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Leptonic Dirac CP violation predictions from residual discrete symmetries

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Abstract

Assuming that the observed pattern of 3-neutrino mixing is related to the existence of a (lepton) flavour symmetry, corresponding to a non-Abelian discrete symmetry group G_f, and that G_f is broken to specific residual symmetries G_e and G_ν of the charged lepton and neutrino mass terms, we derive sum rules for the cosine of the Dirac phase δ of the neutrino mixing matrix U. The residual symmetries considered are: i) $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$; ii) $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$; iii) $G_e = Z_2$ and $G_\nu = Z_2$; iv) G_e is fully broken and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$; and v) $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and G_ν is fully broken. For given G_e and G_ν, the sum rules for $\cos\delta$ thus derived are exact, within the approach employed, and are valid, in particular, for any G_f containing G_e and G_ν as subgroups. We identify the cases when the value of $\cos\delta$ cannot be determined, or cannot be uniquely determined, without making additional assumptions on unconstrained parameters. In a large class of cases considered the value of $\cos\delta$ can be unambiguously predicted once the flavour symmetry G_f is fixed. We present predictions for $\cos\delta$ in these cases for the flavour symmetry groups $G_f = S_4$, A_4, T' and A_5, requiring that the measured values of the 3-neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, taking into account their respective 3σ uncertainties, are successfully reproduced.

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1. Introduction

The discrete symmetry approach to understanding the observed pattern of 3-neutrino mixing (see, e.g., [1]), which is widely explored at present (see, e.g., [2–5]), leads to specific correlations between the values of at least some of the mixing angles of the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) neutrino mixing matrix U and, either to specific fixed trivial or maximal values of the CP violation (CPV) phases present in U (see, e.g., [6–10] and references quoted therein), or to a correlation between the values of the neutrino mixing angles and of the Dirac CPV phase of U [11–15].[2] As a consequence of this correlation the cosine of the Dirac CPV phase δ of the PMNS matrix U can be expressed in terms of the three neutrino mixing angles of U [11–14], i.e., one obtains a sum rule for $\cos\delta$. This sum rule depends on the underlying discrete symmetry used to derive the observed pattern of neutrino mixing and on the type of breaking of the symmetry necessary to reproduce the measured values of the neutrino mixing angles. It depends also on the assumed status of the CP symmetry before the breaking of the underlying discrete symmetry.

The approach of interest is based on the assumption of the existence at some energy scale of a (lepton) flavour symmetry corresponding to a non-Abelian discrete group G_f. Groups that have been considered in the literature include S_4, A_4, T', A_5, D_n (with $n = 10, 12$) and $\Delta(6n^2)$, to name several. The choice of these groups is related to the fact that they lead to values of the neutrino mixing angles, which can differ from the measured values at most by subleading perturbative corrections. For instance, the groups A_4, S_4 and T' are commonly utilised to generate tri-bimaximal (TBM) mixing [18]; the group S_4 can also be used to generate bimaximal (BM) mixing [19][3]; A_5 can be utilised to generate golden ratio type A (GRA) [21–23] mixing; and the groups D_{10} and D_{12} can lead to golden ratio type B (GRB) [24] and hexagonal (HG) [25] mixing.

The flavour symmetry group G_f can be broken, in general, to different symmetry subgroups G_e and G_ν of the charged lepton and neutrino mass terms, respectively. G_e and G_ν are usually called "residual symmetries" of the charged lepton and neutrino mass matrices. Given G_f, which is usually assumed to be discrete, typically there are more than one (but still a finite number of) possible residual symmetries G_e and G_ν. The subgroup G_e, in particular, can be trivial, i.e., G_f can be completely broken in the process of generation of the charged lepton mass term.

The residual symmetries can constrain the forms of the 3×3 unitary matrices U_e and U_ν, which diagonalise the charged lepton and neutrino mass matrices, and the product of which represents the PMNS matrix:

$$U = U_e^\dagger \, U_\nu \, . \tag{1}$$

Thus, by constraining the form of the matrices U_e and U_ν, the residual symmetries constrain also the form of the PMNS matrix U.

In general, there are two cases of residual symmetry G_ν for the neutrino Majorana mass term when a portion of G_f is left unbroken in the neutrino sector. They characterise two possible approaches — direct and semi-direct [4] — in making predictions for the neutrino mixing observables using discrete flavour symmetries: G_ν can either be a $Z_2 \times Z_2$ symmetry (which

[2] In the case of massive neutrinos being Majorana particles one can obtain under specific conditions also correlations between the values of the two Majorana CPV phases present in the neutrino mixing matrix [16] and of the three neutrino mixing angles and of the Dirac CPV phase [11,17].

[3] Bimaximal mixing can also be a consequence of the conservation of the lepton charge $L' = L_e - L_\mu - L_\tau$ (LC) [20], supplemented by a μ–τ symmetry.

sometimes is identified in the literature with the Klein four group), or a Z_2 symmetry. In models based on the semi-direct approach, where $G_\nu = Z_2$, the matrix U_ν contains two free parameters, i.e., one angle and one phase, as long as the neutrino Majorana mass term does not have additional "accidental" symmetries, e.g., the μ–τ symmetry. In such a case as well as in the case of $G_\nu = Z_2 \times Z_2$, the matrix U_ν is completely determined by symmetries up to re-phasing on the right and permutations of columns. The latter can be fixed by considering a specific model. It is also important to note here that in this approach Majorana phases are undetermined.

In the general case of absence of constraints, the PMNS matrix can be parametrised in terms of the parameters of U_e and U_ν as follows [26]:

$$U = U_e^\dagger\, U_\nu = (\tilde{U}_e)^\dagger\, \Psi \tilde{U}_\nu\, Q_0\,. \tag{2}$$

Here $\tilde{U}_e$ and $\tilde{U}_\nu$ are CKM-like 3×3 unitary matrices and Ψ and Q_0 are given by:

$$\Psi = \mathrm{diag}\left(1, \mathrm{e}^{-i\psi}, \mathrm{e}^{-i\omega}\right), \qquad Q_0 = \mathrm{diag}\left(1, \mathrm{e}^{i\frac{\xi_{21}}{2}}, \mathrm{e}^{i\frac{\xi_{31}}{2}}\right), \tag{3}$$

where ψ, ω, ξ_{21} and ξ_{31} are phases which contribute to physical CPV phases. Thus, in general, each of the two phase matrices Ψ and Q_0 contain two physical CPV phases. The phases in Q_0 contribute to the Majorana phases [16] in the PMNS matrix (see further) and can appear in eq. (2) as a result of the diagonalisation of the neutrino Majorana mass term, while the phases in Ψ can result from the charged lepton sector ($U_e^\dagger = (\tilde{U}_e)^\dagger\, \Psi$), from the neutrino sector ($U_\nu = \Psi \tilde{U}_\nu Q_0$), or can receive contributions from both sectors.

As is well known, the 3×3 unitary PMNS matrix U can be parametrised in terms of three neutrino mixing angles and, depending on whether the massive neutrinos are Dirac or Majorana particles, by one Dirac CPV phase, or by one Dirac and two Majorana [16] CPV phases:

$$U = U_e^\dagger\, U_\nu = V\, Q\,, \qquad Q = \mathrm{diag}\left(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}}\right), \tag{4}$$

where $\alpha_{21,31}$ are the two Majorana CPV phases and V is a CKM-like matrix. In the standard parametrisation of the PMNS matrix [1], which we are going to use in what follows, V has the form:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{5}$$

where $0 \le \delta \le 2\pi$ is the Dirac CPV phase and we have used the standard notation $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with $0 \le \theta_{ij} \le \pi/2$. Notice that if CP invariance holds, then we have $\delta = 0, \pi, 2\pi$, with the values 0 and 2π being physically indistinguishable, and $\alpha_{21} = k\pi$, $\alpha_{31} = k'\pi$, $k, k' = 0, 1, 2$.[4] Therefore, the neutrino mixing observables are the three mixing angles, θ_{12}, θ_{13}, θ_{23}, the Dirac phase δ and, if the massive neutrinos are Majorana particles, the Majorana phases α_{21} and α_{31}.

The neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$, which will be relevant for our further discussion, have been determined with a relatively good precision in the recent global analyses of the neutrino oscillation data [28,29]. For the best fit values and the 3σ allowed ranges of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$, the authors of Ref. [28] have obtained:

[4] In the case of the type I seesaw mechanism of neutrino mass generation the range in which α_{21} and α_{31} vary is $[0, 4\pi]$ [27]. Thus, in this case α_{21} and α_{31} possess CP-conserving values for $k, k' = 0, 1, 2, 3, 4$.

$$(\sin^2\theta_{12})_{\rm BF} = 0.308\,, \qquad 0.259 \leq \sin^2\theta_{12} \leq 0.359\,, \tag{6}$$

$$(\sin^2\theta_{23})_{\rm BF} = 0.437\ (0.455)\,, \qquad 0.374\ (0.380) \leq \sin^2\theta_{23} \leq 0.626\ (0.641)\,, \tag{7}$$

$$(\sin^2\theta_{13})_{\rm BF} = 0.0234\ (0.0240)\,, \qquad 0.0176\ (0.0178) \leq \sin^2\theta_{13} \leq 0.0295\ (0.0298)\,. \tag{8}$$

Here the values (values in brackets) correspond to neutrino mass spectrum with normal ordering (inverted ordering) (see, e.g., [1]), denoted further as the NO (IO) spectrum.

In Ref. [11] (see also [12–14]) we have considered the cases when, as a consequence of underlying and residual symmetries, the matrix U_ν, and more specifically, the matrix $\tilde{U}_\nu$ in eq. (2), has the i) TBM, ii) BM, iii) GRA, iv) GRB and v) HG forms. For all these forms we have $\tilde{U}_\nu = R_{23}(\theta^\nu_{23}) R_{12}(\theta^\nu_{12})$ with $\theta^\nu_{23} = -\pi/4$, R_{23} and R_{12} being 3×3 orthogonal matrices describing rotations in the 2–3 and 1–2 planes:

$$\tilde{U}_\nu = R_{23}\left(\theta^\nu_{23}\right) R_{12}\left(\theta^\nu_{12}\right) = \begin{pmatrix} \cos\theta^\nu_{12} & \sin\theta^\nu_{12} & 0 \\ -\frac{\sin\theta^\nu_{12}}{\sqrt{2}} & \frac{\cos\theta^\nu_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta^\nu_{12}}{\sqrt{2}} & \frac{\cos\theta^\nu_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{9}$$

The value of the angle θ^ν_{12}, and thus of $\sin^2\theta^\nu_{12}$, depends on the form of $\tilde{U}_\nu$. For the TBM, BM, GRA, GRB and HG forms we have: i) $\sin^2\theta^\nu_{12} = 1/3$ (TBM), ii) $\sin^2\theta^\nu_{12} = 1/2$ (BM), iii) $\sin^2\theta^\nu_{12} = (2+r)^{-1} \cong 0.276$ (GRA), r being the golden ratio, $r = (1+\sqrt{5})/2$, iv) $\sin^2\theta^\nu_{12} = (3-r)/4 \cong 0.345$ (GRB), and v) $\sin^2\theta^\nu_{12} = 1/4$ (HG).

The TBM form of $\tilde{U}_\nu$, for example, can be obtained from a $G_f = A_4$ symmetry, when the residual symmetry is $G_\nu = Z_2$, i.e. the S generator of A_4 (see Appendix A) is unbroken. In this case there is an additional accidental μ–τ symmetry, which together with the Z_2 symmetry leads to the TBM form of $\tilde{U}_\nu$ (see, e.g., [3]). The TBM form can also be derived from $G_f = T'$ with $G_\nu = Z_2$, provided that the left-handed (LH) charged leptons and neutrinos each transform as triplets of T' and the TST^2 element of T' is unbroken, see Appendix A for further explanation. Indeed when working with 3-dimensional and 1-dimensional representations of T', there is no way to distinguish T' from A_4 [30]. Finally, one can obtain BM mixing from, e.g., the $G_f = S_4$ symmetry, when the residual symmetry is $G_\nu = Z_2$. There is an accidental μ–τ symmetry in this case as well [31].

For all the forms of $\tilde{U}_\nu$ considered in [11] and listed above we have i) $\theta^\nu_{13} = 0$, which should be corrected to the measured value of $\theta_{13} \cong 0.15$, and ii) $\sin^2\theta^\nu_{23} = 0.5$, which might also need to be corrected if it is firmly established that $\sin^2\theta_{23}$ deviates significantly from 0.5. In the case of the BM and HG forms, the values of $\sin^2\theta^\nu_{12}$ lie outside the current 3σ allowed ranges of $\sin^2\theta_{12}$ and have also to be corrected.

The requisite corrections are provided by the matrix U_e, or equivalently, by $\tilde{U}_e$. The approach followed in [11–14] corresponds to the case of a trivial subgroup G_e, i.e., of G_f completely broken by the charged lepton mass term. In this case the matrix $\tilde{U}_e$ is unconstrained and was chosen in [11] on phenomenological grounds to have the following two forms:

$$\tilde{U}_e = R_{12}^{-1}(\theta^e_{12})\,, \tag{10}$$

$$\tilde{U}_e = R_{23}^{-1}(\theta^e_{23})\, R_{12}^{-1}(\theta^e_{12})\,. \tag{11}$$

These two forms appear in a large class of theoretical models of flavour and theoretical studies, in which the generation of charged lepton masses is an integral part (see, e.g., [17,32–37]).

In this setting with $\tilde{U}_\nu$ having one of the five symmetry forms, TBM, BM, GRA, GRB and HG, and $\tilde{U}_e$ given by eq. (11), the Dirac phase δ of the PMNS matrix was shown in [11] to satisfy the following sum rule[5]:

$$\cos\delta = \frac{\tan\theta_{23}}{\sin 2\theta_{12}\sin\theta_{13}}\left[\cos 2\theta^\nu_{12} + \left(\sin^2\theta_{12} - \cos^2\theta^\nu_{12}\right)\left(1 - \cot^2\theta_{23}\sin^2\theta_{13}\right)\right]. \quad (12)$$

Within the approach employed this sum rule is exact.[6] It is valid, in particular, for any value of the angle θ^ν_{23} [14].[7] In [11], by using the sum rule in eq. (12), predictions for $\cos\delta$ and δ were obtained in the TBM, BM, GRA, GRB and HG cases for the best fit values of $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. The results thus obtained permitted to conclude that a sufficiently precise measurement of $\cos\delta$ would allow to discriminate between the different forms of $\tilde{U}_\nu$ considered.

Statistical analyses of predictions of the sum rule given in eq. (12) i) for δ and for the $J_{\rm CP}$ factor, which determines the magnitude of CP-violating effects in neutrino oscillations [38], using the current uncertainties in the determination of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and δ from [28], and ii) for $\cos\delta$ using the prospective uncertainties on $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, were performed in [13] for the five symmetry forms — BM (LC), TBM, GRA, GRB and HG — of $\tilde{U}_\nu$.

In [14] we extended the analyses performed in [11,13] by obtaining sum rules for $\cos\delta$ for the following forms of the matrices $\tilde{U}_e$ and $\tilde{U}_\nu$[8]:

A. $\tilde{U}_\nu = R_{23}(\theta^\nu_{23})R_{12}(\theta^\nu_{12})$ with $\theta^\nu_{23} = -\pi/4$ and θ^ν_{12} as dictated by TBM, BM, GRA, GRB or HG mixing, and i) $\tilde{U}_e = R^{-1}_{13}(\theta^e_{13})$, ii) $\tilde{U}_e = R^{-1}_{23}(\theta^e_{23})R^{-1}_{13}(\theta^e_{13})$, and iii) $\tilde{U}_e = R^{-1}_{13}(\theta^e_{13})R^{-1}_{12}(\theta^e_{12})$;
B. $\tilde{U}_\nu = R_{23}(\theta^\nu_{23})R_{13}(\theta^\nu_{13})R_{12}(\theta^\nu_{12})$ with θ^ν_{23}, θ^ν_{13} and θ^ν_{12} fixed by arguments associated with symmetries, and iv) $\tilde{U}_e = R^{-1}_{12}(\theta^e_{12})$, and v) $\tilde{U}_e = R^{-1}_{13}(\theta^e_{13})$.

The sum rules for $\cos\delta$ were derived first for $\theta^\nu_{23} = -\pi/4$ for the cases listed in point A, and for the specific values of (some of) the angles in $\tilde{U}_\nu$, characterising the cases listed in point B, as well as for arbitrary fixed values of all angles contained in $\tilde{U}_\nu$. Predictions for $\cos\delta$ and $J_{\rm CP}$ ($\cos\delta$) were also obtained performing statistical analyses utilising the current (the prospective) uncertainties in the determination of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and δ ($\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$).

In the present article we extend the analyses performed in [11,13,14] to the following cases:

1) $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$;
2) $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$;
3) $G_e = Z_2$ and $G_\nu = Z_2$;
4) G_e is fully broken and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$;
5) $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and G_ν is fully broken.

5 The sum rule is given in the standard parametrisation of the PMNS matrix (see, e.g., [1]).

6 For the TBM and BM forms of $\tilde{U}_\nu$, and for $\tilde{U}_e$ given in eq. (11), it was first derived in Ref. [12].

7 The two forms of $\tilde{U}_e$ in eqs. (10) and (11) lead, in particular, to different predictions for $\sin^2\theta_{23}$: for $\theta^\nu_{23} = -\pi/4$ in the case of eq. (10) we have $\sin^2\theta_{23} \cong 0.5(1 - \sin^2\theta_{13})$, while if $\tilde{U}_e$ is given by eq. (11), $\sin^2\theta_{23}$ can deviate significantly from 0.5.

8 We performed in [14] a systematic analysis of the forms of $\tilde{U}_e$ and $\tilde{U}_\nu$, for which sum rules for $\cos\delta$ of the type of eq. (12) could be derived, but did not exist in the literature.

In the case of $G_e = Z_2$ ($G_\nu = Z_2$) the matrix U_e (U_ν) is determined up to a $U(2)$ transformation in the degenerate subspace, since the representation matrix of the generator of the residual symmetry has degenerate eigenvalues. On the contrary, when the residual symmetry is large enough, namely, $G_e = Z_n, n > 2$ or $Z_n \times Z_m, n, m \geq 2$ and $G_\nu = Z_2 \times Z_2$ ($G_\nu = Z_n, n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$) for Majorana (Dirac) neutrinos, the matrices U_e and U_ν are fixed (up to diagonal phase matrices on the right, which are either unphysical for Dirac neutrinos, or contribute to the Majorana phases otherwise, and permutations of columns) by the residual symmetries of the charged lepton and neutrino mass matrices. In the case when the discrete symmetry G_f is fully broken in one of the two sectors, the corresponding mixing matrix U_e or U_ν is unconstrained and contains in general three angles and six phases.

Our article is organised as follows. In Section 2 we describe the parametrisations of the PMNS matrix depending on the residual symmetries G_e and G_ν considered above. In Sections 3, 4 and 5 we consider the breaking patterns 1), 2), 3) and derive sum rules for $\cos\delta$. At the end of each of these sections we present numerical predictions for $\cos\delta$ in the cases of the flavour symmetry groups $G_f = A_4$, T', S_4 and A_5. In Section 6 we provide a summary of the sum rules derived in Sections 3–5. Further, in Sections 7 and 8 we derive the sum rules for the cases 4) and 5), respectively. In these cases the value of $\cos\delta$ cannot be fixed without additional assumptions on the unconstrained matrix U_e or U_ν. The cases studied in [14] belong to the ones considered in Section 7, where the particular forms of the matrix U_e, leading to sum rules of interest, have been considered. In Section 9 we present the summary of the numerical results. Section 10 contains the conclusions. Appendices A, B, C, D and E contain technical details related to the study.

2. Preliminary considerations

As was already mentioned in the Introduction, the residual symmetries of the charged lepton and neutrino mass matrices constrain the forms of the matrices U_e and U_ν and, thus, the form of the PMNS matrix U. To be more specific, if the charged lepton mass term is written in the left–right convention, the matrix U_e diagonalises the hermitian matrix $M_e M_e^\dagger$, $U_e^\dagger M_e M_e^\dagger U_e = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)$, M_e being the charged lepton mass matrix. If G_e is the residual symmetry group of $M_e M_e^\dagger$ we have:

$$\rho(g_e)^\dagger M_e M_e^\dagger \rho(g_e) = M_e M_e^\dagger \,, \tag{13}$$

where g_e is an element of G_e, ρ is a unitary representation of G_f and $\rho(g_e)$ gives the action of G_e on the LH components of the charged lepton fields having as mass matrix M_e. As can be seen from eq. (13), the matrices $\rho(g_e)$ and $M_e M_e^\dagger$ commute, implying that they are diagonalised by the same matrix U_e.

Similarly, if G_ν is the residual symmetry of the neutrino Majorana mass matrix M_ν one has:

$$\rho(g_\nu)^T M_\nu \rho(g_\nu) = M_\nu \,, \tag{14}$$

where g_ν is an element of G_ν, ρ is a unitary representation of G_f under which the LH flavour neutrino fields $\nu_{lL}(x)$, $l = e, \mu, \tau$, transform, and $\rho(g_\nu)$ determines the action of G_ν on $\nu_{lL}(x)$. It is not difficult to show that also in this case the matrices $\rho(g_\nu)$ and $M_\nu^\dagger M_\nu$[9] commute, and

[9] The right–left convention for the neutrino mass term is assumed.

therefore they can be diagonalised simultaneously by the same matrix U_ν. In the case of Dirac neutrinos eq. (14) is modified as follows:

$$\rho(g_\nu)^\dagger M_\nu^\dagger M_\nu \rho(g_\nu) = M_\nu^\dagger M_\nu . \tag{15}$$

The types of residual symmetries allowed in this case and discussed below are the same as those of the charged lepton mass term.

In many cases studied in the literature (e.g., in the cases of $G_f = S_4,\ A_4,\ T',\ A_5$) $\rho(g_f)$, g_f being an element of G_f, is assumed to be a 3-dimensional representation of G_f because one aims at unification of the three flavours (e.g., three lepton families) at high energy scales, where the flavour symmetry group G_f is unbroken.

At low energies the flavour symmetry group G_f has necessarily to be broken to residual symmetries G_e and G_ν, which act on the LH charged leptons and LH neutrinos as follows:

$$l_L \to \rho(g_e) l_L\,, \quad \nu_{lL} \to \rho(g_\nu)\nu_{lL}\,,$$

where g_e and g_ν are the elements of the residual symmetry groups G_e and G_ν, respectively, and $l_L = (e_L, \mu_L, \tau_L)^T$, $\nu_{lL} = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})^T$.

The largest possible exact symmetry of the Majorana mass matrix M_ν having three non-zero and non-degenerate eigenvalues, is a $Z_2 \times Z_2 \times Z_2$ symmetry. The largest possible exact symmetry of the Dirac mass matrix M_e is $U(1) \times U(1) \times U(1)$. Restricting ourselves to the case in which G_f is a subgroup of $SU(3)$ instead of $U(3)$, the indicated largest possible exact symmetries reduce respectively to $Z_2 \times Z_2$ and $U(1) \times U(1)$ because of the *special* determinant condition imposed from $SU(3)$. The residual symmetries G_e and G_ν, being subgroups of G_f (unless there are accidental symmetries), should also be contained in $U(1) \times U(1)$ and $Z_2 \times Z_2$ ($U(1) \times U(1)$) for massive Majorana (Dirac) neutrinos, respectively.

If $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$, the matrix U_e is fixed by the matrix $\rho(g_e)$ (up to multiplication by diagonal phase matrices on the right and permutations of columns), $U_e = U_e^\circ$. In the case of a smaller symmetry, i.e., $G_e = Z_2$, U_e is defined up to a $U(2)$ transformation in the degenerate subspace, because in this case $\rho(g_e)$ has two degenerate eigenvalues. Therefore,

$$U_e = U_e^\circ U_{ij}(\theta^e_{ij}, \delta^e_{ij}) \Psi_k \Psi_l\,,$$

where U_{ij} is a complex rotation in the i–j plane and Ψ_k, Ψ_l are diagonal phase matrices,

$$\Psi_1 = \mathrm{diag}\left(e^{i\psi_1}, 1, 1\right), \quad \Psi_2 = \mathrm{diag}\left(1, e^{i\psi_2}, 1\right), \quad \Psi_3 = \mathrm{diag}\left(1, 1, e^{i\psi_3}\right). \tag{16}$$

The angle θ^e_{ij} and the phases δ^e_{ij}, ψ_1, ψ_2 and ψ_3 are free parameters. As an example of the explicit form of $U_{ij}(\theta^a_{ij}, \delta^a_{ij})$, we give the expression of the matrix $U_{12}(\theta^a_{12}, \delta^a_{12})$:

$$U_{12}(\theta^a_{12}, \delta^a_{12}) = \begin{pmatrix} \cos\theta^a_{12} & \sin\theta^a_{12} e^{-i\delta^a_{12}} & 0 \\ -\sin\theta^a_{12} e^{i\delta^a_{12}} & \cos\theta^a_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

where $a = e, \nu, \circ$. The indices e, ν indicate the free parameters, while "$\circ$" indicates the angles and phases which are fixed. The complex rotation matrices $U_{23}(\theta^a_{23}, \delta^a_{23})$ and $U_{13}(\theta^a_{13}, \delta^a_{13})$ are defined in an analogous way. The real rotation matrices $R_{ij}(\theta^a_{ij})$ can be obtained from $U_{ij}(\theta^a_{ij}, \delta^a_{ij})$ setting δ^a_{ij} to zero, i.e., $R_{ij}(\theta^a_{ij}) = U_{ij}(\theta^a_{ij}, 0)$. In the absence of a residual symmetry no constraints are present for the mixing matrix U_e, which can be in general expressed in terms of three rotation angles and six phases.

Similar considerations apply to the neutrino sector. If $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ for Dirac neutrinos, or $G_\nu = Z_2 \times Z_2$ for Majorana neutrinos, the matrix U_ν is fixed up to permutations of columns and right multiplication by diagonal phase matrices by the residual symmetry, i.e., $U_\nu = U_\nu^\circ$. If the symmetry is smaller, $G_\nu = Z_2$, then

$$U_\nu = U_\nu^\circ U_{ij}(\theta_{ij}^\nu, \delta_{ij}^\nu)\Psi_k \Psi_l\,.$$

Obviously, in the absence of a residual symmetry, U_ν is unconstrained. In all the cases considered above where G_e and G_ν are non-trivial, the matrices $\rho(g_e)$ and $\rho(g_\nu)$ are diagonalised by U_e° and U_ν°:

$$(U_e^\circ)^\dagger \rho(g_e) U_e^\circ = \rho(g_e)^{\text{diag}} \quad \text{and} \quad (U_\nu^\circ)^\dagger \rho(g_\nu) U_\nu^\circ = \rho(g_\nu)^{\text{diag}}\,.$$

In what follows we define U° as the matrix fixed by the residual symmetries, which, in general, gets contributions from both the charged lepton and neutrino sectors, $U^\circ = (U_e^\circ)^\dagger U_\nu^\circ$. Since U° is a unitary 3×3 matrix, we will parametrise it in terms of three angles and six phases. These, however, as we are going to explain, reduce effectively to three angles and one phase, since the other five phases contribute to the Majorana phases of the PMNS mixing matrix, unphysical charged lepton phases and/or to a redefinition of the free parameters contained in U_e and U_ν. Furthermore, we will use the notation θ_{ij}^e, θ_{ij}^ν, δ_{ij}^e, δ_{ij}^ν for the free angles and phases contained in U, while the parameters marked with a circle contained in U°, e.g., θ_{ij}°, δ_{ij}°, are fixed by the residual symmetries.

In the case when $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ for massive Dirac neutrinos, or $G_\nu = Z_2 \times Z_2$ for Majorana neutrinos, we have:

$$\begin{aligned} U &= U_{ij}(\theta_{ij}^e, \delta_{ij}^e)\Psi_j^\circ U^\circ(\theta_{12}^\circ, \theta_{13}^\circ, \theta_{23}^\circ, \{\delta_{kl}^\circ\})Q_0 \\ &= \Psi_j^\circ U_{ij}(\theta_{ij}^e, \delta_{ij}^e - \psi_j^\circ)U^\circ(\theta_{12}^\circ, \theta_{13}^\circ, \theta_{23}^\circ, \{\delta_{kl}^\circ\})Q_0\,, \end{aligned} \tag{18}$$

where $(ij) = (12), (13), (23)$ and $\{\delta_{kl}^\circ\} = \{\delta_{12}^\circ, \delta_{13}^\circ, \delta_{23}^\circ\}$. The unitary matrix U° contains three angles and three phases, since the additional three phases can be absorbed by redefining the charged lepton fields and the free parameter δ_{ij}^e (see below). Here Ψ_j° is a diagonal matrix containing a fixed phase in the j-th position. Namely,

$$\Psi_1^\circ = \text{diag}\left(e^{i\psi_1^\circ}, 1, 1\right), \quad \Psi_2^\circ = \text{diag}\left(1, e^{i\psi_2^\circ}, 1\right), \quad \Psi_3^\circ = \text{diag}\left(1, 1, e^{i\psi_3^\circ}\right). \tag{19}$$

The matrix Q_0, defined in eq. (3), is a diagonal matrix containing two free parameters contributing to the Majorana phases. Since the presence of the phase ψ_j° amounts to a redefinition of the free parameter δ_{ij}^e, we denote $(\delta_{ij}^e - \psi_j^\circ)$ as δ_{ij}^e. This allows us to employ the following parametrisation for U:

$$U = U_{ij}(\theta_{ij}^e, \delta_{ij}^e)U^\circ(\theta_{12}^\circ, \theta_{13}^\circ, \theta_{23}^\circ, \delta_{kl}^\circ)Q_0\,, \tag{20}$$

where the unphysical phase matrix Ψ_j° on the left has been removed by charged lepton re-phasing and the set of three phases $\{\delta_{kl}^\circ\}$ reduces to only one phase, δ_{kl}°, since the other two contribute to redefinitions of Q_0, δ_{ij}^e and to unphysical phases. The possible forms of the matrix U°, which we are going to employ, are given in Appendix B.

