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A perfect obstruction theory for moduli of coherent systems

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Introduction

A very classical way to investigate the properties of a smooth algebraic curve C is the study of the linear series on C. A linear series is a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle on C and V is a subspace of the space of sections of $\mathcal{L}$. If $V = \mathrm{H}^0(\mathcal{L})$, the linear series $(\mathcal{L}, V)$ is called a complete linear series. The degree of a linear series $(\mathcal{L}, V)$ is the degree of the line bundle $\mathcal{L}$ and the dimension of $(\mathcal{L}, V)$ is the dimension of the projective space $\mathbb{P}(V)$. As described in [ACGH85], each moduli space W_d^r parametrizing complete linear series of degree d and dimension at least r is called a Brill-Noether locus on C. Brill-Noether loci are strictly related to moduli spaces G_d^r parametrizing linear series of degree d and dimension r. Both the moduli spaces W_d^r and G_d^r have been widely investigated and understood and the collection of results regarding them is known as Brill-Noether theory. In particular, Griffiths and Harris have showed that the moduli G_d^r has expected dimension

$$\rho = g - (r+1)(g-d+r)\,,$$

and so every irreducible component of G_d^r has dimension greater or equal to ρ. The number ρ is called the Brill-Noether number.

A natural way to generalize the notion of linear series is given by considering pairs (E, V) where E is a vector bundle on C and V is a linear subspace of the space of sections of E. Such pairs are called coherent systems (CS) on the curve C and they have been introduced by Le Potier [Le 93], Bertram [Ber93] and Raghavendra and Vishwanath [RV94].

The study of CS moduli spaces has been widely developed by Newstead et al in [KN95] and [BGPMN03]. For a coherent system (E, V) we have a notion of (semi-)stability, distinct from the (semi-)stability of the bundle E; the definition depends on a real parameter $\alpha \in \mathbb{R}$ and leads to a finite family of moduli spaces of α-stable coherent systems. Such moduli spaces are obtained as GIT quotients and the choice of the parameter α is equivalent to the choice of a GIT linearization. The type of a CS is the triple of integers (n, d, k) where n is the rank of E, d is the degree of E and k is the dimension of the subspace V. Different choices of type (n, d, k) give different connected components of CS moduli spaces.

The infinitesimal study of CS moduli spaces has been developed in [HE98]; there He has proved that every moduli space of stable CS has expected dimension equal to the number

$$\beta(n,d,k) = n^2(g-1) + 1 - k(k-d+n(g-1)),$$

which is still called the Brill-Noether number [BGPMN03, Definition 2.7]. Indeed, if $n = 1$ then $\beta = \rho$ and (E, V) is a linear series.

The fact that a moduli space $\mathcal{M}$ has an expected dimension naturally begs the question: is such expected dimension due to a perfect obstruction theory for $\mathcal{M}$ of rank equal to the expected dimension? Inspired by this idea, the goal of this thesis is producing an obstruction theory of rank $\beta(n,d,k)$ for every moduli space of α-stable coherent systems.

A perfect obstruction theory for a moduli space $\mathcal{M}$ is a morphism $E^\bullet \to \tau_{\geq -1} \mathrm{L}_\mathcal{M}$ in the derived category of $\mathcal{M}$, where $\mathrm{L}_\mathcal{M}$ is the cotangent complex of $\mathcal{M}$; the complex $E^\bullet$ is required to be perfect of perfect amplitude contained in $[-1,0]$ and the morphism $E^\bullet \to \tau_{\geq -1} \mathrm{L}_\mathcal{M}$ has to satisfy some technical conditions in cohomology, which implies that $\mathcal{M}$ has expected dimension equal to the rank of $E^\bullet$.

The notion of obstruction theory has been introduced and investigated by Behrend and Fantechi in [BF97]. The motivation for studying obstruction theories for a moduli space $\mathcal{M}$ is that they provide a global point of view on the infinitesimal deformation properties of $\mathcal{M}$. Obstruction theories give the possibility to define enumerative invariants for $\mathcal{M}$, because they produce a virtual fundamental class of the expected dimension in the Chow group of $\mathcal{M}$. The definition of Gromov-Witten invariants [Beh97] is the first example of such procedure for computing enumerative invariants by integrating a virtual fundamental class.

We make the following remarks.

1. The construction of an obstruction theory is based on the existence of a universal family; hence it naturally leads to working with the fine moduli stack instead of the (coarse) GIT quotient.

2. It is often easier to construct a relative obstruction theory for a morphism of stacks $\mathcal{M} \to \mathcal{N}$ with good properties, and use such relative obstruction theory to derive an absolute obstruction theory for $\mathcal{M}$, in case $\mathcal{N}$ is smooth.

3. Some of the hypotheses in the definition of CS are not relevant for the construction of an obstruction theory; so we can work in a more general setting.

