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Painlevé IV equation, Fredholm Determinants and Double-Scaling Limits

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Abstract

This thesis focuses on the Painlevé IV equation and its relationship with double scaling limits in normal matrix models whose potentials exhibit a discrete rotational symmetry. In the first part, we study a special solution of the Painlevé IV equation, which is determined by a particular choice of the monodromy data of the associated linear system, and consider the Riemann-Hilbert problem associated to it. From the Riemann-Hilbert problem we use the theory of integrable operators in order to associate a Fredholm determinant representation, or equivalently a τ–function, to our specific solution. The poles of our Painlevé IV solution are the zeros of this τ–function. We study numerically the τ–function for real values of the independent variable s and locate its zero on the real line.

In the second part of this thesis, we introduce the subject of orthogonal polynomials that appear in the study of statistical quantities related to normal matrix models. We chose, for our normal matrix models, a potential with a discrete rotational symmetry. A potential of this form has different regimes: *pre-critical*, *critical* and *post-critical*, according to the values of its parameter. Such regimes describe the transition of the support of the limiting distribution of the eigenvalues of the normal matrix model, from a connected domain to a domain with several connected components. We are interested in considering the critical case. Our purpose is to consider the orthogonal polynomials associated with this matrix model and study their asymptotic behaviour. We achieve this goal by transforming the orthogonality relations on the complex plane to a Riemann-Hilbert problem on a contour. By following the general method of nonlinear steepest descent of Deift-Zhou, we are able to perform the asymptotic analysis of the orthogonal polynomials as the degree of the polynomials goes to infinity. As a consequence of this procedure, we will find that the Riemann-Hilbert problem obtained after some transformations is the same as the one that we had obtained for our special solution to Painlevé IV equation. We will then fulfil our goal of finding the asymptotic behaviour of the orthogonal polynomials in every region of the complex plane.

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Introduction

The subjects of Painlevé equations, random matrices and Riemann-Hilbert problems are deeply related. The general purpose of this work will be to study a particular stance of this connection. This will be done by considering a special solution of the Painlevé IV equation, analyzing it and establishing its relationship with a particular random matrix-problem via the related orthogonal polynomials. Doing so will require us to use the techniques of Fredholm Determinants, Double-Scaling Limits and to evaluate the associated Riemann-Hilbert problems.

The first subject that will be of interest for us in this work is the one of Painlevé equations.

The classical Painlevé transcendents were originally introduced by Paul Painlevé as the solution of a specific classification problem for second order ODEs of the type

$$u_{xx} = F\left(x, u, u_x\right), \tag{1}$$

where F is a meromorphic function in x and rational in u and u_x. The author intended to find all equations of this form that satisfied the condition of their solutions not having movable singularities. As a consequence of this, essential singularities of the solution and possible branch points should not depend on the initial data. The reason for considering a problem of this kind is the fact that, when there are no movable singularities, every solution of the problem can be meromorphically extended to the entire universal covering of a punctured complex sphere that is only determined by the equation.

When studying this type of equations, Painlevé found out only fifty of them, up to equivalence given by a transformation of the form

$$u \to \frac{\alpha\left(x\right) u + \beta\left(x\right)}{\gamma\left(x\right) u + \delta\left(x\right)}, \quad \text{where } \alpha, \beta, \gamma, \delta \text{ are meromorphic functions in } x. \tag{2}$$

It was later realized that these equations could either be integrated in terms of known functions or mapped to a set of six equations that cannot be integrated in terms of known functions. The six equations in this set are the so-called Painlevé equations

$$\frac{d^2u}{dx^2} = 6u^2 + x, \tag{3}$$

$$\frac{d^2u}{dx^2} = xu + 2u^3 - \alpha, \tag{4}$$

$$\frac{d^2u}{dx^2} = \frac{u_x^2}{u} - \frac{u_x}{x} + \frac{1}{x}\left(\alpha u^2 + \beta\right) + \gamma u^3 + \frac{\delta}{u}, \tag{5}$$

$$\frac{d^2u}{dx^2} = \frac{1}{2}\frac{u_x^2}{u} + \frac{3}{2}u^3 - 2xu^2 + \left(1 + \frac{x^2}{2} - 2\alpha\right)u - \frac{2\beta}{u}, \tag{6}$$

$$\frac{d^2u}{dx^2} = \frac{3u-1}{2u\,(u-1)}u_x^2 - \frac{1}{x}u_x + \frac{(u-1)^2}{x^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{x} + \frac{\delta u\,(u+1)}{u-1}, \tag{7}$$

$$\begin{aligned} \frac{d^2u}{dx^2} =& \frac{u_x^2}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right) - u_x\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right) \\ &+ \frac{u\,(u-1)\,(u-x)}{x^2\,(x-1)^2}\left(\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x\,(x-1)}{(u-x)^2}\right). \end{aligned} \tag{8}$$

In this work, we will be particularly interested in the fourth Painlevé equation, (6). Following the original work of Painlevé, several developments have been made regarding the study of this class of equations, their properties and applications. The study of their solutions, the so-called Painlevé transcendents, has also been an active area of research.

One important fact regarding Painlevé equations is their formulation in terms of an associated isomonodromy deformation for a linear ODE.

To explain the notion in rather more general terms, let us consider the general case of a Fuchsian system

$$\frac{d\Psi}{d\lambda} = \sum_{j=1}^{n}\frac{A_j}{\lambda - a_j}\Psi, \quad \text{where } \Psi, A_j \text{ are } N\times N \text{ matrices}, \tag{9}$$

the monodromy group of this system is defined as a representation of the fundamental group of the punctured Riemann sphere

$$\rho : \pi_1\left(\mathbb{CP}^1 \setminus \{a_1, \ldots, a_n, \infty\}\right) \to \mathbf{GL}\,(N, \mathbb{C})\,, \tag{10}$$

generated by encircling the singular points $a_1, \ldots, a_n, a_\infty = \infty$

$$\Psi\,(\lambda)\,|_{(\lambda - a_j)\to(\lambda-a_j)e^{2\pi i}} = \Psi\,(\lambda)\,M_j, \quad M_\infty M_n \ldots M_1 = \mathbb{1}. \tag{11}$$

These matrices, $M_1, \ldots, M_n, M_\infty$, are called the monodromy matrices and we call monodromy data the set $\mathfrak{m} = \{M_1, \ldots, M_n, M_\infty\}$. This set defines, up to conjugation, the monodromy group, $\mathfrak{M}$, of the Fuchsian equation (9). We can also consider the set of the form $\mathbb{M} = \{a_1, \ldots, a_n; M_1, \ldots, M_n\}$, which is called the extended monodromy data of the Fuchsian system, and the set $\mathbb{A} = \{a_1, \ldots, a_n; A_1, \ldots, A_n\}$, which is the singular data of the system.

The above description for $n = 4$ and $N = 2$ leads to the sixth Painlevé equation; the case which is more relevant to our discussion will be presented a bit below.

The second subject that will be of interest for us in this work is the one of Riemann-Hilbert problems and their relationship with Painlevé equations. The Riemann-Hilbert problem associated with a Fuchsian system consists in proving the existence of such a Fuchsian system with given singular points $a_1, \ldots, a_n$ and monodromy group $\mathfrak{M}$. Considering $\mathcal{A} \equiv \{\mathbb{A}\}$ to be the set of singular data and $\mathcal{M} \equiv \{\mathfrak{M}\}$ to be the set of monodromy data, one has to analyze the direct monodromy map $\mathcal{A} \to \mathcal{M}$ and the inverse monodromy map $\mathcal{M} \to \mathcal{A}$. This is the main problem in the general theory of Fuchsian systems; in the case of relevance to our study, we need a slightly refined notion of "monodromy" that includes the data of Stokes' matrices (see [22]).

Since our purpose is to see the Riemann-Hilbert problem in the context of Painlevé equations, we will now look at how the monodromy can be used to study Painlevé equations. For the purposes of our work, it will suffice to consider the matrix size to be $N = 2$.

The idea now is to consider a general linear system with a number of irregular singularities of the form

$$\frac{d\Psi}{d\lambda} = A(\lambda; s)\,\Psi, \quad A(\lambda; s) = \sum_{k=1}^{m} \sum_{i=1}^{r_k} \frac{A_i^{(k)}}{(\lambda - a_k)^i} + \sum_{i=0}^{r_\infty - 1} A_i^{(\infty)} \lambda^i. \tag{12}$$

Since in this work we are going to refer to the case of Painlevé IV, we should now state that in this case we will have the matrix $A(\lambda; s)$ to be of the form

$$A(\lambda; s) = A_1 \lambda + A_0 + A_{-1} \frac{1}{\lambda}, \tag{13}$$

where A_1, A_0 and A_{-1} are matrices that will be given in Chapter 1. Without loss of generality (up to conjugations/scaling) $A_1 = -\frac{1}{2}\sigma_3$. General theory implies that there is a formal series solution

$$\Psi_f(\lambda) = \left(\mathbb{1} + \mathcal{O}(\lambda^{-1})\right) \lambda^{\Theta_\infty \sigma_3}\, e^{-\theta(\lambda)\sigma_3}, \tag{14}$$

with $\theta = \frac{\lambda^2}{4} + \frac{s}{2}\lambda$ a polynomial of degree 2; here the subscript $_f$ stands for "formal". The equation (14) expresses the asymptotic behaviour of certain solutions $\Psi_{(\mu)}$ in appropriate sectors $\mathcal{S}_{(\mu)}$. Two such solutions are connected to each other by multiplication on the right by invertible matrices, called *the Stokes' matrices*. Each solution $\Psi_{(\mu)}$ has an analogous local expansion near the point $\lambda = 0$ (a *Fuchsian* singularity of the equation) of the form

$$\Psi_{(\mu)}(\lambda) = G(\mathbb{1} + \mathcal{O}(\lambda))\lambda^{\Theta_0 \sigma_3} C_{(\mu)}, \tag{15}$$

where the matrices $C_{(\mu)}$ are called the *connection* matrices; analytic continuation of (15) around the origin yields the same matrix up to the right multiplication by the *monodromy matrix* $M_0 = C_{(\mu)}^{-1}\, \mathrm{e}^{2i\pi\Theta_0\sigma_3}\, C_{(\mu)}$.

In the isomonodromic method, we demand that the Stokes' matrices, together with the connection matrices and the exponents Θ_0, Θ_∞ are independent of s (this implies that the monodromy is also s–independent, whence the term "isomonodromy").

It then follows that these different solutions satisfy the same ODE in s

$$\frac{d}{ds}\Psi_{(\mu)}(\lambda; s) = B(\lambda; s)\Psi_{(\mu)}(\lambda; s), \tag{16}$$

with a matrix $B(\lambda; s)$ that is the same for all of them. Using the asymptotic behaviours of the Ψ's, one can deduce that B is (in our case) a polynomial in λ of degree 1:

$$B(\lambda; s) = B_1 \lambda + B_0. \tag{17}$$

where the exact form of these matrices B_1 and B_0 will be given in Chapter 1.

Therefore, the function Ψ should satisfy the Lax Pair given by equations (12) and (16). Since the compatibility condition of this system

$$\Psi_{\lambda s} = \Psi_{s\lambda} \tag{18}$$

should be satisfied, then it can be seen that this implies the zero curvature function

$$[\partial_\lambda - A, \partial_s - B] \equiv 0. \tag{19}$$

Since we now have that this equation should be satisfied, then we are lead to the Painlevé equation.

As a result of this procedure, it can be seen that the Painlevé equations can be interpreted as the compatibility condition for the Lax Pair. The generalized monodromy data (i.e. Stokes' matrices, connection matrices, exponents) parametrize the general solution of the Painlevé equation.

For any solution $A(\lambda; s), B(\lambda; s)$ of the zero–curvature equations (19) the authors of [22] associated a particular function that they named τ–function and is defined by the first order differential equation that, in our case, reads simply:

$$\frac{\partial}{\partial s} \ln \tau_{JMU}(s) = - \operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(\Psi_f^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda} \Psi_f \frac{\partial}{\partial s} \theta(\lambda; s) \sigma_3 \right) \mathrm{d}\lambda, \tag{20}$$

where the residue is intended as a formal one (the coefficient of the power λ^{-1} in the formal expansion).

In the study that we will be doing, it will also be important for us to consider the so-called *Fredholm Determinant*. This is a quantity that is defined in the following way

$$\det\left(\mathrm{Id} - \rho \hat{\mathcal{K}}\right) = 1 + \sum_{l=1}^{\infty} \frac{(-\rho)^l}{l!} \int_{X^l} \det\left[K\left(x_i, x_j\right)\right]_{i,j=1}^{l} \mathrm{d}\nu\left(x_1\right) \ldots \mathrm{d}\nu\left(x_l\right), \tag{21}$$

where $\hat{\mathcal{K}}$ is an integral operator $\hat{\mathcal{K}} : L^2\left(X, \mathrm{d}\nu\right) \to L^2\left(X, \mathrm{d}\nu\right)$ with kernel $K(z, w) : X \times X \to \mathbb{C}$. The importance of this definition for us is the fact that, through Theorem 1.1, it is possible to establish a connection between the Fredholm Determinant and the Riemann-Hilbert problem that we will be working with. The key to do this will be in defining the kernel $K\left(\lambda, \mu\right)$ of the operator $\hat{\mathcal{K}}$ through the product of two matrix-valued functions $f, g : \Sigma \to \mathrm{Mat}\left(n \times m, \mathbb{C}\right)$

$$K\left(z, w\right) := \frac{f^T\left(z\right).g\left(w\right)}{z - w}. \tag{22}$$

As a result of Theorem 1.1, it is established that the Riemann-Hilbert problem admits a solution if and only if the Fredholm Determinant is non-zero. Furthermore, the jump matrices of the Riemann-Hilbert problem can be obtained through the functions f and g in the following way

$$\mathbb{1} - 2\pi i f\left(z\right) g^T\left(z\right). \tag{23}$$

The Fredholm Determinant is also important in the context of Painlevé equations. This is due to the fact that, if the operator $\hat{\mathcal{K}}$ is defined in the way that we have just specified, then it can also be shown that

$$\tau(\rho) = \det\left(\mathrm{Id} - \rho\hat{\mathcal{K}}\right). \tag{24}$$

In the case of an appropriate operator $\hat{\mathcal{K}}$ (in Chapter 1) where the operator depends analytically on a parameter s, the definition of the tau function as a Fredholm determinant turns out to coincide with the τ-function of the Painlevé equation associated with the Riemann-Hilbert problem as defined by (20). Therefore, by using the Fredholm Determinant for an operator $\hat{\mathcal{K}}$ related to a given Riemann-Hilbert problem, then it is possible to study the τ-function of the Painlevé equation associated with that Riemann-Hilbert problem.

The third subject that will be of interest for us in this work is the one of random matrices.

In the study of random matrices, one considers a set of $m \times m$ matrices, $M = \{M_{ij}\}$, with certain properties and the purpose is to study probability distributions of the form

$$\mu_m(\mathrm{d}M) = \frac{1}{Z_{m,N}} e^{-N\mathrm{Tr}(V(M))}\,\mathrm{d}M, \tag{25}$$

where N is a positive parameter and $Z_{m,N}$ is the normalization constant, or partition function, defined in the following way

$$Z_{m,N} = \int e^{-N\mathrm{Tr}(V(M))}\,\mathrm{d}M, \tag{26}$$

where the integration is over the space of matrices that is being considered. In our case, we will be working with the space of $m \times m$ normal matrices

$$\mathcal{N}_m = \{M : [M, M^*] = 0\} \subset \mathrm{Mat}_{m\times m}(\mathbb{C}). \tag{27}$$

Therefore, in (25) and (26), $\mathrm{d}M$ will refer to the volume form induced on $\mathcal{N}_m$ that is invariant under conjugation by unitary matrices. Regarding $V(M)$, it is a function of the form $V : \mathbb{C} \to \mathbb{R}$ and is assumed to have growth at infinity in such a way that the integral in (26) is bounded. Due to the fact that we are considering only normal matrices, we can now take advantage of the fact that they are diagonalizable by unitary transformations and reduce the probability density to the form

$$\mu_m(\mathrm{d}M) = \frac{1}{Z_{m,N}} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_{j=1}^{m}\mathrm{Tr}(V(\lambda_i))}\,\mathrm{d}A(\lambda_1)\ldots\mathrm{d}A(\lambda_m), \tag{28}$$

where λ_i are the eigenvalues of the normal matrix M and $\mathrm{d}A(z)$ is the area measure. The partition function can now be written as

$$Z_{m,N} = \int_{\mathbb{C}^m} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_{j=1}^{m}\mathrm{Tr}(V(\lambda_i))}\,\mathrm{d}A(\lambda_1)\ldots\mathrm{d}A(\lambda_m), \tag{29}$$

where the integration is over $\mathbb{C}^m$ since the eigenvalues λ_i are complex valued.

In the context of matrix models, one can also introduce the associated orthogonal polynomials. This is done by defining n-th monic polynomials $p_n(z)$ through the following set of relations

$$\int_{\mathbb{C}} p_n(z)\,\overline{z}^{n'} e^{-NV(z)}\,\mathrm{d}A(z) = h_{n,N}\,\delta_{n,n'}, \quad h_{n,N} > 0, \quad n, n' = 0, 1, \ldots m, \tag{30}$$

where p_n are of the form

$$p_n = z^n + c_{n-1} z^{n-1} + \ldots \tag{31}$$

and $h_{n,N}$ is called the norming constant.

0.1 Summary of the Results

The first chapter of this thesis is devoted to the study of a special case of the Painlevé IV equation and the Fredholm Determinant associated to it.

We begin by deriving the general Painlevé IV equation. This is done by introducing the Lax Pair, which consists on the following system of linear differential equations

$$\begin{cases} \Psi_\lambda = A(\lambda; s)\,\Psi \\ \\ \Psi_s = B(\lambda; s)\,\Psi \end{cases} \tag{32}$$

as explained in (12) and (16), and the Painlevé equation is obtained as the compatibility condition between the two equations of the system.

The following step consists in establishing a special solution of the Painlevé equation. This is done by considering specific values for the monodromy data. In our work, we will be interested in considering the following values for the Stokes' parameters (see Chapter 1 for the precise form of the Stokes' matrices), connection matrices and exponents

$$S_1 = 1, \quad S_0 = 0, \quad S_{-1} = -1, \quad S_{-2} = -1, \quad C_{(\mathrm{III})} = C_{(\mathrm{IV})} = \mathbb{1}, \quad \Theta_0 = \frac{\gamma}{2}, \quad \Theta_\infty = \frac{\gamma}{2}. \tag{33}$$

This will allow us to establish a Riemann-Hilbert problem whose jump matrices will be given by the Stokes' matrices with the values of S_i that we specified and behaviour at infinity in each region given by $\Psi_{\text{formal}}(\lambda)$. Our procedure will then require us to modify the rays of the Riemann-Hilbert problem, in a way that will be specified later, and derive a new Riemann-Hilbert problem constructed for a new matrix valued function that is defined as

$$\mathcal{H}(\lambda) := \Psi(\lambda)\, e^{\theta\,\sigma_3}\, \lambda^{-\frac{\gamma}{2}\sigma_3}. \tag{34}$$

We will then make use of the Fredholm Determinant in order to study the τ–function of our special solution of Painlevé IV. The purpose of this will be to see for which values of the parameter s does the τ–function exist. This analysis will initially lead us to obtain the following result

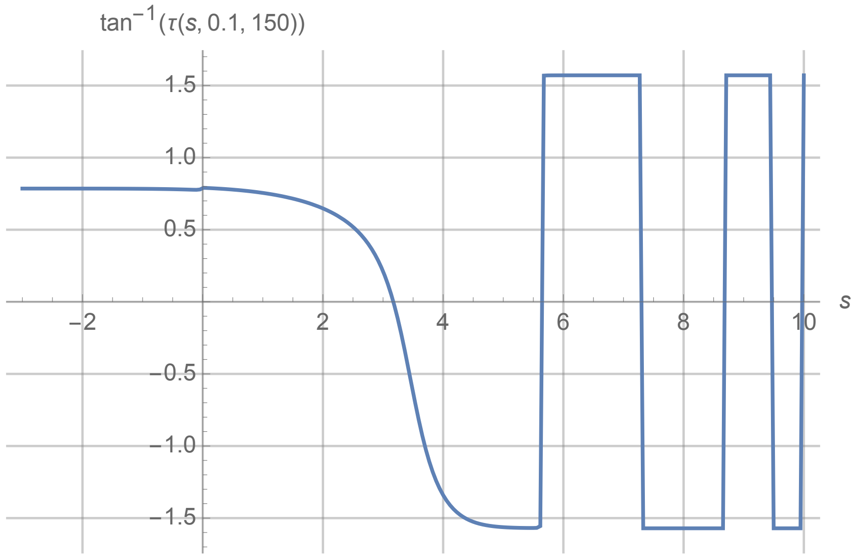


Figure 0.1: $\tau\left(s\right)$ for $\gamma = 0, 1$ and with $n = 150$.

Theorem 0.1. *The Riemann–Hilbert problem for $\mathcal{H}\left(\lambda\right)$ admits a solution for $s \in \left(-\infty, -\tilde{s}_0\right)$, where $\tilde{s_0} = -s_0 = 0.7701449782$. In particular, the solution of the fourth Painlevé equation* (1.6) *for our choice of monodromy data* (33) *is pole–free within that range.*

Since this estimate does not contain information for all possible values of s, we will then proceed to do a numerical analysis of the τ–function. Doing so, will require us to make use of the technique of Gauss-Hermite quadrature in order to have a faster convergence of this numerical estimate. The τ–function here will be a function of the two parameters s and γ, $\tau = \tau\left(s, \gamma\right)$, and due to the method that we are using, the numerical analysis will also depend on the number n of points that are used to compute the function. As a result of this procedure, we will obtain for the function $\tau\left(s, \gamma = 0.1, n = 150\right)$ the results that can be seen in Figure 0.1, where we plotted the arctan of τ because the function takes very large values for $s > 0$. This shows us that the τ–function is always non-zero for s negative, as was stated in the Theorem, and is only zero for certain positive values of s. In fact, it can be seen that, after a certain value of s, the function becomes oscillatory.

The second chapter of this work will be devoted to our random matrices problem. More specifically, we will be considering normal matrix models, where one considers $n \times n$ normal matrices, and their associated orthogonal polynomials, which are defined in the following way

$$\int_{\mathbb{C}} p_n\left(\lambda\right) \overline{p_m\left(\lambda\right)} e^{-NW\left(\lambda\right)} \, \mathrm{d}A\left(\lambda\right) = h_{n,N}\, \delta_{n,m}, \quad h_{n,N} > 0, \quad n, m = 0, 1, \ldots. \tag{35}$$

We will be interested in considering the case when the external potential exhibits a discrete rotational symmetry of the form

$$W\left(\lambda\right) = \left|\lambda\right|^{2d} - t\lambda^d - \bar{t}\bar{\lambda}^d, \quad \lambda \in \mathbb{C}. \tag{36}$$

It can be seen that potentials of this type exhibit different regimes according to the value of the parameter t that is being considered. In this work, we will be interested in studying the asymptotics of the orthogonal polynomials in the so-called critical case.

The first thing that we will do will be to use the symmetry of the external potential, $W(\lambda)$, to perform a reduction of the orthogonal polynomials, $p_n(\lambda)$, to some new polynomials, $\pi_k(z)$, where $z = 1 - \frac{\lambda^d}{t}$. In the critical case, $t = t_c$, the zeros of the orthogonal polynomials behave in the following way

Theorem 0.2. *The zeros of the polynomials $p_n(\lambda)$ defined in* (35) *for $t = t_c = \sqrt{\frac{T}{d}}$, behave in the following way*

- *for $n = kd + d - 1$, let $\omega = e^{\frac{2\pi i}{d}}$. Then $t^{\frac{1}{d}}, \omega t^{\frac{1}{d}}, \dots, \omega^{k-1} t^{\frac{1}{d}}$ are zeros of the polynomials p_{kd+d-1} with multipicity k and $\lambda = 0$ is a zero with multiplicity $r - 1$.*
- *for $n = kd + \ell$, $\ell = 0, \dots, r-2$ the polynomial $p_n(\lambda)$ has a zero in $\lambda = 0$ with multiplicity ℓ and the remaining zeros in the limit $n, N \to \infty$ such that*
$$N = \frac{n - \ell}{T}, \tag{37}$$
accumulate on the level curve $\hat{\mathcal{C}}$ defined in (2.17), namely
$$\hat{\mathcal{C}} := \{\lambda \in \mathbb{C} : \quad \left|(t_c - \lambda^d) \exp\left(\frac{\lambda^d}{t_c}\right)\right| = t_c, \quad |\lambda^d - t| \le t_c\}. \tag{38}$$
The measure $\hat{\nu}$ in (2.18) *is the weak-star limit of the normalized zero counting measure ν_n of the polynomials p_n for $n = kd + \ell$, $\ell = 0, \dots, d-2$.*

We will then reformulate the orthogonality relations for the polynomials $\pi_k(z)$ so that they can be written as an orthogonality relation defined over a contour in the complex plane. The advantage of doing this is the fact that it will allow us to reformulate our problem in terms of a Riemann-Hilbert problem for the polynomials $\pi_k(z)$.

The third chapter will be devoted to the study of the Riemann-Hilbert problem. Doing so, will require us to begin by defining the matrix of $Y(z)$ in such a way that the analysis for large k behaviour can be performed. The matrix is defined as

$$Y(z) = \begin{bmatrix} \pi_k(z) & \dfrac{1}{2\pi i} \displaystyle\int_\Sigma \frac{\pi_k(z')}{z' - z} w_k(z')\, \mathrm{d}z' \\ -2\pi i\, \Pi_{k-1}(z) & -\displaystyle\int_\Sigma \frac{\Pi_{k-1}(z')}{z' - z} w_k(z')\, \mathrm{d}z' \end{bmatrix}, \tag{39}$$

where Y_{11} can be seen to correspond to the reduced orthogonal polynomials whose asymptotics we want to compute. This matrix is the unique solution of a Riemann-Hilbert problem that we will have to define.

In order to analyze the asymptotic behaviour of the Riemann-Hilbert problem, we follow the general method of the nonlinear steepest descent of Deift–Zhou [11, 12], which requires a chain of transformations. This will be detailed in Chapter 3.

As it will appear in due course, the method requires the construction of an auxiliary scalar function (called the "g–function") based on a suitable measure supported on an appropriate contour. The way that this contour is defined will have to take into account the fact that we are in the critical case, where $z_0 = 1$, and all contours should cross this point. It should be noted that the analysis for the pre-critical, $z_0 > 1$, and post-critical, $z_0 < 1$, cases were studied in the work of Balogh, Merzi and Grava [2], where they were able to find the asymptotics for the orthogonal polynomials in these two regimes.

