal elements of the form αI where I denotes the unity of the ring $\mathcal{M}(h,R)$ and $\alpha \in R^*$. The quotient $PGL(h,R) := GL(h,R)/R^*$ is called *projective general linear group.* The image of the subgroup $SL(h,R)$ in this quotient is denoted by $PSL(h,R)$ and called *projective special linear group.* Remark that in some cases we have that $PSL(h,R) = PGL(h,R)$, for instance when $R = \mathbb{C}$ is the field of complex numbers or if $R = \mathbb{F}_{p^m}$ and $h = p$ (this is due to the existence of h-roots of elements of R^*).

Notice that if $R = F$ is a field, the group $PGL(h,F)$ acts on the projective space $F\mathbb{P}^{h-1}$ (see paragraph 1.2). If $h = 2$ these groups are also called *linear fractional groups.*

1.3.2 Theorem: *Let $R = \mathbb{F}_q$. The groups $PSL(h,q)$ are simple for all choices of (h,q) except $(2,2),(2,3),(3,2)$.*

1.3.3 Examples: In the following we shall consider some remarkable projective linear groups and some of their properties.

i) The group of automorphisms of the complex projective line is $PSL(2,\mathbb{C}) = PGL(2,\mathbb{C})$. It can also be identified with the group of biholomorphisms of the 2-sphere $\mathbf{S}^2$ to itself and with the group of orientation-preserving isometries of the hyperbolic space $\mathbf{H}^3$.

ii) The group of automorphisms of the real projective line is $PGL(2,\mathbb{R})$ and it can be identified with the group of isometries of the hyperbolic plane $\mathbf{H}^2$. It is in a natural way a subgroup of $PGL(2,\mathbb{C})$ since the real numbers are a subfield of the complex numbers. The subgroup of the orientation-preserving isometries of the hyperbolic plane is $PSL(2,\mathbb{R})$.

iii) The *modular group* $PSL(2,\mathbb{Z})$ is in a natural way a subgroup of the previous three. It is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. This means that a group is $(2,3)$-generated (i.e. there exists two elements in the group of orders 2 and 3 respectively which generate the whole group) if and only if it is a quotient of the modular group. We shall call $PGL(2,\mathbb{Z})$ *extended modular group*; it is again a subgroup of the groups described in i) and ii).

Notice that, since the real and complex numbers are endowed in a natural way with a topology, so are all the projective linear groups defined on them. In this topology the modular and extended modular groups are discrete subgroups (indeed the integers are a discrete subset of the real line and of the complex plane).

iv) The *Bianchi groups* and *extended Bianchi groups* (see e.g. [15]) generalize the modular and extended modular groups and constitute an important class of discrete subgroups of $PSL(2,\mathbb{C})$. They are of the form $PSL(2,\mathcal{O}_d)$ resp. $PGL(2,\mathcal{O}_d)$. Among them we mention the *Picard group* $PSL(2,\mathcal{O}_1)$.

Assume that q is such that the ring homomorphisms ϕ_j defined in the previous paragraph satisfy $\phi_j(\mathcal{O}_d) \subset \mathbb{F}_q$. In this case they induce group homomorphisms

$$\Phi_j : PSL(2,\mathcal{O}_d)(PGL(2,\mathcal{O}_d)) \longrightarrow PSL(2,q)(PGL(2,q))$$

by applying ϕ_j to the entries of the matrices which represent the elements of $PSL(2,\mathcal{O}_d)$ $(PGL(2,\mathcal{O}_d))$. We shall say that the two homomorphisms are obtained by *reduction of coefficients mod p*.

v) Some groups of type $PSL(2,q)$ and $PGL(2,q)$ are isomorphic to spherical groups (i.e. groups of isometries of $\mathbf{S}^2$) or alternating groups. We recall that the spherical groups are cyclic or dihedral of any finite order or the groups of symmetries of the five platonic solids: tetrahedron, cube, octahedron, dodecahedron, icosahedron. We have
$PSL(2,2) \cong \mathbf{D}_3$ the dihedral group of order 6,
$PSL(2,3) \cong \mathbf{A}_4$ the tetrahedral group or alternating group on four elements.
$PGL(2,3) \cong \mathbf{S}_4$ the octahedral group or symmetric group on four elements.
$PSL(2,2^2) \cong PSL(2,5) \cong \mathbf{A}_5$ the dodecahedral group or alternating group on five elements,
$PSL(2,3^2) \cong \mathbf{A}_6$ the alternating group on six elements.

From now on we shall assume $h = 2$ and $R = \mathbb{F}_q$. For these groups see also [17] and [18].

1.3.4 Theorem: *The order of $PSL(2,q)$ is $q(q^2-1)/2$ if q is odd and $q(q^2-1)$ if q is even. If q is odd the projective special linear group is a subgroup of index 2 in the projective general linear group; else they coincide.*

1.3.5 Definition: The *unitary group* is

$$U(2,q) := \{\begin{pmatrix} a & b \\ -b^q & a^q \end{pmatrix} \mid a,b \in \mathbb{F}_{q^2},\ a^{q+1}+b^{q+1} \in \mathbb{F}_q^*\}$$

and its subgroup

$$SU(2,q) := \{\begin{pmatrix} a & b \\ -b^q & a^q \end{pmatrix} \in U(2,q) \mid a^{q+1}+b^{q+1} = 1\}$$

is the *special unitary group*. The quotients of these two groups by their centres are the groups $PU(2,q)$ resp. $PSU(2,q)$ and are called *projective unitary group* and *projective special unitary group*.

1.3.6 Theorem: *The groups $SL(2,q)$ ($PGL(2,q)$) and $SU(2,q)$ (resp. $PU(2,q)$) are isomorphic.*

Notice that $PSL(2,q)$ and $PSU(2,q)$ are subgroups of $PSL(2,q^2)$. Because every element of $\mathbb{F}_q$ is a square in $\mathbb{F}_{q^2}$ also $PGL(2,q)$ and $PU(2,q)$ can be considered as subgroups of $PSL(2,q^2)$, by normalizing determinants to one (note however that after normalization the elements in $PU(2,q) - PSU(2,q)$ are no longer in unitary form). Moreover the two pairs of isomorphic groups are conjugate in $PSL(2,q^2)$.

Since $SL(2,q)$ is a matrix group, we can consider the trace of an element of $SL(2,q)$ which is an element of $\mathbb{F}_q$. Elements of $PSL(2,q)$ are just elements of $SL(2,q)$ up to sign (indeed if q is even, we have $PSL(2,q) = SL(2,q)$). Thus we can associate to each element of $PSL(2,q)$ an element of $\mathbb{F}_q$, well-defined up to sign, which is the trace of one of the matrices representing our element; we shall call such number again *trace*. There exists an element of order k in $PSL(2,q)$ if and only if k divides either $(q-1)/2$ (resp. $q-1$) or $(q+1)/2$ (resp. $q+1$), or if $k = p$ (where $q = p^m$) if q is odd (resp. even); in this last case the element is parabolic, that is, has trace ± 2. Two non-parabolic elements are conjugate if and only if they have the same trace. For all integers $k \geq 2$ there exists a polynomial $P_k \in \mathbb{Z}[t]$ with the following property (see [49]):
Consider an element of $PSL(2,q)$ of order $k \geq 2$. Then its trace is a root of the polynomial $P_{k,q} \in \mathbb{F}_q[t]$ which is the image of P_k in the canonical unitary ring homomorphism $\mathbb{Z}[t] \longrightarrow \mathbb{F}_q[t]$ (this is an easy consequence of the Cayley-Hamilton theorem, see [26]). Viceversa if there exists a root t_0 of $P_{k,q}$ in $\mathbb{F}_q$ then the image in $PSL(2,q)$ of the matrix

$$\begin{pmatrix} t_0 & 1 \\ -1 & 0 \end{pmatrix}$$

has order a divisor of k. In particular if k is a prime number and a root of the polynomial exists, then elements of order k exist in $PSL(2,q)$.

1.3.7 Examples: We give some examples of polynomials $P_k(t)$.

$P_2(t) = t$
$P_3(t) = t^2 - 1$
$P_4(t) = t^2 - 2$
$P_5(t) = (t^2 + t - 1)(t^2 - t - 1)$
$P_6(t) = t^2 - 3$
$P_p(t) = t^2 - 4$.

Since $PGL(2,q) \subset PSL(2,q^2)$, it will make sense to talk of traces, parabolic elements and so on also for elements of $PGL(2,q)$ and $PU(2,q)$; observe that the trace of an element in $PGL(2,q) - PSL(2,q)$ is an element in $\iota\mathbb{F}_q$. The orders of non-parabolic elements in $PGL(2,q)$ divide either $q-1$ or $q+1$. Non-parabolic elements of $PSL(2,q)$, q odd, (resp. q even or $PGL(2,q)$) are conjugate to diagonal matrices if their orders divide $(q-1)/2$ (resp. $q-1$) -*hyperbolic elements*; they are conjugate to diagonal matrices in $PSU(2,q) \cong PSL(2,q)$, q odd, (resp. q even or $PU(2,q) \cong PGL(2,q)$) if their order divides $(q+1)/2$ (resp. $q+1$) -*elliptic elements*.

1.3.8 Theorem (classification of subgroups of $PSL(2,q)$):

A group is contained in $PSL(2,q)$ if it is a subgroup of one of the following groups:

i) *dihedral of order $q \pm 1$ (resp. $2(q \pm 1)$) if q is odd (resp. even);*
ii) *a group H such that $1 \longrightarrow (\mathbb{Z}_p)^m \longrightarrow H \longrightarrow \mathbb{Z}_n \longrightarrow 1$ where $q = p^m$ and $n = (q-1)/2$ if q is odd or $n = q-1$ if q is even;*
iii) $\mathbf{A}_4$ *if $q \neq 2^{2m+1}$ and it is a maximal subgroup whenever q is a prime number. $q > 3$ and $q \equiv 3, 13, 27, 37 \pmod{40}$;*
iv) $\mathbf{S}_4$ *if $q \equiv \pm 1 \pmod 8$ and it is a maximal subgroup whenever q is a prime number;*
v) $\mathbf{A}_5$ *if $q(q^2-1) \equiv 0 \pmod 5$ and it is maximal whenever $q = 4^m, 5^m$ and m is a prime, $q \equiv \pm 1 \pmod 5$ is a prime number or $q \equiv -1 \pmod 5$ is a square of an odd prime number;*
vi) *$PSL(2,q')$ if q is a power of q' and it is maximal if $q = (q')^m$ with m an odd prime number;*
vii) *$PGL(2,q')$ if q is an even power of q' and it is maximal if $q = (q')^2$.*

In particular $PSL(2,q)$ does not have subgroups of index lesser than $q+1$ unless $q = 2, 3, 5, 7, 3^2, 11$.

The subgroups in ii) *are called $affine$ (as well as the cyclic subgroups); those in* vi) *and* vii) *are called projective and the remaining exceptional.*

1.3.9 Theorem (Macbeath): *The groups $PSL(2,q)$ are $(2,3)$-generated or, equivalently, quotients of the modular group, for all $q \neq 3^2$.*

1.3.10 Theorem (Dickson): *Let $q \neq 2^m, 3^2$ and let $\xi \in \mathbb{F}_q$ be a generator of $\mathbb{F}_q$ over $\mathbb{F}_p$ ($q = p^m$). Then the group $PSL(2,q)$ is generated by the matrices*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -\xi & 1 \end{pmatrix}.$$

For the last two results see [39] and [13] respectively.

1.4. Coxeter, triangle and tetrahedral groups

In this paragraph we shall give the basic definitions concerning some important classes of groups, i.e. Coxeter, triangle and tetrahedral groups. Again we refer to [45], paragraph 3.4. Some results on triangle groups can be found in [6] (see also [54]).

1.4.1 Definition: Let C be a group admitting a system of generators $\{x_j\}_{j\in J} \subset C$ of order 2 (i.e. $\langle\{x_j\}_{j\in J}\rangle = C$ and $x_j^2 = 1$). Let m_{ij} be the order of the element x_ix_j if the order is finite, so that m_{ij} is a positive integer, or $m_{ij} = 0$ if the order of x_ix_j is infinite. Consider the set $\{(x_ix_j)^{m_{ij}}\}$ for all i,j. If it is a complete system of relations for the group C, then we shall say that C is a *Coxeter group*. Notice that $m_{ij} = m_{ji}$. If J is a finite set, we can store all the information concerning a Coxeter group in a symmetric matrix M such that $M = (m_{ij})$ (note that $m_{ii} = 1$ for $x_ix_i = 1$).

1.4.2 Example: The Coxeter groups with two generators are the dihedral groups if $m_{12} \neq 0$; all dihedral groups arise in this way. If $m_{12} = 0$, the group is the free product of two copies of the cyclic group of order 2.

Geometrically, examples of Coxeter groups are given by the groups of reflections in the sides (resp. faces) of a polygon (resp. polyhedron) with (dihedral) angles of width π/n, for some integer $n \geq 2$. Indeed a presentation for these groups can be given using Poincaré's theorem for fundamental polygons (resp. polyhedra) (see [32]), thus proving that these groups are Coxeter groups. Polygons (resp. polyhedra) with (dihedral) angles of this kind are called *Coxeter polygons* (resp. *Coxeter polyhedra*). Their subgroups of index 2 consisting of all orientation-preserving isometries are called *polygonal* (resp. *polyhedral*) *groups.* In particular, if the polygon (resp. polyhedron) is a triangle (resp. tetrahedron) we shall call the subgroup of orientation-preserving isometries *triangle* (resp. *tetrahedral*) *group.* A presentation for these subgroups can be given using the Reidemeister-Schreier's subgroup method (see [54]). Remark that the polygons (polyhedra) (resp. two copies of them) are fundamental domains for the associated Coxeter groups (resp. their subgroups of orientation-preserving isometries).

It is well-known that, given a triplet (m, n, k) of integers $k \geq n \geq m \geq 2$, there exists a triangle with angles $(\pi/m, \pi/n, \pi/k)$ in the 2-sphere, Euclidean plane or hyperbolic plane exactly when $1/m + 1/n + 1/k$ is > 1, $= 1$ or < 1 respectively. This means that we can always realize a Coxeter group with three generators and three relations, all of finite orders, as a group of isometries (in the appropriate space). In this case we shall call the Coxeter group *extended triangle group* and denote it by $[m, n, k]$. A (extended) triangle group is called *spherical, Euclidean* or *hyperbolic* according as it acts as a group of isometries of the 2-sphere, Euclidean plane or hyperbolic plane respectively. The triangle group associated to $[m, n, k]$ will be denoted by (m, n, k).

1.4.3 Examples: Extended triangle groups and triangle groups have the following presentations:

$$[m, n, k] = \langle x_1, x_2, x_3 \mid \{x_j^2\}_{j=1,2,3}, (x_1x_2)^m, (x_1x_3)^n, (x_2x_3)^k \rangle \,,$$

$$(m, n, k) = \langle x = x_1x_2, y^{-1} = x_1x_3 \mid x^m, y^n, (xy)^k \rangle \,.$$

Some triangle groups of special type are

i) the dihedral groups:

$$\mathbf{D}_n = (2, 2, n);$$

ii) the groups of symmetries of the platonic solids:

$$\mathbf{A}_4 = (2, 3, 3),$$

$$\mathbf{S}_4 = (2, 3, 4),$$

$$\mathbf{A}_5 = (2, 3, 5).$$

Notice that the above are exactly the groups associated to all possible spherical triangles.

The Coxeter groups of reflections in the faces of a tetrahedron have four generators. All information concerning them can be summarized in the following way. Draw a tetrahedron and mark its faces with the numbers $1, 2, 3, 4$, then mark the edge common to the faces i and $j \in \{1, 2, 3, 4\}$, $i \neq j$, with the integer $m_{ij} \geq 2$ if the associated dihedral angle has width π/m_{ij}. The associated Coxeter group has presentation

$$\langle x_1, x_2, x_3, x_4 \mid \{x_j^2\}_{j=1,\dots,4}, \{(x_ix_j)^{m_{ij}}\}_{i\neq j} \rangle$$

(x_i denotes the reflection in the face i) while its tetrahedral subgroup has presentation

$$\langle \{x_{ij} := x_i x_j\}_{i \neq j} \mid \{x_{ij}^{m_{ij}}\}_{i,j=1,\ldots,4} \rangle \,.$$

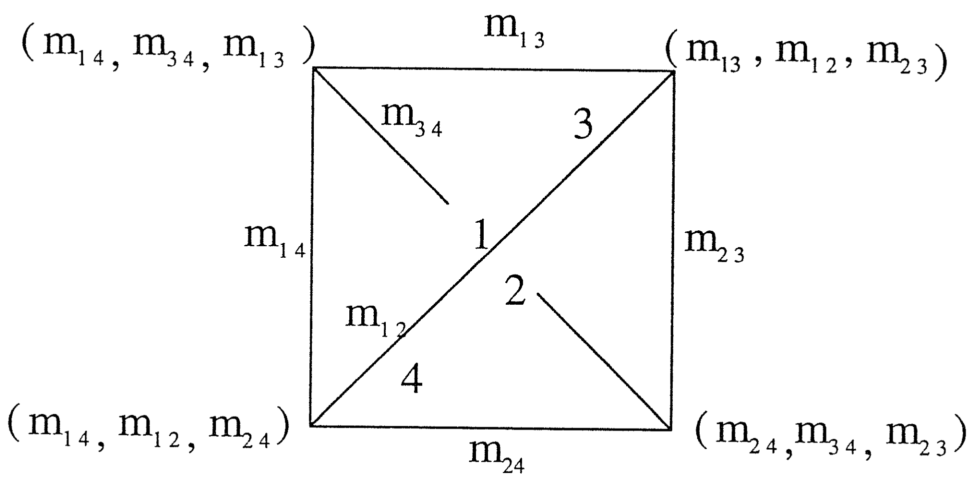


Figure 1

Notice that the labels of the edges are the orders of the rotations along the edges themselves (i.e. the orders of the cyclic groups stabilizing the edges), which generate the tetrahedral group. We can label each vertex with the triangle group generated by the rotations along its three incoming edges (see Figure 1). We shall call a vertex *spherical, Euclidean* or *hyperbolic* if such is the triangle group attached to it. Observe that the 1-skeleton of the tetrahedron plus the labels is a graph of groups, thus we shall call *vertex-groups* (resp. *edge-groups*) the groups attached to vertices (resp. edges) of the graph.

Observe that the extended triangle groups are a particular class of Coxeter groups generated by the reflections in the faces of certain tetrahedra. With the notation of Figure 1, we have that the extended triangle group $[m, n, k]$ is the tetrahedral group associated to the Coxeter tetrahedron with edges labeled by $m_{12} = m_{13} = m_{14} = 2$, $m_{23} = m$, $m_{24} = n$ and $m_{34} = k$.

We are interested in particular in tetrahedra that can be realized in hyperbolic space (see [47]). It turns out that only nine Coxeter tetrahedra can be realized as bounded tetrahedra in $\mathbf{H}^3$. Such tetrahedra are called *Lannér tetrahedra* and all their vertices are spherical. There are then 23 tetrahedra whose vertices either belong to $\mathbf{H}^3$ or to the sphere at infinity (they have at least one vertex of this kind). We shall call these *cusped tetrahedra.* They have finite volume and their ideal vertices are Euclidean (for a list, see [35] or [47]). The remaining hyperbolic Coxeter tetrahedra have infinite volume since some of their vertices (the hyperbolic ones) are beyond the sphere at infinity. We shall call these *unbounded.*

1.4.4 Examples:

i) The extended Picard group is a hyperbolic cusped tetrahedral group.

ii) Other hyperbolic cusped tetrahedral groups are: $PGL(2, \mathbb{Z}[\omega])$, $PSL(2, \mathbb{Z}[\omega])$.

iii) The Picard group is the polyhedral group of the Coxeter group generated by the reflexions in the faces of a hyperbolic quadrangular pyramid.

It is possible to describe Coxeter (tetrahedral) groups by means of *Coxeter diagrams* but we will not make use of them in this work.

1.5. Hyperbolic geometry, orbifolds and volumes

In this paragraph we shall recall some basic definitions concerning hyperbolic geometry. The main references here are [7] and [47]. For group actions one can see also [6].

We recall that by a *hyperbolic h-manifold* we mean a h-dimensional Riemannian manifold of constant negative curvature -1. In the following we shall always assume manifolds to be orientable. The notion of *orbifold* generalizes that of manifold: an orbifold is a space locally modelled on $\mathbb{R}^h$ modulo some finite-group action. We say that a group G *acts* on a topological space M if there is a group homomorphism mapping G to the group of homeomorphisms of M. We shall only consider *faithful actions*, i.e. the homomorphism mapping the group is injective.

1.5.1 Definition: We shall call *hyperbolic G-manifold of dimension h* an orientable complete hyperbolic h-manifold on which a finite group G acts effectively (i.e. faithfully) by orientation-preserving isometries.

A group is discrete if its image has discrete subspace topology. The *stabilizer of a point* is the subgroup of G which leaves the point fixed. A group acts *freely* if the stabilizer of any point is trivial. A group acts *transitively* if for any pair of points there exists an element of G mapping the first point to the second. The set of the images of a point by all elements of G is called *orbit* of that point. The stabilizers of two points in the same orbit are conjugate; in particular, if the action is transitive, there exists exactly one orbit and the stabilizers of any two points of the space are conjugate. A group acts *discontinuously* if for any point there exists a neighbourhood which intersects only finitely many of its images (or *translates*) by elements of G; in particular: if a group acts discontinuously the stabilizers of any point must be finite, a finite group acts discontinuously. The *quotient of a topological space under a group action* is the space consisting of all orbits of G with the quotient topology induced by the natural projection mapping a point to its orbit. It is very easy to see that the orbits form an equivalence relation.

1.5.2 Definition: A *hyperbolic orientable h-orbifold* $\mathfrak{O}_\Gamma$ is the quotient of $\mathbf{H}^h$ by the discontinuous action of a discrete group of orientation-preserving isometries Γ of the hyperbolic h-space. The *singular set* of an orbifold is the set of points which are images in $\mathfrak{O}_\Gamma$ of points of the hyperbolic space fixed by some non-trivial element of Γ. We shall say that a singular point (i.e. a point belonging to the singular set) $P \in \mathfrak{O}_\Gamma$ is of type Γ_P if Γ_P is (up to conjugation) the stabilizer in Γ of a preimage in $\mathbf{H}^h$ of P. If $\Gamma_P \cong \mathbb{Z}_n$ we shall say that *P has order n*. We shall say that Γ *uniformizes* $\mathfrak{O}_\Gamma$ or that Γ is the group of *deck transformations* of $\mathfrak{O}_\Gamma$ acting on its universal covering $\mathbf{H}^h$. Extending the usual concept of covering, we shall call Γ *orbifold fundamental group* of $\mathfrak{O}_\Gamma$ and denote it by $\pi_1^{\mathrm{or}}(\mathfrak{O}_\Gamma)$ or simply by $\pi_1(\mathfrak{O}_\Gamma)$; moreover for all subgroups $\Gamma' \subset \Gamma$ the quotient space $\mathbf{H}^h/\Gamma'$ will be called *orbifold covering* of $\mathfrak{O}_\Gamma$ associated to Γ'. Notice that outside the singular set an orbifold covering is a covering.

We shall say that a group uniformizing an orbifold is *cocompact, of finite covolume, etc.* if the associated orbifold is compact, of finite volume, etc..