For the breaking patterns $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$, valid for both Majorana and Dirac neutrinos, we have:

$$\begin{aligned} U &= U^\circ(\theta_{12}^\circ, \theta_{13}^\circ, \theta_{23}^\circ, \delta_{kl}^\circ)\Psi_i^\circ \Psi_j^\circ U_{ij}(\theta_{ij}^\nu, \delta_{ij}^\nu)Q_0 \\ &= U^\circ(\theta_{12}^\circ, \theta_{13}^\circ, \theta_{23}^\circ, \delta_{kl}^\circ)U_{ij}(\theta_{ij}^\nu, \delta_{ij}^\nu - \psi_i^\circ + \psi_j^\circ)\Psi_i^\circ \Psi_j^\circ Q_0\,, \end{aligned} \tag{21}$$

where $(ij) = (12), (13), (23)$, and the two free phases, which contribute to the Majorana phases of the PMNS matrix if the massive neutrinos are Majorana particles, have been included in the diagonal phase matrix Q_0. Notice that if neutrinos are assumed to be Dirac instead of Majorana, then the matrix Q_0 can be removed through re-phasing of the Dirac neutrino fields. Without loss of generality we can redefine the combination $\delta^{\nu}_{ij} - \psi^{\circ}_i + \psi^{\circ}_j$ as δ^{ν}_{ij} and the combination $\Psi^{\circ}_i \Psi^{\circ}_j Q_0$ as Q_0, so that the following parametrisation of U is obtained:

$$U = U^{\circ}(\theta^{\circ}_{12}, \theta^{\circ}_{13}, \theta^{\circ}_{23}, \delta^{\circ}_{kl}) U_{ij}(\theta^{\nu}_{ij}, \delta^{\nu}_{ij}) Q_0 \,. \tag{22}$$

In the case of $G_e = Z_2$ and $G_\nu = Z_2$ for both Dirac and Majorana neutrinos, we can write

$$\begin{aligned} U &= U_{ij}(\theta^{e}_{ij}, \delta^{e}_{ij}) \Psi^{\circ}_j U^{\circ}(\theta^{\circ}_{12}, \theta^{\circ}_{13}, \theta^{\circ}_{23}, \delta^{\circ}_{kl}) \Psi^{\circ}_r \Psi^{\circ}_s U_{rs}(\theta^{\nu}_{rs}, \delta^{\nu}_{rs}) Q_0 \\ &= \Psi^{\circ}_j U_{ij}(\theta^{e}_{ij}, \delta^{e}_{ij} - \psi^{\circ}_j) U^{\circ}(\theta^{\circ}_{12}, \theta^{\circ}_{13}, \theta^{\circ}_{23}, \delta^{\circ}_{kl}) U_{rs}(\theta^{\nu}_{rs}, \delta^{\nu}_{rs} - \psi^{\circ}_r + \psi^{\circ}_s) \Psi^{\circ}_r \Psi^{\circ}_s Q_0 \,, \end{aligned} \tag{23}$$

with $(ij) = (12), (13), (23)$, $(rs) = (12), (13), (23)$. The phase matrices Ψ°_i are defined as in eq. (19). Similarly to the previous cases, we can redefine the parameters in such a way that U can be cast in the following form:

$$U = U_{ij}(\theta^{e}_{ij}, \delta^{e}_{ij}) U^{\circ}(\theta^{\circ}_{12}, \theta^{\circ}_{13}, \theta^{\circ}_{23}, \delta^{\circ}_{kl}) U_{rs}(\theta^{\nu}_{rs}, \delta^{\nu}_{rs}) Q_0 \,, \tag{24}$$

where Q_0 can be phased away if neutrinos are assumed to be Dirac particles.[10]

If G_e is fully broken and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ for Dirac neutrinos or $G_\nu = Z_2 \times Z_2$ for Majorana neutrinos, the form of U reads

$$U = U(\theta^{e}_{12}, \theta^{e}_{13}, \theta^{e}_{23}, \delta^{e}_{rs}) \Psi_2 \Psi_3 U^{\circ}(\theta^{\circ}_{12}, \theta^{\circ}_{13}, \theta^{\circ}_{23}, \{\delta^{\circ}_{kl}\}) Q_0 \,, \tag{25}$$

where the phase matrices Ψ_2 and Ψ_3 are defined as in eq. (16). Notice that in general we can effectively parametrise U° in terms of three angles and one phase since of the set of three phases $\{\delta^{\circ}_{kl}\}$, two contribute to a redefinition of the matrices Q_0, Ψ_2 and Ψ_3. Furthermore, under the additional assumptions on the form of $U(\theta^{e}_{12}, \theta^{e}_{13}, \theta^{e}_{23}, \delta^{e}_{rs})$ and also taking $\{\delta^{\circ}_{kl}\} = 0$, the form of U given in eq. (25) leads to the sum rules derived in [11,14]. In the numerical analyses performed in [11,13,14], the angles θ°_{ij} have been set, in particular, to the values corresponding to the TBM, BM (LC), GRA, GRB and HG symmetry forms.

Finally for the breaking patterns $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and G_ν fully broken when considering both Dirac and Majorana neutrino possibilities, the form of U can be derived from eq. (25) by interchanging the fixed and the free parameters. Namely,

$$U = U^{\circ}(\theta^{\circ}_{12}, \theta^{\circ}_{13}, \theta^{\circ}_{23}, \delta^{\circ}_{kl}) \Psi_2 \Psi_3 U(\theta^{\nu}_{12}, \theta^{\nu}_{13}, \theta^{\nu}_{23}, \delta^{\nu}_{rs}) Q_0 \,. \tag{26}$$

The cases found in eqs. (20), (22), (24), (25) and (26) are summarised in Table 1. The reduction of the number of free parameters indicated with arrows corresponds to a redefinition of the charged lepton fields.

In the breaking patterns considered, it may be also possible to impose a generalised CP (GCP) symmetry. An example of how imposing a GCP affects the sum rules is shown in Appendix D. In the case in which a GCP symmetry is preserved in the neutrino sector we have for the neutrino Majorana mass matrix [39]:

$$X_i^T M_\nu X_i = M_\nu^* \,. \tag{27}$$

[10] We will not repeat this statement further, but it should be always understood that if the massive neutrinos are Dirac fermions, then two phases in the matrix Q_0 are unphysical and can be removed from U by a re-phasing of the Dirac neutrino fields.

Table 1
Number of effective free parameters, degrees of freedom (d.o.f.), contained in U relevant for the PMNS angles and the Dirac phase (and Majorana phases) in the cases of the different breaking patterns of G_f to G_e and G_ν. Arrows indicate the reduction of the number of parameters, which can be absorbed with a redefinition of the charged lepton fields.

$G_e \subset G_f$	$G_\nu \subset G_f$	U_e d.o.f.	U_ν d.o.f.	U d.o.f.
fully broken	fully broken	$9 \to 6$	$9 \to 8$	$12 \to 4$ (+2)
Z_2	fully broken	$4 \to 2$	$9 \to 8$	$10 \to 4$ (+2)
$\left\{ Z_n,\ n > 2 \text{ or } Z_n \times Z_m,\ n, m \geq 2 \right\}$	fully broken	0	$9 \to 8$	$8 \to 4$ (+2)
fully broken	Z_2	$9 \to 6$	4	$10 \to 4$ (+2)
fully broken	$\left\{ Z_n,\ n > 2 \text{ or } Z_n \times Z_m,\ n, m \geq 2 \right\}$	$9 \to 6$	2	$8 \to 4$ (+2)
Z_2	Z_2	$4 \to 2$	4	4 (+2)
$\left\{ Z_n,\ n > 2 \text{ or } Z_n \times Z_m,\ n, m \geq 2 \right\}$	Z_2	0	4	2 (+2)
Z_2	$\left\{ Z_n,\ n > 2 \text{ or } Z_n \times Z_m,\ n, m \geq 2 \right\}$	$4 \to 2$	2	2 (+2)
$\left\{ Z_n,\ n > 2 \text{ or } Z_n \times Z_m,\ n, m \geq 2 \right\}$	$\left\{ Z_n,\ n > 2 \text{ or } Z_n \times Z_m,\ n, m \geq 2 \right\}$	0	2	0 (+2)

Since the matrix X_i is symmetric there exists a unitary matrix Ω_i such that $X_i = \Omega_i\, \Omega_i^T$ and $\Omega_i^T M_\nu \Omega_i$ is real. Therefore when GCP is preserved in the neutrino sector, the phases in the matrix U_ν can be fixed. An alternative possibility is that GCP is preserved in the charged lepton sector, which leads to the condition [39]:

$$(X_i^e)^\dagger M_e M_e^\dagger X_i^e = (M_e M_e^\dagger)^* \,. \tag{28}$$

Since $(X_i^e)^T = X_i^e$, the phases in the matrix U_e are fixed, because $(\Omega_i^e)^\dagger M_e M_e^\dagger \Omega_i^e$ is real. The fact that the matrices X_i, if GCP is preserved in the neutrino sector, or X_i^e if it is preserved in the charged lepton sector, are symmetric matrices can be proved applying the GCP transformation twice. In the first case, eq. (27) allows one to derive the general form of X_i [40–42]:

$$X_i = U_\nu X_i^{\text{diag}} U_\nu^T \,, \tag{29}$$

while in the latter case

$$X_i^e = U_e (X_i^e)^{\text{diag}} U_e^T \,. \tag{30}$$

Equations (29) and (30) imply that X_i and X_i^e are symmetric matrices.[11]

We note finally that the titles of the following sections refer to the residual symmetries of the charged lepton and neutrino mass matrices, while the titles of the subsections reflect the free

[11] This fact can be also derived from the requirement that the GCP transformations contain the physical CP transformation, i.e., the GCP transformations applied twice to a field should give the field itself [40,43,44]:

$$\phi(x) \to X_{\mathbf{r}} \phi^*(x_{\text{p}}) \to X_{\mathbf{r}} X_{\mathbf{r}}^* \phi(x) = \phi(x) \,, \tag{31}$$

where $x = (x_0, \vec{x})$, $x_p = (x_0, -\vec{x})$. The notation we have used for $X_{\mathbf{r}}$ emphasises the representation $\mathbf{r}$ for the GCP transformations.

complex rotations contained in the corresponding parametrisation of U, eqs. (20), (22), (24), (25) and (26).

3. The pattern $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$

In this section we derive sum rules for $\cos\delta$ for the cases given in eq. (20). Recall that the matrix U_e is fixed up to a complex rotation in one plane by the residual $G_e = Z_2$ symmetry, while U_ν is completely determined (up to multiplication by diagonal phase matrices on the right and permutations of columns) by the $G_\nu = Z_2 \times Z_2$ residual symmetry in the case of neutrino Majorana mass term, or by $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$, residual symmetries if the massive neutrinos are Dirac particles. At the end of this section we will present results of a study of the possibility of reproducing the observed values of the lepton mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ and of obtaining physically viable predictions for $\cos\delta$ in the cases when the residual symmetries $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$, originate from the breaking of the lepton flavour symmetries A_4 (T'), S_4 and A_5.

3.1. *The case with $U_{12}(\theta^e_{12}, \delta^e_{12})$ complex rotation (case A1)*

Employing the parametrisation of the PMNS matrix U given in eq. (20) with $(ij) = (12)$ and the parametrisation of U° given as

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}) = U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13})\,, \tag{32}$$

we get for U (see Appendix B for details):

$$U = U_{12}(\theta^e_{12}, \delta^e_{12}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13}) Q_0\,. \tag{33}$$

The results derived in Appendix B and given in eq. (212) allow us to cast eq. (33) in the form:

$$U = R_{12}(\hat\theta_{12}) P_1(\hat\delta_{12}) R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13}) Q_0\,, \quad P_1(\hat\delta_{12}) = \mathrm{diag}(e^{i\hat\delta_{12}}, 1, 1)\,, \tag{34}$$

with $\hat\delta_{12} = \alpha - \beta$, where $\sin\hat\theta_{12}$, α and β are defined as in eqs. (213) and (214) after setting $i = 1$, $j = 2$, $\theta^a_{12} = \theta^e_{12}$, $\delta^a_{12} = \delta^e_{12}$, $\theta^b_{12} = \theta^\circ_{12}$ and $\delta^b_{12} = \delta^\circ_{12}$. Using eq. (34) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\hat\theta_{12}\sin^2\theta^\circ_{13} + \cos^2\theta^\circ_{13}\sin^2\hat\theta_{12}\sin^2\theta^\circ_{23} + \frac{1}{2}\sin 2\hat\theta_{12}\sin 2\theta^\circ_{13}\sin\theta^\circ_{23}\cos\hat\delta_{12}\,, \tag{35}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{1}{\cos^2\theta_{13}}\left[\sin^2\theta^\circ_{13} - \sin^2\theta_{13} + \cos^2\theta^\circ_{13}\sin^2\theta^\circ_{23}\right], \tag{36}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\cos^2\theta^\circ_{23}\sin^2\hat\theta_{12}}{\cos^2\theta_{13}}\,. \tag{37}$$

From eqs. (35) and (36) we get the following correlation between the values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$:

$$\sin^2\theta_{13} + \cos^2\theta_{13}\sin^2\theta_{23} = \sin^2\theta^\circ_{13} + \cos^2\theta^\circ_{13}\sin^2\theta^\circ_{23}\,. \tag{38}$$

Notice that eq. (37) implies that

$$\sin^2\hat\theta_{12} = \frac{\cos^2\theta_{13}\sin^2\theta_{12}}{\cos^2\theta^\circ_{23}}\,. \tag{39}$$

Table 2
The symmetry forms TBM, BM (LC), GRA, GRB and HG obtained in terms of the three rotations $R_{12}(\theta^\circ_{12})R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})$.

Mixing	θ°_{12}	θ°_{23}	θ°_{13}
TBM	$\pi/4$	$-\sin^{-1}(1/\sqrt{3})$	$\pi/6$
BM	$\sin^{-1}\sqrt{2/3}$	$-\pi/6$	$\sin^{-1}(1/\sqrt{3})$
GRA	$\sin^{-1}\sqrt{(7-\sqrt{5})/11}$	$-\sin^{-1}\sqrt{(5+\sqrt{5})/20}$	$\sin^{-1}\sqrt{(7-\sqrt{5})/22}$
GRB	$\sin^{-1}\sqrt{2(15-2\sqrt{5})/41}$	$-\sin^{-1}\sqrt{(3+\sqrt{5})/16}$	$\sin^{-1}\sqrt{(15-2\sqrt{5})/41}$
HG	$\sin^{-1}\sqrt{2/5}$	$-\sin^{-1}\sqrt{3/8}$	$\sin^{-1}\sqrt{1/5}$

In order to obtain a sum rule for $\cos\delta$, we compare the expressions for the absolute value of the element $U_{\tau 2}$ of the PMNS matrix in the standard parametrisation and in the parametrisation defined in eq. (34),

$$|U_{\tau 2}| = |\cos\theta_{12}\sin\theta_{23} + \sin\theta_{13}\cos\theta_{23}\sin\theta_{12}e^{i\delta}| = |\sin\theta^\circ_{23}|\,. \tag{40}$$

From the above equation we get for $\cos\delta$:

$$\cos\delta = \frac{\cos^2\theta_{13}(\sin^2\theta^\circ_{23} - \cos^2\theta_{12}) + \cos^2\theta^\circ_{13}\cos^2\theta^\circ_{23}(\cos^2\theta_{12} - \sin^2\theta_{12}\sin^2\theta_{13})}{\sin 2\theta_{12}\sin\theta_{13}|\cos\theta^\circ_{13}\cos\theta^\circ_{23}|(\cos^2\theta_{13} - \cos^2\theta^\circ_{13}\cos^2\theta^\circ_{23})^{\frac{1}{2}}}\,. \tag{41}$$

For the considered specific residual symmetries G_e and G_ν, the predicted value of $\cos\delta$ in the case A1 discussed in this subsection depends on the chosen discrete flavour symmetry G_f via the values of the angles θ°_{13} and θ°_{23}.

The method of derivation of the sum rule for $\cos\delta$ of interest employed in the present subsection and consisting, in particular, of choosing adequate parametrisations of the PMNS matrix U (in terms of the complex rotations of U_e and of U_ν) and of the matrix U° (determined by the symmetries G_e, G_ν and G_f), which allows to express the PMNS matrix U in terms of minimal numbers of angle and phase parameters, will be used also in all subsequent sections. The technical details related to the method are given in Appendices B and C.

We note finally that in the case of $\delta^\circ_{12} = 0$, the symmetry forms TBM, BM, GRA, GRB and HG can be obtained from $U^\circ = R_{12}(\theta^\circ_{12})R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})$ for specific values of the angles given in Table 2. In this case, the angles θ°_{ij} are related to the angles θ^ν_{ij} defined in Section 2.1 of Ref. [14] as follows:

$$\sin^2\theta^\circ_{23} = \cos^2\theta^\nu_{12}\sin^2\theta^\nu_{23}\,, \qquad \sin^2\theta^\circ_{13} = \frac{\sin^2\theta^\nu_{23}\sin^2\theta^\nu_{12}}{1-\sin^2\theta^\circ_{23}}\,, \qquad \sin^2\theta^\circ_{12} = \frac{\sin^2\theta^\nu_{12}}{1-\sin^2\theta^\circ_{23}}\,. \tag{42}$$

3.2. *The case with $U_{13}(\theta^e_{13}, \delta^e_{13})$ complex rotation (case A2)*

Using the parametrisation of the PMNS matrix U given in eq. (20) with $(ij) = (13)$ and the following parametrisation of U°,

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{13}) = U_{13}(\theta^\circ_{13}, \delta^\circ_{13})R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})\,, \tag{43}$$

we get for U (for details see Appendix B):

$$U = U_{13}(\theta^e_{13}, \delta^e_{13})U_{13}(\theta^\circ_{13}, \delta^\circ_{13})R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})Q_0\,. \tag{44}$$

The results derived in Appendix B and presented in eq. (212) allow us to recast eq. (44) in the following form:

$$U = R_{13}(\hat{\theta}_{13}) P_1(\hat{\delta}_{13}) R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) Q_0 \,, \quad P_1(\hat{\delta}_{13}) = \mathrm{diag}(e^{i\hat{\delta}_{13}}, 1, 1) \,. \tag{45}$$

Here $\hat{\delta}_{13} = \alpha - \beta$, where $\sin\hat{\theta}_{13}$, α and β are defined as in eqs. (213) and (214) after setting $i = 1$, $j = 3$, $\theta^a_{13} = \theta^e_{13}$, $\delta^a_{13} = \delta^e_{13}$, $\theta^b_{13} = \theta^\circ_{13}$ and $\delta^b_{13} = \delta^\circ_{13}$. Using eq. (45) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\hat{\theta}_{13}\cos^2\theta^\circ_{23} \,, \tag{46}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \frac{\sin^2\theta^\circ_{23}}{1 - \sin^2\theta_{13}} \,, \tag{47}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{1}{1 - \sin^2\theta_{13}}\Big[\cos^2\hat{\theta}_{13}\sin^2\theta^\circ_{12} + \cos^2\theta^\circ_{12}\sin^2\hat{\theta}_{13}\sin^2\theta^\circ_{23} - \frac{1}{2}\sin 2\hat{\theta}_{13}\sin 2\theta^\circ_{12}\sin\theta^\circ_{23}\cos\hat{\delta}_{13}\Big] \,. \tag{48}$$

Thus, in this scheme, as it follows from eq. (47), the value of $\sin^2\theta_{23}$ is predicted once the symmetry group G_f is fixed. This prediction, when confronted with the measured value of $\sin^2\theta_{23}$, constitutes an important test of the scheme considered for any given discrete (lepton flavour) symmetry group G_f, which contains the residual symmetry groups $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ and/or $Z_n \times Z_m$, $n, m \geq 2$ as subgroups.

As can be easily demonstrated, the case under discussion coincides with the one analysed in Section 2.2 of Ref. [14]. The parameters θ^ν_{23} and θ^ν_{12} in [14] can be identified with θ°_{23} and θ°_{12}, respectively. Therefore the sum rule we obtain coincides with that given in eq. (32) in [14]:

$$\cos\delta = -\frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{23} - \cos^2\theta_{12}) + \sin^2\theta^\circ_{23}(\cos^2\theta_{12} - \sin^2\theta_{12}\sin^2\theta_{13})}{\sin 2\theta_{12}\sin\theta_{13}|\sin\theta^\circ_{23}|(\cos^2\theta_{13} - \sin^2\theta^\circ_{23})^{\frac{1}{2}}} \,. \tag{49}$$

The dependence of $\cos\delta$ on G_f in this case is via the values of the angles θ°_{12} and θ°_{23}.

3.3. The case with $U_{23}(\theta^e_{23}, \delta^e_{23})$ complex rotation (case A3)

In the case with $(ij) = (23)$, as can be shown, $\cos\delta$ does not satisfy a sum rule, i.e., it cannot be expressed in terms of the three neutrino mixing angles θ_{12}, θ_{13} and θ_{23} and the other fixed angle parameters of the scheme. Indeed, employing the parametrisation of U° as $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{23}) = U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12})$, we can write the PMNS matrix in the following form:

$$U = U_{23}(\theta^e_{23}, \delta^e_{23}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12}) Q_0 \,. \tag{50}$$

Using the results derived in Appendix B and shown in eq. (212), we can recast eq. (50) as

$$U = R_{23}(\hat{\theta}_{23}) P_2(\hat{\delta}_{23}) R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12}) Q_0 \,, \quad P_2(\hat{\delta}_{23}) = \mathrm{diag}(1, e^{i\hat{\delta}_{23}}, 1) \,, \tag{51}$$

with $\hat{\delta}_{23} = \alpha - \beta$, where $\sin\hat{\theta}_{23}$, α and β are defined as in eqs. (213) and (214) after setting $i = 2$, $j = 3$, $\theta^a_{23} = \theta^e_{23}$, $\delta^a_{23} = \delta^e_{23}$, $\theta^b_{23} = \theta^\circ_{23}$ and $\delta^b_{23} = \delta^\circ_{23}$. Comparing eq. (51) and the standard parametrisation of the PMNS matrix, we find that $\sin^2\theta_{13} = \sin^2\theta^\circ_{13}$, $\sin^2\theta_{23} = \sin^2\hat{\theta}_{23}$, $\sin^2\theta_{12} = \sin^2\theta^\circ_{12}$ and $\cos\delta = \pm\cos\hat{\delta}_{23}$.

It follows from the preceding equations, in particular, that since, for any given G_f compatible with the considered residual symmetries, θ°_{13} and θ°_{12} have fixed values, the values of both $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$ are predicted. The predictions depend on the chosen symmetry G_f. Due to these predictions the scheme under discussion can be tested for any given discrete symmetry candidate G_f, compatible, in particular, with the considered residual symmetries.

We have also seen that δ is related only to an unconstrained phase parameter of the scheme. In the case of a flavour symmetry G_f which, in particular, allows to reproduce correctly the observed values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$, it might be possible to obtain physically viable prediction for $\cos\delta$ by employing a GCP invariance constraint. An example of the effect that GCP invariance has on restricting CPV phases is given in Appendix D. Investigating the implications of the GCP invariance constraint in the charged lepton or the neutrino sector in the cases considered by us is, however, beyond the scope of the present study.

3.4. Results in the cases of $G_f = A_4\ (T'),\ S_4$ and A_5

The cases detailed in Sections 3.1–3.3 can all be obtained from the groups A_4 (T'), S_4 and A_5, when breaking them to $G_e = Z_2$ and $G_\nu = Z_n$ $(n \geq 3)$ in the case of Dirac neutrinos, or $G_\nu = Z_2 \times Z_2$ in the case of both Dirac and Majorana neutrinos.[12] We now give an explicit example of how these cases can occur in A_4.

In the case of the group A_4 (see, e.g., [45]), the structure of the breaking patterns discussed, e.g., in subsection 3.1 can be realised when i) the S generator of A_4 is preserved in the neutrino sector, and when, due to an accidental symmetry, the mixing matrix is fixed to be tri-bimaximal, $U^\circ_\nu = U_{\rm TBM}$, up to permutations of the columns, and ii) a $Z_2^{T^2ST}$ or $Z_2^{TST^2}$ is preserved in the charged lepton sector. The group element generating the Z_2 symmetry is diagonalised by the matrix U°_e. Therefore the angles θ°_{12}, θ°_{13} and θ°_{23} are obtained from the product $U^\circ = (U^\circ_e)^\dagger U^\circ_\nu$. The same structure (the structure discussed in subsection 3.2) can be obtained in a similar manner from the flavour groups S_4 and A_5 (A_4, S_4 and A_5).

We have investigated the possibility of reproducing the observed values of the lepton mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ as well as obtaining physically viable predictions for $\cos\delta$ in the cases of residual symmetries $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$[13] (Dirac neutrinos), or $G_\nu = Z_2 \times Z_2$ (Majorana neutrinos), discussed in subsections 3.1, 3.2 and 3.3 denoted further as A1, A2 and A3, assuming that these residual symmetries originate from the breaking of the flavour symmetries A_4 (T'), S_4 and A_5. The analysis was performed using the current best fit values of the three lepton mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$. The results we have obtained for the symmetries A_4 (T'), S_4 and A_5 are summarised below.

We have found that in the cases under discussion, i.e., in the cases A1, A2 and A3, and flavour symmetries $G_f = A_4$ (T'), S_4 and A_5, with the exceptions to be discussed below, it is impossible either to reproduce at least one of the measured values of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ even taking into account its respective 3σ uncertainty, or to get physically viable values of $\cos\delta$ satisfying $|\cos\delta| \leq 1$. In the cases A1 and A2 and the flavour groups A_4 and S_4, for instance, the values of $\cos\delta$ are unphysical. Using the group $G_f = A_5$ leads either to unphysical values of $\cos\delta$, or to values of $\sin^2\theta_{23}$ which lie outside the corresponding current 3σ allowed

[12] We only consider $Z_2 \times Z_2$ when it is an actual subgroup of G_f.

[13] Note that there are no subgroups of the type $Z_n \times Z_m$ bigger than $Z_2 \times Z_2$ in the cases of A_4, S_4 and A_5.

interval. In the case A3 (discussed in subsection 3.3), the symmetry A_4, for example, leads to $(\sin^2\theta_{12}, \sin^2\theta_{13}) = (0, 0)$ or $(1, 0)$.

As mentioned earlier, there are three exceptions in which we can still get phenomenologically viable results. In the A1 case (A2 case) and S_4 flavour symmetry, one obtains bimaximal mixing corrected by a complex rotation in the 1–2 plane[14] (1–3 plane). The PMNS angle θ_{23} is predicted to have a value corresponding to $\sin^2\theta_{23} = 0.488$ ($\sin^2\theta_{23} = 0.512$). For the best fit values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ we find that $\cos\delta = -1.29$ ($\cos\delta = +1.29$). However, using the value of $\sin^2\theta_{12} = 0.348$, which lies in the 3σ allowed interval, one gets the same value of $\sin^2\theta_{23}$ and $\cos\delta = -0.993$ ($\cos\delta = 0.993$), while in the part of the 3σ allowed interval of $\sin^2\theta_{12}$, $0.348 \leq \sin^2\theta_{12} \leq 0.359$, we have $-0.993 \leq \cos\delta \leq -0.915$ ($0.993 \geq \cos\delta \geq 0.915$).

Also in the A1 case (A2 case) but with an A_5 flavour symmetry and residual symmetry $G_\nu = Z_3$, which is only possible if the massive neutrinos are Dirac particles, we get the predictions $\sin^2\theta_{23} = 0.553$ ($\sin^2\theta_{23} = 0.447$) and $\cos\delta = 0.716$ ($\cos\delta = -0.716$). In the A1 case (A2 case) with an A_5 flavour symmetry and residual symmetry $G_\nu = Z_5$, which can be realised for neutrino Dirac mass term only, for the best fit values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ we get the predictions $\sin^2\theta_{23} = 0.630$ ($\sin^2\theta_{23} = 0.370$), which is slightly outside the current 3σ range) and $\cos\delta = -1.12$ ($\cos\delta = 1.12$). However, using the value of $\sin^2\theta_{12} = 0.321$, which lies in the 1σ allowed interval of $\sin^2\theta_{12}$, one gets the same value of $\sin^2\theta_{23}$ and $\cos\delta = -0.992$ ($\cos\delta = 0.992$). In the part of the 3σ allowed interval of $\sin^2\theta_{12}$, $0.321 \leq \sin^2\theta_{12} \leq 0.359$, one has $-0.992 \leq \cos\delta \leq -0.633$ ($0.992 \geq \cos\delta \geq 0.633$).

4. The pattern $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$

In this section we derive sum rules for $\cos\delta$ in the case given in eq. (22). We recall that for $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$ of interest, the matrix U_e is unambiguously determined (up to multiplication by diagonal phase matrices on the right and permutations of columns), while the matrix U_ν is determined up to a complex rotation in one plane.