Due to the fact that we are dealing with a Riemann-Hilbert problem in the critical case, it will make sense for us to consider a double–scaling regime. In fact, applying the double–scaling procedure in the study of orthogonal polynomials is an idea that was originally introduced in the work of Bleher and Its [7]. Their purpose was to use this when dealing with matrix models with potentials of the form $W(\lambda) = \frac{t}{2}\lambda^2 + \frac{g}{4}\lambda^4$. As a result of their work, it was found that the Painlevé II equation appeared as the governing equation for the double scaling limit. In our work, where we are dealing with a different matrix model, we intend to follow a similar procedure to apply the double scaling limit in the study of orthogonal polynomials. As a result of this, we will be able to see that a particular solution to Painlevé IV is obtained. It should also be noted that, in our case, the relevant measure depends on a real parameter t and the double-scaling will be studied when t is near a critical value t_c.

By establishing this connection with Painlevé IV, we will finally be able to obtain the asymptotics of the orthogonal polynomials. The results obtained are stated in the following theorem.

Theorem 0.3 (Double scaling limit). *The polynomials $p_n(\lambda)$ with $n = kd + \ell$, $\ell = 0, \dots, d-2$, $\gamma = \frac{d-\ell-1}{d} \in (0,1)$, have the following asymptotic behaviour when $n, N \to \infty$ in such a way that $NT = n - \ell$ and*

$$\lim_{k\to\infty, t\to t_c} \sqrt{k}\left(\frac{t^2}{t_c^2} - 1\right) \to \mathcal{S},$$

with $\mathcal{S}$ in compact subsets of the real line so that the solution $Y(\mathcal{S})$ of the Painlevé IV equation does not have poles.

(1) For λ in compact subsets of the exterior of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell \left(\lambda^d - t_c\right)^k \left(\frac{\lambda^d - t_c}{\lambda^d}\right)^\gamma \left(1 - \frac{H(\mathcal{S})\, t_c}{\sqrt{k}\,\lambda^d} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{40}$$

with $H(\mathcal{S})$ the Hamiltonian (1.10) of the Painlevé IV equation (1.6).

(2) For λ in the region near $\hat{\mathcal{C}}$ and away from the point $\lambda = 0$ one has

$$p_n(\lambda) = \lambda^\ell \left(\lambda^d - t_c\right)^k \left(\frac{\lambda^d - t_c}{\lambda^d}\right)^\gamma \left(1 - \frac{H(\mathcal{S})\, t_c}{\sqrt{k}\,\lambda^d} + \frac{Z(\mathcal{S})}{U(\mathcal{S})} \frac{t_c\, e^{-k\hat{\varphi}(\lambda)}}{\lambda^d\, k^{\frac{1+\gamma}{2}}} \left(\frac{\lambda^d - t_c}{\lambda^d}\right)^{-\gamma} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{41}$$

with $\hat{\varphi}(\lambda)$ defined in (2.16) *and the functions Z, U and H are related to the Painlevé IV equation (1.6) by the relations (1.5) and (1.10), respectively.*

(3) For λ in compact subsets of the interior region of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell \left(\lambda^d - t_c\right)^k \frac{e^{-k\hat{\varphi}(\lambda)}}{k^{\frac{1}{2}+\gamma}} \left(\frac{Z(\mathcal{S})}{U(\mathcal{S})} \frac{t_c}{\lambda^d} + \mathcal{O}\left(\frac{1}{k}\right)\right). \tag{42}$$

(4) In the neighbourhood of the point $\lambda = 0$ and in the interior of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell \left(\lambda^d - t\right)^k \left(\frac{\lambda^d - t_c}{\lambda^d}\right)^\gamma e^{-k\hat{\varphi}(\lambda)} \left(\frac{\widehat{\Psi}_{11}(\sqrt{-k\,\hat{\varphi}(\lambda)};\mathcal{S})}{k^{\frac{\gamma}{4}}\sqrt{-\hat{\varphi}(\lambda)}} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{43}$$

where $\widehat{\Psi}_{11}$ is the $1\,1$ entry of the deformed Painlevé IV Riemann-Hilbert problem (3.4.4) obtained by deforming the Riemann-Hilbert problem (1.2.1) with Stokes multipliers specified in (1.15) and (1.16).

(5) In the neighbourhood of the point $\lambda = 0$ and in the exterior of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell \left(\lambda^d - t\right)^k \left(\frac{\lambda^d - t_c}{\lambda^d}\right)^\gamma \left(\frac{\widehat{\Psi}_{11}(\sqrt{-k\,\hat{\varphi}(\lambda)};\mathcal{S})}{k^{\frac{\gamma}{4}}\sqrt{-\hat{\varphi}(\lambda)}} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{44}$$

where the notation is as above in point (4).

0.2 Outlooks

As we have stated, the analysis performed in Chapter 1 allowed us to understand the domain of existence of the τ–function for our special solution of Painlevé IV, i.e., the points where the τ–function is non-zero. Due to the oscillatory behaviour that this function exhibits after a certain value of s, it can be observed that it is non-zero except for a set of infinitely many points. One could now consider the problem of dealing with these points so that this undesired behaviour (τ having zero value) can be tackled. One idea to do this would be by introducing a triple-scaling limit. More concretely, this means the following; suppose that $\mathcal{S} = s_0$ is any of the real zeroes of the Fredholm determinant (24) and hence a pole on the positive real axis for the functions $H(\mathcal{S})\,, Z(\mathcal{S})\,, U(\mathcal{S})$ appearing in (41). Following the ideas pioneered in [6] for the semiclassical analysis of the focusing Nonlinear Schrödinger equation, and refined in [5] for certain other orthogonal polynomials, one introduces a "slow" dependence of $\mathcal{S}$ on k as $\mathcal{S} = s_0 + \epsilon/k^{n_\sharp}$, with

$n_\sharp$ an appropriate scaling power; this allows one to investigate a neighbourhood of the pole s_0 in the plane of the Painlevé variable s. The technique requires one to construct a special interpolating local parametrix in place of the one that we used in this thesis.

The reason why this analysis is of interest is the following; according to the expression (41) for $\mathcal{S} \simeq s_0$, we have that the subleading corrections become divergent and it seems to suggest that the orthogonal polynomials have some sort of singularity. However, the polynomials come from a positive definite measure and therefore they should always exist, for all *real* values of t and hence all *real* values of $\mathcal{S}$. It is thus our expectation that a triple scaling analysis as indicated above will reveal that there is no divergence at $\mathcal{S} \sim s_0$ along the real $\mathcal{S}$–axis. This is the object of our future investigations.

1

Painlevé IV and the Fredholm Determinant

The purpose of this chapter is to introduce the Painlevé IV equation and study various aspects related to it that are connected with the work that we will be doing in the following chapters.

We will begin by introducing and deriving the general Painlevé IV equation following the ideas outlined by Miwa, Jimbo and Ueno [22, 23]. In the sequence of this, we will also discuss the associated Stokes' phenomenon and the monodromy problem (see Wasow [28]).

The second part of this chapter will be devoted to applying the general setting of Painlevé IV in a particular case, in order to obtain a special solution for it. This will correspond to a special Riemann-Hilbert problem whose properties will be outlined. The reason and motivation for introducing this solution is the fact that this Riemann-Hilbert problem will be seen to be useful in the chapters that follow.

The third and final part of this chapter will be devoted to the study of the Fredholm determinant, as it has been established by Its [21], Harnad [19], Bertola and Cafasso [4], among others. This will be used in order to study the τ–function of the special solution of Painlevé IV that we introduced before and analyze its validity with respect to the value of the parameter s.

1.1 The general Painlevé IV

Our first step in this work is to introduce the Painlevé IV equation. In order to derive it, we will begin by establishing its Lax Pair formulation as it was done in the original work of Miwa, Jimbo and Ueno [22, 23].

Consider the matrix–valued function Ψ, associated with Painlevé IV, as the fundamental joint solution of the system of linear differential equations, the Lax Pair, given by

$$\begin{cases} \Psi_\lambda = A\left(\lambda; s\right)\Psi \\ \\ \Psi_s = B\left(\lambda; s\right)\Psi \end{cases} \tag{1.1}$$

where $\Psi_\lambda = \frac{\partial \Psi}{\partial \lambda}$ and $\Psi_s = \frac{\partial \Psi}{\partial s}$. Regarding $A\left(\lambda; s\right)$ and $B\left(\lambda; s\right)$, these are matrices that are given by

$$A(\lambda; s) = -\frac{1}{2}(\lambda + s)\sigma_3 + \begin{pmatrix} 0 & \frac{Z}{U} \\ -U & 0 \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} \Theta_\infty - Z & \frac{(\Theta_\infty - Z)^2 - \Theta_0^2}{YU} \\ -UY & Z - \Theta_\infty \end{pmatrix}, \tag{1.2}$$

$$B(\lambda; s) = -\frac{1}{2}\lambda\,\sigma_3 + \begin{pmatrix} 0 & \frac{Z}{U} \\ -U & 0 \end{pmatrix}, \tag{1.3}$$

where σ_3 is the third Pauli matrix, Y, Z and U are functions of s whose dependence we now derive, while Θ_0, Θ_∞ are arbitrarily chosen constants. If one now applies the operator $\frac{\partial}{\partial\lambda}$ to Ψ_s and the operator $\frac{\partial}{\partial s}$ to Ψ_λ, it can easily be seen that the compatibility condition between these two equations will lead to the zero curvature equation, which we can now write in the following way

$$[\partial_\lambda - A, \partial_s - B] \equiv 0. \tag{1.4}$$

This equation can be computed explicitly using the matrices for $A(\lambda; s)$ and $B(\lambda; s)$ that were written above. Doing so, it can be seen that this leads to the following system of equations for U, Z and Y

$$\begin{cases} U' = U(Y - s) \\[2ex] Z' = ZY + \dfrac{-(Z - \Theta_\infty)^2 + \Theta_0^2}{Y} \\[2ex] Y' = -Y^2 + sY - 2Z + 2\Theta_\infty \end{cases} \tag{1.5}$$

If we now solve the equation of Y' for Z and place it into the equation for Z', it is possible to see that the following result is obtained

$$Y'' = \frac{1}{2}\frac{(Y')^2}{Y} + \frac{3}{2}Y^3 - 2sY^2 + \left(1 + \frac{s^2}{2} - 2\Theta_\infty\right)Y - \frac{2\Theta_0^2}{Y}. \tag{1.6}$$

This is the Painlevé IV equation, which can be interpreted as the compatibility condition for the system (1.1).

1.1.1 Stokes' phenomenon and the Monodromy problem

Considering the differential equation $\Psi_\lambda = A(\lambda; s)\Psi$ in (1.1), it is possible to find a formal solution of $\Psi(\lambda)$ (the dependence on s is understood even if not indicated) of the form

$$\Psi_{\text{formal}}(\lambda) = \left(\mathbb{1} + \frac{V_1}{\lambda} + \frac{V_2}{\lambda^2} + \dots\right)\lambda^{\Theta_\infty\sigma_3}e^{-\theta(\lambda)\sigma_3}, \tag{1.7}$$

where V_j are matrices that are constants in λ but depend on s and $\theta(\lambda)$ is of the form

$$\theta(\lambda) = \frac{\lambda^2}{4} + \frac{s}{2}\lambda. \tag{1.8}$$

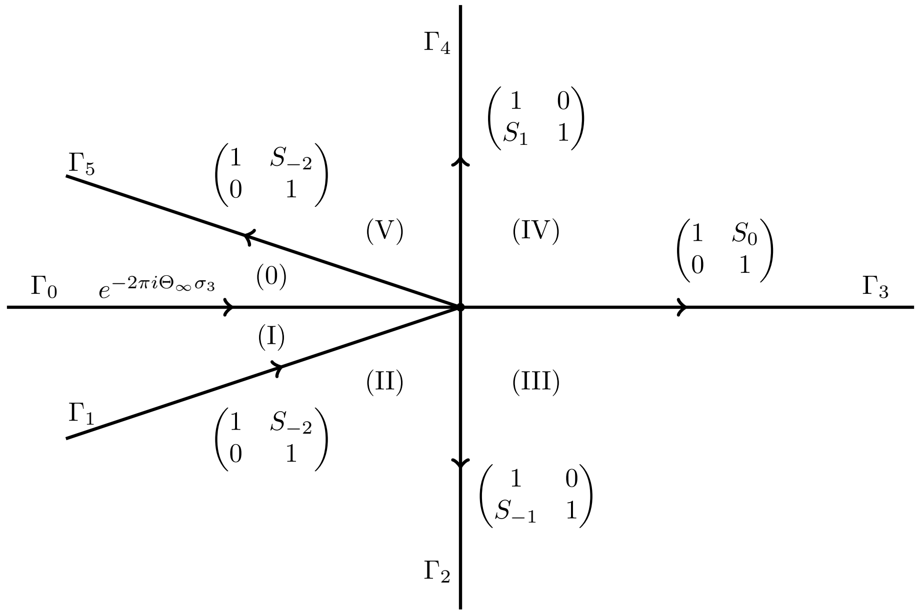


Figure 1.1: Stokes' phenomenon and the corresponding jumps in the $\Psi(\lambda)$ Riemann-Hilbert problem.

According to a general Theorem in the theory of differential equations (see Wasow [28]), these matrices V_j can be found by recursively plugging $\Psi_{\text{formal}}(\lambda)$ into (1.1) and matching the coefficients in λ. As a result of this procedure it is possible to obtain the following expression for $\Psi(\lambda)$

$$\Psi(\lambda) = \left(\mathbb{1} + \frac{1}{\lambda}\begin{bmatrix} H & \frac{Z}{U} \\ U & -H \end{bmatrix} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right) \lambda^{\Theta_\infty \sigma_3} e^{-\theta(\lambda)\sigma_3}, \tag{1.9}$$

where

$$H = \left(s + \frac{2\Theta_\infty}{Y} - Y\right) Z - \frac{\Theta_\infty^2 - \Theta_0^2}{Y} - \frac{Z^2}{Y}. \tag{1.10}$$

Even though the Laurent expansion for $\Psi_{\text{formal}}(\lambda)$ is not convergent, there are analytic solutions $\Psi_{(q)}(\lambda)$ in $\mathbb{C} \setminus \mathbb{R}_-$, where $q = 0$, I, II, $\ldots$ corresponds to the sectors that can be seen in Figure 1.1. These solutions are such that, in the corresponding sector q, they have the asymptotic expansion (1.7) when $\lambda \to \infty$. The solutions $\Psi_{(q)}(\lambda)$ in each of the different sectors are related by the so-called Stokes' matrices and, according to the general theory of Wasow, it is established that these have given triangularity. The Stokes' matrices for each of the different rays are those that can be seen in Figure 1.1.

Each of these solutions of Painlevé IV for each sector q, $\Psi_{(q)}(\lambda)$, has an asymptotic expansion near $\lambda = 0$ with the following behaviour

$$\Psi_{(q)}(\lambda) = G\left(\mathbb{1} + \mathcal{O}(\lambda)\right) \lambda^{\Theta_0 \sigma_3} C_{(q)}, \tag{1.11}$$

where G is a diagonalizing matrix for the $\frac{1}{\lambda}$ coefficient of $A(\lambda; s)$. From this equation, it can also be seen, by making $\lambda^r \to \lambda^r e^{2\pi i r}$, that the analytic continuation of $\Psi_{(0)}$ around $\lambda = 0$ in the counterclockwise way leads to the matrix

$$\tilde{\Psi}_{(0)}(\lambda) = \Psi_{(0)}(\lambda)\, C_{(0)}^{-1}\, e^{2\pi i \Theta_0 \sigma_3}\, C_{(0)}. \tag{1.12}$$

$\tilde{\Psi}_{(0)}(\lambda)$ should be interpreted as the expression for $\Psi(\lambda)$ in the sector (0) after a loop in the counterclockwise way has been performed. If we now consider the Riemann-Hilbert problem associated to $\Psi(\lambda)$ and using the Stokes' matrices that are exposed in Figure 1.1, then it should be obtained that

$$\Psi_{(0)}(\lambda) = \Psi_{(0)}(\lambda) \begin{pmatrix} 1 & -S_{-2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -S_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S_{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & S_{-2} \\ 0 & 1 \end{pmatrix} e^{2\pi i \Theta_\infty \sigma_3}. \tag{1.13}$$

It is clear that (1.12) and (1.13) should be equivalent equations. Therefore, comparing both of them, the following constraint is obtained

$$C_{(0)}^{-1}\, e^{2\pi i \Theta_0 \sigma_3}\, C_{(0)} = \begin{pmatrix} 1 & -S_{-2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -S_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -S_{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & S_{-2} \\ 0 & 1 \end{pmatrix} e^{2\pi i \Theta_\infty \sigma_3}. \tag{1.14}$$

This result is called the *Monodromy relation*. The remaining connection matrices $C_{(q)}$ are obtained from $C_{(0)}$ by multiplying it by an appropriate sequence of the Stokes' matrices; for example $C_{(\mathrm{V})} = C_{(0)} \begin{pmatrix} 1 & S_{-2} \\ 0 & 1 \end{pmatrix}$.

The general solution to the isomonodromic equations (1.5) for given Θ_0, Θ_∞ is parametrized by the choices of parameters $S_{-1}, \ldots$ and connection matrix $C_{(0)}$.

1.2 Special Riemann-Hilbert Problem $\Psi(\lambda)$ and Painlevé IV

We will now use the general setting of the Painlevé IV equation that we have just described and, by giving values to the monodromy data, define a special solution of this equation. The motivation that compels us to do this is the fact that this solution will be seen, in what follows, to be interesting for the remainder of our work.

In order to introduce the special solution of the Riemann-Hilbert problem, we will proceed by giving certain values to the Stokes' Matrices and to the connection matrices. The choice that we are interested in is the following

$$S_1 = 1, \quad S_0 = 0, \quad S_{-1} = -1, \quad S_{-2} = -1, \tag{1.15}$$

$$C_{(\mathrm{III})} = C_{(\mathrm{IV})} = \mathbb{1}. \tag{1.16}$$

Furthermore, we will also give values to the following quantities

$$\Theta_0 = \frac{\gamma}{2}, \quad \Theta_\infty = \frac{\gamma}{2}. \tag{1.17}$$

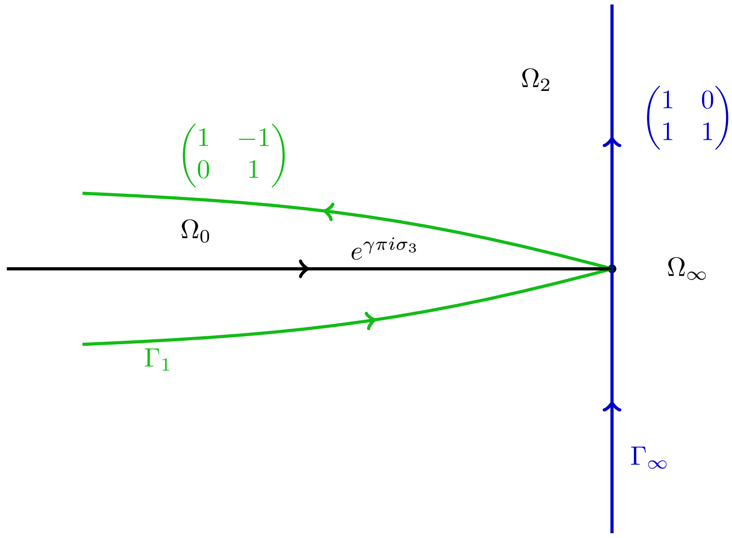


Figure 1.2: Jumps in the $\Psi(\lambda)$ Riemann-Hilbert problem.

Applying all these conditions, this gives rise to a new Riemann-Hilbert problem whose jump matrices are now given by the Stokes' matrices with the values for S_i that we have just specified. The resulting Riemann-Hilbert problem is the one depicted in Figure 1.2, where we have also given the names Γ_1 and Γ_∞ to the contours over which the jump matrices apply, and is established in the following way

Riemann-Hilbert Problem 1.2.1. *1. Piecewise Analyticity:*

$$\Psi(\lambda) \text{ is analytic in } \mathbb{C} \setminus (\Gamma_1 \cup \Gamma_\infty \cup \mathbb{R}_-).$$

2. Jumps on $\Sigma_\Psi = \Gamma_1 \cup \Gamma_\infty \cup \mathbb{R}_-$*:*

$$\Psi_+(\lambda) = \Psi_-(\lambda)\, v_\Psi, \quad \lambda \in \Sigma_\Psi, \tag{1.18}$$

where

$$v_\Psi = \begin{cases} \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, & \lambda \in \Gamma_1 \\ \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, & \lambda \in \Gamma_\infty \\ e^{\gamma\pi i\sigma_3}, & \lambda \in \mathbb{R}_- \end{cases} \tag{1.19}$$

which are the jumps that can be seen in Figure 1.2.

3. *Large λ boundary behaviour:*

$$\Psi(\lambda) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{\lambda}\right)\right)\lambda^{\frac{\gamma}{2}\sigma_3}\, e^{-\theta(\lambda)\sigma_3}, \quad \lambda \to \infty\,. \tag{1.20}$$

4. *Endpoint Behaviour:*

$$\Psi(\lambda) = \mathcal{O}(1)\lambda^{\frac{\gamma}{2}\sigma_3}, \tag{1.21}$$

as $\lambda \to 0$ in the region Ω_∞ (and the implication of this behaviour as $\lambda \to 0$ within the other regions).

We will now proceed to do a further modification of this $\Psi(\lambda)$–Riemann-Hilbert problem that we have just discussed. This will lead to a new Riemann-Hilbert problem that is going to be used in the next section for the study of the Fredholm Determinant.

This new Riemann-Hilbert problem $\mathcal{H}$ is going to be constructed by establishing the following conditions

- Move the jump contour $\Gamma_\infty = i\mathbb{R}$ in the following way $i\mathbb{R} \to i\mathbb{R} + \varepsilon$
- Join together (collapse) the two parts of the contour Γ_1 and $\mathbb{R}_-$

The new Riemann-Hilbert problem $\mathcal{H}$ is now defined in the following way

$$\mathcal{H}(\lambda) := \Psi(\lambda)\; e^{\theta(\lambda)\sigma_3}\, \lambda^{-\frac{\gamma}{2}\sigma_3}. \tag{1.22}$$

As a result of this procedure, we obtain the Riemann-Hilbert problem that is depicted in Figure 1.3, where the dashed line is due to the fact that, as it was stated above, the contour $\Gamma_\infty = i\mathbb{R}$ was moved by a factor of ε. The Riemann-Hilbert problem for $\mathcal{H}(\lambda)$ is therefore established in the following way

Riemann-Hilbert Problem 1.2.2. 1. *Piecewise Analyticity:*

$$\mathcal{H}(\lambda) \;\; \textit{is analytic in } \mathbb{C} \setminus (\Gamma_\infty \cup \mathbb{R}_-)\,.$$

2. *Jumps on $\Sigma_\Psi = \Gamma_\infty \cup \mathbb{R}_-$:*

$$\mathcal{H}_+(\lambda) = \mathcal{H}_-(\lambda)\; v_{\mathcal{H}}, \quad \lambda \in \Sigma_{\mathcal{H}}, \tag{1.23}$$

where

$$v_{\mathcal{H}} = \begin{cases} \begin{pmatrix} 1 & 2i\sin(\pi\gamma)\,|\lambda|^{\gamma} e^{-2\theta(\lambda)} \\ 0 & 1 \end{pmatrix}, & \lambda \in \mathbb{R}_- \\[2em] \begin{pmatrix} 1 & 0 \\ \lambda^{-\gamma} e^{2\theta(\lambda)} & 1 \end{pmatrix}, & \lambda \in \Gamma_\infty \end{cases} \tag{1.24}$$

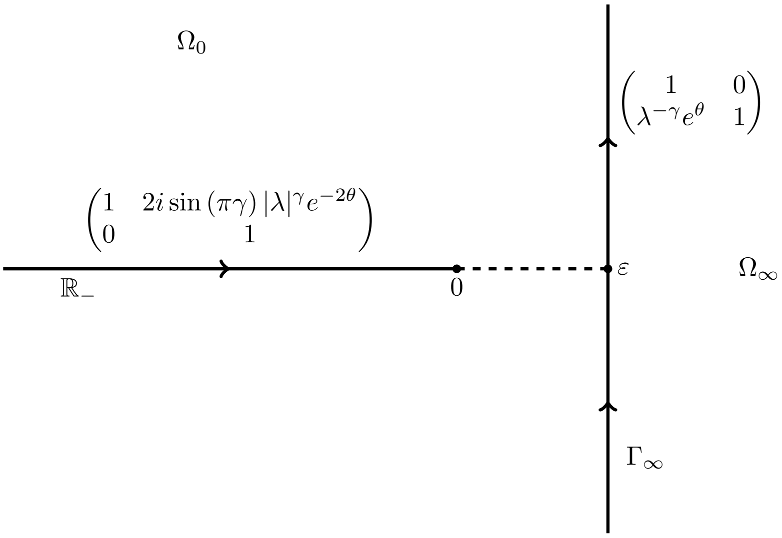


Figure 1.3: Jumps in the $\mathcal{H}(\lambda)$ Riemann-Hilbert problem.

which are the jumps that can be seen in Figure 1.3.

3. *Large λ boundary behaviour:*

$$\mathcal{H}(\lambda) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{\lambda}\right)\right), \quad \lambda \to \infty\,. \tag{1.25}$$

4. *Endpoint Behaviour:*

$$\mathcal{H}(\lambda) \textit{ is bounded near } \lambda = 0.$$

This is a Riemann-Hilbert problem that can be associated to an integral operator on $L^2\big(\mathbb{R}_- \cup (i\mathbb{R} + \varepsilon)\big)$ and will be used in the next section in order to compute the Fredholm Determinant.

1.3 Fredholm determinant and the τ-function of Painlevé IV

We now want to study the τ–function of our special solution of the Painlevé IV equation. In order to do this, we will introduce the notion of Fredholm determinant.

