Physics Letters B 734 (2014) 52–57

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Boundary state in an integrable quantum field theory out of equilibrium

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ARTICLE INFO

Article history:
Received 27 November 2013
Received in revised form 4 March 2014
Accepted 30 April 2014
Available online 13 May 2014
Editor: L. Alvarez-Gaumé

ABSTRACT

We study a quantum quench of the mass and the interaction in the Sinh-Gordon model starting from a large initial mass and zero initial coupling. Our focus is on the determination of the expansion of the initial state in terms of post-quench excitations. We argue that the large energy profile of the involved excitations can be relevant for the late time behaviour of the system and common regularization schemes are unreliable. We therefore proceed in determining the initial state by first principles expanding it in a systematic and controllable fashion on the basis of the asymptotic states. Our results show that, for the special limit of pre-quench parameters we consider, it assumes a squeezed state form that has been shown to evolve so as to exhibit the equilibrium behaviour predicted by the Generalized Gibbs Ensemble.

1. Introduction

Research in non-equilibrium processes of Quantum Field Theory (QFT) and their statistical mechanical properties constitutes a fast developing and widely applicable area of theoretical physics. The correct understanding of phenomena out of equilibrium not only plays a crucial role for our knowledge about as diverse topics as cosmology, heavy-ion collision experiments and cold atom systems (see, for instance, [1–3] and references therein), but it also poses purely theoretical questions in the subject of QFT itself.

This is particularly true in the case of $(1+1)$-dimensional integrable QFTs, i.e. systems which have an infinite number of local integrals of motion [4]: in this case, connecting far-from-equilibrium dynamics at early times with the approach to equilibrium at late times may be a true challenge for the theory. In particular, the experimental evidence of lack of thermalization in a 1d system of bosons with pointlike interactions [5] (a system described by a special limit of the integrable QFT of the Sinh-Gordon model [6,7]) led to the conjecture that quantum integrable systems exhibit equilibration to a *Generalized Gibbs Ensemble* (GGE) rather than the usual Gibbs ensemble of thermal equilibrium [8].

The GGE is associated with a density matrix

$$\rho_{GGE} \propto \exp\left(-\sum_i \lambda_i \mathcal{Q}_i\right)$$

that involves *all* local integrals of motion $\mathcal{Q}_i$ of the integrable model, including the Hamiltonian. The validity of the GGE has been verified with a variety of different approaches and settings in many systems which can be mapped to free boson or fermion systems, even though such mappings are often highly non-trivial [9–19]. For genuine interacting integrable QFT there have been so far only a few studies [20–30], and presently the most general result concerns the time-average of one-point functions of local operators, for which it was shown in [28] that their values can indeed be recovered by the GGE average.

In many non-equilibrium situations of a QFT, the future evolution of the system is entirely encoded into the specification of the initial state $|B\rangle$, also called *boundary state*. This is what happens, for instance, in the global *Quantum Quench* (QQ) process, where a parameter of the Hamiltonian is abruptly changed at $t = t_0$ and the role of the boundary state is played by the ground state of the pre-quench Hamiltonian.

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http://dx.doi.org/10.1016/j.physletb.2014.04.058
0370-2693/

The subject of this paper is the theoretical investigation of the boundary state closely related to Dirichlet boundary conditions of an interacting integrable QFT, in particular its determination according to basic principles. An important issue of our analysis will concern the proper treatment of the ultraviolet unbounded behaviour originally present in the expression of $|B\rangle$, a task that will lead us to a non-trivial set of equations involving the exact matrix elements of the field $\phi(x)$ on the asymptotic states. For convenience we will present our main results through the simplest representative of these interacting theories, i.e. the Sinh-Gordon model based on the bosonic field $\phi(x)$, the generalization to more complicated integrable QFT being straightforward.

