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# Critical asymptotic behavior for the Korteweg–de Vries equation and in random matrix theory

TOM CLAEYS AND TAMARA GRAVA

We discuss universality in random matrix theory and in the study of Hamiltonian partial differential equations. We focus on universality of critical behavior and we compare results in unitary random matrix ensembles with their counterparts for the Korteweg–de Vries equation, emphasizing the similarities between both subjects.

## 1. Introduction

It has been observed and conjectured that the critical behavior of solutions to Hamiltonian perturbations of hyperbolic and elliptic systems of partial differential equations near points of gradient catastrophe is asymptotically independent of the chosen initial data and independent of the chosen equation [Dubrovin 2006; Dubrovin et al. 2009]. A classical example of a Hamiltonian perturbation of a hyperbolic equation which exhibits such universal behavior, is the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \quad \epsilon > 0. \quad (1-1)$$

If one is interested in the behavior of KdV solutions in the small dispersion limit  $\epsilon \rightarrow 0$ , it is natural to study first the inviscid Burgers' or Hopf equation  $u_t + 6uu_x = 0$ . Given smooth initial data  $u(x, 0) = u_0(x)$  decaying at  $\pm\infty$ , the solution of this equation is, for  $t$  sufficiently small, given by the method of characteristics: we have  $u(x, t) = u(\xi(x, t))$ , where  $\xi(x, t)$  is given as the solution to the equation

$$x = 6tu_0(\xi) + \xi. \quad (1-2)$$

It is easily derived from this implicit form of the solution that the  $x$ -derivative of  $u(x, t)$  blows up at time

$$t_c = \frac{1}{\max_{\xi \in \mathbb{R}}(-6u'_0(\xi))},$$

which is called the time of gradient catastrophe. After this time, the Hopf solution  $u(x, t)$  ceases to exist in the classical sense. For  $t$  slightly smaller than the critical time  $t_c$  the KdV solution starts to oscillate as shown numerically in [Grava and Klein 2007]. For  $t > t_c$  the KdV solution develops a train of rapid oscillations of wavelength of order  $\epsilon$ . In general, the asymptotics for the KdV solution as  $\epsilon \rightarrow 0$  can be described in terms of an equilibrium problem, discovered by Lax and Levermore [1983a; 1983b; 1983c; Lax et al. 1993].

The support of the solution of the equilibrium problem, which depends on  $x$  and  $t$ , consists of a finite or infinite union of intervals [Grava 2004; Deift et al. 1998b], and the endpoints evolve according to the Whitham equations [Whitham 1974; Flaschka et al. 1980]. For  $t < t_c$ , the support of the equilibrium problem consists of one interval and the KdV solution as  $\epsilon \rightarrow 0$  is approximated by the Hopf solution. For  $t > t_c$  the support of the equilibrium problem may consist of several intervals and the KdV solution is approximated as  $\epsilon \rightarrow 0$  by Riemann  $\theta$ -functions [Gurevich and Pitaevskii 1973; Lax and Levermore 1983a; 1983b; 1983c; Deift et al. 1997; Venakides 1990].

The  $(x, t)$ -plane can thus be divided into different regions labeled by the number of intervals in the support of the Lax–Levermore minimization problem. Such regions are independent of  $\epsilon$  and depend only on the initial data. Those regions are separated by a collection of breaking curves where the number of intervals in the support changes. We will review recently obtained results concerning the asymptotic behavior of KdV solutions near curves separating a one-interval region from a two-interval region. The two interval region corresponds to the solution of KdV being approximated as  $\epsilon \rightarrow 0$  by the Jacobi elliptic function, the one interval region corresponds to the solution of KdV being approximated by the Hopf solution (1-2).

On the space of  $n \times n$  Hermitian matrices, one can define unitary invariant probability measures of the form

$$\frac{1}{\tilde{Z}_n} \exp(-N \operatorname{Tr} V(M)) dM, \quad dM = \prod_{i=1}^n dM_{ii} \prod_{i < j} d\operatorname{Re} M_{ij} d\operatorname{Im} M_{ij}, \quad (1-3)$$

where  $\tilde{Z}_n = \tilde{Z}_n(N)$  is a normalization constant which depends on the integer  $N$  and  $V$  is a real polynomial of even degree with positive leading coefficient. The eigenvalues of random matrices in such a unitary ensemble follow a determinantal point process defined by

$$\frac{1}{Z_n} \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^n e^{-NV(\lambda_i)} d\lambda_i, \quad (1-4)$$

with correlation kernel

$$K_n(u, v) = \frac{e^{-\frac{N}{2}V(u)}e^{-\frac{N}{2}V(v)}}{u-v} \frac{\kappa_{n-1}}{\kappa_n} (p_n(u)p_{n-1}(v) - p_n(v)p_{n-1}(u)), \quad (1-5)$$

where  $p_k$  is the degree  $k$  orthonormal polynomial with respect to the weight  $e^{-NV}$  defined by

$$\int_{\mathbb{R}} p_j(s)p_k(s)e^{-NV(s)} ds = \delta_{jk}, \quad j, k \in \mathbb{R},$$

and  $\kappa_k > 0$  is the leading coefficient of  $p_k$ . The average counting measure of the eigenvalues has a limit as  $n = N \rightarrow \infty$ . We will denote this limiting mean eigenvalue distribution by  $\mu_V$ . For a general polynomial external field  $V$  of degree  $2m$ , the support of  $\mu_V$  consists of a finite union of at most  $m$  intervals [Deift et al. 1998a]. If  $V$  depends on one or more parameters, the measure  $\mu_V$  will in general also vary with those parameters. Critical phenomena occur when the number of intervals in the support of  $\mu_V$  changes. A decrease in the number of intervals can be caused essentially by three different events:

- (i) shrinking of an interval, which disappears ultimately;
- (ii) merging of two intervals to a single interval;
- (iii) simultaneous merging of two intervals and shrinking of one of them.

Near such transitions, double scaling limits of the correlation kernel are different from the usual sine or Airy kernel. At a type (i) transition, the limiting kernel is built out of Hermite polynomials [Eynard 2006; Claeys 2008; Mo 2008; Bertola and Lee 2009], at a type (ii) transition the limiting kernel is built out of functions related to the Painlevé II equation [Bleher and Its 2003; Claeys et al. 2008], and at a type (iii) transition the limiting kernel is related to the Painlevé I hierarchy [Brézin et al. 1990; Claeys and Vanlessen 2007b]. Higher order transitions, such as the simultaneous merging and/or shrinking of more than two intervals, can also take place but will not be considered here. Rather than on the limiting kernels, we will concentrate on the asymptotic behavior of the recurrence coefficients of the orthogonal polynomials, defined by the three-term recurrence relation

$$sp_n(s) = \gamma_{n+1}p_{n+1}(s) + \beta_n p_n(s) + \gamma_n p_{n-1}(s). \quad (1-6)$$

The recurrence coefficients contain information about the orthogonal polynomials and about the partition function  $Z_n$  of the determinantal point process (1-4) [Bessis et al. 1980; Bleher and Its 2005; Ercolani and McLaughlin 2003]. The large  $n, N$  asymptotics for the recurrence coefficients show remarkable similarities with the asymptotic behavior for KdV solution  $u(x, t, \epsilon)$  as  $\epsilon \rightarrow 0$ .