Considering a space X and an associated measure $\mathrm{d}\nu$, suppose that we have an operator $\hat{\mathcal{K}} : L^2(X, \mathrm{d}\nu) \to L^2(X, \mathrm{d}\nu)$ such that

$$\left(\hat{\mathcal{K}} f\right) = \int_X \mathcal{K}(x, y)\, f(y)\, \mathrm{d}\nu(y)\,. \tag{1.26}$$

Then, the following definition can be made

Definition 1.1. The *Fredholm Determinant* is defined as

$$\tau(\rho) = \det\left(\mathrm{Id} - \rho\hat{\mathcal{K}}\right) = 1 + \sum_{\ell=1}^{\infty} \frac{(-\rho)^{\ell}}{\ell!} \int_{X^{\ell}} \det\left[K\left(x_i, x_j\right)\right]_{i,j=1}^{\ell} \mathrm{d}\nu\left(x_1\right) \ldots \mathrm{d}\nu\left(x_l\right)$$
$$= 1 - \rho \int_X K\left(x, x\right) \mathrm{d}\nu\left(x\right) + \frac{\rho^2}{2} \int_X \int_X \det \begin{bmatrix} K\left(x, x\right) & K\left(x, y\right) \\ K\left(y, x\right) & K\left(y, y\right) \end{bmatrix} \mathrm{d}\nu\left(x\right) \mathrm{d}\nu\left(y\right) \tag{1.27}$$
$$+ \sum_{\ell=3}^{\infty} \frac{(-\rho)^{\ell}}{\ell!} \int_{X^{\ell}} \det\left[K\left(x_i, x_j\right)\right]_{i,j=1}^{\ell} \mathrm{d}\nu\left(x_1\right) \ldots \mathrm{d}\nu\left(x_{\ell}\right).$$

It can be seen, [26], that this series converges if and only if the operator $\hat{\mathcal{K}}$ is *trace-class* and, in this case, the Fredholm determinant is an entire function of ρ. Moreover, since $\hat{\mathcal{K}}$ is also a *compact* operator, its spectrum is purely discrete, the eigenvalues can accumulate only at 0 and all the multiplicities of the nonzero eigenvalues are finite (see e.g. [26]). It can also be said that

$$\tau\left(\rho\right) = 0 \text{ if and only if } \frac{1}{\rho} \text{ is an eigenvalue of } \hat{\mathcal{K}}. \tag{1.28}$$

For values of ρ sufficiently small, i e. such that they satisfy the condition

$$|\rho| < \left(\text{Spectral radius of } \hat{\mathcal{K}}\right)^{-1}, \tag{1.29}$$

then the condition of $\hat{\mathcal{K}}$ being trace-class can equivalently be written as

$$\log\det\left(\mathrm{Id} - \rho\hat{\mathcal{K}}\right) = -\sum_{\ell=1}^{\infty} \frac{1}{\ell} \rho^{\ell}\, \mathrm{Tr}\hat{\mathcal{K}}^{\ell}. \tag{1.30}$$

1.3.1 Riemann-Hilbert problems and Fredholm determinants.

We will now establish the general connection between a Riemann-Hilbert problem and Fredholm Determinants.

We begin by considering a set of contours $\Sigma \subset \mathbb{C}$ and define the following

$$K\left(\lambda, \mu\right) := \frac{f^T\left(\lambda\right).g\left(\mu\right)}{\lambda - \mu}, \tag{1.31}$$

where $f, g : \Sigma \to \mathrm{Mat}\left(n \times m, \mathbb{C}\right)$ are matrix–valued (smooth) functions that satisfy the condition $f^T\left(\lambda\right).g\left(\lambda\right) = 0$. Furthermore, $K\left(\lambda, \mu\right)$ is taken to be the kernel of an integral operator $\hat{\mathcal{K}} : L^2\left(\Sigma, \mathbb{C}^p\right) \to L^2\left(\Sigma, \mathbb{C}^p\right)$.

In order to work with the Fredholm determinant of $\hat{\mathcal{K}}$, we will use the following *Jacobi variational formula*

$$\partial\log\det\left(\mathrm{Id} - \hat{\mathcal{K}}\right) = \mathrm{Tr}_{L^2}\left(\left(\mathrm{Id} + \mathcal{R}\right) \circ \partial\hat{\mathcal{K}}\right), \tag{1.32}$$

where $\mathcal{R}$ is the resolvent operator

$$\mathcal{R} := -\hat{\mathcal{K}} \circ \left(\mathrm{Id} - \hat{\mathcal{K}}\right)^{-1}. \tag{1.33}$$

We now have the tools to state the Theorem, that can be seen in see [21], regarding the general connection between an operator $\hat{\mathcal{K}}$, its Fredholm determinant and a Riemann-Hilbert problem.

Theorem 1.1. *The kernel, $R(\lambda,\mu)$, of the resolvent operator $\mathcal{R}$ is given by*

$$\begin{aligned} R(\lambda,\mu) :&= -K\circ(\mathrm{Id}-K)^{-1}(\lambda,\mu) \\ &= \frac{f^T(\lambda)\,\Gamma^T(\lambda)\,\Gamma^{-T}(\mu)\,g(\mu)}{\lambda-\mu}, \end{aligned} \tag{1.34}$$

where $\Gamma(\lambda)$ solves the Riemann-Hilbert problem

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)\left(\mathbb{1} - 2\pi i f(\lambda)\,g^T(\lambda)\right), \quad \lambda\in\Sigma, \tag{1.35}$$

$$\Gamma(\lambda) = \mathbb{1} + \mathcal{O}\left(\lambda^{-1}\right), \quad \lambda\to\infty. \tag{1.36}$$

Moreover, the above Riemann—Hilbert problem admits a unique solution if and only if the Fredholm determinant $\det\left(\mathrm{Id}-\hat{\mathcal{K}}\right)$ *is non-zero.*

1.3.2 Fredholm determinant and the Riemann-Hilbert problem $\mathcal{H}$

We will now apply these results regarding the Fredholm Determinant to our special solution of Painlevé IV, which resulted in the Riemann-Hilbert problem $\mathcal{H}(z)$.

To do this, we begin by defining the following vectors

$$f(z) = \begin{bmatrix} 2i\sin(\pi\gamma)\,|z|^{\frac{\gamma}{2}}e^{-\theta(z)}\chi_{\mathbb{R}_-}(z) \\ \\ z^{-\frac{\gamma}{2}}e^{\theta(z)}\chi_{i\mathbb{R}}(z) \end{bmatrix}, \tag{1.37}$$

$$g(z) = -\frac{1}{2\pi i}\begin{bmatrix} z^{-\frac{\gamma}{2}}e^{\theta(z)}\chi_{i\mathbb{R}}(z) \\ \\ |z|^{\frac{\gamma}{2}}e^{-\theta(z)}\chi_{\mathbb{R}_-}(z) \end{bmatrix}, \tag{1.38}$$

where $\chi_{i\mathbb{R}}$ and $\chi_{\mathbb{R}_-}$ are the characteristic functions of $i\mathbb{R}$ and $\mathbb{R}_-$, respectively. With this definition, it is easy to see that, according to whether $i\mathbb{R}$ or $\mathbb{R}_-$ is being considered, these matrices $f(z)$ and $g(z)$ lead to the following jump matrices

$$\mathbb{1} - 2\pi i f(z)\,g^T(z) = \begin{cases} \begin{pmatrix} 1 & 2i\sin(\pi\gamma)\,|z|^{\gamma}e^{-2\theta(z)} \\ 0 & 1 \end{pmatrix}, & z\in\mathbb{R}_- \\ \\ \begin{pmatrix} 1 & 0 \\ z^{-\gamma}e^{2\theta(z)} & 1 \end{pmatrix}, & z\in i\mathbb{R} \end{cases} \tag{1.39}$$

Recalling the Riemann-Hilbert problem $\mathcal{H}(z)$ that was stated above, it can now be seen that its jump matrices $v_{\mathcal{H}}$, that were given in (1.24), are the same as the ones obtained in (1.39), for $\mathbb{1} - 2\pi i\, f(z)\, g^T(z)$.

As a consequence of this, we can now write the Riemann-Hilbert problem $\mathcal{H}$ in the following way

$$\mathcal{H}_+(z) = \mathcal{H}_-(z)\left(\mathbb{1} - 2\pi i f(z)\,g^T(z)\right), \quad z\in\Sigma, \tag{1.40}$$

$$\mathcal{H} = \mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty, \tag{1.41}$$

where $\Sigma = i\mathbb{R} \cup \mathbb{R}_-$. It should also be noted that, due to the construction that was done, we have $f^T(z)\, g(z) \equiv 0$.

From what we stated before, it is also clear that the solution to the Riemann-Hilbert problem $\mathcal{H}(z)$ exists if and only if

$$\det\left(\mathbb{1} - \hat{\mathcal{K}}\right) \neq 0. \tag{1.42}$$

Regarding $\hat{\mathcal{K}}$, it is an operator of the form $\hat{\mathcal{K}} : L^2(\Sigma) \to L^2(\Sigma)$ and given by

$$\hat{\mathcal{K}}[\varphi](z) = \int_\Sigma \frac{f^T(z)\, g(w)}{z-w} \varphi(w)\; \mathrm{d}w, \tag{1.43}$$

where $\varphi \in L^2(\Sigma)$. In order to make an analysis of this operator, we begin by splitting $\Sigma = i\mathbb{R} \cup \mathbb{R}_-$, which leads to

$$L^2(\Sigma) = L^2(i\mathbb{R} \cup \mathbb{R}_-) = L^2(i\mathbb{R}) \oplus L^2(\mathbb{R}_-), \tag{1.44}$$

where the last equality is an isomorphism. More specifically, for a function $\varphi \in L^2(\Sigma)$, we can associate the two components along $L^2(\mathbb{R}_-)$, $L^2(i\mathbb{R})$ as follows

$$\varphi \in L^2(i\mathbb{R} \cup \mathbb{R}_-) \to \varphi_0 \in L^2(\mathbb{R}_-), \varphi_1 \in L^2(i\mathbb{R}), \tag{1.45}$$

$$\varphi(z) = \varphi_0(z)\, \chi_{\mathbb{R}_-}(z) + \varphi_1(z)\, \chi_{i\mathbb{R}}(z). \tag{1.46}$$

Due to this fact, the operator $\hat{\mathcal{K}}[\varphi]$ can now be written as

$$\hat{\mathcal{K}}[\varphi] = \begin{bmatrix} \mathcal{K}_{0\,0} & \mathcal{K}_{0\,1} \\ \mathcal{K}_{1\,0} & \mathcal{K}_{1\,1} \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{0\,0}\,\varphi_0 + \mathcal{K}_{0\,1}\,\varphi_1 \\ \mathcal{K}_{1\,0}\,\varphi_0 + \mathcal{K}_{1\,1}\,\varphi_1 \end{bmatrix}, \tag{1.47}$$

where the operators $\mathcal{K}_{i\,j}$ are of the form

$$\mathcal{K}_{0\,0} :\; L^2(\mathbb{R}_-) \to L^2(\mathbb{R}_-), \tag{1.48}$$

$$\mathcal{K}_{0\,1} :\; L^2(i\mathbb{R}) \to L^2(\mathbb{R}_-), \tag{1.49}$$

$$\mathcal{K}_{1\,0} :\; L^2(\mathbb{R}_-) \to L^2(i\mathbb{R}), \tag{1.50}$$

$$\mathcal{K}_{1\,1} :\; L^2(i\mathbb{R}) \to L^2(i\mathbb{R}). \tag{1.51}$$

Regarding the particular case of the Riemann-Hilbert problem $\mathcal{H}(z)$ that we want to consider, recall that in this case we have $\mathbb{1} - 2\pi i f(z)\, g^T(z)$, which leads to (1.39). We now have the operator (1.43) with $f(z)$ and $g(z)$ given by (1.37) and (1.38), respectively, and therefore it is easy to see that for this case

$$\mathcal{K}_{0\,0} = \mathcal{K}_{1\,1} = 0. \tag{1.52}$$

We will now compute the kernel of our integral operator $\hat{\mathcal{K}}$, (1.31):

$$\begin{aligned}\frac{f^T(z)\,g(w)}{z-w} &= -\frac{1}{2\pi i}\left(\frac{2i\sin(\pi\gamma)|z|^{\frac{\gamma}{2}}e^{-\theta(z)}\chi_{\mathbb{R}_-}(z)\,w^{-\frac{\gamma}{2}}e^{\theta(w)}\chi_{i\mathbb{R}}(w)}{z-w}\right)\\ &\quad-\frac{1}{2\pi i}\left(\frac{z^{-\frac{\gamma}{2}}e^{\theta(z)}\chi_{i\mathbb{R}}(z)\,|w|^{\frac{\gamma}{2}}e^{-\theta(w)}\chi_{\mathbb{R}_-}(w)}{z-w}\right)\end{aligned} \tag{1.53}$$

which leads to

$$\begin{aligned}\int_\Sigma \frac{f^T(z)\,g(w)}{z-w}\varphi(w)\,\mathrm{d}w &= -\frac{2i\sin(\pi\gamma)}{2\pi i}\left(\int_{i\mathbb{R}}\frac{w^{-\frac{\gamma}{2}}e^{\theta(w)}\varphi_1(w)}{z-w}\,\mathrm{d}w\right)|z|^{\frac{\gamma}{2}}e^{-\theta(z)}\chi_{\mathbb{R}_-}(z)\\ &\quad-\frac{1}{2\pi i}\left(\int_{\mathbb{R}_-}\frac{|w|^{\frac{\gamma}{2}}e^{-\theta(w)}\varphi_0(w)}{z-w}\,\mathrm{d}w\right)z^{-\frac{\gamma}{2}}e^{\theta(z)}\chi_{i\mathbb{R}}(z)\,.\end{aligned} \tag{1.54}$$

Therefore, the entries of $\hat{\mathcal{K}}$ that we have in (1.47) can be seen to be of the following form

$$\begin{aligned}&K_{0\,1}: L^2(i\mathbb{R})\to L^2(\mathbb{R}_-)\\ &(K_{0\,1}\,\varphi)(z) = -\frac{2i\sin(\pi\gamma)}{(2\pi i)}\left(\int_{i\mathbb{R}}\frac{w^{-\frac{\gamma}{2}}e^{\theta(w)}\varphi_1(w)}{z-w}\,\mathrm{d}w\right)|z|^{\frac{\gamma}{2}}e^{-\theta(z)},\end{aligned} \tag{1.55}$$

$$\begin{aligned}&K_{1\,0}: L^2(\mathbb{R}_-)\to L^2(i\mathbb{R})\\ &(K_{1\,0}\,\varphi)(z) = -\frac{1}{(2\pi i)}\left(\int_{\mathbb{R}_-}\frac{|w|^{\frac{\gamma}{2}}e^{-\theta(w)}\varphi_0(w)}{z-w}\,\mathrm{d}w\right)z^{-\frac{\gamma}{2}}e^{\theta(z)}.\end{aligned} \tag{1.56}$$

Following the Definition 1.1 of the Fredholm Determinant, we can now compute it for the case when $\hat{\mathcal{K}}$ is given by this operator

$$\begin{aligned}\det\left(\mathrm{Id}-\hat{\mathcal{K}}\right) &= \det\left(\begin{bmatrix}\mathbb{1}_0 & 0\\ 0 & \mathbb{1}_1\end{bmatrix}-\begin{bmatrix}0 & K_{0\,1}\\ K_{1\,0} & 0\end{bmatrix}\right)\\ &= \det\left(\begin{bmatrix}\mathbb{1}_0 & K_{0\,1}\\ 0 & \mathbb{1}_1\end{bmatrix}\begin{bmatrix}\mathbb{1}_0 & -K_{0\,1}\\ -K_{1\,0} & \mathbb{1}_1\end{bmatrix}\right)\\ &= \det\begin{bmatrix}\mathbb{1}_0-K_{0\,1}K_{1\,0} & 0\\ -K_{1\,0} & \mathbb{1}_1\end{bmatrix},\end{aligned} \tag{1.57}$$

where $\mathbb{1}_0=\mathrm{Id}_{L^2(\mathbb{R}_-)}$, $\mathbb{1}_1=\mathrm{Id}_{L^2(i\mathbb{R})}$ and we are left with the equality

$$\det\left(\mathrm{Id}-\hat{\mathcal{K}}\right) = \det{}_{L^2(\mathbb{R}_-)}\left[\mathrm{Id}_{L^2(\mathbb{R}_-)}-K_{0\,1}K_{1\,0}\right]. \tag{1.58}$$

We will now use the operators that were defined in (1.55) and (1.56) to compute the product $K_{0\,1}K_{1\,0}$ that appears in (1.58)

$$(K_{0\,1})\,(K_{1\,0})\,(\varphi_0)\,(z) = -\frac{2i\sin(\pi\gamma)}{4\pi^2}|z|^{\frac{\gamma}{2}}e^{-\theta(z)}\int_{i\mathbb{R}}\frac{x^{-\gamma}e^{2\theta(x)}}{z-x}\left(\int_{\mathbb{R}_-}\frac{|w|^{\frac{\gamma}{2}}e^{-\theta(w)}\varphi_0(w)}{x-w}\,\mathrm{d}w\right)\mathrm{d}x. \tag{1.59}$$

This can be written as

$$\left(K_{0\,1}K_{1\,0}\right)\left(\varphi_0\right)(z) = \int_{\mathbb{R}_-} F^{\text{Kernel}}(z,w)\,\varphi_0(w)\,\mathrm{d}w, \tag{1.60}$$

where

$$F^{\text{Kernel}} = \frac{2i\sin(\pi\gamma)}{4\pi^2}|z|^{\frac{\gamma}{2}}e^{-\theta(z)}|w|^{\frac{\gamma}{2}}e^{-\theta(w)}\int_{i\mathbb{R}}\frac{x^{-\gamma}e^{2\theta(x)}}{(z-x)(w-x)}\,\mathrm{d}x. \tag{1.61}$$

1.3.3 Norm estimate for small s

We would like to make an estimate of the value of the Fredholm Determinant. To do this, we begin by using the fact that $\theta(\lambda)$ is given by (1.8) and simplify F^{Kernel}

$$F^{\text{Kernel}} = \frac{2i\sin(\pi\gamma)}{4\pi^2}e^{\frac{-z^2-w^2}{4}}|z|^{\frac{\gamma}{2}}|w|^{\frac{\gamma}{2}}e^{-\frac{s}{2}(z+w)}\frac{1}{z-w}\int_{i\mathbb{R}}\left(x^{-\gamma}e^{\frac{x^2}{2}+sx}\right)\left(\frac{1}{z-x}-\frac{1}{w-x}\right)\mathrm{d}x. \tag{1.62}$$

If we now define

$$H(u) := \int_{i\mathbb{R}} i\frac{x^{-\gamma}e^{\frac{x^2}{2}+sx}}{u-x}\,\mathrm{d}x, \tag{1.63}$$

$$a(u) = |u|^{\frac{\gamma}{2}}e^{-\frac{u^2}{4}-\frac{s}{2}u}, \tag{1.64}$$

then F^{Kernel} can now be written as

$$F^{\text{Kernel}} = K_0\,a(z)\,a(w)\,\frac{H(z)-H(w)}{z-w}, \tag{1.65}$$

where $K_0 = \frac{\sin(\pi\gamma)}{2\pi}$. Since we want to use this to compute (1.60), we can write

$$\int_{\mathbb{R}_-} F^{\text{Kernel}}(z,w)\,\varphi_0(w)\,\mathrm{d}w = K_0\,a(z)\left[H(z)\fint_{\mathbb{R}_-}\frac{a(w)\,\varphi_0(w)}{z-w}\frac{\mathrm{d}w}{\pi} - \fint_{\mathbb{R}_-}\frac{a(w)\,H(w)\,\varphi_0(w)}{z-w}\frac{\mathrm{d}w}{\pi}\right], \tag{1.66}$$

where $\mathcal{P}(\phi) = \fint \frac{\phi(w)}{z-w}\frac{\mathrm{d}w}{\pi}$ denotes the Cauchy principal value, also known as Hilbert transform.

We can now estimate the value of this integral. Given an essentially bounded function $f \in L^\infty(\mathbb{R}_-)$, we denote by M_f the corresponding multiplication operator and recall that

$$|||M_f||| = \|f\|_\infty, \tag{1.67}$$

where $|\cdot\|_\infty$ denotes the $L^\infty(\mathbb{R}_-)$ norm. The Hilbert transform is a bounded operator with norm one

$$|||\mathcal{P}||| = 1, \tag{1.68}$$

which follows from the fact that, in Fourier space, the Hilbert transform is simply the multiplication operator by the sign function multiplied by $\mathrm{e}^{i\pi/2}$. This leads to

$$\begin{aligned}\left\|\int_{\mathbb{R}_-} F^{\text{Kernel}}(z,w)\,\varphi_0(w)\,\mathrm{d}w\right\| &= |K_0|\,\|(M_{aH}\mathcal{P}M_a - M_a\mathcal{P}M_{aH})(\varphi)\|_{L^2} \\ &\le |K_0|\left(\|M_{aH}\mathcal{P}M_a\varphi\|_{L^2} + \|M_a\mathcal{P}M_{aH}\varphi\|_{L^2}\right) \\ &\le |K_0|\left(|||M_{aH}\mathcal{P}M_a||| + |||M_a\mathcal{P}M_{aH}|||\right)\|\varphi\|.\end{aligned} \tag{1.69}$$

As a result of this procedure, we obtain the following estimate for $|||F|||$

$$|||F||| \leq \frac{\sin(\pi\gamma)}{\pi} \|a\|_\infty^2 \|H\|_\infty |||\mathcal{P}|||. \tag{1.70}$$

We should now get estimates for the values of $\|a\|_\infty$, $\|H\|_\infty$ and $|||\mathcal{P}|||$. Since, as we stated before, $|||\mathcal{P}||| = 1$, we will now compute the other two quantities. To do this, we begin by considering that $x = \varepsilon + it$ and the integral will be over $i\mathbb{R} + \varepsilon$

$$H(z) = \int_{i\mathbb{R}+\varepsilon} i\mathrm{d}x \, \frac{x^{-\gamma} e^{\frac{x^2}{2} - \tilde{s}x}}{z - x}, \quad \tilde{s} = -s. \tag{1.71}$$

If we now take $\varepsilon = \max(0, \tilde{s})$, we get the estimate for $|H(z)|$ in the following way for $\tilde{s} > 0$ (i.e. $s < 0$)

$$\begin{aligned}
|H(z)| &\leq \int_{i\mathbb{R}+\tilde{s}} |\mathrm{d}x| \, \frac{|x|^{-\gamma} e^{\mathrm{Re}\left(\frac{x^2}{2} - \tilde{s}x\right)}}{|z - x|}, \qquad (|z - x| \geq \tilde{s}) \\
&\leq \frac{1}{\tilde{s}} \int_{i\mathbb{R}+\tilde{s}} |\mathrm{d}x| \, |x|^{-\gamma} e^{\mathrm{Re}\left(\frac{x^2}{2} - \tilde{s}x\right)} \\
&\leq \frac{1}{\tilde{s}} \int_{\mathbb{R}} \mathrm{d}t \, |\tilde{s} + it|^{-\gamma} e^{-\frac{t^2}{2} - \frac{\tilde{s}^2}{2}} \\
&\leq \frac{e^{-\frac{\tilde{s}^2}{2}}}{\tilde{s}} \tilde{s}^{-\gamma} \int_{\mathbb{R}} \mathrm{d}t \, e^{-\frac{t^2}{2}} \\
&= \frac{e^{-\frac{\tilde{s}^2}{2}}}{\tilde{s}^{1+\gamma}} \sqrt{2\pi}.
\end{aligned} \tag{1.72}$$

A calculus exercise shows that the function $a(u)$, (1.64), satisfies $\|a\|_\infty \leq 1$, for $u \in \mathbb{R}_-$, $\gamma \in [0, 1]$, $\tilde{s} > 0$ and then the estimate for $|||F|||$ becomes

$$|||F||| \leq \frac{\sin(\pi\gamma)}{\pi} \frac{e^{-\frac{\tilde{s}^2}{2}}}{\tilde{s}^{1+\gamma}} \sqrt{2\pi}. \tag{1.73}$$

At this point, we need to see for which values of $\tilde{s}$ the norm is certainly smaller than 1; this guarantees that the determinant (1.58) will not be zero and hence that our Riemann-Hilbert problem 1.2.2 admits a solution. We can estimate then from (1.73) that (recall $\gamma \in [0, 1]$)

$$|||F||| \leq \frac{\sin(\pi\gamma)}{\pi} \frac{\mathrm{e}^{-\frac{\tilde{s}^2}{2}}}{\tilde{s}^{1+\gamma}} \sqrt{2\pi} \leq \begin{cases} \dfrac{\sqrt{2}}{\sqrt{\pi}} \dfrac{\mathrm{e}^{-\frac{\tilde{s}^2}{2}}}{\tilde{s}^2} & \tilde{s} \in [0, 1] \\ \dfrac{\sqrt{2}}{\sqrt{\pi}} \dfrac{\mathrm{e}^{-\frac{\tilde{s}^2}{2}}}{\tilde{s}} & \tilde{s} > 1 \end{cases} \tag{1.74}$$

and we can easily see that the norm is less than one for $\tilde{s} > \tilde{s}_0$ with $\tilde{s}_0 < 1$, which can be approximated by $\tilde{s}_0 = 0.7701449782$. In summary, we have proven

Theorem 1.3.1. *The Riemann–Hilbert problem 1.2.2 admits a solution for $s \in (-\infty, -\tilde{s}_0)$. In particular, the solution of the fourth Painlevé equation* (1.6) *for our choice of monodromy data (1.15, 1.16) is pole–free within that range.*

It would be desirable to show as a result that our solution is pole–free for all real values of s; however, numerical analysis to be performed in the following section, shows that this is not the case and there are a discrete (in principle infinite) number of values of s for which the Fredholm determinant (1.58) vanishes and hence the solution of (1.6) has poles.

1.4 Gaussian Quadrature and Numerics

As we have stated before, see Definition 1.1, the importance of the Fredholm Determinant for our work is due to the fact that it allows us to compute the τ–function. Through the study that we made in the previous section, we were also able to see that computing the Fredholm Determinant requires one to be able to compute the kernel of the operator (1.60), which is given by (1.62). By looking at this operator kernel, it can be seen that the value of the τ–function depends on the values chosen for s and γ. In this section, our purpose is to compute this function numerically according to given values for these two parameters.

1.4.1 The Gauss-Hermite quadrature

The following is an exposition, with some adapted details, of the main idea of numerical evaluation of Fredholm determinants based on Nystrom method, as explained by Bornemann [9]. The simple idea is to suitably "discretize" the integral operator; the simplest approach would be to use Riemann sums in place of the integrals, but this is notoriously slow to converge. The main improvement on this idea is to use some appropriate Gaussian quadrature (see [27]).

More specifically, the type of quadrature formulæ that we will use are the so–called "Gauss–Hermite" quadratures, which are defined in the following way

Definition 1.2. Considering an integral of the form

$$\int_{-\infty}^{\infty} f(x)\, e^{-\Lambda x^2}\, \mathrm{d}x, \tag{1.75}$$

the *Gauss-Hermite quadrature* is an approximation of the value of this integral and is defined as

$$\int_{-\infty}^{\infty} f(x)\, e^{-\Lambda x^2}\, \mathrm{d}x \simeq \sum_{i=1}^{n} f\left(x_i^{(n)}\right) w_i^{(n)}, \tag{1.76}$$

where n is the number of points used in this approximation: the points $\{x_j^{(n)}\}_{j=1}^n$ are called the *nodes* and the coefficients $\{w_j^{(n)}\}_{j=1}^n$ are called the *weights* of the quadrature rule. The nodes x_j are the roots of the n-th Hermite polynomial $H_n\left(\sqrt{\Lambda}x\right)$ and the weights $w_i^{(n)}$ are given by

$$w_i^{(n)} = \frac{2^{n-1} n! \sqrt{\pi}}{\sqrt{\Lambda} n^2 \left(H_{n-1}\left(\sqrt{\Lambda} x_i\right)\right)^2}. \tag{1.77}$$

We will have to adapt this definition so that it is suitable to be applied to our case.