2. Boundary states

In global QQ, $|B\rangle$ is the ground state of the pre-quench Hamiltonian and important step for solving the subsequent out-of-equilibrium dynamics is to express this state in terms of the operators that create the particle excitations of the post-quench Hamiltonian. A familiar and simple example of this procedure, which will be important for our future considerations, is the quench process of the mass term $m_0 \to m$ of a free bosonic massive QFT [31]: in this case, introducing

$$c_\pm(p) = \frac{1}{2}\left(\sqrt{\frac{E_0(p)}{E(p)}} \pm \sqrt{\frac{E(p)}{E_0(p)}}\right),$$

with $E(p) = \sqrt{p^2+m^2}$ and $E_0(p) = \sqrt{p^2+m_0^2}$, and denoting by $(A_0(p), A_0^\dagger(p))$ and $(A(p), A^\dagger(p))$ the pre/post-quench annihilation and creation operators, these two sets are related by a Bogoliubov transformation

$$\begin{aligned} A_0(p) &= c_+(p)A(p) + c_-(p)A^\dagger(-p) \\ A_0^\dagger(p) &= c_+(p)A^\dagger(p) + c_-(p)A(-p) \end{aligned} \tag{1}$$

In this example, the boundary state is identified by the condition $A_0(p)|B\rangle = 0$, which can be expressed in terms of the post-quench operators as

$$\left[c_+(p)A(p) + c_-(p)A^\dagger(-p)\right]|B\rangle = 0. \tag{2}$$

The solution of this equation provides the sought expression of the boundary state in terms of the post-quench operators

$$|B_{\text{free}}\rangle \sim \exp\left[-\int_0^\infty \frac{dp}{2\pi} K_{\text{free}}(p) A^\dagger(-p)A^\dagger(p)\right]|\Omega\rangle \tag{3}$$

where $|\Omega\rangle$ is the ground state of the post-quench Hamiltonian and

$$K_{\text{free}}(p) = \frac{E_0(p) - E(p)}{E_0(p) + E(p)}. \tag{4}$$

From the point of view of the post-quench system, the boundary state is then an infinite superposition of pairs of equal and opposite momentum, each of them weighted with the amplitude $K_{\text{free}}(p)$ (see Fig. 1).

In the case of interacting integrable QFT (for simplicity we are considering integrable QFT with only one type of particle excitations), a generalization of this class of boundary states consisting of an infinite number of pairs of equal and opposite momentum is given by the general expression

$$|B\rangle \sim \exp\left(\int_0^\infty \frac{d\theta}{2\pi} K(\theta) Z^\dagger(-\theta) Z^\dagger(\theta)\right)|\Omega\rangle \tag{5}$$

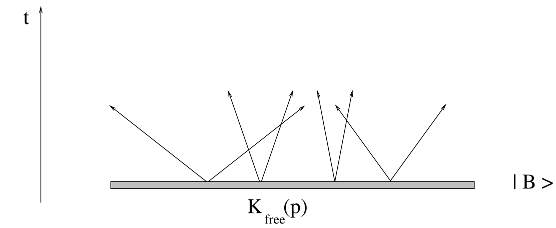

Fig. 1. With respect to the post-quench Hamiltonian, the boundary state $|B\rangle$ appears as a coherent superposition of an infinite number of pairs of particles with equal and opposite momentum.

where $Z^\dagger(\theta)$ are creation operators of the post-quench Hamiltonian, $|\Omega\rangle$ is its ground state and the variable θ conveniently parameterizes the dispersion relation of the particle excitations of mass m, given by $E = m\cosh\theta$ and $p = m\sinh\theta$. The operators $Z(\theta)$ and $Z^\dagger(\theta)$ provide a complete basis of the Hilbert space of the post-quench integrable QFT and satisfy the Zamolodchikov–Faddeev algebra

$$\begin{aligned} Z(\theta_1)Z(\theta_2) &= S(\theta_1-\theta_2)Z(\theta_2)Z(\theta_1) \\ Z(\theta_1)Z^\dagger(\theta_2) &= S(\theta_2-\theta_1)Z^\dagger(\theta_2)Z(\theta_1) + 2\pi\delta(\theta_1-\theta_2) \end{aligned} \tag{6}$$

that involves the exact two-body S-matrix $S(\theta_1-\theta_2)$, function of the rapidity differences. Boundary states of the exponential form (5) have been considered in [20,32], where it was shown that they automatically lead to equilibration of one-point observables according to the GGE (in agreement with the general analysis done in [28]). Such a class of exponential states includes the important examples of *integrable boundary states*, i.e. boundary states that respect the integrability of the bulk [33], like the Dirichlet states that can be prepared by a special QQ where the mass parameter m_0 of the pre-quench Hamiltonian is sent to infinity.

While it is presently not known whether a general QQ in an integrable QFT leads to an exponential state (5), we will argue that this may be the case for a natural class of quench processes in the Sinh-Gordon model. Before facing this aspect of the problem, let us discuss another important issue related to boundary states.