## 2. Phase diagram for the KdV equation

We assume throughout this section that the ( $\epsilon$ -independent) initial data  $u_0(x)$  for the KdV equation are real analytic in a neighborhood of the real line, negative, have a single local minimum  $x_M$  for which  $u_0(x_M) = -1$ , and that they decay sufficiently rapidly as  $x \rightarrow \infty$  in a complex neighborhood of the real line. The neighborhood of the real line where  $u_0$  is analytic and where the decay holds should contain a sector  $\{|\arg x| < \delta\} \cup \{|\arg(-x)| < \delta\}$ . In addition certain generic conditions have to be valid; we refer to [Claeys and Grava 2009] for details about those. A simple example of admissible initial data is given by  $u_0(x) = -\operatorname{sech}^2 x$ .

**2.1. Regular asymptotics for the KdV solution.** Before the time of gradient catastrophe

$$t_c = \frac{1}{\max_{\xi \in \mathbb{R}}(-6u_0'(\xi))},$$

the asymptotics for the KdV solution  $u(x, t, \epsilon)$  as  $\epsilon \rightarrow 0$  are given by

$$u(x, t, \epsilon) = u(x, t) + \mathcal{O}(\epsilon^2),$$

where  $u(x, t)$  is the solution to the Hopf equation with initial data  $u_0(x)$ , that is, the implicit solution  $u_0(\xi(x, t))$  defined by (1-2). The leading term of the above asymptotic expansion was obtained in [Lax and Levermore 1983a; 1983b; 1983c] while the error term was obtained only recently for a larger class of equations and initial data in [Masoero and Raimondo 2013]. Such an expansion still holds true after the time of gradient catastrophe as long as  $x$  is outside the interval where the KdV solution develops oscillations. In the oscillatory region, the oscillations for some time  $t > t_c$  can be approximated as  $\epsilon \rightarrow 0$ , by the elliptic function

$$u(x, t, \epsilon) = \beta_1 + \beta_2 + \beta_3 + 2\alpha + 2\epsilon^2 \frac{\partial^2}{\partial x^2} \log \vartheta \left( \frac{\sqrt{\beta_1 - \beta_3}}{2\epsilon K(s)} [x - 2t(\beta_1 + \beta_2 + \beta_3) - q]; \tau \right) + \mathcal{O}(\epsilon). \quad (2-1)$$

Here

$$\alpha = -\beta_1 + (\beta_1 - \beta_3) \frac{E(s)}{K(s)}, \quad \tau = i \frac{K'(s)}{K(s)}, \quad s^2 = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}, \quad (2-2)$$

where  $K(s)$  and  $E(s)$  are the complete elliptic integrals of the first and second kind,  $K'(s) = K(\sqrt{1-s^2})$ , and  $\vartheta(z; \tau)$  is the Jacobi elliptic theta function. In the formula (2-1) the term  $\beta_1 + \beta_2 + \beta_3 + 2\alpha$  is the weak limit of the solution  $u(x, t, \epsilon)$  of KdV as  $\epsilon \rightarrow 0$  and it was derived in the seminal papers [Lax and Levermore 1983a; 1983b; 1983c]. The asymptotic description of the oscillations by theta-function was obtained in [Venakides 1990]. A heuristic derivation of formula (2-1) without the phase, was first obtained in [Gurevich and Pitaevskii

1973]. The phase  $q$  in the argument of the Jacobi elliptic theta function (2-1) was derived in [Deift et al. 1997]. It depends on  $\beta_1, \beta_2, \beta_3$  and on the initial data and it was observed in [Grava and Klein 2007] that  $q$  satisfies a linear over-determined system of Euler–Poisson–Darboux type derived in [Tian 1994b; Gurevich et al. 1992]. The negative numbers  $\beta_1 > \beta_2 > \beta_3$  depend on  $x$  and  $t$  and solve the genus one Whitham equations [Whitham 1974]. The complete solution of the Whitham equation for the class of initial data considered, was derived in [Tian 1994a].

At later times, the KdV solution can, depending on the initial data, develop multiphase oscillations which can be described in terms of higher genus Whitham equations [Flaschka et al. 1980] and in terms of Riemann  $\theta$  functions [Venakides 1990; Lax and Levermore 1983a; 1983b; 1983c; Deift et al. 1997].

The parameters  $\beta_1, \beta_2, \beta_3$  can be interpreted in terms of the endpoints of the support  $[0, \sqrt{\beta_3 + 1}] \cup [\sqrt{\beta_2 + 1}, \sqrt{\beta_1 + 1}]$  of the minimizer of the Lax–Levermore energy functional [Lax and Levermore 1983a; 1983b; 1983c; 1993; 1997].

A transition from the elliptic asymptotic region to the Hopf region can happen in three different ways:

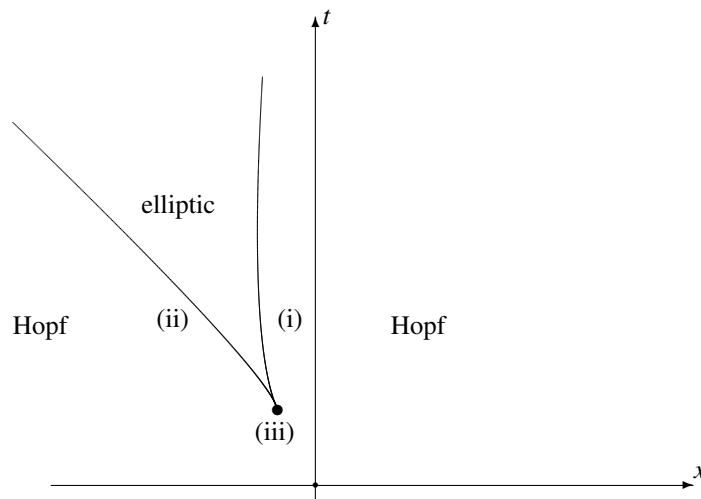
- (i)  $\beta_1$  approaches  $\beta_2$  (shrinking of an interval),
- (ii)  $\beta_2$  approaches  $\beta_3$  (merging of two intervals),
- (iii)  $\beta_1, \beta_2$ , and  $\beta_3$  approach each other (simultaneous shrinking and merging of intervals).