4.1. The case with $U_{13}(\theta^\nu_{13}, \delta^\nu_{13})$ complex rotation (case B1)

Combining the parametrisation of the PMNS matrix U given in eq. (22) with $(ij) = (13)$ and the parametrisation of U° as

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{13}) = R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) \,, \tag{53}$$

we get for U (the details are given again in Appendix B):

$$U = R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) U_{13}(\theta^\nu_{13}, \delta^\nu_{13}) Q_0 \,. \tag{54}$$

The results derived in Appendix B and reported in eq. (212) allow us to recast eq. (54) in the form:

$$U = R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) P_3(\hat\delta_{13}) R_{13}(\hat\theta_{13}) Q_0 \,, \quad P_3(\hat\delta_{13}) = \mathrm{diag}(1, 1, e^{i\hat\delta_{13}}) \,. \tag{55}$$

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[14] For the case A1 it can been shown that

$$\mathrm{diag}(-1, 1, 1) U(\theta^\circ_{12}, \delta^\circ_{12}) R(\theta^\circ_{23}) R(\theta^\circ_{13}) \,\mathrm{diag}(1, -1, 1) = U_{\rm BM} \,, \tag{52}$$

if $\theta^\circ_{23} = \sin^{-1}(1/2)$, $\theta^\circ_{13} = \sin^{-1}(\sqrt{1/3})$, $\theta^\circ_{12} = \tan^{-1}(\sqrt{3/2} + \sqrt{1/2})$ and $\delta^\circ_{12} = 0$. Therefore, one has BM mixing corrected from the left by a $U(2)$ transformation in the degenerate subspace in the 1–2 plane. Note that our results are in agreement with those obtained in [46].

Here $\hat{\delta}_{13} = -\alpha - \beta$ and we have redefined $P_{13}(\alpha,\beta)Q_0$ as Q_0, where $P_{13}(\alpha,\beta) = \mathrm{diag}(e^{i\alpha}, 1, e^{i\beta})$ and the expressions for $\sin^2\hat{\theta}_{13}$, α and β can be obtained from eqs. (213) and (214), by setting $i = 1$, $j = 3$, $\theta^a_{13} = \theta^\circ_{13}$, $\delta^a_{13} = \delta^\circ_{13}$, $\theta^b_{13} = \theta^\nu_{13}$ and $\delta^b_{13} = \delta^\nu_{13}$. Using eq. (55) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\theta^\circ_{12}\sin^2\hat{\theta}_{13}\,, \tag{56}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{1}{\cos^2\theta_{13}}\Big[\cos^2\theta^\circ_{23}\sin^2\hat{\theta}_{13}\sin^2\theta^\circ_{12} + \cos^2\hat{\theta}_{13}\sin^2\theta^\circ_{23} - \frac{1}{2}\sin 2\hat{\theta}_{13}\sin 2\theta^\circ_{23}\sin\theta^\circ_{12}\cos\hat{\delta}_{13}\Big]\,, \tag{57}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\sin^2\theta^\circ_{12}}{\cos^2\theta_{13}}\,. \tag{58}$$

It follows from eq. (58) that in the case under discussion the values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ are correlated.

A sum rule for $\cos\delta$ can be derived by comparing the expressions for the absolute value of the element $U_{\tau 2}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (55):

$$|U_{\tau 2}| = |\cos\theta_{12}\sin\theta_{23} + \sin\theta_{13}\cos\theta_{23}\sin\theta_{12}e^{i\delta}| = |\cos\theta^\circ_{12}\sin\theta^\circ_{23}|\,. \tag{59}$$

From this equation we get

$$\cos\delta = -\frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{23} - \cos^2\theta_{23}) + \sin^2\theta^\circ_{12}(\cos^2\theta_{23} - \sin^2\theta_{13}\sin^2\theta_{23})}{\sin 2\theta_{23}\sin\theta_{13}|\sin\theta^\circ_{12}|(\cos^2\theta_{13} - \sin^2\theta^\circ_{12})^{\frac{1}{2}}}\,. \tag{60}$$

The dependence of the predictions for $\cos\delta$ on G_f is in this case via the values of θ°_{12} and θ°_{23}.

4.2. The case with $U_{23}(\theta^\nu_{23}, \delta^\nu_{23})$ complex rotation (case B2)

Utilising the parametrisation of the PMNS matrix U given in eq. (22) with $(ij) = (23)$ and the following parametrisation of U°,

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{23}) = R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})U_{23}(\theta^\circ_{23}, \delta^\circ_{23})\,, \tag{61}$$

we obtain for U (Appendix B contains the relevant details):

$$U = R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})U_{23}(\theta^\circ_{23}, \delta^\circ_{23})U_{23}(\theta^\nu_{23}, \delta^\nu_{23})Q_0\,. \tag{62}$$

The results given in eq. (212) in Appendix B make it possible to bring eq. (62) to the form:

$$U = R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})P_3(\hat{\delta}_{23})R_{23}(\hat{\theta}_{23})Q_0\,, \quad P_3(\hat{\delta}_{23}) = \mathrm{diag}(1, 1, e^{i\hat{\delta}_{23}})\,. \tag{63}$$

Here $\hat{\delta}_{23} = -\alpha - \beta$ and we have redefined $P_{23}(\alpha,\beta)Q_0$ as Q_0, where $P_{23}(\alpha,\beta) = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$. Using eq. (63) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\theta^\circ_{13}\sin^2\theta^\circ_{12}\sin^2\hat{\theta}_{23} + \sin^2\theta^\circ_{13}\cos^2\hat{\theta}_{23} + \frac{1}{2}\sin 2\hat{\theta}_{23}\sin 2\theta^\circ_{13}\sin\theta^\circ_{12}\cos\hat{\delta}_{23}\,, \tag{64}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{\cos^2\theta^\circ_{12}\sin^2\hat{\theta}_{23}}{\cos^2\theta_{13}}\,, \tag{65}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\cos^2\theta_{13}-\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{13}}{\cos^2\theta_{13}}\,. \tag{66}$$

Equation (66) implies that, as in the case investigated in the preceding subsection, the values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ are correlated.

The sum rule for $\cos\delta$ of interest can be obtained by comparing the expressions for the absolute value of the element $U_{\tau 1}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (63):

$$|U_{\tau 1}| = |\sin\theta_{12}\sin\theta_{23} - \sin\theta_{13}\cos\theta_{12}\cos\theta_{23}e^{i\delta}| = |\cos\theta^\circ_{12}\sin\theta^\circ_{13}|\,. \tag{67}$$

From the above equation we get for $\cos\delta$:

$$\cos\delta = \frac{\cos^2\theta_{13}(\sin^2\theta^\circ_{12}-\cos^2\theta_{23}) + \cos^2\theta^\circ_{12}\cos^2\theta^\circ_{13}(\cos^2\theta_{23}-\sin^2\theta_{13}\sin^2\theta_{23})}{\sin 2\theta_{23}\sin\theta_{13}|\cos\theta^\circ_{12}\cos\theta^\circ_{13}|(\cos^2\theta_{13}-\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{13})^{\frac{1}{2}}}\,. \tag{68}$$

The dependence of $\cos\delta$ on G_f is realised in this case through the values of θ°_{12} and θ°_{13}.

4.3. The case with $U_{12}(\theta^\nu_{12}, \delta^\nu_{12})$ complex rotation (case B3)

In this case, as we show below, $\cos\delta$ does not satisfy a sum rule, and thus is, in general, a free parameter. Indeed, using the parametrisation of U° as $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}) = R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})U_{12}(\theta^\circ_{12}, \delta^\circ_{12})$ we get the following expression for U:

$$U = R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})U_{12}(\theta^\circ_{12}, \delta^\circ_{12})U_{12}(\theta^\nu_{12}, \delta^\nu_{12})Q_0\,. \tag{69}$$

After recasting eq. (69) in the form

$$U = R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})P_2(\hat{\delta}_{12})R_{12}(\hat{\theta}_{12})Q_0\,, \quad P_2(\hat{\delta}_{12}) = \mathrm{diag}(1, e^{i\hat{\delta}_{12}}, 1)\,, \tag{70}$$

where $\hat{\delta}_{12} = -\alpha - \beta$, we find that $\sin^2\theta_{13} = \sin^2\theta^\circ_{13}$, $\sin^2\theta_{23} = \sin^2\theta^\circ_{23}$, $\sin^2\theta_{12} = \sin^2\hat{\theta}_{12}$ and $\cos\delta = \pm\cos\hat{\delta}_{12}$.

It follows from the expressions for the neutrino mixing parameters thus derived that, given a discrete symmetry G_f which can lead to the considered breaking patterns, the values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ are predicted. This, in turn, allows to test the phenomenological viability of the scheme under discussion for any appropriately chosen discrete lepton flavour symmetry G_f.

In what concerns the phase δ, it is expressed in terms of an unconstrained phase parameter present in the scheme we are considering. The comment made at the end of subsection 3.3 is valid also in this case. Namely, given a non-Abelian discrete flavour symmetry G_f which allows one to reproduce correctly the observed values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, it might be possible to obtain physically viable prediction for $\cos\delta$ by employing a GCP invariance constraint in the charged lepton or the neutrino sector.

4.4. Results in the cases of $G_f = A_4$ (T'), S_4 and A_5

The schemes discussed in Sections 4.1–4.3 are realised when breaking $G_f = A_4$ (T'), S_4 and A_5, to $G_e = Z_n$ $(n \geq 3)$ or $Z_2 \times Z_2$ and $G_\nu = Z_2$, for both Dirac and Majorana neutrinos. As a reminder to the reader, we investigate the case of $Z_2 \times Z_2$ when it is an actual subgroup of G_f. As an explicit example of how this breaking can occur, we will consider the case of $G_f = A_4$ (T'). The other cases when $G_f = S_4$ or A_5 can be obtained from the breaking of S_4 and A_5 to the relevant subgroups as given in [46] and [47], respectively.

In the case of the group A_4 (see, e.g., [45]), the structure of the breaking patterns discussed, e.g., in subsection 4.1 can be obtained by breaking A_4 i) in the charged lepton sector to any of the four Z_3 subgroups, namely, Z_3^T, Z_3^{ST}, Z_3^{TS}, Z_3^{STS}, and ii) to any of the three Z_2 subgroups, namely, Z_2^S, $Z_2^{T^2ST}$, $Z_2^{TST^2}$, in the neutrino sector. In this case the matrix $U^\circ = U_{\text{TBM}}$ gets corrected by a complex rotation matrix in the 1–3 plane coming from the neutrino sector.

The results of the study performed by us of the phenomenological viability of the schemes with residual symmetries $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$, discussed in subsections 4.1, 4.2 and 4.3, and denoted further as B1, B2 and B3, when the residual symmetries result from the breaking of the flavour symmetries A_4 (T'), S_4 and A_5, are described below. We present results only in the cases in which we obtain values of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ compatible with their respective measured values (including the corresponding 3σ uncertainties) and physically acceptable values of $\cos\delta$.

For $G_f = A_4$, we find that only the case B1 with $G_e = Z_3$ is phenomenologically viable. In this case we have $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/3, 1/2)$, which leads to the predictions $\sin^2\theta_{12} = 0.341$ and $\cos\delta = 0.570$. We find precisely the same results in the case B1 if $G_f = S_4$ and $G_e = Z_3$. Phenomenologically viable results are obtained for $G_f = S_4$ and $G_e = Z_3$ in the case B2 as well. In this case $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}) = (1/6, 1/5)$, implying the predictions $\sin^2\theta_{12} = 0.317$ and $\cos\delta = -0.269$. If $G_e = Z_4$ or $Z_2 \times Z_2$ results from $G_f = S_4$, we get in the case B1 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/4, 1/3)$ and correspondingly $\sin^2\theta_{12} = 0.256$ (which lies slightly outside the current 3σ allowed range of $\sin^2\theta_{12}$) and the unphysical value of $\cos\delta = -1.19$. These two values are obtained for the best fit values of $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$. However, for $\sin^2\theta_{23} = 0.419$ we find the physical value $\cos\delta = -0.990$, while in the part of the 3σ allowed interval of $\sin^2\theta_{23}$, $0.374 \leq \sin^2\theta_{23} \leq 0.419$, we have $-0.495 \geq \cos\delta \geq -0.990$.

If $G_f = A_5$, we find phenomenologically viable results i) for $G_e = Z_3$, in the case B1, ii) for $G_e = Z_5$, in the cases B1 and B2, and iii) for $G_e = Z_2 \times Z_2$, in the case B2. More specifically, if $G_e = Z_3$, we obtain in the case B1 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/3, 1/2)$ leading to the predictions $\sin^2\theta_{12} = 0.341$ and $\cos\delta = 0.570$. For $G_e = Z_5$ in the case B1 (case B2) we find $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (0.276, 1/2)$ ($(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}) = (0.138, 0.160)$), which leads to the predictions $\sin^2\theta_{12} = 0.283$ and $\cos\delta = 0.655$ ($\sin^2\theta_{12} = 0.259$ and $\cos\delta = -0.229$). Finally, for $G_e = Z_2 \times Z_2$ in the case B2 we have two sets of values for $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13})$. The first one, $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}) = (0.096, 0.276)$, together with the best fit values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, leads to $\sin^2\theta_{12} = 0.330$ and $\cos\delta = -1.36$. However, $\cos\delta$ takes the physical value of $\cos\delta = -0.996$ for $\sin^2\theta_{23} = 0.518$. In the part of the 3σ allowed interval of values of $\sin^2\theta_{23}$, $0.518 \leq \sin^2\theta_{23} \leq 0.641$, we have $-0.996 \leq \cos\delta \leq -0.478$. For the second set of values, $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}) = (1/4, 0.127)$, we get the predictions $\sin^2\theta_{12} = 0.330$ and $\cos\delta = 0.805$.

5. The pattern $G_e = Z_2$ and $G_\nu = Z_2$

In this section we derive sum rules for $\cos\delta$ in the case given in eq. (24). We recall that when the residual symmetries are $G_e = Z_2$ and $G_\nu = Z_2$, each of the matrices U_e and U_ν is determined up to a complex rotation in one plane.

5.1. The case with $U_{12}(\theta^e_{12}, \delta^e_{12})$ and $U_{13}(\theta^\nu_{13}, \delta^\nu_{13})$ complex rotations (case C1)

Similar to the already considered cases we combine the parametrisation of the PMNS matrix U given in eq. (24) with $(ij) = (12)$ and $(rs) = (13)$, with the parametrisation of U° given as

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}, \delta^\circ_{13}) = U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{23}(\theta^\circ_{23}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) \,, \tag{71}$$

and get the following expression for U (as usual, we refer to Appendix B for details):

$$U = U_{12}(\theta^e_{12}, \delta^e_{12}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{23}(\theta^\circ_{23}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) U_{13}(\theta^\nu_{13}, \delta^\nu_{13}) Q_0 \,. \tag{72}$$

Utilising the results derived in Appendix B and reported in eq. (212), we can recast eq. (72) in the form

$$U = R_{12}(\hat\theta^e_{12}) P_1(\hat\delta) R_{23}(\theta^\circ_{23}) R_{13}(\hat\theta^\nu_{13}) Q_0 \,, \quad P_1(\hat\delta) = \mathrm{diag}(e^{i\hat\delta}, 1, 1) \,. \tag{73}$$

Here $\hat\delta = \alpha^e - \beta^e + \alpha^\nu + \beta^\nu$ and we have redefined the matrix Q_0 by absorbing the diagonal phase matrix $P_{13}(-\beta^\nu, -\alpha^\nu) = \mathrm{diag}(e^{-i\beta^\nu}, 1, e^{-i\alpha^\nu})$ in it. Using eq. (73) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\hat\theta^e_{12} \sin^2\hat\theta^\nu_{13} + \cos^2\hat\theta^\nu_{13} \sin^2\hat\theta^e_{12} \sin^2\theta^\circ_{23} + \frac{1}{2} \sin 2\hat\theta^e_{12} \sin 2\hat\theta^\nu_{13} \sin\theta^\circ_{23} \cos\hat\delta \,, \tag{74}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \frac{\sin^2\hat\theta^\nu_{13} - \sin^2\theta_{13} + \cos^2\hat\theta^\nu_{13} \sin^2\theta^\circ_{23}}{1 - \sin^2\theta_{13}} \,, \tag{75}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{\sin^2\hat\theta^e_{12} \cos^2\theta^\circ_{23}}{1 - \sin^2\theta_{13}} \,. \tag{76}$$

The sum rule for $\cos\delta$ of interest can be derived by comparing the expressions for the absolute value of the element $U_{\tau 2}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (73):

$$|U_{\tau 2}| = |\cos\theta_{12} \sin\theta_{23} + \sin\theta_{13} \cos\theta_{23} \sin\theta_{12} e^{i\delta}| = |\sin\theta^\circ_{23}| \,. \tag{77}$$

From the above equation we get for $\cos\delta$:

$$\cos\delta = \frac{\sin^2\theta^\circ_{23} - \cos^2\theta_{12} \sin^2\theta_{23} - \cos^2\theta_{23} \sin^2\theta_{12} \sin^2\theta_{13}}{\sin\theta_{13} \sin 2\theta_{23} \sin\theta_{12} \cos\theta_{12}} \,. \tag{78}$$

Given the assumed breaking pattern, $\cos\delta$ depends on the flavour symmetry G_f via the value of θ°_{23}. Using the best fit values of the standard mixing angles for the NO neutrino mass spectrum and the requirement $|\cos\delta| \le 1$, we find that $\sin^2\theta^\circ_{23}$ should lie in the following interval: $0.236 \le \sin^2\theta^\circ_{23} \le 0.377$. Fixing two of the three angles to their best fit values and varying the third one in its 3σ experimentally allowed range and considering all the three possible combinations, we get that $|\cos\delta| \le 1$ if $0.195 \le \sin^2\theta^\circ_{23} \le 0.504$.

5.2. The case with $U_{13}(\theta^e_{13}, \delta^e_{13})$ and $U_{12}(\theta^\nu_{12}, \delta^\nu_{12})$ complex rotations (case C2)

As in the preceding case, we use the parametrisation of the PMNS matrix U given in eq. (24) but this time with $(ij) = (13)$ and $(rs) = (12)$, and the parametrisation of U° as

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}, \delta^\circ_{13}) = U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) R_{23}(\theta^\circ_{23}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) \,, \tag{79}$$

to get for U (again the details can be found in Appendix B):

$$U = U_{13}(\theta^e_{13}, \delta^e_{13}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) R_{23}(\theta^\circ_{23}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) U_{12}(\theta^\nu_{12}, \delta^\nu_{12}) Q_0 \,. \tag{80}$$

The results derived in Appendix B and reported in eq. (212) allow us to rewrite the expression for U in eq. (80) as follows:

$$U = R_{13}(\hat\theta^e_{13}) P_1(\hat\delta) R_{23}(\theta^\circ_{23}) R_{12}(\hat\theta^\nu_{12}) Q_0 \,, \quad P_1(\hat\delta) = \mathrm{diag}(e^{i\hat\delta}, 1, 1) \,, \tag{81}$$

where $\hat\delta = \alpha^e - \beta^e + \alpha^\nu + \beta^\nu$, and also in this case we have redefined the matrix Q_0 by absorbing the phase matrix $P_{12}(-\beta^\nu, -\alpha^\nu) = \mathrm{diag}(e^{-i\beta^\nu}, e^{-i\alpha^\nu}, 1)$ in it. From eq. (81) and the standard parametrisation of the PMNS matrix U we get:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\theta^\circ_{23} \sin^2\hat\theta^e_{13} \,, \tag{82}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \frac{\sin^2\theta^\circ_{23}}{\cos^2\theta_{13}} \,, \tag{83}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{1}{1 - \sin^2\theta_{13}} \Big[\cos^2\hat\theta^e_{13} \sin^2\hat\theta^\nu_{12} + \cos^2\hat\theta^\nu_{12} \sin^2\hat\theta^e_{13} \sin^2\theta^\circ_{23} - \frac{1}{2} \sin 2\hat\theta^e_{13} \sin 2\hat\theta^\nu_{12} \sin\theta^\circ_{23} \cos\hat\delta \Big] . \tag{84}$$

Given the value of $\sin^2\theta^\circ_{23}$, eq. (83) implies the existence of a correlation between the values of $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$.

Comparing the expressions for the absolute value of the element $U_{\mu 1}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (81), we have

$$|U_{\mu 1}| = |\sin\theta_{12}\cos\theta_{23} + \sin\theta_{13}\sin\theta_{23}\cos\theta_{12} e^{i\delta}| = |\sin\hat\theta^\nu_{12} \cos^2\theta^\circ_{23}| \,. \tag{85}$$

From the above equations we get for $\cos\delta$:

$$\cos\delta = \frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{23}\sin^2\hat\theta^\nu_{12} - \sin^2\theta_{12}) + \sin^2\theta^\circ_{23}(\sin^2\theta_{12} - \cos^2\theta_{12}\sin^2\theta_{13})}{\sin 2\theta_{12}\sin\theta_{13}|\sin\theta^\circ_{23}|(\cos^2\theta_{13} - \sin^2\theta^\circ_{23})^{\frac{1}{2}}} \,. \tag{86}$$

In this case $\cos\delta$ is a function of the known neutrino mixing angles θ_{12} and θ_{13}, of the angle θ°_{23} fixed by G_f and the assumed symmetry breaking pattern, as well as of the phase parameter $\hat\delta$ of the scheme. Predictions for $\cos\delta$ can only be obtained when $\hat\delta$ is fixed by additional considerations of, e.g., GCP invariance, symmetries, etc. In view of this we show in Fig. 1 $\cos\delta$ as a function of $\cos\hat\delta$ for the current best fit values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$, and for the value $\sin^2\theta^\circ_{23} = 1/2$ corresponding to $G_f = S_4$. We do not find phenomenologically viable cases for A_4 (T') and A_5. Therefore we do not present such a plot for these groups.

5.3. The case with $U_{12}(\theta^e_{12}, \delta^e_{12})$ and $U_{23}(\theta^\nu_{23}, \delta^\nu_{23})$ complex rotations (case C3)

We get for the PMNS matrix U,

$$U = U_{12}(\theta^e_{12}, \delta^e_{12}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{13}(\theta^\circ_{13}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) U_{23}(\theta^\nu_{23}, \delta^\nu_{23}) Q_0 \,, \tag{87}$$

utilising the parametrisations of U shown in eq. (24) with $(ij) = (12)$ and $(rs) = (23)$ and that of U° given below (further details can be found in Appendix B),

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}, \delta^\circ_{23}) = U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{13}(\theta^\circ_{13}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) \,. \tag{88}$$

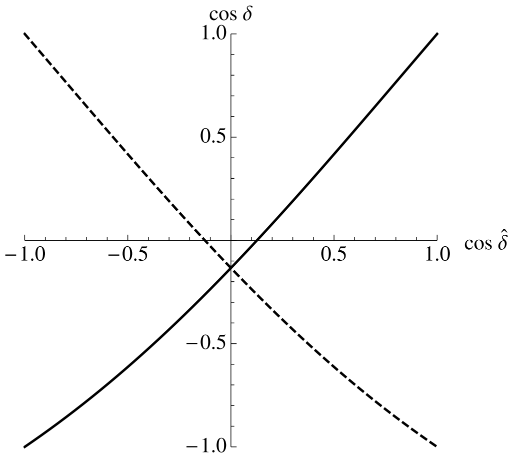

Fig. 1. Dependence of $\cos\delta$ on $\cos\hat{\delta}$ in the case of $G_f = S_4$ with $\sin^2\theta^{\circ}_{23} = 1/2$. The mixing parameters $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ have been fixed to their best fit values for the NO neutrino mass spectrum quoted in eqs. (6) and (8). The solid (dashed) line is for the case when $\sin 2\hat{\theta}^e_{13}\sin 2\hat{\theta}^\nu_{12}$ is positive (negative).

With the help of the results derived in Appendix B and especially of eq. (212), the expression in eq. (87) for the PMNS matrix U can be brought to the form

$$U = R_{12}(\hat{\theta}^e_{12})P_2(\hat{\delta})R_{13}(\theta^{\circ}_{13})R_{23}(\hat{\theta}^\nu_{23})Q_0\,, \quad P_2(\hat{\delta}) = \mathrm{diag}(1, e^{i\hat{\delta}}, 1)\,, \tag{89}$$

where $\hat{\delta} = \beta^e - \alpha^e + \alpha^\nu + \beta^\nu$ and, as in the preceding cases, we have redefined the phase matrix Q_0 by absorbing the phase matrix $P_{23}(-\beta^\nu, -\alpha^\nu) = \mathrm{diag}(1, e^{-i\beta^\nu}, e^{-i\alpha^\nu})$ in it. Using eq. (89) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\hat{\theta}^e_{12}\sin^2\hat{\theta}^\nu_{23} + \cos^2\hat{\theta}^e_{12}\cos^2\hat{\theta}^\nu_{23}\sin^2\theta^{\circ}_{13} + \frac{1}{2}\sin 2\hat{\theta}^e_{12}\sin 2\hat{\theta}^\nu_{23}\sin\theta^{\circ}_{13}\cos\hat{\delta}\,, \tag{90}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{\sin^2\hat{\theta}^\nu_{23} - \sin^2\theta_{13} + \cos^2\hat{\theta}^\nu_{23}\sin^2\theta^{\circ}_{13}}{1-\sin^2\theta_{13}}\,, \tag{91}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\sin^2\hat{\theta}^e_{12} - \sin^2\theta_{13} + \cos^2\hat{\theta}^e_{12}\sin^2\theta^{\circ}_{13}}{1-\sin^2\theta_{13}}\,. \tag{92}$$

The sum rule for $\cos\delta$ of interest can be derived, e.g., by comparing the expressions for the absolute value of the element $U_{\tau 1}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (89):

$$|U_{\tau 1}| = |\sin\theta_{12}\sin\theta_{23} - \sin\theta_{13}\cos\theta_{12}\cos\theta_{23}e^{i\delta}| = |\sin\theta^{\circ}_{13}|\,. \tag{93}$$

For $\cos\delta$ we get:

$$\cos\delta = \frac{\sin^2\theta_{12}\sin^2\theta_{23} - \sin^2\theta^{\circ}_{13} + \cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}}{\sin\theta_{13}\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}}\,. \tag{94}$$

In this case, in contrast to that considered in the preceding subsection, $\cos\delta$ is predicted once the angle θ°_{13}, i.e., the flavour symmetry G_f, is fixed. Using the best fit values of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ for the NO neutrino mass spectrum, we find that physical values of

$\cos\delta$ satisfying $|\cos\delta| \le 1$ can be obtained only if $\sin^2\theta^\circ_{13}$ lies in the following interval: $0.074 \le \sin^2\theta^\circ_{13} \le 0.214$. Fixing two of the three neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ to their best fit values and varying the third one in its 3σ experimentally allowed range and taking into account all the three possible combinations, we get that $|\cos\delta| \le 1$ provided $0.056 \le \sin^2\theta^\circ_{13} \le 0.267$.

5.4. The case with $U_{13}(\theta^e_{13}, \delta^e_{13})$ and $U_{23}(\theta^\nu_{23}, \delta^\nu_{23})$ complex rotations (case C4)

The parametrisation of the PMNS matrix U, to be used further,

$$U = U_{13}(\theta^e_{13}, \delta^e_{13}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) R_{12}(\theta^\circ_{12}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) U_{23}(\theta^\nu_{23}, \delta^\nu_{23}) Q_0\,, \tag{95}$$

is found in this case from the parametrisations of the matrix U given in eq. (24) with $(ij) = (13)$ and $(rs) = (23)$ and that of U° shown below (see Appendix B for details),

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{13}, \delta^\circ_{23}) = U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) R_{12}(\theta^\circ_{12}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23})\,. \tag{96}$$

The results presented in eq. (212) of Appendix B allow us to recast eq. (95) in the form:

$$U = R_{13}(\hat\theta^e_{13}) P_3(\hat\delta) R_{12}(\theta^\circ_{12}) R_{23}(\hat\theta^\nu_{23}) Q_0\,, \quad P_3(\hat\delta) = \mathrm{diag}(1, 1, e^{i\hat\delta})\,. \tag{97}$$

Here $\hat\delta = \beta^e - \alpha^e - \alpha^\nu - \beta^\nu$ and we have absorbed the phase matrix $P_{23}(\alpha^\nu, \beta^\nu) = \mathrm{diag}(1, e^{i\alpha^\nu}, e^{i\beta^\nu})$ in the matrix Q_0. Using eq. (97) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\hat\theta^\nu_{23}\sin^2\hat\theta^e_{13} + \cos^2\hat\theta^e_{13}\sin^2\hat\theta^\nu_{23}\sin^2\theta^\circ_{12} + \frac{1}{2}\sin 2\hat\theta^e_{13}\sin 2\hat\theta^\nu_{23}\sin\theta^\circ_{12}\cos\hat\delta\,, \tag{98}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{\cos^2\theta^\circ_{12}\sin^2\hat\theta^\nu_{23}}{1-\sin^2\theta_{13}}\,, \tag{99}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\sin^2\hat\theta^e_{13} - \sin^2\theta_{13} + \cos^2\hat\theta^e_{13}\sin^2\theta^\circ_{12}}{1-\sin^2\theta_{13}}\,. \tag{100}$$

Comparing the expressions for the absolute value of the element $U_{\mu 1}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (97), we find

$$|U_{\mu 1}| = |\sin\theta_{12}\cos\theta_{23} + \sin\theta_{13}\sin\theta_{23}\cos\theta_{12}e^{i\delta}| = |\sin\theta^\circ_{12}|\,. \tag{101}$$

From the above equation we get for $\cos\delta$:

$$\cos\delta = \frac{\sin^2\theta^\circ_{12} - \cos^2\theta_{23}\sin^2\theta_{12} - \cos^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}}{\sin\theta_{13}\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}}\,. \tag{102}$$

The predicted value of $\cos\delta$ depends on the discrete symmetry G_f through the value of the angle θ°_{12}. Using the best fit values of the standard mixing angles for the NO neutrino mass spectrum and the requirement $|\cos\delta| \le 1$, we find that $\sin^2\theta^\circ_{12}$ should lie in the following interval: $0.110 \le \sin^2\theta^\circ_{12} \le 0.251$. Fixing two of the three neutrino mixing angles to their best fit values and varying the third one in its 3σ experimentally allowed range and accounting for all the three possible combinations, we get that $|\cos\delta| \le 1$ if $0.057 \le \sin^2\theta^\circ_{12} \le 0.281$.