3. The problem of ultraviolet behaviour

The Dirichlet state $|D\rangle$, like all other known integrable boundary states, suffers from an ultraviolet unbounded behaviour. For example, taking literally the expression that comes out in the limit $m_0 \to \infty$ of the free bosonic case (3), one realizes that the corresponding amplitude $K_D(p)$, associated with an idealized Dirichlet state, is a constant and does not decay sufficiently fast for large momenta. This is easily understood since excitations on the initial state are cut-off by m_0 which in this case is taken to be infinite. Similar unbounded behaviour is present in any other integrable QFT and, as a result, gives rise to divergent expectation values.

When the post-quench Hamiltonian is a Conformal Field Theory, a cure of this problem was proposed by Cardy and Calabrese [31,34], who assumed a large but not infinite m_0 and made use of the concept of 'extrapolation length' τ_0, already known from boundary Renormalization Group (RG) theory, to account for the difference of the actual initial state from the idealized Dirichlet state. The parameter τ_0 is a 'small' regularization parameter that plays the role of the inverse of an exponential cut-off and depends on the initial parameters: chosen to be of the order $1/m_0$, it turns out to give indeed a good approximation of the initial state when the post-quench system is critical.

In the case of massive post-quench theories, the obvious generalization of the above idea is to replace $K_D(\theta)$ by $K_D(\theta)e^{-2E(\theta)\tau_0}$

and to postulate that τ_0 is still of order $1/m_0$ as in the conformal case. In order to check the validity of this assumption and estimate a suitable value for τ_0, one may choose an observable and equate its expectation values in the exact and in the approximate initial state. If the τ_0 regularization were consistent, this estimate should be independent of the choice of observable under consideration. However this turns out not to be true, as shown in detail in the Supplementary Material. For instance, for a mass quench in free bosonic theory, choosing as observable the operator $\phi^2(x)$, one arrives at the scaling relation

$$\tau_0 = \frac{\pi}{2e^{\gamma}} m_0^{-1} \approx 0.881938 m_0^{-1}, \tag{7}$$

where γ is the Euler–Mascheroni constant. Although this goes as in the conformal case, $\tau_0 \sim 1/m_0$ [31,34], the prefactor is however different. A similar scaling law, but with another prefactor, is obtained choosing as observable the Hamiltonian: in this case one arrives at

$$\tau_0 = \frac{\pi}{2\sqrt{3}} m_0^{-1} \approx 0.906900 m_0^{-1}. \tag{8}$$

Even though the two scaling laws (7) and (8) are very close numerically, the impossibility to arrive to a universal expression of τ_0 is nevertheless a flaw of the present regularization scheme. This discrepancy can be interpreted as an indication that the effect of higher energy excitations present in the initial state cannot be incorporated in an appropriate and unique definition of an energy cut-off: observables that weigh differently the effect of low and high energy excitations can then reveal different ultraviolet behaviour.

A way out of these difficulties is to assume τ_0 to be not a constant but a quantity that depends on p itself. In fact, such a dependence is perfectly justified from the point of view of boundary RG, according to which the actual boundary state may be constructed involving any boundary irrelevant operator, as recently discussed in [35,36]. In this approach, the introduction of the extrapolation length τ_0 amounts to a perturbation of the boundary state generated by the bulk Hamiltonian i.e. the state becomes $e^{-H\tau_0}|D\rangle$. In addition to the latter, one must in general introduce a different τ_0 for each bulk conserved charge $\mathcal{Q}_s$, which are indeed boundary irrelevant operators. This would lead to a regularized initial state obtained by $e^{-\sum_s \mathcal{Q}_s \tau_{0,s}}|D\rangle$, which turns out to be still of the form (5) but with a τ_0 that is a momentum-dependent function. This is because all charges $\mathcal{Q}_s$ of an IFT can be put in the form $\int d\theta e^{s\theta} Z^\dagger(\theta) Z(\theta)$ [4]. However the problem of how to determine the suitable function $\tau_0(\theta)$ or equivalently $K(\theta)$ remains.

In the following we will study a QQ in the Sinh-Gordon (shG) model in which we start from a large but not infinite mass m_0 and use the exponential form (5) as an Ansatz, providing a series of arguments for such a choice. We then derive, from first principles, a sequence of integral equations that must be satisfied by the function $K(\theta)$ and propose a solution based on an analytical approximation which we verify numerically with a high level of accuracy. This provides a posteriori a non-trivial check of the validity of our initial Ansatz (5).

