REMARKS ON WILMSHURST'S THEOREM

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Abstract. We demonstrate counterexamples to Wilmshurst's conjecture on the valence of harmonic polynomials in the plane, and we conjecture a bound that is linear in the analytic degree for each fixed anti-analytic degree. Then we initiate a discussion of Wilmshurt's theorem in more than two dimensions, showing that if the zero set of a polynomial harmonic field is bounded then it must have codimension at least two. Examples are provided to show that this conclusion cannot be improved.

1. Introduction

Suppose $F : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field with polynomial components, each of degree $n > 0$. For a generic choice of F, intersection theory (Bezout's theorem) implies:

$$N_F \leq n^d, \tag{1}$$

where N_F is the number of zeros of F, points where F vanishes. This bound is sharp in general, but it is natural to ask:

Question: For interesting special classes of F, can (1) be improved?

Consider for example a *proper* F; then it extends to a continuous self-map of the one-point compactification $S^d = \mathbb{R}^d \cup \{\infty\}$, and the topological degree (i.e., Brouwer degree) $\deg F$ can be defined as $F_*(1)$, where $F_* : H_d(S^d) \to H_d(S^d) \cong \mathbb{Z}$ is the homomorphism between homology groups induced by F and 1 is the generator corresponding to a fixed orientation. This is equivalent to the notion of degree from differential topology [14] expressed as a sum of signs of the Jacobian determinant of F evaluated at all preimages of any given non-critical value (a proof of the equivalence of the two notions of degrees is given in [8, Proposition 2.30]).

Example 1: In particular, if the Jacobian determinant of $F : \mathbb{R}^d \to \mathbb{R}^d$ is non-negative, and zero is not a critical value, then the number of zeros N_F coincides with the $\deg F$. For $d = 2$ and $F(z)$ an analytic polynomial of $z \in \mathbb{C} \cong \mathbb{R}^2$ with no singular zeros, we have

$$N_F = \deg F = n.$$

This is the fundamental theorem of algebra.

Example 2: Suppose $F(x) = F_+(x) + F_-(x)$ can be decomposed into an orientation preserving part F_+ of (topological) degree n, and an orientation reversing part F_- of degree $m < n$. Can the bound (1) be improved? What if the components of F are assumed to be harmonic polynomials?

While considering special classes of polynomial vector fields, another question is whether the word "generic" can be removed for F within that class. A. S. Wilmshurst [19, 20] considered the case of harmonic fields. Using complex variable notation $z \in \mathbb{C}$, a harmonic field can be expressed as a sum of analytic and anti-analytic parts:

$$F(z) = p(z) + \overline{q(z)}$$

(this is an instance of a decomposition $F(x) = F_+(x) + F_-(x)$ as described above). In these terms, Wilmshurst showed:

Theorem 1 (A. S. Wilmshurst, 1994)**.** *If* $\deg p = n > m = \deg q$*, then* $N_F \leq n^2$*.*

In other words, for this class of vector fields F, Bezout's bound $N_F \leq n^2$ applies generally, not just generically. This was independently shown in [16].

As to the question of improving (1) given additional information, Wilmshurst made the tantalizing conjecture that

$$N_F \leq 3n - 2 + m(m-1). \tag{2}$$

This conjecture is stated in [20, Remark 2], and was discussed further in [18]. It is also mentioned in the list of open problems in [5]. For $m = n-1$ the upper bound follows from Wilmshurst's theorem, and examples were also given in [20] showing that this bound is sharp (shown independently in [3]). For $m = 1$, the upper bound was proved by D. Khavinson and G. Swiatek [9] using holomorphic dynamics. A proof of the Crofoot-Sarason conjecture given in [7] (cf. [4]) established that this bound is sharp. For $m = n - 3$, the conjectured bound is

$$3n - 2 + m(m-1) = n^2 - 4n + 10.$$

We provide counterexamples for which

$$N_F > n^2 - 3n + \mathcal{O}(1).$$

We state the counterexamples in Section 2 and prove this estimate on the number of zeros.

It is also of interest to mention that an extension of the $m = 1$ result [9] to rational functions [10, 11] (again using holomorphic dynamics) settled a conjecture in gravitational lensing. This setting was recently revisited in [1] with a more real algebraic approach, as we have taken in the current paper.

In Section 3, we give an alternative proof of Wilmshurst's theorem (Corollary 8 for $d = 2$) that relies more heavily on real algebraic geometry and readily generalizes to harmonic vector fields in higher dimensions but with a weaker conclusion: the zero set has codimension at least two (for $d = 2$ this implies the number of zeros is finite). As we show by example, this cannot be improved without adding additional assumptions on F. Even though the number of zeros may not be finite, the number of connected components is, and the bound due to J. Milnor [13] can be applied to estimate the number of connected components (see Section 3).