1.5.3 Remarks: In a similar fashion one defines *spherical* and *Euclidean orbifolds.* Indeed a more general definition of the notion of orbifold can be given (see [47]). Notice that we are considering only *good orbifolds.*

Notice that according to a theorem by Hopf-Killing (see [53, pag 69]) any hyperbolic manifold is in particular an orbifold. The theorem states that any hyperbolic h-manifold is the quotient of $\mathbf{H}^h$ by the free and discontinuous action of a discrete group of isometries of $\mathbf{H}^h$. Trivially the singular set of an orbifold which is also a manifold is empty. Viceversa an orbifold is a manifold whenever it is uniformized by a group acting freely.

Recall that for $h = 3$ a group acts discontinuously if and only if it is discrete (see [6]). As a consequence of this fact, the group of isometries of a hyperbolic 3-manifold (orbifold) closed or with totally geodesic boundary is finite.

Naively an orbifold is a topological manifold endowed with a metric which becomes singular at certain points. Thus an orbifold is determined by a topological space plus a subspace (the singular set) whose points are labeled by groups describing the type of singularity. In some sense the orbifolds are analogous to branched coverings in algebraic geometry. For this reason in some cases we shall also say that an orbifold covering is a *branched covering.*

Observe that the orbifold fundamental group and the fundamental group of the underlying topological space are different in general (unless the orbifold is a manifold). Consider, for instance, the group of rotations of order n about a point in the hyperbolic plane. The associated orbifold is a cone, i.e. topologically it is simply a disk. Its orbifold fundamental group is $\mathbb{Z}_n$ while the fundamental group of the underlying space is trivial. For a hyperbolic orbifold one can prove that $\pi_1^{\mathrm{or}}(\mathfrak{O}_\Gamma)/\langle \mathrm{Tor}(\pi_1^{\mathrm{or}}(\mathfrak{O}_\Gamma))\rangle \cong \pi_1(\mathfrak{O}_\Gamma)$ where the quotient is over the normal closure of the torsion elements in the group (compare [23]; in the hyperbolic case only elliptic elements have finite order and fixed points -see [6]).

Assume from now on that $h = 3$. In this case, the singular set of an orbifold is an embedded graph of groups whose edges are always labeled with finite cyclic groups. The orbifold fundamental group can be recovered by applying either the orbifold version of the Seifert-Van Kampen theorem (see [23]) or, if the orbifold is topologically $\mathbf{S}^3$, the Wirtinger's method for knots to the complement of the graph (see [40], [58, page 202]). Considering orbifold coverings of $\mathbf{S}^3$ is not restrictive, in a certain sense, as the following result shows (see [4]):

1.5.4 Theorem (Alexander): *Every 3-manifold is a branched covering of* S^3.

The orbifolds which are uniformized by a hyperbolic (cusped, Lannér, etc.) tetrahedral group (see section 1.4) are called *hyperbolic* (*cusped*, *Lannér*, etc.) *tetrahedral* orbifolds. Topologically they are $\mathbf{S}^3$ minus a finite number of points (corresponding to the ideal and hyperbolic vertices) with singular set a graph which is the 1-skeleton of the tetrahedron without the ideal and hyperbolic vertices. More precisely, the graph is the image of the 1-skeleton of the Coxeter tetrahedron whose double is a fundamental domain for the tetrahedral group. The interior points of an edge have exactly the order of the rotation along that edge. Indeed such rotation stabilizes the edge. The vertices are instead of type (m, n, k) if the three incoming edges have orders m, n, k (compare also the previous paragraph). When we are not considering Lannér tetrahedra, we can truncate the hyperbolic space along horospheres centred at the ideal vertices and along hyperbolic planes orthogonal to the three faces meeting at the hyperbolic vertices of our Coxeter tetrahedron. We thus construct a (convex) subspace of $\mathbf{H}^3$ which is invariant for the action of the corresponding tetrahedral group. The quotient of this subspace by the action of the tetrahedral group is a compact orbifold with non-empty boundary. The boundary consists of Euclidean (in the case of ideal vertices) or totally geodesic hyperbolic triangular 2-orbifolds (i.e. 2-spheres with three branch points).

For tetrahedral orbifolds the orbifold fundamental group is given by the graph amalgamation or polygonal product over the graph of groups associated to the singular set (see [9], [52]), that is the iterated free-product of the vertex groups amalgamated over the edge groups. Notice that this is the quotient of the fundamental group of the graph of groups obtained by setting the HNN-generators equal to 1.

1.5.5 Definition: A group homomorphism defined from a graph amalgamation product to a finite group will be called *admissible* if it preserves the orders of torsion elements in the vertex groups. In particular, if the graph amalgamation product has infinite order an admissible homomorphism has torsion-free kernel and if has a finite vertex-group then the homomorphism restricted to such vertex-group is injective.

We want to recall some basic results concerning hyperbolic volumes. As a consequence of Mostow's rigidity theorem (see [7]) which states that two hyperbolic closed manifolds of dimension $h \geq 3$ are isometric if and only if they are homotopically equivalent (in particular if and only if they are homeomorphic), the hyperbolic volume of a manifold is a topological invariant and so plays an important role in hyperbolic geometry (in dimension 2 this follows from Gauss-Bonnet theorem -see again [7]). We state here some important results holding in dimension 3.

1.5.6 Theorem (Jørgsen-Thurston): *The volumes of complete orientable hyperbolic* 3*-orbifolds form a closed non-discrete subset of the real line. The set is well-ordered (of ordinal type* ω^ω*). There are only finitely many orbifolds of a given volume.*

Remark that the number of orbifolds having the same volume can be arbitrary large.

1.5.7 Theorem (Meyerhoff): *All complete orientable hyperbolic* 3*-orbifolds have volume at least* 0.0000013. *All cusped complete orientable hyperbolic* 3*-orbifolds have volume at least* 0.07217.

For these two results see [47] and [35] respectively.

The Thurston-Jørgsen theorem implies that there must exist an orbifold of minimal volume and that if we consider the class of orbifolds satisfying certain given properties, there must be an orbifold of minimal volume belonging to the class.

1.5.8 Examples: The manifold of smallest volume (≈ 0.94) was constructed by Weeks (see [51]) and by Matveev and Fomenko (see [33]). Manifolds of second and third known smallest volumes were constructed by Thurston (see [47]) and Meyerhoff and Neumann (see [36]).

Among all hyperbolic 3-manifolds with totally geodesic boundary there are six of minimal volume (see [26]). They are quotients of a double triangular pyramid (the suspension of a triangle) whose vertices (which are hyperbolic) are truncated along planes orthogonal to the faces. One of these manifolds was investigated by Thurston in [47].

The non-compact manifold of minimal volume is the figure-eight knot complement (see [1]).

We stress that the closed orbifold of minimal volume is not known yet. Lannér tetrahedral orbifolds have actually volumes among the smallest known. Indeed the smallest known orbifold volume is attained by an orbifold which is double covered by a Lannér tetrahedral orbifold (see [35], a list can be found also in [47]). Recall that in dimension 2 the orbifold of minimal volume is uniformized by the triangle group $(2,3,7)$.

Among cusped orbifolds, the smallest volume is attained by the orbifold uniformized by the group $PGL(2,\mathbb{Z}[\omega])$ (see [34]) which is a tetrahedral group (see Examples 1.3.3 iv) and 1.4.4 ii)). The second and third smallest volumes of cusped orbifolds are again attained by tetrahedral cusped orbifolds (see [3], [38]).

The Picard group (see Examples 1.3.3 iv) and 1.4.4 iii)) uniformizes the limit orbifold of minimal volume (i.e. its volume is a limit of other volumes of orbifolds -see [2]).

1.6. Heegaard splittings and Dehn surgery

In this paragraph we shall consider some facts concerning low-dimensional topology. We refer to [38] (for Heegaard splittings and Dehn surgery) and [42] (for triangulations and handles).

1.6.1 Definition: Let M be a h-manifold with non-empty boundary. A h-ball $\mathrm{B}^r \times \mathrm{B}^{h-r}$ such that $\partial\mathrm{B}^r \times \mathrm{B}^{h-r}$ is glued homeomorphically into ∂M will be called an *r-handle of dimension h.* The subspace $\partial\mathrm{B}^r \times \mathrm{B}^{h-r}$ is called the *attaching sphere* while the subspace $\mathrm{B}^r \times \partial\mathrm{B}^{h-r}$ is called the *belt sphere.*

It is a standard fact that, attaching to a manifold an r-handle and an $(r+1)$-handle in such a way that the belt sphere of the former intersects transversally in exactly one point the attaching sphere of the latter, the manifold does not change. In this case the handles are said to be *complementary* and they *cancel.*

From now on let $h = 3$.

1.6.2 Definition: An *handlebody of dimension* 3 *and genus* g is a 3-ball with g 1-handles attached. Notice that g is the genus of the boundary (an orientable surface) and that a handlebody of genus g is the connected sum of g solid tori. A handlebody of genus g can also be seen as the product of a 2-disk with g holes times a closed interval.

1.6.3 Definition: For all integers $k \geq 2$ let

$$\mathrm{G}_k = \mathrm{D}_k *_{\mathbb{Z}_k} (2,3,k).$$

We shall call G_k*-group* any finite admissible (see Definition 1.5.5) quotient of the group G_k defined above.

Notice that the group $\mathbf{G}_2$ is the extended modular group $PGL(2, \mathbb{Z})$ (see Example 1.3.3 iii)).

We recall the well-known theorem by Hurwitz (1893) (see e.g. [54]) which gives a bound on the order of a group acting orientation-preservingly on a closed Riemann surface of genus $g > 1$. Any such group must have order not larger than $84(g-1)$. In analogy in [55] the following result was proved

1.6.4 Theorem (Zimmermann): *If the finite group G acts orientation-preservingly on a handlebody of genus $g > 1$, then $|G| \leq 12(g-1)$. If equality holds, then G is a surjective image with torsion-free kernel of the group $\mathbf{G}_k$, $k = 2, 3, 4, 5$, moreover any such quotient give rise to an action of maximal order.*

In other words, a group G of order $12(g-1)$ acts orientation-preservingly on a handlebody of genus $g > 1$ if and only if it is a $\mathbf{G}_k$-group ($k = 2, 3, 4, 5$). These groups are also called *handlebody groups* (see also [57]).

Remark that the $\mathbf{G}_2$-groups are also the finite symmetry groups of maximal possible order (i.e. $12(g-1)$) of compact bounded (non-closed) surfaces of algebraic genus $g > 1$ (compare Definition 1.6.1; given an action on a 2-disk with g holes we recover the action on the handlebody by letting the action be trivial on the interval).

1.6.5 Definition: Assume that a 3-manifold M can be decomposed into two handlebodies of genus g identified along their boundaries. Such a decomposition is called *Heegaard splitting* (or *decomposition* or *diagram*) *of genus g* for the manifold M. The *genus* of M is the minimal genus of all possible Heegaard splittings for M.

1.6.6 Remarks: The only 3-manifold of genus 0 is $\mathbf{S}^3$.

Every triangulable 3-manifold admits a Heegaaard splitting where one of the two handlebodies is given by a tubular neighbourhood of the 1-skeleton of a triangulation. According to a theorem of Moise-Bing (see [37]) every 3-manifold is triangulable and hence every 3-manifold admits a Heegaard splitting.

Heegaard diagrams can be represented by the curves on one of the two handlebodies along which the meridians of the other are glued.

One can extend the definition of Heegaard splitting to the orbifold case. Topologically nothing changes but in this case the handlebodies are endowed with some singular set.

As in [57] we give the following

1.6.7 Definition: Let G be a finite group acting on a closed 3-manifold M. M is called *G-manifold of genus g* if g is the minimal genus of a Heegaard splitting for M which is left invariant under the G action.

If $g > 1$ and $|G| = 12(g-1)$ (i.e. G has maximal possible order) we say that the G-manifold and the G-action are *maximally symmetric* (see [57] and [58]).

1.6.8 Definition: Let M be a 3-manifold perhaps with non-empty boundary. Let $L = \cup_j L_j$ a (finite) link in the interior of M. Let U_j's be disjoint tubular neighbourhoods of the L_j's (i.e. U_j is homeomorphic to the product of the component L_j of the link times an open 2-ball). Let $\mathcal{C}_j$ a simple closed curve in ∂U_j. Remove from M the tubular neighbourhoods U_j and glue them back in such a way that a meridian of U_j is identified with $\mathcal{C}_j$. The resulting manifold is said to be obtained by *Dehn surgery* on M along L. It can be proved that the resulting manifold does not depend on the particular identification chosen. Moreover if we substitute the curves $\mathcal{C}_j$ with other curves in the same homology class (in $H_1(\partial U_j; \mathbb{Z})$) the resulting manifold does not change.

The following is a fundamental result of 3-dimensional topology (see [29] and [50]).

1.6.9 Theorem (Lickorish-Wallace): *Any closed* 3*-manifold can be obtained by Dehn surgery on* $\mathbf{S}^3$.

We thus see that it is not restrictive to assume $M = \mathbf{S}^3$. Recall that the homology $H_1(\partial U_j; \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 2 with basis given by a preferred frame, i.e. a longitude (a curve which is null-homologous in the solid torus $M - U_j$) and a meridian (a curve which is null-homologous in the solid torus U_j). Any curve $\mathcal{C}_j$ is then determined by a pair of coprime integers, i.e. its coefficients in terms of the preferred frame. These pair of integers are called *surgery coefficients.*

Let us now go back to hyperbolic geometry. If we consider a cusped hyperbolic 3-manifold (with finite volume) we can truncate the cusps obtaining a manifold with boundary consisting of tori. Along these tori it is possible to perform Dehn surgery. In this setting we have the following result (see [47])

1.6.10 Theorem (Thurston's hyperbolic Dehn surgery): *Let M be a cusped hyperbolic* 3*-manifold with finite volume. For almost all surgery coefficients the manifold obtained by Dehn surgery along the truncation of M is hyperbolic with volume not greater than the volume of M.*

The theory can be extended to the orbifold case (see [14]) but we will not give details here.

2. TETRAHEDRAL CASE

In this chapter we classify admissible quotients of type $PSL(2, q)$ and of type $PGL(2, q)$ of some hyperbolic tetrahedral groups. In paragraph 2.1 working with matrices we obtain necessary and sufficient conditions for three classes of tetrahedral groups to have an admissible homomorphism with image in $PSL(2, q)$ or $PGL(2, q)$. This result is applied in the remaining paragraphs to obtain the complete classification of quotients of linear fractional type for some groups belonging to the classes considered in 2.1.

2.1. Technical results

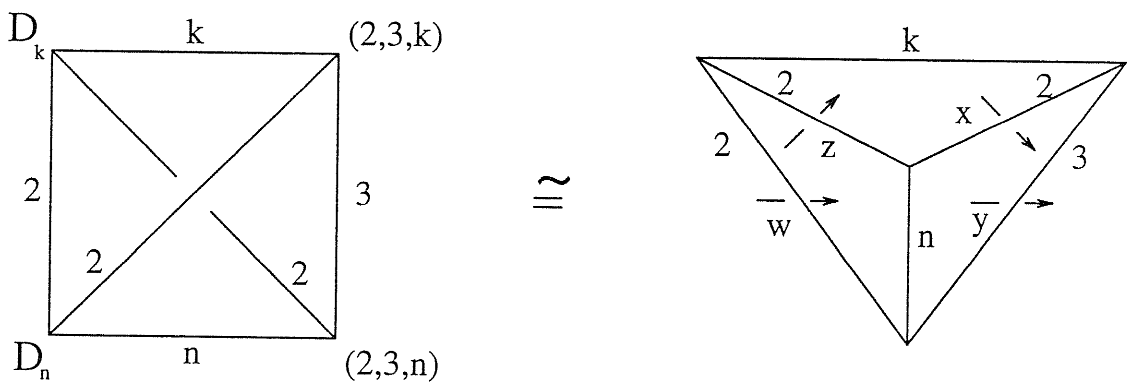


Figure 2

In this paragraph we shall discuss some technical results giving necessary and sufficient conditions for the existence of an admissible homomorphism (see Definition 1.5.5) from three classes of tetrahedral groups (see paragraph 1.4) to the linear fractional groups $PSL(2, q)$ and $PGL(2, q)$ (see paragraph 1.3). We also give necessary and sufficient conditions for another class of tetrahedral groups to be mapped admissibly to the groups $PSL(2, q)$. The first two classes of tetrahedral groups depend on two integer parameters k and n. We shall denote these classes by $T_{k,n}$ and $\tilde{T}_{k,n}$. In Figures 2 and 3 the singular sets of the tetrahedral orbifolds associated to the two classes are drawn; a set of generators for the groups is also pictured. Notice that the Figures represent also the Coxeter tetrahedra whose doubles are fundamental polyhedra for the associated tetrahedral groups (see again paragraph 1.4).

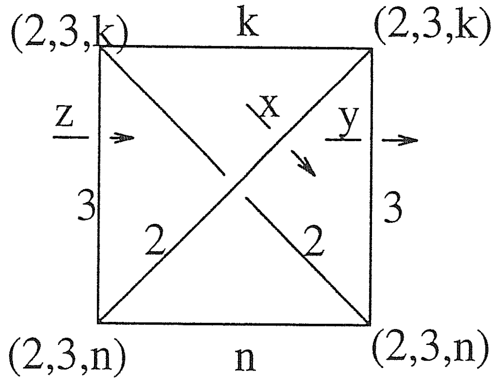


Figure 3

The two classes of groups have the following presentations in terms of the given generators:

$$T_{k,n} = \langle x, y, z \mid x^2, y^3, z^2, (xy)^k, (xyz)^2, (zx)^n \rangle$$

$$\widetilde{T}_{k,n} = \langle x, y, z \mid x^2, y^3, z^3, (xy)^k, (xyz)^2, (zx)^n \rangle .$$

In particular, they are quotients of the free-products with amalgamation of their upper vertex-groups. The vertex groups are the following triangle groups:

$$(2,3,k) = \langle x, y \mid x^2, y^3, (xy)^k \rangle$$

right-upper vertex-group

$$\mathrm{D}_k = \langle z, w \mid z^2, w^2, (zw)^k \rangle$$

$$(2,3,k) = \langle z, w \mid z^3, w^2, (zw)^k \rangle$$

left-upper vertex-groups. We have

$$T_{k,n} = (\mathrm{D}_k *_{\mathbb{Z}_k} (2,3,k))/\mathbb{Z}_n$$

$$\widetilde{T}_{k,n} = ((2,3,k) *_{\mathbb{Z}_k} (2,3,k))/\mathbb{Z}_n$$

where $\mathbb{Z}_k = \langle xy \rangle = \langle (zw)^{-1} \rangle$ and $\mathbb{Z}_n = \langle zx \rangle$.

Observe that, for a symmetry matter, $T_{k,n} \cong T_{n,k}$ and $\widetilde{T}_{k,n} \cong \widetilde{T}_{n,k}$.

Remark that $T_{k,n} = \mathrm{G}_k/\mathbb{Z}_n$ (see Definition 1.6.2). This implies that the tetrahedral orbifolds associated to the tetrahedral groups $T_{k,n}$ are those which admit a Heegaard splitting along a 2-orbifold which is a 2-sphere with four branch points of orders 2, 2, 2 and 3. This is the minimal possibility for Heegaard splittings of 3-orbifolds which is not too special (i.e. along hyperbolic Heegaard 2-orbifolds), see [56] and paragraph 1.6.

We have thus seen that the groups $T_{n,k}$ are $\mathbf{G}_k$-groups (and $\mathbf{G}_n$-groups as well). In particular for $k = 2$ these groups are quotients of the extended modular group (see Example 1.3.3 iii)) and it is not difficult to see that $T_{2,n} \cong [2,3,n]$ (compare paragraph 1.4).

Among the tetrahedra associated to these groups there are two Lannér tetrahedra (associated to $T_{4,5}$ and $T_{5,5}$) and four cusped hyperbolic tetrahedra (associated to $T_{k,6}$ with $k = 3,4,5,6$). For all larger values of k, n the associated tetrahedra are unbounded hyperbolic, while for smaller values they belong to other geometries. The tetrahedron associated to $T_{4,5}$ has minimal volume among the Lannér tetrahedra (see [35]), and finite admissible quotients of the tetrahedral groups $T_{4,5}$ and $T_{5,5}$ have been investigated in [20]. As we have already pointed out in Examples 1.5.8, the tetrahedral orbifold associated to $T_{3,6}$ is the unique cusped hyperbolic 3-orbifold of minimal volume ([34]).

All but one of the remaining Lannér tetrahedra are associated to groups in the second class; more precisely to $\widetilde{T}_{4,n}$ with $n = 3,4$ and $\widetilde{T}_{5,n}$ with $n = 2,3,4,5$. The tetrahedral orbifold associated to $\widetilde{T}_{5,2}$ is the double covering of the hyperbolic 3-orbifold of minimal known volume (see again [35]).

To study the admissible homomorphisms from these tetrahedral groups to the groups of linear fractional type we give necessary and sufficient conditions to the existence of matrices of proper orders to which the generators of the tetrahedral group can be mapped. We first give conditions for the existence of an admissible homomorphism from a vertex group (which is a particular triangle group) to $PSL(2,q)$ and $PGL(2,q)$ and then we try to extend such homomorphism to the whole tetrahedral group. We shall consider the triangle vertex-group

$$(2,3,k) = \langle x, y \mid x^2, y^3, (xy)^k \rangle .$$

2.1.1 Lemma: *Let γ be the trace of an element of order k in $PSL(2,q^2)$.*

There exists an admissible homomorphism Φ from the triangle group $(2,3,k)$ to $PSL(2,q)$ (resp. $PGL(2,q)$) such that the trace of $\Phi(xy)$ is γ if and only if $\gamma \in \mathbb{F}_q$ (resp. $\gamma \in \iota\mathbb{F}_q^*$; in this case assume $q \neq 2^m$), unless $k = 2$ and $q = 3^{2m+1}$. In this case the image is in $PGL(2,q)$. Moreover if $\gamma^2 \neq 3$, up to conjugation in $PGL(2,q)$ there is exactly one such homomorphism.

Proof:

In the following we shall write $X := \Phi(x)$ and $Y := \Phi(y)$.

First of all notice that the condition on the trace is trivially necessary. Moreover if we want the image of our triangle group to be in $PGL(2,q)$ but not in $PSL(2,q)$ we must require the element of order k to belong to $PGL(2,q) - PSL(2,q)$ (compare paragraph 1.3).

Assume that $\gamma \neq \pm 2$ and suppose there exists an admissible homomorphism $\Phi : (2,3,k) \longrightarrow PSL(2,q)$. Since XY is not parabolic, up to conjugation in

$PSL(2,q)$, we can assume that it is in diagonal form either working in $PSL(2,q)$ or in $PSU(2,q)$. So

$$XY = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where $\gamma = \lambda + \lambda^{-1}$ and $\lambda \in \mathbb{F}_q^*$ if we are working in $PSL(2,q)$ or $\lambda^{-1} = \lambda^q$ if we are working in $PSU(2,q)$. Remark that the matrices $\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $\pm \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ are conjugated by the matrix $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now we must be able to find an element of order 2 or equivalently trace 0 such that the element $Y = XXY$ has order 3 or equivalently has trace $\pm 1 =: \epsilon$. An element of order 2 must be of the form

$$X := \pm \begin{pmatrix} \alpha & \beta \\ \delta & -\alpha \end{pmatrix}$$

with $\alpha, \beta, \delta \in \mathbb{F}_q$ if we are working in $PSL(2,q)$ or $\alpha^q = -\alpha$ and $\delta = -\beta^q$ if we are working in $PSU(2,q)$. Multiplying we obtain

$$Y = \pm \begin{pmatrix} \alpha\lambda & \beta\lambda^{-1} \\ \delta\lambda & -\alpha\lambda^{-1} \end{pmatrix}.$$

Imposing the trace of Y to be ϵ we get $\alpha = \epsilon/(\lambda - \lambda^{-1}) \neq 0$ and $\beta\delta = (3 - \gamma^2)/(\gamma^2 - 4)$. Notice that the condition $\alpha^q = -\alpha$ is automatically satisfied when working in $PSU(2,q)$ since $\lambda^q = \lambda^{-1}$ in this case.