4. The Sinh-Gordon model

The shG Hamiltonian is

$$H = \frac{1}{2}\pi^2(x) + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{\mu^2}{g^2}(\cosh g\phi - 1) \tag{9}$$

where $\phi = \phi(x,t)$ is a real scalar field, $\pi(x)$ its conjugate momentum, μ the mass and g the coupling constant. In this integrable field theory there is only one type of particle with physical renormalized mass m given by $m^2 = \mu^2 \sin\alpha\pi/\alpha\pi$ where α is the dimensionless renormalized coupling constant $\alpha = g^2/(8\pi + g^2)$. Particle scattering is fully determined by the two-particle S-matrix given by [37,38]

$$S(\theta) = \frac{\sinh\theta - i\sin\alpha\pi}{\sinh\theta + i\sin\alpha\pi}, \tag{10}$$

where θ is the rapidity difference between the particles.

Let us consider a QQ in the shG model starting from a large initial mass m_0 and, for reasons that become clear soon, zero interaction $\alpha_0 = 0$: the final quantities are finite values of m and α. Such a quench may be regarded as made of a sequence of processes: an initial quench of the mass in free bosonic theory, $m_0 \to m$, swiftly followed by a switching on of the coupling, $\alpha_0 \to \alpha$.

To determine the boundary state $|B\rangle$ for this QQ in terms of the post-quench Hamiltonian, let us use the condition that $|B\rangle$ is annihilated by the pre-quench annihilation operator $Z_0(p)$ [39]. The choice of the initial coupling value $\alpha_0 = 0$ is particularly convenient because in this case $Z_0(p)$ is just the annihilation operator of the free bosonic theory, easily expressible in terms of the physical field operator $\phi(x)$ and its conjugate momentum $\pi(x)$ as

$$Z_0(p) = \sqrt{\frac{E_{0p}}{2}}\left(\tilde{\phi}(p) + i\frac{\tilde{\pi}(p)}{E_{0p}}\right), \tag{11}$$

where $\tilde{\phi}(p) \equiv \int dx\, e^{-ipx}\phi(x)$ is the Fourier transform of $\phi(x)$ and $\tilde{\pi}(p)$ of $\pi(x)$. Since in a QFT we have $\pi = \dot{\phi} = -i[\phi, H]$, we arrive to the following equation

$$\left(\tilde{\phi}(p) + \left[\tilde{\phi}(p), H\right]/E_{0p}\right)|B\rangle = 0. \tag{12}$$

To make progress in the solution of this equation, let us first expand the state in the post-quench basis in the most general way

$$|B\rangle = \left(1 + \sum_{\substack{s=2\\ s\ \text{even}}}^{\infty} \prod_{r=1}^{s} \int_{-\infty}^{+\infty} \frac{d\theta_r}{2\pi} 2\pi\delta\left(\sum_{r=1}^{s} p(\theta_r)\right) \tilde{K}_s(\{\theta_r\})\right) \times |\theta_1, \ldots, \theta_s\rangle \tag{13}$$

where $|\theta_1, \ldots, \theta_s\rangle \equiv Z^\dagger(\theta_1)\ldots Z^\dagger(\theta_s)|\Omega\rangle$ ($\theta_1 \ge \theta_2 \ge \theta_3 \ldots$) is the post-quench eigenstate containing s particles with rapidities $\theta_1, \ldots, \theta_s$ and $p(\theta_r) = m\sinh\theta_r$ is the momentum corresponding to rapidity θ_r.

Additional constraints on $|B\rangle$ come by exploiting the symmetries of the quench process and the boundary state. Since this is the ground state of the pre-quench free Hamiltonian, it is invariant under parity and translation invariance. Moreover both symmetries are preserved by the quench process: hence, for parity reason, the sum runs over even integer numbers of particles only, while, for translation invariance, each term in the sum has zero total momentum, as ensured by the δ-function.