In order to use this technique for the computation of $F^{\mathrm{Kernel}}(x,y)$, we will begin by recalling this operator kernel and writing it in the following way

$$\begin{aligned} F^{\mathrm{Kernel}}(x,y) &= \frac{\sin(\pi\gamma)}{2\pi^2} e^{\frac{-x^2-y^2}{4}} |x|^{\frac{\gamma}{2}} |y|^{\frac{\gamma}{2}} e^{-\frac{s}{2}(x+y)} \int_{i\mathbb{R}} i\mathrm{d}z \, \frac{z^{-\gamma} e^{\frac{z^2}{2}+sz}}{(x-z)(y-z)} \\ &= \frac{\sin(\pi\gamma)}{2\pi^2} e^{\frac{-x^2-y^2}{4}} |x|^{\frac{\gamma}{2}} |y|^{\frac{\gamma}{2}} e^{-\frac{s}{2}(x+y)} \int_{\mathbb{R}} \mathrm{d}t \, \frac{(it+\varepsilon)^{-\gamma} e^{-\frac{t^2}{2}} e^{it(s+\varepsilon)}}{(x-\varepsilon-it)(y-\varepsilon-it)} e^{\frac{\varepsilon^2}{2}+s\varepsilon}, \end{aligned} \tag{1.78}$$

where we have used $x=\varepsilon+it$. It appears from (1.78) that the relevant scaling parameter to be used in Definition 1.2 is $\Lambda=\frac{1}{2}$. Moreover, since we consider the semi-axis $\mathbb{R}_-$, it is convenient to use the Gauss-Hermite quadrature with an even number, $2n$, of nodes, so that (by symmetry) the first n are in the negative axis. Since we want to apply the procedure of the Gauss-Hermite quadrature, it is convenient to write the operator kernel as

$$F^{\mathrm{Kernel}}(x,y) = e^{\frac{-x^2-y^2}{4}} H(x,y)\,, \tag{1.79}$$

where it was defined

$$H(x,y) = \frac{\sin(\pi\gamma)}{2\pi^2} |x|^{\frac{\gamma}{2}} |y|^{\frac{\gamma}{2}} e^{-\frac{s}{2}(x+y)} \int_{\mathbb{R}} \mathrm{d}t \, \frac{(it+\varepsilon)^{-\gamma} e^{-\frac{t^2}{2}} e^{it(s+\varepsilon)}}{(x-\varepsilon-it)(y-\varepsilon-it)} e^{\frac{\varepsilon^2}{2}+s\varepsilon}. \tag{1.80}$$

Since our goal with this procedure is to compute the τ–function, we will now apply the Gauss-Hermite quadrature to compute the operator $K_{01}K_{10}$ that appears in (1.58). This is done in the following way

$$\begin{aligned} \det{}_{L^2(\mathbb{R}_-)}\left[\mathrm{Id}_{L^2(\mathbb{R}_-)} - K_{01}K_{10}\right] &= \det{}_{L^2(\mathbb{R}_-)}\left[\mathrm{Id}_{L^2(\mathbb{R}_-)} - F^{\mathrm{Kernel}}(x,y)\right] \\ &\simeq \det{}_{n\times n}\left[\mathrm{Id}_n - \left[H\left(x_i^{(2n)}, x_j^{(2n)}\right)\sqrt{w_i^{(2n)} w_j^{(2n)}}\right]_{i,j=1}^{n}\right]. \end{aligned} \tag{1.81}$$

In order to understand why applying the Gauss-Hermite quadrature leads to this result, we will now consider the following procedure to do the discretization of a function (smooth) $\varphi_0(x) \in L^2(\mathbb{R}_-)$

$$\varphi_0(x) \to \mathbb{V}[\varphi_0] := \begin{bmatrix} \varphi_0(x_1)\sqrt{w_1} e^{\frac{x_1^2}{4}} \\ \varphi_0(x_2)\sqrt{w_2} e^{\frac{x_2^2}{4}} \\ \vdots \\ \varphi_0(x_n)\sqrt{w_n} e^{\frac{x_n^2}{4}} \end{bmatrix}. \tag{1.82}$$

This correspondence provides an approximate isometry between $L^2(\mathbb{R}_-, |\mathrm{d}x|)$ and $\mathbb{C}^n$ since

$$\int_{\mathbb{R}_-} \mathrm{d}x |\varphi_0(x)|^2 = \int_{\mathbb{R}_-} |\varphi_0(x) \mathrm{e}^{\frac{x^2}{4}}|^2 \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x \simeq \sum_{j=1}^{n} |\mathbb{V}_j[\varphi_0]|^2\,,$$

where the last approximation is valid thanks to the definition of the vector $\mathbb{V}$ and the quadrature rule (1.76). If we now apply the same procedure to do the discretization of the operator

$\left(\hat{F}\varphi_0\right)(x)$, the result will be

$$\left(\hat{F}\varphi_0\right)(x) = \int F(x,y)\,\varphi_0(y)\,\mathrm{d}y \to \left[\int \mathrm{d}y\, F(x_j,y)\,\varphi_0(y)\,\sqrt{w_j}\,e^{\frac{x_j^2}{4}}\right]_{j=1,\ldots,n}. \tag{1.83}$$

The definition of the Gauss-Hermite quadrature now comes into play and is used to compute the integral that we have on the right hand side of this equation. By applying this procedure, the variable y is now decomposed into n discrete x_k points

$$\begin{aligned}
&\left[\int \mathrm{d}y\, F\left(x_j^{(2n)},y\right)\varphi_0(y)\sqrt{w_j^{(2n)}}e^{\frac{(x_j^{(2n)})^2}{4}}\right]_{j=1,\ldots,n} \simeq \\
&\simeq \left[\sum_{k=1}^{n} F\left(x_j^{(2n)},x_k^{(2n)}\right)\varphi_0\left(x_k^{(2n)}\right)w_k^{(2n)}e^{\frac{\left(x_k^{(2n)}\right)^2}{2}}\sqrt{w_j^{(2n)}}e^{\frac{\left(x_j^{(2n)}\right)^2}{4}}\right]_{j=1,\ldots,n} \\
&= \left[\sum_{k=1}^{n} F\left(x_j^{(2n)},x_k^{(2n)}\right)e^{\frac{\left(x_k^{(2n)}\right)^2}{4}+\frac{\left(x_j^{(2n)}\right)^2}{4}}\sqrt{w_k^{(2n)}}\sqrt{w_j^{(2n)}}\varphi_0\left(x_k^{(2n)}\right)\sqrt{w_k^{(2n)}}e^{\frac{\left(x_k^{(2n)}\right)^2}{4}}\right]_{j=1,\ldots,n},
\end{aligned} \tag{1.84}$$

and it can be seen that in this equation we have

$$H\left(x_j^{(2n)},x_k^{(2n)}\right) = F\left(x_j^{(2n)},x_k^{(2n)}\right)e^{\frac{x_k^2}{4}+\frac{x_j^2}{4}}, \tag{1.85}$$

which is in accordance with the definition that had been made in (1.79). This means that, as a result of this discretization, we get (we drop the superscript $^{(2n)}$ for readability)

$$\left(\hat{F}\varphi_0\right)(x) \to \begin{bmatrix} H(x_1,x_1)\,w_1 & H(x_1,x_2)\sqrt{w_1w_2} & \ldots & H(x_1,x_n)\sqrt{w_1w_n} \\ H(x_2,x_1)\sqrt{w_2w_1} & H(x_2,x_2)\,w_2 & \ldots & H(x_2,x_n)\sqrt{w_2w_n} \\ \vdots & \vdots & \ldots & \vdots \\ H(x_n,x_1)\sqrt{w_nw_1} & H(x_n,x_2)\sqrt{w_nw_2} & \ldots & H(x_n,x_n)\,w_n \end{bmatrix}\begin{bmatrix} \varphi_0(x_1)\sqrt{w_1}e^{\frac{x_1^2}{4}} \\ \varphi_0(x_2)\sqrt{w_2}e^{\frac{x_2^2}{4}} \\ \vdots \\ \varphi_0(x_n)\sqrt{w_n}e^{\frac{x_n^2}{4}} \end{bmatrix}. \tag{1.86}$$

Therefore, the matrix that we are interested in computing is indeed given by

$$K_{0\,1}K_{1\,0} = \left[H\left(x_j^{(2n)},x_k^{(2n)}\right)\sqrt{w_j^{(2n)}w_k^{(2n)}}\right]_{j,k=1}^{n} \equiv \left[H_{j\,k}\right]_{j,k=1}^{n} \tag{1.87}$$

and we need to know how to compute each entry of this matrix. To do that, we recall the definition of $H(x,y)$, (1.80), which leads to the following

$$H_{j\,k} = \frac{e^{\frac{\varepsilon^2}{2}+s\varepsilon}\sin(\pi\gamma)}{2\pi^2}\left|x_j\,x_k\right|^{\frac{\gamma}{2}}e^{-\frac{s}{2}(x_j+x_k)}\sqrt{w_j\,w_k}\int_{\mathbb{R}}\mathrm{d}t\,\frac{(it+\varepsilon)^{-\gamma}\,e^{it(s+\varepsilon)}}{(x_j-\varepsilon-it)(x_k-\varepsilon-it)}e^{-\frac{t^2}{2}}. \tag{1.88}$$

In order to compute this integral, we once again apply the Gauss-Hermite quadrature, which gives the following result

$$\int_{\mathbb{R}}\mathrm{d}t\,\frac{(it+\varepsilon)^{-\gamma}\,e^{it(s+\varepsilon)}}{(x_j-\varepsilon-it)(x_k-\varepsilon-it)}e^{-\frac{t^2}{2}} = \sum_{\ell=1}^{2n}\frac{(ix_\ell+\varepsilon)^{-\frac{\gamma}{2}}\,e^{\frac{ix_\ell}{2}(s+\varepsilon)}}{(x_j-\varepsilon-ix_\ell)}\sqrt{w_\ell}\frac{(ix_\ell+\varepsilon)^{-\frac{\gamma}{2}}\,e^{\frac{ix_\ell}{2}(s+\varepsilon)}}{(x_k-\varepsilon-ix_\ell)}\sqrt{w_\ell}. \tag{1.89}$$

Note that, in the above, we use all the $2n$ nodes because the integral is over the whole $\mathbb{R}$. The matrix given by $K_{0\,1}K_{1\,0}$ can now be written as

$$\left[H_{j\,k}\right]_{j,k=1}^{n} = \frac{e^{\frac{\varepsilon^2}{2}+s\varepsilon}}{2\pi^2}\sin\left(\pi\gamma\right)\mathbb{A}\,.\,\mathbb{A}^T, \tag{1.90}$$

where $\mathbb{A}$ is a matrix whose entries are defined defined as

$$\mathbb{A}_{j\,\ell} = |x_j|^{\frac{\gamma}{2}}\,\sqrt{w_j}\,e^{-\frac{s}{2}x_j}\,\frac{\left(ix_\ell+\varepsilon\right)^{-\frac{\gamma}{2}}\,e^{i\frac{x_\ell}{2}(s+\varepsilon)}}{\left(x_j-\varepsilon-ix_\ell\right)}\sqrt{w_\ell}, \tag{1.91}$$

and $1 \leq j \leq n$ and $1 \leq \ell \leq 2n$. Regarding ε, this is a small parameter which we will have to assign a number to when doing the numerical estimates.

As a result of the procedure that has just been described, when performing numerical computations, our efforts will be centered in computing the term $\left[H\left(x_i^{(2n)},x_j^{(2n)}\right)\sqrt{w_i^{(2n)}w_j^{(2n)}}\right]_{i,j=1}^{n}$. This will allow us to compute the τ–function in terms of the parameters s, $\gamma\in[0,1)$ and n, which is the number of points that we want to consider for our estimate. Therefore, the final equation that we have for τ is

$$\tau\left(s,\gamma,n\right) = \det\left(\mathbb{1}_{n\times n} - \frac{e^{\frac{\varepsilon^2}{2}+s\varepsilon}}{2\pi^2}\sin\left(\pi\gamma\right)\mathbb{A}\,.\,\mathbb{A}^T\right). \tag{1.92}$$

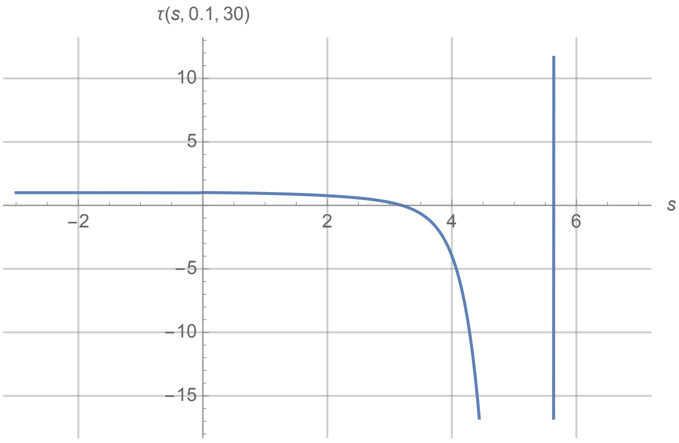
It should be noted that, since ε is a small number, we will consider that it will be such that $\varepsilon = \max\left[0,-s\right]$.

1.4.2 Numerical study of the τ-function

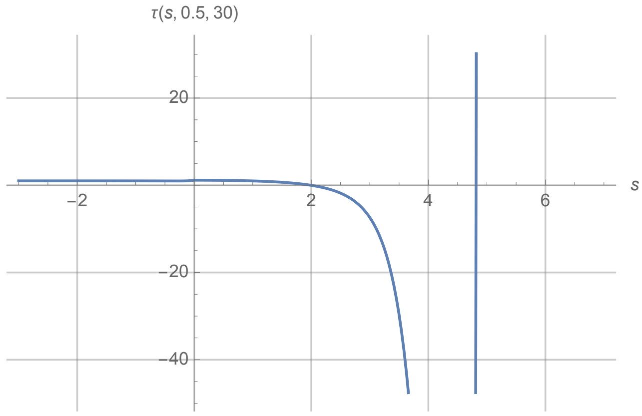
Having obtained (1.92), we will now use this as a tool to study numerically the τ–function of our special solution to Painlevé IV.

The numerical study that is going to be done in what follows will be accomplished using *Mathematica* and the code of the program that we used can be seen in Appendix A. The purpose of this study is to understand for which values of s the Fredholm Determinant, or equivalently the τ–function, is non-zero. As we may recall from Theorem 1.1, the Riemann-Hilbert problem has a solution if and only if the Fredholm Determinant is non-zero. Since its value depends on the parameters γ and s, we will study the τ–function according to these two parameters. We should also notice that, as it becomes clear from (1.92), since we are doing a numerical study, the values that we get for the τ–function will also be dependent on the number n of points that are considered.

We began by fixing a number of points $n = 30$. Our purpose was to see the behaviour of the τ–function over a range of values of s for different values of the parameter γ. Doing this for $\gamma = 0,1$ and $\gamma = 0,5$, we got the results for the cases that can be seen in Figures 1.4a and 1.4b, respectively. It is clear that, for s negative, irrespective of the value of γ, the τ–function is always non-zero and positive. However, after a certain value of s, the τ–function oscillates



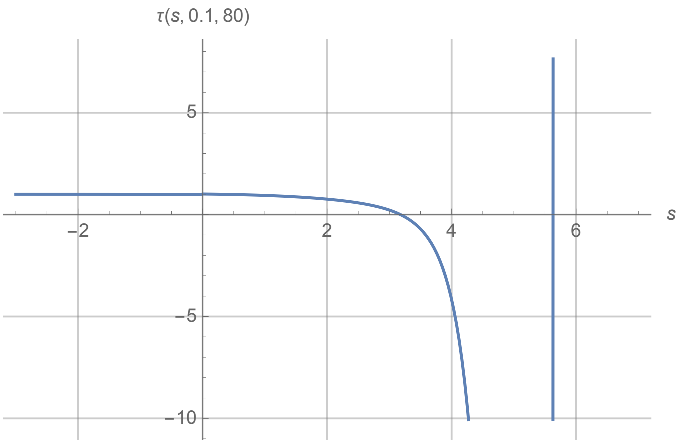
(a) $\gamma = 0,1$.



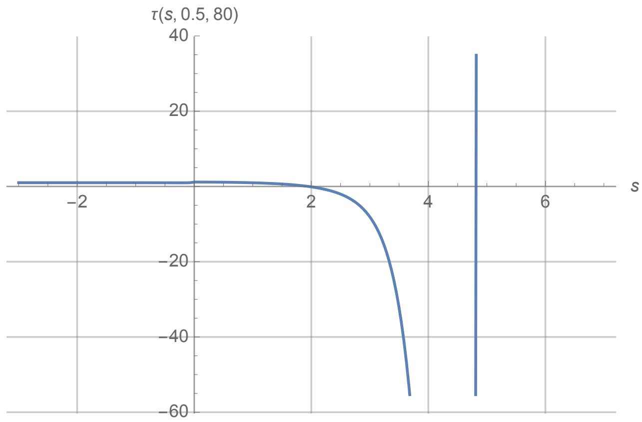
(b) $\gamma = 0,5$.

Figure 1.4: τ–function with $n = 30$ points.

and crosses the axis, which obviously means that, every time it is crossed, the τ–function will be 0 in that point. Therefore, it can be said that the Riemann-Hilbert problem has a solution for almost all values of s. It can also be seen that the exact values of the τ–function and the exact points where the s-axis is crossed depends on the value of γ that is considered. However, the general behaviour of the function seems to hold for both cases of γ that we considered.



(a) $\gamma = 0,1$.



(b) $\gamma = 0,5$.

Figure 1.5: τ–function with $n = 80$ points

We also wanted to know whether these results depended on the number of points that were considered in the evaluation. To do this, we repeated the same computations but now for the case of $n = 80$. Doing this for both $\gamma = 0,1$ and $\gamma = 0,5$ we got the results that can be seen in Figures 1.5a and 1.5b, respectively. By looking at these Figures, it is clear that the behaviour of the τ–function is the same. However, if one computes the value of τ at a given (s, γ, n)–point, it can be seen that the numeric values may change slightly. This is to be expected since increasing the number of points n corresponds to increasing the precision of the computation and therefore

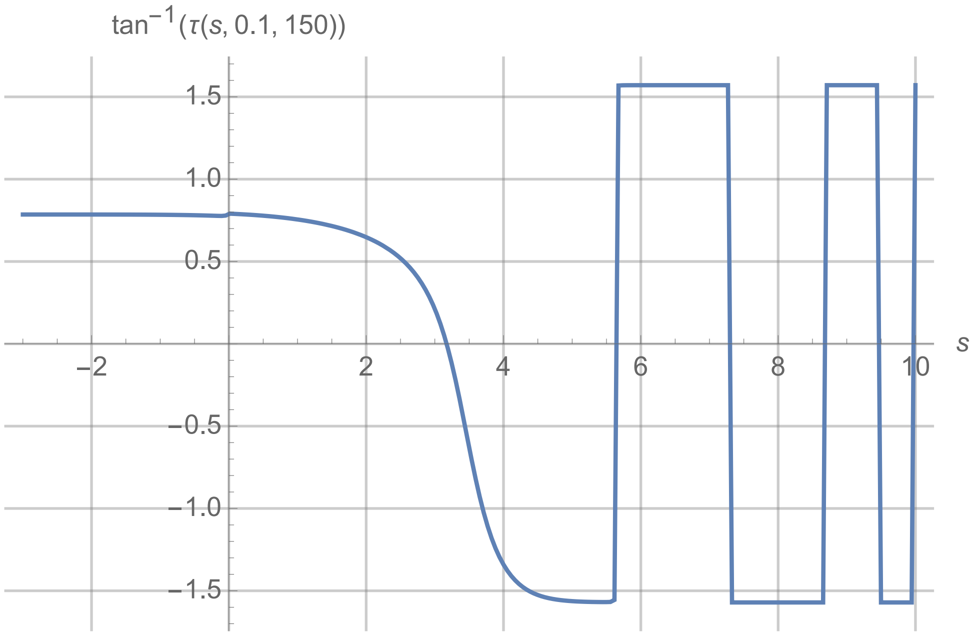


Figure 1.6: Arctan of the τ–function for $\gamma = 0,1$ and with $n = 150$ with an extended range of values for s being considered.

the exact values will change.

It should also be noticed that, after a given value of s, the τ–function becomes highly oscillatory. In order to see this, we increased the range of values of s being evaluated and considered the arctan of the τ–function, leading to the result that can be see in Figure 1.6. The reason for doing the arctan is the fact that the τ–function has very high (negative and positive) values. Therefore, taking the arctan allows us to see the results in a more compact way, since the values always have to be in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. From this we can see that, indeed, the τ–function becomes very oscillatory and it should be noted that, every time it crosses the s-axis, it will obviously have the value 0 at that point, meaning that it will not exist at such a point. Dealing with this situation would require us to introduce a triple-scaling limit.

2

Asymptotics of Orthogonal Polynomials

In the study of matrix models, it is natural to consider their associated *orthogonal polynomials*. For a given $N \in \mathbb{N}$ and an associated weight $e^{-NW(\lambda)}$, one can consider the sequence of *orthogonal polynomials* $\{p_n\}$ as being defined by the condition that p_j is a monic polynomial of degree n that satisfies the following set of relations

$$\int_{\mathbb{C}} p_n(\lambda)\,\overline{p_m(\lambda)} e^{-NW(\lambda)}\,\mathrm{d}A(\lambda) = h_{n,N}\,\delta_{n,m}, \quad h_{n,N} > 0, \quad n, m = 0, 1, \ldots, \tag{2.1}$$

where $\mathrm{d}A(z)$ is the area measure in the complex plane, $V : \mathbb{C} \to \mathbb{R}$ is the external potential and $h_{n,N}$ is called the norming constant. Regarding the external potential, we assume that its growth at infinity is such that the integrals in (2.1) are bounded.

Planar orthogonal polynomials satisfying (2.1) appear naturally in the context of normal matrix models [10], where one studies probability distributions of the form

$$M \to \frac{1}{\mathcal{Z}_{n,N}} e^{-N\,\mathrm{Tr}(W(M))}\,\mathrm{d}M, \qquad \mathcal{Z}_{n,N} = \int_{\mathcal{N}_n} e^{-N\,\mathrm{Tr}(W(M))}\,\mathrm{d}M, \tag{2.2}$$

where $\mathcal{N}_n$ is the algebraic variety of $n \times n$ normal matrices

$$\mathcal{N}_n = \{M \;:\; [M, M^\star] = 0\} \subset \mathrm{Mat}_{n\times n}(\mathbb{C}) \tag{2.3}$$

and $\mathrm{d}M$ is the volume form induced on $\mathcal{N}_n$, which is invariant under conjugation by unitary matrices. Since normal matrices are diagonalizable by unitary transformations, the probability density (2.2) can be reduced to the form [24]

$$\frac{1}{Z_{n,N}} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_{j=1}^n W(\lambda_i)}\,\mathrm{d}A(\lambda_1)\ldots\mathrm{d}A(\lambda_n),$$

where λ_j are the complex eigenvalues of the normal matrix M and the normalizing factor $Z_{n,N}$, called partition function, is given by

$$Z_{n,N} = \int_{\mathbb{C}^n} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_{j=1}^n W(\lambda_i)}\,\mathrm{d}A(\lambda_1)\ldots\mathrm{d}A(\lambda_n).$$

The statistical quantities related to eigenvalues can be expressed in terms of the orthogonal polynomials $p_n(\lambda)$ defined in (2.1). In particular, the average density of eigenvalues is

$$\rho_{n,N}(\lambda) = \frac{1}{n} e^{-NW(\lambda)} \sum_{j=0}^{n-1} \frac{1}{h_{j,N}} |p_j(\lambda)|^2 . \tag{2.4}$$

The density of eigenvalues, $\rho_{n,N}(\lambda)$, converges in the limit

$$n \to \infty , \quad N \to \infty , \quad \frac{N}{n} \to \frac{1}{T} , \tag{2.5}$$

to the unique probability measure μ^* in the plane which minimizes the functional [13], [20]

$$I(\mu) = \iint \log |\lambda - \eta|^{-1} \, \mathrm{d}\mu(\lambda) \, \mathrm{d}\mu(\eta) + \frac{1}{T} \int W(\lambda) \, \mathrm{d}\mu(\lambda) . \tag{2.6}$$

The functional $I(\mu)$ in (2.6) is the Coulomb energy functional in two dimensions and the existence of a unique minimizer is a well-established fact under mild assumptions on the potential $W(\lambda)$, [25]. If W is twice continuously differentiable and its Laplacian, ΔW, is non-negative, the equilibrium measure is given by

$$\mathrm{d}\mu^*(\lambda) = \Delta W(\lambda) \, \chi_D(\lambda) \, \mathrm{d}A(\lambda) ,$$

where χ_D is the characteristic function of the compact support set $D = \mathrm{supp}(\mu^*)$. When $W(\lambda) = \lambda^2$, one has the Ginibre ensemble [18] and the measure $\mathrm{d}\mu^*(\lambda)$ is the uniform measure on the disk of radius $\sqrt{T}$.