It would be interesting to carry out a random study for $d > 2$ similar to what was done in $d = 2$ dimensions [12]. For any reasonable distribution of probability on the space of harmonic polynomials, the number of zeros N_F is finite with probability 1. This is because a generic F does satisfy the Bezout bound, which we explain in Section 3, Proposition 9 (similar in spirit to the genericity results in [1]). Thus, it makes sense to ask what is the expected number of zeros of a random harmonic polynomial field F.

Returning to the deterministic setting, the spirit of Wilmshurst's conjecture — that the maximum number of zeros is linear in n for each fixed m — still seems plausible. We do not venture a conjecture as to the exact maximum (and perhaps it is not described by a simple formula), but we are willing to make the following conjecture.

Conjecture: Let $F(z) = p(z) + \overline{q(z)}$ be as above (with $n > m$). Then the number N_- of orientation reversing zeros of F satisfies the upper bound

$$N_- \leq m(n-1).$$

The number $m(n-1)$ is the degree of q times the maximum possible number of components of the set where F reverses orientation. The Conjecture implies that $N_F \leq 2m(n-1) + n$ when F is free of singular zeros (which is worse than the Bezout bound for large m, but it is linear in n for fixed m and reduces to $3n - 2$ when $m = 1$). This can be seen using basic degree theory (the argument principle for harmonic functions). Suppose $F(z) = p(z) + \overline{q(z)}$ is free of singular zeros (zeros where the Jacobian of F, $|p'(z)|^2 - |q'(z)|^2$ vanish). Let N_+ count the orientation-preserving zeros and N_- the orientation-reversing zeros of F. Suppose Ω is a domain with smooth boundary $\partial\Omega$ without zeros on $\partial\Omega$. The argument principle for harmonic functions [6] states that the winding number $\text{Ind}_{\partial\Omega}F(z)$ around the boundary of a domain Ω counts the number of orientation preserving zeros inside Ω minus the number of orientation reversing zeros. Thus, for a large enough circle C:

$$\text{Ind}_C F(z) = N_+ - N_-.$$

If $\deg p = n > m = \deg q$ then $\text{Ind}_C F(z) = n$ for C large enough, since the z^n term dominates. If $N_- \leq m(n-1)$ then $N_+ = N_- + n$, and thus, as claimed above, the conjecture implies that $N_F = N_+ + N_- \leq 2m(n-1) + n$.

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2. Counterexamples to Wilmshurst's conjecture

Theorem 2. *For $m = n-3$ and $n \geq 4$, there exist polynomials $p(z)$ and $q(z)$ of degrees n and m respectively, such that the number of roots to the equation, $\overline{p(z)} = q(z)$, is at least*

$$n^2 - 4n + 4\left\lfloor \frac{n-2}{\pi} \arctan \frac{\sqrt{n^2-2n}}{n} \right\rfloor + 2. \tag{3}$$

Remark 1. As n goes to ∞, the above expression is $n^2 - 3n + \mathcal{O}(1)$ and, therefore, provides infinitely many counterexamples to Wilmshurst's conjecture, with a discrepency that grows linearly in n.

Remark 2. We note that (3) is *not* the maximal number of roots. In fact, the explicit examples that we will give in the proof of the theorem have more roots than stated in (3). We state our theorem with the modest estimate shown in (3) as this is enough for our purpose which is to provide counterexamples. Also, we do not know whether our examples in the proof give the maximal number of roots for a given n.

Remark 3. To mention a specific case, for $n = 8$ with $a = 1 + 0.04\mathrm{i}$ Figure 1 shows the plot of real and imaginary parts having 44 roots. This exceeds the conjectured bound of 42 by two. Note however that 44 does not follow from the above theorem (see *Remark* 2). The above theorem begins to produce more roots than the conjectured bound at exactly $n \geq 15$. For $n = 15$, the above bound gives 179 roots, while the conjectured bound $n^2 - 4n + 10$ gives 175.

The explicit construction of counterexamples will be done similarly as the construction of the examples for $m = n-1$ by Wilmshurst [Wilm95].

Let us consider the polynomial of degree n given by

$$f(z) = (z-a)^{n-2}P(z), \quad P(z) = z^2 + (n-2)\,a\,z + \frac{(n-2)(n-1)}{2}a^2. \tag{4}$$

The quadratic factor $P(z)$ is chosen such that $f(z) = z^n + \mathcal{O}(z^{n-3})$ as $z \to \infty$.