Applying suitable test states on the left of (12), we can derive integral equations satisfied by the amplitudes $\tilde{K}_s$ of the excitations present in $|B\rangle$. However our investigation drastically simplifies if we assume that the state is of the exponential form (5). If we apply first the assumption that the state consists only of pairs of particles with opposite rapidities, we have

$$|B\rangle = \sum_{s=0}^{\infty} \prod_{r=1}^{s} \int_{-\infty}^{+\infty} \frac{d\theta_r}{2\pi} K_s(\theta_1, \ldots, \theta_s)|-\theta_1, \theta_1, \ldots, -\theta_s, \theta_s\rangle \tag{14}$$

where, due to the algebra (6), the amplitudes K_s satisfy the properties

$K_s(\dots,-\theta_i,\dots) = S(-2\theta_i) K_s(\dots,\theta_i,\dots),$

$K_s(\dots,\theta_i,\dots,\theta_j,\dots) = K_s(\dots,\theta_j,\dots,\theta_i,\dots).$

Assuming further that the state is of the more special form (5), the amplitudes K_s are related to each other by

$$K_s(\theta_1,\dots,\theta_s) = \frac{1}{s!}\prod_{r=1}^{s} K_1(\theta_r). \tag{15}$$

The plausibility of such an Ansatz comes from a series of reasons: first of all, from the vanishing of the expectation values on the state $|B\rangle$ of all infinite conserved charges $\mathcal{Q}_a^-$ $(a = 1, 3, 5, \dots)$ of the Sinh-Gordon model which are odd under parity transformation. Indeed, if P is the parity operator which is conserved in the quench process, then $PQ_a^- P = -Q_a^-$ and since $|B\rangle$ is an even state, $P|B\rangle = |B\rangle$. Therefore

$$\langle B|\mathcal{Q}_a^-|B\rangle = \langle B|P^2\mathcal{Q}_a^- P^2|B\rangle = -\langle B|\mathcal{Q}_a^-|B\rangle = 0.$$

Since on the asymptotic states such charges act as $\mathcal{Q}_a^-|\theta_1,\dots,\theta_n\rangle = \sum_{k=1}^n \sinh(a\theta_k)|\theta_1,\dots,\theta_n\rangle$ a pair-wise structure of the boundary state automatically guarantees the vanishing of the expectation values on the state $|B\rangle$. Secondly, imagine to realize the overall quench in terms of a sequence of quenches, the first QQ_1 in which we change only the mass (at $\alpha_0 = 0$), the second QQ_2 in which we switch on the coupling. After QQ_1, the resulting boundary state is $|B_{\text{free}}\rangle$ given in (3), which is made of pairs of equal and opposite particles created by the free operators $Z_0^\dagger(p)$ with mass m. After QQ_2, where we have switched on the coupling constant α, the infinite number of pairs present in $|B_{\text{free}}\rangle$ start interacting between them. However the interaction provided by the Sinh-Gordon model cannot create or destroy particles since it is integrable and when the particles cross each other, they just experience a time-delay dictated by the elastic S-matrix given in (10). It is therefore conceivable that the only effect of interaction is to "dress" both the free particle amplitude $K_{\text{free}}(\theta) \to K(\theta)$ and the free creation operators $Z_0^\dagger(p) \to Z^\dagger(p)$, preserving though the pair-wise structure of the boundary state.

Assuming the validity of the pair-wise structure of the initial state and the exponentiation of the amplitudes, i.e. assuming the form (5), let us start our analysis from the limit $m_0 \to \infty$ which corresponds to the Dirichlet state $|D\rangle$ satisfying the condition

$$\tilde{\phi}(p)|D\rangle = 0. \tag{16}$$

Such a boundary state is known to be of the exponential form (5) with amplitude $K_D(\theta)$ given by [40]

$$K_D(\theta) = i\tanh(\theta/2)\left(\frac{1+\cot(\pi\alpha/4 - i\theta/2)}{1-\tan(\pi\alpha/4 + i\theta/2)}\right). \tag{17}$$

Such a known case provides a non-trivial check of the approach we are going to propose. Indeed, if we now take as test state an arbitrary 1-particle excitation $\langle\theta|$ applied to the left of (16), substitute (5) and expand, we obtain in this way the following integral equation that must be satisfied by the amplitude $K_D(\theta)$

$$\sum_{s=0}^{\infty}\frac{1}{s!}\left(\prod_{i=0}^{s}\int_{C_i}\frac{d\theta_i'}{2\pi}K_D(\theta_i')\right) \times F_{2s+1}(\theta+i\pi, -\theta_1', \theta_1', \dots, -\theta_s', \theta_s') = 0, \tag{18}$$

where $F_n(\{\theta_j\})$ are the matrix elements (the so-called *Form Factors*) of the field ϕ defined by $F_n(\{\theta_j\}) \equiv \langle 0|\phi|\{\theta_j\}\rangle$. In the derivation of (18) we have exploited both the crossing symmetry and the analytical properties of QFT [4,41] which have allowed us to write the matrix elements $\langle\theta|\phi|\{-\theta_i', +\theta_i'\}\rangle$ in terms of the Form Factors above. The exact expressions of the Form Factors of the Sinh-Gordon model were computed in [37,38] (for convenience, their exact expressions can be found in the Supplementary Material). Moreover since the numerical value of the F_n's decreases with the order n, the series (18) shows a fast convergent behaviour and can be approximated to the desired order of accuracy simply restricting to the lowest terms.