The transitions (i), (ii) and (iii) will lead to an asymptotic description of the KdV solution which is similar to the asymptotic description for the recurrence coefficients of orthogonal polynomials when the number of intervals in the support of the limiting mean eigenvalue density of random matrix ensembles changes. A transition of type (iii) takes place at the point of gradient catastrophe. In the  $(x, t)$  plane after the time of gradient catastrophe, the oscillations asymptotically develop in a  $V$ -shape region that does not depend on  $\epsilon$ ; see Figure 1. At the left boundary (the leading edge), a transition of type (ii) takes place, and at the right boundary (the trailing edge) we have a type (i) transition. Given  $t$  sufficiently short after the time of gradient catastrophe  $t_c$ , the leading edge  $x^-(t)$  is characterized by the system of equations

$$x^-(t) = 6tu(t) + f_L(u(t)), \quad (2-3)$$

$$6t + \theta(v(t); u(t)) = 0, \quad (2-4)$$

$$\partial_v \theta(v(t); u(t)) = 0, \quad (2-5)$$



**Figure 1.** Sketch of the phase diagram for the equilibrium problem associated to the KdV equation. Outside the cusp-shaped region, the support of the Lax–Levermore minimizer consists of one interval; inside of two intervals. At the cusp point, we have a type (iii) transition, at the left breaking curve one of type (ii), and at the right breaking curve, one of type (i).

where  $u(t) > v(t)$ ,  $f_L(u)$  is the inverse of the decreasing part of  $u_0(x)$ , and  $\theta$  is given by

$$\theta(\lambda; u) = \frac{1}{2\sqrt{2}} \int_{-1}^1 \frac{f'_L\left(\frac{1+m}{2}\lambda + \frac{1-m}{2}u\right) dm}{\sqrt{1-m}}. \quad (2-6)$$

This corresponds to the confluent case where the elliptic solution (2-1) degenerates formally to linear oscillations, namely  $\beta_2 = \beta_3 = v$  and  $\beta_1 = u$ . The trailing edge on the other hand is characterized by

$$x^+(t) = 6tu(t) + f_L(u(t)), \quad (2-7)$$

$$6t + \theta(v(t); u(t)) = 0, \quad (2-8)$$

$$\int_{u(t)}^{v(t)} (6t + \theta(\lambda; u(t))) \sqrt{\lambda - u(t)} d\lambda = 0, \quad (2-9)$$

with  $u(t) < v(t)$ , and  $\theta(\lambda; u)$  defined in (2-6). In this case we have  $\beta_1 = \beta_2 = v$  and  $\beta_3 = u$ . In this case the solution (2-1) degenerates formally to a soliton.

## 2.2. Critical asymptotics for the KdV solution.

**2.2.1. Point of gradient catastrophe.** Near the first break-up time, the KdV solution starts developing oscillations for small  $\epsilon$ . These oscillations are modeled

by a Painlevé transcendent  $U(X, T)$ , defined as the unique real smooth solution to the fourth order ODE

$$X = T U - \left[ \frac{1}{6} U^3 + \frac{1}{24} (U_X^2 + 2U U_{XX}) + \frac{1}{240} U_{XXXX} \right], \quad (2-10)$$

with asymptotic behavior given by

$$U(X, T) = \mp (6|X|)^{1/3} \mp \frac{1}{3} 6^{2/3} T |X|^{-1/3} + \mathcal{O}(|X|^{-1}), \quad \text{as } X \rightarrow \pm\infty, \quad (2-11)$$

for each fixed  $T \in \mathbb{R}$ . The existence of a pole free solution of (2-10) with asymptotic conditions (2-11) was conjectured in [Dubrovin 2006]) and proved in [Claeys and Vanlessen 2007a]. Let us denote  $t_c$  for the time of gradient catastrophe,  $x_c$  for the point where the  $x$ -derivative of the Hopf solution blows up, and  $u_c = u(x_c, t_c)$ . We take a double scaling limit where we let  $\epsilon \rightarrow 0$  and at the same time we let  $x \rightarrow x_c$  and  $t \rightarrow t_c$  in such a way that, for fixed  $X, T \in \mathbb{R}$ ,

$$\lim \frac{x - x_c - 6u_c(t - t_c)}{(8k\epsilon^6)^{1/7}} = X, \quad \lim \frac{6(t - t_c)}{(4k^3\epsilon^4)^{1/7}} = T, \quad (2-12)$$

where

$$k = -f_L'''(u_c).$$

In this double scaling limit the solution  $u(x, t, \epsilon)$  of the KdV Equation (1-1) has the expansion

$$u(x, t, \epsilon) = u_c + \left( \frac{2\epsilon^2}{k^2} \right)^{1/7} U \left( \frac{x - x_c - 6u_c(t - t_c)}{(8k\epsilon^6)^{1/7}}, \frac{6(t - t_c)}{(4k^3\epsilon^4)^{1/7}} \right) + \mathcal{O}(\epsilon^{4/7}). \quad (2-13)$$

The idea that the solution of KdV near the point of gradient catastrophe can be approximated by the solution of (2-10) appeared first in [Suleĭmanov 1994; 1993] and in a more general setting in [Dubrovin 2006], and was confirmed rigorously in [Claeys and Grava 2009]. In [Claeys and Grava 2012] the correction term of order  $\epsilon^{4/7}$  was determined.

**2.2.2. Leading edge.** Near the leading edge, the onset of the oscillations is described by the Hastings–McLeod solution to the Painlevé II equation

$$q''(s) = sq + 2q^3(s). \quad (2-14)$$

The Hastings–McLeod solution is characterized by the asymptotics

$$q(s) = \sqrt{-s/2}(1 + o(1)), \quad \text{as } s \rightarrow -\infty, \quad (2-15)$$

$$q(s) = \text{Ai}(s)(1 + o(1)), \quad \text{as } s \rightarrow +\infty, \quad (2-16)$$

where  $\text{Ai}(s)$  is the Airy function. The leading edge  $x^-(t)$  is, for  $t$  sufficiently short after  $t_c$ , determined by the system of equations (2-3)–(2-5). Let us consider



a double scaling limit where we let  $\epsilon \rightarrow 0$  and at the same time we let  $x \rightarrow x^-(t)$  in such a way that

$$\lim \frac{x - x^-(t)}{\epsilon^{2/3}} = X \in \mathbb{R}, \quad (2-17)$$

for  $t > t_c$  fixed. In this double scaling limit, the solution  $u(x, t, \epsilon)$  of the KdV equation with initial data  $u_0$  has the asymptotic expansion

$$u(x, t, \epsilon) = u - \frac{4\epsilon^{1/3}}{c^{1/3}} q[s(x, t, \epsilon)] \cos \frac{\Theta(x, t)}{\epsilon} + O(\epsilon^{2/3}), \quad (2-18)$$

where

$$\Theta(x, t) = 2\sqrt{u-v}(x - x^-) + 2 \int_v^u (f'_L(\xi) + 6t)\sqrt{\xi - v} d\xi, \quad (2-19)$$

and

$$c = -\sqrt{u-v} \frac{\partial^2}{\partial v^2} \theta(v; u) > 0, \quad s(x, t, \epsilon) = -\frac{x - x^-}{c^{1/3} \sqrt{u-v} \epsilon^{2/3}}, \quad (2-20)$$

with  $\theta$  defined by (2-6), and  $q$  is the Hastings–McLeod solution to the Painlevé II equation. Here  $x^-$  and  $v < u$  (each of them depending on  $t$ ) solve the system (2-3)–(2-5). The above result was proved in [Claeys and Grava 2010a], confirming numerical results in [Grava and Klein 2007]. In [Claeys and Grava 2010a], an explicit formula for the correction term of order  $\epsilon^{2/3}$  was obtained as well. We remark that a connection between leading edge asymptotics and the Painlevé II equation also appeared in [Kudashev and Suleĭmanov 1999].