5.5. The case with $U_{23}(\theta^e_{23}, \delta^e_{23})$ and $U_{13}(\theta^\nu_{13}, \delta^\nu_{13})$ complex rotations (case C5)

The parametrisation of the PMNS matrix U, which is convenient for our further analysis,

$$U = U_{23}(\theta^e_{23}, \delta^e_{23}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) R_{12}(\theta^\circ_{12}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) U_{13}(\theta^\nu_{13}, \delta^\nu_{13}) Q_0\,, \tag{103}$$

can be obtained in this case utilising the parametrisations of the matrix U given in eq. (24) with $(ij) = (23)$ and $(rs) = (13)$ and that of the matrix U° given below (for details see Appendix B),

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{13}, \delta^\circ_{23}) = U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) R_{12}(\theta^\circ_{12}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13})\,. \tag{104}$$

The expression in eq. (103) for U can further be cast in a "minimal" form with the help of eq. (212) in Appendix B:

$$U = R_{23}(\hat\theta^e_{23}) P_3(\hat\delta) R_{12}(\theta^\circ_{12}) R_{13}(\hat\theta^\nu_{13}) Q_0\,, \quad P_3(\hat\delta) = \mathrm{diag}(1, 1, e^{i\hat\delta})\,, \tag{105}$$

where $\hat\delta = \beta^e - \alpha^e - \alpha^\nu - \beta^\nu$ and we have absorbed the matrix $P_{13}(\alpha^\nu, \beta^\nu) = \mathrm{diag}(e^{i\alpha^\nu}, 1, e^{i\beta^\nu})$ in the phase matrix Q_0. Using eq. (105) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\theta^\circ_{12} \sin^2\hat\theta^\nu_{13}\,, \tag{106}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2} = \frac{1}{1 - \sin^2\theta_{13}} \Big[\cos^2\hat\theta^\nu_{13} \sin^2\hat\theta^e_{23} + \cos^2\hat\theta^e_{23} \sin^2\hat\theta^\nu_{13} \sin^2\theta^\circ_{12} - \frac{1}{2} \sin 2\hat\theta^e_{23} \sin 2\hat\theta^\nu_{13} \sin\theta^\circ_{12} \cos\hat\delta \Big]\,, \tag{107}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{\sin^2\theta^\circ_{12}}{1 - \sin^2\theta_{13}}\,. \tag{108}$$

We note that, given G_f, the values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ are correlated. This allows one to perform a critical test of the scheme under study once the discrete symmetry group G_f has been specified.

The sum rule for $\cos\delta$ of interest can be derived, e.g., by comparing the expressions for the absolute value of the element $U_{\tau2}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (105):

$$|U_{\tau2}| = |\cos\theta_{12} \sin\theta_{23} + \sin\theta_{13} \cos\theta_{23} \sin\theta_{12} e^{i\delta}| = |\cos\theta^\circ_{12} \sin\hat\theta^e_{23}|\,. \tag{109}$$

This leads to

$$\cos\delta = \frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{12} \sin^2\hat\theta^e_{23} - \sin^2\theta_{23}) + \sin^2\theta^\circ_{12}(\sin^2\theta_{23} - \cos^2\theta_{23} \sin^2\theta_{13})}{\sin 2\theta_{23} \sin\theta_{13} |\sin\theta^\circ_{12}| (\cos^2\theta_{13} - \sin^2\theta^\circ_{12})^{\frac{1}{2}}}\,. \tag{110}$$

Similar to the case C2 analysed in subsection 5.2, $\cos\delta$ is a function of the known neutrino mixing angles θ_{13} and θ_{23}, of the angle θ°_{12} fixed by G_f and the assumed symmetry breaking pattern, as well as of the phase parameter $\hat\delta$ of the scheme. Predictions for $\cos\delta$ can be obtained if $\hat\delta$ is fixed by additional considerations of, e.g., GCP invariance, symmetries, etc. In view of this we show in Fig. 2 $\cos\delta$ as a function of $\cos\hat\delta$ for the current best fit values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, and for the value $\sin^2\theta^\circ_{12} = 1/4$ corresponding to $G_f = S_4$ and A_5. We do not find phenomenologically viable cases for A_4 (T'). Therefore we do not present such a plot for these groups.

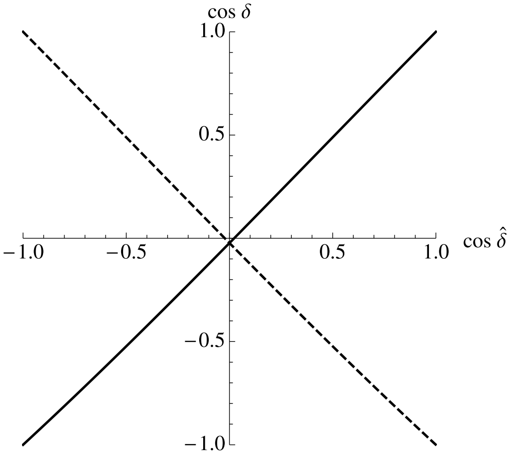

Fig. 2. Dependence of $\cos\delta$ on $\cos\hat{\delta}$ in the case of $G_f = S_4$ or A_5 with $\sin^2\theta^\circ_{12} = 1/4$. The mixing parameters $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ have been fixed to their best fit values for the NO neutrino mass spectrum quoted in eqs. (7) and (8). The solid (dashed) line is for the case when $\sin 2\hat{\theta}^e_{23} \sin 2\hat{\theta}^\nu_{13}$ is positive (negative).

5.6. *The case with $U_{23}(\theta^e_{23}, \delta^e_{23})$ and $U_{12}(\theta^\nu_{12}, \delta^\nu_{12})$ complex rotations (case C6)*

We show below that in this case $\cos\delta$ coincides (up to a sign) with the cosine of an unconstrained CPV phase parameter of the scheme and therefore cannot be determined from the values of the neutrino mixing angles and of the angles determined by the residual symmetries. Indeed, using the parametrisation of the matrix U given in eq. (24) with $(ij) = (23)$ and $(rs) = (12)$ and the parametrisation of U° as follows (see Appendix B for details),

$$U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}, \delta^\circ_{23}) = U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) R_{13}(\theta^\circ_{13}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12})\,, \tag{111}$$

we get for U:

$$U = U_{23}(\theta^e_{23}, \delta^e_{23}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) R_{13}(\theta^\circ_{13}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) U_{12}(\theta^\nu_{12}, \delta^\nu_{12}) Q_0\,. \tag{112}$$

The results derived in Appendix B in eq. (212) make it possible to recast eq. (112) in the form:

$$U = R_{23}(\hat{\theta}^e_{23}) P_2(\hat{\delta}) R_{13}(\theta^\circ_{13}) R_{12}(\hat{\theta}^\nu_{12}) Q_0\,, \quad P_2(\hat{\delta}) = \mathrm{diag}(1, e^{i\hat{\delta}}, 1)\,. \tag{113}$$

Here $\hat{\delta} = \alpha^e - \beta^e - \alpha^\nu - \beta^\nu$ and, as in the preceding cases, we have redefined the phase matrix Q_0 by absorbing the phase matrix $P_{12}(\alpha^\nu, \beta^\nu) = \mathrm{diag}(e^{i\alpha^\nu}, e^{i\beta^\nu}, 1)$ in it. Using eq. (113) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\theta^\circ_{13}\,, \tag{114}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \sin^2\hat{\theta}^e_{23}\,, \tag{115}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \sin^2\hat{\theta}^\nu_{12}\,. \tag{116}$$

Comparing the absolute value of the element $U_{\tau 1}$ allows us to find that $\cos\delta = \pm\cos\hat{\delta}$.

It follows from eq. (114) that for a given flavour symmetry G_f, the value of $\sin^2\theta_{13}$ is predicted. This allows to test the phenomenological viability of the case under discussion, since the value of $\sin^2\theta_{13}$ is known experimentally with a relatively high precision.

A comment, analogous to those made in similar cases considered in subsections 3.3 and 4.3, is in order. Namely, for a non-Abelian flavour symmetry G_f which allows to reproduce correctly the observed values of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, it might be possible to obtain physically viable prediction for $\cos\delta$ by employing GCP invariance in the charged lepton or the neutrino sector.

5.7. The case with $U_{12}(\theta^e_{12}, \delta^e_{12})$ and $U_{12}(\theta^\nu_{12}, \delta^\nu_{12})$ complex rotations (case C7)

Using the following parametrisation of U°,

$$U^\circ(\theta^\circ_{12}, \tilde{\theta}^\circ_{12}, \theta^\circ_{23}, \delta^\circ_{12}, \tilde{\delta}^\circ_{12}) = U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{23}(\theta^\circ_{23}) U_{12}(\tilde{\theta}^\circ_{12}, \tilde{\delta}^\circ_{12})\,, \tag{117}$$

we have for U:

$$U = U_{12}(\theta^e_{12}, \delta^e_{12}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{23}(\theta^\circ_{23}) U_{12}(\tilde{\theta}^\circ_{12}, \tilde{\delta}^\circ_{12}) U_{12}(\theta^\nu_{12}, \delta^\nu_{12}) Q_0\,. \tag{118}$$

Utilising the results derived in Appendix B and reported in eq. (212), we can recast eq. (118) in the form:

$$U = R_{12}(\hat{\theta}^e_{12}) P_1(\hat{\delta}) R_{23}(\theta^\circ_{23}) R_{12}(\hat{\theta}^\nu_{12}) Q_0\,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1)\,. \tag{119}$$

Here $\hat{\delta} = \alpha^e - \beta^e + \alpha^\nu + \beta^\nu$ and we have redefined the matrix Q_0 by absorbing the diagonal phase matrix $P_{12}(-\beta^\nu, -\alpha^\nu) = \mathrm{diag}(e^{-i\beta^\nu}, e^{-i\alpha^\nu}, 1)$ in it. Using eq. (119) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\theta^\circ_{23}\sin^2\hat{\theta}^e_{12}\,, \tag{120}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{\sin^2\theta^\circ_{23}\cos^2\hat{\theta}^e_{12}}{1-\sin^2\theta_{13}}\,, \tag{121}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{1}{1-\sin^2\theta_{13}}\Big[\cos^2\theta^\circ_{23}\cos^2\hat{\theta}^\nu_{12}\sin^2\hat{\theta}^e_{12} + \cos^2\hat{\theta}^e_{12}\sin^2\hat{\theta}^\nu_{12} + \frac{1}{2}\sin 2\hat{\theta}^e_{12}\sin 2\hat{\theta}^\nu_{12}\cos\theta^\circ_{23}\cos\hat{\delta}\Big]\,. \tag{122}$$

From eqs. (120) and (121) we see that the angles θ_{13} and θ_{23} are correlated:

$$\sin^2\theta_{23} = \frac{\sin^2\theta^\circ_{23} - \sin^2\theta_{13}}{1-\sin^2\theta_{13}}\,. \tag{123}$$

Comparing the expressions for the absolute value of the element $U_{\tau 1}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (119), we have

$$|U_{\tau 1}| = |\sin\theta_{12}\sin\theta_{23} - \sin\theta_{13}\cos\theta_{23}\cos\theta_{12}e^{i\delta}| = |\sin\hat{\theta}^\nu_{12}\sin\theta^\circ_{23}|\,. \tag{124}$$

From the above equations we get for $\cos\delta$:

$$\cos\delta = \frac{\sin^2\theta_{13}(\cos^2\theta_{12}\cos^2\theta^\circ_{23} - \sin^2\theta_{12}) + \sin^2\theta^\circ_{23}(\sin^2\theta_{12} - \cos^2\theta_{13}\sin^2\hat{\theta}^\nu_{12})}{\sin 2\theta_{12}\sin\theta_{13}|\cos\theta^\circ_{23}|(\sin^2\theta^\circ_{23} - \sin^2\theta_{13})^{\frac{1}{2}}}\,. \tag{125}$$

In this case $\cos\delta$ is a function of the known neutrino mixing angles θ_{12} and θ_{13}, of the angle θ°_{23} fixed by G_f and the assumed symmetry breaking pattern, as well as of the phase parameter $\hat{\delta}$ of the scheme. Predictions for $\cos\delta$ can only be obtained when $\hat{\delta}$ is fixed by additional

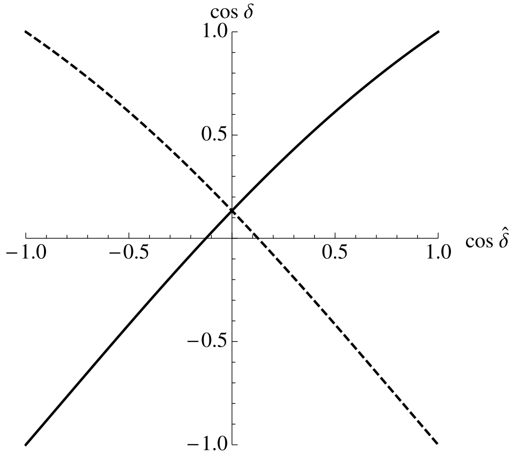

Fig. 3. Dependence of $\cos\delta$ on $\cos\hat{\delta}$ in the case of $G_f = S_4$ with $\sin^2\theta^\circ_{23} = 1/2$. The mixing parameters $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ have been fixed to their best fit values for the NO neutrino mass spectrum quoted in eqs. (6) and (8). The solid (dashed) line is for the case when $\sin 2\hat{\theta}^e_{12} \sin 2\hat{\theta}^\nu_{12}$ is positive (negative).

considerations of, e.g., GCP invariance, symmetries, etc. In view of this we show in Fig. 3 $\cos\delta$ as a function of $\cos\hat{\delta}$ for the current best fit values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$, and for the value $\sin^2\theta^\circ_{23} = 1/2$ corresponding to $G_f = S_4$. We do not find phenomenologically viable cases for $G_f = A_4$ (T') and A_5.

5.8. *The case with $U_{13}(\theta^e_{13}, \delta^e_{13})$ and $U_{13}(\theta^\nu_{13}, \delta^\nu_{13})$ complex rotations (case C8)*

Using the following parametrisation of U°,

$$U^\circ(\theta^\circ_{13}, \tilde{\theta}^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{13}, \tilde{\delta}^\circ_{13}) = U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) R_{23}(\theta^\circ_{23}) U_{13}(\tilde{\theta}^\circ_{13}, \tilde{\delta}^\circ_{13}) \,, \tag{126}$$

we have for U:

$$U = U_{13}(\theta^e_{13}, \delta^e_{13}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) R_{23}(\theta^\circ_{23}) U_{13}(\tilde{\theta}^\circ_{13}, \tilde{\delta}^\circ_{13}) U_{13}(\theta^\nu_{13}, \delta^\nu_{13}) Q_0 \,. \tag{127}$$

Utilising the results derived in Appendix B and reported in eq. (212), we can recast eq. (127) in the form:

$$U = R_{13}(\hat{\theta}^e_{13}) P_1(\hat{\delta}) R_{23}(\theta^\circ_{23}) R_{13}(\hat{\theta}^\nu_{13}) Q_0 \,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1) \,. \tag{128}$$

Here $\hat{\delta} = \alpha^e - \beta^e + \alpha^\nu + \beta^\nu$ and we have redefined the matrix Q_0 by absorbing the diagonal phase matrix $P_{13}(-\beta^\nu, -\alpha^\nu) = \mathrm{diag}(e^{-i\beta^\nu}, 1, e^{-i\alpha^\nu})$ in it. Using eq. (128) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\theta^\circ_{23} \cos^2\hat{\theta}^\nu_{13} \sin^2\hat{\theta}^e_{13} + \cos^2\hat{\theta}^e_{13} \sin^2\hat{\theta}^\nu_{13} + \frac{1}{2} \sin 2\hat{\theta}^e_{13} \sin 2\hat{\theta}^\nu_{13} \cos\theta^\circ_{23} \cos\hat{\delta} \,, \tag{129}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \frac{\sin^2\theta^\circ_{23} \cos^2\hat{\theta}^\nu_{13}}{1 - \sin^2\theta_{13}} \,, \tag{130}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{\sin^2\theta^\circ_{23} \sin^2\hat{\theta}^e_{13}}{1 - \sin^2\theta_{13}} \,. \tag{131}$$

The sum rule for $\cos\delta$ of interest can be derived by comparing the expressions for the absolute value of the element $U_{\mu 2}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (128):

$$|U_{\mu 2}| = |\cos\theta_{12}\cos\theta_{23} - \sin\theta_{13}\sin\theta_{23}\sin\theta_{12}e^{i\delta}| = |\cos\theta^{\circ}_{23}|\,. \tag{132}$$

From the above equation we get for $\cos\delta$:

$$\cos\delta = \frac{\cos^2\theta_{12}\cos^2\theta_{23} - \cos^2\theta^{\circ}_{23} + \sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}}{\sin\theta_{13}\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}}\,. \tag{133}$$

Given the assumed breaking pattern, $\cos\delta$ depends on the flavour symmetry G_f via the value of θ°_{23}. Using the best fit values of the standard mixing angles for the NO neutrino mass spectrum and the requirement $|\cos\delta| \leq 1$, we find that $\sin^2\theta^{\circ}_{23}$ should lie in the following interval: $0.537 \leq \sin^2\theta^{\circ}_{23} \leq 0.677$. Fixing two of the three angles to their best fit values and varying the third one in its 3σ experimentally allowed range and considering all the three possible combinations, we get that $|\cos\delta| \leq 1$ if $0.496 \leq \sin^2\theta^{\circ}_{23} \leq 0.805$.

5.9. The case with $U_{23}(\theta^e_{23}, \delta^e_{23})$ and $U_{23}(\theta^\nu_{23}, \delta^\nu_{23})$ complex rotations (case C9)

Using the following parametrisation of U°,

$$U^{\circ}(\theta^{\circ}_{23}, \tilde{\theta}^{\circ}_{23}, \theta^{\circ}_{12}, \delta^{\circ}_{23}, \tilde{\delta}^{\circ}_{23}) = U_{23}(\theta^{\circ}_{23}, \delta^{\circ}_{23})R_{12}(\theta^{\circ}_{12})U_{23}(\tilde{\theta}^{\circ}_{23}, \tilde{\delta}^{\circ}_{23})\,, \tag{134}$$

we have for U:

$$U = U_{23}(\theta^e_{23}, \delta^e_{23})U_{23}(\theta^{\circ}_{23}, \delta^{\circ}_{23})R_{12}(\theta^{\circ}_{12})U_{23}(\tilde{\theta}^{\circ}_{23}, \tilde{\delta}^{\circ}_{23})U_{23}(\theta^\nu_{23}, \delta^\nu_{23})Q_0\,. \tag{135}$$

Utilising the results derived in Appendix B and reported in eq. (212), we can recast eq. (135) in the form:

$$U = R_{23}(\hat{\theta}^e_{23})P_2(\hat{\delta})R_{12}(\theta^{\circ}_{12})R_{23}(\hat{\theta}^\nu_{23})Q_0\,, \quad P_2(\hat{\delta}) = \mathrm{diag}(1, e^{i\hat{\delta}}, 1)\,. \tag{136}$$

Here $\hat{\delta} = \alpha^e - \beta^e + \alpha^\nu + \beta^\nu$ and we have redefined the matrix Q_0 by absorbing the diagonal phase matrix $P_{23}(\alpha^\nu, \beta^\nu) = \mathrm{diag}(1, e^{i\alpha^\nu}, e^{i\beta^\nu})$ in it. Using eq. (136) and the standard parametrisation of the PMNS matrix U, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\theta^{\circ}_{12}\sin^2\hat{\theta}^\nu_{23}\,, \tag{137}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{1}{1-\sin^2\theta_{13}}\Big[\cos^2\theta^{\circ}_{12}\cos^2\hat{\theta}^e_{23}\sin^2\hat{\theta}^\nu_{23} + \cos^2\hat{\theta}^\nu_{23}\sin^2\hat{\theta}^e_{23} + \frac{1}{2}\sin 2\hat{\theta}^e_{23}\sin 2\hat{\theta}^\nu_{23}\cos\theta^{\circ}_{12}\cos\hat{\delta}\Big]\,, \tag{138}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\sin^2\theta^{\circ}_{12}\cos^2\hat{\theta}^\nu_{23}}{1-\sin^2\theta_{13}}\,. \tag{139}$$

From eqs. (137) and (139) we find that the angles θ_{13} and θ_{12} are correlated:

$$\sin^2\theta_{12} = \frac{\sin^2\theta^{\circ}_{12} - \sin^2\theta_{13}}{1-\sin^2\theta_{13}}\,. \tag{140}$$

Comparing the expressions for the absolute value of the element $U_{\tau 1}$ of the PMNS matrix in the standard parametrisation and in the one obtained using eq. (136), we have

$$|U_{\tau 1}| = |\sin\theta_{12}\sin\theta_{23} - \sin\theta_{13}\cos\theta_{23}\cos\theta_{12}e^{i\delta}| = |\sin\hat{\theta}^e_{23}\sin\theta^{\circ}_{12}|\,. \tag{141}$$

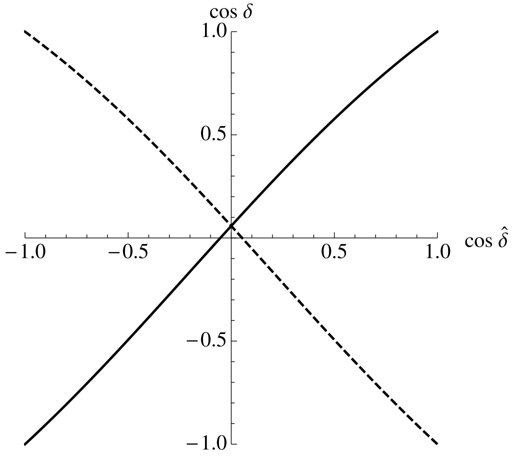

Fig. 4. Dependence of $\cos\delta$ on $\cos\hat{\delta}$ in the case of $G_f = A_5$ with $\sin^2\theta^\circ_{12} = (r+2)/(4r+4) \cong 0.345$. The mixing parameters $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ have been fixed to their best fit values for the NO neutrino mass spectrum quoted in eqs. (7) and (8). The solid (dashed) line is for the case when $\sin 2\hat{\theta}^e_{23} \sin 2\hat{\theta}^\nu_{23}$ is positive (negative).

From the above equations we get for $\cos\delta$:

$$\cos\delta = \frac{\sin^2\theta_{13}(\cos^2\theta_{23}\cos^2\theta^\circ_{12} - \sin^2\theta_{23}) + \sin^2\theta^\circ_{12}(\sin^2\theta_{23} - \cos^2\theta_{13}\sin^2\hat{\theta}^e_{23})}{\sin 2\theta_{23}\sin\theta_{13}|\cos\theta^\circ_{12}|(\sin^2\theta^\circ_{12} - \sin^2\theta_{13})^{\frac{1}{2}}}\,. \tag{142}$$

In this case $\cos\delta$ is a function of the known neutrino mixing angles θ_{23} and θ_{13}, of the angle θ°_{12} fixed by G_f and the assumed symmetry breaking pattern, as well as of the phase parameter $\hat{\delta}$ of the scheme. Predictions for $\cos\delta$ can only be obtained when $\hat{\delta}$ is fixed by additional considerations of, e.g., GCP invariance, symmetries, etc. In view of this we show in Fig. 4 $\cos\delta$ as a function of $\cos\hat{\delta}$ for the current best fit values of $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$, and for the value $\sin^2\theta^\circ_{12} = (r+2)/(4r+4) \cong 0.345$ corresponding to $G_f = A_5$. We do not find phenomenologically viable cases for $G_f = A_4$ (T') and S_4.

5.10. Results in the cases of $G_f = A_4$ (T'), S_4 and A_5

The schemes considered in Sections 5.1–5.9 can be applied when considering the breaking G_f to $G_e = Z_2$ and $G_\nu = Z_2$, for both Majorana and Dirac neutrinos. As explicit examples of this, we now consider $G_f = A_4$ (T'), S_4 and A_5 broken to $G_e = Z_2$ and $G_\nu = Z_2$. As such, we have considered all possible combinations of residual Z_2 symmetries for a given flavour symmetry group, namely, $G_e = Z_2$ and $G_\nu = Z_2$ for $G_f = A_4$ (T'), S_4, A_5. For instance, in the cases of the schemes described in subsections 5.1–5.5, and $G_f = S_4$ broken to $G_e = Z_2^a$ and $G_\nu = Z_2^b$ with $(a,b) = (T^2U, U)$, (T^2U, SU), (T^2U, TU), $(T^2U, STSU)$, etc. (a total of 24 combinations of order two elements), the value of the relevant parameter contained in the fixed matrix U° yields $\sin^2\theta^\circ_{23} = 1/4$, $\sin^2\theta^\circ_{23} = 1/2$, $\sin^2\theta^\circ_{13} = 1/4$, $\sin^2\theta^\circ_{12} = 1/4$, and $\sin^2\theta^\circ_{12} = 1/4$, respectively. In A_5 for the cases C1, C3, C4 and C5 we find the sine square of the corresponding fixed angle in the matrix U° to be 1/4, e.g., for $G_e = Z_2^a$ and $G_\nu = Z_2^b$ with $(a,b) = (S, ST^2ST^3S)$, (S, ST^3ST^2S), (S, T^2ST^3), (S, T^3ST^2), etc. (in total, for 60 combinations of order two elements).

For the symmetry group A_4 we find that none of the combinations of the residual symmetries $G_e = Z_2$ and $G_\nu = Z_2$ provide physical values of $\cos\delta$ and phenomenologically viable results for the neutrino mixing angles simultaneously.

For $G_f = S_4$, using the best fit values of the mixing angles θ_{12}, θ_{13} and θ_{23}, we get $\cos\delta = -0.806$, -1.52 and 0.992 in the cases C1, C3 and C4, respectively. Physically acceptable value of $\cos\delta$ in the case C3 can be obtained for $\sin^2\theta_{23} = 0.562$ allowed at 3σ, for which $\cos\delta = -0.996$. In the part of the 3σ allowed range of $\sin^2\theta_{23}$, $0.562 \leq \sin^2\theta_{23} \leq 0.641$, we have $-0.996 \leq \cos\delta \leq -0.690$. Further, in the case C2, in which the relevant parameter $\sin^2\theta^\circ_{23} = 1/2$, the value of $\cos\delta$ is not fixed, while the atmospheric angle is predicted to have a value corresponding to $\sin^2\theta_{23} = 0.512$. Similarly, in the case C5 the value of $\cos\delta$ is not fixed, while $\sin^2\theta_{12} = 0.256$ (which is slightly outside the corresponding 3σ interval). In the case C7 we find that $\cos\delta$ is not fixed and $\sin^2\theta_{23} = 0.488$. Finally, for C8 with $\sin^2\theta^\circ_{23} = 1/2$ and $3/4$, using the best fit values of the neutrino mixing angles for the NO spectrum, we have $\cos\delta = -1.53$ and 2.04, respectively. The physical values of $\cos\delta$ can be obtained, using, e.g., the values of $\sin^2\theta_{23} = 0.380$ and 0.543, for which $\cos\delta = -0.995$ and 0.997, respectively. In the parts of the 3σ allowed range of $\sin^2\theta_{23}$, $0.374 \leq \sin^2\theta_{23} \leq 0.380$ and $0.543 \leq \sin^2\theta_{23} \leq 0.641$, we have $-0.938 \geq \cos\delta \geq -0.995$ and $0.997 \geq \cos\delta \geq 0.045$, respectively.

For the A_5 symmetry group the cases C1 with $\sin^2\theta^\circ_{23} = 1/4$, C3 with $\sin^2\theta^\circ_{13} = 1/4$ and C4 with $\sin^2\theta^\circ_{12} = 1/4$ lead to the same predictions obtained with $G_f = S_4$, namely, $\cos\delta = -0.806$, -1.52 and 0.992, respectively. Moreover, in the case C3 (case C4) the value of $\sin^2\theta^\circ_{13} = 0.096$ ($\sin^2\theta^\circ_{12} = 0.096$) is found, which along with the best fit values of the mixing angles gives $\cos\delta = 0.688$ ($\cos\delta = -1.21$). Using the value of $\sin^2\theta_{23} = 0.487$ allowed at 2σ, one gets in the case C4 $\cos\delta = -0.997$, while in the part of the 3σ allowed range of $\sin^2\theta_{23}$, $0.487 \leq \sin^2\theta_{23} \leq 0.641$, we have $-0.997 \leq \cos\delta \leq -0.376$. Note also, if $\sin^2\theta_{23}$ is fixed to its best fit value, one can obtain the physical value of $\cos\delta = -0.999$ using $\sin^2\theta_{12} = 0.277$. For the part of the 3σ allowed range of $\sin^2\theta_{12}$, $0.259 \leq \sin^2\theta_{12} \leq 0.277$, one gets $-0.871 \geq \cos\delta \geq -0.999$. The cases C5 and C8 are the same as for the S_4 symmetry group. Finally, in the case C9 the value of $\cos\delta$ is not fixed, while using the best fit value of the reactor angle, we get $\sin^2\theta_{12} = 0.330$.

6. Summary of the results of Sections 3, 4 and 5

The sum rules derived in Sections 3, 4 and 5 are summarised in Tables 3 and 4. The formulae for $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, which lead to predictions for the indicated neutrino mixing parameters once the discrete flavour symmetry G_f is fixed, are given in Tables 5 and 6. In the cases in Tables 5 and 6 in which $\cos\delta$ is unconstrained, a relatively precise measurement of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ or $\sin^2\theta_{23}$ can provide a critical test of the corresponding schemes due to constraints satisfied by the indicated neutrino mixing parameters.