We define the level lines of $\operatorname{Im} f$ by

$$\Gamma = \{z \,|\, \operatorname{Im} f(z) = 0\}. \tag{5}$$

For $z \in \mathcal{Z} := \Gamma \cap \{z \,|\, \operatorname{Re} z^n = 0\}$, the following equation is satisfied.

$$z^n + \overline{z^n} = f(z) - \overline{f(z)} \quad \Longrightarrow \quad -z^n + f(z) = \overline{z^n} + \overline{f(z)}. \tag{6}$$

The left hand side is a holomorphic polynomial of degree $n-3$, and the right hand side is a polynomial of degree n. Therefore the set $\mathcal{Z}$ consists of the roots of the equation $\overline{p(z)} = q(z)$ where $q(z) = -z^n + f(z)$ and $p(z) = z^n + f(z)$.

The following lemma shows properties of Γ that are useful in counting the points in $\mathcal{Z}$.

Lemma 3. *The level set Γ satisfies the following:*

- Γ *is the union of* n *smooth curves that diverge to* ∞ *in the directions* $\mathrm{e}^{\mathrm{i}\pi j/n}$ *and* $\mathrm{e}^{\mathrm{i}\pi k/n}$ *for some* $j, k \in \{0, 1, \cdots, 2n-1\}$ *such that* $j \neq k$. *(Let us use the notations* $\infty \times \mathrm{e}^{\mathrm{i}\theta}$ *for "the limit taken in the direction* $\mathrm{e}^{\mathrm{i}\theta}$*". Then we may say that "*Γ *interpolates* $\infty \times \mathrm{e}^{\mathrm{i}\pi j/n}$ *and* $\infty \times \mathrm{e}^{\mathrm{i}\pi k/n}$*".) For any* $j \in \mathbb{Z}$, $\infty \times \mathrm{e}^{\mathrm{i}\pi j/n}$ *is visited by a single such curve. Let us call each of these* n *curves by a "line".*
- *For each* n *there exists an* $a \in \mathbb{C}$ *such that the only intersections between the lines occur at* a *where* $n-2$ *lines intersect.*

Proof. The asymptotic behavior stated in the first property follows from the leading order term for $f(z)$. To be more explicit, $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a ramified cover with ∞ being the ramification point of multiplicity n. Hence the map f is locally n-to-1 around ∞, and analytically isomorphic to $z \to z^n$.

Intersection of the lines can occur only at a critical point. The critical points that are not a are given by the roots of the equation

$$
\begin{aligned}
0 &= f'(z)/(z-a)^{n-3} \\
&= (n-2)\left(z^2 + (n-2)\,a\,z + \frac{(n-2)(n-1)}{2}a^2\right) + \big(2z + (n-2)\,a\big)(z-a) \\
&= n\,z^2 + (n^2-3n)\,a\,z + \frac{(n-2)n(n-3)}{2}a^2.
\end{aligned} \tag{7}
$$

When a is purely real the above two critical points are not on the real axis and complex conjugate to each other because the discriminant is negative, i.e.

$$
(n^2-3n)^2 - 2n^2(n-2)(n-3) < 0. \tag{8}
$$

These two critical points, say $\zeta_\pm$ with $\overline{\zeta}_+ = \zeta_-$, are given by

$$
\zeta_\pm = \left(-\frac{n-3}{2} \pm \mathrm{i}\sqrt{\frac{(n-3)(n-1)}{4}}\right) a. \tag{9}
$$

For purely real a, they are either both on Γ or none on Γ by symmetry. To find out which, one may calculate the argument of

$$
\frac{f(\zeta_+)}{a} = (\zeta_+ - a)^{n-2}\big(\zeta_+ + (n-2)a\big) \tag{10}
$$

where the second factor is obtained by simplifying $P(\zeta_+)$ using the equation (7) for the critical points. The argument is given by

(11)
$$\begin{aligned}\arg f(\zeta_+) &= (n-2)\arctan\left(\frac{\sqrt{(n-3)(n-1)}}{-n+1}\right) + \arctan\left(\frac{\sqrt{(n-3)(n-1)}}{n-1}\right)\\ &= (n-3)\arctan\left(\frac{\sqrt{(n-3)(n-1)}}{-n+1}\right).\end{aligned}$$

If $f(\zeta_+)$ is purely real then one should have

$$\tan\frac{\pi k}{n-3} = \frac{\sqrt{(n-3)(n-1)}}{-n+1} \tag{12}$$

for some integer k.