**2.2.3. Trailing edge.** The trailing edge  $x^+(t)$  of the oscillatory interval (i.e., the right edge of the cusp-shaped region in Figure 2) is determined by the equations (2-7)–(2-9). As  $\epsilon \rightarrow 0$ , we have, for fixed  $y$  and  $t$ ,

$$u\left(x^+ + \frac{\epsilon \ln \epsilon}{2\sqrt{v-u}} y, t, \epsilon\right) = u + 2(v-u) \sum_{k=0}^{\infty} \operatorname{sech}^2 X_k + \mathcal{O}(\epsilon \ln^2 \epsilon), \quad (2-21)$$

where

$$\begin{aligned} X_k &= \frac{1}{2} \left( \frac{1}{2} - y + k \right) \ln \epsilon - \ln(\sqrt{2\pi} h_k) - \left( k + \frac{1}{2} \right) \ln \gamma, \\ h_k &= \frac{2^{k/2}}{\pi^{1/4} \sqrt{k!}}, \quad \gamma = 4(v-u)^{5/4} \sqrt{-\partial_v \theta(v; u)}, \end{aligned} \quad (2-22)$$

and  $\theta$  is given by (2-6) [Claeys and Grava 2010b]. It should be noted in this context that the KdV equation admits soliton solutions of the form  $a \operatorname{sech}^2(bx - ct)$ . This means that the last oscillations of the KdV solution resemble, at the local scale, solitons.

### 3. Phase diagram for unitary random matrix ensembles

**3.1. Equilibrium problem.** In unitary random matrix ensembles of the form (1-3), the limiting mean eigenvalue density is characterized as the equilibrium measure minimizing the logarithmic energy

$$I_V(\mu) = \iint \log \frac{1}{|s-y|} d\mu(s) d\mu(y) + \int V(s) d\mu(s), \quad (3-1)$$

among all probability measures on  $\mathbb{R}$ . For a polynomial external field of degree  $2m$ , the equilibrium measure is supported on a union  $S_V$  of at most  $m$  disjoint intervals. Its density can be written in the form [Deift et al. 1998a]

$$\psi_V(s) = \prod_{j=1}^k \sqrt{(b_j - s)(s - a_j)} h(s), \quad s \in \bigcup_{j=1}^k [a_j, b_j], \quad k \leq m, \quad (3-2)$$

where  $h$  is a polynomial of degree at most  $2(m-k)$ . The equilibrium measure is characterized by the variational conditions

$$2 \int \log |s-y| d\mu(y) - V(s) = \ell_V, \quad s \in \bigcup_{j=1}^k [a_j, b_j], \quad (3-3)$$

$$2 \int \log |s-y| d\mu(y) - V(s) \leq \ell_V, \quad s \in \mathbb{R}. \quad (3-4)$$

The external field  $V$  is called  $k$ -cut regular if  $h(s)$  in (3-2) is strictly positive on  $\bigcup_{j=1}^k [a_j, b_j]$  and if (3-4) is strict for  $s \in \mathbb{R} \setminus \bigcup_{j=1}^k [a_j, b_j]$ . In other words, it is singular if

- (i) equality in (3-4) holds at a point  $s^* \in \mathbb{R} \setminus \bigcup_{j=1}^k [a_j, b_j]$ ,
- (ii)  $h(s^*) = 0$  with  $s^* \in \bigcup_{j=1}^k (a_j, b_j)$ ,
- (iii)  $h(s^*) = 0$  with  $s^* = a_j$  or  $s^* = b_j$ .

**3.2. Example: quartic external field.** Let us now study a two-parameter family of quartic external fields

$$V_{x,t}(s) = e^x \left[ (1-t) \frac{s^2}{2} + t \left( \frac{s^4}{20} - \frac{4s^3}{15} + \frac{s^2}{5} + \frac{8}{5}s \right) \right]. \quad (3-5)$$

For  $t = 0$ , we have  $V_{x,0}(s) = e^x s^2/2$ , which means that the random matrix ensemble is a rescaled Gaussian Unitary Ensemble. The equilibrium measure  $\mu_{x,0}$  is then given by

$$d\mu_{x,0}(s) = \frac{e^x}{2\pi} \sqrt{4e^{-x} - s^2} ds, \quad s \in [-2e^{-x/2}, 2e^{-x/2}]. \quad (3-6)$$

It can indeed be verified directly that this measure satisfies the variational conditions (3-3)–(3-4). For  $x = 0$  and  $0 < t \leq 1$ , one can verify that

$$d\mu_{0,t}(s) = \frac{1}{2\pi(5+\gamma^2)} \sqrt{s^2 - 4((s-2)^2 + \gamma^2)} ds, \quad s \in [-2, 2], \gamma = \sqrt{\frac{5}{t} - 5}. \quad (3-7)$$

This shows that  $V_{0,1}$  has a singular point of type (iii) at  $s = 2$ . On the line  $t = 9$ ,  $V_{x,9}(s)$  is symmetric around  $s^* = \frac{4}{3}$ . The external field is one-cut regular for  $x < x^* =: -\log \frac{245}{9}$ , and presumably two-cut for  $x > x^*$ . For  $x \leq x^*$ , the equilibrium measure is given by

$$d\mu_{x,9}(s) = \frac{8}{\pi b^2(b^2 + 4C)} \sqrt{(s - \frac{4}{3} + b)(\frac{4}{3} + b - s)((s - \frac{4}{3})^2 + C)} ds, \quad s \in [s^* - b, s^* + b], \quad (3-8)$$

where

$$b = \sqrt{\frac{140}{27} + \frac{4}{27}\sqrt{5}e^{-x}\sqrt{27e^x + 245e^{2x}}}, \quad C = \frac{e^{-x}}{36b^2}(80 - 9b^4e^x). \quad (3-9)$$

At  $x = x^*$ , the equilibrium measure is given by

$$d\mu_{x^*,9}(s) = \frac{8}{\pi b^4} \sqrt{(s - \frac{4}{3} + b)(\frac{4}{3} + b - s)(s - \frac{4}{3})^2} ds, \quad s \in [\frac{4}{3} - b, \frac{4}{3} + b], \quad b = \frac{2}{3}\sqrt{35},$$

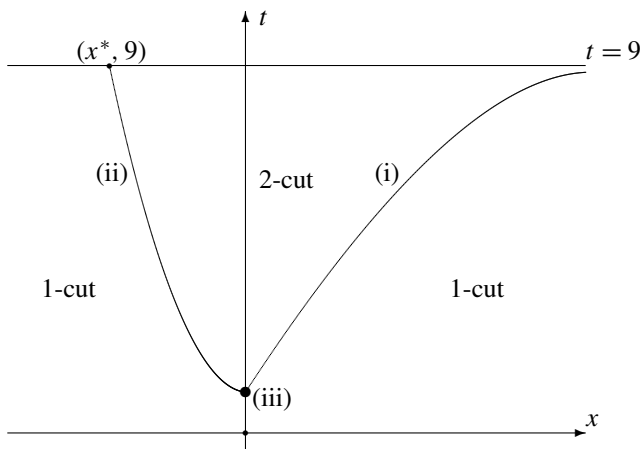
which means that there is a type (ii) singular point at  $s^* = \frac{4}{3}$ .