In this thesis we study a generalization of the notion of coherent systems. A generalized coherent system (GCS) is a triple (E, F, φ) where E, F are vector bundles on an algebraic curve C and φ is a morphism of vector bundles $E \to F$. Differently to the usual assumptions on coherent systems, in this contest the curve C is not necessarily smooth, but it is projective and Gorestein. A standard coherent system (E, V) induces a generalized coherent system where F is the the trivial vector bundle $V \otimes \mathcal{O}_C$ and φ is induced by the injection $V \hookrightarrow \mathrm{H}^0(E)$.

We fix a flat family of Gorestein projective curves over an algebraic stack $\mathcal{M}$ and we define a moduli stack $\mathcal{S}$ of families of GCS on curves in $\mathcal{M}$. Then we prove that $\mathcal{S}$ is algebraic in the sense of Artin. The stack $\mathcal{S}$ has a universal family $(\mathcal{E}, \mathcal{F}, \phi)$, where $\mathcal{E}, \mathcal{F}$ are vector bundles over the universal curve $\pi : \mathcal{S}' \to \mathcal{S}$ and $\phi : \mathcal{F} \to \mathcal{E}$ is a morphism of bundles. The curve $\mathcal{S}'$ is relatively Gorestein and we call $\omega \in \mathrm{Pic}(\mathcal{S}')$ its dualizing sheaf. There is a natural forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ from the GCS moduli stack $\mathcal{S}$ to the moduli stack $\mathcal{N}$ of pairs of vector bundles.

The central result of this thesis is the construction of a perfect relative obstruction theory for the morphism $G : \mathcal{S} \to \mathcal{N}$.

Theorem 3.2.3. *Let* $E^\bullet := \mathbb{R}\pi_*(\mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega[1])$. *There is a canonical morphism*

$$E^\bullet \longrightarrow \tau_{\geq -1} \mathrm{L}_G$$

which is a perfect relative obstruction theory for the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$.

The construction of a relative obstruction theory for the morphism $G : \mathcal{S} \to \mathcal{N}$ allows us to define a perfect obstruction theory for the moduli spaces of simple coherent systems (i.e. CS whose group of automorphisms is the scalars). As α-stable coherent systems are simple (see Proposition 1.5.10), such an obstruction theory induces a perfect obstruction theory for every moduli space of α-stable coherent systems.

Theorem 4.3.7. *Fix* $\alpha \in \mathbb{R}$. *Let* C *be a smooth, projective, genus* g *curve and let* (n, d, k) *be a suitable triple of positive integers. Let* $\beta := \beta(n, d, k)$ *be the Brill Noether number [BGPMN03, 2.7]. Then the moduli space of* α*-stable coherent systems of type* (n, d, k) *has a perfect obstruction theory of rank* β.

This thesis is organized in four chapters. The first chapter is a collection of preliminary results that we need in order to construct an obstruction theory for GCS. In this chapter we recall the definition and the basic properties of algebraic stacks; we give a short review of deformation theory; we recall the definition of obstruction theory; we recall the definition of coherent systems and the basic results about CS moduli spaces.

In the second chapter we introduce the moduli stack $\mathcal{S}$ of generalized coherent systems (for a family of projective Gorestein curves over an algebraic stack $\mathcal{M}$) and the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$, where $\mathcal{N}$ is the moduli stack of pairs of vector bundles; we prove that $\mathcal{S}$ is algebraic and that G is strongly representable; we prove that G factors as the composition of a closed embedding followed by a smooth morphism and we deduce an explicit formulation for the cotangent complex of G.

In the third chapter we construct a perfect relative obstruction theory for the morphism G:

$$E^{\bullet} \longrightarrow \tau_{\geq -1} \mathrm{L}_G ;$$

in particular we prove that the complex $E^{\bullet}$ is perfect of perfect amplitude contained in $[-1, 0]$.

In the fourth chapter we study the rigidification of $G : \mathcal{S} \to \mathcal{N}$ with respect to $\mathbb{C}^*$-automorphisms and we prove that the obstruction theory $E^{\bullet} \to \tau_{\geq -1} \mathrm{L}_G$ descends to such rigidification. Then we use this result to derive a perfect obstruction theory for moduli spaces of α-stable coherent systems.

The main results collected in this thesis are based on the definition of GCS, (E, F, φ), and in particular on the assumption that E and F are locally free sheaves. Indeed, it is our future intention to generalize Theorem 3.2.3 replacing GCS's with morphisms of torsion free sheaves $F \to E$. We would also like to investigate the case E and F are defined on surfaces or higher dimensional varieties. For this purpose some of the preliminary results introduced in Chapter 1 (for example Proposition 1.8.9) are formulated in a higher generality than strictly required.

Contents

Chapter 1

Notation and preliminaries

This chapter is a collection of preliminary results that we will use in this thesis in order to construct a perfect obstruction theory for moduli spaces of coherent systems. Most of the material collected here cannot be found in the standard texts about Algebraic Geometry. On account of this, we will provide detailed references everytime it is needed.