The purpose of this chapter will be to study the strong asymptotics of the polynomials $p_n(\lambda)$, which are orthogonal with respect to an associated weight $e^{-NW(\lambda)}$, following the method developed in [2]. In what follows, we will consider the external potential to be of the form

$$W(\lambda) = |\lambda|^{2d} - t\,\lambda^d - \bar{t}\,\bar{\lambda}^d, \quad \lambda \in \mathbb{C}, \tag{2.7}$$

where r is a positive integer and $t \in \mathbb{C}^*$. As it can be seen, this potential has a discrete rotational $\mathbb{Z}_d$-symmetry, therefore, by rotating the variable λ, the analysis of the problem can be reduced for the case where t is real and positive. Without loss of generality, we can now take $t \in \mathbb{R}_+$ and have the external potential written as

$$W(\lambda) = |\lambda|^{2d} - t\left(\lambda^d + \bar{\lambda}^d\right), \quad \lambda \in \mathbb{C}, \tag{2.8}$$

leading to the associated orthogonality measure

$$e^{-N\left(|\lambda|^{2d} - t\left(\lambda^d + \bar{\lambda}^d\right)\right)} \, \mathrm{d}A(\lambda) . \tag{2.9}$$

It is now useful to recall some results regarding potentials with discrete rotational symmetries that were observed in [3] and [14]. Considering a potential with discrete d-symmetry, such as the one we wrote in (2.7), if it can be written in the form

$$W(\lambda) = \frac{1}{d} Q\left(\lambda^d\right) , \tag{2.10}$$

then the equilibrium measure for W can be obtained from the equilibrium measure of Q by an unfolding procedure. Considering the particular case of (2.7) that we are interested in, the potential Q can be written as

$$\begin{aligned} Q(u) &= d\,|u|^2 - d\,t\,(u+\bar{u}) \\ &= d\,|u-t|^2 - t^2\,d, \end{aligned}$$

which corresponds to the Ginibre ensemble, being the equilibrium measure of the potential Q the normalized area measure of the disk

$$|u-t| = t_c, \quad t_c = \sqrt{\frac{T}{d}}. \tag{2.11}$$

Regarding the equilibrium measure of W, it is given by

$$\mathrm{d}\mu_W = \frac{d}{\pi\,t_c^2}\,|\lambda|^{2(d-1)}\,\chi_D\,\mathrm{d}A(\lambda)\,, \tag{2.12}$$

where $\mathrm{d}A$ is the area measure and χ_D is the characteristic function of the domain D. This is a domain that is established by

$$D := \left\{\lambda\in\mathbb{C},\, |\lambda^d - t| \le t_c\right\}. \tag{2.13}$$

A plot of the domain D for different values of t is shown in figure 2.1. This set D is the support

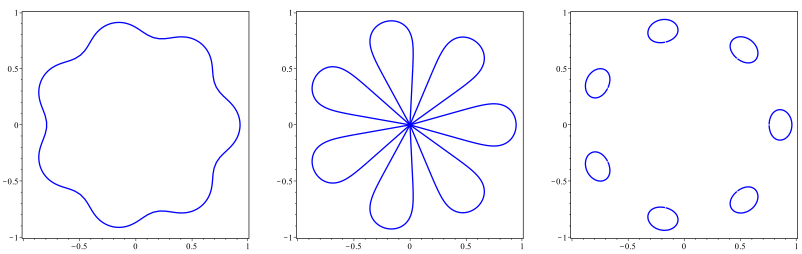


Figure 2.1: The domain D for three different values of t, on the left $t < t_c$, in the center $t = t_c$ and on the right $t > t_c$.

set of μ_W and can be described for the following two different regimes

- pre-critical $|t| < t_c$: (2.13) defines a simply connected domain in the complex plane described by the following uniformizing map from the exterior of the unit circle to the exterior of D

$$f(\zeta) = t_c^{1/d}\zeta\left(1+\frac{t}{t_c}\frac{1}{\zeta^d}\right)^{\frac{1}{d}}, \quad f^{-1}(\zeta) = F(\lambda) = \frac{\lambda}{t_c^{\frac{1}{d}}}\left(1-\frac{t}{\lambda^d}\right)^{\frac{1}{d}}. \tag{2.14}$$

- post-critical $|t| > t_c$: (2.13) defines a multiply connected domain that is made of d components that have discrete rotational symmetry. The boundary of the domain D can be described by

$$\bar{\lambda} = S(\lambda)\,, \quad S(\lambda) = \left(t + \frac{t_c^2}{\lambda^d - t}\right)^{\frac{1}{d}}. \tag{2.15}$$

As we have just seen, the problem that we have contains two different regimes for the pre-critical, $t < t_c$, and post-critical, $t > t_c$, cases. In the work of Balogh, Grava and Merzi [2], the strong asymptotics of the orthogonal polynomials were studied in both of these regimes.

In the work we will be presenting in this chapter, we will be considering the asymptotic behaviour of solutions in the critical case, $t = t_c$, in order to understand how the distribution of the zeros of the orthogonal polynomials behaves in this situation.

Before stating our main results, let us introduce the function

$$\hat{\varphi}(\lambda) = \log\left(t_c - \lambda^d\right) + \frac{\lambda^d}{t_c} - \log t_c \tag{2.16}$$

and let us consider the level curve $\hat{\mathcal{C}}$

$$\hat{\mathcal{C}} := \left\{\lambda \in \mathbb{C},\ \mathrm{Re}\ \hat{\varphi}(\lambda) = 0,\ \left|\lambda^d - t_c\right| \leq t_c\right\}. \tag{2.17}$$

Observe that the level curve $\hat{\mathcal{C}}$ consists of a closed contour contained in the set D for $t = t_c$, where D has been defined in (2.13). Define the measure $\hat{\nu}$ associated with this family of curves given by

$$\mathrm{d}\hat{\nu}(\lambda) = \frac{1}{2\pi i d}\mathrm{d}\hat{\varphi}(\lambda) \tag{2.18}$$

and supported on $\hat{\mathcal{C}}$.

Lemma 2.1. *The a-priori complex measure* $\mathrm{d}\hat{\nu}$ *in* (2.18) *is a probability measure on the contour* $\hat{\mathcal{C}}$ *defined in* (2.17).

Our result is summarized in the following theorem.

Theorem 2.2. *The zeros of the polynomials* $p_n(\lambda)$ *defined in* (2.1) *for* $t = t_c = \sqrt{T/d}$, *behave as follows*

- *for* $n = kd + d - 1$, *let* $\omega = e^{\frac{2\pi i}{d}}$. *Then* $t^{\frac{1}{d}}, \omega t^{\frac{1}{d}}, \ldots, \omega^{k-1} t^{\frac{1}{d}}$ *are zeros of the polynomials* p_{kd+d-1} *with multipicity* k *and* $\lambda = 0$ *is a zero with multiplicity* $r - 1$.

- *for* $n = kd + \ell$, $\ell = 0, \ldots, r-2$ *the polynomial* $p_n(\lambda)$ *has a zero in* $\lambda = 0$ *with multiplicity* ℓ *and the remaining zeros in the limit* n, $N \to \infty$ *such that*

$$N = \frac{n - \ell}{T} \tag{2.19}$$

accumulates on the level curve $\hat{\mathcal{C}}$ *defined in (2.17), namely*

$$\hat{\mathcal{C}} := \{\lambda \in \mathbb{C} : \quad \left|\left(t_c - \lambda^d\right)\exp\left(\frac{\lambda^d}{t_c}\right)\right| = t_c, \quad \left|\lambda^d - t\right| \leq t_c\}. \tag{2.20}$$

The measure $\hat{\nu}$ *in* (2.18) *is the weak-star limit of the normalized zero counting measure* ν_n *of the polynomials* p_n *for* $n = kd + \ell$, $\ell = 0, \ldots, d-2$.

Remark 2.3. We observe that the curve (2.20) in the rescaled variable $z = 1 - \lambda^d/t_c$ takes the form

$$\mathcal{C} := \left\{ z \in \mathbb{C} : \quad \left| z\, e^{1-z} \right| = 1 \right\} . \tag{2.21}$$

The curve $\mathcal{C}$ is the Szegő curve that was first observed in relation to the zeros of the Taylor polynomials of the exponential function [27].

Our next result gives strong uniform asymptotics as $n \to \infty$ for the polynomials $p_n(\lambda)$ in the complex plane. We consider the double scaling limit such that

$$t \to t_c, \quad k \to \infty, \quad k = \frac{n-\ell}{d},$$

in such a way that the quantity

$$\lim_{k\to\infty,\, t\to t_c} \sqrt{k} \left(\frac{t^2}{t_c^2} - 1 \right) \to \mathcal{S},$$

with $\mathcal{S}$ in compact subsets of the complex plane. In the description of the asymptotic behaviour of the orthogonal polynomials, $p_n(\lambda)$, in this double-scaling limit, the Painlevé IV transcendent with $\Theta_0 = \Theta_\infty = \frac{\gamma}{2}$ and $\gamma = \frac{d-\ell-1}{d} \in (0,1)$ will play a major role.

Theorem 2.4 (Double scaling limit)**.** *The polynomials $p_n(\lambda)$ with $n = k\,d + \ell$, $\ell = 0, \dots, d-2$, $\gamma = \frac{d-\ell-1}{d} \in (0,1)$, have the following asymptotic behaviour when n, $N \to \infty$ in such a way that $N\,T = n - \ell$ and*

$$\lim_{k\to\infty,\, t\to t_c} \sqrt{k} \left(\frac{t^2}{t_c^2} - 1 \right) \to \mathcal{S},$$

with $\mathcal{S}$ in compact subsets of the real line so that the solution $Y(\mathcal{S})$ of the Painlevé IV equation (1.6) *does not have poles. Below, the function $Z = Z(\mathcal{S})$, $U = U(\mathcal{S})$ and the Hamiltonian $H = H(\mathcal{S})$ are related to the Painlevé IV equation (1.6) by the relations (1.5) and (1.10), respectively.*

(1) For λ in compact subsets of the exterior of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell \left(\lambda^d - t_c \right)^k \left(\frac{\lambda^d - t_c}{\lambda^d} \right)^\gamma \left(1 - \frac{H(\mathcal{S}) t_c}{\sqrt{k}\, \lambda^d} + \mathcal{O}\left(\frac{1}{k} \right) \right), \tag{2.22}$$

with $H(\mathcal{S})$ the Hamiltonian (1.10) of the Painlevé IV equation (1.6).

(2) For λ in the region near $\hat{\mathcal{C}}$ and away from the point $\lambda = 0$ one has

$$p_n(\lambda) = \lambda^\ell \left(\lambda^d - t_c \right)^k \left(\frac{\lambda^d - t_c}{\lambda^d} \right)^\gamma \left(1 - \frac{H(\mathcal{S}) t_c}{\sqrt{k}\, \lambda^d} + \frac{Z(\mathcal{S})}{U(\mathcal{S})} \frac{t_c\, e^{-k\,\hat{\varphi}(\lambda)}}{\lambda^d\, k^{\frac{1+\gamma}{2}}} \left(\frac{\lambda^d - t_c}{\lambda^d} \right)^{-\gamma} + \mathcal{O}\left(\frac{1}{k} \right) \right), \tag{2.23}$$

with $\hat{\varphi}(\lambda)$ defined in (2.16).

(3) For λ in compact subsets of the interior region of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell(\lambda^d - t_c)^k \frac{e^{-k\hat{\varphi}(\lambda)}}{k^{\frac{1}{2}+\gamma}} \left(\frac{Z(\mathcal{S})}{U(\mathcal{S})} \frac{t_c}{\lambda^d} + \mathcal{O}\left(\frac{1}{k}\right)\right). \tag{2.24}$$

(4) In the neighbourhood of the point $\lambda = 0$ and in the interior of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell(\lambda^d - t)^k \left(\frac{\lambda^d - t_c}{\lambda^d}\right)^\gamma e^{-k\hat{\varphi}(\lambda)} \left(\frac{\widehat{\Psi}_{11}(\sqrt{-k\,\hat{\varphi}(\lambda)};\mathcal{S})}{k^{\frac{\gamma}{4}}\sqrt{-\hat{\varphi}(\lambda)}} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{2.25}$$

where $\widehat{\Psi}_{11}$ is the 11 entry of the deformed Painlevé IV Riemann-Hilbert problem (3.4.4) obtained by deforming the Riemann-Hilbert problem (1.2.1) with Stokes multipliers specified in (1.15) and (1.16).

(5) In the neighbourhood of the point $\lambda = 0$ and in the exterior of $\hat{\mathcal{C}}$ one has

$$p_n(\lambda) = \lambda^\ell(\lambda^d - t)^k \left(\frac{\lambda^d - t_c}{\lambda^d}\right)^\gamma \left(\frac{\widehat{\Psi}_{11}(\sqrt{-k\,\hat{\varphi}(\lambda)};\mathcal{S})}{k^{\frac{\gamma}{4}}\sqrt{-\hat{\varphi}(\lambda)}} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{2.26}$$

where the notation is as above in point (4).

We observe that, in compact subsets of the exterior of $\hat{\mathcal{C}}$, there are no zeros of the polynomials $p_n(\lambda)$. The only possible zeros are located in $\lambda = 0$ and in the region where the second term in parenthesis in the expression (2.23) is of order one. Since $\operatorname{Re}\hat{\varphi}(\lambda)$ is negative inside $\hat{\mathcal{C}}$ and positive outside $\hat{\mathcal{C}}$, it follows that the possible zeros of $p_n(\lambda)$ lie inside $\hat{\mathcal{C}}$ and are determined by the condition

$$\operatorname{Re}\hat{\varphi}(\lambda) = -\frac{1+\gamma}{2}\frac{\log k}{k} + \frac{1}{k}\log\left(\left|\frac{\lambda^d}{\lambda^d - t_k}\right|^\gamma \left|\frac{t_c\,Z(\mathcal{S})}{\lambda^d\,U(\mathcal{S})}\right|\right) + \frac{1}{k^{\frac{3}{2}}}\operatorname{Re}\left(\frac{t_c\,H(\mathcal{S})}{\lambda^d}\right). \tag{2.27}$$

We conclude with the following proposition.

Proposition 2.5. *The support of the counting measure of the zeros of the polynomials $p_n(\lambda)$ in the limit $n \to \infty$ outside an arbitrary small disk surrounding the point $\lambda = 0$ tends uniformly to the curve $\hat{\mathcal{C}}$ defined in* (2.20). *The zeros are within a distance $\mathcal{O}(1/k)$ from the curve defined by* (2.20).

In order to prove Theorem 2.2, Theorem 2.4 and Proposition 2.5 we will achieve the following results:

- exploiting the symmetry of the exponential weight $W(\lambda)$ we will reduce the polynomials $p_n(\lambda)$ to some polynomials $\pi_k(z)$ with $z = 1 - \lambda^d/t$;
- we will reformulate the orthogonality relations for the polynomials $\pi_k(z)$ to an orthogonality relation on a contour in the complex plane;

- we will introduce a Riemann-Hilbert problem for the polynomials $\pi_k(z)$;
- we will perform a double scaling limit $t \to t_c$ and $k \to \infty$ for the polynomials $\pi_k(z)$ using the Deift-Zhou steepest descent method;
- we solve the asymptotic Riemann-Hilbert problem obtained in the double scaling limit using the Painlevé IV isomonodromic problem for specific values of the Stokes multipliers and connection matrices.

2.1 Symmetry and Contour Integral Orthogonality

The purpose of this section will be to simplify the orthogonal polynomials that we are working with and to replace the two-dimensional integral conditions that we have in (2.1) with an equivalent set of orthogonality relations that are written in terms of contour integrals.

As we have seen above, the external potential (2.8) that we are considering leads to an associated orthogonality measure that exhibits a $\mathbb{Z}_d$ discrete rotational symmetry. The fact that such a symmetry exists will be reflected in the corresponding orthogonal polynomials. Considering the non-trivial orthogonality relations

$$\int_{\mathbb{C}} p_n(\lambda)\, \bar{\lambda}^{jd+\ell} e^{-NW(\lambda)}\, \mathrm{d}A(\lambda) = 0, \quad j = 0, \ldots, k-1, \tag{2.28}$$

where k and ℓ are such that

$$n = kd + \ell, \qquad 0 \le \ell \le d-1, \tag{2.29}$$

meaning that the n-th monic orthogonal polynomial satisfies the relation

$$p_n(e^{\frac{2\pi i}{d}}\lambda) = e^{\frac{2\pi i n}{d}} p_n(\lambda). \tag{2.30}$$

Therefore, there is a monic polynomial $q_k^{(\ell)}$ of degree k that can be defined in the following way

$$p_n(\lambda) = \lambda^{\ell}\, q_k^{(\ell)}\left(\lambda^d\right). \tag{2.31}$$

We can now use this in order to split the initial sequence of orthogonal polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ in d sub-sequences, each of which labelled by the remainder $\ell \equiv n \bmod d$. Using this procedure, the asymptotics of the orthogonal polynomials can be studied through the sequences of reduced polynomials

$$\left\{q_k^{(\ell)}(u)\right\}_{k=0}^{\infty}, \qquad \ell = 0, 1, \ldots, d-1. \tag{2.32}$$

We can now use this sequence of monic orthogonal polynomials and through a change of coordinates see that they satisfy the orthogonality relations

$$\int_{\mathbb{C}} q_k^{(\ell)}(u)\, \bar{u}^j |u|^{-2\gamma} e^{-N\left(|u|^2 - tu - t\bar{u}\right)}\, \mathrm{d}A(u) = 0, \qquad j = 0, \ldots, k-1, \tag{2.33}$$

where we can see that there is orthogonality with respect to the measure

$$|u|^{-2\gamma}e^{-N(|u|^2-tu-t\bar{u})}\,\mathrm{d}A\,(u)\,,\quad \gamma:=\frac{d-\ell-1}{d}\in[0,1). \tag{2.34}$$

This procedure of symmetry reduction shows that, for a class of measures that exhibit $\mathbb{Z}_d$-symmetry such as (2.9), it is sufficient to consider polynomials that are orthogonal with respect to a family of measures of the form (2.34).

As we have stated above, the purpose of this chapter is to study the asymptotic behaviour of the orthogonal polynomials, $p_n\,(\lambda)$, in a particular regime. However, from the symmetry reduction procedure that we have just shown, it was seen that these polynomials can be split into d families of orthogonal polynomials, $q_k^{(\ell)}\left(\lambda^d\right)$. Therefore, it suffices to study the asymptotic behaviour of the orthogonal polynomials $q_k^{(\ell)}(u)$ that satisfy the orthogonality relation (2.33).

Having done this symmetry reduction, the following step will be to replace the two dimensional integral conditions that we have in (2.33) by an equivalent set of constraints written in terms of contour integrals. In order to do this, we begin by considering the change of coordinate

$$u=-t\,(z-1)\,,\quad z\in\mathbb{C}, \tag{2.35}$$

which leads to the transformed monic polynomial

$$\pi_k\,(z):=\frac{(-1)^k}{t^k}\,q_k^{(\ell)}\,(-t\,(z-1))\,, \tag{2.36}$$

which is defined in terms of a new variable z and is also a monic polynomial. These new polynomials are an equivalent characterization of $q_k(u)$ and are useful to establish the following Theorem.

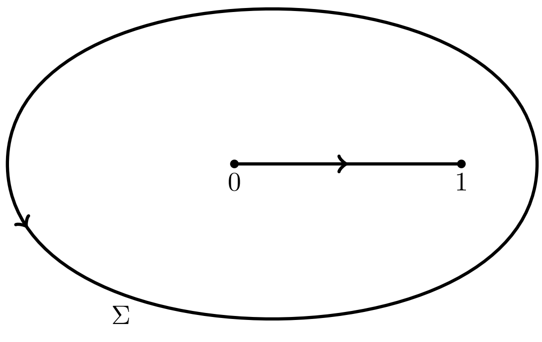


Figure 2.2: The contour Σ

Theorem 2.6. *[2] Consider $q_k^{(\ell)}\,(u)$ to be the monic polynomial of degree k with the orthogonality relations*

$$\int_{\mathbb{C}} q_k^{(\ell)}(u)\,\bar{u}^j|u|^{-2\gamma}e^{-N\left(|u|^2-tu-t\bar{u}\right)}\,\mathrm{d}A\,(u)=0,\qquad j=0,\dots,k-1$$

where $\gamma \in [0,1)$ *and* $t \in \mathbb{C}$*. The transformed monic polynomial of degree* k *given by*

$$\pi_k(z) := \frac{(-1)^k}{t^k}\, q_k^{(\ell)}\left(-t\,(z-1)\right),$$

is characterized by the non-hermitian orthogonality relations

$$\oint_\Sigma \pi_k(z)\, z^j \frac{e^{-Nt^2 z}}{z^k}\left(\frac{z}{z-1}\right)^\gamma \mathrm{d}z = 0, \qquad j = 0,1,\ldots,k-1, \tag{2.37}$$

where Σ *is a simple positively oriented contour encircling* $z=0$ *and* $z=1$*, as it can be seen in Figure 2.2, and the function* $\left(\frac{z}{z-1}\right)^\gamma$ *is analytic in* $\mathbb{C}\backslash[0,1]$ *and tends to one for* $|z| \to \infty$*.*

Proof. In order to prove the theorem we will have to use *Stokes' Theorem* to reduce the integration on the plane that we have in the orthogonality relation of the polynomials $q_k^\ell(u)$, (2.33), into an integration on a contour. This is done by introducing a function $\chi_j(u,\bar{u})$ that we require to be such that it solves the $\bar{\partial}$-problem

$$\partial_{\bar{u}}\, \chi_j(u,\bar{u}) = \bar{u}^j |u|^{-2\gamma} e^{-N\left(|u|^2 - tu - t\bar{u}\right)}. \tag{2.38}$$

Using this function and the exterior derivative $\mathrm{d}f(u,\bar{u})$ we will have for any polynomial $q(u)$ the following equality

$$\begin{aligned}
\mathrm{d}\left[q(u)\, \chi_j(u,\bar{u})\, \mathrm{d}u\right] &= q(u)\, \partial_{\bar{u}}\, \chi_j(u,\bar{u})\ \mathrm{d}\bar{u} \wedge \mathrm{d}u \\
&= q(u)\, \bar{u}^j |u|^{-2\gamma} e^{-N\left(|u|^2 - tu - t\bar{u}\right)}\, \mathrm{d}\bar{u} \wedge \mathrm{d}u.
\end{aligned}$$

Provided that such a function $\chi_j(u,\bar{u})$ exists, *Stokes' Theorem* can now be applied in order to reduce the orthogonality relation to one defined on an adequate contour. In fact, the contour integral solution of (2.38) is given by

$$\begin{aligned}
\chi_j(u,\bar{u}) &= u^{-\gamma} e^{Ntu} \int_0^{\bar{u}} s^{j-\gamma} e^{-Nus+Nts}\, \mathrm{d}s \\
&= \frac{1}{N^{j-\gamma+1}} \left(1 - \frac{t}{u}\right)^\gamma \frac{e^{Ntu}}{(u-t)^{j+1}} \int_0^{N\bar{u}(u-t)} \xi^{j-\gamma} e^{-\xi}\, \mathrm{d}\xi \\
&= \frac{1}{N^{j-\gamma+1}} \left(1 - \frac{t}{u}\right)^\gamma \frac{e^{Ntu}}{(u-t)^{j+1}} \left[\Gamma(j-\gamma+1) - \int_{N\bar{u}(u-t)}^\infty \xi^{j-\gamma} e^{-\xi}\, \mathrm{d}\xi\right] \\
&= \frac{\Gamma(j-\gamma+1)}{N^{j-\gamma+1}} \left(1 - \frac{t}{u}\right)^\gamma \frac{e^{Ntu}}{(u-t)^{j+1}} \left[1 - \mathcal{O}\left(e^{-N\bar{u}(u-t)}\right)\right], \quad |u| \to \infty\,.
\end{aligned}$$

This leads to the fact that, for any polynomial $q(u)$, the following integral identity holds

$$\begin{aligned}\int_{\mathbb{C}} q(u)\,\bar{u}^j |u|^{-2\gamma} e^{-N\left(|u|^2-tu-t\bar{u}\right)}\,\mathrm{d}A\,(u) &= \frac{1}{2i}\lim_{R\to\infty}\int_{|u|\le R} q(u)\,\bar{u}^j|u|^{-2\gamma}e^{-N\left(|u|^2-tu-t\bar{u}\right)}\,\mathrm{d}\bar{u}\wedge\mathrm{d}u\\ &= \frac{1}{2i}\lim_{R\to\infty}\oint_{|u|=R} q(u)\,\chi_j(u,\bar{u})\,\mathrm{d}u\\ &= \frac{1}{2i}\lim_{R\to\infty}\oint_{|u|=R} q(u)\left[G_j(u)-\mathcal{O}\left(e^{-\bar{u}(u-t)}\right)\right]\,\mathrm{d}u\\ &= \frac{1}{2i}\oint_{|z|=R_0} q(u)\,G_j(u)\,\mathrm{d}u,\end{aligned}$$

where R and R_0 are sufficiently large and

$$G_j(u) = \frac{\Gamma(j-\gamma+1)}{N^{j-\gamma+1}}\left(1-\frac{t}{u}\right)^{\gamma}\frac{e^{Ntu}}{(u-t)^{j+1}}. \tag{2.39}$$

Using these results it follows that, for any polynomial $q(u)$, the following identity is satisfied

$$\int_{\mathbb{C}} q(u)\,\bar{u}^j|u|^{-2\gamma}e^{-N\left(|u|^2-tu-t\bar{u}\right)}\,\mathrm{d}A\,(u) = \frac{\pi\,\Gamma(j-\gamma+1)}{N^{j-\gamma+1}}\frac{1}{2\pi i}\oint_{\tilde{\Sigma}} q(u)\,\frac{e^{Ntu}}{(u-t)^{j+1}}\left(1-\frac{t}{u}\right)^{\gamma}\mathrm{d}u, \tag{2.40}$$

where $\gamma\in(0,1)$, j is an arbitrary non-negative integer, and $\tilde{\Sigma}$ is a positively oriented simple closed loop enclosing $u=0$ and $u=t$. If we now make the change of coordinates $u=-t\,(z-1)$, the statement of the Theorem is achieved. □

2.2 The Riemann-Hilbert Problem

As we have said before, the purpose of this chapter is to study the asymptotic behaviour of the orthogonal polynomials. This amounts to computing the polynomials $\pi_k(z)$ in the limit $k\to\infty$ and $N\to\infty$ in such a way that, for $n=kd+\ell$, one has

$$T=\frac{n-\ell}{N}>0. \tag{2.41}$$

By looking at the terms in (2.37), it can be seen that it makes sense to introduce the function

$$V(z)=\frac{z}{z_0}+\log z, \tag{2.42}$$

where

$$z_0=\frac{t_c^2}{t^2},\quad t_c^2=\frac{T}{d}, \tag{2.43}$$

so that the following weight function can be defined

$$w_k(z):=e^{-kV(z)}\left(\frac{z}{z-1}\right)^{\gamma}. \tag{2.44}$$

Using this weight function and plugging it back in (2.37), the orthogonality relations can be written as

$$\oint_{\Sigma} \pi_k(z)\, z^j\, w_k(z)\; \mathrm{d}z = 0 \qquad j = 0, 1, \ldots, k-1. \tag{2.45}$$

When the limit $k \to \infty$ is taken, three different regimes arise

- *pre-critical* case: $0 < t < t_c$ leading to $z_0 > 1$
- *critical* case: $t = t_c$ leading to $z_0 = 1$
- *post-critical* case: $t > t_c$ leading to $z_0 < 1$

and in what follows we will be interested in analyzing the critical case. The method that we will follow to study the asymptotics of the orthogonal polynomials will be by establishing a matrix-valued Riemann-Hilbert problem and writing the polynomial $\pi_k(z)$ as a particular entry of the unique solution to this problem. We will now see how this is done.

We begin by defining the so-called complex moments as

$$\nu_j := \oint_{\Sigma} z^j\, w_k(z)\; \mathrm{d}z, \tag{2.46}$$

where the dependency on k is omitted in order to simplify notation, and use this to introduce the auxiliary polynomial

$$\Pi_{k-1}(z) := \frac{1}{\det\left[\nu_{i+j}\right]_{0\le i,j\le k-1}} \det \begin{bmatrix} \nu_0 & \nu_1 & \ldots & \nu_{k-1} \\ \nu_1 & \nu_2 & \ldots & \nu_k \\ \vdots & & & \vdots \\ \nu_{k-2} & \ldots & & \nu_{2k-3} \\ 1 & z & \ldots & z^{k-1} \end{bmatrix}. \tag{2.47}$$

It can be seen that this polynomial is not necessarily monic and its degree may be less than $k-1$. In order to guarantee the existence of such a polynomial, it has to be established that the determinant in the denominator does not vanish.