For  $t$  fixed and  $x$  sufficiently large and positive, it follows from results in [Kuijlaars and McLaughlin 2000] that the number of intervals is equal to the number of global minima of  $V_{x,t}$ , which is one for  $t < 9$  and two for  $t = 9$ . For  $t$  fixed and  $x$  sufficiently large negative, one can show that the equilibrium measure is supported on a single interval. Also, for any  $t$ , when  $x$  decreases, the support of the equilibrium measure increases. This suggests that there are, as shown in Figure 2, two curves in the  $(x, t)$ -plane where  $V_{x,t}$  is singular: one connecting  $(0, 1)$  with  $(x^*, 9)$  where a singular point of type (ii) is present, and one connecting  $(0, 1)$  with  $(+\infty, 9)$  where a singular point of type (i) occurs.

**Remark.** In [Bertola and Tovbis 2011], orthogonal polynomials with respect to complex weights of the form  $e^{-nV(x)}$  were considered, with  $V$  quartic symmetric with complex-valued leading coefficient. This lead to a phase diagram which shows certain similarities with ours, but also with breaking curves of a different nature.

**3.3. Regular asymptotics.** If  $V$  is a one-cut regular external field, the leading term of the asymptotics for the recurrence coefficients depends in a very simple way on the endpoints  $a$  and  $b$ : we have [Deift et al. 1999]

$$\gamma_n = \frac{b-a}{4} + \mathcal{O}(n^{-2}) \quad \text{as } n \rightarrow \infty \quad (3-10)$$



**Figure 2.** Sketch of the phase diagram for the equilibrium measure in external field  $V_{x,t}$ . The one-cut region and the two-cut region are separated by two curves, at the left curve a type (ii) singular point is present, at the right curve a type (i) singular point, and at the intersection point  $(0, 1)$  there is a type (iii) singular point.

and

$$\beta_n = \frac{b+a}{2} + \mathcal{O}(n^{-2}) \quad \text{as } n \rightarrow \infty. \quad (3-11)$$

If  $V$  is a two-cut regular external field, the leading order term in the asymptotic expansion for the recurrence coefficients is still determined by the endpoints  $a_1, b_1, a_2, b_2$ , but the dependence is somewhat more complicated, and the leading term is oscillating with  $n$ . An explicit formula for the leading order asymptotics was given and proved in [Deift et al. 1999] for  $k$ -cut regular external fields  $V$ , with  $k$  arbitrary. We will not give details about those asymptotics, but we note that the expansion is of a similar nature as (2-1) in the two-cut case.

**3.4. Critical asymptotics.** We will now describe the critical asymptotics for the recurrence coefficients  $\gamma_n(x, t)$  and  $\beta_n(x, t)$  of the orthogonal polynomials with respect to the weight  $e^{-nV_{x,t}}$ . It should be noted that critical asymptotics near type (ii) and type (iii) singular points are known for more general deformations of external fields  $V_{x,t}$  than only the one defined by (3-5).

**3.4.1. Singular interior points.** Assume that  $V_{x^*, t^*}(s)$  is a singular external field with a singular point  $s^*$  of type (ii) (a singular interior point), and with support  $[a, b]$  of the equilibrium measure. Asymptotics for the recurrence coefficients were obtained in [Bleher and Its 2003] for quartic symmetric  $V$  and in [Claeys et al. 2008] for real analytic  $V$ . Let us specialize the results to our example

where  $V_{x,t}$  is given by (3-5). Since  $V_{x^*,t^*}$  is quartic, this implies that  $\psi_{x^*,t^*}$  has the form

$$\psi_{x^*,t^*}(s) = C\sqrt{(s-a)(b-s)}(s-s^*)^2. \quad (3-12)$$

Then as  $n \rightarrow \infty$  simultaneously with  $x \rightarrow x^*$  such that  $x - x^* = \mathcal{O}(n^{-2/3})$ , we have the asymptotic expansions [Claeys et al. 2008]

$$\gamma_n(x, t^*) = \frac{b-a}{4} - \frac{1}{2c}q(s_{x,n})\cos(2\pi n\omega(x))n^{-1/3} + \mathcal{O}(n^{-2/3}), \quad (3-13)$$

$$\beta_n(x, t^*) = \frac{b+a}{2} + \frac{1}{c}q(s_{x,n})\sin(2\pi n\omega(x) + \theta)n^{-1/3} + \mathcal{O}(n^{-2/3}), \quad (3-14)$$

where

$$s_{x,n} = n^{2/3}(e^{x^*-x} - 1)\frac{1}{c\sqrt{(s^*-a)(b-s^*)}},$$

and where  $c$ ,  $\theta$ , and  $\omega$  are given by

$$c = \left(\frac{\pi C\sqrt{(s^*-a)(b-s^*)}}{4}\right)^{1/3}, \quad \theta = \arcsin \frac{b+a}{b-a},$$

$$\omega(x) = \int_0^b \psi_{x^*,t^*}(s) ds + \mathcal{O}(n^{-2/3}) \quad \text{as } n \rightarrow \infty.$$

An exact formula for  $\omega$  can be given in terms of a modified equilibrium problem. When  $x$  approaches  $x^*$ , we observe that the recurrence coefficients develop oscillations. The envelope of the oscillations is described by the Hastings–McLeod solution  $q$ . One should compare formulas (3-13)–(3-14) with (2-13) and note that the scalings correspond after identifying  $\epsilon$  with  $1/n$ .