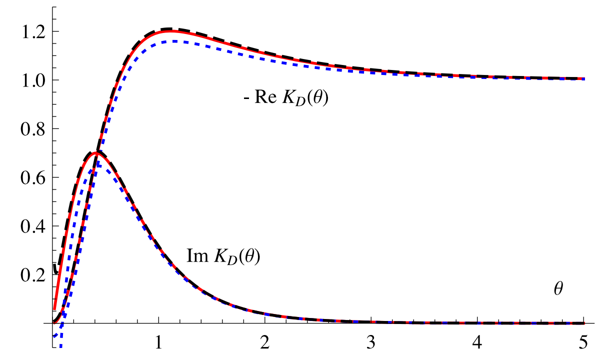

Fig. 2. Plot of the numerical solution of (19) truncated after the first 3 (blue dotted line) or 5 terms (black dashed line), along with the analytical result (17) (red full line) for $m = 1, \alpha = 0.4$.

There is however a technical issue to take care of: since the Form Factors have poles whenever an in- and an out-rapidity coincide, one needs to choose a suitable prescription on how to pass around the poles, in order that the equation above makes sense. This prescription is encoded in the integration contours C_i which can be determined by means of a *finite volume regularization* [42–45] (details are discussed in the Supplementary Material). Once this prescription is implemented, the first few terms of the series give as a result the equation

$$\begin{aligned} 0 = {} & F_1 + \frac{1}{2}F_1 K_D(\theta)\big(1+S(-2\theta)\big) \\ & + \frac{1}{2}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{d\theta'}{2\pi}F_3(\theta+i\pi, -\theta', \theta')K_D(\theta') \\ & + \frac{1}{4}\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\big(S(-2\theta)K_D(\theta) + S(2\theta)S(\theta-\theta')S(\theta+\theta') \\ & \times K_D(-\theta)\big)F_3(-\theta, -\theta', \theta')K_D(\theta') \\ & + \frac{1}{8}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{d\theta_1'}{2\pi}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{d\theta_2'}{2\pi}F_5(\theta+i\pi, -\theta_1', \theta_1', -\theta_2', \theta_2') \\ & \times K_D(\theta_1')K_D(\theta_2') + \dots \end{aligned} \tag{19}$$

In Fig. 2 we plot the numerical solution of (19) when we keep its first three and five terms, along with the analytical result (17). As shown in Fig. 2, the agreement is quite satisfactory even when the series is truncated up to the first three terms and it improves significantly once the next two terms are included.

Using multi-particle test states applied on the left of (16), one obtains a series of equations that must all be satisfied by the amplitude $K_D(\theta)$. More details of such computations will be presented elsewhere [46].

Supported by this positive check for the case $m_0 \to \infty$, let us now address the problem of determining the amplitude $K(\theta)$ in the case of large but finite m_0. The equation that defines the initial state is now (12). Considering a 1-particle test state as before and

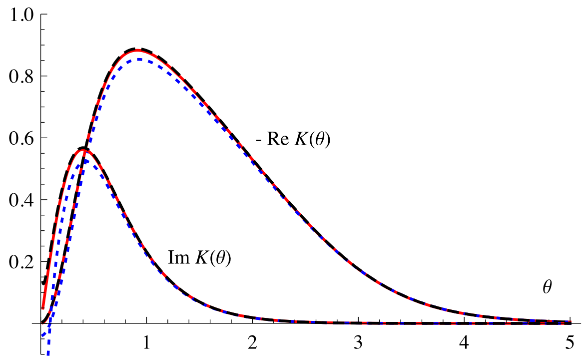

Fig. 3. Plot of the numerical solution of (21) truncated after the first 3 (blue dotted line) or 5 terms (black dashed line), along with the analytical Ansatz (22) (red full line) for $m = 1$, $m_0 = 10$, $\alpha = 0.4$.