A general comment on the results derived in Sections 3, 4 and 5 is in order. Since we do not have any information on the mass matrices, we have the freedom to permute the columns of the matrices U_e and U_ν, or equivalently, the columns and the rows of the PMNS matrix U. The results in Tables 3 and 4 cover all the possibilities because, as we demonstrate below, the permutations bring one of the considered cases into another considered case. For example, consider the case of $U = U_{13}(\theta^e_{13}, \delta^e_{13}) U^\circ U_{23}(\theta^\nu_{23}, \delta^\nu_{23}) Q_0$. The permutation of the second and the third rows of U is given by $\pi_{23} U = \pi_{23} U_{13}(\theta^e_{13}, \delta^e_{13}) \pi_{23} \pi_{23} U^\circ U_{23}(\theta^\nu_{23}, \delta^\nu_{23}) Q_0$, where we have defined

Table 3
Summary of the sum rules for $\cos\delta$. The cases A1, A2 and A3 correspond to $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$, while B1, B2 and B3 correspond to $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$. See text for further details.

Case	Parametrisation of U	Sum rule for $\cos\delta$
A1	$U_{12}(\theta^e_{12}, \delta^e_{12}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13}) Q_0$	$\frac{\cos^2\theta_{13}(\sin^2\theta^\circ_{23}-\cos^2\theta_{12})+\cos^2\theta^\circ_{13}\cos^2\theta^\circ_{23}(\cos^2\theta_{12}-\sin^2\theta_{12}\sin^2\theta_{13})}{\sin 2\theta_{12}\sin\theta_{13}\lvert\cos\theta^\circ_{13}\cos\theta^\circ_{23}\rvert(\cos^2\theta_{13}-\cos^2\theta^\circ_{13}\cos^2\theta^\circ_{23})^{\frac{1}{2}}}$
A2	$U_{13}(\theta^e_{13}, \delta^e_{13}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) Q_0$	$-\frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{23}-\cos^2\theta_{12})+\sin^2\theta^\circ_{23}(\cos^2\theta_{12}-\sin^2\theta_{12}\sin^2\theta_{13})}{\sin 2\theta_{12}\sin\theta_{13}\lvert\sin\theta^\circ_{23}\rvert(\cos^2\theta_{13}-\sin^2\theta^\circ_{23})^{\frac{1}{2}}}$
A3	$U_{23}(\theta^e_{23}, \delta^e_{23}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12}) Q_0$	$\pm\cos\hat{\delta}_{23}$
B1	$R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) U_{13}(\theta^\circ_{13}, \delta^\circ_{13}) U_{13}(\theta^\nu_{13}, \delta^\nu_{13}) Q_0$	$-\frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{23}-\cos^2\theta_{23})+\sin^2\theta^\circ_{12}(\cos^2\theta_{23}-\sin^2\theta_{13}\sin^2\theta_{23})}{\sin 2\theta_{23}\sin\theta_{13}\lvert\sin\theta^\circ_{12}\rvert(\cos^2\theta_{13}-\sin^2\theta^\circ_{12})^{\frac{1}{2}}}$
B2	$R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12}) U_{23}(\theta^\circ_{23}, \delta^\circ_{23}) U_{23}(\theta^\nu_{23}, \delta^\nu_{23}) Q_0$	$\frac{\cos^2\theta_{13}(\sin^2\theta^\circ_{12}-\cos^2\theta_{23})+\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{13}(\cos^2\theta_{23}-\sin^2\theta_{13}\sin^2\theta_{23})}{\sin 2\theta_{23}\sin\theta_{13}\lvert\cos\theta^\circ_{12}\cos\theta^\circ_{13}\rvert(\cos^2\theta_{13}-\cos^2\theta^\circ_{12}\cos^2\theta^\circ_{13})^{\frac{1}{2}}}$
B3	$R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13}) U_{12}(\theta^\circ_{12}, \delta^\circ_{12}) U_{12}(\theta^\nu_{12}, \delta^\nu_{12}) Q_0$	$\pm\cos\hat{\delta}_{12}$

Table 4
Summary of the sum rules for $\cos\delta$. The cases C1–C9 correspond to $G_e = Z_2$ and $G_\nu = Z_2$. See text for further details.

Case	Parametrisation of U	Sum rule for $\cos\delta$
C1	$U_{12}(\theta^e_{12},\delta^e_{12})U_{12}(\theta^\circ_{12},\delta^\circ_{12})R_{23}(\theta^\circ_{23})U_{13}(\theta^\circ_{13},\delta^\circ_{13})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	$\frac{\sin^2\theta^\circ_{23}-\cos^2\theta_{12}\sin^2\theta_{23}-\cos^2\theta_{23}\sin^2\theta_{12}\sin^2\theta_{13}}{\sin\theta_{13}\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}}$
C2	$U_{13}(\theta^e_{13},\delta^e_{13})U_{13}(\theta^\circ_{13},\delta^\circ_{13})R_{23}(\theta^\circ_{23})U_{12}(\theta^\circ_{12},\delta^\circ_{12})U_{12}(\theta^\nu_{12},\delta^\nu_{12})Q_0$	$\frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{23}\sin^2\hat\theta^\nu_{12}-\sin^2\theta_{12})+\sin^2\theta^\circ_{23}(\sin^2\theta_{12}-\cos^2\theta_{12}\sin^2\theta_{13})}{\sin 2\theta_{12}\sin\theta_{13}\lvert\sin\theta^\circ_{23}\rvert(\cos^2\theta_{13}-\sin^2\theta^\circ_{23})^{\frac{1}{2}}}$
C3	$U_{12}(\theta^e_{12},\delta^e_{12})U_{12}(\theta^\circ_{12},\delta^\circ_{12})R_{13}(\theta^\circ_{13})U_{23}(\theta^\circ_{23},\delta^\circ_{23})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	$\frac{\sin^2\theta_{12}\sin^2\theta_{23}-\sin^2\theta^\circ_{13}+\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}}{\sin\theta_{13}\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}}$
C4	$U_{13}(\theta^e_{13},\delta^e_{13})U_{13}(\theta^\circ_{13},\delta^\circ_{13})R_{12}(\theta^\circ_{12})U_{23}(\theta^\circ_{23},\delta^\circ_{23})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	$\frac{\sin^2\theta^\circ_{12}-\cos^2\theta_{23}\sin^2\theta_{12}-\cos^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}}{\sin\theta_{13}\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}}$
C5	$U_{23}(\theta^e_{23},\delta^e_{23})U_{23}(\theta^\circ_{23},\delta^\circ_{23})R_{12}(\theta^\circ_{12})U_{13}(\theta^\circ_{13},\delta^\circ_{13})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	$\frac{\cos^2\theta_{13}(\cos^2\theta^\circ_{12}\sin^2\hat\theta^e_{23}-\sin^2\theta_{23})+\sin^2\theta^\circ_{12}(\sin^2\theta_{23}-\cos^2\theta_{23}\sin^2\theta_{13})}{\sin 2\theta_{23}\sin\theta_{13}\lvert\sin\theta^\circ_{12}\rvert(\cos^2\theta_{13}-\sin^2\theta^\circ_{12})^{\frac{1}{2}}}$
C6	$U_{23}(\theta^e_{23},\delta^e_{23})U_{23}(\theta^\circ_{23},\delta^\circ_{23})R_{13}(\theta^\circ_{13})U_{12}(\theta^\circ_{12},\delta^\circ_{12})U_{12}(\theta^\nu_{12},\delta^\nu_{12})Q_0$	$\pm\cos\hat\delta$
C7	$U_{12}(\theta^e_{12},\delta^e_{12})U_{12}(\theta^\circ_{12},\delta^\circ_{12})R_{23}(\theta^\circ_{23})U_{12}(\tilde\theta^\circ_{12},\tilde\delta^\circ_{12})U_{12}(\theta^\nu_{12},\delta^\nu_{12})Q_0$	$\frac{\sin^2\theta_{13}(\cos^2\theta_{12}\cos^2\theta^\circ_{23}-\sin^2\theta_{12})+\sin^2\theta^\circ_{23}(\sin^2\theta_{12}-\cos^2\theta_{13}\sin^2\hat\theta^\nu_{12})}{\sin 2\theta_{12}\sin\theta_{13}\lvert\cos\theta^\circ_{23}\rvert(\sin^2\theta^\circ_{23}-\sin^2\theta_{13})^{\frac{1}{2}}}$
C8	$U_{13}(\theta^e_{13},\delta^e_{13})U_{13}(\theta^\circ_{13},\delta^\circ_{13})R_{23}(\theta^\circ_{23})U_{13}(\tilde\theta^\circ_{13},\tilde\delta^\circ_{13})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	$\frac{\cos^2\theta_{12}\cos^2\theta_{23}-\cos^2\theta^\circ_{23}+\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}}{\sin\theta_{13}\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}}$
C9	$U_{23}(\theta^e_{23},\delta^e_{23})U_{23}(\theta^\circ_{23},\delta^\circ_{23})R_{12}(\theta^\circ_{12})U_{23}(\tilde\theta^\circ_{23},\tilde\delta^\circ_{23})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	$\frac{\sin^2\theta_{13}(\cos^2\theta_{23}\cos^2\theta^\circ_{12}-\sin^2\theta_{23})+\sin^2\theta^\circ_{12}(\sin^2\theta_{23}-\cos^2\theta_{13}\sin^2\hat\theta^e_{23})}{\sin 2\theta_{23}\sin\theta_{13}\lvert\cos\theta^\circ_{12}\rvert(\sin^2\theta^\circ_{12}-\sin^2\theta_{13})^{\frac{1}{2}}}$

Table 5
Summary of the formulae for $\sin^2\theta_{12}$ and/or $\sin^2\theta_{13}$ and/or $\sin^2\theta_{23}$. The cases A1, A2 and A3 correspond to $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$, while B1, B2 and B3 correspond to $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$. See text for further details.

Case	Parametrisation of U	Sum rule for $\sin^2\theta_{12}$ and/or $\sin^2\theta_{13}$ and/or $\sin^2\theta_{23}$
A1	$U_{12}(\theta^e_{12}, \delta^e_{12})U_{12}(\theta^\circ_{12}, \delta^\circ_{12})R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})Q_0$	$\sin^2\theta_{23} = \dfrac{\sin^2\theta^\circ_{13} - \sin^2\theta_{13} + \cos^2\theta^\circ_{13}\sin^2\theta^\circ_{23}}{1 - \sin^2\theta_{13}}$
A2	$U_{13}(\theta^e_{13}, \delta^e_{13})U_{13}(\theta^\circ_{13}, \delta^\circ_{13})R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})Q_0$	$\sin^2\theta_{23} = \dfrac{\sin^2\theta^\circ_{23}}{1 - \sin^2\theta_{13}}$
A3	$U_{23}(\theta^e_{23}, \delta^e_{23})U_{23}(\theta^\circ_{23}, \delta^\circ_{23})R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})Q_0$	$\sin^2\theta_{13} = \sin^2\theta^\circ_{13}$, $\sin^2\theta_{12} = \sin^2\theta^\circ_{12}$
B1	$R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})U_{13}(\theta^\circ_{13}, \delta^\circ_{13})U_{13}(\theta^\nu_{13}, \delta^\nu_{13})Q_0$	$\sin^2\theta_{12} = \dfrac{\sin^2\theta^\circ_{12}}{1 - \sin^2\theta_{13}}$
B2	$R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})U_{23}(\theta^\circ_{23}, \delta^\circ_{23})U_{23}(\theta^\nu_{23}, \delta^\nu_{23})Q_0$	$\sin^2\theta_{12} = \dfrac{\cos^2\theta_{13} - \cos^2\theta^\circ_{12}\cos^2\theta^\circ_{13}}{1 - \sin^2\theta_{13}}$
B3	$R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})U_{12}(\theta^\circ_{12}, \delta^\circ_{12})U_{12}(\theta^\nu_{12}, \delta^\nu_{12})Q_0$	$\sin^2\theta_{13} = \sin^2\theta^\circ_{13}$, $\sin^2\theta_{23} = \sin^2\theta^\circ_{23}$

$$\pi_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{143}$$

Since the combination $\pi_{23}U_{13}(\theta^e_{13}, \delta^e_{13})\pi_{23}$ gives a unitary matrix $U_{12}(\theta^e_{13}, \delta^e_{13})$, the result after the redefinition, $\theta^e_{13} \to \theta^e_{12}$, $\delta^e_{13} \to \delta^e_{12}$ and $\pi_{23}U^\circ \to U^\circ$, yields

$$U = U_{12}(\theta^e_{12}, \delta^e_{12})U^\circ U_{23}(\theta^\nu_{23}, \delta^\nu_{23})Q_0 ,$$

which represents another case present in Table 4. It is worth noting that the freedom in redefining the matrix U° follows from the fact that U° is a general 3×3 unitary matrix and hence can be parametrised as described in Section 2 and in Appendix B. All the other permutations should be treated in the same way and lead to similar results.

7. The case of fully broken G_e

If the discrete flavour symmetry G_f is fully broken in the charged lepton sector the matrix U_e is unconstrained and includes, in general, three rotation angle and three CPV phase parameters. It is impossible to derive predictions for the mixing angles and CPV phases in the PMNS matrix in this case. Therefore, we will consider in this section forms of U_e corresponding to one of the rotation angle parameters being equal to zero. Some of these forms of U_e correspond to a class of models of neutrino mass generation (see, e.g., [17,32–36]) and lead, in particular, to sum rules for $\cos\delta$.

We give in Appendix C the most general parametrisations of U under the assumption that in the case of fully broken G_e one rotation angle in the matrix U_e vanishes. The second case in Table 14 with $\theta^\circ_{13} = 0$ have been analysed in [11,13,14], while the third case with $U_{12}(\theta^e_{12}, \delta^e_{12})U_{13}(\theta^e_{13}, \delta^e_{13})$ has been investigated in [14].

Table 6
Summary of the formulae for $\sin^2\theta_{12}$ and/or $\sin^2\theta_{13}$ and/or $\sin^2\theta_{23}$. The cases C1–C9 correspond to $G_e = Z_2$ and $G_\nu = Z_2$. See text for further details.

Case	Parametrisation of U	Sum rule for $\sin^2\theta_{12}$ and/or $\sin^2\theta_{13}$ and/or $\sin^2\theta_{23}$
C1	$U_{12}(\theta^e_{12},\delta^e_{12})U_{12}(\theta^\circ_{12},\delta^\circ_{12})R_{23}(\theta^\circ_{23})U_{13}(\theta^\circ_{13},\delta^\circ_{13})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	not fixed
C2	$U_{13}(\theta^e_{13},\delta^e_{13})U_{13}(\theta^\circ_{13},\delta^\circ_{13})R_{23}(\theta^\circ_{23})U_{12}(\theta^\circ_{12},\delta^\circ_{12})U_{12}(\theta^\nu_{12},\delta^\nu_{12})Q_0$	$\sin^2\theta_{23} = \frac{\sin^2\theta^\circ_{23}}{1-\sin^2\theta_{13}}$
C3	$U_{12}(\theta^e_{12},\delta^e_{12})U_{12}(\theta^\circ_{12},\delta^\circ_{12})R_{13}(\theta^\circ_{13})U_{23}(\theta^\circ_{23},\delta^\circ_{23})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	not fixed
C4	$U_{13}(\theta^e_{13},\delta^e_{13})U_{13}(\theta^\circ_{13},\delta^\circ_{13})R_{12}(\theta^\circ_{12})U_{23}(\theta^\circ_{23},\delta^\circ_{23})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	not fixed
C5	$U_{23}(\theta^e_{23},\delta^e_{23})U_{23}(\theta^\circ_{23},\delta^\circ_{23})R_{12}(\theta^\circ_{12})U_{13}(\theta^\circ_{13},\delta^\circ_{13})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	$\sin^2\theta_{12} = \frac{\sin^2\theta^\circ_{12}}{1-\sin^2\theta_{13}}$
C6	$U_{23}(\theta^e_{23},\delta^e_{23})U_{23}(\theta^\circ_{23},\delta^\circ_{23})R_{13}(\theta^\circ_{13})U_{12}(\theta^\circ_{12},\delta^\circ_{12})U_{12}(\theta^\nu_{12},\delta^\nu_{12})Q_0$	$\sin^2\theta_{13} = \sin^2\theta^\circ_{13}$
C7	$U_{12}(\theta^e_{12},\delta^e_{12})U_{12}(\theta^\circ_{12},\delta^\circ_{12})R_{23}(\theta^\circ_{23})U_{12}(\tilde{\theta}^\circ_{12},\tilde{\delta}^\circ_{12})U_{12}(\theta^\nu_{12},\delta^\nu_{12})Q_0$	$\sin^2\theta_{23} = \frac{\sin^2\theta^\circ_{23}-\sin^2\theta_{13}}{1-\sin^2\theta_{13}}$
C8	$U_{13}(\theta^e_{13},\delta^e_{13})U_{13}(\theta^\circ_{13},\delta^\circ_{13})R_{23}(\theta^\circ_{23})U_{13}(\tilde{\theta}^\circ_{13},\tilde{\delta}^\circ_{13})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	not fixed
C9	$U_{23}(\theta^e_{23},\delta^e_{23})U_{23}(\theta^\circ_{23},\delta^\circ_{23})R_{12}(\theta^\circ_{12})U_{23}(\tilde{\theta}^\circ_{23},\tilde{\delta}^\circ_{23})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	$\sin^2\theta_{12} = \frac{\sin^2\theta^\circ_{12}-\sin^2\theta_{13}}{1-\sin^2\theta_{13}}$

7.1. The scheme with $U_{23}(\theta^e_{23}, \delta^e_{23})U_{12}(\theta^e_{12}, \delta^e_{12})$ (case D1)

We consider the following parametrisation of the PMNS matrix (see Appendix C, first case in Table 14):

$$U = U_{23}(\theta^e_{23}, \delta^e_{23}) R_{12}(\hat{\theta}_{12}) P_1(\hat{\delta}) R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13}) Q_0\,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1). \tag{144}$$

We find that:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\theta_{13}(\hat{\theta}_{12}, \hat{\delta}, \theta^\circ_{13}, \theta^\circ_{23})\,, \tag{145}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \sin^2\theta_{23}(\hat{\theta}_{12}, \hat{\delta}, \theta^e_{23}, \delta^e_{23}, \theta^\circ_{13}, \theta^\circ_{23})\,, \tag{146}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\cos^2\theta^\circ_{23}\sin^2\hat{\theta}_{12}}{\cos^2\theta_{13}}\,. \tag{147}$$

As it can be seen from the previous equations and the absolute value of the element $U_{\mu 2}$,

$$|U_{\mu 2}| = |\cos\theta^e_{23}\cos\hat{\theta}_{12}\cos\theta^\circ_{23} - e^{-i\delta^e_{23}}\sin\theta^e_{23}\sin\theta^\circ_{23}|\,, \tag{148}$$

a sum rule for $\cos\delta$ might be derived in the case of fixed δ^e_{23}. In the general case of free δ^e_{23} we find that $\cos\delta$ is a function of δ^e_{23}. Since in this case the analytical expression of $\cos\delta$ in terms of δ^e_{23} is rather complicated, we do not present this result here. Note that imposing either $\theta^\circ_{23} = 0$ or $\theta^\circ_{13} = 0$ is not enough to fix the value of $\cos\delta$. As eqs. (145) and (146) suggest, in the case of fixed δ^e_{23} there exist multiple solutions for the value of $\cos\delta$ for any given value of δ^e_{23}. This is demonstrated in Fig. 5, in which we plot $\cos\delta$ versus δ^e_{23}, assuming that the angles θ°_{13} and θ°_{23} have the values corresponding to the TBM, GRA, GRB and HG symmetry forms given in Table 2. The figure is obtained for $\hat{\theta}_{12}$ belonging to the first quadrant. The solid lines correspond to $\hat{\delta} = \cos^{-1}(\cos\hat{\delta})$, where $\cos\hat{\delta}$ is the solution of eq. (145), while the dashed lines correspond to $\hat{\delta} = 2\pi - \cos^{-1}(\cos\hat{\delta})$. Multiple lines reflect the fact that eq. (146) for θ^e_{23} has several solutions. We note that Fig. 5 remains the same for $\hat{\theta}_{12}$ belonging to the third quadrant, while for $\hat{\theta}_{12}$ lying in the second or fourth quadrant the solid and dashed lines interchange. For the BM (LC) symmetry form $\cos\hat{\delta}$ has an unphysical value, which indicates that the considered scheme with the BM (LC) form of the matrix diagonalising the neutrino mass matrix does not provide a good description of the current data on the neutrino mixing angles [12].[15] Thus, we do not present such a plot in this case. If δ^e_{23} turns out to be fixed (by GCP invariance, symmetries, etc.), then, as can be seen from Fig. 5, $\cos\delta$ is predicted to take a value from a discrete set. For instance, when $\delta^e_{23} = 0$ or π, we have

$$\cos\delta = \{-0.135, 0.083\} \quad \text{for TBM;} \tag{149}$$

$$\cos\delta = \{-0.317, 0.269\} \quad \text{for GRA;} \tag{150}$$

$$\cos\delta = \{-0.221, 0.170\} \quad \text{for GRB;} \tag{151}$$

$$\cos\delta = \{-0.500, 0.459\} \quad \text{for HG.} \tag{152}$$

In the case of $\delta^e_{23} = \pi/2$ or $3\pi/2$, we find

[15] Note that the scheme under discussion corresponds to inverse ordering of the charged lepton corrections, i.e., $U_e^\dagger = U_{23}(\theta^e_{23}, \delta^e_{23})U_{12}(\theta^e_{12}, \delta^e_{12})$ (see [12]).

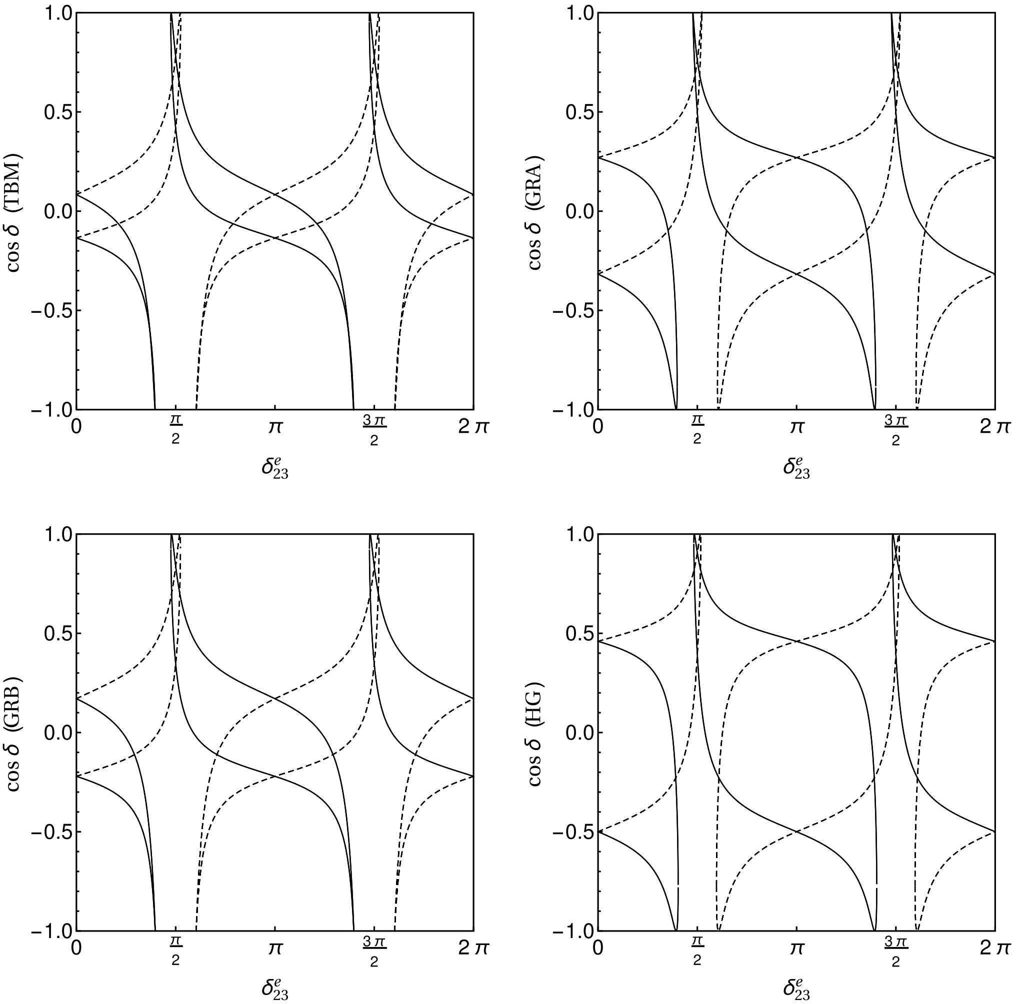

Fig. 5. Dependence of $\cos\delta$ on δ^e_{23} in the cases of the TBM, GRA, GRB and HG symmetry forms. The mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ have been fixed to their best fit values for the NO neutrino mass spectrum quoted in eqs. (6)–(8). The angle $\hat{\theta}_{12}$ is assumed to belong to the first quadrant. The solid lines correspond to $\hat{\delta} = \cos^{-1}(\cos\hat{\delta})$, where $\cos\hat{\delta}$ is the solution of eq. (145), while the dashed lines correspond to $\hat{\delta} = 2\pi - \cos^{-1}(\cos\hat{\delta})$. See text for further details.

$$\cos\delta = \{0.418, 0.779\} \quad \text{for TBM;} \tag{153}$$

$$\cos\delta = \{0.498, 0.761\} \quad \text{for GRA;} \tag{154}$$

$$\cos\delta = \{0.346, 0.837\} \quad \text{for GRB;} \tag{155}$$

$$\cos\delta = \{0.394, 0.906\} \quad \text{for HG.} \tag{156}$$

7.2. The scheme with $U_{13}(\theta^e_{13}, \delta^e_{13})U_{12}(\theta^e_{12}, \delta^e_{12})$ (case D2)

We consider the following parametrisation of the PMNS matrix (see Appendix C, first case in Table 14):

$$U = U_{13}(\theta^e_{13}, \delta^e_{13}) R_{12}(\hat{\theta}_{12}) P_1(\hat{\delta}) R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13}) Q_0 \,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1) \,. \tag{157}$$

A sum rule for $\cos\delta$ is obtained in the cases of either $\theta^\circ_{23} = k\pi$, $k = 0, 1, 2$, or $\theta^\circ_{13} = q\pi/2$, $q = 0, 1, 2, 3, 4$. For the general form of U we find for the absolute value of the element $U_{\mu 2}$:

$$|U_{\mu 2}| = |\cos\hat{\theta}_{12} \cos\theta^\circ_{23}| \,, \tag{158}$$

which in each of the two limits indicated above is fixed because $|\cos\hat{\theta}_{12}|$ can be expressed in terms of the PMNS neutrino mixing angles. This can be seen from the following relation, which is obtained using the expressions for $|U_{\mu 3}|^2$ in the standard parametrisation of the PMNS matrix U and in the parametrisation given in eq. (157):

$$\cos^2\theta_{13} \sin^2\theta_{23} = |-e^{i\hat{\delta}} \sin\hat{\theta}_{12} \sin\theta^\circ_{13} + \cos\hat{\theta}_{12} \cos\theta^\circ_{13} \sin\theta^\circ_{23}|^2 \,. \tag{159}$$

Equating the expression for $|U_{\mu 2}|$ given in eq. (158) with the one in the standard parametrisation, we find

$$\cos\delta = \frac{\cos^2\theta_{23}\cos^2\theta_{12} + \sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23} - \cos^2\hat{\theta}_{12}\cos^2\theta^\circ_{23}}{\sin 2\theta_{23} \sin\theta_{12} \cos\theta_{12} \sin\theta_{13}} \,. \tag{160}$$

7.3. *The scheme with $U_{12}(\theta^e_{12}, \delta^e_{12}) U_{23}(\theta^e_{23}, \delta^e_{23})$ (case D3)*

We consider the following parametrisation of the PMNS matrix (see Appendix C, second case in Table 14):

$$U = U_{12}(\theta^e_{12}, \delta^e_{12}) R_{23}(\hat{\theta}_{23}) P_2(\hat{\delta}) R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12}) Q_0 \,, \quad P_2(\hat{\delta}) = \mathrm{diag}(1, e^{i\hat{\delta}}, 1) \,. \tag{161}$$

A sum rule for $\cos\delta$ can be derived in the cases of either $\theta^\circ_{13} = k\pi$, $k = 0, 1, 2$, or $\theta^\circ_{12} = q\pi/2$, $q = 0, 1, 2, 3, 4$. Indeed, the relation $\cos^2\theta_{13}\cos^2\theta_{23} = \cos^2\hat{\theta}_{23}\cos^2\theta^\circ_{13}$ (which can be obtained from the expressions for the element $U_{\tau 3}$ of the PMNS matrix U in the standard parametrisation and in the one given in eq. (161)), allows us to express $\cos^2\hat{\theta}_{23}$ in terms of the known product $\cos^2\theta_{13}\cos^2\theta_{23}$ and the parameter $\cos^2\theta^\circ_{13}$ which, in principle, is fixed by the symmetries G_f and G_ν. We have also

$$|U_{\tau 2}| = |e^{i\hat{\delta}} \cos\theta^\circ_{12} \sin\hat{\theta}_{23} + \cos\hat{\theta}_{23} \sin\theta^\circ_{12} \sin\theta^\circ_{13}| \,. \tag{162}$$

In the limits of either $\theta^\circ_{13} = k\pi$, $k = 0, 1, 2$, or $\theta^\circ_{12} = q\pi/2$, $q = 0, 1, 2, 3, 4$, $|U_{\tau 2}|$ does not depend on $\hat{\delta}$ and is also fixed. This makes it possible to derive a sum rule for $\cos\delta$. In the general case, $\cos\delta$ is a function of $\hat{\delta}$:

$$\begin{aligned}
\cos\delta = \frac{2}{\sin 2\theta_{12} \sin 2\theta_{23} \sin\theta_{13} \cos^2\theta^\circ_{13}} \Big[& \cos^2\theta^\circ_{12} \left(\cos^2\theta^\circ_{13} - \cos^2\theta_{13}\cos^2\theta_{23}\right) \\
& - \cos^2\theta_{12}\sin^2\theta_{23}\cos^2\theta^\circ_{13} \\
& + \cos^2\theta_{23} \left(\cos^2\theta_{13}\sin^2\theta^\circ_{12}\sin^2\theta^\circ_{13} - \sin^2\theta_{12}\sin^2\theta_{13}\cos^2\theta^\circ_{13}\right) \\
& + \kappa \cos\hat{\delta} \cos\theta_{13}\cos\theta_{23}\sin 2\theta^\circ_{12}\sin\theta^\circ_{13} \left(\cos^2\theta^\circ_{13} - \cos^2\theta_{13}\cos^2\theta_{23}\right)^{\frac{1}{2}} \Big] \,,
\end{aligned} \tag{163}$$

where $\kappa = 1$ if $\hat{\theta}_{23}$ belongs to the first or third quadrant, and $\kappa = -1$ otherwise. For $\theta^\circ_{13} = 0$ the sum rule reduces to the one derived in [11] and discussed in detail in [11,13,14].