**3.4.2. Singular edge points.** Asymptotics for the recurrence coefficients for general one-cut external fields  $V$  with a singular endpoint were obtained in [Claeys and Vanlessen 2007b]. Let  $V_0$  be an external field such that the equilibrium measure is supported on  $[a, b]$  and such that the density  $\psi_0$  behaves like  $\psi_0(s) \sim c(b-s)^{5/2}$  as  $s \rightarrow b$ ,  $c \neq 0$ . Double scaling asymptotics were obtained for external fields of the form  $V_0 + SV_1 + TV_2$  with real  $S, T \rightarrow 0$ , where  $V_1$  is arbitrary and  $V_2$  satisfies the condition

$$\int_a^b \sqrt{\frac{s-a}{b-s}} V_2'(s) ds = 0.$$

We can write  $V_{x,t}$  in the form

$$V_{x,t}(s) = V_{0,1}(s) + (e^x - 1)V_{0,1}(s) + e^x(t-1)(V_{0,1}(s) - V_{0,0}(s)). \quad (3-15)$$

Since

$$\int_{-2}^2 \sqrt{\frac{s+2}{2-s}} (V_{0,1}'(s) - V_{0,0}'(s)) ds = 0,$$

we can apply the results of [Claeys and Vanlessen 2007b]. In the double scaling limit where  $n \rightarrow \infty$  and simultaneously  $x \rightarrow 0$ ,  $t \rightarrow 1$  in such a way that  $\lim n^{6/7}(e^x - 1)$  and  $\lim n^{4/7}e^x(t - 1)$  exist, we have

$$\gamma_n(x, t) = 1 + \frac{1}{2c} U(c_1 n^{6/7}(e^x - 1), c_2 n^{4/7}e^x(t - 1)) n^{-2/7} + \mathcal{O}(n^{-4/7}), \quad (3-16)$$

$$\beta_n(x, t) = \frac{1}{c} U(c_1 n^{6/7}(e^x - 1), c_2 n^{4/7}e^x(t - 1)) n^{-2/7} + \mathcal{O}(n^{-4/7}). \quad (3-17)$$

The constants  $c$ ,  $c_1$ ,  $c_2$  are given by

$$c = 6^{2/7} > 0, \quad c_1 = \frac{1}{2\pi c^{1/2}} \int_{-2}^2 \sqrt{\frac{u-2}{2-u}} V'_{0,1}(u) du = 6^{-1/7},$$

$$c_2 = \frac{1}{4\pi i c^{3/2}} \int_{\gamma} \sqrt{\frac{2+u}{(2-u)^3}} (V'_{0,1}(u) - V'_{0,0}(u)) du = 2 \cdot 6^{-3/7},$$

where  $\gamma$  is a counterclockwise oriented contour encircling  $[-2, 2]$ .

**Remark.** Applying the results from [Claeys and Vanlessen 2007b] directly, one has an error term  $\mathcal{O}(n^{-3/7})$  in (3-16) and (3-17), but going through the calculations, it can be verified that the error term is actually  $\mathcal{O}(n^{-4/7})$ . The analogy between (3-16)–(3-17) and (2-13) is obvious.

**3.4.3. Singular exterior points.** Asymptotics for the recurrence coefficients in the vicinity of a singular exterior point have not appeared in the literature to the best of our knowledge. Asymptotics for orthogonal polynomials associated to an external field  $V$  with a singular exterior point and for the correlation kernel (1-5) have been studied in [Claeys 2008; Bertola and Lee 2009; Mo 2008] using the Riemann–Hilbert approach. We are convinced that the same analysis can be used, with some additional effort, to compute asymptotics for the recurrence coefficients. If  $V_{x^*, t^*}$  is an external field with a singular exterior point, the analogy with the KdV asymptotics suggests asymptotic expansions of the form

$$\gamma_n\left(x^* - y \frac{\ln n}{c_0 n}, t\right) = \frac{b(x^*, t) - a(x^*, t)}{4} + c_1 \sum_{k=0}^{\infty} \operatorname{sech}^2 X_k + \mathcal{O}(n^{-1} \ln^2 n) \quad (3-18)$$

$$\beta_n\left(x^* - y \frac{\ln n}{c_0 n}, t\right) = \frac{b(x^*, t) + a(x^*, t)}{2} + c_1 \sum_{k=0}^{\infty} \operatorname{sech}^2 X_k + \mathcal{O}(n^{-1} \ln^2 n), \quad (3-19)$$

as  $n \rightarrow \infty$ , where

$$X_k = -c_2(y, k) \ln n + c_3(k).$$

#### 4. The problem of matching

Asymptotic expansions for KdV solutions are known in the regular regions and in critical regions, but we do not have uniform asymptotics for  $u(x, t, \epsilon)$  in  $x$  and  $t$ . Indeed, the critical asymptotics are only valid in shrinking neighborhoods of the breaking curves: a neighborhood of size  $\mathcal{O}(\epsilon^{2/3})$  near the leading edge, a neighborhood of size  $\mathcal{O}(\epsilon \ln \epsilon)$  near the trailing edge, and a neighborhood of size  $\mathcal{O}(\epsilon^{4/7})$  at the point of gradient catastrophe. On the other hand, the regular asymptotics are only proved to hold uniformly for  $x$  and  $t$  at a fixed distance away from the breaking curves. However one can see easily that (2-13) and (2-21) match formally with the regular asymptotics for  $x$  close to the breaking curves but outside the cusp-shaped region. Indeed for (2-13) this follows from the decay of the Hastings–McLeod solution  $q$  at  $+\infty$ . When  $x$  is close to the boundary but inside the cusp-shaped region, the situation is more complicated. One can hope that the regular asymptotics can be improved in such a way that they hold also when  $x, t$  approach a breaking curve sufficiently slowly when  $\epsilon$  tends to 0, and that the critical asymptotics can be improved to hold in a slightly bigger neighborhood of the breaking curves. It would be of interest to see if such an approach could provide uniform asymptotics for the KdV solution as  $\epsilon \rightarrow 0$ .

The problem of obtaining uniform asymptotics in  $x$  and  $t$  for the recurrence coefficients  $\gamma_n(x, t)$  and  $\beta_n(x, t)$  may seem an artificial one at first sight, since one is often interested in a random matrix with a fixed external field  $V$  instead of letting  $V$  vary. However, it becomes more relevant when studying the partition function

$$Z_n = \int_{\mathbb{R}^n} \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^n e^{-nV(\lambda_i)} d\lambda_i.$$

It is well-known that

$$Z_n = n! \prod_{j=1}^{n-1} \kappa_j^{-2},$$

where  $\kappa_j$  is the leading coefficient of the normalized orthogonal polynomial  $p_j$  with respect to the weight  $e^{-nV}$ . A consequence of this formula is that, if one lets  $V$  vary with a parameter  $\tau$  in a convenient way, it is possible to derive various identities for  $\tau$ -derivatives of  $\ln Z_n$  in terms of the recurrence coefficients  $\gamma_k(\tau)$  and  $\beta_k(\tau)$  for  $k$  large [Bleher and Its 2005; Ercolani and McLaughlin 2003]. A possible strategy to obtain asymptotics for the partition function, is to let the  $\tau$ -dependence be such that  $V$  interpolates between the Gaussian  $V(z; \tau_0) = \frac{z^2}{2}$  and  $V(z; \tau_1) = V(z)$ . Integrating the differential identity then requires asymptotics for the Gaussian partition function (which are known) and uniform asymptotics for the recurrence coefficients  $\gamma_n(\tau)$  and  $\beta_n(\tau)$  over the whole range  $[\tau_0, \tau_1]$ .