If $q \equiv 0 \pmod 3$, we must check that Y is not the trivial element in order for Φ to be admissible. This is equivalent to require that either β or δ is not 0. This is always the case if $3 - \gamma^2 \neq 0$. In this situation, moreover, the solution is also unique up to conjugation in $PGL(2,q)$. Indeed, up to conjugation with an element in the centralizer of XY, either β or δ can be rescaled to be 1, if we are working in $PSL(2,q)$, or to be a fixed $(q+1)$-root of $(3-\gamma^2)/(\gamma^2-4)$, if we are working in $PSU(2,q)$. Elements in the centralizer of XY are of the form

$$\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$$

and we must choose those where ρ^2 belongs to $\mathbb{F}_q^*$ if we are working in $PSL(2,q)$ (and the centralizer is contained in $PGL(2,q)$) or $\rho^{q+1} = 1$ if we are working in $PSU(2,q)$ (and the centralizer is contained in $PSU(2,q)$). If $\gamma = 0$, the centralizer contains also the element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This shows that the entries of the matrix X are completely determined up to conjugation in $PGL(2,q)$.

On the other hand, if $\gamma^2 = 3$ we have three different choices (up to conjugation with an element in the centralizer of XY) according as $\beta = \delta = 0$, $\beta = 0, \delta \neq 0$ or $\beta \neq 0, \delta = 0$. Note, however, that the latter two cases cannot happen when working in $PSU(2,q)$. In order to have Φ admissible then, we must require that in the case $q \equiv 0 \pmod 3$ and $\gamma = 0$ we are not in $PSU(2,q)$, i.e. $q \neq 3^{m+1}$.

Suppose now that $\gamma = \pm 2$. In this case at least one of X and Y must be non-parabolic and can be put in diagonal form. Suppose that Y is non-parabolic. Then

$$Y = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda + \lambda^{-1} = \epsilon$. X has again the same form as above and we must impose that the trace of the product XY is γ. One gets $\alpha = \gamma/(\lambda - \lambda^{-1})$ and $\beta\delta = (-\gamma^2 - \epsilon^2 + 4)/(\epsilon^2 - 4) = -(3-\gamma^2)/3$, the same condition as in the previous case. If Y is parabolic (i.e. $\gamma = \epsilon = \pm 2$ and $q \equiv 0 \pmod 3$), then we can assume that

$$X = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda + \lambda^{-1} = 0$. The matrix Y must now be of the form

$$Y = \pm \begin{pmatrix} \alpha & \beta \\ \delta & \epsilon - \alpha \end{pmatrix} .$$

The product turns out to be

$$XY = \pm \begin{pmatrix} \alpha\lambda & \beta\lambda^{-1} \\ \delta\lambda & (\epsilon - \alpha)\lambda^{-1} \end{pmatrix} .$$

Imposing its trace to be γ we obtain $\alpha = (\gamma - \epsilon\lambda^{-1})/(\lambda - \lambda^{-1})$ and $\beta\delta = -(3 - \gamma^2)/4 = 1$, again the same condition.

Suppose now that we want to find an admissible homomorphism Φ with image contained in $PGL(2,q)$ but not in $PSL(2,q)$. We can assume that $\gamma \in \iota\mathbb{F}_q^*$ or that $\gamma = 0$ when $q = 3^{2m+1}$. The matrix XY is never parabolic and can be put in diagonal form

$$XY = \pm \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1\lambda_2$ is a non-square in $\mathbb{F}_q^*$ and $\gamma = (\lambda_1 + \lambda_2)/\sqrt{\lambda_1\lambda_2}$. Notice that $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$ if the matrix is diagonalizable in $PGL(2,q)$ or $\lambda_2 = \lambda_1^q$ if the matrix is diagonalizable in $PU(2,q)$. Now we look for the element X which must be of the form

$$X = \pm \begin{pmatrix} \alpha & \beta \\ \delta & -\alpha \end{pmatrix}$$

with $\alpha^q = -\alpha$ and $\delta = -\delta^q$ if we are working in $PU(2,q)$, else all entries belong to $\mathbb{F}_q$. Remark that we can always choose X in such a way that $-\alpha^2 - \beta\delta = (\lambda_1\lambda_2)^{-1}$. Imposing that $Y = XXY$ has trace ϵ we obtain $\alpha = \epsilon/(\lambda - \lambda^{-1})$ and $\beta\delta = (3-\gamma^2)/(\lambda_1\lambda_2(\gamma^2-4))$. One now concludes as before. Observe that if Y is parabolic (i.e. $q \equiv 0 \pmod 3$) and $\beta\delta = 0$ we can choose one of β, δ to be non zero. To convince oneself that there exists an admissible homomorphism from $(2,3,2)$ to $PGL(2,3^{2m+1})$ it is enough to observe that $\mathbf{D}_3 \subset \mathbf{S}_4 \subset PGL(2,3^{2m+1})$ (see 1.3.8).

2.1.2 Theorem: *Let $\gamma, \tau \in \mathbb{F}_q \cup \iota\mathbb{F}_q^*$ be traces of elements of order k resp. n in $PSL(2,q^2)$ where q is odd and $\gamma \neq \pm 2$. Let $Q(\gamma,\tau) := \gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 + 12 \in \mathbb{F}_q^*$.*

Then, except in the case $\gamma = \tau = 0$ and $q \equiv 0 \pmod 3$, there exists an admissible homomorphism Φ from $T_{k,n}$ to $PSL(2,q^2)$ such that $\Phi(xy)$ has trace γ and $\Phi(zx)$has trace τ. Such a homomorphism has image in $PSL(2,q)$ if and only if $\gamma, \tau \in \mathbb{F}_q$ and $Q(\gamma,\tau)$ is a square in $\mathbb{F}_q$. It has image in $PGL(2,q)$ if and only if one of the following conditions holds:

i) *$\gamma, \tau \in \iota\mathbb{F}_q^*$ and $Q(\gamma,\tau)$ is a square in $\mathbb{F}_q$;*

ii) *one of γ, τ is in $\mathbb{F}_q^*$, the other in $\iota\mathbb{F}_q^*$ and $Q(\gamma,\tau)$ is a non-square in $\mathbb{F}_q$;*

iii) *$\gamma, \tau \in \mathbb{F}_q$, one is 0, and $Q(\gamma,\tau)$ is a non-square in $\mathbb{F}_q$;*

iv) *one of γ, τ is 0 and the other is in $\iota\mathbb{F}_q^*$;*

v) *at least one of $\gamma, \tau$is in $\iota\mathbb{F}_q^*$ and $Q(\gamma,\tau) = 0$.*

In each case, up to conjugation in $PSL(2,q^2)$ resp. $PGL(2,q)$ there are at most two admissible homomorphisms Φ from $T_{k,n}$ to $PSL(2,q^2)$ resp. $PSL(2,q)$ or $PGL(2,q)$ such that $\Phi(xy)$ has trace γ and $\Phi(zx)$ has trace τ. In particular if $Q(\gamma,\tau) = 0$ there is exactly one homomorphism Φ.

Proof:

Again we shall write $X := \Phi(x)$, $Y := \Phi(y)$ and $Z := \Phi(z)$. We shall exploit the result and the proof of Lemma 2.1.1: we try to extend the admissible homomorphism Φ defined from the vertex group $(2,3,k)$ to the whole tetrahedral group. Remark that XY is non-parabolic by hypothesis this time. We shall use that same notation as in Lemma 2.1.1.

First we want to see when we have an admissible homomorphism to $PSL(2,q)$. For sure we must assume that $\Phi((2,3,k)) \subset PSL(2,q)$. Now we must find an element Z of order 2 or equivalently trace 0 such that XYZ has trace 0 and ZX has trace τ. Obviously it is necessary that τ belongs to $\mathbb{F}_q$. We have that Z must be of the form

$$Z = \pm \begin{pmatrix} \mu & \varphi \\ \psi & -\mu \end{pmatrix}.$$

We find

$$XYZ = \pm \begin{pmatrix} \lambda\mu & \lambda\varphi \\ \lambda^{-1}\psi & -\lambda^{-1}\mu \end{pmatrix}$$

which has order 2 if and only if $\mu = 0$ since $\lambda - \lambda^{-1} \neq 0$ for the element XY is non-trivial. Next we compute

$$ZX = \pm \begin{pmatrix} \varphi\delta & -\varphi\alpha \\ \psi\alpha & \psi\beta \end{pmatrix}.$$

We have $\psi = -\varphi^{-1}$. Let us impose the trace to be τ. Note that even if $\tau = \pm 2$, ZX is non-trivial since $\varphi\alpha \neq 0$. If $3 - \gamma^2 \neq 0$, we are led to solve a second degree equation

$$\delta\varphi^2 - \tau\varphi - \beta = 0$$

in the unknown φ (note that since $\beta \neq 0$ we have that $\varphi \neq 0$). Since q is odd by hypothesis we can consider the discriminant of the equation (for q even one must try to solve the equation directly case by case). The discriminant is

$$\Delta = \frac{\gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 + 12}{\gamma^2 - 4}.$$

If we are working in $PSL(2,q)$, we need to find a solution in $\mathbb{F}_q$, so Δ must be a square in $\mathbb{F}_q$. Since $\gamma^2 - 4 = (\lambda - \lambda^{-1})^2$ is a square in $\mathbb{F}_q$ we obtain the given condition with $Q(\gamma, \tau) := \Delta(\gamma^2 - 4)$.

If we are working in $PSU(2,q)$, again we have to solve the same equation which always admits a solution in $\mathbb{F}_{q^2}$, but we have to check that $\varphi^{-1} = \varphi^q$. This is easily seen to be verified whenever Δ is 0 or a non-square in $\mathbb{F}_q$. Indeed one compares the two expressions for ψ given by $(\tau - \delta\varphi)/\beta$ and $-\varphi^q$ where $\varphi = (\tau \pm \sqrt{\Delta})/2\delta$. Since in this case $\gamma^2 - 4$ is a non-square in $\mathbb{F}_q$ (see Remark 1.1.6), we get the same condition found for $PSL(2,q)$.

If now $3 - \gamma^2 = 0$, we have $Q(\gamma, \tau) = \tau^2(\gamma^2 - 4) = -\tau^2$. There are different cases to consider.

If $\tau = 0$ then to find a solution we must choose X such that $\beta = \delta = 0$. In this case any $\varphi \neq 0$ is an admissible solution. Anyway, up to conjugation with an element in the centralizer of XY (note that this elements centralize also X) the choice is unique (and $Q(\gamma, \tau) = 0$ in this case). We can assume $\varphi = 1$ and this solution is acceptable for both $PSL(2,q)$ and $PSU(2,q)$. However if $q \equiv 0 \pmod 3$, the homomorphism is not admissible since the element Y is trivial and we must exclude this case.

If $\tau \neq 0$ instead, we cannot choose X in such a way that $\beta = \delta = 0$ and we have exactly one solution for both possible choices of X. Indeed if $\gamma = 0$, the two possible choices for X are conjugated by the extra element in the centralizer of XY. Remark that the elements Z that we find are in $PSL(2,q)$ but not in $PSU(2,q)$. The latter case, however, happens exactly when $Q(\gamma, \tau)$ is a non-square in $\mathbb{F}_q$.

If we want to define an admissible homomorphism to $PGL(2,q)$ (whose image is not contained in $PSL(2,q)$), we have to consider different situations.

In the first case we assume that the triangle group $(2,3,k)$ is contained in $PSL(2,q)$. Then we must require Z to sit in $PGL(2,q)-PSL(2,q)$ which implies that $ZX \in PGL(2,q)-PSL(2,q)$ so $\tau \in \iota\mathbb{F}_q$. Again Z has the same form as before (in particular $\mu = 0$) but now $-\psi\varphi =: \zeta$ is a non-square in $\mathbb{F}_q$. Remember that $\psi, \varphi \in \mathbb{F}_q^*$ if XY can be chosen in diagonal form in $PSL(2,q)$ or $\psi^q = -\varphi$ if XY can be chosen in diagonal form in $PSU(2,q)$. We have to solve the following second degree equation (assume $\gamma^2 \neq 3$)

$$\delta\varphi^2 - \tau\sqrt{\zeta}\varphi - \beta\zeta = 0$$

whose discriminant is $\Delta = \sqrt{\zeta}(\gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 + 12)/(\gamma^2 - 4)$. As before we have to consider two cases according as we are working in $PGL(2,q)$ or in $PU(2,q)$ and in both cases the condition turns out to be $Q(\gamma,\tau)$ a non-square in $\mathbb{F}_q$ or 0. Anyway if $\tau = 0$ we must require $Q(\gamma,\tau) \neq 0$ otherwise the image would be contained in $PSL(2,q)$. We are in cases ii), iii), iv), v).

If we have that the vertex-group $(2,3,k)$ maps to $PGL(2,q)$ (second part of Lemma 2.1.1) we have again two situations to consider: either Z belongs to $PSL(2,q)$ or to $PGL(2,q)-PSL(2,q)$. The matrix has always the same form and in the latter case we require its determinant to be $\lambda_1\lambda_2$ (i.e. the same determinant as XY). As before we are led to solve a second degree equation when we impose the trace of ZX to be τ. Note that if Z is in $PSL(2,q)$ then we have to rescale the trace since the determinant of X is not 1 (just like in the previous case). In the other case, we have already chosen determinants in such a way that the determinant of ZX is 1. We see that if Z is in $PGL(2,q)-PSL(2,q)$ then ZX is in $PSL(2,q)$ and τ must belong to $\mathbb{F}_q$, else ZX is in $PGL(2,q)-PSL(2,q)$ and τ must belong to $\iota\mathbb{F}_q$. One now concludes as before, paying attention to distinguish the case $PGL(2,q)$ from the case $PU(2,q)$. We have that if $Z \in PSL(2,q)$ then $Q(\gamma,\tau)$ must be a square in $\mathbb{F}_q$ (cases i),v)), else $Q(\gamma,\tau)$ must be a non-square in $\mathbb{F}_q$ or 0 (cases ii), iii), iv), v)). We only have to exclude the cases when the image is already contained in $PSL(2,q)$.

We omitted the case $3-\gamma^2 = 0$ since it is enough to repeat the same considerations made previously.

We can now conclude that since $Q(\gamma,\tau)$ belongs to $\mathbb{F}_q$ we always find a solution to our equation in $\mathbb{F}_{q^2}$, thus an admissible homomorphism to $PSL(2,q^2)$ always exists under our hypotheses.

Note now that, if $3-\gamma^3 \neq 0$, up to conjugation, the entries of the generators X and Y are completely determined and we have two possible choices for the entries of the element Z defined by the two possible solutions of the second degree equation. As remarked, we have at most two possible homomorphisms Φ even if $3-\gamma^2 = 0$. This proves the last statement of the Theorem.

2.1.3 Theorem: *Let $\gamma, \tau \in \mathbb{F}_q \cup \iota\mathbb{F}_q^*$ be traces of elements of order k resp. n in $PSL(2,q^2)$ where q is odd and $\gamma \neq \pm 2$. Let $\widetilde{Q}(\gamma,\tau) := \gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 \pm 2\gamma\tau + 9$.*

Then, if $\widetilde{Q}(\gamma,\tau)$ is a square in $\mathbb{F}_{q^2}$, there exists an admissible homomorphism Φ from $\widetilde{T}_{k,n}$ to $PSL(2,q^2)$ such that $\Phi(xy)$ has trace γ and $\Phi(zx)$ has trace τ. Such a homomorphism has image in $PSL(2,q)$ if and only if $\gamma,\tau \in \mathbb{F}_q$ and $\widetilde{Q}(\gamma,\tau)$ is a square in $\mathbb{F}_q$ except for the case $\gamma=\tau=0$ when $q=3^{2m+1}$. It has image in $PGL(2,q)$ if and only if $\gamma,\tau \in \iota\mathbb{F}_q$ and $\widetilde{Q}(\gamma,\tau)$ is 0 or a non-square in $\mathbb{F}_q$ except for the case when all $\gamma,\tau,\widetilde{Q}(\gamma,\tau)$ are 0 and $q \neq 3^{2m+1}$.

Proof:

In the following, as usual, we shall write $X := \Phi(x)$, $Y := \Phi(y)$ and $Z := \Phi(z)$. We shall again exploit Lemma 2.1.1 and keep on using the notation adopted there. In this case, however, since both Y and Z have order 3, we shall denote by ϵ_1 (instead of ϵ) the trace of Y and by ϵ_2 the trace of Z.

Assume that an admissible homomorphism Φ is defined on the vertex-group $(2,3,k)$ with image in $PSL(2,q)$. We want to extend it to the whole group in such way that the image is still contained in $PSL(2,q)$. Now we must find an element Z of order 3 such that XYZ has order 2 or equivalently trace 0 and ZX has trace τ. We have

$$Z := \pm \begin{pmatrix} \mu & \varphi \\ \psi & \nu \end{pmatrix}$$

where $\mu+\nu=\epsilon_2 := \pm 1$ and all elements are in $\mathbb{F}_q$ if working in $PSL(2,q)$ or $\nu=\mu^q$, $\psi=-\varphi^q$ if working in $PSU(2,q)$. For Φ to be admissible, we must also require that at least one of φ and ψ is not 0 if $q \equiv 0 \pmod 3$. Computing XYZ we see that it has order 2 if and only if $\mu = -\epsilon_2\lambda^{-1}/(\lambda-\lambda^{-1})$ from which we obtain $\nu=\epsilon_2\lambda/(\lambda-\lambda^{-1})$ and $\varphi\psi=(3-\gamma^2)/(\gamma^2-4)$. Next we compute

$$ZX = \pm \begin{pmatrix} \mu\alpha+\varphi\delta & \mu\beta-\varphi\alpha \\ \psi\alpha+\nu\delta & \psi\beta-\nu\alpha \end{pmatrix}.$$

Let us impose the trace to be τ:

$$\alpha(\mu-\nu)+\delta\varphi+\beta\psi=\tau\,.$$

Note that even if $\tau=\pm 2$, ZX is non-trivial since if $\mu\beta-\varphi\alpha=0$ and $\psi\alpha+\nu\delta=0$ we can express φ and ψ in terms of $\alpha,\beta,\delta,\mu,\nu$ (being $\alpha\neq 0$) and substitute them back in the expression for τ. Simplifying one gets $\tau=-\epsilon_1\epsilon_2\gamma$ which is impossible.

If $3-\gamma^2 \neq 0$, we are led to solve a second degree equation

$$\delta\varphi^2 - \left(\tau + \frac{\epsilon_1\epsilon_2\gamma}{\gamma^2-4}\right)\varphi + \beta\frac{3-\gamma^2}{\gamma^2-4} = 0$$

in the unknown φ (notice that since β and $3-\gamma^2$ are not 0, we have $\varphi \neq 0$) whose discriminant is $\Delta = (\tau^2\gamma^2-4\tau^2-4\gamma^2+2\epsilon_1\epsilon_2\gamma\tau+9)/(\gamma^2-4)$. Again we are assuming q odd so it makes sense to consider the discriminant. If we are working in $PSL(2,q)$ we need to find a solution in $\mathbb{F}_q$, so Δ must be a square in $\mathbb{F}_q$. Since $\gamma^2-4=(\lambda-\lambda^{-1})^2$ is a square in $\mathbb{F}_q$ we obtain the given condition with $\widetilde{Q}(\gamma,\tau):=\Delta(\gamma^2-4)$.

If we are working in $PSU(2,q)$, again we have to solve the same equation which always admits a solution in $\mathbb{F}_{q^2}$ since $\widetilde{Q}(\gamma,\tau)\in\mathbb{F}_q$ in this situation, but we have to check that $\varphi^{-1}=\varphi^q$. This is easily seen to be verified whenever Δ is 0 or a non-square in $\mathbb{F}_q$: just like in Theorem 2.1.2, compare the two expressions for ψ given by $(\tau+\epsilon_1\epsilon_2\gamma/(\gamma^2-4)-\delta\varphi)/\beta$ and $-\varphi^q$ where $\varphi=(\tau+\epsilon_1\epsilon_2\gamma/(\gamma^2-4)\pm\sqrt{\Delta})/2\delta$. Since in this case γ^2-4 is a non-square in $\mathbb{F}_q$, we get the same condition found for $PSL(2,q)$.

If $3-\gamma^2=0$ we have $\widetilde{Q}(\gamma,\tau)=-(\tau-\epsilon_1\epsilon_2\gamma)^2$. Suppose $q\not\equiv 0 \pmod 3$ so that $k=6$. If we are working in $PSL(2,q)$ ($q\equiv 1 \pmod{12}$), we can assume that either β or δ is not 0 and in both cases we find a solution since our second degree equation becomes a first degree equation. We can choose $\beta=\delta=0$ only if $\tau=\epsilon_1\epsilon_2\gamma$. Anyway we see that in this case $\widetilde{Q}(\gamma,\tau)$ is always a square in $\mathbb{F}_q$. If we are working in $PSU(2,q)$ (i.e. $q\equiv -1 \pmod{12}$), then we are compelled to choose $\beta=\delta=0$ and so the trace must satisfy $\tau=\epsilon_1\epsilon_2\gamma$ so that $\widetilde{Q}(\gamma,\tau)=0$ and this is the sole case when $\widetilde{Q}(\gamma,\tau)$ is a square. We can now conclude that, even in this case, the condition given in Theorem holds.

Suppose now that $q\equiv 0 \pmod 3$. In this case $k=2$ and $\widetilde{Q}(\gamma,\tau)=-\tau^2$. We must require that one of β and δ is not 0. This means that we cannot have a solution in $PSU(2,q)$ (i.e. for all odd powers of 3). This is again the condition stated by the Theorem if $\tau\neq 0$. Moreover, if $\tau\neq 0$ exactly one of φ,ψ is not 0 and Φ is admissible. We must then discard just the case $\gamma=\tau=0$ and q an odd power of 3 since a homomorphism exists but it is not admissible.

It is now obvious that the given condition is necessary and sufficient.

Now we want to define an admissible homomorphism to $PGL(2,q)$ whose image is not contained in $PSL(2,q)$.

We must assume that the group $(2,3,k)$ lies in $PGL(2,q)-PSL(2,q)$; indeed if it lay in $PSL(2,q)$, the image of the whole group would be contained in $PSL(2,q)$. We have $\gamma\in\iota\mathbb{F}_q$. Let

$$Z:=\pm\begin{pmatrix}\mu & \varphi\\ \psi & \nu\end{pmatrix}.$$

Notice that $Z\in PSL(2,q)$. As before, we are led to solve a second degree equation when we impose the trace of ZX to be τ. Note that since Z is in $PSL(2,q)$ we have to rescale the trace for the determinant of X is not 1. We see that Z is in $PGL(2,q)-PSL(2,q)$ and τ must belong to $\iota\mathbb{F}_q$. One now concludes as before, paying attention to distinguish the case $PGL(2,q)$ from the case $PU(2,q)$. We have that $\widetilde{Q}(\gamma,\tau)$ must be a non-square in $\mathbb{F}_q$ or 0. We only have to exclude the cases when the image is already contained in $PSL(2,q)$.