Writing this equation using sine, we have

$$\sin^2\frac{\pi k}{n-3} = \frac{n-3}{2n-4}.$$

By the double angle identity

$$\cos\frac{2\pi k}{n-3} = 1 - 2\sin^2\frac{\pi k}{n-3} = \frac{1}{n-2}. \tag{13}$$

Let $x = \frac{2\pi k}{n-3}$, and suppose (13) has an integer solution k. Then x/π and $\cos x$ are both rational, so Niven's theorem [15, Corollary 3.12] applies stating that $\cos x = 0, \pm 1/2, \pm 1$. It is easy to see that none of these possibilities can be realized in (13).

This shows that, when a is purely real, the only critical points on Γ occur at a. □

From Lemma 3 one easily observes that, for a purely real,

- Γ has the symmetry with respect to the real axis;
- the real axis belongs to Γ;
- there are exactly two (symmetrically located with respect to the real axis) lines that do not go through the point a and each of the two lines connect the consecutive infinities, i.e. $\infty \times e^{i\pi j/n}$ and $\infty \times e^{i\pi(j+1)/n}$ for some j.

An example of the resulting configuration is in the left picture of Figure 1. The thick lines are the set Γ. There are two lines that do not pass a. Let us name them each by Γ_+ and Γ_-. As we saw in the proof of Lemma 3, $\Gamma_\pm$ are away from any critical point of f, i.e. $f' \neq 0$ on $\Gamma_\pm$. This gives that $\operatorname{Re} f$ is strictly monotonic along $\Gamma_\pm$ since $\operatorname{Im} f$ is constant along each of $\Gamma_\pm$. Since $\operatorname{Re} f$ approaches $+\infty$ at one end of Γ_+ (resp. Γ_-) and $-\infty$ at the other end of Γ_+ (resp. Γ_-), each line must contain exactly one root of f. Therefore $\Gamma_\pm$ must pass through the two roots

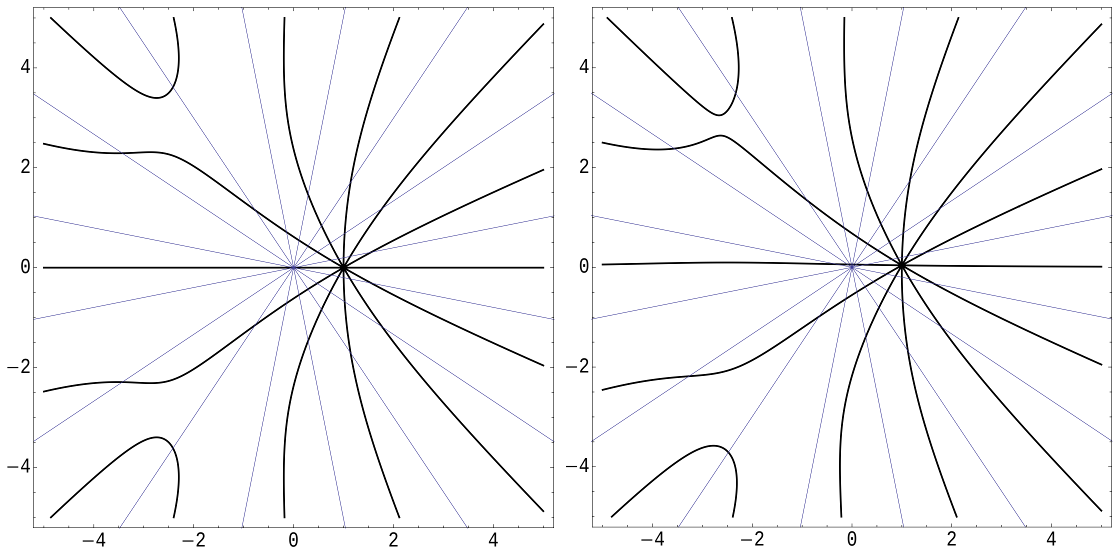

FIGURE 1. $n = 8$. The left is for $a = 1$; the right is for $a = 1 + 0.04\mathrm{i}$

of f that are not a, i.e. the two roots are the roots of $P(z) = 0$; let us denote the two roots by

$$z_{\pm} = -a\left(\frac{n-2}{2} \mp \mathrm{i}\frac{\sqrt{n^2-2n}}{2}\right). \tag{14}$$

The following fact shows how these two lines, $\Gamma_{\pm}$, behave near ∞, namely, how many lines "pass between" Γ_+ and Γ_-. (For example, in Figure 1 there are 3 lines that pass in between $\Gamma_{\pm}$.)