Depending on the chosen deformation, this could require uniform asymptotics for the recurrence coefficients near a singular point of type (i), (ii), or (iii). The results presented in the previous section do not provide sufficiently detailed asymptotics for the recurrence coefficients: they are not uniform near the breaking curves. For example near a critical point of type (ii), formulas (3-13)–(3-14) are only valid for  $x - x^* = \mathcal{O}(n^{-2/3})$  as  $n \rightarrow \infty$ , whereas the asymptotic formula in the two-cut region is valid only at a fixed distance away from a critical point.

### 5. The Toda lattice and KdV

It is well-known that recurrence coefficients for orthogonal polynomials follow the time flows of the Toda hierarchy. In this section, following [Dubrovin 2009; Dubrovin et al. 2013] we will formally derive the KdV equation as a scaling limit of the continuum limit of the Toda lattice. This gives a heuristic argument why asymptotics for KdV and the recurrence coefficients show similarities.

The Toda lattice is a Hamiltonian system described by the equations

$$\frac{du_n}{dt} = v_n - v_{n-1}, \quad \frac{dv_n}{dt} = e^{u_{n+1}} - e^{u_n}, \quad n \in \mathbb{Z}. \quad (5-1)$$

The Toda lattice is a prototypical example of a completely integrable system [Flaschka 1974]. Let

$$V(\xi) = V_0(\xi) + \sum_{j=1}^{2d} t_j \xi^j, \quad t_{2d} > 0, \quad (5-2)$$

where  $V_0(\xi)$  is a fixed polynomial of even degree with positive leading coefficient, and let  $p_j$  be the orthogonal polynomials defined by

$$\int_{-\infty}^{\infty} p_n(\xi) p_m(\xi) e^{-\frac{1}{\epsilon} V(\xi)} d\xi = \delta_{nm}, \quad (5-3)$$

where  $\epsilon = \frac{1}{N}$  is a small positive parameter. As mentioned before, the polynomials  $p_n(\xi)$  satisfy a three term recurrence relation of the form (1-6).

The recurrence coefficients  $\gamma_n$  and  $\beta_n$  in (1-6) evolve with respect to the times  $t_k$  defined in (5-2) according to the equations [Eynard 2001; Douglas and Shenker 1990; Fokas et al. 1992; Bertola et al. 2003]

$$\epsilon \frac{\partial \gamma_n}{\partial t_k} = \frac{\gamma_n}{2} ([Q^k]_{n-1, n-1} - [Q^k]_{nn}), \quad (5-4)$$

$$\epsilon \frac{\partial \beta_n}{\partial t_k} = \gamma_n [Q^k]_{n, n-1} - \gamma_{n+1} [Q^k]_{n+1, n}, \quad (5-5)$$



where  $[Q^k]_{n,m}$  denotes the  $n, m$ -th element of the matrix  $Q^k$  and  $Q$  is the tridiagonal matrix

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & 0 & 0 & \dots \\ \gamma_1 & \beta_1 & \gamma_2 & 0 & 0 & \dots \\ 0 & \gamma_2 & \beta_2 & \gamma_3 & 0 & \dots \\ 0 & 0 & \gamma_3 & \beta_3 & \gamma_4 & \dots \\ 0 & 0 & 0 & \gamma_4 & \beta_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (5-6)$$

The equations (5-4)–(5-5) are the Toda lattice hierarchy in the Flaschka variables [Flaschka 1974]. In particular the first flow of the hierarchy takes the form

$$\begin{aligned} \epsilon \frac{\partial \gamma_n}{\partial t_1} &= \frac{\gamma_n}{2} (\beta_{n-1} - \beta_n), \\ \epsilon \frac{\partial \beta_n}{\partial t_1} &= \gamma_n^2 - \gamma_{n+1}^2. \end{aligned} \quad (5-7)$$

These equations correspond to the Toda lattice (5-1) by identifying  $t_1 = t$ ,  $\beta_n = -v_n$  and

$$u_n = \log \gamma_n^2. \quad (5-8)$$

In addition to the Toda equations, the recurrence coefficients for the orthogonal polynomials satisfy a constraint that is given by the discrete string equation which takes the form [Fokas et al. 1992]

$$\gamma_n [V'(Q)]_{n,n-1} = n\epsilon, \quad [V'(Q)]_{n,n} = 0. \quad (5-9)$$

For example, choosing  $V_0(\xi) = \frac{1}{2}\xi^2$  one obtains

$$\beta_n(\mathbf{t} = 0) = 0, \quad \gamma_n^2(\mathbf{t} = 0) = n\epsilon, \quad \mathbf{t} = (t_1, t_2, \dots, t_{2d}). \quad (5-10)$$

To obtain the continuum limit of the Toda lattice, let us assume that  $u(x)$  and  $v(x)$  are smooth functions that interpolate the sequences  $u_n, v_n$  in the following way:  $u(\epsilon n) = u_n$  and  $v(\epsilon n) = v_n$  for some small  $\epsilon > 0$ ,  $n > 0$ ,  $x = \epsilon n$ . Then the Toda lattice (5-1) reduces to an evolutionary PDE of the form [Eguchi and Yang 1994; Deift and McLaughlin 1998]

$$\begin{aligned} u_t &= \frac{1}{\epsilon} [v(x) - v(x - \epsilon)] = v_x - \frac{1}{2}\epsilon v_{xx} + O(\epsilon^2), \\ v_t &= \frac{1}{\epsilon} [e^{u(x+\epsilon)} - e^{u(x)}] = e^u u_x + \frac{1}{2}\epsilon (e^u)_{xx} + O(\epsilon^2). \end{aligned} \quad (5-11)$$

In order to write the continuum limit of the Toda lattice in a canonical Hamiltonian form, following [Dubrovin and Zhang 2001], we introduce  $w(x)$  by

$$w(x) = \epsilon \partial_x [1 - e^{-\epsilon \partial_x}]^{-1} u(x) = u + \frac{\epsilon}{2} u_x + \frac{\epsilon^2}{12} u_{xx} + \dots \quad (5-12)$$

In the coordinates  $v, w$  the continuum limit of the Toda lattice equations takes the form

$$\begin{aligned} w_t &= v_x, \\ v_t &= e^w \left( w_x + \frac{\epsilon^2}{24} (2w_{xxx} + 4w_x w_{xx} + w_x^3) \right) + O(\epsilon^4), \end{aligned} \quad (5-13)$$

with the corresponding Hamiltonian given by

$$H = \int \left( \frac{v^2}{2} + e^w - \frac{\epsilon^2}{24} e^w w_x^2 + \dots \right) dx$$

and Poisson bracket  $\{v(x), w(y)\} = \delta'(x - y)$  where  $\delta(x)$  is the Dirac  $\delta$  function. We remark that in these coordinates the continuum limit of the Toda equation contains only even terms in  $\epsilon$ . For  $\epsilon = 0$ , (5-13) reduces to

$$w_t = v_x, \quad v_t = e^w w_x. \quad (5-14)$$

The solution of equations (5-14) can be obtained by the method of characteristics. The initial data relevant to us should satisfy the continuum limit of the string equation (5-9) for  $t = 0$ . The Riemann invariants of (5-14) are

$$r_{\pm} = v \pm 2e^{w/2},$$

so that (5-14) takes the form

$$\frac{\partial}{\partial t} r_{\pm} + \lambda_{\pm} \frac{\partial}{\partial x} r_{\pm} = 0, \quad \lambda_{\pm} = \mp e^{w/2} = \mp \frac{r_+ - r_-}{4}.$$