We omitted the case when $3 - \gamma^2 = 0$ since it is enough to repeat the same considerations made previously.

We can now conclude that if $\widetilde{Q}(\gamma, \tau)$ is a square in $\mathbb{F}_q$ we always find a solution to our equation in $\mathbb{F}_{q^2}$ thus an admissible homomorphism to $PSL(2, q^2)$ always exists under our hypotheses.

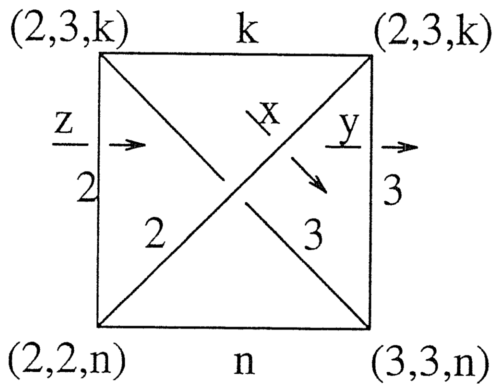


Figure 4

Consider now the tetrahedron depicted in Figure 4. For $k = 5$ and $n = 2$ we obtain the only Lannér tetrahedron which does not belong to the classes studied before. Notice that in this case the roles of k and n cannot be interchanged. Denote by $T'_{k,n}$ the associated tetrahedral group whose generators x, y and z are represented in Figure 4. It has presentation

$$\langle x, y, z \mid x^2, y^3, z^2, (xy)^k, (xyz)^3, (zx)^n \rangle$$

Just like in Theorems 2.1.2 and 2.1.3 and exploiting Lemma 2.1.1, we can prove the following

2.1.4 Theorem: *Let $\gamma, \tau \in \mathbb{F}_q \cup \iota\mathbb{F}_q^*$ be traces of elements of order k resp. n in $PSL(2, q^2)$ where q is odd and $\gamma \neq \pm 2$. Let $Q'(\gamma, \tau) := \gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 \pm 4\tau + 8$.*

Then, if $Q'(\gamma, \tau)$ is a square in $\mathbb{F}_{q^2}$, there exists an admissible homomorphism Φ from $T'_{k,n}$ to $PSL(2, q^2)$ such that $\Phi(xy)$ has trace γ and $\Phi(zx)$ has trace τ unless $\gamma^2 = 3$ and $\tau = \pm 2$. Such a homomorphism has image in $PSL(2, q)$ if and only if $\gamma, \tau \in \mathbb{F}_q$ and $Q'(\gamma, \tau)$ is a square in $\mathbb{F}_q$ except for the case $\gamma = \tau = 0$ and $q = 3^{2m+1}$. It has image in $PGL(2, q)$ if and only if $\gamma \in \iota\mathbb{F}_q^*$, $\tau \in \mathbb{F}_q$ and $Q'(\gamma, \tau)$ is a square in $\mathbb{F}_q$ or $\gamma = \tau = 0$, $q = 3^{2m+1}$.

Remark that if $\tau = \pm 2$, for Φ to be admissible we must require $\tau = -\epsilon_1\epsilon_2$ where ϵ_1, ϵ_2 are the traces of the images of y and xyz respectively.

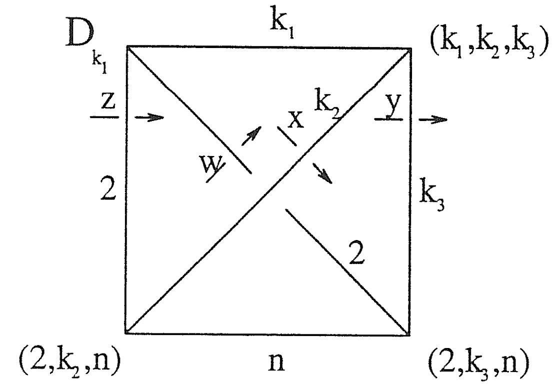


Figure 5

Consider now the tetrahedron represented in Figure 5. We shall denote the tetrahedral group associated to it by $T(k_1, k_2, k_3, n)$. The groups $T(4, 2, 4, 3)$ and $T(3, 3, 3, 3)$ are cusped hyperbolic tetrahedral groups associated to the orbifolds of second and third smallest volumes among hyperbolic cusped 3-orbifolds (compare Examples 1.5.8). We give the following generalization of part of the previous Theorems (generalizing also Lemma 2.1.1) and we omit the proof. Again one has to look for elements of the appropriate orders and check that defining relations for $T(k_1, k_2, k_3, n)$ are satisfied paying attention that no torsion element is trivial. Generators for $T(k_1, k_2, k_3, n)$ are represented in Figure 5 and a presentation for the group is

$$\langle x, y, z \mid x^{k_1}, y^{k_2}, z^2, (xy)^{k_3}, (xyz)^2, (zx)^n \rangle$$

2.1.5 Theorem: *For $j = 1, 2, 3$, let γ_j resp. $\tau \in \mathbb{F}_q \cup \iota\mathbb{F}_q^*$ be traces of elements of orders k_j resp. n in $PSL(2, q^2)$, where q is odd and $\gamma_1 \neq \pm 2$. Let $Q(\gamma_1, \gamma_2, \gamma_3, \tau) := \gamma_1^2\tau^2 + 4\gamma_1\gamma_2\gamma_3 - 4\gamma_1^2 - 4\gamma_2^2 - 4\gamma_3^2 - 4\tau^2 + 16$.*

If $Q(\gamma_1, \gamma_2, \gamma_3, \tau)$ is a square in $\mathbb{F}_{q^2}$, then there exists an admissible homomorphism $\Phi : T(k_1, k_2, k_3, n) \longrightarrow PSL(2, q^2)$ such that $\Phi(xy)$ has trace γ_1, $\Phi(x)$ has trace γ_2, $\Phi(y)$ has trace γ_3 and $\Phi(zx)$ has trace τ except in the two cases $\gamma_2 = \gamma_3 = 0$, $\tau = \pm 2$ and γ_2 or $\gamma_3 = \pm 2$, $\gamma_1\gamma_2\gamma_3 - \gamma_2^2 - \gamma_3^2 = 0$, $\tau = 0$.

The image of Φ lies inside $PSL(2, q)$ if and only if γ_j and τ are in $\mathbb{F}_q$ and $Q(\gamma_1, \gamma_2, \gamma_3, \tau)$ is a square in $\mathbb{F}_q$ $(j = 1, 2, 3)$.

If $\gamma_j \in \mathbb{F}_q$, $j = 1, 2, 3$, while either $\tau \in \iota\mathbb{F}_q$ and $Q(\gamma_1, \gamma_2, \gamma_3, \tau)$ is a non-square in $\mathbb{F}_q$ or τ is in $\iota\mathbb{F}_q^*$ and $Q(\gamma_1, \gamma_2, \gamma_3, \tau) = 0$, then the image of Φ lies in $PGL(2, q)$ (but not in $PSL(2, q)$).

Note that we do not give a complete classification of admissible homomorphisms with image contained in $PGL(2, q)$ as in the previous Theorems, since there are too many cases to be considered according to where the images of the elements x, y and z lie. In this Theorem we only consider the case when both $\Phi(x)$ and $\Phi(y)$ belong to $PSL(2, q)$ while $\Phi(z)$ belongs to $PGL(2, q) - PSL(2, q)$. Note moreover that for $\gamma_2 = 0$ and $\gamma_3 = \pm 1$ we obtain part of Theorem 2.1.2, instead for $\gamma_1 = \tau$, $\gamma_2 = \pm 1$, $\gamma_3 = \pm 1$ and $\tau = \gamma$ we obtain part of Theorem 2.1.4 (this can be easily seen by rotating the tetrahedron in Figure 4 upside down).

2.2. Lannér case

In this paragraph we shall apply Theorems 2.1.3 and 2.1.4 to classify the admissible quotients of linear fractional type of some hyperbolic tetrahedral groups associated to Lannér tetrahedra. In [20] this problem is solved for the groups $T_{4,5}$ and $T_{5,5}$ except for the quotients of type $PSL(2,q)$ with q even. Here we shall discuss the omitted cases for these two groups and give a complete classification for the groups $\widetilde{T}_{2,5}$, $\widetilde{T}_{4,3}$, $\widetilde{T}_{4,4}$ and for the Lannér tetrahedral group $T'_{5,2}$. We shall see that $PSL(2,q)$ is an admissible quotient of the Lannér tetrahedral groups only for q prime, a square of a prime or the forth power of a prime.

For the remaining three tetrahedral groups the expression for $\widetilde{Q}(\gamma,\tau)$ assumes too many different values or is the algebraic sum of different square-roots. This means that we are not able to decide in general whether $\widetilde{Q}(\gamma,\tau)$ is a square but we can only establish it case by case.

Remark that if at least one between k and n is odd, we cannot have admissible quotients of type $PGL(2,q)$ for the groups $\widetilde{T}_{k,n}$. Indeed, by a symmetry matter it is not restrictive to assume that k is odd. In this case then, if X, Y and Z (with the notation of Theorem 2.1.3) are in $PGL(2,q)$, they do belong to $PSL(2,q)$ and thus the image of Φ is contained in $PSL(2,q)$ as well.

2.2.1 Theorem: *The group $PSL(2,q)$ is an admissible quotient of the group $\widetilde{T}_{k,n}$ exactly the in following cases:*

$k = 2, n = 5$:

i) $q = p$, $p \equiv \pm 1 \pmod{10}$ *and some value of* $3 + 2\sqrt{5}$ *is a square in* $\mathbb{F}_p$;

ii) $q = p^2$, $p \equiv \pm 1 \pmod{10}$ *and some value of* $3 + 2\sqrt{5}$ *is a non-square in* $\mathbb{F}_p$ *or* $p \equiv \pm 3 \pmod{10}$ *and some value of* $3 + 2\sqrt{5}$ *is a square in* $\mathbb{F}_{p^2}$ *or* $p = 5$;

iii) $q = p^4$, $p \equiv \pm 3 \pmod{10}$ *and some value of* $3 + 2\sqrt{5}$ *is a non-square in* $\mathbb{F}_{p^2}$ *or* $p = 2$.

$k = 4, n = 3$:

i) $q = p$, $p \equiv \pm 1 \pmod 8$ *and some value of* $-1 + 2\sqrt{2}$ *is a square in* $\mathbb{F}_p$;

ii) $q = p^2$, $p \equiv \pm 1 \pmod 8$ *and some value of* $-1 + 2\sqrt{2}$ *is a non-square in* $\mathbb{F}_p$ *or* $p \equiv \pm 3 \pmod 8$ *and some value of* $-1 + 2\sqrt{2}$ *is a square in* $\mathbb{F}_{p^2}$;

iii) $q = p^4$, $p \equiv \pm 3 \pmod 8$ *and some value of* $-1 + 2\sqrt{2}$ *is a non-square in* $\mathbb{F}_{p^2}$.

$k = 4, n = 4$:

i) $q = p$, $p \equiv \pm 1 \pmod 8$ *and* $\left(\frac{p}{7}\right) = 1$;
ii) $q = p^2$, $p \equiv \pm 1 \pmod 8$ *and* $\left(\frac{p}{7}\right) = -1$ *or* $p \equiv \pm 3 \pmod 8$ *and* $\left(\frac{p}{7}\right) = 1$.

The group $PGL(2,q)$ *is an admissible quotient of the group* $\widetilde{T}_{k,n}$ *exactly in the following cases:*

$k = 4, n = 4$

$q = p$, $p \equiv \pm 3 \pmod 8$ *and* $\left(\frac{p}{7}\right) = -1$

There are no admissible quotients of type $PGL(2,q)$ *in the other two cases.*

Proof:

$k = 2, n = 5$:
If $p \equiv \pm 1 \pmod{10}$ (resp. $p \equiv \pm 3 \pmod{10}$) there exists an injection of $\mathbf{A}_5$ in $PSL(2,p)$ (resp. $PSL(2,p^2)$) -compare Theorem 1.3.8. According to Theorem 2.1.3, we can extend this injection to $\widetilde{T}_{2,5}$ if and only if $\widetilde{Q}(\gamma,\tau)$ is a square in $\mathbb{F}_p$ (resp. $\mathbb{F}_{p^2}$). In our situation we have $\tau = 0$ and $\gamma = \pm(1 \pm \sqrt{5})/2$ so that $\widetilde{Q}(\gamma,\tau) = 3 \pm 2\sqrt{5}$ and we want at least one of the two values to be a square. Whenever the $\widetilde{Q}(\gamma,\tau)$ is a square we have an extension to $PSL(2,q)$, otherwise we are able to find a solution to our equation in $\mathbb{F}_{q^2}$ that is an extension to $PSL(2,q^2)$.

We observe that, since $(3 + 2\sqrt{5})(3 - 2\sqrt{5}) = -11$, if $q = p$ and $\left(\frac{p}{11}\right) = -1$ then exactly one of the two values of $\widetilde{Q}(\gamma,\tau)$ is a square (this means that we find admissible homomorphisms to both $PSL(2,p)$ and $PSL(2,p^2)$), else if $q = p$ and $\left(\frac{p}{11}\right) = 1$ or $q = p^2$ either both possible values are squares or neither (see section 1.1).

Notice that if $p = 5$, $\widetilde{Q}(\gamma,\tau) = 3$ is a non-square and we do not have an admissible homomorphism to $PSL(2,5)$.

If q is even we can work with $\widetilde{T}_{5,2}$. We need to solve the following second degree equation (compare the proof of Theorem 2.1.2 and note that $\beta\delta \neq 0$)

$$\delta\varphi^2 + (\tau + \frac{1}{\gamma})\varphi + \beta(\frac{1+\gamma^2}{\gamma^2}) = 0$$

where γ satisfies the relation $\gamma^2 + \gamma + 1$ and $\tau = 0$ (see Examples 1.3.7). Notice that $\mathbb{F}_{2^2} = \{0, 1, \gamma, \gamma^2\}$. Using these relations the equation becomes

$$(\delta\varphi)^2 + \gamma^2(\delta\varphi) + \gamma = 0.$$

Substituting to $\delta\varphi$ all elements $\mathbb{F}_{2^2}$ we see if we have a non trivial solution. Remember that we always have a solution to the equation in $PSL(2,2^4)$.

It remains to prove that such extensions are surjective. This is trivial if we have an extension to $PSL(2,p)$ since $\mathbf{A}_5$ is a maximal subgroup and $T_{2,5}$ does not have an admissible extension on $\mathbf{A}_5$ itself. By the classification of subgroups of $PSL(2,q)$ (Theorem 1.3.8), we see that the only possible images of our extensions are either $PSL(2,q)$ or $PGL(2,q)$ since they must contain $\mathbf{A}_5$ as a proper subgroup.

Uniqueness depends on the fact that all immersions of $\mathbf{A}_5$ into $PSL(2,q)$ are conjugate to an immersion in $PSL(2,p)$ -if $p \equiv \pm 1 \pmod{10}$- or in $PSL(2,p^2)$ -if $p \equiv \pm 3 \pmod{10}$.

$k = 4, n = 3$:

This is proved exactly as in the previous case.

$k = 4, n = 4$:

This case is slightly different since we always have a trivial extension onto $\mathbf{S}_4 \cong PGL(2,3) \subset PSL(2,q)$. Computing $\widetilde{Q}(\gamma,\tau)$ by substituting $\gamma = \pm\sqrt{2}$ and $\tau = \pm\sqrt{2}$ one finds two values: 1 which is always a square and -7. The first gives the trivial solutions while the other does not. To see this, one has to check that the two solutions relative to the first value (call them φ_1 and φ_2) are not the solutions obtained with the second (call them φ_1' and φ_2') or their opposites. Considering the equation for φ, we note that the product of the two solutions is the same in both cases. This means that we could have $\varphi_1 = \pm\varphi_j'$ and $\varphi_2 = \pm\varphi_{j+1}'$ (indices (mod 2)) but not $\varphi_1 = \pm\varphi_j'$ and $\varphi_2 = \mp\varphi_{j+1}'$ (indices (mod 2)). An easy computation shows that the sums of the two pairs of solutions cannot be equal or opposite.

2.2.2 Theorem: *The admissible quotients of type $PSL(2,q)$ of $T_{5,2}'$ are exactly the following:*

i) $q = p$, $p \equiv 1, 9 \pmod{20}$ *and some value of* $2 + 2\sqrt{5}$ *is a square in* $\mathbb{F}_p$ *or* $p \equiv 11, 19 \pmod{20}$;

ii) $q = p^2$, $p \equiv 1, 9 \pmod{20}$ *and some value of* $2 + 2\sqrt{5}$ *is a non-square in* $\mathbb{F}_p$ *or* $p \equiv 5, 11, 19 \pmod{20}$ *or* $p \equiv \pm 3 \pmod{10}$ *and some value of* $2 + 2\sqrt{5}$ *is a square in* $\mathbb{F}_{p^2}$;

iii) $q = p^4$, $p \equiv \pm 3 \pmod{10}$ *and some value of* $2 + 2\sqrt{5}$ *is a non-square in* $\mathbb{F}_{p^2}$.

There are no admissible quotients of type $PGL(2,q)$ of $T_{5,2}'$.

Proof:

Apply Theorem 2.1.4. Here $Q'(\gamma,\tau) = 2 \pm 2\sqrt{5}$. The case when q is even is checked directly as in the proof of Theorem 2.2.1; it turns out that a homomorphism to $PSL(2,2^2)$ exists but it is not admissible. For $q \equiv 0 \pmod{5}$, one can apply Theorem 2.1.5 with $k_1 = 2$, $k_2 = k_3 = 3$ and $n = 5$. Else one can work with matrices and start with the element XY of order 5 in upper triangular form

$$XY = \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

with $\lambda \neq 0$ and seek elements X and Z of the needed orders satisfying the given relations.

For completeness we give the classification also for the groups $T_{4,5}$ and $T_{5,5}$. The result, apart from the case $q \equiv 0 \pmod 2$, can be found in [19] where the two groups are denoted by T and T' respectively.

2.2.3 Theorem: *The group $PSL(2,q)$ is an admissible quotient of the group $T_{k,5}$ exactly in the following cases:*

$k = 4$*:*

i) $q = p$, $p \equiv 1, 9 \pmod{40}$ *and some value of* $1 + \sqrt{5}$ *is a square in* $\mathbb{F}_p$ *or* $p \equiv 31, 19 \pmod{40}$*;*

ii) $q = p^2$, $p \equiv 1, 9 \pmod{40}$ *and some value of* $1 + \sqrt{5}$ *is a non-square in* $\mathbb{F}_p$ *or* $p \equiv 5, 31, 19 \pmod{20}$ *or* $p \equiv 21, 29 \pmod{40}$ *and some value of* $1 + \sqrt{5}$ *is a square in* $\mathbb{F}_p$ *or* $p \equiv \pm 3 \pmod{10}$ *and some value of* $1 + \sqrt{5}$ *is a square in* $\mathbb{F}_{p^2}$*;*

iii) $q = p^4$, $p \equiv \pm 3 \pmod{10}$ *and some value of* $1 + \sqrt{5}$ *is a non-square in* $\mathbb{F}_{p^2}$*.*

$k = 5$*:*

i) $q = p$, $p \equiv \pm 1 \pmod{10}$ *and some value of* $(7 + 5\sqrt{5})/2$ *is a square in* $\mathbb{F}_p$ *or* $p = 5$*;*

ii) $q = p^2$, $p \equiv \pm 1 \pmod{10}$ *and some value of* $(7 + 5\sqrt{5})/2$ *is a non-square in* $\mathbb{F}_p$ *or* $p \equiv \pm 3 \pmod{10}$ *and some value of* $(7 + 5\sqrt{5})/2$ *is a square in* $\mathbb{F}_{p^2}$ *or* $p = 2$*;*

iii) $q = p^4$, $p \equiv \pm 3 \pmod{10}$ *and some value of* $(7 + 5\sqrt{5})/2$ *is a non-square in* $\mathbb{F}_{p^2}$ *or* $p = 2$*.*

The group $PGL(2,q)$ is an admissible quotient of the group $T_{k,5}$ exactly in the following cases:

$k = 4$*:*

$q = p$, $p \equiv 21, 29 \pmod{40}$ *and some value of* $(7 + 5\sqrt{5})/2$ *is a non-square in* $\mathbb{F}_p$ *or* $p \equiv 11, 19 \pmod{40}$

There are no admissible quotients of type $PGL(2,q)$ of the group $T_{5,5}$.

Proof:

For the case q even just note that there cannot be admissible quotients of type $PSL(2, 2^m)$ if $k = 4$ (see Theorem 1.3.8), while for $k = 5$ one reasons as in Theorem 2.2.1.

The Theorems of this and the previous paragraph allow us to list all the Lannér tetrahedral groups -if any- which have $PSL(2,q)$ as an admissible quotient for all $q \leq 16$. For the groups $\widetilde{T}_{5,3}$ $\widetilde{T}_{5,4}$ and $\widetilde{T}_{5,5}$ the conditions given in Theorem 2.1.3 are checked case by case, moreover we cannot apply Theorem 2.1.3 to determine the quotients of $\widetilde{T}_{5,5}$ if $q \equiv 0 \pmod 5$. In this case computations are made directly as well as for the case q even. Note that $PSL(2,q)$ may be an admissible quotient of some tetrahedral group only if q is a prime, a square of a prime or a fourth power of a prime.

2.2.4 Corollary: *The group $PSL(2,q)$, $q \leq 16$ is an admissible quotient of exactly the following Lannér tetrahedral groups:*

$$
\begin{aligned}
PSL(2,2) \cong \mathbf{D}_3 &: \\
PSL(2,3) \cong \mathbf{A}_4 &: \\
PSL(2,2^2) \cong PSL(2,5) \cong \mathbf{A}_5 &: \widetilde{T}_{3,5}, \widetilde{T}_{5,5}, T_{5,5} \\
PSL(2,7) &: \widetilde{T}_{4,3}, \widetilde{T}_{4,4} \\
PSL(2,2^3) &: \\
PSL(2,3^2) \cong \mathbf{A}_6 &: \widetilde{T}_{5,2}, \widetilde{T}_{5,4}, \widetilde{T}_{5,5} \\
PSL(2,11) &: \widetilde{T}_{5,2}, \widetilde{T}_{5,3}, \widetilde{T}_{5,5}, T'_{5,2} \\
PSL(2,13) &: \\
PSL(2,2^4) &: \widetilde{T}_{5,2}, T_{5,5}
\end{aligned}
$$

2.3. Cusped case

The next Theorem gives the classification of all finite admissible quotients of linear fractional type of the three tetrahedral groups $T_{k,n}$ associated to three of the nine Coxeter tetrahedra with exactly one cusp, and in particular of the tetrahedral group $T_{3,6}$ uniformizing the smallest orientable cusped hyperbolic 3-orbifold (see Examples 1.5.8).

2.3.1 Theorem: *The group $PSL(2,q)$ is an admissible quotient of the tetrahedral group $T_{k,6}$ exactly in the following cases:*

$k=3$:

i) $q=p$, $p \equiv 1 \pmod{12}$;
ii) $q=p^2$, $p \equiv -1 \pmod 6$.

$k=4$:

i) $q=p$, $p \equiv 1 \pmod{24}$;
ii) $q=p^2$, $p \equiv 13, 17, 19, 23 \pmod{24}$.