7.4. The scheme with $U_{13}(\theta^e_{13}, \delta^e_{13})U_{23}(\theta^e_{23}, \delta^e_{23})$ (case D4)

We consider the following parametrisation of the PMNS matrix (see Appendix C, second case in Table 14):

$$U = U_{13}(\theta^e_{13}, \delta^e_{13}) R_{23}(\hat{\theta}_{23}) P_2(\hat{\delta}) R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12}) Q_0\,, \quad P_2(\hat{\delta}) = \mathrm{diag}(1, e^{i\hat{\delta}}, 1)\,. \tag{164}$$

In this case a sum rule for $\cos\delta$ exists provided either $\theta^\circ_{13} = k\pi$, $k = 0, 1, 2$, or $\theta^\circ_{12} = q\pi/2$, $q = 0, 1, 2, 3, 4$. This follows from the relation $|U_{\mu 3}|^2 = \cos^2\theta_{13}\sin^2\theta_{23} = \cos^2\theta^\circ_{13}\sin^2\hat{\theta}_{23}$ and the expression for $|U_{\mu 2}|$:

$$|U_{\mu 2}| = |e^{i\hat{\delta}}\cos\theta^\circ_{12}\cos\hat{\theta}_{23} - \sin\hat{\theta}_{23}\sin\theta^\circ_{12}\sin\theta^\circ_{13}|\,. \tag{165}$$

The sum rule of interest for $\cos\delta$ reads

$$\begin{aligned}\cos\delta = &-\frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{13}}\Big[\cos^2\theta^\circ_{12}\left(\cos^2\theta^\circ_{13} - \cos^2\theta_{13}\sin^2\theta_{23}\right)\\ &- \cos^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^\circ_{13}\\ &+ \sin^2\theta_{23}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{12}\sin^2\theta^\circ_{13} - \sin^2\theta_{12}\sin^2\theta_{13}\cos^2\theta^\circ_{13}\right)\\ &- \kappa\cos\hat{\delta}\cos\theta_{13}\sin\theta_{23}\sin 2\theta^\circ_{12}\sin\theta^\circ_{13}\left(\cos^2\theta^\circ_{13} - \cos^2\theta_{13}\sin^2\theta_{23}\right)^{\frac{1}{2}}\Big]\,,\end{aligned} \tag{166}$$

where $\kappa = 1$ if $\hat{\theta}_{23}$ belongs to the first or third quadrant, and $\kappa = -1$ otherwise. As in the previous case, $\cos\delta$ is a function of $\hat{\delta}$. For $\theta^\circ_{13} = 0$ the sum rule in eq. (166) reduces to the one derived in [14].

7.5. The scheme with $U_{23}(\theta^e_{23}, \delta^e_{23})U_{13}(\theta^e_{13}, \delta^e_{13})$ (case D5)

In this case we consider the following parametrisation of the PMNS matrix (see Appendix C, third case in Table 14):

$$U = U_{23}(\theta^e_{23}, \delta^e_{23}) R_{13}(\hat{\theta}_{13}) P_1(\hat{\delta}) R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) Q_0\,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1)\,. \tag{167}$$

We find that:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\theta^\circ_{23}\sin^2\hat{\theta}_{13}\,, \tag{168}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \sin^2\theta_{23}(\hat{\theta}_{13}, \theta^e_{23}, \delta^e_{23}, \theta^\circ_{23})\,, \tag{169}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \sin^2\theta_{12}(\hat{\theta}_{13}, \hat{\delta}, \theta^\circ_{12}, \theta^\circ_{23})\,. \tag{170}$$

Since, as can be shown, $|U_{\mu 2}|$ is a function of the parameters θ^e_{23}, δ^e_{23}, $\hat{\delta}$, $\hat{\theta}_{13}$, θ°_{12} and θ°_{23}, and $\hat{\theta}_{13}$, and $\cos\hat{\delta}$ can be extracted from eqs. (168) and (170), respectively, it might be possible to find a sum rule for $\cos\delta$ in the case of fixed δ^e_{23}. Since in this case the analytical expression of $\cos\delta$ in terms of δ^e_{23} is rather complicated, we do not present it here. Note that imposing either $\theta^\circ_{12} = 0$ or $\theta^\circ_{23} = 0$ is not enough to fix the value of $\cos\delta$. Even in the case of fixed δ^e_{23} it follows from eqs. (169) and (170) that for any given value of δ^e_{23}, $\cos\delta$ can take several values. This can be understood, e.g., from eq. (170) which allows to fix $\cos\hat{\delta}$, but not $\sin\hat{\delta}$. This ambiguity, in particular, leads to multiple solutions for $\cos\delta$. In Fig. 6 we show these solutions in the cases of the TBM, GRA, GRB and HG symmetry forms. We remind that for these forms $\theta^\circ_{23} = -\pi/4$ and $\theta^\circ_{12} = \sin^{-1}(1/\sqrt{3})$ (TBM), $\theta^\circ_{12} = \sin^{-1}(1/\sqrt{2+r})$ (GRA), $r = (1+\sqrt{5})/2$ being the golden ratio, $\theta^\circ_{12} = \sin^{-1}(\sqrt{3-r}/2)$ (GRB), and $\theta^\circ_{12} = \pi/6$ (HG). We assume $\hat{\theta}_{13}$ to lie in the first quadrant. The solid lines correspond to $\hat{\delta} = \cos^{-1}(\cos\hat{\delta})$, where $\cos\hat{\delta}$ is the solution of eq. (170), while the dashed lines correspond to $\hat{\delta} = 2\pi - \cos^{-1}(\cos\hat{\delta})$. Multiple lines reflect the fact that eq. (169) for θ^e_{23} has several solutions. We note that Fig. 6 does not change in the case of $\hat{\theta}_{13}$ belonging to the third quadrant, while for $\hat{\theta}_{13}$ lying in the second or fourth quadrant the solid and dashed lines interchange. For $\delta^e_{23} = 0$ or π, we find

$$\cos\delta = \{-0.114, 0.114\} \quad \text{for TBM;} \tag{171}$$

$$\cos\delta = \{-0.289, 0.289\} \quad \text{for GRA;} \tag{172}$$

$$\cos\delta = \{-0.200, 0.200\} \quad \text{for GRB;} \tag{173}$$

$$\cos\delta = \{-0.476, 0.476\} \quad \text{for HG.} \tag{174}$$

It is worth noting that in the scheme under consideration the values of δ^e_{23} in a vicinity of $\pi/2$ $(3\pi/2)$ do not provide physical values of $\cos\delta$ (see Fig. 6).

7.6. *The scheme with* $U_{12}(\theta^e_{12}, \delta^e_{12})U_{13}(\theta^e_{13}, \delta^e_{13})$ *(case D6)*

It is convenient to consider the following parametrisation of the PMNS matrix U (see Appendix C, third case in Table 14):

$$U = U_{12}(\theta^e_{12}, \delta^e_{12}) R_{13}(\hat{\theta}_{13}) P_1(\hat{\delta}) R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12}) Q_0\,, \qquad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1)\,. \tag{175}$$

We find that a sum rule for $\cos\delta$ can be derived if either $\theta^\circ_{12} = q\pi/2$, $q = 0, 1, 2, 3, 4$, or $\theta^\circ_{23} = k\pi$, $k = 0, 1, 2$. Indeed, the relation $|U_{\tau 3}|^2 = \cos^2\theta_{13}\cos^2\theta_{23} = \cos^2\hat{\theta}_{13}\cos^2\theta^\circ_{23}$, allows us to determine $\cos^2\hat{\theta}_{13}$ in terms of the known quantity $\cos^2\theta_{13}\cos^2\theta_{23}$ and the parameter $\cos^2\theta^\circ_{23}$, which is fixed once G_f and G_ν are fixed. Further, we have

$$|U_{\tau 2}| = |e^{i\hat{\delta}} \sin\theta^\circ_{12} \sin\hat{\theta}_{13} + \cos\hat{\theta}_{13}\cos\theta^\circ_{12}\sin\theta^\circ_{23}|\,, \tag{176}$$

where the only unconstrained parameter is the phase $\hat{\delta}$. In the cases indicated above with either $\theta^\circ_{12} = q\pi/2$, $q = 0, 1, 2, 3, 4$, or $\theta^\circ_{23} = k\pi$, $k = 0, 1, 2$, the absolute value of the element $U_{\tau 2}$ does not depend on $\hat{\delta}$, which in turn allows a sum rule for $\cos\delta$ to be derived. In general, $\cos\delta$ is a function of $\hat{\delta}$:

$$\begin{aligned}\cos\delta = \frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{23}} \Big[&\sin^2\theta^\circ_{12}\left(\cos^2\theta^\circ_{23} - \cos^2\theta_{13}\cos^2\theta_{23}\right) \\ &- \cos^2\theta_{12}\sin^2\theta_{23}\cos^2\theta^\circ_{23} \\ &+ \cos^2\theta_{23}\left(\cos^2\theta_{13}\cos^2\theta^\circ_{12}\sin^2\theta^\circ_{23} - \sin^2\theta_{12}\sin^2\theta_{13}\cos^2\theta^\circ_{23}\right) \\ &+ \kappa\cos\hat{\delta}\cos\theta_{13}\cos\theta_{23}\sin 2\theta^\circ_{12}\sin\theta^\circ_{23}\left(\cos^2\theta^\circ_{23} - \cos^2\theta_{13}\cos^2\theta_{23}\right)^{\frac{1}{2}} \Big],\end{aligned} \tag{177}$$

where $\kappa = 1$ if $\hat{\theta}_{13}$ belongs to the first or third quadrant, and $\kappa = -1$ otherwise. In this case the sum rule for $\cos\delta$ has been derived first in [14] assuming $\theta^\circ_{13} = 0$, but as we can see this

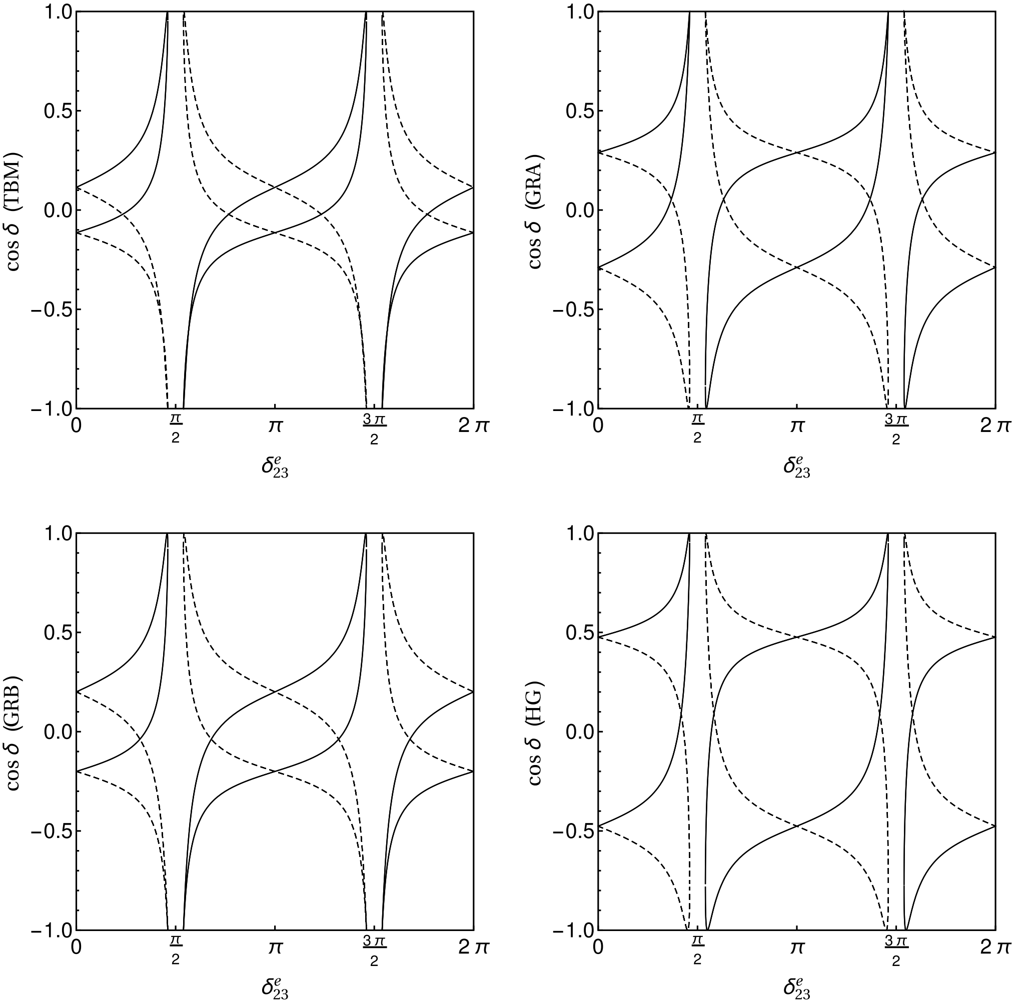

Fig. 6. Dependence of $\cos\delta$ on δ^e_{23} in the cases of the TBM, GRA, GRB and HG symmetry forms. The mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ have been fixed to their best fit values for the NO neutrino mass spectrum quoted in eqs. (6)–(8). The angle $\hat{\theta}_{13}$ is assumed to belong to the first quadrant. The solid lines correspond to $\hat{\delta} = \cos^{-1}(\cos\hat{\delta})$, where $\cos\hat{\delta}$ is the solution of eq. (170), while the dashed lines correspond to $\hat{\delta} = 2\pi - \cos^{-1}(\cos\hat{\delta})$. See text for further details.

result holds also for any fixed value of θ°_{13}, since the parametrisation given in eq. (175) and the corresponding one in [14] are the same after a redefinition of the parameters.

The sum rules derived in Section 7 are summarised in Table 7.

8. The case of fully broken G_ν

When the discrete flavour symmetry G_f is fully broken in the neutrino sector, the matrix U_ν is unconstrained and includes, in general, three complex rotations and three phases, i.e., three angle and six CPV phase parameters. It is impossible to derive predictions for the mixing angles and

Table 7
Summary of the sum rules for $\cos\delta$ in the case of fully broken G_e under the assumption that the matrix U_e consists of two complex rotation matrices. The parameter $\kappa = 1$ if the corresponding hat angle belongs to the first or third quadrant, and $\kappa = -1$ otherwise. The cases D3 and D4 have been analysed for $\theta^\circ_{13} = 0$ in [11,14]. In the case D6 the sum rule for $\cos\delta$ has been derived first in [14] assuming $\theta^\circ_{13} = 0$, but this result holds also for any fixed value of θ°_{13}. See text for further details.

Case	Parametrisation of U	Sum rule for $\cos\delta$
D2	$U_{13}(\theta^e_{13},\delta^e_{13})R_{12}(\hat\theta_{12})P_1(\hat\delta)R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})Q_0$	$\dfrac{\cos^2\theta_{23}\cos^2\theta_{12}+\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}-\cos^2\hat\theta_{12}\cos^2\theta^\circ_{23}}{\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}\sin\theta_{13}}$
D3	$U_{12}(\theta^e_{12},\delta^e_{12})R_{23}(\hat\theta_{23})P_2(\hat\delta)R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})Q_0$	$\dfrac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{13}}\Big[\cos^2\theta^\circ_{12}\left(\cos^2\theta^\circ_{13}-\cos^2\theta_{13}\cos^2\theta_{23}\right) - \cos^2\theta_{12}\sin^2\theta_{23}\cos^2\theta^\circ_{13} + \cos^2\theta_{23}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{12}\sin^2\theta^\circ_{13} - \sin^2\theta_{12}\sin^2\theta_{13}\cos^2\theta^\circ_{13}\right) + \kappa\cos\hat\delta\cos\theta_{13}\cos\theta_{23}\sin 2\theta^\circ_{12}\sin\theta^\circ_{13}\left(\cos^2\theta^\circ_{13}-\cos^2\theta_{13}\cos^2\theta_{23}\right)^{\frac{1}{2}}\Big]$
D4	$U_{13}(\theta^e_{13},\delta^e_{13})R_{23}(\hat\theta_{23})P_2(\hat\delta)R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})Q_0$	$-\dfrac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{13}}\Big[\cos^2\theta^\circ_{12}\left(\cos^2\theta^\circ_{13}-\cos^2\theta_{13}\sin^2\theta_{23}\right) - \cos^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^\circ_{13} + \sin^2\theta_{23}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{12}\sin^2\theta^\circ_{13} - \sin^2\theta_{12}\sin^2\theta_{13}\cos^2\theta^\circ_{13}\right) - \kappa\cos\hat\delta\cos\theta_{13}\sin\theta_{23}\sin 2\theta^\circ_{12}\sin\theta^\circ_{13}\left(\cos^2\theta^\circ_{13}-\cos^2\theta_{13}\sin^2\theta_{23}\right)^{\frac{1}{2}}\Big]$
D6	$U_{12}(\theta^e_{12},\delta^e_{12})R_{13}(\hat\theta_{13})P_1(\hat\delta)R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})Q_0$	$\dfrac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{23}}\Big[\sin^2\theta^\circ_{12}\left(\cos^2\theta^\circ_{23}-\cos^2\theta_{13}\cos^2\theta_{23}\right) - \cos^2\theta_{12}\sin^2\theta_{23}\cos^2\theta^\circ_{23} + \cos^2\theta_{23}\left(\cos^2\theta_{13}\cos^2\theta^\circ_{12}\sin^2\theta^\circ_{23} - \sin^2\theta_{12}\sin^2\theta_{13}\cos^2\theta^\circ_{23}\right) + \kappa\cos\hat\delta\cos\theta_{13}\cos\theta_{23}\sin 2\theta^\circ_{12}\sin\theta^\circ_{23}\left(\cos^2\theta^\circ_{23}-\cos^2\theta_{13}\cos^2\theta_{23}\right)^{\frac{1}{2}}\Big]$

CPV phases in the PMNS matrix in this case. Therefore we will consider in this section forms of U_ν corresponding to one of the rotation angle parameters being equal to zero. Some of these forms of U_ν correspond to a class of models of neutrino mass generation or phenomenological studies (see, e.g., [48]) and lead, in particular, to sum rules for $\cos\delta$. Since in this case G_f is fully broken in the neutrino sector, the $Z_2 \times Z_2$ symmetry of the Majorana mass term does arise accidentally. Therefore the matrix U_ν is not constrained by the symmetry group G_f. We give in Table 14 in Appendix C the most general parametrisations of U under the assumption that for fully broken G_ν one rotation angle vanishes in the matrix U_ν.

8.1. The scheme with $U_{12}(\theta^\nu_{12}, \delta^\nu_{12})U_{13}(\theta^\nu_{13}, \delta^\nu_{13})$ (case E1)

It proves convenient to consider the following parametrisation of the PMNS matrix U in this case (see Appendix C, fourth case in Table 14):

$$U = R_{23}(\theta^\circ_{23}) R_{13}(\theta^\circ_{13}) P_1(\hat{\delta}) R_{12}(\hat{\theta}_{12}) U_{13}(\theta^\nu_{13}, \delta^\nu_{13}) Q_0 \,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1)\,. \tag{178}$$

Consider first the case of $\theta^\circ_{13} = 0$. In this case the phase $\hat{\delta}$ is unphysical. Comparing this parametrisation of U with the standard parametrisation, we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\theta^\nu_{13}\cos^2\hat{\theta}_{12}\,, \tag{179}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{1}{\cos^2\theta_{13}}\Big[\sin^2\theta^\circ_{23}\cos^2\theta^\nu_{13} + \cos^2\theta^\circ_{23}\sin^2\theta^\nu_{13}\sin^2\hat{\theta}_{12} - \frac{1}{2}\sin 2\theta^\circ_{23}\sin 2\theta^\nu_{13}\sin\hat{\theta}_{12}\cos\delta^\nu_{13}\Big]\,, \tag{180}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\sin^2\hat{\theta}_{12}}{\cos^2\theta_{13}}\,. \tag{181}$$

From the ratio

$$\left|\frac{U_{\tau 2}}{U_{\mu 2}}\right|^2 = \tan^2\theta^\circ_{23}\,, \tag{182}$$

we get the following sum rule for $\cos\delta$:

$$\cos\delta = -\frac{\tan\theta_{12}}{\sin 2\theta_{23}\sin\theta_{13}}\left[\cos 2\theta^\circ_{23}\sin^2\theta_{13} + \left(\sin^2\theta_{23} - \sin^2\theta^\circ_{23}\right)\left(\cot^2\theta_{12} - \sin^2\theta_{13}\right)\right]. \tag{183}$$

Substituting the best fit values of the neutrino mixing angles for the NO neutrino mass spectrum and the value of $\theta^\circ_{23} = -\pi/4$, which corresponds to the TBM, BM, GRA, GRB and HG symmetry forms, we obtain $\cos\delta = 0.616$. We note that in the considered scheme the predictions for $\cos\delta$ are all the same for the symmetry forms mentioned above, since these forms are characterised by different values of the angle θ°_{12}, which has been absorbed by the free parameter $\hat{\theta}_{12}$. This "degeneracy" can be lifted in specific models in which the value of θ^ν_{12} is fixed. Using the best fit values and the requirement $|\cos\delta| \leq 1$, we find that the allowed values of $\sin^2\theta^\circ_{23}$ belong to the following interval: $0.338 \leq \sin^2\theta^\circ_{23} \leq 0.538$.

In order to give the general result for $\cos\delta$ in the case of $\theta^\circ_{13} \neq 0$, we use the expression for $\sin^2\theta_{12}$ for non-zero θ°_{13}:

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\cos^2\theta^\circ_{13}\sin^2\hat{\theta}_{12}}{\cos^2\theta_{13}}\,. \tag{184}$$

Employing this relation in the expression for $|U_{\tau 2}|^2$, we get

$$\begin{aligned}\cos\delta = &-\frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{13}}\Big[\cos^2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13}-\sin^2\theta_{12}\cos^2\theta_{13}\right)\\ &-\cos^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^\circ_{13}\\ &+\sin^2\theta_{12}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{13}\sin^2\theta^\circ_{23}-\sin^2\theta_{13}\sin^2\theta_{23}\cos^2\theta^\circ_{13}\right)\\ &-\kappa\cos\hat{\delta}\sin\theta_{12}\cos\theta_{13}\sin\theta^\circ_{13}\sin 2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13}-\sin^2\theta_{12}\cos^2\theta_{13}\right)^{\frac{1}{2}}\Big],\end{aligned} \tag{185}$$

where $\kappa = 1$ if $\hat{\theta}_{12}$ belongs to the first or third quadrant, and $\kappa = -1$ otherwise.

Similar to the cases C2, C5, C7 and C9 analysed in subsections 5.2, 5.5, 5.7 and 5.9, $\cos\delta$ is a function of the known neutrino mixing angles θ_{12}, θ_{13} and θ_{23}, of the angles θ°_{13} and θ°_{23} fixed by G_f and the assumed symmetry breaking pattern, as well as of the phase parameter $\hat{\delta}$ of the scheme. Predictions for $\cos\delta$ can be obtained if $\hat{\delta}$ is fixed by additional considerations of, e.g., GCP invariance, symmetries, etc.

For $\theta^\circ_{13} = k\pi$, $k = 0, 1, 2$, and/or $\theta^\circ_{23} = k'\pi/2$, $k' = 0, 1, 2, 3, 4$, $\cos\delta$ does not depend on $\hat{\delta}$ and κ. In the first case the expression in eq. (185) reduces to the sum rule given in eq. (183).

8.2. The scheme with $U_{12}(\theta^\nu_{12}, \delta^\nu_{12})U_{23}(\theta^\nu_{23}, \delta^\nu_{23})$ (case E2)

In this case it is convenient to use another possible parametrisation of the PMNS matrix, the fourth case in Table 14 given in Appendix C. Namely,

$$U = R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})P_1(\hat{\delta})R_{12}(\hat{\theta}_{12})U_{23}(\theta^\nu_{23}, \delta^\nu_{23})Q_0\,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1)\,. \tag{186}$$

Consider first the possibility of $\theta^\circ_{13} = 0$. Under this assumption we find:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\theta^\nu_{23}\sin^2\hat{\theta}_{12}\,, \tag{187}$$

$$\begin{aligned}\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{1}{\cos^2\theta_{13}}\big[&\sin^2\theta^\circ_{23}\cos^2\theta^\nu_{23}+\cos^2\theta^\circ_{23}\sin^2\theta^\nu_{23}\cos^2\hat{\theta}_{12}\\ &+\frac{1}{2}\sin 2\theta^\circ_{23}\sin 2\theta^\nu_{23}\cos\hat{\theta}_{12}\cos\delta^\nu_{23}\big],\end{aligned} \tag{188}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{\cos^2\theta^\nu_{23}\sin^2\hat{\theta}_{12}}{\cos^2\theta_{13}}\,. \tag{189}$$

The sum rule of interest for $\cos\delta$ can be derived in this case using the ratio

$$\left|\frac{U_{\tau 1}}{U_{\mu 1}}\right|^2 = \tan^2\theta^\circ_{23}\,. \tag{190}$$

We get

$$\cos\delta = \frac{\cot\theta_{12}}{\sin 2\theta_{23}\sin\theta_{13}}\left[\cos 2\theta^\circ_{23}\sin^2\theta_{13} + \left(\sin^2\theta_{23} - \sin^2\theta^\circ_{23}\right)\left(\tan^2\theta_{12} - \sin^2\theta_{13}\right)\right]. \tag{191}$$

This sum rule can be formally obtained from the r.h.s. of eq. (183) by interchanging $\tan\theta_{12}$ and $\cot\theta_{12}$ and by multiplying it by (-1). Substituting the best fit values of the neutrino mixing angles for the NO neutrino mass spectrum and the value of $\theta^\circ_{23} = -\pi/4$, we get $\cos\delta = -0.262$. Using the best fit values and the requirement $|\cos\delta| \le 1$, we find that the allowed values of $\sin^2\theta^\circ_{23}$ belong to the following interval: $0.227 \le \sin^2\theta^\circ_{23} \le 0.659$.

In order to find a general result for $\cos\delta$ for arbitrary fixed $\theta^\circ_{13} \ne 0$, we use the following relation:

$$\cos^2\theta_{12}\cos^2\theta_{13} = \cos^2\hat{\theta}_{12}\cos^2\theta^\circ_{13}\,, \tag{192}$$

which follows from the expressions for $|U_{e1}|^2$ in the standard parametrisation and in the parametrisation given in eq. (186). With the help of this relation, using $|U_{\mu 1}|$, we get

$$\begin{aligned}\cos\delta = {} & \frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{13}}\Bigg[\cos^2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13} - \cos^2\theta_{12}\cos^2\theta_{13}\right)\\ & - \sin^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^\circ_{13}\\ & + \cos^2\theta_{12}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{13}\sin^2\theta^\circ_{23} - \sin^2\theta_{13}\sin^2\theta_{23}\cos^2\theta^\circ_{13}\right)\\ & + \kappa\cos\hat{\delta}\cos\theta_{12}\cos\theta_{13}\sin\theta^\circ_{13}\sin 2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13} - \cos^2\theta_{12}\cos^2\theta_{13}\right)^{\frac{1}{2}}\Bigg], \end{aligned} \tag{193}$$

where $\kappa = 1$ if $\hat{\theta}_{12}$ belongs to the first or third quadrant, and $\kappa = -1$ otherwise. Also in this case $\cos\delta$ is a function of the unconstrained phase parameter $\hat{\delta}$ of the scheme. Predictions for $\cos\delta$ can be obtained if $\hat{\delta}$ is fixed by additional considerations (e.g., GCP invariance, symmetries, etc.).

As like in the case E1, for $\theta^\circ_{13} = k\pi$, $k = 0, 1, 2$, and/or $\theta^\circ_{23} = k'\pi/2$, $k' = 0, 1, 2, 3, 4$, $\cos\delta$ does not depend on $\hat{\delta}$ and κ. For $\theta^\circ_{13} = 0, \pi, 2\pi$, the sum rule in eq. (193) coincides with the sum rule given in eq. (191).