substituting (5), we find, after some algebra, that the new equation is

$$\sum_{s=0}^{\infty}\frac{1}{s!}\left(\prod_{i=0}^{s}\int_{C_i}\frac{d\theta_i'}{2\pi}K(\theta_i')\right)\left(E_0(\theta)-E(\theta)+\sum_{i=1}^{s}E(\theta_i')\right)$$
$$\times F_{2s+1}(\theta+i\pi,-\theta_1',\theta_1',\ldots,-\theta_s',\theta_s')=0 \qquad (20)$$

which, after a truncation of the series to the same order as before, becomes

$$0=F_1\left(\frac{E_0(\theta)-E(\theta)}{E_0(\theta)+E(\theta)}\right)+\frac{1}{2}F_1K(\theta)\big(1+S(-2\theta)\big)$$
$$+\frac{1}{2}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{d\theta'}{2\pi}\left(\frac{E_0(\theta)-E(\theta)+2E(\theta')}{E_0(\theta)+E(\theta)}\right)$$
$$\times F_3(\theta+i\pi,-\theta',\theta')K(\theta')$$
$$+\frac{1}{4}\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}\left(\frac{E_0(\theta)+E(\theta)+2E(\theta')}{E_0(\theta)+E(\theta)}\right)F_3(-\theta,-\theta',\theta')$$
$$\times\big(S(-2\theta)K(\theta)+S(2\theta)S(\theta-\theta')S(\theta+\theta')K(-\theta)\big)K(\theta')$$
$$+\frac{1}{8}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{d\theta_1'}{2\pi}$$
$$\times\int_{-\infty+i\epsilon}^{+\infty+i\epsilon}\frac{d\theta_2'}{2\pi}\left(\frac{E_0(\theta)-E(\theta)+2E(\theta_1')+2E(\theta_2')}{E_0(\theta)+E(\theta)}\right)$$
$$\times F_5(\theta+i\pi,-\theta_1',\theta_1',-\theta_2',\theta_2')K(\theta_1')K(\theta_2')+\ldots \qquad (21)$$

One way to obtain the solution of this equation is to notice that, for a smooth function K, the integral of the first line is dominated by the contribution of the kinematical poles of the Form Factor at $\theta=\pm\theta'$. At these poles, the prefactor $(E_0(\theta)-E(\theta)+2E(\theta'))/(E_0(\theta)+E(\theta))$ of the integration kernel becomes equal to unit. This suggests the approximate solution

$$K(\theta)\approx K_D(\theta)\left(\frac{E_0(\theta)-E(\theta)}{E_0(\theta)+E(\theta)}\right) \qquad (22)$$

since then the first line of (21) becomes approximately the same as the first line of (19). The second and third lines contribute only small corrections to the solution.

The correctness of our approximate solution can be verified numerically. Fig. 3 shows a typical plot of a numerical solution of (21) truncated to the 3rd or 5th terms, along with the Ansatz (22), for some values of the ratio m_0/m and interaction α. The agreement is quite satisfactory, even when we include the contribution of the second and third lines. Further comparative plots for a wide range of parameter values will be presented elsewhere [46].

Obviously, the proposed solution (22) is expected to be more accurate when the parameters m_0, m and α are such that the domination of the poles is more prominent and the higher order terms of the series give smaller contributions. The second condition is satisfied for example when both masses m_0 and m are large or when the interaction α is small.

In analogy to the Dirichlet state case, using multi-particle test states we can derive a series of equations that must be satisfied by the amplitudes K_s of the initial state in the form (14). Based on the same combination of arguments used above (truncation of the form factor series and pole dominance of the integrals), it is possible to show that also these equations reduce approximately to the ones corresponding to the Dirichlet case when the K_s are chosen to be $K_s(\theta_1,\ldots,\theta_s)=1/s!\prod_{r=1}^{s}K_1(\theta_r)$ with $K_1(\theta)\approx K_D(\theta)K_{\text{free}}(\theta)$. In this way the exponential form of the Dirichlet state leads also to approximate exponentiation of the QQ initial state.