$k=5$:

i) $q=p$, $p \equiv 1, 49 \pmod{60}$;
ii) $q=p$, $p \not\equiv 1, 2, 3, 19 \pmod{30}$.

The group $PGL(2,q)$ is an admissible quotient of the tetrahedral group $T_{k,6}$ exactly in the following cases:

$k = 3$:

$q = p$, $p \equiv 7 \pmod{12}$.

$k = 4$:

$q = p$, $p \equiv 5, 7, 11 \pmod{24}$.

$k = 5$:

$q = p$, $p \equiv 19, 31 \pmod{60}$.

Up to conjugation in $PGL(2,q)$ resp. in $PSL(2,q^2)$ there are at most two admissible surjections of $T_{k,6}$ if $k = 2, 4$ and at most four admissible surjections if $k = 5$.

Proof:

$k = 3$:

Since we can find elements of order 3 in $PSL(2,q)$ for all q, up to conjugation, we can assume that the image of the element xy sits in $PSL(2,p)$ ($q = p^m$). For $n = 6$ we have $\tau^2 = 3$ and $Q(\gamma, \tau) = -1$ for $\gamma = \pm 1$. We can apply Theorem 2.1.2 with $q := p$ to see where the images of the admissible homomorphisms lie.

Surjectivity follows from the fact that an element of order 6 cannot belong to $\mathbf{S}_4$ or to $\mathbf{A}_5$ the only other possible subgroups in $PSL(2,p)$ containing the vertex group $\mathbf{A}_4$ (compare Theorem 1.3.8). Note that up to conjugation there are no other admissible homomorphisms.

$k = 4$:

Note that an element of order 4 exists in $PSL(2,q)$ if and only if $q \equiv \pm 1 \pmod 8$. Thus if $p \equiv \pm 1 \pmod 8$ we can assume that the image of the element xy sits in $PSL(2,p)$ and as before we can apply Theorem 2.1.2 to the case $\gamma^2 = 2$, $\tau^2 = 3$ and $Q(\gamma, \tau) = -2$ with $q := p$. If $p \not\equiv \pm 1 \pmod 8$, we can find an element of order 4 in $PGL(2,p) - PSL(2,p)$. Applying again Theorem 2.1.2 we are again able to say when the image contained in $PGL(2,p)$ itself. To proof surjectivity one exploits the maximality of the vertex group $\mathbf{S}_4$ in $PSL(2,p)$ if $p \equiv \pm 1 \pmod 8$ or in $PGL(2,p)$ if $p \equiv \pm 3 \pmod 8$.

$k = 5$:

We can find elements of order 5 in $PSL(2,q)$ if and only if $q(q^2 - 1) \equiv \pm 1 \pmod 5$. Moreover elements of odd order cannot lie in $PGL(2,q) - PSL(2,q)$. This means that we can apply Theorem 2.1.2 to the case $q := p$ if $p \equiv \pm 1 \pmod 5$ or $q := p^2$ if $p \equiv \pm 3 \pmod 5$ with $\gamma^2 = (3 \pm \sqrt{5})/2$, $\tau^2 = 3$ and $Q(\gamma, \tau) = -(\pm(1 \pm \sqrt{5})/2)^2$. The only case that must be considered aside is when $q = 5^m$ and $k = 5$, since in this situation the element of order k is parabolic. This can be done, for instance, exchanging the roles of n and k.

Surjectivity follows from the maximality of the vertex group $\mathbf{A}_5$. Note that in this case we can have four possible quotients because we find two differents values for $Q(\gamma, \tau)$ associated to the two different traces of non conjugate elements of order 5.

This finishes the proof of Theorem.

Remark that we do not consider the tetrahedral group $T_{6,6}$ associated to a hyperbolic tetrahedron with two cusps since in this case it is not easy to establish the surjectivity of an admissible homomorphism with image in $PSL(2,q)$ or $PGL(2,q)$ since we cannot exploit the maximality of vertex-groups of exceptional type (compare Theorem 1.3.8).

Up to conjugation, the tetrahedral group $T_{3,6}$ uniformizing the smallest cusped 3-orbifold is equal to the extended Bianchi group $PGL(2,\mathbb{Z}[\omega])$, considered as a subgroup of the isometry group $PSL(2,\mathbb{C})$ of hyperbolic 3-space (see Examples 1.4.4 ii) and 1.5.8), where ω is a primitive cubic root of unity and thus satisfies $\omega^2+\omega+1=0$. Suppose $p \neq 3$. We have the two group homomorphisms $\Phi_j : T_{3,6} = PGL(2,\mathbb{Z}[\omega]) \longrightarrow PGL(2,p^2)$ obtained by reduction of coefficients mod p (compare Example 1.3.3 iv) for definition and notation). Remark that if 3 divides $p-1$ the image of Φ_j is in $PGL(2,q)$.

2.3.2 Theorem: *Let p be a prime different from $2,3$. Then, by reduction of coefficients mod p, we obtain two admissible surjections Φ_j, $j=1,2$, from $T_{3,6} = PGL(2,\mathbb{Z}[\omega])$ onto one of the following groups:*

i) $$p \equiv 1 \pmod{12} : PSL(2,p);$$

ii) $$p \equiv 7 \pmod{12} : PGL(2,p);$$

iii) $$p \equiv 5, 11 \pmod{12} : PSL(2,p^2).$$

Up to conjugation, all admissible homomorphisms from $T_{3,6}$ to a linear fractional group $PSL(2,q)$ or $PGL(2,q)$ are obtained by reduction of coefficients mod p and, for each p, there are exactly two such homomorphisms. For each of the above finite groups, the kernels of the corresponding surjections are the universal covering groups of the cusped hyperbolic 3-manifolds of minimal volume admitting an action of the group.

Proof:

First we show that, for every p different from 2 and 3, the group homomorphisms Φ_j, $j=1,2$, are admissible, that is have torsion-free kernel. An element of finite order in $T_{3,6} = PGL(2,\mathbb{Z}[\omega])$ has order 2, 3 or 6. An element of order 2 in $PGL(2,\mathbb{Z}[\omega])$ has trace zero, so also its image has trace zero. Because $p \neq 2$, it cannot lie in the kernel of Φ_j. An element of order 3 is in $PSL(2,\mathbb{Z}[\omega])$, and the square of its trace is equal to one. Again, because $p \neq 3$, it does not lie in the kernel of Φ_j which is therefore torsion-free.

We determine the image of Φ_j in $PGL(2,p^2)$. Note that the determinants of elements of $PGL(2,\mathbb{Z}[\omega])$ are $\pm 1, \pm\omega$ and $\pm\omega^2$ (these are the units in $\mathbb{Z}[\omega]$)), therefore all elements have representatives with determinant 1 or -1. Now -1 is a square in $\mathbb{F}_p$ if and only if $p \equiv 1 \pmod 4$, otherwise it is a square in $\mathbb{F}_{p^2}$. It follows that the image of Φ_j lies in the groups listed in the three cases of the Theorem, and it remains to prove surjectivity. This follows from Theorem 1.3.10 putting $\xi := \sigma^j$. Remark that the matrices belong to the image of Φ_j, and Φ_j results surjective in each of the three cases.

The two homomorphisms Φ_1 and Φ_2 are not conjugate since the images of the element

$$\begin{pmatrix} 0 & 1 \\ -1 & \omega \end{pmatrix},$$

of determinant one, have different traces σ and σ^2 and thus are not conjugate. Now, by Theorem 2.3.1, each admissible homomorphism is obtained by reduction mod p.

Now we shall apply Theorem 2.1.5 to the hyperbolic cusped tetrahedral groups $T(4,2,4,3)$ and $T(3,3,3,n)$ with $n = 3,4,5$. First of all we shall consider the group $T(4,2,4,3)$ which is the extended Picard group $PGL(2,\mathbb{Z}[i])$ (see [9] and Example 1.4.4 i)).

2.3.3 Theorem: *The group $PSL(2,q)$ is an admissible quotient of the group $T(4,2,4,3)$ exactly in the following cases:*

i) $q = p$, $p \equiv 1 \pmod 8$;
ii) $q = p^2$, $p \equiv -1 \pmod 4$.

The group $PGL(2,q)$ is an admissible quotient of the group $T(4,2,4,3)$ exactly in the following cases:

$q = p$, $p \equiv 5 \pmod 8$.

All admissible quotients of type $PSL(2,q)$ and $PGL(2,q)$ of $T(4,2,4,3) \cong PGL(2,\mathbb{Z}[i])$ are obtained by reduction of coefficients mod p.

Proof:

If an element of order 4 exists in $PSL(2,p)$ (which is the case if and only if $p \equiv \pm 1 \pmod 8$) then we can apply Theorem 2.1.5. We obtain $Q(\gamma_1,\gamma_2,\gamma_3,\tau) = -2$. Since 2 is a square under our hypotheses, we must check when -1 is a square. If $p \equiv 1 \pmod 8$ it is a square and we have a solution in $PSL(2,p)$. Surjectivity follows because of the maximality of S_4. If $p \equiv -1 \pmod 8$, we have a solution in $PSL(2,p^2)$ -the image is not $PGL(2,q)$ because $\tau = \pm 1$ is in $\mathbb{F}_p$.

If $p \equiv \pm 3 \pmod 8$, an element of order 4 exists in $PGL(2,p)$. With the notation of Figure 5, we have that the images of the elements y and xy belong to $PGL(2,p) - PSL(2,p)$ and we can repeat the same reasoning as in the second part of the proofs of Theorems 2.1.2 and 2.1.3. Note that ZX is in $PSL(2,p)$ since it has order 3 forcing also Z to sit in $PSL(2,p)$. Up to conjugation in $PSL(2,p^2)$, we can assume that

$$YX = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

with $i^2 = -1$. Once more we are led to solve a second degree equation whose discriminant is 1. We must require it to be a square (resp. a non-square) if -1 is a square (resp. a non-square) in $\mathbb{F}_p$ (in this case we are working in $PU(2,p)$). Since 1 is always a square, we have a solution in $PGL(2,p)$ if $p \equiv 5 \pmod 8$. Otherwise the solution lies in $PSL(2,p^2)$.

Note that there are no admissible quotients of type $PSL(2,q)$ when q is even, since these groups do not contain elements of order 4.

The last statement of the Theorem is proved analogously to the case of $PGL(2,\mathbb{Z}[\omega])$ in Theorem 2.3.2.

The group $T(3,3,3,3)$ is isomorphic to $PSL(2,\mathbb{Z}[\omega])$ (see [9]). We have the following

2.3.4 Theorem: *The group $PSL(2,q)$ is an admissible quotient of the group $T(3,3,3,n)$ exactly in the following cases:*

$n = 3$:

i) $q = p$, $p \equiv 1 \pmod 6$ *or* $p = 5$;

ii) $q = p^2$, $p \not\equiv 1, 3 \pmod 6$.

$n = 4$:

i) $q = p$, $p \equiv 1, 7 \pmod{24}$;

ii) $q = p^2$, $p \equiv 13, 17, 19, 23 \pmod{24}$.

$n = 5$:

i) $q = p$, $p \equiv 1, 5, 19 \pmod{30}$;

ii) $q = p^2$, $p \not\equiv 1, 19 \pmod{30}$.

The group $PGL(2,q)$ is an admissible quotient of the group $T(3,3,3,n)$ exactly in the following cases:

$n = 4$

$q = p$, $p \equiv 3, 5, 11 \pmod{24}$.

There are no admissible quotients of type $PGL(2,q)$ of the groups $T(3,3,3,n)$ if $n = 3$ or $n = 5$.

In the case $n = 3$, all admissible quotients of $T(3,3,3,3)$ are obtained by reduction of coefficients mod p, with the exception of the group $\mathbf{A}_5$ corresponding to the case $p = 5$ in case i) *and $p = 2$ in case* ii).

Proof:

We begin with the case $n = 3$. Since the elements x and y have order 3, their images must belong to $PSL(2,q)$ and so we do not have to consider an analogue to the situation described in the last part of the proofs of Theorems 2.1.2 and 2.1.3; applying Theorem 2.1.5 we are able to classify also the admissible quotients of type $PGL(2,q)$. Moreover since $\tau^2 = 1$ is always a square we cannot have solutions in $PGL(2,q)$ but only in $PSL(2,q)$.

Computing $Q(\gamma_1, \gamma_2, \gamma_3, \tau)$ one finds the values -3 and 5. Note that all admissible quotients found in Theorem 2.3.1 case $k = 3$ restrict to admissible quotients of $T(3,3,3,3) \subset T_{3,6}$ -this follows because of Theorem 1.3.2. These are exactly the solutions obtained when $Q(\gamma_1, \gamma_2, \gamma_3, \tau) = -3$.

At this point one can prove directly that there is an admissible surjection from $T(3,3,3,3)$ to $\mathbf{A}_5$ simply by considering an immersion of one of the two vertex groups of type $\mathbf{A}_4$ and extending it to the whole of $T(3,3,3,3)$.

For $Q(\gamma_1, \gamma_2, \gamma_3, \tau) = 5$ we do not have any new solution apart from $\mathbf{A}_5 \cong PSL(2,5)$. In fact this condition gives an admissible homomorphism whose image is $\mathbf{A}_5$. Remark that the condition obtained is exactly the condition to have a subgroup of type $\mathbf{A}_5$ inside $PSL(2,q)$.

The exceptional case when the element of order $k = 3$ is parabolic must be checked aside working with permutations in $\mathbf{A}_6$.

The cases $n = 4$ and 5 are again an easy application of Theorem 2.1.5. Just observe that for $n = 4$ resp. $n = 5$, the values 2 resp. $(7 \pm 3\sqrt{5})/2 = ((3 \pm \sqrt{5})/2)^2$ for $Q(\gamma_1, \gamma_2, \gamma_3, \tau)$ give the trivial solutions $\mathbf{S}_4 \cong PGL(2,3)$ resp. $\mathbf{A}_5 \cong PSL(2,2^2) \cong PSL(2,5)$. We remark once more that for $n = 5$ we do not have solutions of type $PGL(2,q)$ because an element of odd order (in this case 5) cannot belong to $PGL(2,q) - PSL(2,q)$.

3. SOME OTHER APPLICATIONS

Exploiting again the techniques used in paragraph 2.1, we study admissible quotients for some other interesting groups. In paragraph 3.1 we consider the orbifold fundamental group of the hyperbolic 3-orbifold of minimal known volume. In paragraph 3.2 we study admissible quotients of type $PSL(2,q) \times \mathbb{Z}_2$ for the Coxeter groups associated to the tetrahedral groups considered in chapter 2.

3.1. An extension of $\widetilde{T}_{5,2}$

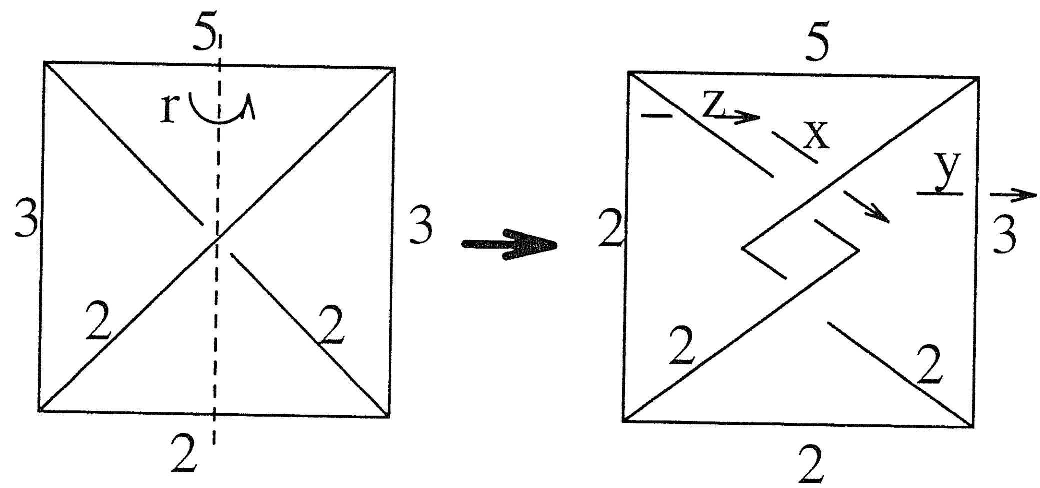


Figure 6

Consider now the graph drawn in Figure 6. It represents the singular set of a hyperbolic orbifold which is topologically the sphere $\mathbf{S}^3$. Moreover this is the hyperbolic 3-orbifold of minimal known volume (see [35]).

It is not difficult to see that this orbifold is double-covered by the orbifold $\mathbf{H}^3/\widetilde{T}_{5,2}$ (i.e. the quotient of the hyperbolic space by the action of one of the tetrahedral groups we considered before). The latter admits a rotational symmetry of order 2, as shown again in Figure 6.

As before we intend to study admissible $PSL(2,q)$ quotients of its orbifold fundamental group. We start with a presentation for the fundamental group that we shall denote by $\widetilde{T}$. Such presentation is derived in the same way as the presentations of the tetrahedral groups (see paragraph 1.5). We have

$$\widetilde{T} = \langle x, y, z \mid x^2, y^3, z^2, (xy)^5, (zxy)^2, (y^{-1}xzxzx)^2 \rangle$$

where the generators are shown in Figure 6. Note that the group is an extension of the group $\widetilde{T}_{5,2}$ and so we have an exact sequence of groups

$$1 \longrightarrow \widetilde{T}_{5,2} \longrightarrow \widetilde{T} \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

where the group $\mathbb{Z}_2$ is generated by the rotation r shown in Figure 6. From the above exact sequence we deduce that $\widetilde{T} = \langle \widetilde{T}_{5,2}, r\rangle$ and in terms of r and the generators of $\widetilde{T}_{5,2}$ shown in Figure 6 $\widetilde{T}$ has presentation

$$\langle t, v, w, r \mid t^2, v^3, w^3, (tv)^5, (tvw)^2, (wt)^2, r^2, rvrw^{-1}, (rtv)^2\rangle$$

(the last two relations tell us how r acts on the generators of $\widetilde{T}_{5,2}$ by conjugation). Eliminating the generator w in the previous presentation one obtains

$$\langle t, v, r \mid t^2, v^3, (tv)^5, (tvrvr)^2, (rvrt)^2, r^2, (rtv)^2\rangle\,.$$

The following Tietze substitutions give the connection between the latter presentation and the first one given for $\widetilde{T}$:

$$t \mapsto x$$

$$v \mapsto y$$

$$r \mapsto y^{-1}xz\,.$$

We have

$$\widetilde{T} = \mathbf{A}_5 *_{\mathbb{Z}_5} \mathbf{D}_5/\langle (y^{-1}xzxzx)^2\rangle = \mathbf{G}_5/\langle (y^{-1}xzxzxi)^2\rangle\,.$$

This will ensure us injectivity on finite subgroups of $\widetilde{T}$ whenever we find in $PSL(2,q)$ generators for two subgroups $\mathbf{A}_5$ and $\mathbf{D}_5$ satisfying the given relations.

Suppose that an admissible homomorphism of $\widetilde{T}$ to $PSL(2,q)$ is given. Such homomorphism must restrict to an admissible homomorphism defined on $T_{5,2}$ and $PSL(2,q)$ must contain one of the groups listed in Theorem 2.2.1 case $k = 2, n = 5$. With the same technique used for the tetrahedral groups, we find necessary and sufficient conditions for the existence of an admissible homomorphism $\widetilde{T} \longrightarrow PSL(2,q)$. We shall restrict our attention to the case $q - 1 \equiv 0 \pmod{10}$ since the case $q + 1 \equiv 0 \pmod{10}$ presents some difficulty. The cases $q \equiv 0 \pmod 5$ and $q \equiv 0 \pmod 2$ will be treated aside.

3.1.1 Lemma: *Let q be odd and such that $q \equiv 1 \pmod 5$. Let $\lambda \in \mathbb{F}_q^*$ be a primitive fifth root of unity (which exists thanks to the previous assumption and Theorem 1.1.3) so that $\gamma := \lambda + \lambda^{-1}$ is the trace of an element of order 5 in $PSL(2,q)$.*

Then there exists an admissible homomorphism from $\widetilde{T}$ to $PSL(2,q)$ if and only if the following two values are squares in $\mathbb{F}_q$:

$$\widetilde{Q} := 9 - 4\gamma^2$$

$$\rho := \frac{-\gamma + \sqrt{-4\gamma^4 + 25\gamma^2 - 36}}{2}.$$

Proof:

Under our assumptions, we see that the element of order 5, which is non-parabolic, can be chosen in the following diagonal form

$$XY = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

As in Theorem 2.1.3 we search for matrices X and Z of the form

$$X = \pm \begin{pmatrix} \alpha & \beta \\ \delta & -\alpha \end{pmatrix}$$

$$Z = \pm \begin{pmatrix} \mu & \varphi \\ \psi & -\mu \end{pmatrix}$$

and impose that the traces of $Y = XXY$ and ZXY are $\epsilon := \pm 1$ and 0 respectively thus obtaining the relations $\alpha = \epsilon/(\lambda - \lambda^{-1})$, $\beta\delta = (3 - \gamma^2)/(\gamma^2 - 4)$, $\mu = 0$, $\varphi\psi = -1$. Computing $Y^{-1}XZXZX = (XY)^{-1}(ZX)^2$ and requiring that its trace is 0, we are led to solve a second degree equation in the unknown φ^2. To have a solution φ, both the discriminant of this equation and one of the two solutions must be squares in $\mathbb{F}_q$. The equation is

$$\lambda^{-1}\delta^2\varphi^4 + \frac{\gamma}{\gamma^2 - 4}\varphi^2 + \lambda\beta^2 = 0$$

and has the same discriminant found in Theorem 2.1.3 when $\tau = 0$ so it is a square if and only if $\widetilde{Q}$ is (compare Theorem 2.1.3). We thus find the same condition of Theorem 2.1.3 and this is what we expected since the existence of an admissible homomorphism from $\widetilde{T}$ to $PSL(2,q)$ implies the existence of an admissible homomorphism from $\widetilde{T}_{5,2}$ to the same group.

The product of the two solutions of the equation is $\lambda^2\beta^2/\delta^2$ which is always a square. So we deduce that either both solutions are squares or neither. Since the solutions are

$$\varphi_{1,2}^2 = \frac{-\gamma \pm \sqrt{-4\gamma^4 + 25\gamma^2 - 36}}{2\lambda^{-1}\delta^2(\gamma^2 - 4)}$$

observing that trivially δ^2 is a square, that λ is a square -it is a primitive fifth root of unity so it must be the square of λ^3- and that $\gamma^2 - 4$ is a square (see the proof of Theorem 2.1.3), we get the condition given in the statement of the Lemma.

For $q \equiv 0 \pmod 5$, we can rephrase the proof of Lemma 3.1.1 exchanging the roles of the elements of order 2 and 5 exploiting the symmetry of the singular set as usual. We obtain an equation that admits a solution which guarantees the existence of an admissible surjection of $\widetilde{T}$ onto $PSL(2,5^2)$.

Note that if q is even, we still have to solve the same equation. By a direct check we see that we have two solutions $\varphi = \sqrt{\lambda^{-1}}\delta^{-1}, \sqrt{\lambda^{-1}}\delta^{-1}\gamma^2$ and we can conclude that there is an admissible surjection $\widetilde{T} \longrightarrow PSL(2,2^4)$.