Proposition 2.7. *[2] The determinant* $\det[\nu_{i+j}]_{0\le i,j\le k-1}$ *does not vanish.*

Proof. Using the definition (2.46), it can be seen that

$$\begin{aligned} \det[\nu_{i+j}]_{0\le i,j\le k-1} &= \det\left[\oint_{\Sigma} z^{i+j}\frac{e^{-Nt^2 z}}{z^k}\left(\frac{z}{z-1}\right)^{\gamma} dz\right]_{0\le i,j\le k-1} \\ &= (-1)^{k(k-1)/2}\det\left[\oint_{\Sigma} z^{i-j}\frac{e^{-Nt^2 z}}{z}\left(\frac{z}{z-1}\right)^{\gamma} dz\right]_{0\le i,j\le k-1}, \end{aligned} \tag{2.48}$$

where, to obtain the last identity, the reflection of the column index $j \to k-1-j$ has been used. If we now use Theorem 2.6, it can be seen that

$$\begin{aligned}&\int_{\mathbb{C}} \pi(z)\,(\bar{z}-1)^j\,|z-1|^{-2\gamma} e^{-Nt^2|z|^2}\,\mathrm{d}A(z) = \\ &= t^{2-2j-2\gamma}\frac{\pi\,\Gamma(j-\gamma+1)}{N^{j-\gamma+1}}\frac{1}{2\pi i}\oint_{\Sigma}\pi(z)\,\frac{e^{-Nt^2z}}{z^{j+1}}\left(\frac{z}{z-1}\right)^{\gamma}\,\mathrm{d}z,\end{aligned} \tag{2.49}$$

and therefore the second determinant is given by

$$\begin{aligned}&\det\left[\oint_{\Sigma} z^{i-j}\frac{e^{-Nt^2z}}{z}\left(\frac{z}{z-1}\right)^{\gamma}\,\mathrm{d}z\right] = \\ &= \det\left[\iint_{\mathbb{C}} z^i\,(\bar{z}-1)^j\,|z-1|^{-2\gamma}e^{-Nt^2|z|^2}\,\mathrm{d}A(z)\right]\prod_{j=0}^{k-1}\frac{2\,i\,t^{2j+2\gamma-2}\,N^{j-\gamma+1}}{\Gamma\,(j-\gamma+1)}.\end{aligned} \tag{2.50}$$

It can now be seen that the determinant on the right-hand side is strictly positive since

$$\det\left[\iint_{\mathbb{C}} z^i\,(\bar{z}-1)^j\,|z-1|^{-2\gamma}e^{-Nt^2|z|^2}\,\mathrm{d}A(z)\right] = \det\left[\iint_{\mathbb{C}} z^i\bar{z}^j|z-1|^{-2\gamma}e^{-Nt^2|z|^2}\,\mathrm{d}A(z)\right] > 0, \tag{2.51}$$

where the equality is due to the fact that the columns of the two matrices are related by a unimodular triangular matrix, while the inequality follows from the positivity of the measure. Since $\Gamma(z)$ has no zeros (and no poles since $j-\gamma+1>0$), the non-vanishing follows from (2.50). □

Remark 2.8. Equivalently, $\Pi_{k-1}(z)$ is defined in the following way

$$\oint_{\Sigma}\Pi_{k-1}(z)\,z^l\,w_k(z)\,\mathrm{d}z = \begin{cases} 0, & \ell \le k-2 \\ \\ 1, & \ell = k-1 \end{cases} \tag{2.52}$$

3

Riemann–Hilbert analysis

3.1 Establishing the Riemann-Hilbert Problem

In this section we reformulate the condition of orthogonality for the polynomials $\pi_k(z)$ as a Riemann-Hilbert boundary value problem. This reformulation is essential to do the asymptotics analysis as $k \to \infty$.

We begin by defining the matrix

$$Y(z) = \begin{bmatrix} \pi_k(z) & \dfrac{1}{2\pi i} \displaystyle\int_\Sigma \frac{\pi_k(z')}{z'-z}\, w_k\big(z'\big)\, \mathrm{d}z' \\ -2\pi i\, \Pi_{k-1}(z) & -\displaystyle\int_\Sigma \frac{\Pi_{k-1}(z')}{z'-z}\, w_k\big(z'\big)\, \mathrm{d}z' \end{bmatrix}, \tag{3.1}$$

where the weight $w_k(z)$ has been defined in (2.44).

This matrix is the unique solution of following Riemann-Hilbert problem for orthogonal polynomials [15].

Riemann-Hilbert Problem 3.1.1. *1. Piecewise Analyticity: $Y(z)$ is analytic in $\mathbb{C} \setminus \Sigma$, where Σ is the oriented curve in Figure 3.1 and we identify the $+$ with the left side of the contour and $-$ with the right side. The limits $Y_\pm(z)$ exist and are continuous along Σ.*

2. Jump on Σ: The continuous boundary values $Y_\pm(z)$ are such that

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_k(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma. \tag{3.2}$$

This jump can be seen in Figure 3.1.

3. Behaviour at infinity: $Y(z)$ has the following behaviour as $z \to \infty$

$$Y(z) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{k\sigma_3}. \tag{3.3}$$

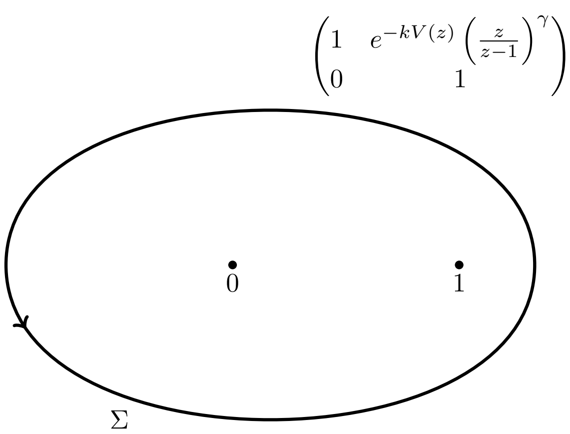


Figure 3.1: Jump in the $Y(z)$ Riemann-Hilbert problem

If the solution of the above Riemann-Hilbert problem exists, it is necessarily unique. Indeed, considering the determinant of (3.2), we obtain that

$$\det Y_+(z) = \det Y_-(z)\,, \quad z \in \Sigma.$$

It follows that $\det Y(z)$ is analytic across Σ and so it is analytic on the whole complex plane. Furthermore, since $\det Y(z) \to 1$ for $z \to \infty$, it follows, by Liouville Theorem, that $\det Y(z) = 1$ for $z \in \mathbb{C}$. Therefore, $Y(z)$ is an invertible matrix for $z \in \mathbb{C}$. Now assume that there are two solutions Y and $\tilde{Y}$ of the above Riemann-Hilbert problem. Then, the ratio $Y(z)\,\tilde{Y}^{-1}(z)$ has no jumps on the complex plane, and goes to the identity at infinity. It follows that $Y(z)\,\tilde{Y}^{-1}(z) = \mathbb{1}$, namely $Y(z) = \tilde{Y}(z)$.

Next, let us show that the solution of the above Riemann-Hilbert problem is given indeed by the matrix (3.1). Let us define $Y = (Y_1, Y_2)$ where Y_1 is the first column of Y and Y_2 is the second column. Then, from the relation (3.2), we have that

$$Y_{1+}(z) = Y_{1-}(z)\,, \quad z \in \Sigma,$$

and

$$Y_1(z) = \begin{pmatrix} z^k \\ \mathcal{O}\big(z^{k-1}\big) \end{pmatrix}.$$

Since $Y_1(z)$ is analytic in the complex plane and has the above behaviour at infinity, it follows that

$$Y_1(z) = \begin{pmatrix} P_k(z) \\ Q_{k-1}(z) \end{pmatrix}$$

for some polynomials $P_k(z) = z^k + \dots$ and $Q_{k-1}(z)$. Next we have

$$Y_{2+}(z) = Y_{2-}(z) + w_k(z)\begin{pmatrix} P_k(z) \\ Q_{k-1}(z) \end{pmatrix},$$

which implies, using Sokhotski-Plemelj formula

$$Y_2(z) = \frac{1}{2\pi i}\int_\Sigma \frac{w_k(z')\begin{pmatrix} P_k(z') \\ Q_{k-1}(z') \end{pmatrix}}{z'-z} dz'. \tag{3.4}$$

We also have the asymptotic condition

$$Y_2(z) = \begin{pmatrix} \mathcal{O}\left(z^{-k-1}\right) \\ z^{-k} + \mathcal{O}\left(z^{-k-1}\right) \end{pmatrix},$$

which is satisfied, by using (3.4), if

$$\int_\Sigma w_k(z)\, P_k(z)\, z^j \, \mathrm{d}z = 0, \quad j = 0, \dots k-1$$

and

$$\int_\Sigma w_k(z)\, Q_k(z)\, z^j \, \mathrm{d}z = 0, \quad j = 0, \dots k-2, \quad \int_\Sigma w_k(z)\, Q_k(z)\, z^{k-1} \, \mathrm{d}z = 1.$$

The above two relations coincide with the orthogonality relations (2.45) and (2.52) for the polynomials $\pi_k(z)$ and $\Pi_k(z)$, respectively. It follows that $P_k(z) \equiv \pi_k(z)$ and $Q_{k-1}(z) \equiv \Pi_{k-1}(z)$.

3.1.1 First Undressing Step

We will now proceed with a simplification of the Riemann-Hilbert problem that we have just established. This will be necessary in order to simplify the procedure that will follow.

We define a new matrix-valued function $\tilde{Y}(z)$ by

$$\tilde{Y}(z) := Y(z)\left(1 - \frac{1}{z}\right)^{-\frac{\gamma}{2}\sigma_3}, \quad z \in \setminus \left(\Sigma \cup [0,1]\right). \tag{3.5}$$

This matrix satisfies the following Riemann-Hilbert problem

Riemann-Hilbert Problem 3.1.2. *1. Piecewise Analyticity:*

$$\tilde{Y}(z) \text{ is analytic in } \mathbb{C} \setminus \left(\Sigma \cup [0,1]\right).$$

2. Jumps on Σ and $[0,1]$: Due to the definition of $\tilde{Y}(z)$ in (3.5), the jump in the Riemann Hilbert problem of $Y(z)$, (3.2), will now be transformed into

$$\tilde{Y}_+(z) = \tilde{Y}_-(z)\begin{cases} \begin{pmatrix} 1 & e^{-kV(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma \\ \\ e^{-\gamma\pi i \sigma_3}, & z \in (0,1) \end{cases} \tag{3.6}$$

where $V(z) = \frac{z}{z_0} + \log z$. This jump can be seen in Figure 3.2.

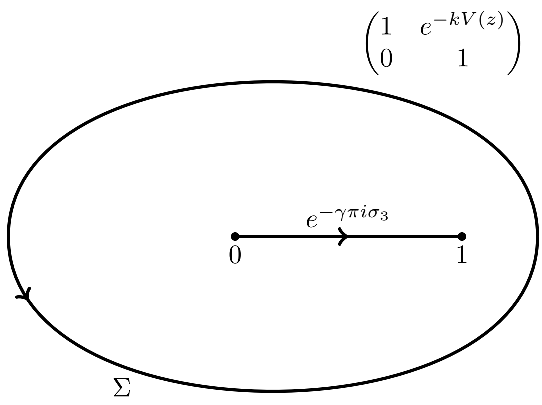


Figure 3.2: Jumps in the $\tilde{Y}(z)$ Riemann-Hilbert problem

3. Large z boundary behaviour:

$$\tilde{Y}(z) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{k\sigma_3}, \quad z \to \infty\,. \tag{3.7}$$

4. Endpoint Behaviour:

$$\tilde{Y}(z) = \mathcal{O}\,(1)\, z^{\frac{\gamma}{2}\sigma_3}\,, \quad z \to 0\ , \tag{3.8}$$

$$\tilde{Y}(z) = \mathcal{O}\,(1)\,(z-1)^{-\frac{\gamma}{2}\sigma_3}\ , \quad z \to 1\ . \tag{3.9}$$

It is easy to see that from this Riemann-Hilbert problem we can recover the polynomials

$$\pi_k(z) = \tilde{Y}_{11}(z)\left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}}\,. \tag{3.10}$$

The reason why we are doing this simplification is because we want to perform the analysis of the orthogonal polynomials in the large k limit, but this cannot be done through the matrix $Y(z)$. Therefore, we will have to apply several transformations to our problem, of which establishing $\tilde{Y}(z)$ is the first step. In what follows, we will proceed with further transformations and simplifications of our problem.

3.2 Transforming the Riemann-Hilbert Problem

We are now interested in performing transformations of the Riemann-Hilbert problem of the matrix $\tilde{Y}(z)$ that would allow us to perform the analysis of the large k behaviour. This will require us to use the steepest descent method of Deift-Zhou [12].

The first requirement that we want for the new Riemann-Hilbert problem, $U(z)$, is that it is normalized to the identity as $|z| \to \infty$. In order to do this, a new function $g(z)$ that is analytic off Γ, which is a new contour homotopically equivalent to Σ in $\mathbb{C} \setminus [0,1]$, is introduced, being both the function and the new contour unknown. We consider that the function $g(z)$ is of the form

$$g(z) = \int_\Gamma \log(z-s)\,\mathrm{d}\nu(s)\,, \tag{3.11}$$

where $\mathrm{d}\nu(s)$ is a positive measure with support on Γ that satisfies

$$\int_\Gamma \mathrm{d}\nu(s) = 1. \tag{3.12}$$

It can be seen that, in the limit $|z| \to \infty$, this function has the behaviour

$$g(z) = \log z + \mathcal{O}\left(z^{-1}\right), \tag{3.13}$$

where the logarithm is branched in the positive real axis.

3.2.1 Transformation $\tilde{Y} \to U$

We now want to proceed to establish the transformation $\tilde{Y} \to U$. In order to do this, we begin by deforming the contour Σ, which we had in the Riemann-Hilbert problem for $\tilde{Y}$, into the contour Γ. This can be done because the new contour Γ is homotopically equivalent to Σ in $\mathbb{C} \setminus [0,1]$.

We define the modified matrix U as

$$U(z) = e^{-k\frac{\ell}{2}\sigma_3}\,\tilde{Y}(z)\,e^{-k\,g(z)\,\sigma_3}\,e^{k\frac{\ell}{2}\sigma_3}, \quad z \in \mathbb{C} \setminus (\Gamma \cup [0,1])\,, \tag{3.14}$$

where ℓ is a real number to be determined. The matrix $U(z)$ solves the following Riemann-Hilbert problem

Riemann-Hilbert Problem 3.2.1. 1. *Piecewise Analyticity:*

$$U(z) \text{ is analytic in } \mathbb{C} \setminus (\Gamma \cup [0,1])\,.$$

2. *Jumps on Γ and $[0,1]$: As it can be seen in Figure 3.3, the Riemann-Hilbert problem has the following jumps*

$$U_+(z) = U_-(z) \begin{cases} \begin{pmatrix} e^{-k(g_+-g_-)} & e^{k(g_++g_--\ell-V)} \\ 0 & e^{k(g_+-g_-)} \end{pmatrix} & , \quad z \in \Gamma \\ \\ e^{-\gamma\pi i\sigma_3} & , \quad z \in (0,1) \end{cases} \tag{3.15}$$

3. *Large z boundary behaviour:*

$$U(z) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right)\right), \quad z \to \infty\,. \tag{3.16}$$

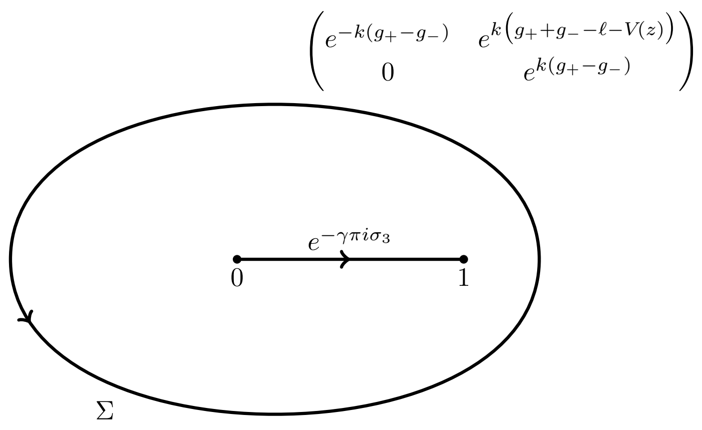


Figure 3.3: Jumps in the $U(z)$ Riemann-Hilbert problem

4. Endpoint Behaviour:

$$U(z) = \mathcal{O}(1)\, z^{\frac{\gamma}{2}\sigma_3}\,, \quad z \to 0\;, \tag{3.17}$$

$$U(z) = \mathcal{O}(1)\,(z-1)^{-\frac{\gamma}{2}\sigma_3}\;, \quad z \to 1\;. \tag{3.18}$$

The polynomials $\pi_k(z)$ can be recovered from this Riemann-Hilbert problem

$$\pi_k(z) = U_{1\,1}(z)\, e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}}\;. \tag{3.19}$$

It is now necessary to find the function $g(z)$ and the contour Γ. We want this to be done in a way such that the jump matrix (3.15) for $z \in \Gamma$ becomes purely oscillatory in the limit of large k. This can be done by requiring that the following conditions are satisfied

$$\begin{aligned} g_+(z) + g_-(z) - \ell - V(z) = 0 \\ \mathrm{Re}\,(g_+(z) - g_-(z)) = 0 \end{aligned} \qquad z \in \Gamma. \tag{3.20}$$

We will now show that such a function $g(z)$ and contour Γ that satisfy these conditions can be found. To do that, we will refer to [17] and use the following Lemma

Lemma 3.1. *Let L be a simple closed contour dividing the complex plane in two regions D_+ and D_-, where $D_+ = \mathrm{Int}(L)$ and $D_- = \mathrm{Ext}(L)$. Suppose that a function $\varphi(\zeta)$ defined on L can be represented in the form*

$$\varphi(\zeta) = \psi_+(\zeta) + \psi_-(\zeta)\,, \quad \zeta \in L, \tag{3.21}$$

where $\psi_+(\zeta)$ is analytic for $z \in D_+$ and continuous on L and $\psi_-(\zeta)$ is analytic for $z \in D_-$, continuous on L and such that $\psi_-(\infty) = 0$. Then, the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \tag{3.22}$$

can be represented in the form

$$\begin{aligned} \Phi_+(z) &= \psi_+(z)\,, \quad z \in D_+, \\ \Phi_-(z) &= -\psi_-(z)\,, \quad z \in D_- \end{aligned} \tag{3.23}$$

and the boundary values of the function Φ *on the two sides of the contour* L *satisfy*

$$\begin{aligned} \Phi_+(\zeta) + \Phi_-(\zeta) &= \psi_+(\zeta) - \psi_-(\zeta) \\ \Phi_+(\zeta) - \Phi_-(\zeta) &= \psi_+(\zeta) + \psi_-(\zeta) \end{aligned} \qquad \zeta \in L. \tag{3.24}$$

Our purpose is to use this Lemma in order to determine $g(z)$. More precisely, we consider $g'(z)$ as being the function that satisfies the differentiated boundary condition

$$g'_+(\zeta) + g'_-(\zeta) = V'(\zeta) = \frac{1}{z_0} + \frac{1}{\zeta}, \quad \zeta \in \Gamma. \tag{3.25}$$

By relating this with the function ψ in (3.21), we have that $\psi_+ = \frac{1}{z_0}$ and $\psi_- = -\frac{1}{z}$. Applying the statement of the Lemma, this means that $g'(z)$ is given by

$$g'(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\frac{1}{z_0} - \frac{1}{\tau}}{\tau - z}\, \mathrm{d}\tau = \begin{cases} \dfrac{1}{z_0}, & z \in \mathrm{Int}(\Gamma) = D_+ \\[2ex] \dfrac{1}{z}, & z \in \mathrm{Ext}(\Gamma) = D_- \end{cases} \tag{3.26}$$

and it is now easy to see that the measure $\mathrm{d}\nu$ in $g(z)$ is given by

$$\mathrm{d}\nu(z) = \frac{1}{2\pi i} \left(\frac{1}{z} - \frac{1}{z_0} \right) \mathrm{d}z, \quad z \in \Gamma. \tag{3.27}$$

One can now perform the integration of (3.26) and use the asymptotic behaviour of $g(z)$, which is given in (3.13), in order to obtain the function of $g(z)$ on both sides of the contour

$$g(z) = \begin{cases} g_+ = \dfrac{z}{z_0} + \ell, & z \in \mathrm{Int}(\Gamma) \\[2ex] g_- = \log z, & z \in \mathrm{Ext}(\Gamma) \end{cases} \tag{3.28}$$

where ℓ is an integration constant and $\log z$ is analytic in $\mathbb{C} \setminus \mathbb{R}^+$. The value of ℓ can now be found by doing the integral in (3.11) for a specific value of $z \in \Gamma$, which we take to be $z = 0$, and deforming Γ to a circle of radius r

$$\ell = \log r - \frac{r}{z_0}, \quad r > 0. \tag{3.29}$$

If we now look at ℓ as a function $\ell(r)$, it can be seen that it has a maximum at $r = z_0$ and diverges to $-\infty$ for $r \to 0$ and $r \to \infty$. Therefore, it suffices to consider r such that $0 < r \leq z_0$.

Regarding the measure $\mathrm{d}\nu(z)$, it is normalized to one on any closed contour that contains the point $z = 0$ by the residue Theorem. It should be noted that the contour Γ has to be defined

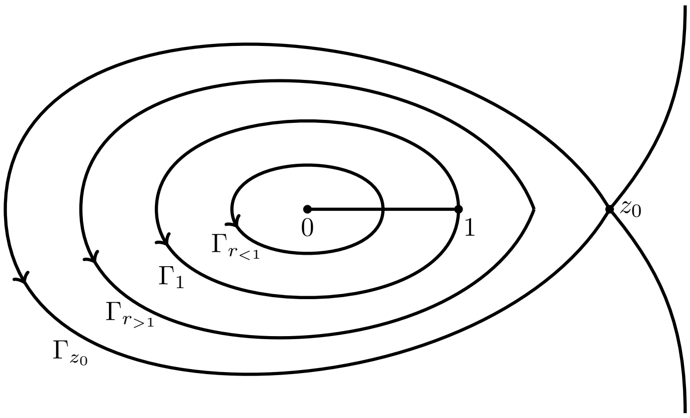


Figure 3.4: Family of contours Γ_r as defined in (3.34). Four possible cases are considered here according to their value in r: $r < 1$, $r = 1$, $1 < r < z_0$ and $r = z_0$

in such a way that the measure is real and positive on it. In order to do this, the following function is introduced

$$\varphi(z; r) = \log r - \frac{r}{z_0} + V - 2g(z)\,, \tag{3.30}$$

which can be seen to have the following behaviour depending on its location with respect to the contour

$$\varphi(z; r) = \begin{cases} \log z - \dfrac{z}{z_0} - \log r + \dfrac{r}{z_0}, & z \in \mathrm{Int}(\Gamma) \\ \\ \dfrac{z}{z_0} - \log z + \log r - \dfrac{r}{z_0}, & z \in \mathrm{Ext}(\Gamma) \end{cases}. \tag{3.31}$$

It can be seen that

$$\frac{1}{2}\left(\varphi_- - \varphi_+\right) = \ell + \frac{z}{z_0} - \log z = g_+ - g_-, \quad z \in \Gamma. \tag{3.32}$$

We now recall the condition (3.20) that we imposed before. Applied to this equation we are lead to

$$\begin{aligned} \mathrm{Re}\left(\varphi(z; r)\right) &= \mathrm{Re}\left(\ell + \frac{z}{z_0} - \log z\right) \\ &= \log r - \frac{r}{z_0} + \frac{\mathrm{Re}\,(z)}{z_0} - \log|z| = 0. \end{aligned} \tag{3.33}$$

It can be seen that this equation defines a family of contours that are closed in the case $|z| \le r$. We define the contour Γ_r as

$$\Gamma_r = \{z \in \mathbb{C} : \mathrm{Re}\left(\varphi(z; r)\right) = 0,\, |z| \le z_0\}\,, \quad 0 < r \le z_0. \tag{3.34}$$

The contours Γ_r for several values of r are plotted in Figure 3.4. The measure $\mathrm{d}\nu$ is clearly real on the contour Γ_r. In order to show that the measure $\mathrm{d}\nu$ is positive on such contours, we observe that by defining $\psi_r = e^{\varphi_r}$ one has that $|\psi_r| = 1$ on the contour Γ_r and

$$\mathrm{d}\nu = \frac{1}{2\pi i}\frac{\mathrm{d}\psi_r}{\psi_r}.$$

Therefore, in the variable ψ_r the measure $\mathrm{d}\nu$ is a uniform measure on the circle and clearly positive. Since the map $\psi_r = e^{\varphi_r}$ is a univalent conformal map from the interior of Γ_r to the interior of the circle, it follows that $\mathrm{d}\nu$ is a positive measure on Γ_r.

Summarizing, we have got the measure $\mathrm{d}\nu$ and a family of contours Γ_r. In order to select uniquely a contour among all possible values of r we need to make further analysis.

3.2.2 Transformation $U \to T$

In order to proceed with the transformation of the Riemann-Hilbert problem, we will look at the jump matrix of $U(z)$ on $z \in \Gamma$ given by (3.15). It can be seen that it is not feasible for our purposes to take the limit $k \to \infty$ in this jump matrix. Therefore, we will now use the fact that this matrix can be factorized in the following way

$$\begin{aligned}\begin{pmatrix} e^{-k(g_+-g_-)} & e^{k(g_++g_--\ell-V)} \\ 0 & e^{k(g_+-g_-)} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ e^{k(\ell+V-2g_-)} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{k(\ell+V-2g_+)} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix},\end{aligned} \tag{3.35}$$

where, in order to write the second line, we have used the definition of the $\varphi(z)$ function defined in (3.30) and the function $g(z)$ defined in (3.28). Since we have performed this factorization of the jump matrix, we can now see each of these matrices as corresponding to a jump on a different but homotopic contour.