The generic solution of (5-14) can be written in the form [Tsarëv 1990; Whitham 1974]

$$x = \lambda_{\pm} t + f_{\pm}(r_+, r_-), \quad (5-15)$$

where  $f_{\pm}(r_+, r_-)$  are two functions that satisfy the equations [Tsarëv 1990]

$$\frac{\partial}{\partial r_-} f_+ = \frac{\partial \lambda_+}{\partial r_-} \frac{f_+ - f_-}{\lambda_+ - \lambda_-} = -\frac{f_+ - f_-}{2(r_+ - r_-)} = \frac{\partial}{\partial r_+} f_-. \quad (5-16)$$

From (5-16) one can conclude that there exists a function  $f = f(r_+, r_-)$  such that

$$f_{\pm} = \frac{\partial f}{\partial r_{\pm}}.$$

The explicit dependence of  $f$  for a certain class of initial data can be found in [Deift and McLaughlin 1998]. To obtain  $f$  in the random matrix case we impose that the equations (5-15) are consistent with the continuum limit of the discrete string equation (5-9) for  $t_1 = t \geq 0$  and  $t_j = 0$  for  $j > 1$ . At the leading order in  $\epsilon$  the string equation (5-9) in the Riemann invariants  $r_{\pm} = -\beta \pm 2\gamma$  gives after straightforward but long calculations, the following expression for the function  $f(r_+, r_-)$ :

$$f(r_+, r_-) = -\text{Res}_{\xi=\infty} [V_0'(\xi) \sqrt{(\xi - r_+)(\xi - r_-)} d\xi]. \quad (5-17)$$

**Remark.** The equations (5-15) with  $f$  given in (5-17), coincide with the equations that define the support of the equilibrium measure for the variational problem

$$\inf_{\int_{\mathbb{R}} d\nu(\xi)=1} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} \log \frac{1}{|\xi - \eta|} d\nu(\xi) d\nu(\eta) + \frac{1}{x} \int_{\mathbb{R}} V(\xi) d\nu(\xi) \right]$$

in the case where the equilibrium measure is supported on one interval. The Riemann invariants  $r_+$  and  $r_-$  can thus be interpreted as the end-points of the support of the equilibrium measure.

In what follows, we are going to show that the solution of the Equation (5-13) in the vicinity of a singular point of type (iii) reduces to the KdV equation, in agreement with [Dubrovin 2009]. First we consider the solution of the hodograph Equation (5-15) near a singular point of type (iii); namely, let  $(x_c, t_c)$  be a point of gradient catastrophe for the Riemann invariant  $r_+$ , which means that  $\partial_x r_+$  goes to infinity at the critical point  $(x_c, t_c)$ . We define  $r_{\pm}(x_c, t_c) = r_{\pm}^c$ . Such a critical point is characterized by the conditions

$$\lambda_{+,+}^c t_c + f_{+,+}^c = 0, \quad \lambda_{+,++}^c t_c + f_{+,++}^c = 0,$$

and the critical point is generic if

$$\lambda_{+,+++}^c t_c + f_{+,+++}^c \neq 0, \quad \lambda_{+,-}^c t_c + f_{+,-}^c \neq 0,$$

where we have used the notation  $\lambda_{-,-}^c = \frac{\partial}{\partial r_-} \lambda_-(r_+ = r_+^c, r_- = r_-^c)$  and consistently for the other terms.

Expanding in power series (5-15) near  $(x_c, t_c)$  and using (5-16) after the rescalings

$$\begin{aligned} x_- &= k^{-2/3}(x - x_c - \lambda_-^c(t - t_c)), & x_+ &= k^{-1}(x - x_c - \lambda_+^c(t - t_c)) \\ \bar{r}_- &= k^{-2/3}(r_- - r_-^c), & \bar{r}_+ &= k^{-1/3}(r_+ - r_+^c), \end{aligned} \quad (5-18)$$

one obtains, letting  $k \rightarrow 0$ ,

$$x_- = c_1 \bar{r}_-, \quad x_+ = c_2 x_- \bar{r}_+ + c_3 \bar{r}_+^3, \quad (5-19)$$

where

$$c_1 = (f_{-,-}^c + \lambda_{-,-}^c t_c), \quad c_2 = \frac{\lambda_{+,+}^c}{\lambda_+^c - \lambda_-^c}, \quad c_3 = \frac{1}{6}(\lambda_{+,+++}^c t_c + f_{+,+++}^c). \quad (5-20)$$

We observe that (5-19) describes a Whitney singularity in the neighborhood of  $(0, 0)$  [Dubrovin 2009]. Performing the same rescalings (5-18) to the equations (5-14) and letting  $k \rightarrow 0$  one obtains

$$\frac{\partial \bar{r}_-}{\partial x_+} = 0, \quad \frac{\partial \bar{r}_+}{\partial x_-} + c_2 \bar{r}_+ \frac{\partial \bar{r}_+}{\partial x_+} = 0,$$

with  $c_2$  as in (5-20). Clearly (5-19) represents a solution of the above equations with singularity in  $(x_+ = 0, x_- = 0)$  and at  $\bar{r}_\pm = 0$ . The next step is to perform the rescaling (5-18) to (5-13) and letting  $\epsilon \rightarrow k^{7/6}\epsilon$ . One obtains, in the limit  $k \rightarrow 0$ ,

$$\begin{aligned} \bar{r}_- &= \frac{x_-}{c_1} + c_4 \epsilon^2 \frac{\partial^2}{\partial x_+^2} \bar{r}_+, \quad c_4 = \frac{r_+^c - r_-^c}{192(\lambda_-^c - \lambda_+^c)} = \frac{1}{96}, \\ \frac{\partial \bar{r}_+}{\partial x_-} + c_2 \bar{r}_+ \frac{\partial \bar{r}_+}{\partial x_+} + c_4 \epsilon^2 \frac{\partial^3}{\partial x_+^3} \bar{r}_+ &= 0. \end{aligned} \quad (5-21)$$

The first of the above equations has been obtained after integration with respect to  $x_+$  using (5-19). The second one is the KdV equation for  $r_+$  with time variable  $x_-$  and space variable  $x_+$ . Such derivation has been obtained in a more general setting in [Dubrovin et al. 2013]. On the formal level, the above calculations explain why the asymptotic behavior of the solution of the continuum limit of Toda lattice and in particular of the recurrence coefficients of orthogonal polynomials near the point of gradient catastrophe is of a similar nature as the KdV case. However, a rigorous proof of the generic behavior of the solution of the continuum limit of Toda lattice near the point of gradient catastrophe cannot be derived from the KdV case but a separate proof is needed.

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