3.1.2 Remark: If 5 divides $q+1$ instead, the condition becomes $\varphi^q = \varphi^{-1}$ because now we work in $PSU(2,q)$. In particular $(\varphi^2)^q = \varphi^{-2}$ and as in Theorem 2.1.3 we have that the discriminant of the equation must be a non-square in $\mathbb{F}_q$ or 0. Equivalently $\widetilde{Q}$ must be a square in $\mathbb{F}_q$. Note that $\varphi_{1,2}^2$ is always a square in $\mathbb{F}_{q^2}$ (since the cyclic multiplicative group of this field has order q^2-1 while the order of $\varphi_{1,2}^2$ divides $q+1$) but it is not easy to exclude the case when $\varphi^q = -\varphi^{-1}$.

For the group $\widetilde{T}$, however, we cannot exhibit a complete classification since we cannot decide whether ρ is a square in $\mathbb{F}_q$. This is due to the presence of square roots of quantities which contain square roots in their expression.

3.1.3 Corollary: *If $q \equiv 1 \pmod{10}$ then every admissible surjection of $\widetilde{T}_{5,2}$ onto $PSL(2,q)$ extends to an admissible surjection from $\widetilde{T}$ onto $PSL(2,q)$ or $PGL(2,q)$.*

Proof:

If there is an admissible surjection from $\widetilde{T}_{5,2}$ onto $PSL(2,q)$ and φ^2 is a square in $\mathbb{F}_q$ then we have a solution to our problem. If φ^2 is not a square then we have a surjective extension to $\widetilde{T}$ onto $PGL(2,q)$. This is easily seen since $\widetilde{T}_{5,2}$ has index 2 in $\widetilde{T}$ and the image of the subgroup must be normal in the image of the group.

Using GAP (see [17]) we are able to list all solutions of type $PSL(2,p)$ and $PGL(2,p)$ for the primes $p \equiv 1 \pmod{10}$ lesser then 1000.

3.1.4 Corollary: *Let $p \equiv 1 \pmod{10}$ be a prime lesser than* 1000.

i) *$PSL(2,p)$ is an admissible quotient of $\widetilde{T}$ exactly for $p =$* 71, 101, 131, 151, 211, 251, 271, 311, 461, 541, 631, 691, 751, 761, 941.

ii) *$PGL(2,p)$ is an admissible quotient of $\widetilde{T}$ exactly for $p =$* 11, 31, 41, 61, 71, 241, 251, 281, 311, 431, 491, 571, 601, 631, 661, 691, 701, 751, 811, 821, 881.

Since there are two possible values for the trace γ, there are two possible values for $\widetilde{Q}$. If only one of the two values of $\widetilde{Q}$ is a square in $\mathbb{F}_q$, we have an admissible surjection from $\widetilde{T}$ onto either $PSL(2,q)$ or $PGL(2,q)$ (since both solutions φ^2 are squares or neither). If both values of $\widetilde{Q}$ are squares, we might have both types of admissible quotients at the same time. Indeed all situations happen: both quotients are of type $PSL(2,q)$ (e.g. $q = 1061$), both are of type $PGL(2,q)$ (e.g. $q = 661, 821$) or one is of type $PSL(2,q)$ and the other of type $PGL(2,q)$ (e.g. $q = 71, 251, 311, 631, 691, 751$).

3.2. The Coxeter case

At this point we wish to consider the problem to find admissible quotients for the Coxeter groups whose subgroups of index 2 of orientation preserving elements are the tetrahedral groups we discussed in chapter 1. We shall denote by $C_{k,n}$, $\widetilde{C}_{k,n}$, $C'(n,k)$, resp. $C(k_1,k_2,k_3,n)$ the Coxeter groups whose tetrahedral groups are $T_{k,n}$ $\widetilde{T}_{k,n}$, $T'_{n,k}$ resp. $T(k_1,k_2,k_3,n)$. The presentations are as follows:

$$C_{k,n} = \langle a,b,c,d \mid a^2, b^2, c^2, d^2, (ab)^k, (ac)^2, (ad)^2, (bc)^2, (bd)^3, (cd)^n \rangle$$

$$\widetilde{C}_{k,n} = \langle a,b,c,d \mid a^2, b^2, c^2, d^2, (ab)^k, (ac)^3, (ad)^2, (bc)^2, (bd)^3, (cd)^n \rangle$$

$$C'_{k,n} = \langle a,b,c,d \mid a^2, b^2, c^2, d^2, (ab)^k, (ac)^2, (ad)^2, (bc)^3, (bd)^3, (cd)^n \rangle$$

$$C(k_1,k_2,k_3,n) = \langle a,b,c,d \mid a^2, b^2, c^2, d^2, (ab)^{k_1}, (ac)^2, (ad)^{k_2}, (bc)^2, (bd)^{k_3}, (cd)^n \rangle.$$

By means of Tietze substitutions we derive a presentation for these groups in terms of the generators of the tetrahedral subgroup and just one reflection (compare paragraph 1.4). Let $x := ad$, $y := db$, $z := ca$ and $a = a$ and substitute obtaining

$$C_{k,n} = \langle a,x,y,z \mid x^2, y^3, z^2, (xy)^k, (xyz)^2, (zx)^n, a^2, (axy)^2, (za)^2, (ax)^2 \rangle$$

$$\widetilde{Q}_{k,n} = \langle a,x,y,z \mid x^2, y^3, z^3, (xy)^k, (xyz)^2, (zx)^n, a^2, (axy)^2, (za)^2, (ax)^2 \rangle$$

$$C'_{k,n} = \langle a,x,y,z \mid x^2, y^3, z^2, (xy)^k, (xyz)^3, (zx)^n, a^2, (axy)^2, (za)^2, (ax)^2 \rangle$$

$$C(k_1,k_2,k_3,n) =$$

$$\langle a,x,y,z \mid x^{k_2}, y^{k_3}, z^2, (xy)^{k_1}, (xyz)^2, (zx)^n, a^2, (axy)^2, (za)^2, (ax)^2 \rangle\ .$$

We shall work with groups $PSL(2,q)$ which are images (always with torsion-free kernel) of the tetrahedral group of index 2 in the Coxeter group we are considering. Working as usual with matrices we shall try to find under which conditions there exist extensions to the Coxeter group.

3.2.1 Lemma:

i) *Assume that the hypotheses of Theorem 2.1.2 are satisfied and that there exists an admissible surjection from the group $T_{k,n}$ onto the group $PSL(2,q)$.*

Then such surjection extends to $C_{k,n}$ *if and only if either* $3-\gamma^2$ *is a square in* $\mathbb{F}_q$ *and the following equation is identically satisfied:*

$$Q(\gamma,\tau) := \gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 + 12 = 0$$

or $\gamma = 0$.

ii) *Assume that the hypotheses of Theorem 2.1.3 are satisfied and that there exists an admissible surjection from the group* $\widetilde{T}_{k,n}$ *onto the group* $PSL(2,q)$. *Then such surjection extends to* $\widetilde{C}_{k,n}$ *if and only if either* $3-\gamma^2 \neq 0$ *is a square in* $\mathbb{F}_q$ *and the following equation is identically satisfied:*

$$\widetilde{Q}(\gamma,\tau) := \gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 \pm 2\gamma\tau + 9 = 0$$

or $\gamma^2 = \tau^2 = 3$ *and* $q \not\equiv 0 \pmod 3$.

iii) *Assume that the hypotheses of Theorem 2.1.4 are satisfied and that there exists an admissible surjection from the group* $T'_{k,n}$ *onto the group* $PSL(2,q)$. *Then such surjection extends to* $C'_{k,n}$ *if and only if either* $3-\gamma^2$ *is a square in* $\mathbb{F}_q$ *and the following equation is identically satisfied:*

$$Q'(\gamma,\tau) := \gamma^2\tau^2 - 4\gamma^2 - 4\tau^2 \pm 4\tau + 8 = 0$$

or $\gamma = 0$.

iv) *Assume that the hypotheses of Theorem 2.1.5 are satisfied and that there exists an admissible surjection from the group* $T(k_1,k_2,k_3,n)$ *to the group* $PSL(2,q)$. *Then such surjection extends to* $C(k_1,k_2,k_3,n)$ *if and only if either* $\gamma_1\gamma_2\gamma_3 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 + 4$ *is a square in* $\mathbb{F}_q$ *and the following equation is identically satisfied:*

$$Q(\gamma_1,\gamma_2,\gamma_3,\tau) := \gamma_1^2\tau^2 + 4\gamma_1\gamma_2\gamma_3 - 4\gamma_1^2 - 4\gamma_2^2 - 4\gamma_3^2 - 4\tau^2 + 16 = 0$$

or $\gamma_1 = \gamma_2 = 0$.

Proof:

We shall discuss in detail only case i), the remaining being proved in the same fashion.

By assumption we can find three matrices

$$XY = \pm\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

$$X = \pm\begin{pmatrix} \alpha & \beta \\ \delta & -\alpha \end{pmatrix}$$

$$Z = \pm \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}$$

satisfying $\gamma := \lambda + \lambda^{-1}$, $\alpha = \epsilon/(\lambda - \lambda^{-1})$, $\beta\delta = (3-\gamma^2)/(\gamma^2-4)$, $\psi\varphi = -1$ $\delta\varphi + \beta\psi = \tau$ where $\epsilon = \pm 1$ and we must remember that if k divides $(q-1)/2$ we are working in $PSL(2,q)$ else we are working in $PSU(2,q)$. We now look for a matrix A of order 2 such that AXY, ZA and AX have order 2 or are trivial (we are no more looking for an admissible homomorphism). Since A is of order 2 it must be of the form

$$A = \pm \begin{pmatrix} \eta & \kappa \\ \theta & -\eta \end{pmatrix}$$

where $-\eta^2 - \kappa\theta = 1$. Indeed we can allow A to be the trivial element. From the group presentation one deduces that this can be the case if and only if $k = 2$. Assume now that A is non trivial of the above form. Computing AXY we find

$$\pm \begin{pmatrix} \eta\lambda & \kappa\lambda^{-1} \\ \theta\lambda & -\eta\lambda^{-1} \end{pmatrix}$$

and since the trace must be 0 while $\lambda - \lambda^{-1} \neq 0$ we must have $\eta = 0$. We have

$$A = \pm \begin{pmatrix} 0 & \kappa \\ \theta & 0 \end{pmatrix}$$

Next we compute

$$AX = \pm \begin{pmatrix} \kappa\delta & -\kappa\alpha \\ \theta\alpha & \theta\beta \end{pmatrix}$$

$$ZA = \pm \begin{pmatrix} \varphi\theta & 0 \\ 0 & \psi\kappa \end{pmatrix}$$

from which we obtain the relations $\kappa\delta + \theta\beta = 0$ and either $\varphi\theta + \psi\kappa = 0$ or $\varphi\theta + \psi\kappa = \pm 2$ (this latter case happens when ZA is the trivial element; note that since $\kappa\alpha \neq 0$ AX cannot be the trivial element). We must thus solve a system of equations

$$\begin{cases} \beta\delta = (3-\gamma^2)/(\gamma^2-4) \\ \psi\varphi = \theta\kappa = -1 \\ \delta\varphi + \beta\psi = \tau \\ \kappa\delta + \theta\beta = 0 \\ \varphi\theta + \psi\kappa = 0, \pm 2 \end{cases}$$

Remember that all elements belong to $\mathbb{F}_q$ when working with $PSL(2,q)$, while, when working with $PSU(2,q)$, we have $\kappa^q = \kappa^{-1}$ (in this case $\beta^q = -\delta$, $\varphi^q = \varphi^{-1}$ and $\lambda^q = \lambda^{-1}$).

Assume first that $\gamma^2 = 3$ then, from the fourth equation, we deduce that both β and δ must be 0, so $\tau = 0$ and a solution always exists $\kappa = \pm\varphi = \pm 1$ -choose the second possibility for the last equation.

Assume now that $\gamma^2 \neq 3$. From the fourth equation one can express $\kappa^2 = (3-\gamma^2)/(\gamma^2-4)$ and thus $3-\gamma^2$ must be a square in $\mathbb{F}_q$. Remark that one obtains the same condition even when working with $PSU(2,q)$. Just express $\theta = -\delta\kappa/\beta$ and $\theta = -\kappa^q$ with $\kappa = \pm\sqrt{\beta\delta}/\delta$ and reason as in the proofs of Theorems 2.1.2 and 2.1.3. Remark also that in the previous case $3-\gamma^2 = 0$ is a square and $Q(\gamma,\tau) = 0$. From the last equation we can find an expression for $\varphi = \pm\delta^{-1}\sqrt{(\gamma^2-3)/(\gamma^2-4)}$ (or $\varphi = \pm\delta^{-1}\sqrt{(3-\gamma^2)/(\gamma^2-4)}$). If we substitute this expression in the third equation we must get an identity. This gives exactly the condition $Q(\gamma,\tau) = 0$ after squaring and simplifying the term $\gamma^2 - 4 \neq 0$ (the second possibility gives $\tau = 0$, again the same condition of Lemma).

Remark that we never exploit the fact that q is odd here. In fact for q even again we need to find a solution to the above system of equations. For completeness we give also the systems obtained in the other three cases.

ii)
$$\begin{cases} \beta\delta = \psi\varphi = (3-\gamma^2)/(\gamma^2-4) \\ \psi\varphi = \theta\kappa = -1 \\ \delta\varphi + \beta\psi \pm \gamma/(\gamma^2-4) = \tau \\ \kappa\delta + \theta\beta = 0 \\ \varphi\theta + \psi\kappa = 0 \end{cases}$$

iii)
$$\begin{cases} \beta\delta = \psi\varphi = (3-\gamma^2)/(\gamma^2-4) \\ \theta\kappa = -1 \\ \delta\varphi + \beta\psi \pm (\gamma^2-4)^{-1} = \tau \\ \kappa\delta + \theta\beta = 0 \\ \varphi\theta + \psi\kappa = 0 \end{cases}$$

iv)
$$\begin{cases} \beta\delta = (\gamma_1\gamma_2\gamma_3 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 + 4)/(\gamma_1^2-4) \\ \psi\varphi = \theta\kappa = -1 \\ \delta\varphi + \beta\psi = \tau \\ \kappa\delta + \theta\beta = 0 \\ \varphi\theta + \psi\kappa = 0, \pm 2 \end{cases}$$

Note that for all groups considered in sections 2.2 and 2.3 the associated Coxeter groups have the extended tetrahedral group $\mathbf{A}_4 \times \mathbb{Z}_2$, the extended octahedral group $\mathbf{S}_4 \times \mathbb{Z}_2$ or the extended dodecahedral group $\mathbf{A}_5 \times \mathbb{Z}_2$ as subgroups. However these groups are not subgroups of $PSL(2,q)$ (see Theorem 1.3.8), so the extensions we are going to find will not be admissible. Once we have found an extension onto $PSL(2,q)$ we can conclude that there is an admissible surjection of the Coxeter group onto $PSL(2,q) \times \mathbb{Z}_2$ where the element of order 2 commuting with $\mathbf{A}_4$, $\mathbf{S}_4$ or $\mathbf{A}_5$ is mapped to the generator of $\mathbb{Z}_2$.

On the other hand, assume that an admissible surjection of a Coxeter group onto $PSL(2,q) \times \mathbb{Z}_2$ is given, consider the intersection of the image of the tetrahedral group with $PSL(2,q)$. This is a normal subgroup and so it must coincide with $PSL(2,q)$ itself if $q \neq 2,3$ (see Theorem 1.3.2). Yet we cannot be sure that the surjection obtained by restricting our map to the tetrahedral group and then projecting onto $PSL(2,q)$ has torsion-free kernel. Any torsion elements in the kernel must be of order 2. One can see that if any of the elements of order 2 in the groups considered in sections 2.2 and 2.3 (apart from $T'_{5,2}$) are mapped to the identity, the whole group itself is mapped to the identity. For $T'_{5,2}$ we can map the element zx of order 2 to the trivial element, but in this case the image of the group would be $\mathbf{A}_5$ and this homomorphism does not extend to an admissible surjection from $T'_{5,2}$ to $PSL(2,5) \times \mathbb{Z}_2$. This means that the only possible admissible surjections from a Coxeter group associated to any of the tetrahedral groups studied in paragraphs 2.2 and 2.3 onto $PSL(2,q) \times \mathbb{Z}_2$ restrict to an admissible surjections of the tetrahedral group to $PSL(2,q)$. Thanks to these remarks we can prove

3.2.2 Theorem:

i) *There is an admissible surjection* $\widetilde{C}_{5,2} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$ *exactly for* $q = 11$.

ii) *There is an admissible surjection* $\widetilde{C}_{4,3} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$ *exactly for* $q = 7$.

iii) *There are no admissible surjections* $\widetilde{C}_{4,4} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

iv) *There are no admissible surjections* $C'_{5,2} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

v) *There are no admissible surjections* $C_{4,5} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

vi) *There is an admissible surjection* $C_{5,5} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$ *exactly for* $q = 19$.

vi) *There are no admissible surjections* $C_{3,6} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

vii) *There are no admissible surjections* $C_{4,6} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

viii) *There are no admissible surjections* $C_{5,6} \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

ix) *There are no admissible surjections* $C(4,2,4,3) \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

x) *There are is an admissible surjection* $C(3,3,3,3) \longrightarrow PSL(2,q) \times \mathbb{Z}_2$ *exactly for* $q = 5$.

xi) *There are no admissible surjections* $C(3,3,3,4) \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

xii) *There are no admissible surjections* $C(3,3,3,5) \longrightarrow PSL(2,q) \times \mathbb{Z}_2$.

Proof:

This is proved by applying Lemma 3.2.1 and exploiting the results found in paragraphs 2.2 and 2.3. Just note that one must fix the values for the traces and then verify that the conditions are satisfied altogether. The cases when Lemma 3.2.1 does not apply (e.g. $C(3,3,3,n)$ and $q \equiv 0 \pmod 3$) are checked directly. For q even one uses the remark made at the end of the proof of Lemma 3.2.1.

According to the cases listed in [25], there are other admissible surjections, precisely

in i) $q = 2^4, 5^2$,

in ii) $q = 7^2$,

in iv) $q = 5^2$,

in vi) $q = 2^4$.

It is important to point out that if the Coxeter groups have admissible quotients of type $PSL(2, q)$ they must be in finite number.

4. POLYHEDRAL CASE

In the following we shall classify admissible quotients of type $PSL(2,q)$ for the Picard group and another polyhedral group (paragraph 4.2). Even if we rely on the results of section 2.1, the classification here is carried out adopting a method which differs from that used in paragraphs 2.2 and 2.3. This method will allow us to study also some infinite series of tetrahedral groups (paragraph 4.1). We conclude the chapter with some considerations on the construction of closed hyperbolic 3-manifolds with large group actions.

4.1. Some infinite series of tetrahedral groups $T_{k,n}$

In this section, for a fixed $2 \leq k \leq 5$ we shall consider the series of tetrahedral groups $T_{k,n}$ for arbitrary n. For $n > 6$ these tetrahedral groups have exactly one hyperbolic vertex group which is the triangle group $(2,3,n)$. For any single $n > 6$ it is rather difficult to give a complete classification of the finite quotients of $T_{k,n}$ of linear fractional type (see [21] for the "Hurwitz-case" $n = 7$), so we will discuss simultaneously the whole series of groups. As noted in paragraph 2.1, the groups $T_{2,n}$ are isomorphic to the extended triangle groups $[2,3,n]$. The groups $T_{3,n}$ have been considered in detail in [12]. Both the groups $T_{2,n}$ and $T_{3,n}$ will be used in the next section to determine the finite quotients of the Picard group.

Another reason for considering these series is that, for a fixed k and arbitrary n, the finite admissible quotients of the groups $T_{k,n}$ are exactly the $\mathbf{G}_k$-groups. We refer to paragraph 1.6 for definition and general properties of $\mathbf{G}_k$-groups.

The group $\mathbf{G}_2 = \mathbf{D}_2 *_{\mathbb{Z}_2} \mathbf{D}_3$ is isomorphic to the extended modular group $PGL(2,\mathbb{Z})$ (see Example 1.3.3 iii)), and its finite quotients of linear fractional type have been classified in [43]. This is the case $k = 2$ of the following Theorem whose proof, like that in [43], relies on the result of Macbeath stated in Theorem 1.3.9.

4.1.1 Theorem: *Let $2 \leq k \leq 5$ be fixed. Then $PSL(2,q)$ is an admissible quotient of $T_{k,n}$, for some n, exactly in the following cases; equivalently, $PSL(2,q)$ is a $\mathbf{G}_k$-group exactly in the following cases:*

$k = 2$: $\quad q \neq 2, 7, 3^2, 11, 3^{2m+1}$;

$k = 3$: $\quad q \neq 2, 7, 3^2, 11, 3^{2m+1}$;

$k = 4$: $\quad q \equiv \pm 1 \pmod 8$ *but* $q \neq 7, 3^2$;

$k = 5$: $\qquad q(q^2-1) \equiv 0 \pmod 5$ *but* $q \neq 3^2$.

Proof:

Assume first that $q = p^n$ is odd. From Theorem 2.1.2 we know that, if $\gamma \neq \pm 2$, an admissible homomorphism from $T_{k,n}$ to $PSL(2,q)$ exists if and only if we can find traces $\gamma, \tau \in \mathbb{F}_q$ of elements of order k resp. n in $PSL(2,q)$ such that $Q(\gamma,\tau)$ is a square in $\mathbb{F}_q$. If we want a solution for some n all we need to check is that, fixed an appropriate γ according to the cases $k = 2, 3, 4$ resp. 5, there exists $\tau \in \mathbb{F}_q$ such that $Q(\gamma,\tau)$ is a square in $\mathbb{F}_q$. The trace γ of an element of order $k = 2, 3, 4$ resp. 5 satisfies $\gamma = 0$, $\gamma^2 = 1$, $\gamma^2 = 2$ resp. $\gamma^2 \pm \gamma - 1 = 0$ (see Examples 1.3.7). In particular, there are always elements of order $k = 2$ and 3 in $PSL(2,q)$, and there are elements of order 4 (resp. 5) if and only if $q \equiv \pm 1 \pmod 8$ (resp. $q(q^2-1) \equiv 0 \pmod 5$) (compare Theorem 1.3.8). In the following assume that $\gamma \neq \pm 2$ and that, fixed k, $PSL(2,q)$ contains elements of order k. In the different cases $Q(\gamma,\tau)$ is given by:

$k = 2$: $\quad -4\tau^2 + 12$, which is a square in $\mathbb{F}_q$ if and only if $3 - \tau^2$ is, since 4 is always a square;

$k = 3$: $\quad 8 - 3\tau^2$;

$k = 4$: $\quad -2\tau^2 + 4$, which is a square in $\mathbb{F}_q$ if and only if $2 - \tau^2$ is a square, since in this case $2 = \gamma^2$ is a square;

$k = 5$: $\quad (-5 \pm \sqrt{5}/2)\tau^2 \mp 2\sqrt{5}$.

By Theorem 2.1.2, there exists an admissible homomorphism from $T_{k,n}$ to $PSL(2,q)$ if and only if the affine conic defined over $\mathbb{F}_q$

$k = 2$: $\quad a^2 + \tau^2 = 3$;

$k = 3$: $\quad a^2 + 3\tau^2 = 8$;

$k = 4$: $\quad a^2 + \tau^2 = 2$;

$k = 5$: $\quad a^2 + ((5 \mp \sqrt{5})/2)\tau^2 = \mp 2\sqrt{5}$;

has got $\mathbb{F}_q$-rational points (see paragraph 1.2). We can apply Theorem 1.2.12 once we assume $q \not\equiv 0 \pmod 3$ if $k = 2$ so that no coefficient is 0.