We will now consider three different loops Γ_i, Γ_r and Γ_e, so that the space is split into four different domains Ω_0, Ω_1, Ω_2 and Ω_∞, as it can be seen in Figure 3.5. Γ_i is in the interior of Γ_r and Γ_e is in the exterior. Using this we will now define a new matrix-valued function $T(z)$ in the following way

$$T(z) = \begin{cases} U(z)\,, & z \in \Omega_0 \cup \Omega_\infty \\ U(z)\begin{pmatrix} 1 & 0 \\ -e^{k\varphi(z)} & 1 \end{pmatrix}, & z \in \Omega_1 \\ U(z)\begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}, & z \in \Omega_2\,. \end{cases} \tag{3.36}$$

This matrix $T(z)$ satisfies the following Riemann-Hilbert problem

Riemann-Hilbert Problem 3.2.2. *1. Piecewise Analyticity:*

$$T(z) \ \textit{is analytic in}\ \mathbb{C} \setminus (\Gamma_i \cup \Gamma_r \cup \Gamma_e \cup [0,1])\,.$$

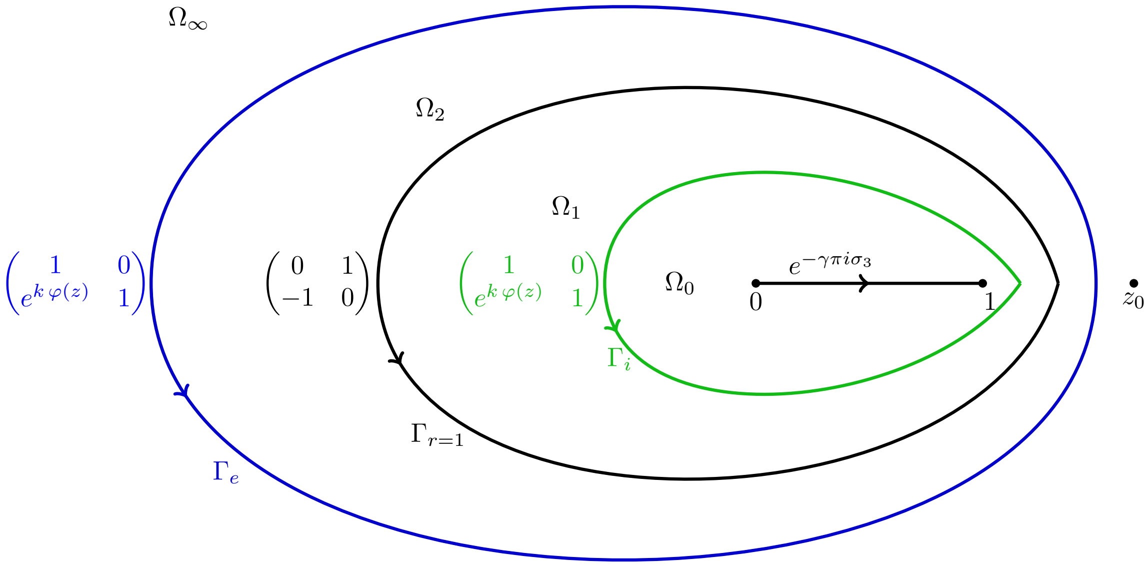


Figure 3.5: Jumps in the $T(z)$ Riemann-Hilbert problem

2. *Jumps on* $\Sigma_T = \Gamma_i \cup \Gamma_r \cup \Gamma_e \cup [0,1]$*:*

$$T_+(z) = T_-(z)\, v_T, \quad z \in \Sigma_T \tag{3.37}$$

where

$$v_T = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}, & z \in \Gamma_i \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Gamma_r \\ \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}, & z \in \Gamma_e \\ e^{-\gamma\pi i\sigma_3}, & z \in (0,1) \end{cases} \tag{3.38}$$

3. *Large z boundary behaviour:*

$$T(z) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right)\right), \quad z \to \infty\,. \tag{3.39}$$

4. *Endpoint Behaviour:*

$$T(z) = \mathcal{O}(1)\, z^{\frac{\gamma}{2}\sigma_3}, \quad z \to 0\,, \tag{3.40}$$

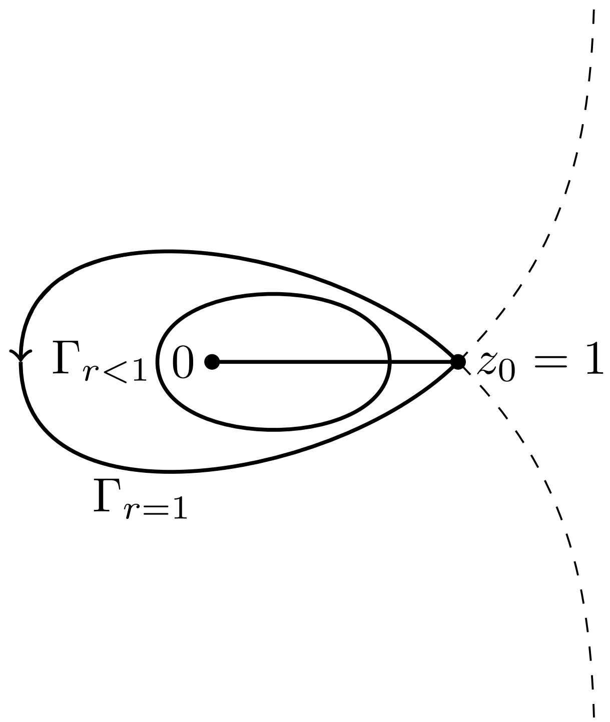


Figure 3.6: The contours Γ_r for $z_0 = 1$ and different values of $0 < r \leq 1$.

$$T(z) = \mathcal{O}(1)(z-1)^{-\frac{\gamma}{2}\sigma_3}, \quad z \to 1 . \tag{3.41}$$

It can be seen that $e^{\pm k\varphi(z)}$ is analytic in $\mathbb{C} \setminus \{0\}$. The goal is to choose the contour Γ_r in such a way that the jumps of the Riemann-Hilbert problems are either constant or exponentially decreasing in k as $k \to \infty$.

3.3 Choice of Contour

A crucial component in the analysis of the Riemann-Hilbert problems that we have been considering is the choice of the contours where the jumps are located. In fact, when introducing the Riemann-Hilbert problem $T(z)$, we have seen that there were three distinct contours, each of which with a distinct jump matrix assigned to it, that have to be considered Γ_i, Γ_r and Γ_e. However, up to this point, the three contours have remained mostly arbitrary, provided that they were around $[0, 1]$ and didn't go over z_0, as it can be seen in Figure 3.5. Since our analysis will be performed in the critical case when

$$z_0 \to 1,$$

we can see from Figure 3.6 that the choice of the contour Γ_r with $0 < r \leq z_0 = 1$ is uniquely defined. Indeed, the contours for $0 < r < 1$ cross the segment $[0, 1]$ and they have to discarded. So the only possible contour remains the case $r = 1$.

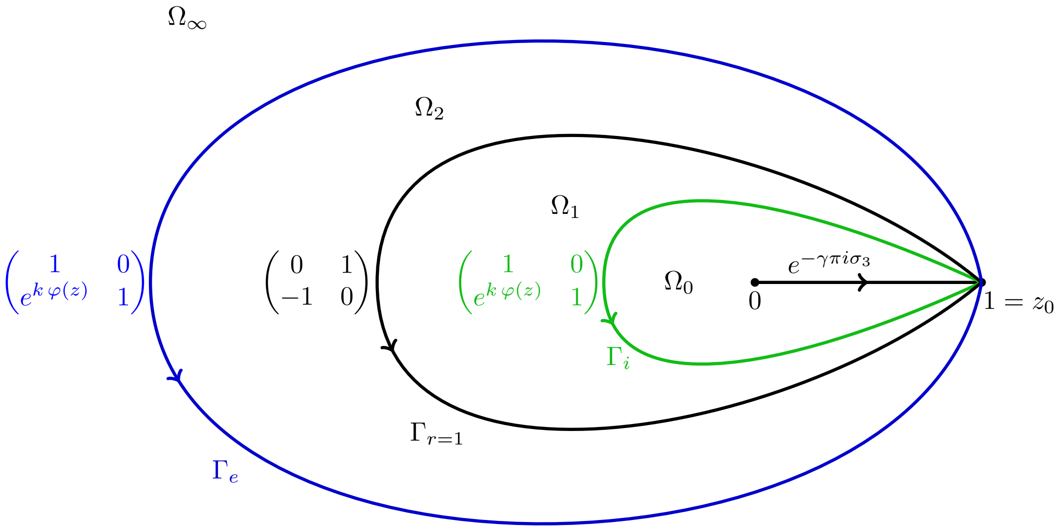


Figure 3.7: Contours in the critical case $z_0 = 1$.

The corresponding contours Γ_i and Γ_e have to be chosen in such a way that the jump matrix v_T defined in (3.38) goes exponentially fast to a constant. We make the choice of the deformed contours Γ_i and Γ_e as in Figure 3.7. In what follows, we argue that this is the right choice of contours. Indeed, we need to study the sign of $\operatorname{Re}\varphi(z)$ on the contours Γ_i and Γ_e. This is accomplished in Figure 3.8 where the region where $\operatorname{Re}\varphi(z) < 0$ is plotted. It can be concluded that the jump matrix on Γ_i and Γ_e goes exponentially fast to a constant matrix, except in a neighbourhood of the point $z = 1$ where $\varphi(z = 1) = 0$. For $z_0 > 1$ the same choice of contour $\Gamma_{r=1}$ has been considered in [2]. In this thesis we will consider the case $z_0 \to 1$, and we can stick to the set of contours considered in Figure 3.7.

Remark 3.2. We observe that the curve $\Gamma_{r=1}$ in (3.34) for $z_0 = 1$ takes the form

$$\mathcal{C} := \left\{ z \in \mathbb{C} : \quad \left| z e^{1-z} \right| = 1 \right\}. \tag{3.42}$$

The curve $\mathcal{C}$ is the Szegő curve that was first observed in relation to the zeros of the Taylor polynomials of the exponential function [27].

With this choice of contours we arrive to the following theorem

Theorem 3.3. *The jump matrix v_T converges exponentially fast as $k \to \infty$ to the constant*

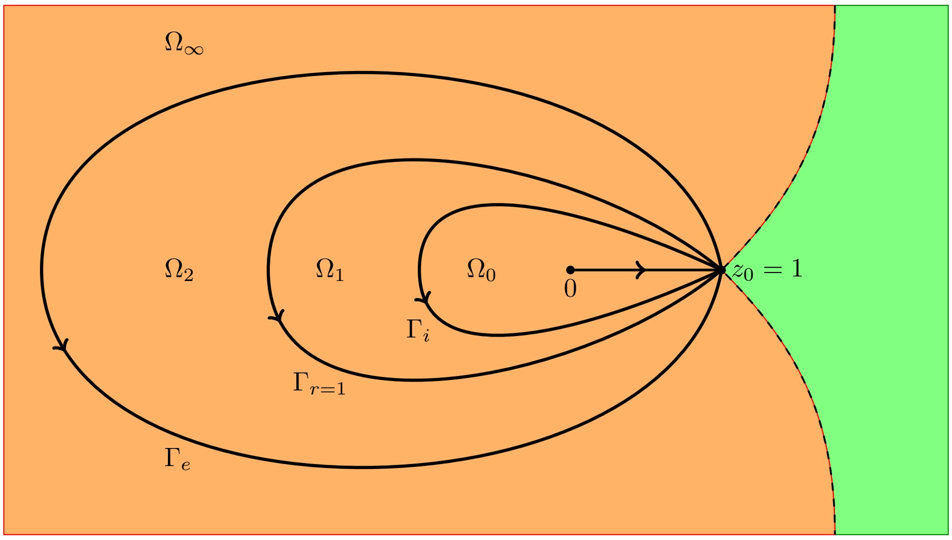


Figure 3.8: Contours Γ chosen for the critical case $z_0 = r = 1$. The orange region corresponds to the part of the complex plane where $\mathrm{Re}\,(\varphi(z)) < 0$ and the green contour is the region where $\mathrm{Re}\,(\varphi(z)) > 0$.

jump matrix

$$v_\infty = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in (\Gamma_e \cup \Gamma_i) \backslash \mathbb{D} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Gamma_1 \\ e^{-\gamma\pi i \sigma_3}, & z \in (0,1), \end{cases} \tag{3.43}$$

where $\mathbb{D}$ is a small circle surrounding the point $z = 1$.

Proof. The jumps of $v_T\,(z)$ on $\Gamma_{r=1}$ and $(0,1)$ are constant. The only thing we need to prove is the exponential convergence of the jumps on $(\Gamma_e \cup \Gamma_i)\backslash\mathbb{D}$ to the identity. Since $\mathrm{Re}\,\varphi(z) < 0$ for $z \in (\Omega_0 \cup \Omega_1 \cup \Omega_2)\,\backslash\mathbb{D}$ there exists a constant $c_0 > 0$ such that

$$v_T\,(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(e^{-kc}), \quad z \in (\Gamma_e \cup \Gamma_i)\backslash\mathbb{D}.$$

From this consideration it follows that

$$v_T\,(z) = v_\infty\,(z) + \mathcal{O}(e^{-kc}), \quad \text{as } \; k \to \infty, \quad z \in \Sigma_T\backslash\mathbb{D}.$$

□

3.4 Approximate solutions to $T(z)$

Now we are ready to approximate the matrix $T(z)$ with two solutions, one outside a neighbourhood of $z = 1$ and one inside. We call the exterior parametrix the Riemann-Hilbert problem solved by the matrix $M(z)$ with jump $v_\infty(z)$. We call the local parametrix the solution $P(z)$ of the Riemann-Hilbert problem obtained within a neighbourhood of $z = 1$. These two solutions are approximations of the exact solution $T(z)$ in the limit $k \to \infty$. In order to obtain the asymptotics of the orthogonal polynomials $\pi_k(z)$, we need sub-leading corrections to the matrices $M(z)$ and $P(z)$ and this will be accomplished by evaluating perturbatively the error matrix $E(z)$, which is defined as

$$E(z) = \begin{cases} T(z)\, M(z)^{-1}\,, & z \in \mathbb{C}\backslash\mathbb{D} \\ T(z)\, P(z)^{-1}\,, & z \in \mathbb{D}. \end{cases}$$

We will first construct the matrix $M(z)$ and then the matrix $P(z)$.

The exterior parametrix $M(z)$ is a 2×2 matrix that solves the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 3.4.1. *1. Piecewise Analyticity:*

$$M(z) \;\; \textit{is analytic in}\; \mathbb{C} \setminus \Sigma_M, \quad \Sigma_M = (\Gamma_{r=1} \cup [0,1])\,.$$

2.

$$M_+(z) = M_-(z)\; v_\infty, \quad z \in \Sigma_M \tag{3.44}$$

where v_∞ has been defined in (3.43).

3. Large z boundary behaviour:

$$M(z) = \left(\mathbb{1} + \mathcal{O}\left(\frac{1}{z}\right)\right), \quad z \to \infty\,. \tag{3.45}$$

4. Endpoint Behaviour:

$$M(z) = \mathcal{O}\,(1)\, z^{\frac{\gamma}{2}\sigma_3}\,, \quad z \to 0\;, \tag{3.46}$$

$$M(z) = \mathcal{O}\,(1)\, (z-1)^{-\frac{\gamma}{2}\sigma_3}\;, \quad z \to 1\;. \tag{3.47}$$

The solution of the above Riemann-Hilbert problem is given by the expression

$$M(z) = \begin{cases} \left(1 - \dfrac{1}{z}\right)^{\frac{\gamma}{2}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \text{ inside } \Gamma_{r=1} \\ \\ \left(1 - \dfrac{1}{z}\right)^{\frac{\gamma}{2}\sigma_3}, & z \text{ outside } \Gamma_{r=1}. \end{cases} \tag{3.48}$$

3.4.1 Local Parametrix and double scaling limit

The local parametrix near the point $z = 1$ is the solution $P(z)$ of a matrix Riemann-Hilbert problem that has the same jumps as $T(z)$ and matches $M(z)$ on the boundary of a disc centered at $z = 1$. Let us define as $\mathbb{D}$ a closed disc centered at $z = 1$.

Riemann-Hilbert Problem 3.4.2. *1. Piecewise Analyticity:*

$$P(\zeta) \text{ is analytic in } \mathbb{D} \setminus \Sigma_P, \quad \Sigma_P = \Sigma_e \cup \Sigma_{r=1} \cup \Sigma_i \cup [0,1]$$

2. Jumps on Σ_P:

$$P_+(z) = P_-(\zeta)\, v_P\,(z)\,, \quad \zeta \in \Sigma_P \cap \mathbb{D} \tag{3.49}$$

where

$$v_P\,(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}, & z \in \Gamma_i \cap \mathbb{D} \\[2ex] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Gamma_r \cap \mathbb{D} \\[2ex] \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}, & z \in \Gamma_e \cap \mathbb{D} \\[2ex] e^{-\gamma\pi i\sigma_3}, & z \in (0,1) \cap \mathbb{D} \end{cases} \tag{3.50}$$

which are the jumps that can be seen in Figure 3.9 .

3. Behaviour at the boundary $\partial\mathbb{D}$:

$$P(z) = M(z)\,(\mathbb{1} + \mathcal{O}(1))\,, \quad \text{as } k \to \infty \text{ and } z \in \partial\mathbb{D}. \tag{3.51}$$

In order to solve the Riemann-Hilbert problem for the matrix $P(z)$ we will use the Riemann-Hilbert problem for the Painlevé IV equation. Before doing this, some observations are useful.

Considering the fact that we have taken $r = z_0 = 1$, the function $\varphi(z, z_0 = 1)$, according to the region where it is located, is given by

$$\varphi(z; z_0 = 1) = \begin{cases} \log(z) - z + 1, & z \in \mathrm{Int}(\Gamma_{r=1}) \\[2ex] z - 1 - \log(z), & z \in \mathrm{Ext}(\Gamma_{r=1})\,. \end{cases} \tag{3.52}$$

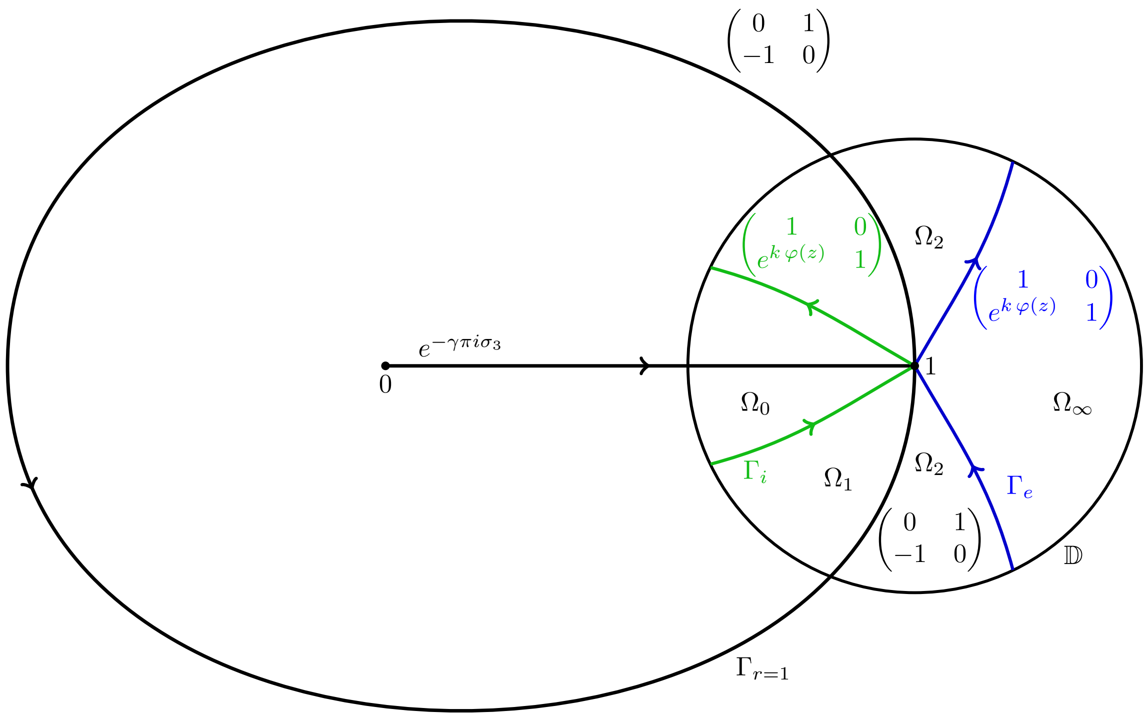


Figure 3.9: Jumps of the Riemann-Hilbert problem for the matrix $P(z)$.

Since the point $z = 1$ is the one that will give the biggest contribution to $e^{k\varphi(z)}$, we will now Taylor expand $\varphi(z)$ around it

$$k\,\varphi(z; z_0 = 1) = k\,(\log(z) - z + 1) = -\frac{k}{2}\,(z-1)^2\,(1 + \mathcal{O}\,(z-1)) = -k\frac{\zeta^2(z)}{2}, \quad z \in \mathrm{Int}(\Gamma_{r=1})$$

$$k\,\varphi(z; z_0 = 1) = k\,(z - 1 - \log(z)) = \frac{k}{2}\,(z-1)^2\,(1 + \mathcal{O}\,(z-1)) = k\frac{\zeta^2(z)}{2}, \qquad z \in \mathrm{Ext}(\Gamma_{r=1}). \tag{3.53}$$

It follows that $\zeta(z)$ has an expansion of the form

$$\zeta(z) = (z-1)\left(1 - \frac{1}{3}(z-1) + \mathcal{O}\,(z-1)^2\right). \tag{3.54}$$

Note that $\zeta(z)$ is a conformal map from the z-plane to the ζ-plane so that the contours for the jump matrix v_P are conformally deformed to new contours. This procedure allows us to transform the Riemann-Hilbert problem $P(z)$, defined in the z-plane, into a new one that depends on the variable ζ.

3.4.2 The Double Scaling limit for $k\,\varphi(z)$

Our purpose now will be to consider $k\,\varphi(z;z_0)$ for $z_0 \to 1$. The key to do this is that now we will consider $z_0 \sim 1$ but not equal to 1, which was what we did in the case above. Therefore, we will now do the Taylor expansion of $k\,\varphi(z)$ around $z = 1$. To do this, we begin by recalling the expression of $k\,\varphi(z;z_0)$ where we make the explicit the dependence on z_0:

$$\varphi(z;z_0) = \left(\frac{z-1}{z_0} - \log z\right) \tag{3.55}$$

and then perform the expansion

$$\begin{aligned} \varphi(z) &= \left(\frac{z-1}{z_0} + (1-z) + \frac{1}{2}(1-z)^2 + \dots\right) \\ &= \left((z-1)\left(\frac{1}{z_0} - 1\right) + \frac{1}{2}(z-1)^2 + \dots\right). \end{aligned} \tag{3.56}$$

From this expansion we define the quantity A as

$$A := \frac{\varphi(z;z_0) - \varphi(z;z_0 = 1)}{\zeta(z)} = \frac{1}{z_0} - 1 \tag{3.57}$$

and establish the following equality

$$\varphi(z;z_0) = \frac{1}{2}\zeta^2(z) + A\,\zeta(z)\,. \tag{3.58}$$

Definition 3.1 (Double-Scaling Limit)**.** The Double-Scaling Limit is defined by taking $k \to \infty$ and $A \to 0$ (or equivalently $z_0 \to 1$) so that

$$\lim_{k\to\infty, z_0\to 1} \sqrt{k}\,A = \mathcal{S} \tag{3.59}$$

with $\mathcal{S}$ in compact subsets of the complex plane. Equivalently, it can be defined as

$$A \sim \frac{\mathcal{S}}{\sqrt{k}} \quad \text{or} \quad z_0 \sim \frac{\sqrt{k}}{\sqrt{k}+\mathcal{S}}. \tag{3.60}$$

Remark 3.4. As it was stated above, from the general definition of the Double-Scaling Limit, one has $\mathcal{S} \in \mathbb{C}$. However, in our case it was established that k is a Real constant and since A was defined as in (3.57), where $z_0 \sim 1$ is also Real. Therefore, it is clear that for the case we are working in, it suffices to consider only real values of $\mathcal{S}$.

3.4.3 Model problem and the Painlevé IV equation

We recall the Riemann-Hilbert problem for the function $\Psi(\lambda;s)$ associated to the Painlevé IV equation. For convenience, in this section we replace the contour Γ_1 with $\hat{\Gamma}_i$ and the contour Γ_∞ with $\hat{\Gamma}_e$ to be consistent with the notation we are using in this section.

Riemann-Hilbert Problem 3.4.3. *1. Piecewise Analyticity:*

$$\Psi(\lambda; s) \text{ is analytic in } \mathbb{C} \setminus \Sigma_\Psi, \quad \Sigma_\Psi = \left(\hat{\Gamma}_i \cup \hat{\Gamma}_e \cup \mathbb{R}_-\right).$$

2. *Jumps on Σ_Ψ:*

$$\Psi_+(\lambda; s) = \Psi_-(\lambda; s)\, v_\Psi, \quad \lambda \in \Sigma_\Psi, \tag{3.61}$$

where

$$v_\Psi = \begin{cases} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & \lambda \in \hat{\Gamma}_i \\ \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \lambda \in \hat{\Gamma}_e \\ \\ e^{\gamma \pi i \sigma_3}, & \lambda \in \mathbb{R}_- \end{cases} \tag{3.62}$$

which are the jumps that can be seen in Figure 1.2.

3. *Large λ boundary behaviour:*

$$\Psi(\lambda; s) = \left(\mathbb{1} + \frac{\Psi_1}{\lambda} + \frac{\Psi_2}{\lambda^2} + \mathcal{O}(\lambda^{-3})\right) \lambda^{\frac{\gamma}{2}\sigma_3}\, e^{-\theta\,\sigma_3}, \quad \lambda \to \infty, \tag{3.63}$$

where $\theta = \dfrac{\lambda^2}{4} + s\dfrac{\lambda}{2}$ *and the matrices* $\Psi_1 = \Psi_1(s)$ *and* $\Psi_2 = \Psi_2(s)$ *take the form*

$$\Psi_1(s) = \begin{bmatrix} H(s) & \dfrac{Z(s)}{U(s)} \\ U(s) & -H(s) \end{bmatrix} \tag{3.64}$$

$$\Psi_2(s) = \begin{bmatrix} \dfrac{1}{2}\left(H(s)^2 + Z(s) - sH(s)\right) & \dfrac{Z(s)\,(Z(s) - \gamma - Y(s)\,s - H(s)\,Y(s))}{U(s)\,Y(s)} \\ U(s)\,H(s) + U(s)Y(s) - s\,U(s) & \dfrac{1}{2}\left(H(s)^2 + Z(s) + s\,H(s)\right) \end{bmatrix} \tag{3.65}$$

The scalar complex functions $H(s)$, $Z(s)$ *and* $U(s)$ *are given by*

$$\begin{cases} U' = U\,(Y - s) \\ \\ Z = \dfrac{1}{2}\left(sY + \gamma - Y' - Y^2\right) \\ \\ H = \left(s + \dfrac{\gamma}{Y} - Y\right) Z - \dfrac{Z^2}{Y} \end{cases} \tag{3.66}$$

where $Y = Y(s)$ solves the Painlevé IV equation (1.6) *with $\Theta_\infty = \Theta_0 = \gamma/2$, namely*

$$Y'' = \frac{1}{2}\frac{(Y')^2}{Y} + \frac{3}{2}Y^3 - 2sY^2 + \left(1 + \frac{s^2}{2} - 2\frac{\gamma}{2}\right)Y - \frac{\gamma^2}{2Y}. \tag{3.67}$$

4. *Endpoint Behaviour*

$$\Psi(\lambda; s) = \mathcal{O}(1)\,\lambda^{\frac{\gamma}{2}\sigma_3}, \quad \lambda \to 0\,, \tag{3.68}$$

as $\lambda \to 0$ in the region Ω_∞ (and the implication of this behaviour as $\lambda \to 0$ within the other regions).