Lemma 4. *We say that a smooth curve "passes between z_- and z_+" if the curve intersects the straight path from z_- to z_+ (let us assume that the straight path does not include the points z_- and z_+) exactly an odd number of times without counting any tangential intersection. Let K be the number of lines in Γ that pass between z_+ and z_-. Then K satisfies*

$$2\left\lfloor \frac{n-2}{\pi} \arctan \frac{\sqrt{n^2-2n}}{n} \right\rfloor - 1 \le K \le n. \tag{15}$$

Proof. The upper bound is obvious. Let us consider the variation of

(16)

$$\arg f(z) = \mathrm{Im}\Big[(n-2)\log(z-a)\Big] + \mathrm{Im}\left[\log\left(z^2 + (n-2)\,a\,z + \frac{(n-2)(n-1)}{2}a^2\right)\right]$$

over the straight path that connects z_+ and z_-. The variation of the second term is zero. So we have

(17)

$$\Delta \arg f = 2(n-2)(\pi - \arg(z_- - a)) = 2(n-2)\left(\pi - \arg \frac{a}{2}\Big(-n + \mathrm{i}\sqrt{n^2-2n}\Big)\right) = 2(n-2)\arctan\frac{\sqrt{n^2-2n}}{n},$$

where $\Delta \arg f$ is the angular variation of f from z_+ to z_-.

Let z_0 be the intersection of straight path between z_- and z_+ with the real axis. The number of lines in Γ that are not $\Gamma_\pm$ and pass between z_0 and z_+ is at least

$$\left\lfloor \frac{\Delta \arg f}{2\pi} \right\rfloor - 1. \tag{18}$$

The minus 1 above comes because there can be an intersection of the vertical path between z_0 and z_+ with Γ_+. (This in fact occurs at $n = 19$.)

Counting the real axis, the total number of smooth curves in Γ that pass between z_+ and z_- is at least

$$1 + 2\left(\left\lfloor \frac{\Delta \arg f}{2\pi} \right\rfloor - 1\right) = 2\left\lfloor \frac{\Delta \arg f}{2\pi} \right\rfloor - 1, \tag{19}$$

and this proves our lemma. □

To give an example for $n = 8$ in Figure 1, K is 3 and it satisfies the inequalities (15) in Lemma 4, which gives $1 \le K \le 8$.

Now let us consider a small perturbation of a, see the right picture in Figure 1. For $\operatorname{Im} a$ close to zero, the above K is an odd number (by the symmetry under x axis). By a small perturbation of a, Γ no longer contains the origin.

Finally, we count the number of points in $\mathcal{Z} = \Gamma \cap \{z \,|\, \operatorname{Re} z^n = 0\}$. Let us call each smooth curve in Γ that starts at a and escapes to infinity a "ray" such that each line that passes through a consists of two rays. We count the intersections of each ray with $\{z \,|\, \operatorname{Re} z^n = 0\}$.

Lemma 4 says that there is a ray that escapes to $\infty \times (-\mathrm{e}^{\pi \mathrm{i} j/n})$ for all j in $-n < j \le n$ except for

$$|j| = \frac{K-1}{2} + 1 \quad \text{and} \quad |j| = \frac{K-1}{2} + 2. \tag{20}$$

Lemma 5. *The ray that starts at a and escapes to $\infty \times (-\mathrm{e}^{\pi \mathrm{i} j/n})$ (without passing the origin) intersects the set $\{z \,|\, \operatorname{Re} z^n = 0\}$ at least $n - |j|$ times.*

Proof. For a close to the positive real axis, we have a is in the sector $S_0 := \{z | -\frac{\pi}{2n} < \arg z < \frac{\pi}{2n}\}$. The ray under consideration starts in the sector S_0 at a and ends in the sector $S_1 := \{z | \frac{\pi(n-j)}{n} - \frac{\pi}{2n} < \arg z < \frac{\pi(n-j)}{n} + \frac{\pi}{2n}\}$, hence it crosses each of the $n-|j|$ lines in the set $\{z \,|\, \operatorname{Re} z^n = 0\}$ that separate these two sectors. Namely, the lines are $\{z = te^{\mathrm{i}\left(\frac{\pi}{2n}+\frac{k\pi}{n}\right)} : t \in \mathbb{R}\}$, for $k = 0, 1, 2, .., n-|j|-1$. $\square$

Therefore the total number of intersections is given by the sum of intersections for all rays:

$$\left(n+2\sum_{j=1}^{n-1}(n-j)\right)-2\left(n-\frac{K-1}{2}-1\right)-2\left(n-\frac{K-1}{2}-2\right) = n^2-4n+2K+4. \tag{21}$$

Using Lemma 4 the number of roots to (6) is given at least by

$$n^2-4n+2K+4 \geq n^2-4n+4\left\lfloor \frac{n-2}{\pi}\arctan\frac{\sqrt{n^2-2n}}{n}\right\rfloor+2. \tag{22}$$

This proves Theorem 2.