However we want to know when this solution is surjective. This will always be the case if the elements of order 2 and 3 whose product has order n are a generating pair. All we must do is discard the cases when they generate a proper subgroup, in particular we shall assume $q \neq 3^2$. Using Theorem 1.3.8, we see that if the elements of order 2 and 3 generate a proper subgroup then we are in one of the following situations:

i) $\tau^2 = 0, 1, 2, 3$ or $\tau^2 \pm \tau - 1 = 0$ where they generate an exceptional subgroup or an affine subgroup;

ii) $\tau \in \mathbb{F}_{q'} \subset \mathbb{F}_q$ where they generate a subgroup contained in $PSL(2,q') \subset PSL(2,q)$;

iii) $\tau^2 \in \mathbb{F}_{\sqrt{q}}$ (this condition makes sense only if q is an even power of p) where the image is a group inside $PGL(2,\sqrt{q})$.

Following [43], we shall call τ *admissible* if it is not of one of the forms in i), ii) and iii). The idea now is to count all solutions τ and see when their number is larger than the number of non-admissible solutions.

The number of solutions τ is at least $(q-1)/2$ (see Theorem 1.2.12) because with (a, τ) also $(-a, \tau)$ is on the conic, while the number of τ's satisfying i), ii) or iii) is at most $11+\sqrt{q}+\varepsilon\sqrt{q}$ where ε is 0 if q is an odd power of p and 1 otherwise. Note that if $q = p$ is a prime there are at most 11 non-admissible τ's belonging all to case i). Now as in [42] we see that there is a solution for all $q > 25$, $q \neq 7^2$. Just consider the function $f(q) := q - 2(1+\epsilon)\sqrt{q} - 23$ (resp. $g(q) := q - 23$ when q is prime). We want the function to be positive. Redefining $t := \sqrt{q}$ in the first case and studying the functions for all positive real numbers, it is not difficult to see that they are non-positive only for the given values. This means that for $q > 25$ but $q \neq 7^2$ $PSL(2,q)$ is an admissible quotient of $T_{k,n}$, $k = 2,3,4,5$, for some n. In the cases $q \leq 25$, $q = 7^2$ as well as $q = 3^2$, an admissible τ is looked for directly. We shall discuss some cases as an example at the end.

Note that if $k = 2$ the condition that the elements of order 2 and 3 are a generating pair is not only sufficient but also necessary since $(2,3,n)$ has index 2 in $T_{2,n}$ and the linear fractional groups are simple (apart from $PSL(2,2)$, $PSL(2,3)$ -see Theorem 1.3.2).

Let us now study the case $k = 2$, $p = 3$. The conic becomes: $a^2 = -\tau^2$. There is only the point (0,0) on it if -1 is a non-square (i.e. q is an odd power of 3), while there are $2q-1$ points if -1 is a square (see again Theorem 1.2.12). If $q \equiv 1 \pmod 4$ we can repeat the same estimate as above and conclude that we always find an admissible τ apart from $q = 3^2$ (this last case must be treated aside anyway). If $q \equiv -1$ instead, there is no admissible homomorphism from $T_{2,2}$ to $PSL(2,3^m)$ (see Theorem 2.1.2).

Suppose now that $\gamma = \pm 2$. If we require $\tau \neq \pm 2$ then we can exchange the roles of γ and τ. We obtain $Q(\gamma, \tau) = -4$ which is a square if and only if -1 is, independently of τ. We conclude that there are always admissible homomorphism for all $\tau \neq \pm 2$. To ensure surjectivity it is enough to see if $q-2 > (1+\varepsilon)\sqrt{q}+11$ and this is always the case for $q \neq 3,5$. In these two cases the existence of an admissible homomorphism is sufficient to ensure surjectivity since we are mapping $T_{k,n}$ to one of its vertex groups.

We still have to study the cases when $q = 2^m$. We do not need to consider the case $k = 4$ since there are no elements of order 4 in $PSL(2,2^m)$.

Suppose $k = 3,5$ first. To see if there is an admissible homomorphism to $PSL(2,2^m)$ we can repeat the same considerations made in the proof of Theorem 2.1.2. Again we need to solve the following second degree equation

$$\delta\varphi^2 - \tau\varphi - \beta = 0\,.$$

Observe that if $k = 5$, $\delta \neq 0$ and we can put $t := \delta\varphi$ after multiplying by δ both sides of the equation thus obtaining

$$t^2 + \tau t + \gamma^2 = 0$$

(recall that $\gamma^2 + \gamma + 1 = 0$ in this case). Dividing up by γ^2 we obtain the following equation (where the new t stands for t/γ)

$$t^2 + \frac{\tau}{\gamma} t + 1 = 0$$

whose roots are $t_0 \in \mathbb{F}_q^*$ such that $t_0 + t_0^{-1} = \tau/\gamma$. We thus see that there are at least $(q-1)/2$ possible τ's obtained varying t_0 in $\mathbb{F}_q^*$.

If $k = 3$ we have $\beta\delta = 0$ but anyway all $\tau \in \mathbb{F}_q^*$ are solutions to our equation. We have only to discard the case $q = 2$ since in this case the vertex group $\mathbf{A}_4$ is not contained in $PSL(2, 2)$.

Now again one concludes checking that there exist admissible solutions τ. Note that here we have only 4 values in i) and case iii) is meaningless so the conditions become $q - 1 > 4 + \sqrt{q}$ if $k = 3$ and $(q-1)/2 > 4 + \sqrt{q}$ if $k = 5$. They are satisfied for all $q \geq 4$ resp. $q \geq 16$. The only situation that we must check aside is for $k = 5$ $q = 2^2$ but in this case $PSL(2, 2^2) \cong PSL(2, 5)$.

If $k = 2$ the element XY is parabolic. In any case it must be of the form

$$XY = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

where $\lambda \in \mathbb{F}_q^*$ -note that $2 \equiv 0 \pmod 2$. Repeating the same type of procedure used in Theorem 2.1.2, we see that we always have a solution for any $\tau \neq 0, 1$ (if $\tau = 0$ there are no homomorphisms, if $\tau = 1$ a homomorphism exists but it is not admissible). This is enough to ensure existence of admissible quotients of $T_{2,n}$ of type $PSL(2, 2^m)$ for every $m > 1$.

4.1.2 Example: For completeness we show now how one can reason in the cases when $q \leq 25$, $q = 7^2$.

First of all, it will be useful to consider in detail the groups $T_{k,n}$ for k and n varying in the set $\{2, 3, 4, 5\}$ and their admissible quotients of spherical type. We have (see [59])

i) $T_{2,2} \cong \mathbf{D}_6$ but it cannot have an admissible homomorphism onto any of the groups $\mathbf{D}_3 \cong PSL(2, 2)$ or to $\mathbf{D}_2$ which are its vertex groups.

ii) $T_{3,2} \cong \mathbf{S}_4 \cong PGL(2, 3)$ but it does not have an admissible homomorphism to $\mathbf{A}_4 \cong PSL(2, 3)$; one can check this applying Theorem 2.1.2 to $\gamma = 0$, $\tau = \pm 1$ and $q = 3$.

iii) $T_{3,3} \cong \mathbf{A}_5 \cong PSL(2, 2^2) \cong PSL(2, 5)$ but it does not have an admissible homomorphism to $\mathbf{A}_4$ or $\mathbf{S}_4$; this can be seen by a direct computation with permutations. Indeed, consider $T_{3,n}$: the element XY would be parabolic. Working with permutations instead, up to conjugation in $\mathbf{S}_4$ we can fix $XY = (1,2,3)$. Since the upper-right vertex group is $\mathbf{A}_4$, X must be the product of two disjoint transpositions. The possible X's are: $(1,2)(3,4)$, $(1,3)(2,4)$ and $(1,4)(2,3)$; they all have products of order 3 with XY. Now we choose Z. Since XYZ must have order 2, Z cannot be of one of the forms above and so must lie in $\mathbf{S}_4$ (we thus see that $\mathbf{A}_4$ is not an admissible quotient of $T_{3,n}$). The possible choices for Z are: $(1,2)$, $(1,3)$, $(2,3)$. The element ZX has order 2, 4 and 4 respectively.

iv) $T_{4,2} \cong \mathbf{S}_4 \times \mathbb{Z}_2$ and it has an admissible surjection onto $\mathbf{S}_4$.

v) $T_{4,3}$ is an extension of $(\mathbb{Z}_2)^3$ by $\mathbf{S}_4$ and it has an admissible surjection onto $\mathbf{S}_4$.

vi) $T_{4,4}$ is Euclidean and it has an admissible surjection onto $\mathbf{S}_4$.

vii) $T_{5,2} \cong \mathbf{A}_5 \times \mathbb{Z}_2$ and it has an admissible surjection onto $\mathbf{A}_5$.

viii) $T_{5,3}$ is an extension of $\mathbb{Z}_2$ by $(\mathbf{A}_5)^2$ and it has an admissible surjection onto $\mathbf{A}_5$.

ix) $T_{5,4}$ is closed hyperbolic and does not have any admissible surjection onto spherical groups of linear fractional type (see Theorem 2.2.3).

x) $T_{5,5}$ is closed hyperbolic and has an admissible surjection onto $\mathbf{A}_5$ (see Theorem 2.2.3).

Now let us fix k and q. We shall consider all τ's for which we have a solution to our equation and then check if the homomorphism is surjective. We only consider the cases for $k = 3, 4, 5$ since the case for $k = 2$ can be found in [43].

$k = 3$

Let $q = 5$. An admissible solution exists according to case iii) above: it corresponds to $\tau^2 = 1$ and $a^2 = 0$.

Let $q = 7$. We have $\tau^2 = 0,\ 1,\ 4,\ 2$ corresponding to $a^2 = 1,\ 5,\ 3,\ 2$. We must discard the first and last cases because they give the trivial solution $\mathbf{S}_4$ (cases ii) and v) above) and the second and third because the image cannot be in $PSL(2,7)$ by Theorem 2.1.2.

Let $q = 11$. We have $\tau^2 = 0,\ 1,\ 4,\ 9,\ 5$ corresponding to $a^2 = 8,\ 5,\ 7,\ 3,\ 4$. We must discard the first and the third case because they do not give a solution in $PSL(2,11)$ (the image is S_4 and is contained in $PGL(2,11)$ -by case ii) and Theorem 2.1.2) and the remaining cases because they correspond to $n = 3, 5$ and the image is $\mathbf{A}_5$ (cases iii) and viii)).

Let $q = 13$. We have $\tau^2 = 0,\ 1,\ 4,\ 9,\ 3,\ 12,\ 10$ corresponding to $a^2 = 8,\ 5,\ 9,\ 7,\ 12,\ 11,\ 4$. Observe that $\tau^2 = 4,\ 9,\ 12,\ 10$ are admissible and the first and last value give a solution.

For $q = 17, 19, 23$ one reasons as in case $q = 13$: a direct computation shows that there are admissible τ's which are solutions.

We now discuss case $q = 5^2$. Case $q = 7^2$ follows the same pattern. We must discard the cases when the image is contained in $PGL(2, q)$. Let ξ a generator of the extension $\mathbb{F}_{5^2}$ of $\mathbb{F}_5$, such that $\xi^2 = 2$. The possibilities for τ^2 are 0, 1, $2 = \xi^2$, $3 = (2\xi)^2$, 4, $1 + 2\xi = (\xi + 2)^2$, $2 + 4\xi$, $3 + \xi$, $4 + 3\xi$, $3 + 2\xi = (\xi + 2)^2$, $1 + 4\xi$, $4 + \xi$, $2 + 3\xi$ corresponding to $a^2 = 3$, 0, 2, 4, 1, 4ξ, $2 + 3\xi$, $4 + 2\xi$, $1 + \xi$, $4 + 4\xi$, 3ξ, $1 + 2\xi$, $2 + \xi$. The last eight τ^2's are admissible and the seventh and twelfth values for a^2 are indeed squares in $\mathbb{F}_{5^2}$.

If $q = 3^2$, the possible values for n are 2,3,4 and 5. From the above discussion we see that $T_{3,2}$ and $T_{3,4}$ map to $\mathbf{S}_4 \cong PGL(2, 3)$ while $T_{3,3}$ and $T_{3,5}$ map to $\mathbf{A}_5$.

$k = 4$

We have to consider only the cases $q = 7$ and 3^2.

If $q = 7$, the solutions to our equation are $\tau^2 = 0, 1, 2$ corresponding to $n = 2, 3, 4$. Comparing the given list we see that the admissible homomorphisms are not surjective.

If $q = 3^2$, we can reason as in the case $k = 3$.

$k = 5$

The only case to consider is for $q = 3^2$ and we can proceed as before.

4.1.3 Remark: Note that by reduction of coefficients mod p for the extended modular group $PSL(2, \mathbb{Z}) \cong \mathbf{G}_2$ we obtain admissible quotients of type $PGL(2, p)$ if $p \equiv -1 \pmod 4$, i.e. -1 is a non-square in $\mathbb{F}_p$, and of type $PSL(2, p)$ if $p \equiv 1$ (mod 4). Now the case $k = 2$ of Theorem 4.1.1 implies that most admissible quotients of the extended modular group are not obtained by reduction of coefficients mod p.

4.1.4 Corollary: *$PSL(2, q)$ is a maximal handlebody group (see paragraph 1.6) of order $12(g - 1)$ exactly for all q different from 2, 7, 3^2 and 3^{2m+1}.*

4.2. The Picard group

In the next Theorem we shall classify the groups $PSL(2, q)$ which are admissible quotients of the Picard group $PSL(2, \mathcal{O}_1) = PSL(2, \mathbb{Z}[i])$. The Picard group is a polygonal product (see Figure 7) isomorphic to the free product with amalgamation

$$\mathbf{G}_2 *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathbf{G}_3 = (\mathbf{D}_2 *_{\mathbb{Z}_2} \mathbf{D}_3) *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} (\mathbf{D}_3 *_{\mathbb{Z}_2} \mathbf{A}_4),$$

where the amalgam $(\mathbb{Z}_2 = \langle w \rangle) * (\mathbb{Z}_3 = \langle y \rangle)$, isomorphic to the modular group $PSL(2, \mathbb{Z})$, identifies the generators called w and y in the above presentation of $\mathbf{G}_2$ and $\mathbf{G}_3$ (actually, of $\mathbf{G}_k$), see [52], [15] and [16].

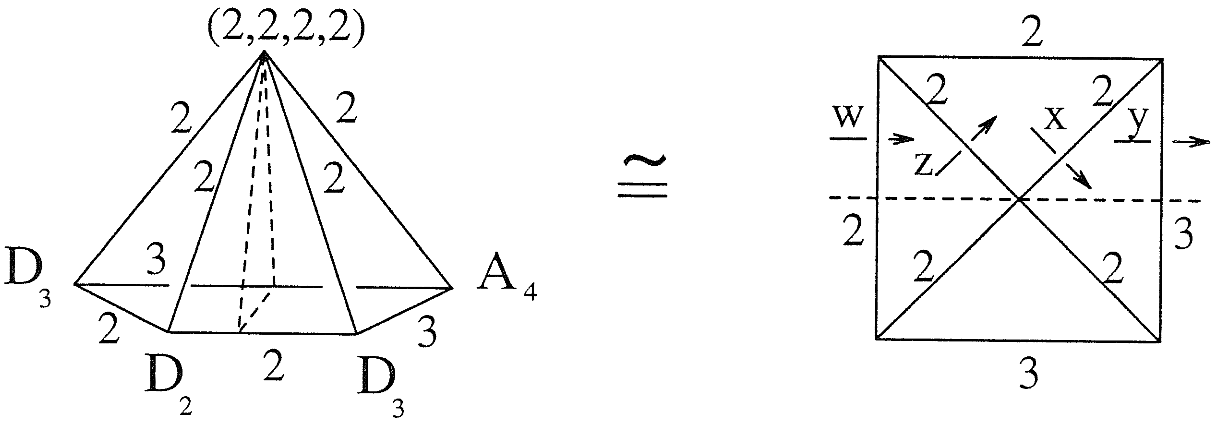


Figure 7

Given a homomorphism of the Picard group $\mathbf{G}_2 *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathbf{G}_3$ onto a finite group, the image of the element wy^{-1} has a certain order n, and then the homomorphism factors through the free product with amalgamation

$$T_{2,n} *_{(2,3,n)} T_{3,n}$$

over the triangle group $(2,3,n) = \langle w, y \mid w^2, y^3, (wy^{-1})^n \rangle$ (note that, in the above generators of both $\mathbf{G}_2$ and $\mathbf{G}_3$, one has $wy^{-1} = zx$, and that $T_{k,n} = \mathbf{G}_k / \langle (zx)^n \rangle \rangle$). Thus we are in a situation where we can apply Theorems 2.1.2 and 4.1.1 to the factors $T_{2,n}$ and $T_{3,n}$ of the product. The result is as follows.

4.2.1 Theorem: *The group $PSL(2,q)$ is an admissible quotient of the Picard group exactly for the values of q different from $2, 7, 3^2, 11$ and 3^{2m+1}.*

Proof:
We assume for the moment that q is not a power of 2 or 3.
Suppose there exists an admissible homomorphism from the Picard group to $PSL(2,q)$. As noted above, the homomorphism factors through $T_{2,n} *_{(2,3,n)} T_{3,n}$ and induces admissible homomorphisms from both $T_{2,n}$ and $T_{3,n}$ to $PSL(2,q)$. By Theorem 2.1.2 resp. the proof of Theorem 4.1.1, we find a solution of the following system of equations

$$\begin{cases} a^2 = 3 - \tau^2 \\ b^2 = 8 - 3\tau^2 \end{cases}$$

where τ is the trace of the element of order n in $PSL(2,q)$ which is the image of zx.

Conversely, suppose we have a solution of this system. By Theorem 2.1.2, there exist admissible homomorphisms from both $T_{2,n}$ and $T_{3,n}$ to $PSL(2,q)$ such that the image of the element zx has trace τ. By Lemma 2.1.1 the restrictions of these homomorphisms to the common triangle subgroup $(2,3,n)$ generated by z and x are the same up to conjugation whenever $\tau^2 \neq 3$. Then the two homomorphisms combine giving an admissible homomorphism from $T_{2,n} *_{(2,3,n)} T_{3,n}$ to $PSL(2,q)$, and thus also from the Picard group $\mathbf{G}_2 *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathbf{G}_3$. So we want to find a solution of the above system which implies also surjectivity (as in the proof of Theorem 4.1.1).

The solutions of the system are the $\mathbb{F}_q$-rational points of an affine variety in $\mathbb{A}^3_{\overline{\mathbb{F}}_q}$ defined over $\mathbb{F}_q$. According to paragraph 1.2, it is easy to see that this variety is a curve of degree 4 and genus 1.

-The Jacobian associated to the polynomials defining the variety is

$$\begin{pmatrix} 2a & 0 & 2\tau \\ 0 & 2b & 6\tau \end{pmatrix}$$

and it is straightforward to verify that if q is not a power of 2 or 3 the Jacobian for any point (a, b, τ) of the variety has rank 2, so that the variety is a smooth curve.

-The intersection of the curve with a general 2-plane (take for instance the planes defined by $a = 0$, $b = 0$ or $\tau = 0$) consists of four points

-The genus can be computed considering the projection of the curve on the plane $a = 0$ and using Riemann-Hurwitz formula (see Theorem 1.2.11). The image of the projection, which has degree 2, is a smooth conic and thus has genus 0. Then there are four points where the fibre has cardinality 1 instead of 2: these are the points of intersection of the curve with the 2-plane defined by $a = 0$.

Applying Theorem 1.2.13, the number N of $\mathbb{F}_q$-rational points on the projective closure of our curve can be estimated by the following inequality:

$$|N - q| \leq 2\sqrt{q} + 1 \,.$$

The number of solutions to our system is then grater or equal to $q - 2\sqrt{q} - 5$ (here we are excluding the points at infinity which are four at the most) and since the curve has degree 4, the number of possible τ's is not less than $(q - 2\sqrt{q} - 5)/4$: fix τ (that is, intersect the curve with the 2-plane τ =const.) there are at the most four possible (a, b) such that (τ, a, b) is on the curve. We want this number to be larger than $(1 + \varepsilon)\sqrt{q} + 11$ (equivalently $q - (6 + 4\varepsilon)\sqrt{q} - 49 > 0$) or 11 (equivalently $q - 2\sqrt{q} - 49 > 0$) if $q = p$ is a prime, just like in Theorem 4.1.1. Studying the functions $f(q) := q - 6\sqrt{q} - 49$ (for q an odd, non-trivial power of a prime), $f'(q) := f(q) - 4\sqrt{q}$ (for q an even power of a prime) and $g(q) := q - 2\sqrt{q} - 49$ (q prime) defined over the positive real numbers (one can replace $t := \sqrt{q}$ for simplicity), we see that they are positive for all $q \geq 67$ but $q \neq 11^2, 13^2$ and in these cases an admissible τ does exist.

We want now to see what happens when the functions are non-positive. First we remark some facts. First of all, by reduction of coefficients mod p we find admissible quotients of the Picard group of type $PSL(2,p)$ if $p \equiv 1 \pmod 4$ and of type $PSL(2,p^2)$ if $p \equiv -1 \pmod 4$ (compare Theorem 2.3.3).

Secondarily it is easy to see that our system of equations is equivalent to

$$\begin{cases} \tau^2 = 3 - a^2 \\ 3a^2 - b^2 = 1 \end{cases}$$

and we are able to solve the second equation and give an explicit expression for a and b (as in the proof of Theorem 1.2.12). Substituting a in the first equation we obtain $\tau^2 = (34 - \alpha^2 - \alpha^{-2})/12$ where $\alpha \in \mathbb{F}_q^*$ if 3 is a square in $\mathbb{F}_q$ or $\alpha \in \mathbb{F}_{q^2}^*$ is a $(q+1)$-root of unity otherwise. Note that in both cases we can put $\alpha = 1$ obtaining $\tau^2 = 8/3$ which is a square if and only if 2 and 3 are both either squares or non-squares (i.e. $q \equiv \pm 1, \pm 5 \pmod{24}$). Assume now that q is a prime. We have to discard the possibility for τ^2 to be non-admissible, i.e. one of those given in case i). We see that we have an admissible solution τ (i.e. $\tau^2 \neq 0, 1, 2, 3$ and $\tau^2 \pm \tau - 1 \neq 0$) for all prime numbers $p \neq 5$, $p \equiv \pm 1, \pm 5 \pmod{24}$.

At this point we only need to check if $PSL(2,q)$ is an admissible quotient of the Picard group for $q = 7, 11, 5^2, 31, 59, 13^2$ and this is done by direct computation as in Example 4.1.2. Here one must remember that there might be admissible surjections from the Picard group onto $PSL(2,q)$ which do not restrict to surjections from any of the two factor groups $T_{2,n}$ and $T_{3,n}$. We do not find an admissible τ only when $q = 7, 11$.

For the cases when q is a power of 2 or 3, we just make the same sort of considerations of Theorem 4.1.1. We conclude that we can find an n_q such that $PSL(2,q)$ is an admissible quotient of both T_{2,n_q} and T_{3,n_q} if and only if $q \neq 2$ or q is an even power of 3 but $q \neq 3^2$.

This finishes the proof of Theorem.