For our purpose we need to modify the Riemann-Hilbert problem for $\Psi(\lambda)$ to be able to match the Riemann-Hilbert problem for $P(z)$. Let us introduce the contour $\Gamma_{r=1}$ on the λ plane, which is a contour between $\hat{\Gamma}_i$ and $\hat{\Gamma}_e$ and passing through $\lambda = 0$. Then let us define

$$\widehat{\Psi}(\lambda) = \Psi(\lambda)\, e^{\theta\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\chi_L}, \tag{3.69}$$

where χ_L is the characteristic function that is one on the left to the contour $\hat{\Gamma}_{r=1}$ and is zero otherwise. Then it is straightforward to check that the Riemann-Hilbert problem for $\widehat{\Psi}(\lambda)$ is given by

Riemann-Hilbert Problem 3.4.4. 1. *Piecewise Analyticity:*

$$\widehat{\Psi}(\lambda) \text{ is analytic in } \mathbb{C}\backslash\Sigma_{\widehat{\Psi}}, \quad \Sigma_{\widehat{\Psi}} = \hat{\Gamma}_e \cup \hat{\Gamma}_i \cup \hat{\Gamma}_{r=1} \cup \mathbb{R}^-$$

2. *Jumps on $\Sigma_{\widehat{\Psi}}$:*

$$\widehat{\Psi}_+(\lambda) = \widehat{\Psi}_-(\lambda)\, v_{\widehat{\Psi}}(\lambda)\,, \quad \zeta \in \Sigma_{\widehat{\Psi}}, \tag{3.70}$$

where

$$v_{\widehat{\Psi}}(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-2\theta(\lambda)} & 1 \end{pmatrix}, & z \in \hat{\Gamma}_i \\[2ex] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \hat{\Gamma}_{r=1} \\[2ex] \begin{pmatrix} 1 & 0 \\ e^{2\theta(\lambda)} & 1 \end{pmatrix}, & z \in \hat{\Gamma}_e \\[2ex] e^{-\gamma\pi i\sigma_3}, & z \in \mathbb{R}^-. \end{cases} \tag{3.71}$$

3. *Behaviour for* $\lambda \to \infty$

$$\widehat{\Psi}(\lambda) = \left(\mathbb{1} + \frac{\Psi_1}{\lambda} + \frac{\Psi_2}{\lambda^2} + \mathcal{O}(\lambda^{-3}) \right) \lambda^{\frac{\gamma}{2}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\chi_L} . \tag{3.72}$$

Comparing the jump matrices for $P(z)$ and $\widehat{\Psi}(\lambda)$, we are now ready to obtain the local parametrix $P(z)$ defined by the Riemann-Hilbert problem (3.4.2), which is given by

$$P(z) = M(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\chi_L} \left(\sqrt{k}\, \zeta(z) \right)^{-\frac{\gamma}{2}\sigma_3} \widehat{\Psi}\left(\sqrt{k}\, \zeta\,(z)\,; \sqrt{k}\, A \right). \tag{3.73}$$

We observe that the product of the first three terms of (3.73) is holomorphic in the neighbourhood of $z = 1$ and therefore does not change the Riemann-Hilbert problem. Furthermore, the first three terms have been inserted in order to have the behaviour (3.51) for $z \in \partial\mathbb{D}$ in the limit $k \to \infty$ and $A \to 0$ in such a way that

$$\lim_{k\to\infty} \sqrt{k} A = \mathcal{S}. \tag{3.74}$$

In the analysis above and below, we assume that $\mathcal{S}$ belongs to compact sets where the solution of the Painlevé IV equation does not have poles. Using (3.72), (3.64) and (3.65), we obtain in the limit $k \to \infty$

$$\begin{aligned} P(z) &= M(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\chi_L} \left(\sqrt{k}\, \zeta(z) \right)^{-\frac{\gamma}{2}\sigma_3} \left(\mathbb{1} + \frac{1}{\sqrt{k}\,\zeta} \begin{bmatrix} H & \frac{Z}{U} \\ U & -H \end{bmatrix} + \mathcal{O}(k^{-1}) \right) \left(\sqrt{k}\, \zeta \right)^{\frac{\gamma}{2}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\chi_L} \\ &= M(z) \left(\mathbb{1} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\chi_L} \frac{1}{\sqrt{k}\zeta} \begin{bmatrix} H & (\sqrt{k}\zeta)^{-\gamma} \frac{Z}{U} \\ (\sqrt{k}\zeta)^{\gamma} U & -H \end{bmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\chi_L} + \mathcal{O}\left(k^{-1+\gamma/2} \right) \right) \\ &= M(z) \left(\mathbb{1} + \mathcal{O}\left(k^{\frac{\gamma-1}{2}} \right) \right). \end{aligned} \tag{3.75}$$

From the above expansion we can see that the subleading terms are not uniformly small for $k \to \infty$ since $\gamma \in [0, 1)$. Note that the source of the non-uniformity in the error analysis is the element $(2, 1)$ in (3.75). For this reason we need to introduce an improved parametrix in the next section.

3.4.4 Improved parametrix

To construct an improved parametrix we define

$$\widetilde{M}(z) = \left(\mathbb{1} + \frac{B\sigma_-}{z-1} \right) M(z), \qquad \sigma_- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{3.76}$$

where B is to be determined. In the same way we define

$$\widetilde{P}(z) = \widetilde{M}(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\chi_L} \left(\sqrt{k}\, \zeta(z) \right)^{-\frac{\gamma}{2}\sigma_3} \left(\mathbb{1} - \frac{U\sigma_-}{\sqrt{k}\, \zeta(z)} \right) \widehat{\Psi}(\sqrt{k}\, \zeta(z)\,; \sqrt{k}\, A), \tag{3.77}$$

where U is the 2 1 entry of the subleading term of the expansion of $\widehat{\Psi}(\lambda)$ for $\lambda \to \infty$. Now we have in the limit $k \to \infty$

$$\begin{aligned}\widetilde{P}(z) &= \widetilde{M}(z)\left(\mathbb{1} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\chi_L} \frac{1}{\sqrt{k}\zeta} \begin{bmatrix} H & (\sqrt{k}\,\zeta)^{-\gamma}\dfrac{Z}{U} \\ 0 & -H \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{\chi_L} + \mathcal{O}(k^{-1+\gamma/2})\right) \\ &= \widetilde{M}(z)\left(\mathbb{1} + \mathcal{O}(k^{-\frac{1}{2}})\right).\end{aligned}$$

The improved parametrics $\widetilde{M}(z)$ and $\widetilde{P}(z)$ have the same jump discontinuities as before, but $\widetilde{P}(z)$ might have poles at $z = 1$. The constant B in (3.76) is then determined by the requirement that $\widetilde{P}(z)$ is bounded at $z = 1$. This gives the constant B as

$$B = U\,k^{\frac{\gamma-1}{2}}. \tag{3.78}$$

We are now ready to compute the error matrix $E(z)$.

3.4.5 Error matrix

The error matrix $E(z)$ is defined as

$$E(z) = \begin{cases} T(z)\,\widetilde{M}(z)^{-1}, & z \in \mathbb{C}\backslash\mathbb{D} \\ T(z)\,\widetilde{P}(z)^{-1}, & z \in \mathbb{D} \end{cases} \tag{3.79}$$

where the boundary of $\mathbb{D}$ is oriented clockwise. Then, the matrix $E(z)$ satisfies the Riemann-Hilbert problem

$$E_+(z) = E_-(z)\,v_E(z)\,, \quad z \in \partial\mathbb{D},$$

where the jump v_E is given by

$$v_E(z) = \widetilde{M}(z)\,\widetilde{P}^{-1}(z)\,. \tag{3.80}$$

In the double scaling limit $k \to \infty$, using (3.72), (3.64) and (3.65), $v_E(z)$ takes the form

$$\begin{aligned} v_E(z) &= \left(\mathbb{1} + \frac{Uk^{\frac{\gamma-1}{2}}}{z-1}\sigma_-\right)\left(\frac{z-1}{\sqrt{k}\,z\zeta}\right)^{\frac{\gamma}{2}\sigma_3} \times \\ &\times \left(\mathbb{1} - \frac{\begin{pmatrix} H & \dfrac{Z}{U} \\ 0 & -H \end{pmatrix}}{\sqrt{k}\,\zeta} + \frac{\begin{bmatrix} \frac{H^2+sH-Z}{2} & \frac{HZY+HY+ZY^2}{UY} \\ sU-UY & \frac{H^2-sH+Z}{2} \end{bmatrix}}{k\,\zeta^2} + \mathcal{O}(k^{-\frac{3}{2}})\right)\left(\frac{z-1}{\sqrt{k}\,z\,\zeta}\right)^{-\frac{\gamma}{2}\sigma_3}\left(\mathbb{1} - \frac{Uk^{\frac{\gamma-1}{2}}}{z-1}\sigma_-\right) \\ &= \mathbb{1} + \frac{v_E^{(1)}}{\sqrt{k}} + \frac{v_E^{(2)}}{k^{\frac{1}{2}+\frac{\gamma}{2}}} + \frac{v_E^{(3)}}{k^{1-\frac{\gamma}{2}}} + \mathcal{O}\big(k^{-1}\big)\,, \end{aligned}$$

where

$$
\begin{aligned}
v_E^{(1)} &= -\frac{H}{\zeta}\sigma_3 \\
v_E^{(2)} &= -\left(\frac{z-1}{z\zeta}\right)^{\gamma}\left(\frac{Z}{U\zeta}\right)\sigma_+ \\
v_E^{(3)} &= \left(\left(\frac{z-1}{z\zeta}\right)^{-\gamma}\frac{(s-Y)U}{\zeta^2} - 2\frac{HU}{(z-1)\zeta}\right)\sigma_- .
\end{aligned}
$$

By the standard theory of small norm Riemann-Hilbert problems, one has a similar expansion for $E(z)$, namely

$$
E(z) = \mathbb{1} + \frac{E^{(1)}}{\sqrt{k}} + \frac{E^{(2)}}{k^{\frac{1}{2}+\frac{\gamma}{2}}} + \frac{E^{(3)}}{k^{1-\frac{\gamma}{2}}} + \mathcal{O}\big(k^{-1}\big)\,, \tag{3.81}
$$

so that

$$
\begin{aligned}
E_+^{(1)}(z) &= E_-^{(1)}(z) + v_E^{(1)}(z)\,, \quad z \in \partial\mathbb{D}, \\
E_+^{(2)}(z) &= E_-^{(2)}(z) + v_E^{(2)}(z)\,, \quad z \in \partial\mathbb{D}, \\
E_+^{(3)}(z) &= E_-^{(3)}(z) + v_E^{(3)}(z)\,, \quad z \in \partial\mathbb{D}.
\end{aligned}
$$

By solving the corresponding Riemann-Hilbert problem, we obtain, using the Plemelj-Sokhtski formula

$$
E^{(j)}(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{v_E^{(i)}(\xi)}{\xi - z}d\xi, \quad j = 1,2,3,
$$

which gives

$$
\begin{aligned}
E^{(1)}(z) &= -\frac{\mathrm{Res}_{\xi=1} v_E^{(1)}(\xi)}{z-1} = \frac{H\sigma_3}{z-1}, \quad z \in \mathbb{C}\backslash\mathbb{D}, \\
E^{(2)}(z) &= -\frac{\mathrm{Res}_{\xi=1} v_E^{(2)}(\xi)}{z-1} = \frac{(Z/U)\,\sigma_+}{z-1}, \quad z \in \mathbb{C}\backslash\mathbb{D}, \\
E^{(3)}(z) &= -\frac{\mathrm{Res}_{\xi=1} v_E^{(3)}(\xi)}{(z-1)} - \frac{\mathrm{Res}_{\xi=1}(\xi-1)\, v_E^{(2)}(\xi)}{(z-1)^2} \\
&= \left(\frac{2}{3}\frac{\gamma\,(2H+\gamma-s)\,\gamma + H + \gamma - s}{z-1} + \frac{U\,(2H+Y-s)}{z-1}\right)\sigma_-, \quad z \in \mathbb{C}\backslash\mathbb{D}
\end{aligned} \tag{3.82}
$$

and

$$
\begin{aligned}
E^{(1)}(z) &= v_E^{(1)} - \frac{\mathrm{Res}_{\xi=1} v_E^{(1)}(\xi)}{z-1} = v_E^{(1)} + \frac{H\sigma_3}{z-1}, \quad z \in \mathbb{D}, \\
E^{(2)}(z) &= v_E^{(2)} - \frac{\mathrm{Res}_{\xi=1} v_E^{(2)}(\xi)}{z-1} = v_E^{(2)} + \frac{(Z/U)\,\sigma_+}{z-1}, \quad z \in \mathbb{D} \\
E^{(3)}(z) &= v_E^{(3)} + \left(\frac{2}{3}\frac{\gamma\,(2H+\gamma-s)\,\gamma + H + \gamma - s}{z-1} + \frac{U\,(2H+Y-s)}{z-1}\right)\sigma_-, \quad z \in \mathbb{D}.
\end{aligned} \tag{3.83}
$$

3.4.6 Asymptotics for the polynomials $\pi_k(z)$ and proof of Theorem 2.2 and Theorem 2.4

We are now ready to determine the asymptotic expansions for the orthogonal polynomials $\pi_k(z)$. Using (3.19), (3.36) and (3.79) we have

$$\pi_k(z) = e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}} [U_k(z)]_{1\,1} \tag{3.84}$$

$$= e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}} \begin{cases} [T_k(z)]_{1\,1} & z \in \Omega_\infty \cup \Omega_0 \\ \left[T_k(z) \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}\right]_{1\,1} & z \in \Omega_1 \\ \left[T_k(z) \begin{pmatrix} 1 & 0 \\ -e^{k\varphi(z)} & 1 \end{pmatrix}\right]_{1\,1} & z \in \Omega_2 \end{cases} \tag{3.85}$$

$$= e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{2}} \begin{cases} \left[E(z)\, \widetilde{M}(z)\right]_{1\,1} & z \in (\Omega_\infty \cup \Omega_0) \setminus \mathbb{D} \\ \left[E(z)\, \widetilde{M}(z) \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}\right]_{1\,1} & z \in \Omega_1 \setminus \mathbb{D} \\ \left[E(z)\, \widetilde{M}(z) \begin{pmatrix} 1 & 0 \\ -e^{k\varphi(z)} & 1 \end{pmatrix}\right]_{1\,1} & z \in \Omega_2 \setminus \mathbb{D} \\ \left[E(z)\, \widetilde{P}(z)\right]_{1\,1} & z \in (\Omega_0 \cup \Omega_\infty) \cap \mathbb{D} \\ \left[E(z)\, \widetilde{P}(z) \begin{pmatrix} 1 & 0 \\ e^{k\varphi(z)} & 1 \end{pmatrix}\right]_{1\,1} & z \in \Omega_1 \cap \mathbb{D} \\ \left[E(z)\, \widetilde{P}(z) \begin{pmatrix} 1 & 0 \\ -e^{k\varphi(z)} & 1 \end{pmatrix}\right]_{1\,1} & z \in \Omega_2 \cap \mathbb{D}\,. \end{cases} \tag{3.86}$$

We want to analyze each distinct region. Using (3.82), (3.83), (3.77) and (3.76) we obtain the following expressions

The region $\Omega_\infty \setminus \mathbb{D}$

$$\begin{aligned} \pi_k(z) &= e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\gamma} \left(1 + \frac{H}{\sqrt{k}\,(z-1)} + \mathcal{O}\left(\frac{1}{k}\right)\right) \\ &= z^k \left(1 - \frac{1}{z}\right)^{\gamma} \left(1 + \frac{H}{\sqrt{k}(z-1)} + \mathcal{O}\left(\frac{1}{k}\right)\right), \end{aligned} \tag{3.87}$$

with H defined in (3.66).

The region $\Omega_0 \setminus \mathbb{D}$

$$\pi_k(z) = e^{kg(z)} \left(1 - \frac{1}{z}\right)^{\gamma/2} \left(-\frac{Z}{U\,(z-1)\,k^{\frac{1}{2}+\gamma}} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{3.88}$$

with Z and U defined in (3.66).

The region $\Omega_1 \setminus \mathbb{D}$

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-1}{z}\right)^{\gamma} \left(e^{k\varphi(z)} \left(1 + \frac{H}{\sqrt{k}\,(z-1)}\right) - \frac{Z}{U(z-1)k^{\frac{1+\gamma}{2}}} \left(\frac{z-1}{z}\right)^{-\gamma} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{3.89}$$

with H, U and Z defined in (3.66). In a similar way we can obtain the expansion in the region $\Omega_2 \setminus \mathbb{D}$.

The region $\Omega_2 \setminus \mathbb{D}$

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-1}{z}\right)^{\gamma} \left(1 + \frac{H}{\sqrt{k}\,(z-1)} - \frac{Ze^{k\varphi(z)}}{U\,(z-1)\,k^{\frac{1+\gamma}{2}}} \left(\frac{z-1}{z}\right)^{-\gamma} + \mathcal{O}\left(\frac{1}{k}\right)\right). \tag{3.90}$$

The region $\mathbb{D}$

In the region $(\Omega_0 \cup \Omega_\infty) \cap \mathbb{D}$ we have

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-1}{z}\right)^{\gamma} \left(\frac{\widehat{\Psi}_{11}\left(\sqrt{k}\,\zeta(z)\,;\sqrt{k}\,A\right)}{k^{\frac{\gamma}{4}}\,\zeta(z)} + \mathcal{O}\left(\frac{1}{k^{\frac{1}{2}+\frac{\gamma}{4}}}\right)\right)$$

where $\widehat{\Psi}_{11}$ is the $1\,1$ entry of the Painlevé isomonodromic problem (3.4.4).

In the region $\Omega_1 \cap \mathbb{D}$ and $\Omega_2 \cap \mathbb{D}$ we have

$$\pi_k(z) = e^{kg(z)} \left(\frac{z-1}{z}\right)^{\gamma} \left(\frac{\widehat{\Psi}_{11}\left(\sqrt{k}\,\zeta(z)\,;\sqrt{k}\,A\right) \pm e^{k\varphi(z)}\widehat{\Psi}_{12}\left(\sqrt{k}\,\zeta(z)\,;\sqrt{k}\,A\right)}{k^{\frac{\gamma}{4}}\,\zeta(z)} + \mathcal{O}\left(\frac{1}{k^{\frac{1}{2}+\frac{\gamma}{4}}}\right)\right),$$

where $\pm$ refers to the region Ω_1 and Ω_2, respectively, and $\widehat{\Psi}_{12}$ is the $1\,2$ entry of the Painlevé isomonodromic problem (3.4.4). Making the change of variables

$$z = 1 - \frac{\lambda^d}{t_c},$$

the proof of Theorem 2.4 follows in a straightforward way from the above expansions. With these expansions, we are able to locate the zeros of the orthogonal polynomials.

Proposition 3.5. *The support of the counting measure of the zeros of the polynomials $\pi_k(z)$ outside an arbitrary small disk $\mathbb{D}$ surrounding the point $z = 1$ tends uniformly to the curve $\Gamma_{r=1}$*

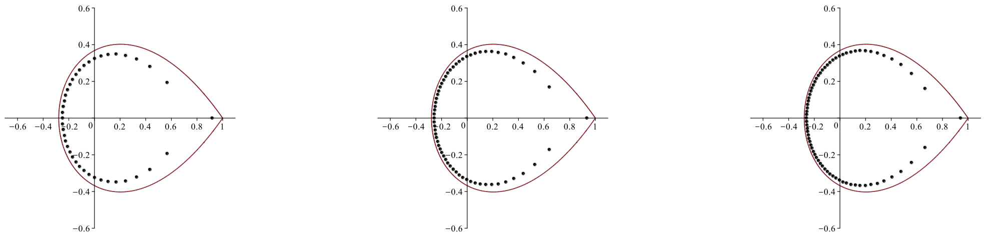


Figure 3.10: Zeros of the polynomials $\pi_k(z)$ for $k = 40, 60, 70$. The values of the parameters are $z_0 = 1$ and $t = 2$, $d = 3$ and $l = 0$. The plot of the support of the limiting measure (Szegö curve) of the zeros of the orthogonal polynomials $\pi_k(z)$ is in red.

defined in (3.34) *for* $z_0 = 1$. *The zeros are within a distance* $o(1/k)$ *from the curve defined by*

$$\log|z| - \frac{|z-1|}{|z_0|} = -\frac{1+\gamma}{2}\frac{\log k}{k} + \frac{1}{k}\log\left(\left|\frac{z}{z-1}\right|^{\gamma}\left|\frac{Z(\mathcal{S})}{(z-1)\,U(\mathcal{S})}\right|\right) \tag{3.91}$$

where we recall from (3.60) *that* $z_0 = \frac{\sqrt{k}}{\sqrt{k}+\mathcal{S}}$, *and that the function* $Z = Z(\mathcal{S})$ *and* $U = U(\mathcal{S})$ *are related to the Painlevé IV equation via* (3.66). *The curves in* (3.91) *approach* $\Gamma_{r=1}$ *at the rate* $\mathcal{O}(\log k/k)$ *and lies in* $\mathrm{Int}(\Gamma_{r=1})$. *The normalized counting measure of the zeros of* $\pi_k(z)$ *converges to the probability measure* ν *defined in* (3.27).

Proof. Observing the asymptotic expansion (3.87) of $\pi_k(z)$ in $\Omega_\infty \setminus \mathbb{D}$, it is clear that $\pi_k(z)$ does not have any zeros in that region, since $z = 0$ and $z = 1$ do not belong to $\Omega_\infty \setminus \mathbb{D}$. The same reasoning applies to the region $\Omega_0 \setminus \mathbb{D}$, where there are no zeros of $\pi_k(z)$ for k sufficiently large.

From the relations (3.89) and (3.90), one has that in $\Omega_1 \cup \Omega_2$ using the explicit expression of $g(z)$ defined in (3.28)

$$\pi_k(z) = z^k\left(\frac{z-1}{z}\right)^{\gamma}\left(1 + \frac{H}{\sqrt{k}\,(z-1)} - \frac{e^{\pm k\varphi(z)}Z}{U\,(z-1)k^{\frac{1+\gamma}{2}}}\left(\frac{z-1}{z}\right)^{-\gamma} + \mathcal{O}\left(\frac{1}{k}\right)\right), \tag{3.92}$$

where $\pm$ refers to Ω_2 and Ω_1, respectively. The zeros of $\pi_k(z)$ may only lie asymptotically where the expression

$$1 + \frac{H}{\sqrt{k}\,(z-1)} = \frac{e^{\pm k\varphi(z)}Z}{U\,(z-1)\,k^{\frac{1+\gamma}{2}}}\left(\frac{z-1}{z}\right)^{-\gamma}, \quad z \in \Omega_1 \cup \Omega_2.$$

Since $\Omega_2 \cup \Omega_1 \subset \{\mathrm{Re}(\varphi) \leq 0\}$, it follows that the zeros of $\pi_k(z)$ may lie only in the region Ω_1 and such that $\mathrm{Re}\,\varphi(z) = \mathcal{O}\left(\log k/k\right)$ (where z_0 is the value given by the double scaling (3.60)). Taking the logarithm of the modulus of the above equality, we obtain

$$\mathrm{Re}\,\varphi(z) = -\frac{1+\gamma}{2}\frac{\log k}{k} + \frac{1}{k}\log\left(\left|\frac{z}{z-1}\right|^{\gamma}\left|\frac{Z(\mathcal{S})}{(z-1)\,U(\mathcal{S})}\right|\right) + \frac{1}{k^{\frac{3}{2}}}\,\mathrm{Re}\left(\frac{H(\mathcal{S})}{z-1}\right) + \mathcal{O}\left(\frac{1}{k^2}\right). \quad (3.93)$$

Namely, the zeros of the polynomials $\pi_k(z)$ lie on the curve given by (3.91) with an error of order $\mathcal{O}\left(1/k^2\right)$. Such curves converge to the curve $\Gamma_{r=1}$ defined (3.34) with $z_0 = 1$ at a rate $\mathcal{O}\left(\log k/k\right)$. □

The proof of Proposition 2.5 follows immediately from the proof of Proposition 3.5. The rest of the proof of Theorem 2.2 follows the steps obtained in [2] and for this reason we omit it.

A

Source code for the computation of the τ–function

In what follows we show the code that was used in *Mathematica* in order to compute the Fredholm Determinant. The purpose of this was to make a numerical study of the τ–function using the technique of Gaussian-Hermite quadrature, which lead to the τ–function being given as in (1.92).

We begin by defining the precision that we want in the computations

```
Precise = 100
```

Then we compute the nodes x_j, which are the roots of the n-th Hermite polynomial $H_n\left(\sqrt{\Lambda}x\right)$ for $\Lambda = \frac{1}{2}$

```
Nodes[n_] := x /. NSolve[HermiteH[n, x/Sqrt[2]] == 0, x, WorkingPrecision ->
    Precise]
```

and the weights of the quadrature rule

```
weight[n_, x_] := (2^(n - 1/2) n! Sqrt[\[Pi]])/(n^2 (HermiteH[n - 1, x/Sqrt
    [2]])^2)


wei[n_] := weight[n, #] & /@ (x /. NSolve[HermiteH[n, x/Sqrt[2]] == 0, x,
    WorkingPrecision -> Precise])
```

It should be noted that both in the computation of the nodes and the wheights, we are applying a procedure so that the results come in a list of n elements, where the i–th element of the list is the i–th node or weight.

In order to compute the τ–function for chosen values of the parameters s and γ and for a chose number of n points we use the following Module

```
Tau[s_, \[Gamma]_, n_] := Module[{X, W, A, AM},
```

```
X = Nodes[2 n];
W = wei[2 n];
\[Epsilon] = Max[0, -s];

A[j_, l_] :=
 Abs[X[[j]]]^(\[Gamma]/2) Sqrt[W[[j]]]
   e^(-s/2 X[[j]]) ((i X[[l]] + \[Epsilon])^(-(\[Gamma]/2)) e^(
   i/2 X[[l]] (s + \[Epsilon])) Sqrt[
   W[[l]]])/(X[[j]] - \[Epsilon] - i X[[l]]);

AM = Table[A[j, l], {j, n}, {l, 2 n}];

Det[IdentityMatrix[n] -
  E^((\[Epsilon])^2/2 + s \[Epsilon])/(2 \[Pi]^2)
    Sin[\[Pi] \[Gamma]] Dot[AM, Transpose[AM]]
]
```

The plots of the τ–function in terms of the parameter s, that we list in Subsection 1.4.2 are done in the following way

```
Plot[Tau[s, 0.5, 80], {s, -3, 7}, PlotPoints -> 30, MaxRecursion -> 3,
    GridLines -> Automatic, AxesLabel -> {s, \[Tau][s, 0.5, 80]}]
```

where this is the example for the case of $\gamma = 0.5$ and with $n = 80$ points.

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