1.1 Notation

In this thesis $\mathbb{C}$ denotes the field of complex numbers; both the symbols (Sch) and $(Sch/\mathbb{C})$ denote the category of schemes of finite type over $\mathbb{C}$. If we say that T is a scheme, we always mean that $T \in (Sch/\mathbb{C})$; in particular, any scheme that we consider is noetherian.

We denote the category of finitely generated, commutative, unitary $\mathbb{C}$-algebras by $(Alg/\mathbb{C})$. We denote the category of local artinian $\mathbb{C}$-algebras by (Art) or by $(Art/\mathbb{C})$; if we say that R is an artinian ring, we always mean that $R \in (Art/\mathbb{C})$. Notice that the residue field of an artinian $\mathbb{C}$-algebra is canonically isomorphic to $\mathbb{C}$.

We denote the category of sets by $(Sets)$ and the category of abelian groups by $(Grps)$.

If $\mathcal{A}$ is any category, $\mathcal{A}_0$ denotes the collection of objects of $\mathcal{A}$ and $\mathcal{A}_1$ denotes the collection of morphisms of $\mathcal{A}$. Sometimes we omit the subscript and let $\mathcal{A}$ also denote the collection of objects of the category $\mathcal{A}$.

A groupoid is a category in which every morphism is an isomorphism. A rigid groupoid is a groupoid $\mathcal{G}$ such that for any $X, Y \in \mathcal{G}_0$ the collection of morphisms $\mathrm{Hom}(X, Y)$ is a set which either contains one element or it is empty. We denote the 2-category of groupoids by $(Grpds)$. We denote the 2-category of categories by (Cat). In a 2-category there are two different composition laws for 2-morphisms:

we denote the vertical composition by $\circ$ and the horizontal composition by $\star$. For a detailed reference about 2-categories and groupoids, you can consult [Sta17, Tag 003G].

Let X be a scheme. If we refer to X as a stack, we mean the stack h_X defined in Example 1.2.5.

We use a specific notation to denote the pullback of a morphism in (Sch). Namely, given a cartesian diagram in (Sch):

$$\begin{tikzcd} H \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\square"] & S \arrow[d, "g"] \\ T \arrow[r, "f"] & B, \end{tikzcd}$$

we write $(f)_g$ meaning f' and $(g)_f$ meaning g'.

Let $\mathcal{A}$ be an algebraic stack (see Definition 1.2.22) and let $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{A})$. We use the notation $\mathbb{V}(\mathcal{F})$ meaning $\mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F}$. For a detailed reference about the algebraic stack $\mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F}$ you can consult [Ols16, Section 10.2].

1.2 Stacks

In this section we provide a short review of the language concerning algebraic stacks. In particular, we recall the definition of stack, of (Artin) algebraic stack and of Deligne Mumford stack; we recall the definition of morphism of stacks and of fiber product of stacks; we give some details about descent and the stack condition; we recall the definition of coherent sheaf and of quasi-coherent sheaf on a stack. Good references for the material introduced in this section are the book of Martin Olsson [Ols16] and the Stacks-Project [Sta17].

We assume that the reader is familiar with the notion of étale topology on (Sch). References for Grothendieck topologies are [Ols16, Chapter 2] and [FGI$^+$05, Part 1].

Definition 1.2.1. A contravariant *pseudo functor* $\mathcal{A}$ from an ordinary category $\mathcal{C}$ to the 2-category $(Grpds)$ is given by the following data:

1. a map $\mathcal{A} : \mathcal{C}_0 \to (Grpds)_0$;
2. for every pair $X, Y \in \mathcal{C}_0$ and every morphism $f : X \to Y$ a morphism of groupoids $f^* : \mathcal{A}(Y) \to \mathcal{A}(X)$;
3. for every $X \in \mathcal{C}_0$ a 2-isomorphism $\alpha_X : \mathrm{id}_{\mathcal{A}(X)} \Rightarrow \mathrm{id}_X^*$;

4. for every pair of composable morphisms $f : X \to Y$, $g : Y \to Z$ of $\mathcal{C}$ a 2-isomorphism $\alpha_{g,f} : (g \circ f)^* \Rightarrow f^* \circ g^*$.

These data are subject to compatibility conditions, as described in [Tag 003G].

Definition 1.2.2. A *prestack* is a contravariant pseudo functor $\mathcal{A} : (Sch)^{op} \to (Grpds)$.
A *morphism of prestacks* $F : \mathcal{A} \to \mathcal{B}$ is given by the following data:

1. for any $X \in (Sch)$ a morphism of groupoids $F_X : \mathcal{A}(X) \to \mathcal{B}(X)$;
2. for any morphism $f : X \to Y$ in (Sch) a 2-isomorphism $F_f : F_X \circ f^*_{\mathcal{A}} \Rightarrow f^*_{\mathcal{B}} \circ F_Y$, such that
3. if $f : X \to Y$ and $g : Y \to Z$ are morphisms in (Sch), the following compatibility condition is satisfied:

$$(\mathrm{id}_{F_Z} \star \beta_{g,f}) \circ F_{g\circ f} = ((id_{f^*_{\mathcal{B}}} \star F_g) \circ (F_f \star \mathrm{id}_{g^*_{\mathcal{A}}})) \circ (\alpha_{g,f} \star \mathrm{id}_{F_X})\,,$$

where $\alpha_{g,f}$ and $\beta_{g,f}$ are the isomorphisms given in Definition 1.2.1(4) (respectively for $\mathcal{A}$ and for $\mathcal{B}$).