8.3. The scheme with $U_{23}(\theta^\nu_{23}, \delta^\nu_{23})U_{12}(\theta^\nu_{12}, \delta^\nu_{12})$ (case E3)

The convenient parametrisation for U to use in this case is that of the fifth case in Table 14 given in Appendix C:

$$U = R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})P_2(\hat{\delta})R_{23}(\hat{\theta}_{23})U_{12}(\theta^\nu_{12}, \delta^\nu_{12})Q_0\,, \quad P_2(\hat{\delta}) = \mathrm{diag}(1, e^{i\hat{\delta}}, 1)\,.$$

We find that:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \sin^2\theta_{13}(\hat{\theta}_{23}, \hat{\delta}, \theta^\circ_{12}, \theta^\circ_{13})\,, \tag{194}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \frac{\cos^2\theta^\circ_{12}\sin^2\hat{\theta}_{23}}{\cos^2\theta_{13}}\,, \tag{195}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \sin^2\theta_{12}(\hat{\theta}_{23}, \hat{\delta}, \theta^\nu_{12}, \delta^\nu_{12}, \theta^\circ_{12}, \theta^\circ_{13})\,. \tag{196}$$

However, a sum rule for $\cos\delta$ cannot be obtained because $\cos\delta$ turns out to depend, in particular, on δ^{ν}_{12} which is an unconstrained phase parameter of the scheme considered, which can be seen from the expression for $|U_{\mu1}|$:

$$|U_{\mu1}| = |\cos\theta^{\nu}_{12}\sin\theta^{\circ}_{12} + e^{i(\hat{\delta}+\delta^{\nu}_{12})}\cos\hat{\theta}_{23}\cos\theta^{\circ}_{12}\sin\theta^{\nu}_{12}|\,. \quad (197)$$

The situation here is analogous to the cases analysed in subsections 7.1 and 7.5. Namely, considering a certain residual symmetry group G_e, from eq. (195) we find that $\sin^2\hat{\theta}_{23}$ is fixed. Then, $\cos\hat{\delta}$ is fixed (up to a sign) by eq. (194). Hence, θ^{ν}_{12} can be expressed in terms of δ^{ν}_{12} by virtue of eq. (196). Thus, numerical predictions for $\cos\delta$ can be obtained if δ^{ν}_{12} is fixed.

8.4. The scheme with $U_{23}(\theta^{\nu}_{23}, \delta^{\nu}_{23})U_{13}(\theta^{\nu}_{13}, \delta^{\nu}_{13})$ (case E4)

Employing the parametrisation for U given in Appendix C, namely the fifth case in Table 14,

$$U = R_{13}(\theta^{\circ}_{13})R_{12}(\theta^{\circ}_{12})P_2(\hat{\delta})R_{23}(\hat{\theta}_{23})U_{13}(\theta^{\nu}_{13}, \delta^{\nu}_{13})Q_0\,, \qquad P_2(\hat{\delta}) = \mathrm{diag}(1, e^{i\hat{\delta}}, 1)\,,$$

we find that $\cos\delta$ is a function of $\hat{\theta}_{23}$, θ°_{12} and the PMNS mixing angles. Therefore, $\cos\delta$ can be determined only in those cases when $\hat{\theta}_{23}$ is fixed. Using the result

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{1}{\cos^2\theta_{13}}\bigg[\cos^2\hat{\theta}_{23}\cos^2\theta^{\circ}_{13}\sin^2\theta^{\circ}_{12} + \sin^2\hat{\theta}_{23}\sin^2\theta^{\circ}_{13} - \frac{1}{2}\cos\hat{\delta}\sin 2\hat{\theta}_{23}\sin 2\theta^{\circ}_{13}\sin\theta^{\circ}_{12}\bigg]\,, \quad (198)$$

we find these cases to be, for example: i) $\theta^{\circ}_{12} = 0, \pi$, leading to the relation $\sin^2\theta_{12}\cos^2\theta_{13} = \sin^2\hat{\theta}_{23}\sin^2\theta^{\circ}_{13}$, ii) $\theta^{\circ}_{13} = 0, \pi$, implying $\sin^2\theta_{12}\cos^2\theta_{13} = \cos^2\hat{\theta}_{23}\sin^2\theta^{\circ}_{12}$, iii) $\theta^{\circ}_{13} = \pi/2$, $3\pi/2$, giving $\sin^2\theta_{12}\cos^2\theta_{13} = \sin^2\hat{\theta}_{23}$. For this reason we give $\cos\delta$ as a function of the angle $\hat{\theta}_{23}$. Namely, the sum rule of interest, which is obtained using $|U_{\mu2}| = |\cos\hat{\theta}_{23}\cos\theta^{\circ}_{12}|$, reads

$$\cos\delta = \frac{\cos^2\theta_{12}\cos^2\theta_{23} + \sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23} - \cos^2\hat{\theta}_{23}\cos^2\theta^{\circ}_{12}}{\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}\sin\theta_{13}}\,. \quad (199)$$

The dependence of $\cos\delta$ on G_f is realised via the values of the angles θ°_{12} and θ°_{13}.

8.5. The scheme with $U_{13}(\theta^{\nu}_{13}, \delta^{\nu}_{13})U_{12}(\theta^{\nu}_{12}, \delta^{\nu}_{12})$ (case E5)

The parametrisation for the PMNS matrix U employed by us in this subsection is the sixth case in Table 14 given in Appendix C:

$$U = R_{23}(\theta^{\circ}_{23})R_{12}(\theta^{\circ}_{12})P_1(\hat{\delta})R_{13}(\hat{\theta}_{13})U_{12}(\theta^{\nu}_{12}, \delta^{\nu}_{12})Q_0\,, \qquad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1)\,.$$

We find that:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \cos^2\theta^{\circ}_{12}\sin^2\hat{\theta}_{13}\,, \quad (200)$$

$$\sin^2\theta_{23} = \frac{|U_{\mu3}|^2}{1-|U_{e3}|^2} = \sin^2\theta_{23}(\hat{\theta}_{13}, \hat{\delta}, \theta^{\circ}_{12}, \theta^{\circ}_{23})\,, \quad (201)$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \sin^2\theta_{12}(\hat{\theta}_{13}, \hat{\delta}, \theta^{\nu}_{12}, \delta^{\nu}_{12}, \theta^{\circ}_{12})\,. \quad (202)$$

However, a sum rule for $\cos\delta$ cannot be obtained because $\cos\delta$ turns out to depend, in particular, on δ^{ν}_{12} which is an unconstrained phase parameter of the scheme considered. This can be seen, e.g., from the expression for $|U_{\mu 1}|$:

$$|U_{\mu 1}| = |\cos\theta^{\nu}_{12}(e^{i\hat{\delta}}\sin\theta^{\circ}_{12}\cos\theta^{\circ}_{23}\cos\hat{\theta}_{13} + \sin\hat{\theta}_{13}\sin\theta^{\circ}_{23}) + e^{i\delta^{\nu}_{12}}\cos\theta^{\circ}_{12}\cos\theta^{\circ}_{23}\sin\theta^{\nu}_{12}|. \quad (203)$$

Similarly to the case analysed in subsection 8.3, for a certain residual symmetry group G_e, from eq. (200) we find that $\sin^2\hat{\theta}_{13}$ is fixed. Then, $\cos\hat{\delta}$ is fixed (up to a sign) by eq. (201), and so the angle θ^{ν}_{12} can be expressed in terms of δ^{ν}_{12} by virtue of eq. (202). Therefore, numerical predictions for $\cos\delta$ can be obtained if δ^{ν}_{12} is fixed.

8.6. The scheme with $U_{13}(\theta^{\nu}_{13}, \delta^{\nu}_{13})U_{23}(\theta^{\nu}_{23}, \delta^{\nu}_{23})$ (case E6)

The parametrisation of the PMNS matrix U utilised by us in the present subsection is that of the sixth case in Table 14 given in Appendix C:

$$U = R_{23}(\theta^{\circ}_{23})R_{12}(\theta^{\circ}_{12})P_1(\hat{\delta})R_{13}(\hat{\theta}_{13})U_{23}(\theta^{\nu}_{23}, \delta^{\nu}_{23})Q_0\,, \quad P_1(\hat{\delta}) = \mathrm{diag}(e^{i\hat{\delta}}, 1, 1)\,.$$

A sum rule and predictions for $\cos\delta$ can be derived in the cases of either $\theta^{\circ}_{23} = q\pi/2$, $q = 0, 1, 2, 3, 4$, or $\theta^{\circ}_{12} = k\pi$, $k = 0, 1, 2$. Indeed, using the relation

$$|U_{e1}|^2 = \cos^2\theta_{12}\cos^2\theta_{13} = \cos^2\hat{\theta}_{13}\cos^2\theta^{\circ}_{12}\,, \quad (204)$$

we can express $\cos^2\hat{\theta}_{13}$ in terms of the product of PMNS neutrino mixing parameters $\cos^2\theta_{12}$ $\cos^2\theta_{13}$ and, the fixed by G_f parameter, $\cos^2\theta^{\circ}_{12}$. The sum rule of interest for $\cos\delta$ can be derived, e.g., from the expression for the absolute value of the element $U_{\mu 1}$:

$$|U_{\mu 1}| = |e^{-i\hat{\delta}}\cos\hat{\theta}_{13}\cos\theta^{\circ}_{23}\sin\theta^{\circ}_{12} + \sin\hat{\theta}_{13}\sin\theta^{\circ}_{23}|\,, \quad (205)$$

since in any of the two limits indicated above, $\theta^{\circ}_{23} = q\pi/2$, $q = 0, 1, 2, 3, 4$, or $\theta^{\circ}_{12} = k\pi$, $k = 0, 1, 2$, $|U_{\mu 1}|$ does not depend on $\hat{\delta}$. In fact, it is given only in terms of the known PMNS neutrino mixing parameters and an angle (either θ°_{23} or θ°_{12}) which is fixed by the symmetry G_f. In the general case, $\cos\delta$ is a function of $\hat{\delta}$. Using eqs. (204) and (205), we get

$$\begin{aligned}\cos\delta = \frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^{\circ}_{12}}\Big[&\sin^2\theta^{\circ}_{23}\left(\cos^2\theta^{\circ}_{12} - \cos^2\theta_{12}\cos^2\theta_{13}\right)\\ &- \sin^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^{\circ}_{12}\\ &+ \cos^2\theta_{12}\left(\cos^2\theta_{13}\sin^2\theta^{\circ}_{12}\cos^2\theta^{\circ}_{23} - \sin^2\theta_{13}\sin^2\theta_{23}\cos^2\theta^{\circ}_{12}\right)\\ &+ \kappa\cos\hat{\delta}\cos\theta_{12}\cos\theta_{13}\sin\theta^{\circ}_{12}\sin 2\theta^{\circ}_{23}\left(\cos^2\theta^{\circ}_{12} - \cos^2\theta_{12}\cos^2\theta_{13}\right)^{\frac{1}{2}}\Big]\,,\end{aligned} \quad (206)$$

where $\kappa = 1$ if $\hat{\theta}_{13}$ lies in the first or third quadrant, and $\kappa = -1$ otherwise. For $\theta^{\circ}_{12} = k\pi$, $k = 0, 1, 2$, and/or $\theta^{\circ}_{23} = k'\pi/2$, $k' = 0, 1, 2, 3, 4$, $\cos\delta$ does not depend on $\hat{\delta}$ and κ.

The sum rules derived in Section 8 are summarised in Table 8.

Table 8
Summary of the sum rules for $\cos\delta$ in the case of fully broken G_ν under the assumption that the matrix U_ν consists of two complex rotation matrices. The parameter $\kappa = 1$ if the corresponding hat angle belongs to the first or third quadrant, and $\kappa = -1$ otherwise. See text for further details.

Case	Parametrisation of U	Sum rule for $\cos\delta$
E1	$R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})P_1(\hat{\delta})R_{12}(\hat{\theta}_{12})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	$-\frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{13}}\Big[\cos^2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13}-\sin^2\theta_{12}\cos^2\theta_{13}\right) - \cos^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^\circ_{13} + \sin^2\theta_{12}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{13}\sin^2\theta^\circ_{23} - \sin^2\theta_{13}\sin^2\theta_{23}\cos^2\theta^\circ_{13}\right) - \kappa\cos\hat{\delta}\sin\theta_{12}\cos\theta_{13}\sin\theta^\circ_{13}\sin 2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13}-\sin^2\theta_{12}\cos^2\theta_{13}\right)^{\frac{1}{2}}\Big]$
E2	$R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})P_1(\hat{\delta})R_{12}(\hat{\theta}_{12})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	$\frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{13}}\Big[\cos^2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13}-\cos^2\theta_{12}\cos^2\theta_{13}\right) - \sin^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^\circ_{13} + \cos^2\theta_{12}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{13}\sin^2\theta^\circ_{23} - \sin^2\theta_{13}\sin^2\theta_{23}\cos^2\theta^\circ_{13}\right) + \kappa\cos\hat{\delta}\cos\theta_{12}\cos\theta_{13}\sin\theta^\circ_{13}\sin 2\theta^\circ_{23}\left(\cos^2\theta^\circ_{13}-\cos^2\theta_{12}\cos^2\theta_{13}\right)^{\frac{1}{2}}\Big]$
E4	$R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})P_2(\hat{\delta})R_{23}(\hat{\theta}_{23})U_{13}(\theta^\nu_{13},\delta^\nu_{13})Q_0$	$\frac{\cos^2\theta_{12}\cos^2\theta_{23}+\sin^2\theta_{12}\sin^2\theta_{13}\sin^2\theta_{23}-\cos^2\hat{\theta}_{23}\cos^2\theta^\circ_{12}}{\sin 2\theta_{23}\sin\theta_{12}\cos\theta_{12}\sin\theta_{13}}$
E6	$R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})P_1(\hat{\delta})R_{13}(\hat{\theta}_{13})U_{23}(\theta^\nu_{23},\delta^\nu_{23})Q_0$	$\frac{2}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}\cos^2\theta^\circ_{12}}\Big[\sin^2\theta^\circ_{23}\left(\cos^2\theta^\circ_{12}-\cos^2\theta_{12}\cos^2\theta_{13}\right) - \sin^2\theta_{12}\cos^2\theta_{23}\cos^2\theta^\circ_{12} + \cos^2\theta_{12}\left(\cos^2\theta_{13}\sin^2\theta^\circ_{12}\cos^2\theta^\circ_{23} - \sin^2\theta_{13}\sin^2\theta_{23}\cos^2\theta^\circ_{12}\right) + \kappa\cos\hat{\delta}\cos\theta_{12}\cos\theta_{13}\sin\theta^\circ_{12}\sin 2\theta^\circ_{23}\left(\cos^2\theta^\circ_{12}-\cos^2\theta_{12}\cos^2\theta_{13}\right)^{\frac{1}{2}}\Big]$

Table 9
The phenomenologically viable case for the symmetry group A_4. The values of $\cos\delta$ and $\sin^2\theta_{12}$ predicted by the scheme B1, which refers to the corresponding parametrisation in Tables 3 and 5, have been obtained using the best fit values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ for the NO spectrum quoted in eqs. (6)–(8).

$(G_e, G_\nu) = (Z_3, Z_2)$	$\cos\delta$	$\sin^2\theta_{12}$
B1 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/3, 1/2)$	0.570	0.341

9. Summary of the predictions for $G_f = A_4\ (T'),\ S_4$ and A_5

In this section we summarise the numerical results obtained in the cases of the discrete flavour symmetry groups A_4 (T'), S_4 and A_5, which have been already discussed in subsections 3.4, 4.4 and 5.10. In Tables 9–11 we give the values of the fixed angles, obtained from the diagonalisation of the corresponding group elements which lead to physical values of $\cos\delta$ and phenomenologically viable results for the "standard" mixing angles θ_{12}, θ_{13} and θ_{23}. In the cases when the standard mixing angles are not fixed by the schemes in Tables 9–11, we use their best fit values for the NO spectrum quoted in eqs. (6)–(8). For the cases in the tables marked with an asterisk, physical values of $\cos\delta$, i.e., $|\cos\delta| \leq 1$, cannot be obtained employing the best fit values of the neutrino mixing angles θ_{12}, θ_{13} and θ_{23}, but they can be achieved for values of the relevant mixing parameters allowed at 3σ. Note that unphysical values of $\cos\delta$, $|\cos\delta| > 1$, occur when the relations between the parameters of the scheme and the standard parametrisation mixing angles cannot be fulfilled for given values of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$. Indeed the parameter space of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ is reduced by these constraints coming from the schemes.

For the symmetry group A_4 we find that the residual symmetries

- $(G_e, G_\nu) = (Z_2, Z_2)$ in the cases C1–C9;
- $(G_e, G_\nu) = (Z_3, Z_2)$ in the cases B2 and B3;
- $(G_e, G_\nu) = (Z_2 \times Z_2, Z_2)$ in the cases B1, B2 and B3;
- $(G_e, G_\nu) = (Z_2, Z_3)$ or $(Z_2, Z_2 \times Z_2)$ in the cases A1, A2 and A3

do not provide phenomenologically viable results for $\cos\delta$ and/or the standard mixing angles. It is worth noticing that the predicted value of $\sin^2\theta_{12} = 0.341$ in Table 9 is within the 2σ allowed range. Varying $\sin^2\theta_{13}$, which enters into the expression for $\sin^2\theta_{12}$, within its respective 3σ allowed range for the NO neutrino mass spectrum, we find $0.339 \leq \sin^2\theta_{12} \leq 0.343$.

For the symmetry group S_4 we find that the residual symmetries

- $(G_e, G_\nu) = (Z_2, Z_2)$ in the cases C6 and C9;
- $(G_e, G_\nu) = (Z_3, Z_2)$ in the case B3;
- $(G_e, G_\nu) = (Z_4, Z_2)$ or $(Z_2 \times Z_2, Z_2)$ in the cases B2 and B3;
- $(G_e, G_\nu) = (Z_2, Z_3)$ in the cases A1, A2 and A3;
- $(G_e, G_\nu) = (Z_2, Z_4)$ or $(Z_2, Z_2 \times Z_2)$ in the case A3

do not provide phenomenologically viable results for $\cos\delta$ and/or for the standard mixing angles.

The cases in Table 10 marked with an asterisk are discussed below. Firstly, using the best fit values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ we get a physical value of $\cos\delta$ in the case C3 for the minimal value of $\sin^2\theta_{23} = 0.562$, for which $\cos\delta = -0.996$. For C8 with $\sin^2\theta^\circ_{23} = 1/2$ and $3/4$, using the best fit values of the neutrino mixing angles for the NO spectrum, we have $\cos\delta = -1.53$

Table 10
The phenomenologically viable cases for the symmetry group S_4. The values of $\cos\delta$ and $\sin^2\theta_{12}$ or $\sin^2\theta_{23}$ predicted by the schemes A1, A2, etc., which refer to the corresponding parametrisations in Tables 3–6, have been obtained using the best fit values for the NO spectrum of the other two (not fixed) neutrino mixing parameters ($\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, or $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$) quoted in eqs. (6)–(8). In the cases marked with an asterisk, physical values of $\cos\delta$ cannot be obtained employing the best fit values of the mixing angles, but are possible for values of the relevant neutrino mixing parameters lying in their respective 3σ allowed intervals. See text for further details.

$(G_e, G_\nu) = (Z_2, Z_2)$	$\cos\delta$	$\sin^2\theta_{ij}$
C1 $\sin^2\theta^\circ_{23} = 1/4$	-0.806	not fixed
C2 $\sin^2\theta^\circ_{23} = 1/2$	not fixed	$\sin^2\theta_{23} = 0.512$
C3 $\sin^2\theta^\circ_{13} = 1/4$	-1*	not fixed
C4 $\sin^2\theta^\circ_{12} = 1/4$	0.992	not fixed
C5 $\sin^2\theta^\circ_{12} = 1/4$	not fixed	$\sin^2\theta_{12} = 0.256$
C7 $\sin^2\theta^\circ_{23} = 1/2$	not fixed	$\sin^2\theta_{23} = 0.488$
C8 $\sin^2\theta^\circ_{23} = \{1/2, 3/4\}$	$\{-1^*, 1^*\}$	not fixed
$(G_e, G_\nu) = (Z_3, Z_2)$	$\cos\delta$	$\sin^2\theta_{12}$
B1 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/3, 1/2)$	0.570	0.341
B2 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}) = (1/6, 1/5)$	-0.269	0.317
$(G_e, G_\nu) = (Z_4, Z_2),\ (Z_2 \times Z_2, Z_2)$	$\cos\delta$	$\sin^2\theta_{12}$
B1 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/4, 1/3)$	-1*	0.256
$(G_e, G_\nu) = (Z_2, Z_4),\ (Z_2, Z_2 \times Z_2)$	$\cos\delta$	$\sin^2\theta_{23}$
A1 $(\sin^2\theta^\circ_{13}, \sin^2\theta^\circ_{23}) = (1/3, 1/4)$	-1*	0.488
A2 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/2, 1/2)$	1*	0.512

and 2.04, respectively. The physical values of $\cos\delta$ can be obtained, using, e.g., the values of $\sin^2\theta_{23} = 0.380$ and 0.543, for which $\cos\delta = -0.995$ and 0.997, respectively. In the parts of the 3σ allowed range of $\sin^2\theta_{23}$, $0.374 \leq \sin^2\theta_{23} \leq 0.380$ and $0.543 \leq \sin^2\theta_{23} \leq 0.641$, we have $-0.938 \geq \cos\delta \geq -0.995$ and $0.997 \geq \cos\delta \geq 0.045$, respectively. Secondly, in the case B1 we obtain $\cos\delta = -0.990$ employing the best fit value of $\sin^2\theta_{13}$ and the maximal value of $\sin^2\theta_{23} = 0.419$. Finally, utilising the best fit value of $\sin^2\theta_{13}$, we get physical values of $\cos\delta$ in the cases A1 and A2 for the minimal value of $\sin^2\theta_{12} = 0.348$, for which $\cos\delta = -0.993$ and 0.993, respectively. Note that for the cases in which $\sin^2\theta_{23}$ is fixed, the predicted values are within the corresponding 2σ range, while in the cases in which $\sin^2\theta_{12}$ is fixed, the values of $\sin^2\theta_{12} = 0.341$ and 0.317 are within 2σ and 1σ, respectively. The value of $\sin^2\theta_{12} = 0.256$ lies slightly outside the current 3σ allowed range.

For the symmetry group A_5 we find that the residual symmetries

- $(G_e, G_\nu) = (Z_2, Z_2)$ in the cases C2, C6 and C7;
- $(G_e, G_\nu) = (Z_3, Z_2)$ in the cases B2 and B3;
- $(G_e, G_\nu) = (Z_5, Z_2)$ in the case B3;
- $(G_e, G_\nu) = (Z_2 \times Z_2, Z_2)$ in the cases B1 and B3;
- $(G_e, G_\nu) = (Z_2, Z_3)$ or (Z_2, Z_5) in the case A3;
- $(G_e, G_\nu) = (Z_2, Z_2 \times Z_2)$ in the cases A1, A2 and A3

Table 11
The phenomenologically viable cases for the symmetry group A_5. The values of $\cos\delta$ and $\sin^2\theta_{12}$ or $\sin^2\theta_{23}$ predicted by the schemes A1, A2, etc., which refer to the corresponding parametrisations in Tables 3–6, have been obtained using the best fit values of the other standard mixing angles for the NO spectrum quoted in eqs. (6)–(8). In the cases marked with an asterisk, the predicted values of $\cos\delta$, obtained for the best fit values of the neutrino mixing angles θ_{12}, θ_{13} and θ_{23}, are unphysical; physical values of $\cos\delta$ can be obtained for values of the neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ lying in their respective 3σ allowed intervals. See text for further details.

$(G_e, G_\nu) = (Z_2, Z_2)$	$\cos\delta$	$\sin^2\theta_{ij}$
C1 $\sin^2\theta^\circ_{23} = 1/4$	-0.806	not fixed
C3 $\sin^2\theta^\circ_{13} = 0.0955,\ 1/4$	$0.688,\ -1^*$	not fixed
C4 $\sin^2\theta^\circ_{12} = 0.0955,\ 1/4$	$-1^*,\ 0.992$	not fixed
C5 $\sin^2\theta^\circ_{12} = 1/4$	not fixed	$\sin^2\theta_{12} = 0.256$
C8 $\sin^2\theta^\circ_{23} = 3/4$	1^*	not fixed
C9 $\sin^2\theta^\circ_{12} = 0.3455$	not fixed	$\sin^2\theta_{12} = 0.330$
$(G_e, G_\nu) = (Z_3, Z_2)$	$\cos\delta$	$\sin^2\theta_{12}$
B1 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (1/3, 1/2)$	0.570	0.341
$(G_e, G_\nu) = (Z_5, Z_2)$	$\cos\delta$	$\sin^2\theta_{12}$
B1 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (0.2764, 1/2)$	0.655	0.283
B2 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}) = (0.1382, 0.1604)$	-0.229	0.259
$(G_e, G_\nu) = (Z_2 \times Z_2, Z_2)$	$\cos\delta$	$\sin^2\theta_{12}$
B2 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}) = (0.0955, 0.2764)$	-1^*	0.330
$(1/4, 0.1273)$	0.805	0.330
$(G_e, G_\nu) = (Z_2, Z_3)$	$\cos\delta$	$\sin^2\theta_{23}$
A1 $(\sin^2\theta^\circ_{13}, \sin^2\theta^\circ_{23}) = (0.2259, 0.4363)$	0.716	0.553
A2 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (0.2259, 0.4363)$	-0.716	0.447
$(G_e, G_\nu) = (Z_2, Z_5)$	$\cos\delta$	$\sin^2\theta_{23}$
A1 $(\sin^2\theta^\circ_{13}, \sin^2\theta^\circ_{23}) = (0.4331, 0.3618)$	-1^*	0.630
A2 $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{23}) = (0.4331, 0.3618)$	1^*	0.370

do not provide phenomenologically viable results for $\cos\delta$ and/or for the standard mixing angles θ_{12}, θ_{13} and θ_{23}.

We will describe next the cases in Table 11 marked with an asterisk, apart from those which have also been found for $G_f = S_4$ and discussed earlier. Using the best fit values of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$ we get a physical value of $\cos\delta$ in the case C4 for the minimal value of $\sin^2\theta_{23} = 0.487$, for which $\cos\delta = -0.997$. Instead using the best fit values of $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ one gets the physical values of $\cos\delta = -1$ for the maximal value of $\sin^2\theta_{12} = 0.277$. Employing the best fit value of $\sin^2\theta_{13}$ we find a physical value of $\cos\delta$ in the case B2 with residual symmetries $(G_e, G_\nu) = (Z_2 \times Z_2, Z_2)$ for the minimal value of $\sin^2\theta_{23} = 0.518$, for which $\cos\delta = -0.996$. Similarly for the cases A1 and A2 with residual symmetries $(G_e, G_\nu) = (Z_2, Z_5)$, the values of $\cos\delta = -0.992$ and 0.992 are obtained using the minimal value of $\sin^2\theta_{12} = 0.321$.

The values of $\sin^2\theta^\circ_{ij}$ in Table 11 used to compute $\cos\delta$ and $\sin^2\theta_{ij}$ are the following ones: $1/(4r^2) \cong 0.0955$, $(3-r)/4 \cong 0.3455$, $1/(2+r) \cong 0.2764$, $1/(4+2r) \cong 0.1382$, $1/(3+2r) \cong 0.1604$, $1/(3+3r) \cong 0.1273$, $2/(4r^2-r) \cong 0.2259$, $r/(6r-6) \cong 0.4363$, $(6r-4)/(10r-3) \cong 0.4331$, $(1-r)/(8-6r) \cong 0.3618$.

10. Conclusions

In the present article we have employed the discrete symmetry approach to understanding the observed pattern of 3-neutrino mixing and, within this approach, have derived sum rules and predictions for the Dirac phase δ present in the PMNS neutrino mixing matrix U. The approach is based on the assumption of the existence at some energy scale of a (lepton) flavour symmetry corresponding to a non-Abelian discrete group G_f. The flavour symmetry group G_f can be broken, in general, to different "residual symmetry" subgroups G_e and G_ν of the charged lepton and neutrino mass terms, respectively. Given G_f, typically there are more than one (but still a finite number of) possible residual symmetries G_e and G_ν. The residual symmetries can constrain the forms of the 3×3 unitary matrices U_e and U_ν, which diagonalise the charged lepton and neutrino mass matrices, and the product of which represents the PMNS neutrino mixing matrix U, $U = U_e^\dagger U_\nu$. Thus, by constraining the form of the matrices U_e and U_ν, the residual symmetries constrain also the form of the PMNS matrix U. This can lead, in particular, to a correlation between the values of the PMNS neutrino mixing angles θ_{12}, θ_{13} and θ_{23}, which have been determined experimentally with a rather good precision, and the value of the cosine of the Dirac CP violation phase δ present in U, i.e., to a "sum rule" for $\cos\delta$. The sum rule for $\cos\delta$ thus obtained depends on residual symmetries G_e and G_ν and in some cases can involve, in addition to θ_{12}, θ_{13} and θ_{23}, parameters which cannot be constrained even when G_f is fixed. For a given fixed G_f, unambiguous predictions for the value of $\cos\delta$ can be derived in the cases when, apart from the parameters determined by G_f (and G_e and G_ν), only θ_{12}, θ_{13} and θ_{23} enter into the expression for the respective sum rule.