3. The higher dimensional case $d > 2$

With $x \in \mathbb{R}^d$, let:

$$F(x_1, x_2, .., x_d) = \langle h_1(x), h_2(x), .., h_d(x)\rangle$$

be a vector field in $\mathbb{R}^d$ having harmonic polynomial components h_k so that the largest degree appearing is n. Let $F = F_n + F_L$ be a decomposition of F into a vector field F_n containing the leading homogeneous terms and F_L containing the lower order terms. Using this set up, Wilmshurst's theorem can be stated in a form that is free of complex variable notation.

Theorem 6. *Let $d = 2$. For a harmonic polynomial vector field $F(x) = F_n(x) + F_L(x)$, with F_n the vector field of leading degree terms, if F_n does not vanish on the unit circle S^1 then N_F is finite and $N_F \leq n^2$.*

This immediately implies the theorem as stated in the introduction, since the leading part of the harmonic field

$$F(x,y) = p(z) + \overline{q(z)} = \left(\operatorname{Re}\{p(x+iy)+q(x+iy)\}, \operatorname{Im}\{p(x+iy)-q(x+iy)\}\right),$$

is, for some nonzero constant c_n,

$$F_n(x,y) = c_n\left(\operatorname{Re}\{(x+iy)^n\}, \operatorname{Im}\{(x+iy)^n\}\right),$$

and this does not vanish on the unit circle S^1.

This naturally leads to the question of whether Theorem 2 is true in $d > 2$ dimensions. We give a generalization showing that the zero set of F has codimension at least two (which implies N_F is finite when $d = 2$). Examples with $d > 2$ and codimension exactly two are described below.

Recall that given a compact semialgebraic subset $S \subset \mathbb{R}^d$ there exists a semialgebraic triangulation $\phi : |K| \to S$, where $|K|$ is the total space of a simplicial complex K and ϕ is a continuous semialgebraic homeomorphism (see [2, Theorem 9.2.1])

In particular the dimension of S (as defined in [2, Ch. 2.8]) is the maximum over the dimension of the simplices in K.

In fact it is even possible to stratify $S = \coprod_{\alpha=1}^{s} S_\alpha$ in such a way that each stratum S_α is a smooth manifold and the dimension of S equals $\max\{\dim(S_\alpha)\}$, where $\dim S_\alpha$ is the dimension as a smooth manifold [2, Proposition 9.1.8].

Theorem 7. *Suppose a real algebraic set $X \subset \mathbb{R}^d$ is defined by harmonic polynomials. If X is bounded then the codimension of X in $\mathbb{R}^d$ is at least two.*

Corollary 8 (Generalization of Wilmshurst's theorem)**.** *For a harmonic polynomial vector field $F(x) = F_n(x) + F_L(x)$ as above, if F_n does not vanish on the unit sphere S^{d-1} then the zero set $\{x \in \mathbb{R}^d : F(x) = 0\}$ is of codimension at least two.*

For $d = 2$ this reduces to Theorem 6.

Proof of Corollary 8. The assumption on F_n implies that $\{x \in \mathbb{R}^d : F(x) = 0\}$ is bounded so that Theorem 7 applies. Indeed, for all $x \in \mathbb{R}^d$ with $|x|$ large enough,

$$|F(x)| \geq |F_n(x)| - |F_L(x)| \geq |x|^n \min_{\theta \in S^{d-1}} |F_n(\theta)| - |x|^{n-1} \max_{\theta \in S^{d-1}} |F_L(\theta)| > 0.$$

□

Proof of Theorem 7. Suppose $X = \{F = 0\}$ has a component of codimension one, i.e. $\dim(X) = d - 1$. We will prove that this implies:

$$\mathbb{R}^d \backslash X \quad \text{has at least one bounded component } A, \tag{23}$$

and this will give an absurdity. In fact, assuming (23), pick a component $h = h_i$ of $F = (h_1, \dots, h_d)$ that doesn't vanish identically on A. By replacing h with $-h$ if necessary, we can assume that h has a maximum $h(x_0) > 0$ with $x_0 \in A$. The Maximum Principle for harmonic functions implies that $h \equiv h(x_0)$ on A, against the assumption that it vanishes on a point $x \in X$ that is also a limit point of A.