If $F, G : \mathcal{A} \to \mathcal{B}$ are morphisms of prestacks, a *2-isomorphism* or *natural equivalence* $\alpha : F \Rightarrow G$ is given by the following data:

1. for any $X \in (Sch)$ a natural equivalence $\alpha_X : F_X \Rightarrow G_X$, such that
2. for any morphism $f : X \to Y$ in (Sch) the following compatibility condition is satisfied:
$$G_f \circ (\alpha_X \star \mathrm{id}_{f^*_{\mathcal{A}}}) = (\mathrm{id}_{f^*_{\mathcal{B}}} \star \alpha_Y) \circ F_f\,.$$

With such a structure the category of prestacks is a 2-category in which every 2-morphism is an isomorphism.

In the book of Olsson prestacks are introduced as categories fibered in groupoids with a fixed choice of pullbacks. Indeed, there is an equivalence between pseudo functors and categories fibered in groupoids with a choice of pullbacks. For a detailed reference about the definition of stacks as categories fibered in groupoids you can consult [Ols16, Chapter 3].

Definition 1.2.3. Let $\mathcal{A}, \mathcal{B}$ be prestacks. We denote by $\mathrm{HOM}(\mathcal{A}, \mathcal{B})$ the groupoid whose objects are morphisms of prestacks $\mathcal{A} \to \mathcal{B}$ and whose isomorphisms are natural equivalences of morphisms of prestacks.

Remark 1.2.4. More generally, if $\mathcal{A}, \mathcal{B}$ are pseudo functors $(Sch)^{op} \to (Cat)$ we can define a category $\mathrm{HOM}(\mathcal{A}, \mathcal{B})$ in a similar way. ♢

Example 1.2.5. Let $X \in (Sch)$ and let $\mathrm{h}_X := (Sch/X)$ be the category of morphisms $T \to X$. The functor $p_X : \mathrm{h}_X \to (Sch), [T \to X] \mapsto T$, gives h_X a structure of fibered category in groupoids over (Sch). Moreover, any fiber $\mathrm{h}_X(T)$ is a set.

In the language of prestacks we have that

$$\begin{aligned} \mathrm{h}_X : (Sch) &\to (Grpds)\,, \\ T &\mapsto \mathrm{Hom}(T, X)\,, \end{aligned}$$

where the set $\mathrm{Hom}(T, X)$ is given the structure of a (rigid) groupoid in the natural way (i. e. $\mathrm{Hom}(f, g) = \varnothing$ for any $f, g \in \mathrm{Hom}(T, X)$ such that $f \neq g$, and $\mathrm{Hom}(f, f) = \{\mathrm{id}_f\}$ for any $f \in \mathrm{Hom}(T, X)$).

Definition 1.2.6. A prestack $\mathcal{A}$ is *represented by a scheme* X if there is an equivalence of prestacks

$$\mathcal{A} \to \mathrm{h}_X\ .$$

Proposition 1.2.7 (2-Yoneda Lemma)**.** *Let $\mathcal{A}$ be a prestack and let $X \in (Sch)$. There is a natural functor*

$$\xi : \mathrm{HOM}(\mathrm{h}_X, \mathcal{A}) \to \mathcal{A}(X)$$

sending a morphism of prestacks $F : \mathrm{h}_X \to \mathcal{A}$ to $F_X(\mathrm{id}_X)$.
The natural functor ξ is an equivalence.

Proof. Refer to [Ols16, Proposition 3.2.2] or to [Vis05, 3.6.2]. □

Remark 1.2.8. As usual if X is a scheme, we say "the prestack X" meaning h_X. Furthermore, If $\mathcal{A}$ is a prestack, we will often write $X \to \mathcal{A}$ for an object in $\mathcal{A}(X)$. This is justified by Proposition 1.2.7, which shows that we can also think of objects of $\mathcal{A}(X)$ as morphisms of prestacks $\mathrm{h}_X \to \mathcal{A}$. ♢

Definition 1.2.9. Let $F : \mathcal{A} \to \mathcal{Y}$ and $G : \mathcal{B} \to \mathcal{Y}$ be morphisms of prestacks. The 2-fiber product $\mathcal{A} \times_{\mathcal{Y}} \mathcal{B}$ is given by the following data:

1. a prestack $\mathcal{X}$;
2. a morphism $\bar{F} : \mathcal{X} \to \mathcal{B}$;
3. a morphism $\bar{G} : \mathcal{X} \to \mathcal{A}$;
4. a 2-isomorphism $\lambda : F \circ \bar{G} \Rightarrow G \circ \bar{F}$.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\bar{F}"] \arrow[d, "\bar{G}"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"] \arrow[ur, Rightarrow, "\lambda"] & \mathcal{Y} \end{tikzcd}$$

These data are subject to the following conditions:

1. for any scheme T we have that
$$\mathcal{X}(T) = \{(a, b, \sigma) \mid a \in \mathcal{A}(T), b \in \mathcal{B}(T), \sigma : F_T(a) \Rightarrow G_T(b)\}\,;$$
an isomorphism $(a, b, \sigma) \Rightarrow (a', b', \sigma')$ is a pair $(\alpha : a \Rightarrow a', \beta : b \Rightarrow b')$ such that $(\mathrm{id}_G \star \beta) \circ \sigma = \sigma' \circ (\mathrm{id}_F \star \alpha)$.

2. $\bar{F}_T(a, b, \sigma) = b$, $\bar{F}_T(\alpha, \beta) = \beta$; $\bar{G}_T(a, b, \sigma) = a$, $\bar{G}_T(\alpha, \beta) = \alpha$;

3. for any $t := (a, b, \sigma) \in \mathcal{X}(T)$ we have that $\lambda \star \mathrm{id}_t = \sigma$.

Remark 1.2.10. The data $(\mathcal{X}, \bar{F}, \bar{G}, \lambda)$ defining the 2-fiber product $\mathcal{A} \times_{\mathcal{Y}} \mathrm{B}$ have the following universal property: suppose that $\mathcal{H}$ is a prestack, that $L : \mathcal{H} \to \mathcal{A}$, $M : \mathcal{H} \to \mathrm{B}$ are morphisms of prestacks and that $\gamma : F \circ L \Rightarrow G \circ M$ is a 2-isomorphism. Then there exists a collection of data
$$(H : \mathcal{H} \to \mathcal{X}, \delta_1, \delta_2)\,,$$
where H is morphism of prestacks, $\delta_1 : L \Rightarrow \bar{G} \circ H$ and $\delta_2 : M \Rightarrow \bar{F} \circ H$ are 2-isomorphisms, and these data satisfy the condition
$$(\mathrm{id}_F \star \delta_1) \circ (\lambda \star \mathrm{id}_H) \circ (\mathrm{id}_G \star \delta_2^{-1}) = \gamma\,.$$
The data (H, δ_1, δ_2) are unique up to unique isomorphism. ◊

Definition 1.2.11. Let $\mathcal{A}$ be a prestack and let (X, R, s, t, m) be a groupoid in (Sch) [Sta17, Tag 0230]. We define a groupoid
$$\mathcal{A}(R \rightrightarrows X)$$
as follows. The objects of $\mathcal{A}(R \rightrightarrows X)$ are pairs (x, σ), where

1. $x \in \mathcal{A}(X)$;

2. $\sigma : s^*x \Rightarrow t^*x$ is an isomorphism in $\mathcal{A}(R)$;

such that the following condition is satisfied: let $p : R\,{}_s\!\times_t R \to R$ and $q : R\,{}_s\!\times_t R \to R$ be the canonical projections; then
$$\sigma \star \mathrm{id}_m = (\sigma \star \mathrm{id}_q) \circ (\sigma \star \mathrm{id}_p)\,.$$
In a picture:
$$\begin{tikzcd}
R\,{}_s\!\times_t R \arrow[rr, "q"] \arrow[dr, "m"] \arrow[dd, "p"'] & & R \arrow[dr, "t"] \arrow[dd, "s"] & \\
& R \arrow[rr, "t"] \arrow[dd, "s"'] & & X \arrow[dd, "x"] \\
R \arrow[rr, "t"] \arrow[dr, "s"'] & & X \arrow[dr, "x"] & \\
& X \arrow[rr, "x"'] \arrow[ur, Rightarrow, "\sigma"'] & & \mathcal{A}\,.
\end{tikzcd}$$

An isomorphism $(x, \sigma) \to (y, \tau)$ is an isomorphism $\alpha : x \Rightarrow y$ in $\mathcal{A}(X)$ such that

$$(\alpha \star \mathrm{id}_t) \circ \sigma = \tau \circ (\alpha \star \mathrm{id}_s) .$$

For an object $(x, \sigma) \in \mathcal{A}(R \rightrightarrows X)$ we refer to the isomorphism σ as *descent data* for the object x.