The proof of Theorem 4.2.1 works for different amalgams of the groups $\mathbf{G}_k$ generalizing the Picard group (some of the corresponding orbifolds and their volumes occur in [12, pages 169-170]). For example the following Theorem deals with the admissible quotients of the hyperbolic polyhedral group uniformizing the orbifold represented in Figure 8. Observe that this group is also an extended Bianchi group.

4.2.2 Theorem: *The group $PSL(2,q)$ is an admissible quotient of the extended Bianchi group $PGL(2, \mathcal{O}_2) = PGL(2, \mathbb{Z}[i\sqrt{2}]) \cong \mathbf{G}_2 *_{(\mathbb{Z}_2 * \mathbb{Z}_3)} \mathbf{G}_4$ exactly for all $q \equiv \pm 1 \pmod 8$ different from $7, 3^2$.*

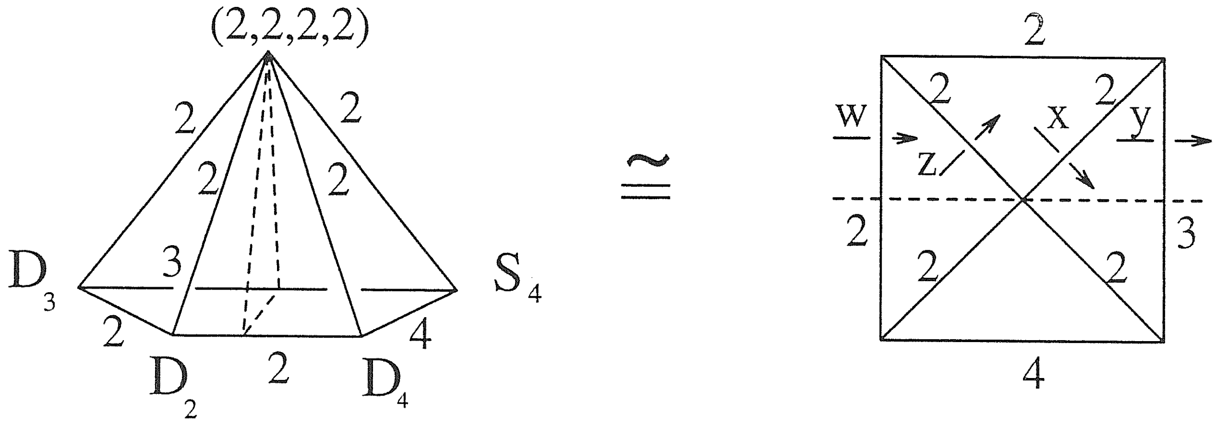


Figure 8

Proof:
We only make some considerations. The system of equations now is

$$\begin{cases} a^2 = 3 - \tau^2 \\ b^2 = 2 - \tau^2 \end{cases}$$

and since one of the vertex-groups is $\mathbf{S}_4$ we must require $q \equiv \pm 1 \pmod 8$. The system defines again a smooth curve of degree 4 and genus 1 (if q is not a power of 3) and the same estimate as in the previous Theorem holds, giving a solution for all $q \geq 67$, $q \neq 11^2, 13^2$.

Reduction mod p again gives the following admissible quotients:

$$\begin{aligned} p &\equiv 1 \pmod 8) : PSL(2,p) \\ p &\equiv -1 \pmod 8 : PGL(2,p) \\ p &\equiv \pm 3 \pmod 8 : PSL(2,p^2)\,. \end{aligned}$$

The system is equivalent to

$$\begin{cases} \tau^2 = 3 - a^2 \\ a^2 - b^2 = (a+b)(a-b) = 1 \end{cases}$$

and one can express $\tau^2 = (10 - \alpha^2 - \alpha^{-2})/4$ with $\alpha \in \mathbb{F}_q^*$. Substituting $\alpha = 3$ one gets $\tau^2 = 2/9$ which is always a square (since 2 is) and one has to exclude only the case when $2/9$ is a non-admissible value for τ^2. This consideration proves that we have a surjection also for $q = 23, 47$. The only cases that must be verified directly are $q = 31, 7^2$.

For $q \equiv 0 \pmod 3$ one reasons as in the previous Theorem.

4.2.3 Remark: As a corollary to Theorem 4.2.2 we see that $PSL(2,q)$ is an admissible quotient of $PSL(2, \mathcal{O}_2)$ for all $q \equiv \pm 1 \pmod 8$ different from 7 and 3^2.

The Picard group has various torsion-free subgroups of small index uniformizing the complements of hyperbolic links in the 3-sphere, for example the Whitehead link and the Borromean rings (see [8] and Figure 9). Recall that the group of a link is defined as the fundamental group of its complement (see [40]). The group of the Whitehead link is a subgroup of index 12 in the Picard group. By restricting the surjections from Theorem 4.2.1 to this subgroup we get

4.2.4 Corollary: *The group $PSL(2,q)$ is a quotient of the group of the Whitehead link for all values $q > 11$ different from 3^{2m+1}.*

Proof:

Surjectivity of the restrictions follows from the fact that, for $q > 11$, a proper subgroup of $PSL(2,q)$ has index at least $q+1$ (see Theorem 1.3.8).

A similar result holds for the group of the Borromean rings which has index 24 in the Picard group. However here one can say much more. The group of the Borromean rings has the free group of rank 2 as a quotient (see the proof of Theorem 4.3.1), and hence also every 2-generator group is a quotient, in particular every finite simple group. Note that the group of the 2-bridge Whitehead link is 2-generated and does not have the free group of rank 2 as a quotient. The group of the figure-8-knot (see Figure 9) is a subgroup of index 12 in the tetrahedral Bianchi group $PSL(2,\mathcal{O}_3)$ which we studied in section 2.3. As for the group $PGL(2,\mathcal{O}_3)$ considered always in section 2.3, all finite admissible quotients of linear fractional type of this tetrahedral group are obtained by reduction mod p, so one obtains quite a restricted set of quotients in this way. It would be interesting to know which finite simple groups are quotients of the group of the figure-8-knot (see also [46]).

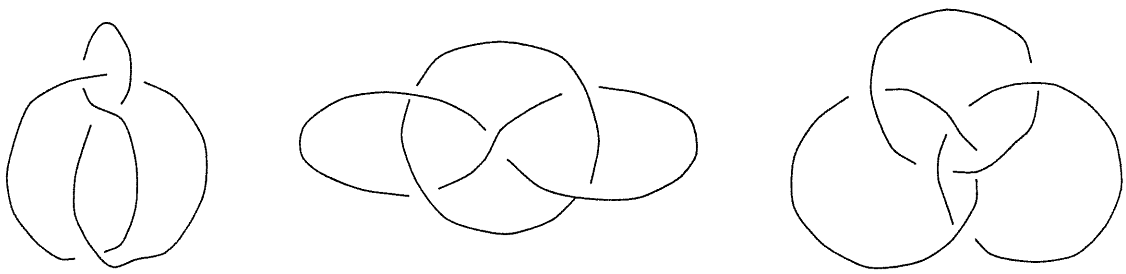


Figure-8-knot Whitehead-link Borromean rings

Figure 9

4.3. Some geometrical remarks

By [2] the Picard orbifold $\mathbf{H}^3/PSL(2,\mathcal{O}_1)$ is the smallest hyperbolic 3-orbifold whose volume is a limit of other volumes or, equivalently, the smallest hyperbolic 3-orbifold with a non-rigid cusp on which Dehn surgery can be performed (see also [14] for the notion of Dehn surgery on orbifolds). Thus hyperbolic Dehn surgery on the Picard orbifold can be used to construct small hyperbolic 3-manifolds admitting $PSL(2,q)$-actions, i.e. the quotient of the volume of the manifold by the order $|PSL(2,q)|$ of the group is small.

The Picard group $PSL(2,\mathbb{Z}[i])$ has the following presentation (generators are represented in Figure 10 a))

$$\langle x,y,z,w \mid x^3, y^3, z^2, w^2, (yx)^2, (x^{-1}w)^2, (wz)^2, (zy^{-1})^2\rangle$$

whose generators (see [52]) as elements of $PSL(2,\mathbb{C})$ are

$$x = \begin{pmatrix} 0 & i \\ i & -1 \end{pmatrix} \quad y = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As we know, reduction of coefficients mod p of the Picard group gives admissible quotients of type $PSL(2,p)$ resp. $PSL(2,p^2)$ if $p \equiv 1$ resp. $-1 \pmod 4$ (see Theorems 2.3.3 and 4.2.1).

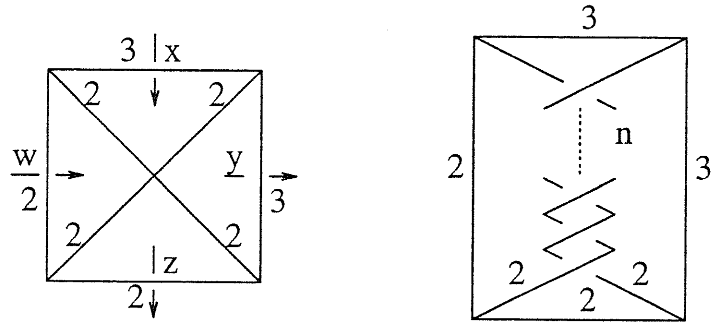
If we compactify the Picard orbifold by means of Dehn surgery along its cusp we obtain for almost all surgery coefficients (n,k) a new hyperbolic orbifold of smaller volume $\mathfrak{O}(n,k)$ (compare Theorem 1.6.10). Denote by $(\ell,\mathfrak{m})$ a longitude-meridian pair of generators for the fundamental group of the border of the Picard orbifold truncated along its cusp. If we glue in a solid torus in such a way that the element $\ell^k\mathfrak{m}^n$ (with n and k coprime) becomes trivial, we perform an (n,k)-surgery (see paragraph 1.6 for definitions). The fundamental group of the new orbifold is a quotient of the Picard group by the extra relation $\ell^k\mathfrak{m}^n = 1$.

We want to see when the admissible surjections defined as reductions mod p from the Picard group to $PSL(2,q)$ factor through the fundamental groups of the orbifolds obtained by Dehn surgery.

In our case we can choose $\ell = yw$ and $\mathfrak{m} = xz$. The element $\ell^k\mathfrak{m}^n$ is given by

$$(-1)^{k+n}\begin{pmatrix} 1 & k+in \\ 0 & 1 \end{pmatrix}$$

and we have the required factorization if and only if the image of this matrix is trivial in $PSL(2,q)$. This is equivalent to ask $k+in = 0$ in $\mathbb{F}_q$. Notice that if $q = p$, for every given n we are able to find an infinite number of k's such that a factorization exists. On the other hand, if $q = p^2$ we must choose $k, n \equiv 0 \pmod{p}$ since both k, n have image in $\mathbb{F}_p$ while i belongs to $\mathbb{F}_{p^2} - \mathbb{F}_p$.



a) b)

Figure 10

In the case of $(n, 1)$-surgery the orbifold obtained has singular set represented in Figure 10 b). Using the Wirtinger method for orbifolds, we see that the relation in this case becomes

$$(yw)^{\frac{n}{2}}(zw)(yw)^{\frac{n}{2}}w = x^{-1}$$

if n is even or

$$(yw)^{\frac{n-1}{2}}(zw)^{-1}(yw)^{\frac{n+1}{2}}w = x^{-1}$$

if r is odd. In terms of matrices the relation reads

$$\begin{pmatrix} in & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}$$

giving the condition $n = i$, which can be satisfied by infinite values of n if and only if $q \equiv 1 \pmod{4}$ is a prime number. What we have said can be summarized in the following

4.3.1 Theorem: *Let $\pi_1^{\mathrm{or}}(\mathfrak{D}(n,k))$ denote the fundamental group of the orbifold $\mathfrak{D}(n,k)$, obtained by hyperbolic Dehn surgery along the cusp of the Picard orbifold with surgery coefficients (n, k). An admissible surjection $PSL(2, \mathbb{Z}[i]) \longrightarrow PSL(2, q)$ obtained by reduction of coefficients mod p factors through $\pi_1(\mathfrak{D}(n,k))$ if and only if $k + in = 0$ in $\mathbb{F}_q$ (here i denotes the image of the imaginary unity in $\mathbb{F}_q$).*

As we have already stressed, closed hyperbolic 3-orbifolds of minimal volumes are not known. The probable candidates in the orientable case are the tetrahedral orbifolds associated to some of the nine Lannér tetrahedra resp. quotients of these by involutions. Finite quotients of type $PSL(2, q)$ and $PGL(2, q)$ of the corresponding tetrahedral groups have been studied in [20] and in paragraph 2.2.

As in the case of cusped tetrahedral groups only a restricted set of values of q occurs. So, similar as in section 4.1 it seems reasonable to consider infinite series of small volume hyperbolic 3-orbifolds simultaneously. As the smallest limit volume is that of the Picard orbifold we shall consider closed hyperbolic 3-orbifolds obtained by generalized hyperbolic Dehn surgery (see paragraph 1.6) on the cusp of the Picard orbifold. The volumes of these closed orbifolds are smaller than the volume of the Picard orbifold which they have as a limit value. Also, the closed hyperbolic 3-orbifolds of smallest known volumes are obtained in this way. We denote by $V = 0,30532\ldots$ the volume of the Picard orbifold. We shall also consider 3-orbifolds obtained by surgery on the Borromean rings (see Figure 9).

The next result should be compared with [46, Theorem 5] where surgery on the complement of the figure-8 knot is considered.

4.3.2 Theorem:

i) *For q different from $2, 7, 3^2$ and 3^{2m+1}, the minimal volume of a closed hyperbolic $PSL(2,q)$-manifold (see Definition 1.5.1) is smaller than $V|PSL(2,q)|$. Moreover, for each fixed q this is the smallest value which is a limit of volumes of hyperbolic $PSL(2,q)$-manifolds.*

ii) *For any finite r-generator group G, there exist hyperbolic G-manifolds of volume smaller than and arbitrarily close to $24V(r-1)|G|$. Given any real constant c, there exist finite groups G such that the volume of any hyperbolic G-manifold is larger than $c|G|$.*

Proof:

We start with the proof of part ii) of the Theorem.

The group of the Borromean rings is a subgroup of index 24 in the Picard group (see [8]), and so their complement has volume $24V$. Computing the Wirtinger presentation from the standard projection of the link one obtains a group presentation with six generators and six defining relations (one of which may be deleted). Three of the relations can be used to eliminate three of the generators. Setting one of the remaining three generators equal to 1, one obtains a free group of rank two which is therefore a quotient of the group of the Borromean rings. Hence every 2-generator group is a quotient of the group of the Borromean rings.

Consider a surjection ϕ of the group of the Borromean rings onto a 2-generator group G. We perform generalized hyperbolic surgeries of the following types on the three components of the Borromean rings. If ℓ and $\mathfrak{m}$ denote a preferred frame for a component of the link, and if ϕ maps $\ell^k\mathfrak{m}^m$, with $(k,m) = 1$, to an element of order n in G then we may perform (np, nq)-surgery on that component, i.e. $\ell^{nk}\mathfrak{m}^{nm}$ becomes trivial after the surgery. The result is a 3-orbifold where the central curve of the added solid torus has branching order n. By Thurston's hyperbolic surgery theorem (Theorem 1.6.10), excluding finitely many surgeries for each component, the resulting closed 3-orbifolds are hyperbolic, and their volumes are smaller than the volume $24V$ of the complement of the Borromean rings which they have as a limit value. By construction, the surjection ϕ induces admissible surjections of the fundamental groups of these closed 3-orbifolds onto the group G. The G-manifolds of the Theorem are now the regular coverings of the orbifolds corresponding to the kernels of these surjections.

For arbitrary r, we note that the free group of rank r is a subgroup of index $r-1$ in the free group of rank 2 (see [54]). Thus the free group of rank r is a quotient of the fundamental group of an r-fold covering of the complement of the Borromean rings. This covering is a hyperbolic 3-manifold with a finite number of cusps, and the proof is now similar as in the case $r = 2$.

For any hyperbolic G-manifold M, the quotient M/G is a hyperbolic 3-orbifold $\mathfrak{O}$ of volume $\mathrm{vol}(M)/|G|$ and M is the covering of $\mathfrak{O}$ corresponding to the kernel of an admissible surjection of $\pi_1(\mathfrak{O})$ onto G. By [14, Prop.5.5], all hyperbolic 3-orbifolds whose volumes are smaller than a constant c are obtained by Dehn surgery on one of a finite set of hyperbolic 3-orbifolds. If r denotes the maximal rank of the fundamental groups of these finitely many 3-orbifolds, then also the fundamental group of any 3-orbifold obtained by surgery on one of these has rank less or equal to r. Hence, if the finite group G has rank larger than r, any hyperbolic G-manifold has volume at least $c|G|$.

This finishes the proof of part ii) of the Theorem.

The proof of part i) is similar using surgery on the cusp of the Picard orbifold. Such a surgery is indicated in Figure 11 a) where a 3-ball orbifold is glued along its boundary to the boundary-horosphere of the compactified Picard orbifold. Denote by $\mathcal{C}$ the curve on the horosphere to which the meridional curve $\mathcal{C}'$ on the boundary of the 3-ball orbifold is glued. The result is a closed 3-orbifold $\mathfrak{O}(\mathcal{C}, n)$. Excluding finitely many isotopy classes of curves $\mathcal{C}$, these 3-orbifold are hyperbolic, and their volumes are smaller than the volume of the Picard orbifold which they have as a limit (see [14]).

By Theorem 4.2.1, for the above values of q there exists an admissible surjection Φ of the Picard group onto $PSL(2, q)$. If Φ maps the curve $\mathcal{C}$ to an element of order n then it induces an admissible surjection of the fundamental group of the hyperbolic 3-orbifold $\mathfrak{O}(\mathcal{C}, n)$ onto $PSL(2, q)$. Part i) of the Theorem is proved now by considering the closed hyperbolic $PSL(2, q)$-manifolds which are the coverings of the orbifolds $\mathfrak{O}(\mathcal{C}, n)$ corresponding to the kernels of these induced surjections.

This finishes the proof of the Theorem.

The simplest of the orbifolds $\mathfrak{O}(\mathcal{C}, n)$ in the proof of the previous Theorem are the polyhedral orbifolds shown in Figure 11 b) which are hyperbolic for $n > 6$. In this case the curve C of the surgery is represented by the element zx in the Picard group, and ϕ induces an admissible surjection of the corresponding hyperbolic polyhedral group onto $PSL(2,q)$ if and only if $\phi(zx)$ has order n. The volumes of these polyhedra are equal to those of the truncated tetrahedra associated to the groups $T_{3,n}$ and can be found in [12].

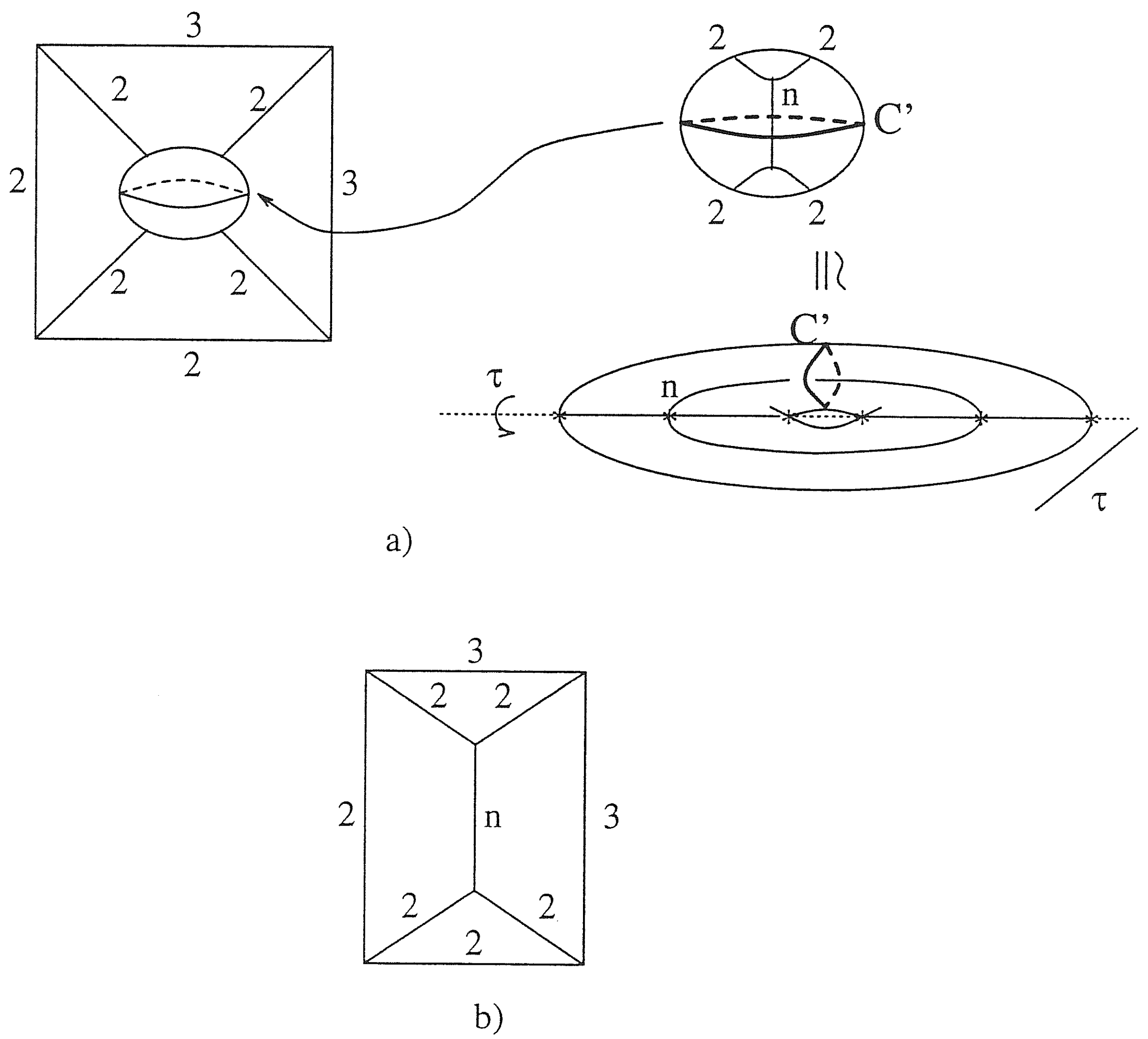


Figure 11

We believe that, at least asymptotically, the orbifolds in the proof of Theorem 4.3.2 i) obtained by surgery on the Picard orbifold give the best realizability results for $PSL(2,q)$-manifolds with respect to minimal volume, in the sense that the supremum

sup $\{\mathrm{vol}(M)/|PSL(2,q)|$: M is a minimal volume hyperbolic $PSL(2,q)$-manifold, for some $q \neq 3^2, 3^{2m+1}\}$

is equal to the volume V of the Picard orbifold. By the above it is certainly not larger than V but at present we do not have a proof of the equality. Also, we think that the Picard orbifold plays a similar role for other classes of finite groups. In particular, it would be interesting to know which finite simple groups are admissible quotients of the Picard group (for some classes of alternating groups see [30, page 153]). In view of the proof of Theorem 4.3.2 ii), also the following question is of interest: which is the smallest volume hyperbolic 3-manifold or 3-orbifold whose fundamental group has the free group of rank 2 as a quotient (or a free product of two finite cyclic groups, most interestingly $\mathbb{Z}_2 * \mathbb{Z}_3$)? Also, what is

sup $\{\mathrm{vol}(M)/|G|$: G is a finite 2-generator group and M a minimal volume hyperbolic G-manifold $\}$?

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