In order to prove that $\dim(X) = d-1$ implies (23) we proceed as follows. First we consider the *one point compactification* $S^d = \mathbb{R}^d \cup \{\infty\}$ of $\mathbb{R}^d$. Since X is closed and bounded, then the number of connected components of $S^d \backslash X$ is the same of $\mathbb{R}^d \backslash X$; moreover the existence of at least two components of $S^d \backslash X$ implies one of them does not contain $\{\infty\}$ and this component will also be a *bounded* component of $\mathbb{R}^d \backslash X$.

To compute the number of connected components of $S^d \backslash X$, recall that this number equals $b_0(S^d \backslash X) = \mathrm{rk} H_0(S^d \backslash X; \mathbb{Z}_2)$ (the zero-th homology group with coefficients in the field $\mathbb{Z}_2$; the choice of $\mathbb{Z}_2$ coefficients does not change the rank of H_0 and will be useful for the study of the homology of the *real* algebraic set X). Moreover, since X is compact, Alexander duality (see [8], Theorem 3.44) implies:

$$\tilde{H}_0(S^d \backslash X; \mathbb{Z}_2) \simeq H^{d-1}(X; \mathbb{Z}_2) \simeq H_{d-1}(X; \mathbb{Z}_2)$$

where the first group is the *reduced* zero-th homology group, whose rank equals $b_0(S^d \backslash X) - 1$, and the last isomorphism comes from the fact that homology and cohomology with coefficients in a field are isomorphic.

Let us now triangulate X as for the definition of its dimension (see above). Proposition 11.1.1 from [2] says that the sum of the simplices of dimension $d-1$ with coefficients in $\mathbb{Z}_2$ is a cycle in X; this cycle cannot be a boundary, since there are no d-dimensional simplices. Hence $H_{d-1}(X) \neq 0$ and the conclusion follows from

$$b_0(S^d \backslash X) = \mathrm{rk}\tilde{H}_0(S^d \backslash X; \mathbb{Z}_2) + 1 = \mathrm{rk}H_{d-1}(X; \mathbb{Z}_2) + 1 \geq 2.$$

□

In this setting, given that "codimension two" is equivalent to "finite" when $d = 2$ yet becomes a weaker conclusion in higher dimensions, it is natural to ask whether it is possible to show when $d > 2$ the stronger conclusion that N_F is finite in Corollary 8. The next example shows that "codimension two" cannot be improved. For simplicity, we describe the example for $d = 3$ but a similar construction works in higher dimensions to exhibit a harmonic field satisfying the hypothesis of Corollary 8 and having zero set of codimension exactly two.

Example: For simplicity of notation we consider a vector field in $\mathbb{R}^3$. Take

$$F(x, y, z) = \langle u(x, y, z), v(x, y, z), w(x, y, z)\rangle,$$

with the following harmonic polynomials as components

$$u(x, y, z) = xy(6z^2 - x^2 - y^2),$$
$$v(x, y, z) = (x^2 - y^2)(6z^2 - x^2 - y^2),$$
$$w(x, y, z) = 35(z-1)^4 - 30(z-1)^2(x^2 + y^2 + (z-1)^2) + 3(x^2 + y^2 + (z-1)^2)^2.$$

The first two are spherical harmonics of fourth degree, and the latter is a spherical harmonic of the same degree, shifted along the z-axis. The leading part F_4 of the field consists of $\langle u(x, y, z), v(x, y, z), w_4(x, y, z)\rangle$, where

$$w_4(x, y, z) = 35z^4 - 30z^2(x^2 + y^2 + z^2) + 3(x^2 + y^2 + z^2)^2,$$

is the homogeneous fourth-degree part of w.

Claim 1: The hypothesis of Corollary 8 is satisfied.

Note that u, v, and w_4 are three elements from the standard basis for spherical harmonics of degree four. Consider the nodal lines of u, v, and w_4 on the sphere (which are well-studied). We will see that there are no points in S^2 common to all three. We first notice that u and v have no meridional lines in common; $xy = 0$ only intersects $x^2 - y^2 = 0$ at the North and South poles, where w_4 is non-vanishing. It remains to check that the horizontal nodal lines, which are identical for u and v, do not intersect those of w_4. With respect to the angle θ from the North pole, w_4 restricted to the sphere is a constant times $P_4(\cos(\theta))$, where P_4 is a Legendre polynomial, whereas the zeros with respect to θ of u and v are given by those of $P_{4,2}(cos(\theta))$, with $P_{4,2}$ the associated Legendre polynomial. The zeros of these polynomials are well known and none are in common.

Claim 2: The zero set of F has codimension exactly two.