Remark 1.2.12. Let $\mathcal{A}$ be a prestack and let $f : X \to Y$ be a morphism of schemes. Consider the following data:

1. $R := X \times_Y X$;
2. $s : R \to X$ and $t : R \to X$ are the canonical projection;
3. $m : R\,{}_s\!\times_t R \to R$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd} R \arrow[r, "s"] & X \\ R\,{}_s\!\times_t R \arrow[u] \arrow[d] \arrow[r, dashed] & R = X \times_Y X \arrow[u] \arrow[d] \\ R \arrow[r, "t"] & X . \end{tikzcd}$$

Then (X, R, s, t, m) is a groupoid in (Sch) and we can define

$$\mathcal{A}(X \to Y) := \mathcal{A}(R \rightrightarrows X) .$$

There is a natural functor of groupoids

$$\epsilon : \mathcal{A}(Y) \to \mathcal{A}(X \to Y) .$$

Namely, given an object $y \in \mathcal{A}(Y)$ we have that $s^*(f^*y) = t^*(f^*y)$, since $f \circ s = f \circ t$. The functor ϵ is defined by sending y to (f^*y, id). $\diamondsuit$

Definition 1.2.13. Let X be a scheme. A *covering of* X (in the étale topology) is an étale surjective morphism of schemes $U \to X$.

Definition 1.2.14. A prestack $\mathcal{A}$ is a *stack* if for every scheme X and for every covering $U \to X$, the natural functor

$$\mathcal{A}(X) \to \mathcal{A}(U \to X)$$

is an equivalence of groupoids. Morphisms of stacks are morphism of prestacks and 2-isomorphisms of stacks are 2-isomorphisms of prestacks.

Remark 1.2.15. If $\mathcal{A} \to \mathcal{Y}$ and $\mathcal{B} \to \mathcal{Y}$ are morphisms of stacks, the prestack $\mathcal{A} \times_{\mathcal{Y}} \mathcal{B}$ is a stack [Ols16, Proposition 4.6.4]. $\Diamond$

Definition 1.2.16. (1) A morphism of stacks $\mathcal{A} \to \mathcal{B}$ is *strongly representable* if for any scheme T and for any morphism $T \to \mathcal{B}$ the fiber product $\mathcal{A} \times_{\mathcal{B}} T$ is represented by a scheme.

(2) Let $f : F \to G$ be a morphism of sheaves on (Sch) with the étale topology; f is strongly representable if for any scheme T and for any morphism $T \to G$ the fiber product $F \times_G T$ is represented by a scheme.

Definition 1.2.17. Let $F : \mathcal{A} \to \mathcal{B}$ be a strongly representable morphism of stacks. We say that F is *smooth* [resp. *étale*] if for every scheme T and morphism $T \to \mathcal{B}$ the morphism of schemes $\mathcal{A} \times_{\mathcal{B}} T \to T$ is smooth [resp. étale].

Definition 1.2.18. An *algebraic space* is a functor

$$X : (Sch)^{op} \to (Sets)$$

such that the following hold:

1. X is a sheaf with respect to the étale topology;
2. the diagonal morphism $\Delta : X \to X \times_{\mathbb{C}} X$ is strongly representable;
3. there exists a scheme U and a étale surjective morphism $U \to X$.

Remark 1.2.19. Let X be a sheaf with respect to the étale topology and assume that the diagonal $\Delta : X \to X \times_{\mathbb{C}} X$ is strongly representable; then if T is a scheme, any morphism $T \to X$ is strongly representable [Ols16, Lemma 5.1.9]. It therefore makes sense to talk about étale surjective morphism as in 1.2.18(3). $\Diamond$

Definition 1.2.20. Let $F : X \to Y$ be a morphism of algebraic spaces. We say that F is *smooth* [resp. *étale*] if there exist coverings $V \to Y$ and $U \to X$ such that the projection $U \times_Y V \to V$ is smooth [resp. étale].

Definition 1.2.21. A morphism of stacks $\mathcal{A} \to \mathcal{B}$ is *representable* if for any scheme T and for any morphism $T \to \mathcal{B}$ the fiber product $\mathcal{A} \times_{\mathcal{B}} T$ is represented by an algebraic space.

Definition 1.2.22. A stack $\mathcal{A}$ is an *algebraic stack* or *Artin stack* if the following hold:

1. the diagonal morphism $\Delta : \mathcal{A} \to \mathcal{A} \times_{\mathbb{C}} \mathcal{A}$ is representable;
2. there exists a scheme U and a smooth surjective morphism $U \to \mathcal{A}$.

A morphism of algebraic stacks is a morphism of stacks.

Remark 1.2.23. Let $\mathcal{A}$ be a stack and assume that the diagonal $\Delta : \mathcal{A} \to \mathcal{A} \times_{\mathbb{C}} \mathcal{A}$ is representable; then if T is a scheme, any morphism $T \to X$ is representable [Ols16, Remark 8.1.6]. It therefore makes sense to talk about smooth surjective morphism as in 1.2.22(2). ♢

Definition 1.2.24. A stack $\mathcal{A}$ is a *Deligne Mumford stack* if the following hold:

1. the diagonal morphism $\Delta : \mathcal{A} \to \mathcal{A} \times_{\mathbb{C}} \mathcal{A}$ is representable;
2. there exists a scheme U an étale surjective morphism $U \to \mathcal{A}$.