5. Observables at large times

According to the analysis done above, a first order approximation of the initial state $|B\rangle$ for the QQ under consideration is given by the exponential form (5) with amplitude $K(\theta)$ given by (22). This decays for large momenta as a power law ($\sim p^{-2}$) and ensures a smooth ultraviolet behaviour through a momentum dependent τ_0-regularization. An initial state of this form belongs to the class studied in [20,28,32] and therefore at least the one-point functions of local observables equilibrate according to the GGE: their long time values are given by

$$\mathcal{O}(x,t\to\infty)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{+\infty}\prod_{i=1}^{n}\frac{d\theta_i}{2\pi}\left(\frac{|\bar K(\theta_i)|^2}{1+|\bar K(\theta_i)|^2}\right)$$
$$\times\langle\theta_n,\ldots,\theta_1|\mathcal{O}(x)|\theta_1,\ldots,\theta_n\rangle_c \qquad (23)$$

where $\bar K$ is given by the solution of the generalized Thermodynamic Bethe Ansatz (TBA) equation

$$|\bar K(\theta)|^2=|K(\theta)|^2\exp\left[\int\frac{d\theta'}{2\pi}\varphi(\theta-\theta')\log\big(1+|\bar K(\theta')|^2\big)\right] \qquad (24)$$

where $\varphi(\theta)\equiv -i\,d(\log S(\theta))/d\theta$, as explained in [20]. From the above equations we can calculate numerically, for instance, the GGE prediction for the operator $\exp(k\phi)$ from which one can also derive all field fluctuations ϕ^{2n} by differentiation with respect to k at $k=0$. In Fig. 4, the three curves represent three successive partial sums of the series (23). The convergence of this series is particularly fast near the point $k=0$, even though to compute the higher moments ϕ^{2n} with sufficient accuracy one needs to employ more terms of the series.

6. Conclusions

In this paper we have studied a QQ of the mass and coupling constant in the Sinh-Gordon model, in the special case of a large initial mass and zero initial interaction. We have seen that the ultraviolet regularization of this state is a non-trivial and physically relevant problem. This has led us to develop a systematic method to determine the expansion of the boundary state in the post-quench basis: this consists in solving integral equations for

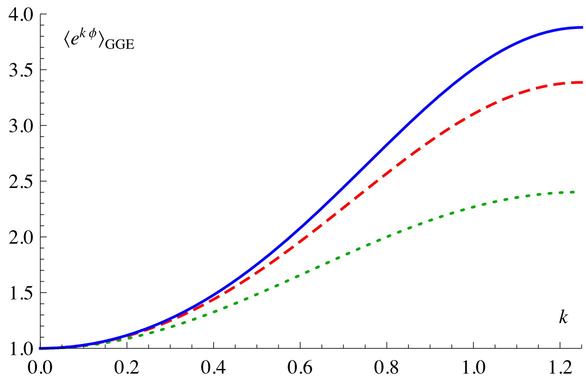

Fig. 4. Plot of the GGE prediction $\langle e^{k\phi}\rangle_{\text{GGE}}$ as a function of k ($m = 1$, $m_0 = 10$, $\alpha = 0.4$). The three curves represent partial sums of the series (23): the dotted (green) line corresponds to the sum of the first two terms, the dashed (red) to the first three and the solid (blue) to the first four terms.

the excitation amplitudes K_s which involve the finite-volume prescription of the exact Form Factors of the elementary field. Assuming that the boundary state is of the exponential form (5), we have obtained a first but quite accurate approximation of the solution by truncating the related Form Factor series. The proposed solution is used to derive the large time behaviour of observables.

The fact that the large energy behaviour of excitation amplitudes in the initial state is relevant for the calculation of physical observables at large times, means that models that are effectively equivalent as far as their ground state or thermal equilibrium properties are concerned, may not be equivalent out-of-equilibrium. More generally, we conclude that RG methods and concepts that are valid at equilibrium cannot always be applied directly to out-of-equilibrium problems.

It would be quite interesting to extend the analysis done in this paper to other QQ protocols and determine the relevant amplitudes K_s of the corresponding boundary state from first principles.

Acknowledgements

Spyros Sotiriadis acknowledges financial support by SISSA – International School for Advanced Studies under the “Young SISSA Scientists Research Projects” scheme 2011–2012 and by the ERC under Starting Grant 279391 EDEQS. G.T. was partially supported by a Hungarian Academy of Sciences “Momentum” grant LP2012-50/2012. The work of G.M. is supported by the IRSES grants QICFT. This work was also supported by the CNR-MTA (grant SNK-84/2013) joint project “Nonperturbative field theory and strongly correlated systems”. We would also like to thank the Max Planck Institute for the Physics of Complex Systems (Dresden, Germany) for their hospitality during the international workshop QSOE’13.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.physletb.2014.04.058.

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