In the present article we have derived sum rules for $\cos\delta$ considering the following discrete residual symmetries: i) $G_e = Z_2$ and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ (Section 3); ii) $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and $G_\nu = Z_2$ (Section 4); iii) $G_e = Z_2$ and $G_\nu = Z_2$ (Section 5); iv) G_e is fully broken and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ (Section 7); and v) $G_e = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$ and G_ν is fully broken (Section 8). The sum rules are summarised in Tables 3, 4, 7 and 8. For given G_e and G_ν, the sum rules for $\cos\delta$ we have derived are exact, within the approach employed, and are valid, in particular, for any G_f containing G_e and G_ν as subgroups. We have identified the cases when the value of $\cos\delta$ cannot be determined, or cannot be uniquely determined, from the sum rule without making additional assumptions on unconstrained parameters (cases A3 in Section 3 and B3 in Section 4 (see also Table 3); cases C2, C5, C6, C7 and C9 in Section 5 (see also Table 4); the cases discussed in Sections 7 and 8). In the majority of the phenomenologically viable cases we have considered the value of $\cos\delta$ can be unambiguously predicted once the flavour symmetry G_f is fixed. In certain cases of fixed G_f, G_e and G_ν, correlations between the values of some of the measured neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, are predicted, and/or the values of some of these parameters, typically of $\sin^2\theta_{12}$ or $\sin^2\theta_{23}$, are fixed. These correlations and "predictions" are summarised in Tables 5 and 6. We have found that a relatively large number of these cases are not phenomenologically viable, i.e., they lead to results which are not compatible with the existing data on neutrino mixing. We have derived predictions for $\cos\delta$ for the flavour symmetry groups $G_f = S_4$, A_4, T' and A_5 using the best fit values of $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, when $\cos\delta$ is unambiguously determined by the corresponding sum rule. We have presented the predictions for $\cos\delta$ only in the phenomenologically viable cases, i.e., when the measured values of the 3-neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, taking into account their respective 3σ uncertainties, are successfully reproduced. These predictions, together with the predictions

Table 12
3-dimensional representation of the generators of A_4, T', S_4 and A_5. We have defined $\omega = e^{2\pi i/3}$, $r = (1+\sqrt{5})/2$ and $\rho = e^{2\pi i/5}$.

Group	3-dimensional representation of the generators		
A_4	$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	
T'	$S = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega^2 & -1 & 2\omega \\ 2\omega & 2\omega^2 & -1 \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	
S_4	$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$T = \frac{1}{2}\begin{pmatrix} i & -\sqrt{2}i & -i \\ \sqrt{2} & 0 & \sqrt{2} \\ i & \sqrt{2}i & -i \end{pmatrix}$	$U = \begin{pmatrix} 0 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 0 \end{pmatrix}$
A_5	$S = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -r & 1/r \\ -\sqrt{2} & 1/r & -r \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^4 \end{pmatrix}$	

for the value of one of the mixing parameters $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$, in the cases when it is fixed by the symmetries, are summarised in Tables 9–11.

The results derived in the present study show, in particular, that with the accumulation of more precise data on the PMNS neutrino mixing parameters $\sin^2\theta_{12}$, $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$, and with the measurement of the Dirac phase δ present in the neutrino mixing matrix U, it will be possible to critically test the predictions of the current phenomenologically viable theories, models and schemes of neutrino mixing based on different non-Abelian discrete (lepton) flavour symmetries G_f and sets of their non-trivial subgroups of residual symmetries G_e and G_ν, operative respectively in the charged lepton and neutrino sectors, and thus critically test the discrete symmetry approach to understanding the observed pattern of neutrino mixing.

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Appendix A. The discrete groups A_4, T', S_4 and A_5

A_4 is the symmetry group of even permutations of four objects (see, e.g., [2]). It is isomorphic to the tetrahedral symmetry group, i.e., the group of rotational symmetries of a regular tetrahedron. As such it can be defined in terms of two generators S and T, satisfying $S^2 = T^3 = (ST)^3 = 1$. In this work, we choose to work in the Altarelli–Feruglio basis [45] for the 3-dimensional representation of the S and T generators, see Table 12.

The group T' is the double covering group of A_4 (see, e.g., [2]), which can be defined in terms of two generators S and T through the algebraic relations: $R^2 = T^3 = (ST)^3 = 1$, $RT = TR$, where $R = S^2$. We use the basis for the 3-dimensional representation of the generators S and T from [30], summarised in Table 12. Since we restrict ourselves to the triplet representation for the LH charged lepton and neutrino fields, there is no way to distinguish T' from A_4 [30].[16] Note that matrices representing S and T in Table 12 for A_4, are related with those for T' by the following redefinition $S \to TST^2$, $T \to T^2$, where S and T before (after) the arrows are the matrices presented in Table 12 for A_4 (T').

S_4 is the group of permutations of four objects, i.e., the rotational symmetry group of a cube (see, e.g., [2]). It can be defined in terms of three generators S, T and U, satisfying [46]: $S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1$. We employ for the 3-dimensional representation of the S, T and U generators the basis given in [46] and summarised in Table 12. As it was also shown in [46], this basis is equivalent to the basis widely used in the literature [31].

A_5 is the group of even permutations of five objects (see, e.g., [2]), i.e., the rotational symmetry group of an icosahedron, which can be defined in terms of two generators S and T, satisfying $S^2 = T^5 = (ST)^3 = 1$. We employ the basis defined in [47], which for the 3-dimensional representation of the generators S and T is summarised in Table 12.

We conclude this appendix by noting that a list of the Abelian subgroups of A_4, T', S_4 and A_5 can be found in [49], [17], [46] and [47], respectively.

Appendix B. Parametrisations of a 3 × 3 unitary matrix

Parametrisations of a 3×3 unitary matrix W (see, e.g., [50–52]) can be obtained, e.g., from one of the six permutations of a product of three complex rotations and diagonal phase matrices, e.g., as follows:

$$W = \Psi_1 \Psi_2 \Psi_3 \overline{W} = \Psi_1 \Psi_2 \Psi_3 \, U_{ij} \, U_{kl} \, U_{rs} \,, \tag{207}$$

where we have assumed $ij \neq kl \neq rs$. It is worth noticing that sometimes it is convenient to use the parametrisations of $\overline{W}$ of the following form:

$$\overline{W} = U_{ij} \, U_{kl} \, \tilde{U}_{ij} \,. \tag{208}$$

As shown in [50], the number of distinctive parametrisations of a CKM-like matrix is nine. We have defined the phase matrices Ψ_i in eq. (16) and the complex rotation matrix in the i–j plane $U_{ij} \equiv U_{ij}(\theta_{ij}, \delta_{ij})$ in eq. (17). The latter can be always parametrised as a product of diagonal phase matrices and the rotation matrix $R_{ij} \equiv R_{ij}(\theta_{ij}) = U_{ij}(\theta_{ij}, 0)$, i.e.,

$$U_{ij} = P_i(\delta)^* \, R_{ij} \, P_i(\delta) = P_j(-\delta)^* \, R_{ij} \, P_j(-\delta) \,, \tag{209}$$

where $P_i(\delta)$ are diagonal matrices defined as follows:

$$P_1(\delta) = \mathrm{diag}(e^{i\delta}, 1, 1) \,, \quad P_2(\delta) = \mathrm{diag}(1, e^{i\delta}, 1) \,, \quad P_3(\delta) = \mathrm{diag}(1, 1, e^{i\delta}) \,. \tag{210}$$

Defining $P_{ij}(\alpha, \beta)$ as a product $P_{ij}(\alpha, \beta) \equiv P_i(\alpha) P_j(\beta)$, the following relation holds:

$$U_{ij}(\theta_{ij}, \delta_{ij}) \, P_{ij}(\alpha, \beta) = P_{ij}(\alpha, \beta) U_{ij}(\theta_{ij}, \delta'_{ij}) \,, \tag{211}$$

with $\delta'_{ij} = \delta_{ij} + \alpha - \beta$.

[16] It is worth noting that A_4 is not a subgroup of T'.

Table 13
Equivalent parametrisations of $\overline{W}$ obtained using the result in eq. (211), which allows us to find the convenient form of the matrix $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \{\delta^\circ_{kl}\})$ defined in Section 2.

Case	Initial form of $\overline{W}$	Final parametrisation of $\overline{W}$
A1	$U_{12}U_{23}U_{13}$	$P^*_{12}(\delta_{13},\delta_{23})U_{12}(\theta_{12},\delta_{12}-\delta_{13}+\delta_{23})R_{23}R_{13}P_{12}(\delta_{13},\delta_{23})$
A2	$U_{13}U_{23}U_{12}$	$P^*_{13}(\delta_{12},-\delta_{23})U_{13}(\theta_{13},\delta_{13}-\delta_{12}-\delta_{23})R_{23}R_{12}P_{13}(\delta_{12},-\delta_{23})$
A3	$U_{23}U_{13}U_{12}$	$P_{23}(\delta_{12},\delta_{13})U_{23}(\theta_{23},\delta_{23}+\delta_{12}-\delta_{13})R_{13}R_{12}P^*_{23}(\delta_{12},\delta_{13})$
B1	$U_{23}U_{12}U_{13}$	$P^*_{13}(\delta_{12},-\delta_{23})R_{23}R_{12}U_{13}(\theta_{13},\delta_{13}-\delta_{12}-\delta_{23})P_{13}(\delta_{12},-\delta_{23})$
B2	$U_{13}U_{12}U_{23}$	$P_{23}(\delta_{12},\delta_{13})R_{13}R_{12}U_{23}(\theta_{23},\delta_{23}+\delta_{12}-\delta_{13})P^*_{23}(\delta_{12},\delta_{13})$
B3	$U_{23}U_{13}U_{12}$	$P^*_{12}(\delta_{13},\delta_{23})R_{23}R_{13}U_{12}(\theta_{12},\delta_{12}-\delta_{13}+\delta_{23})P_{12}(\delta_{13},\delta_{23})$
C1	$U_{12}U_{23}U_{13}$	$P_3(\delta_{23})U_{12}(\theta_{12},\delta_{12})R_{23}U_{13}(\theta_{13},\delta_{13}-\delta_{23})P^*_3(\delta_{23})$
C2	$U_{13}U_{23}U_{12}$	$P_3(\delta_{23})U_{13}(\theta_{13},\delta_{13}-\delta_{23})R_{23}U_{12}(\theta_{12},\delta_{12})P^*_3(\delta_{23})$
C3	$U_{12}U_{13}U_{23}$	$P_3(\delta_{13})U_{12}(\theta_{12},\delta_{12})R_{13}U_{23}(\theta_{23},\delta_{23}-\delta_{13})P^*_3(\delta_{13})$
C4	$U_{13}U_{12}U_{23}$	$P_2(\delta_{12})U_{13}(\theta_{13},\delta_{13})R_{12}U_{23}(\theta_{23},\delta_{23}+\delta_{12})P^*_2(\delta_{12})$
C5	$U_{23}U_{12}U_{13}$	$P_2(\delta_{12})U_{23}(\theta_{23},\delta_{23}+\delta_{12})R_{12}U_{13}(\theta_{13},\delta_{13})P^*_2(\delta_{12})$
C6	$U_{23}U_{13}U_{12}$	$P_3(\delta_{13})U_{23}(\theta_{23},\delta_{23}-\delta_{13})R_{13}U_{12}(\theta_{12},\delta_{12})P^*_3(\delta_{13})$
C7	$U_{12}U_{23}\tilde{U}_{12}$	$P_3(\delta_{23})U_{12}(\theta_{12},\delta_{12})R_{23}U_{12}(\tilde{\theta}_{12},\tilde{\delta}_{12})P^*_3(\delta_{23})$
C8	$U_{13}U_{23}\tilde{U}_{13}$	$P^*_2(\delta_{23})U_{13}(\theta_{13},\delta_{13})R_{23}U_{13}(\tilde{\theta}_{13},\tilde{\delta}_{13})P_2(\delta_{23})$
C9	$U_{23}U_{12}\tilde{U}_{23}$	$P^*_1(\delta_{12})U_{23}(\theta_{23},\delta_{23})R_{12}U_{23}(\tilde{\theta}_{23},\tilde{\delta}_{23})P_1(\delta_{12})$

Starting from the general parametrisation of W in eq. (207) and the relation in eq. (211), we find convenient parametrisations for $\overline{W}$. They are summarised in Table 13. The parametrisations of the matrix $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \{\delta^\circ_{kl}\})$ defined in Section 2 have been obtained from Table 13 after a redefinition of the phases $\{\delta^\circ_{kl}\}$. For example, in the first case when $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \{\delta^\circ_{kl}\})$ is represented by the product $U_{12}(\theta^\circ_{12}, \delta^\circ_{12})U_{23}(\theta^\circ_{23}, \delta^\circ_{23})U_{13}(\theta^\circ_{13}, \delta^\circ_{13})$ the following redefinition is used: $\delta^\circ_{12} - \delta^\circ_{13} + \delta^\circ_{23} \rightarrow \delta^\circ_{12}$.

The product of two complex rotations in the i–j plane can always be written as

$$\begin{aligned} U_{ij}(\theta^a_{ij}, \delta^a_{ij})U_{ij}(\theta^b_{ij}, \delta^b_{ij}) &= P_{ij}(\beta, -\alpha)R_{ij}(\hat{\theta}_{ij})P_i(\alpha-\beta) = P_j(-\alpha-\beta)R_{ij}(\hat{\theta}_{ij})P_{ij}(\alpha,\beta) \\ &= P_{ij}(\alpha, -\beta)R_{ij}(\hat{\theta}_{ij})P_j(\beta-\alpha) \\ &= P_i(\alpha+\beta)R_{ij}(\hat{\theta}_{ij})P_{ij}(-\beta,-\alpha)\,, \end{aligned} \tag{212}$$

where we have defined the angle $\hat{\theta}_{ij}$ as

$$\sin\hat{\theta}_{ij} = |s^a_{ij}c^b_{ij}e^{-i\delta^a_{ij}} + c^a_{ij}s^b_{ij}e^{-i\delta^b_{ij}}|\,, \tag{213}$$

and the phases α, β as

$$\alpha = \arg\left[c^a_{ij}c^b_{ij} - s^a_{ij}s^b_{ij}e^{i(\delta^b_{ij}-\delta^a_{ij})}\right], \quad \beta = \arg\left[s^a_{ij}c^b_{ij}e^{-i\delta^a_{ij}} + c^a_{ij}s^b_{ij}e^{-i\delta^b_{ij}}\right], \tag{214}$$

with $s^{a(b)}_{ij} = \sin\theta^{a(b)}_{ij}$ and $c^{a(b)}_{ij} = \cos\theta^{a(b)}_{ij}$.

Appendix C. The case of fully broken G_e or G_ν

In the case when the group G_e is fully broken and $G_\nu = Z_n$, $n > 2$ or $Z_n \times Z_m$, $n, m \geq 2$, there are cases in which one can express $\cos\delta$ as a function of θ_{12}, θ_{13}, θ_{23} and θ°_{12}, θ°_{13}, θ°_{23}. In the cases

i) $U_e^\dagger = U_{23(13)}(\theta^e_{23(13)}, \delta^e_{23(13)})U_{12}(\theta^e_{12}, \delta^e_{12})$,

ii) $U_e^\dagger = U_{12(13)}(\theta^e_{12(13)}, \delta^e_{12(13)})U_{23}(\theta^e_{23}, \delta^e_{23})$,

Table 14
Upper (lower) part. Parametrisations of U in the case of fully broken G_e (G_ν) and $G_\nu = Z_n, n > 2$ or $Z_n \times Z_m, n, m \geq 2$ ($G_e = Z_n, n > 2$ or $Z_n \times Z_m, n, m \geq 2$) when U_e (U_ν) has particular forms.

$U(\theta^e_{12}, \theta^e_{13}, \theta^e_{23}, \{\delta^e_{kl}\})$	Parametrisation of U for fully broken G_e
$U_{23(13)}(\theta^e_{23(13)}, \delta^e_{23(13)})U_{12}(\theta^e_{12}, \delta^e_{12})$	$U_{23(13)}(\theta^e_{23(13)}, \delta^e_{23(13)})R_{12}(\hat{\theta}_{12})P_1(\hat{\delta})R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})Q_0$
$U_{12(13)}(\theta^e_{12(13)}, \delta^e_{12(13)})U_{23}(\theta^e_{23}, \delta^e_{23})$	$U_{12(13)}(\theta^e_{12(13)}, \delta^e_{12(13)})R_{23}(\hat{\theta}_{23})P_2(\hat{\delta})R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})Q_0$
$U_{23(12)}(\theta^e_{23(12)}, \delta^e_{23(12)})U_{13}(\theta^e_{13}, \delta^e_{13})$	$U_{23(12)}(\theta^e_{23(12)}, \delta^e_{23(12)})R_{13}(\hat{\theta}_{13})P_1(\hat{\delta})R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})Q_0$
$U(\theta^\nu_{12}, \theta^\nu_{13}, \theta^\nu_{23}, \{\delta^\nu_{kl}\})$	Parametrisation of U for fully broken G_ν
$U_{12}(\theta^\nu_{12}, \delta^\nu_{12})U_{13(23)}(\theta^\nu_{13(23)}, \delta^\nu_{13(23)})$	$R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})P_1(\hat{\delta})R_{12}(\hat{\theta}_{12})U_{13(23)}(\theta^\nu_{13(23)}, \delta^\nu_{13(23)})Q_0$
$U_{23}(\theta^\nu_{23}, \delta^\nu_{23})U_{12(13)}(\theta^\nu_{12(13)}, \delta^\nu_{12(13)})$	$R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})P_2(\hat{\delta})R_{23}(\hat{\theta}_{23})U_{12(13)}(\theta^\nu_{12(13)}, \delta^\nu_{12(13)})Q_0$
$U_{13}(\theta^\nu_{13}, \delta^\nu_{13})U_{12(23)}(\theta^\nu_{12(23)}, \delta^\nu_{12(23)})$	$R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})P_1(\hat{\delta})R_{13}(\hat{\theta}_{13})U_{12(23)}(\theta^\nu_{12(23)}, \delta^\nu_{12(23)})Q_0$

iii) $U_e^\dagger = U_{23(12)}(\theta^e_{23(12)}, \delta^e_{23(12)})U_{13}(\theta^e_{13}, \delta^e_{13})$,

we choose for convenience, respectively:

i) $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}) = U_{12}(\theta^\circ_{12}, \delta^\circ_{12})R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})$,

ii) $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{23}) = U_{23}(\theta^\circ_{23}, \delta^\circ_{23})R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})$,

iii) $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{13}) = U_{13}(\theta^\circ_{13}, \delta^\circ_{13})R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})$.

The possible parametrisations of U presented in Table 14 can be obtained from i), ii) and iii) using eqs. (212)–(214). The angles θ^e_{ij}, $\hat{\theta}_{ij}$ and the phases δ^e_{ij}, $\hat{\delta}$ are free parameters. It can be seen from Table 14 that if one of the fixed angles turns out to be zero, the number of free parameters reduces from four to three. The same situation happens if one of the two free phases is fixed. Thus, in some of these cases a sum rule for $\cos\delta$ can be derived.

In the case when the group $G_e = Z_n, n > 2$ or $Z_n \times Z_m, n, m \geq 2$ and G_ν is fully broken, we consider the following forms of the matrix U_ν,

iv) $U_\nu = U_{12}(\theta^\nu_{12}, \delta^\nu_{12})U_{13(23)}(\theta^\nu_{13(23)}, \delta^\nu_{13(23)})Q_0$,

v) $U_\nu = U_{23}(\theta^\nu_{23}, \delta^\nu_{23})U_{12(13)}(\theta^\nu_{12(13)}, \delta^\nu_{12(13)})Q_0$,

vi) $U_\nu = U_{13}(\theta^\nu_{13}, \delta^\nu_{13})U_{12(23)}(\theta^\nu_{12(23)}, \delta^\nu_{12(23)})Q_0$,

for which we choose for convenience, respectively:

iv) $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{12}) = R_{23}(\theta^\circ_{23})R_{13}(\theta^\circ_{13})U_{12}(\theta^\circ_{12}, \delta^\circ_{12})$,

v) $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{23}) = R_{13}(\theta^\circ_{13})R_{12}(\theta^\circ_{12})U_{23}(\theta^\circ_{23}, \delta^\circ_{23})$,

vi) $U^\circ(\theta^\circ_{12}, \theta^\circ_{13}, \theta^\circ_{23}, \delta^\circ_{13}) = R_{23}(\theta^\circ_{23})R_{12}(\theta^\circ_{12})U_{13}(\theta^\circ_{13}, \delta^\circ_{13})$.

The parametrisations of U in the cases iv), v) and vi) presented in Table 14 have been obtained with eqs. (212)–(214). The angles θ^ν_{ij}, $\hat{\theta}_{ij}$ and the phases δ^ν_{ij}, $\hat{\delta}$ are free parameters. It can be seen from Table 14 that if one of the fixed angles turns out to be zero, the number of free parameters reduces from four to three. The same situation happens if one of the two free phases is fixed. Thus, in some of these cases a sum rule for $\cos\delta$ can be derived.

Appendix D. Results for $G_f = A_5$ and generalised CP

Models with A_5 and GCP symmetry have been recently developed by several authors [8–10]. We show that our results for the symmetry group A_5 under the same assumptions of [10] and the same breaking patterns reduce to the one derived in [10]. The results in eqs. (10), (11), (12) and (14) in [10] lead to the following phenomenologically viable cases:

$$\begin{aligned}
&\text{i) } U = \mathrm{diag}(1, i, -i) R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12})\, \mathrm{diag}(1, -i, i) R_{13}(\theta^\nu_{13})\,, \text{ for } G_e = Z_3,\ G_\nu = Z_2,\\
&\text{ii) } U = \mathrm{diag}(1, i, -i) R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12})\, \mathrm{diag}(1, -i, i) R_{13}(\theta^\nu_{13})\,, \text{ for } G_e = Z_5,\ G_\nu = Z_2,\\
&\text{iii) } U = \mathrm{diag}(1, 1, -1) R_{23}(\theta^\circ_{23}) R_{12}(\theta^\circ_{12})\, \mathrm{diag}(1, 1, -1) R_{13}(\theta^\nu_{13})\,, \text{ for } G_e = Z_5,\ G_\nu = Z_2,\\
&\text{iv) } U = R_{13}(\theta^\circ_{13}) R_{12}(\theta^\circ_{12}) R_{23}(\theta^\circ_{23})\, \mathrm{diag}(1, 1, -1) R_{23}(\theta^\nu_{23})\,, \text{ for } G_e = Z_2 \times Z_2,\ G_\nu = Z_2,
\end{aligned}$$

where we have in i) $\theta^\circ_{12} = \sin^{-1}(1/\sqrt{3})$ and $\theta^\circ_{23} = -\pi/4$, ii) $\theta^\circ_{12} = \sin^{-1}(1/\sqrt{2+r})$ and $\theta^\circ_{23} = -\pi/4$, iii) $\theta^\circ_{12} = \sin^{-1}(1/\sqrt{2+r})$ and $\theta^\circ_{23} = -\pi/4$, iv) $\theta^\circ_{12} = \sin^{-1}(1/(2r))$, $\theta^\circ_{13} = \sin^{-1}(1/\sqrt{2+r})$ and $\theta^\circ_{23} = \sin^{-1}(r/\sqrt{2+r})$.

Using $(\sin^2\theta^\circ_{12}, \sin^2\theta^\circ_{13}, \sin^2\theta^\circ_{23}) = (1/3, 0, 1/2)$ in the case i), the results in eqs. (56)–(58), after defining $\hat\theta_{13} = \theta^\nu_{13} = \theta$ and setting $\hat\delta_{13} = \delta^\nu_{13} = \pi/2$, reduce to

$$\sin^2\theta_{13} = \frac{2}{3}\sin^2\theta\,, \quad \sin^2\theta_{12} = \frac{1}{3 - 2\sin^2\theta}\,, \quad \sin^2\theta_{23} = \frac{1}{2} \quad \text{and} \quad \cos\delta = 0\,.$$

Denoting $\hat\theta_{13} = \theta^\nu_{13} = \theta$ and setting $\hat\delta_{13} = \delta^\nu_{13} = \pi/2$ in the case ii), the results in eqs. (56)–(58) reduce to

$$\sin^2\theta_{13} = \frac{\sin^2\theta}{1 + (1-r)^2}\,, \quad \sin^2\theta_{12} = \frac{1}{1 + r^2\cos^2\theta}\,, \quad \sin^2\theta_{23} = \frac{1}{2} \quad \text{and} \quad \cos\delta = 0\,.$$

The difference between the case iii) and the case ii) consists only in the phase $\hat\delta_{13}$ which now is equal to π, $\hat\delta_{13} = \delta^\nu_{13} = \pi$. Therefore while $\sin^2\theta_{13}$ and $\sin^2\theta_{12}$ remain unchanged, we find

$$\sin^2\theta_{23} = \frac{1}{2}\frac{(\sin\theta - \sqrt{1+r^2}\cos\theta)^2}{1 + r^2\cos^2\theta} \quad \text{and} \quad |\cos\delta| = 1\,.$$

Finally, in the case iv) from eqs. (64)–(66), defining $\hat\theta_{23} = \theta^\circ_{23} - \theta^\nu_{23} = \theta^\circ_{23} - \theta$ and $\hat\delta_{23} = 0$, we get:

$$\begin{aligned}
&\sin^2\theta_{13} = \frac{1 + (1-r) f(\theta)}{4}\,, \quad \sin^2\theta_{23} = \frac{1 + r(\cos^2\theta - \sin 2\theta)}{3 - (1-r) f(\theta)}\,,\\
&\sin^2\theta_{12} = \frac{1 + (1-r)(\cos^2\theta + \sin 2\theta)}{3 - (1-r) f(\theta)} \quad \text{and} \quad |\cos\delta| = 1\,,
\end{aligned}$$

where $f(\theta) = (\sin^2\theta - \sin 2\theta)$. Therefore the general results derived in Sections 4.1 and 4.2 with the choices as in i), ii), iii) and iv) and the additional restriction of the parameters due to the presence of GCP allow one to find the formulae derived in [10].

Appendix E. General statement

In this appendix we prove the general statement that Z_2 *symmetries preserved in the neutrino and charged lepton sectors can lead to phenomenologically viable predictions, only if their generators do not belong to the same* $Z_2 \times Z_2$ *subgroup of the original flavour symmetry group*. We compute the form of U° in a model independent way. Given a $Z_2 \times Z_2$ symmetry with elements $Z_2 \times Z_2 = \{1, g_1, g_2, g_3\}$ and a unitary matrix V such that $V^\dagger g_1 V = \mathrm{diag}(1, -1, -1)$, $V^\dagger g_2 V = \mathrm{diag}(-1, 1, -1)$, $V^\dagger g_3 V = \mathrm{diag}(-1, -1, 1)$, we consider first the case of $G_e = Z_2 = \{1, g_i\}$ and

$G_\nu = Z_2 = \{1, g_j\}$ with $i, j = 1, 2, 3$ for all the cases C1–C9 in Table 4. In the case C1 (C2) we find that the matrix U° reads

$$U^\circ = \pi_{23} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ for } i = j \,, \quad U^\circ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } i \neq j \,, \tag{215}$$

defined up to permutations of the 1st and 3rd (1st and 2nd) columns and the 1st and 2nd (1st and 3rd) rows. These permutations are not relevant because they correspond to a redefinition of the free parameters in the transformations $U_{12}(\theta^e_{12}, \delta^e_{12})$, $U_{13}(\theta^\nu_{13}, \delta^\nu_{13})$ and phase matrices contributing to the Majorana phases or removed with a redefinition of the charged lepton fields. In the case C3 (C6) we find that the matrix U° reads

$$U^\circ = \pi_{13} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ for } i = j \,, \quad U^\circ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } i \neq j \,, \tag{216}$$

defined up to permutations of the 2nd and 3rd (1st and 2nd) columns and the 1st and 2nd (2nd and 3rd) rows. For the case C4 (C5) we find that the matrix U° reads

$$U^\circ = \pi_{12} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } i = j \,, \quad U^\circ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } i \neq j \,, \tag{217}$$

defined up to permutations of the 2nd and 3rd (1st and 3rd) columns and the 1st and 3rd (2nd and 3rd) rows. The freedom in permuting the columns and rows as we described above does not have physical implications because it represents the freedom to perform a fixed $U(2)$ transformation in the degenerate subspace of the generator of the corresponding Z_2 symmetry. For the other cases we find similar results. Namely,

$$U^\circ = \mathrm{diag}(1, 1, 1) \text{ for } i = j \text{ and } U^\circ = \pi_{23(13)} \text{ for } i \neq j \text{ for case C7,} \tag{218}$$

$$U^\circ = \mathrm{diag}(1, 1, 1) \text{ for } i = j \text{ and } U^\circ = \pi_{23(12)} \text{ for } i \neq j \text{ for case C8,} \tag{219}$$

$$U^\circ = \mathrm{diag}(1, 1, 1) \text{ for } i = j \text{ and } U^\circ = \pi_{13(12)} \text{ for } i \neq j \text{ for case C9.} \tag{220}$$

The cases in eqs. (215)–(220) do not lead to phenomenologically viable results because some of the elements of the resulting PMNS mixing matrix equal zero. The cases when a) $G_e = Z_2 \times Z_2 = \{1, g_1, g_2, g_3\}$ and $G_\nu = Z_2 = \{1, g_j\}$, b) $G_\nu = Z_2 \times Z_2 = \{1, g_1, g_2, g_3\}$ and $G_e = Z_2 = \{1, g_i\}$, c) $G_e = Z_2 \times Z_2 = \{1, g_1, g_2, g_3\}$ and $G_\nu = Z_2 \times Z_2 = \{1, g_1, g_2, g_3\}$ are not phenomenologically viable as well. This can be seen trivially setting one or two of the free rotation angles, θ^e_{ij}, θ^ν_{kl}, to zero.

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