A morphism of Deligne Mumford stacks is a morphism of stacks.

Definition 1.2.25. A morphism of algebraic stacks $\mathcal{A} \to \mathcal{B}$ is a *Deligne Mumford morphism* if for any scheme T and for any morphism $T \to \mathcal{B}$ the fiber product $\mathcal{A} \times_{\mathcal{B}} T$ is a Deligne Mumford stack.

Remark 1.2.26. The ordinary definitions of quasi-coherent sheaves and of coherent sheaves on a scheme X can be reformulated with the language of 2-categories and of pseudo functors. Namely, there are two pseudo functors

$$\mathrm{Qcoh} : (Sch) \to (Cat)\,, \qquad \mathrm{Coh} : (Sch) \to (Cat)\,.$$

such that, for any scheme X, $\mathrm{Qcoh}(X)$ is the category of quasi-coherent sheaves on X and $\mathrm{Coh}(X)$ is the category of coherent sheaves on X. ♢

Definition 1.2.27. Let $\mathcal{A}$ be a stack; then $\mathcal{A}$ is a pseudo functor in groupoids and therefore it is a pseudo functor in categories. A *quasi-coherent sheaf on* $\mathcal{A}$ is a morphism of pseudo functors in categories $\mathcal{A} \to \mathrm{Qcoh}$; a *coherent sheaf on* $\mathcal{A}$ is a morphism of pseudo functors in categories $\mathcal{A} \to \mathrm{Coh}$. Therefore

$$\mathrm{Qcoh}(\mathcal{A}) = \mathrm{HOM}(\mathcal{A}, \mathrm{Qcoh})\,, \qquad \mathrm{Coh}(\mathcal{A}) = \mathrm{HOM}(\mathcal{A}, \mathrm{Coh})\,.$$

Notice that $\mathrm{HOM}(\mathcal{A}, \mathrm{Qcoh})$ and $\mathrm{HOM}(\mathcal{A}, \mathrm{Coh})$ are categories.

Remark 1.2.28. Let $\mathcal{A}$ be an algebraic stack and let $U \to \mathcal{A}$ be a smooth surjective morphism with U a scheme. Let $R := U \times_{\mathcal{A}} U$; then R is an algebraic space and there is an induced canonical groupoid in algebraic spaces: (U, R, s, t, m). The construction of (U, R, s, t, m) is analogous to the one given in Remark 1.2.12. ♢

Proposition 1.2.29. *Let $\mathcal{A}$ be an algebraic stack and let $U \to \mathcal{A}$ be a smooth surjective morphism with U a scheme. There are canonical equivalences of categories*

$$\mathrm{Qcoh}(U \to \mathcal{A}) \to \mathrm{Qcoh}(\mathcal{A})\,, \qquad \mathrm{Coh}(U \to \mathcal{A}) \to \mathrm{Coh}(\mathcal{A})\,.$$

Proof. That is a consequence of [Sta17, Tag 06WS]. □

Remark 1.2.30. Let $\mathcal{A}$ be an algebraic stack and let $U \to \mathcal{A}$ be a smooth surjective morphism with U a scheme. Define $R := U \times_{\mathcal{A}} U$; then R is an algebraic space, since $\mathcal{A}$ is algebraic (compare Remark 1.2.23). Denote by $s : R \to U$ and by $t : R \to U$ the canonical projections of the fiber product $R = U \times_{\mathcal{A}} U$. Denote by $p : R\,{}_s{\times}_t\,R \to R$ and by $q : R\,{}_s{\times}_t\,R \to R$ the canonical projection of the fiber product $R\,{}_s{\times}_t\,R$.
The smooth surjective morphism $U \to \mathcal{A}$ naturally induces a groupoid in algebraic spaces (U, R, s, t, m); the definition of groupoid in algebraic spaces can be found in [Sta17, Tag 043V]; for the construction of (U, R, s, t, m) compare Remark 1.2.12. In particural, recall that m is a morphism $m : R\,{}_s{\times}_t\,R \to R$.
Then $\mathrm{Qcoh}(\mathcal{A}) \cong \mathrm{Qcoh}(U \to \mathcal{A}) = \mathrm{Qcoh}(R \rightrightarrows U)$. That means that a sheaf $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{A})$ is induced by a pair $(\bar{\mathcal{F}}, \sigma)$, where $\bar{\mathcal{F}} \in \mathrm{Qcoh}(U)$ and $\sigma : s^*\bar{\mathcal{F}} \to t^*\bar{\mathcal{F}}$ is an isomorphism in $\mathrm{Qcoh}(R)$ such that $m^*\sigma = q^*\sigma \circ p^*\sigma$ ("cocycle condition"). ♢

1.3 Deformation Theory

In this section we provide a short review of some basic results concerning the deformation theory of schemes. In particular, we recall the definition of tangent and obstruction spaces for a natural transformation of functors $(Art) \to (Sets)$ and the definition of formally smooth natural transformation. Good references for the material introduced in this section are the book of Hartshorne [Har09], the book of Sernesi [Ser10] and the notes of Stefano Maggiolo [Mag10].
We recall that in this thesis we always denote the category of local artinian $\mathbb{C}$-algebras by (Art) or by $(Art/\mathbb{C})$.