The zero set of u consists of a cone C with vertex at the origin along with a pair of orthogonal planes containing the z-axis. The zero set of v has the same description except the pair of orthogonal planes is different (but the cone C is the same). The zero set of w is a set of two cones C_1, C_2 each with vertex at the point $(0, 0, 1)$. The slopes of C, C_1, and C_2 are all distinct. This is a consequence of the fact used above that P_4 and $P_{4,2}$ don't share any zeros. Indeed, the zeros of P_4 determine the slopes of the cones in the zero set of w while the zeros of $P_{4,2}$ determine the slopes for u and v. It follows that each cone of $\{w = 0\}$ intersects the cone $C = \{u = v = 0\}$ in two circles, as is the case for any intersection of two cones with common axis but different vertices and different slopes.

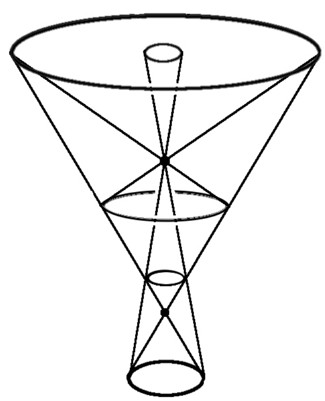

FIGURE 2. The zero set of F includes four circles where C intersects the union of C_1 and C_2.

This example shows that Bezout's theorem cannot be applied when $d > 2$. However, it is still possible to give an upper bound on the number $b_0(F)$ of *connected components* of F. The bound due to J. Milnor [13] for the total Betti number of an intersection in particular gives an upper bound on the number of connected components (zeroth Betti number). Applying this to our situation gives

$$b_0(F) \leq n(2n - 1)^{d-1}.$$

We note that in the generic case it is possible to apply the Bezout bound.

Proposition 9. *For a generic harmonic polynomial vector field $F(x)$, the number of zeros is finite and bounded by the product of the degrees of its component functions.*

Proof. Let $F = (h_1, \dots, h_d)$ with each h_i of degree n_i; if we denote by $H_{l,k}$ the space of real harmonic polynomials of degree l in k variables, in such a way that $H_{l,k} \subset \mathbb{R}[x_1, \dots, x_k]_l = P_{l,k}$, then we have:

$$F \in H_{n_1,d} \oplus \cdots \oplus H_{n_d,d} = V.$$

In this way V is a finite dimensional subspace of $W = P_{n_1,d} \oplus \cdots P_{n_d,d}$. An element of W can be considered as a "system of polynomial equations", in the sense that given $P = (P_1, \dots, P_d) \in W$ we can write the system $\{P_1 = \cdots = P_d = 0\}$. It is well known that for the generic $P \in W$ the above system has a finite number of solutions (being that the number of variables equals to the number of equations), and such number is bounded by (this is simply Bezout's theorem):

$$N_P \leq n_1 \cdots n_d = N_W.$$

Thus, given $F \in V \subset W$, we can perturb it to a $\tilde{F}$ with finitely many zeros, but we do not know (yet) that we can do this perturbation without leaving V. To show that this is indeed possible, consider the "discriminant" $\Delta \subset W$:

$$\Delta = \{P \in W \mid \text{ the system of equations associated to } P \text{ is degenerate}\}.$$

We claim that Δ is a semialgebraic set of dimension:

$$\dim(\Delta) \leq \dim(W) - 1.$$

Semialgebraicity is clear: it is the projection on the second factor of the semialgebraic set $S = \{(x, P) \in \mathbb{R}^d \times W \,|\, P(x) = 0, \det(D_x P) = 0\}$, where $\det(D_x P)$ denotes the Jacobian determinant. (Projections of semialgebraic sets are semialgebraic, see [2].) The fact that Δ has codimension one (according to the above definition) is equivalent to the the fact that it doesn't contain any open subsets, which is clear from the genericity of regular systems. Consider now the *Zariski* closure $\overline{\Delta}$ of Δ in W: Proposition 2.8.2 of [2] implies $\overline{\Delta}$ has the same dimension as Δ, hence $\overline{\Delta}$ is a *proper* algebraic set. Notice that if $P \notin \overline{\Delta}$, then P is regular.

Finally consider the intersection $V \cap \overline{\Delta}$: it is an algebraic set and, if proper, its complement will be an *open dense* set of regular elements of V. To show that $V \cap \overline{\Delta}$ is proper, it is enough to exhibit *one* regular $F \in V$: then for every P in a neighborhood U of F in W such P will be regular; hence $U \cap V$ will be a nonempty open set of regular elements. The existence of a regular F is left as an exercise. □

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