Definition 1.3.1. 1. Let $R \in (Art)$. There is a canonical covariant functor $\mathrm{h}_R : (Art) \to (Sets)$ defined by $\mathrm{h}_R(A) := \mathrm{Hom}(R, A)$.

2. A covariant functor $F : (Art) \to (Sets)$ is called *representable* if there exists $R \in (Art)$ such that $F \cong \mathrm{h}_R$.

3. Let X be a scheme and let $x \in X(\mathbb{C})$. For any $A \in (Art)$ let $\mathrm{h}_{X,x}(A)$ be the set of morphisms of $\mathrm{Spec}\, A$ to X sending the closed point to x. That defines a covariant functor $\mathrm{h}_{X,x} : (Art) \to (Sets)$, $A \mapsto \mathrm{h}_{X,x}(A)$.

4. Let $R \in (Art)$. We denote by $\rho_R : \mathrm{Spec}\, \mathbb{C} \to \mathrm{Spec}\, R$ the canonical embedding of $\mathrm{Spec}\, \mathbb{C}$ into the only point of $\mathrm{Spec}\, R$.

Definition 1.3.2. Let $F, H : (Art) \to (Sets)$ be covariant functors. A natural transformation $H \Rightarrow F$ is called *strongly surjective* if for every $A \in (Art)$ the map

$H(A) \to F(A)$ is surjective and for every surjection $A \twoheadrightarrow B$ in (Art), the map $H(A) \to H(B) \times_{F(B)} F(A)$ is also surjective [Har09, p. 108].

Remark 1.3.3. Let $G : X \to Y$ be a morphism of schemes. Let $x \in X(\mathbb{C})$ and let $y := G(x)$. Composing with G gives a natural transformation $G_* : \mathrm{h}_{X,x} \Rightarrow \mathrm{h}_{Y,y}$. $\diamondsuit$

Proposition 1.3.4 ([Har09, Exercise 15.4]). *Let $G : X \to Y$ be a flat morphism of schemes, let $x \in X(\mathbb{C})$, and let $y := G(x)$. Then the natural transformation $G_* : \mathrm{h}_{X,x} \Rightarrow \mathrm{h}_{Y,y}$ is formally smooth if and only if the morphism $G : X \to Y$ is smooth at x.* $\square$

Definition 1.3.5. A *semismall extension* in (Art) is a surjective map $A \twoheadrightarrow B$ whose kernel I satisfies $I \cdot m_A = 0$.

Definition 1.3.6. Let $S \to M$ be a map in $(Sets)$. If G_1 and G_2 are two groups such that G_1 acts on S and M maps in G_2 , we say that

$$G_1 \to S \to M \to G_2$$

is a *sequence of groups and sets*.

1. The sequence of groups and sets $G_1 \to S \to M \to G_2$ *is exact* if
   (a) two points of S have the same image in M if and only if they are in the same orbit;
   (b) an element $m \in M$ goes to zero in G_2 if and only if $f^{-1}(m) \neq \varnothing$.
2. The sequence of groups and sets

$$0 \to G_1 \to S \to M \to G_2$$

   is exact[1] if
   (a) $G_1 \to S \to M \to G_2$ is exact;
   (b) the group G_1 acts freely on S.

Definition 1.3.7. Let $H \Rightarrow F$ be a natural transformation of covariant functors $(Art) \to (Sets)$. We say that two vector spaces T^1 and T^2 are *the tangent space and an obstruction space for $H \Rightarrow F$* if for every semismall extension $0 \to I \to A \to B \to 0$ in (Art) there is an exact sequence of groups and sets

$$0 \to T^1 \otimes_{\mathbb{C}} I \to H(A) \to H(B) \times_{F(B)} F(A) \to T^2 \otimes_{\mathbb{C}} I$$

which is functorial in the semismall extension.

[1] Notice that this is not a standard notation.

Remark 1.3.8. If $H \Rightarrow F$ is formally smooth, then $T^2 = \{0\}$ is an obstruction space for $H \Rightarrow F$, that is the natural transformation $H \Rightarrow F$ is unobstructed. $\Diamond$

Remark 1.3.9. Let $G : X \to Y$ be a morphism of schemes; let $x \in X$, $y := G(x)$ and let $G_* : \mathrm{h}_{X,x} \Rightarrow \mathrm{h}_{Y,y}$ be the natural transformation induced by G at x. Let $0 \to I \to A \to B \to 0$ be a semismall extension in (Art) and let $\iota : \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced closed embedding. An element $(b, a) \in \mathrm{h}_{X,x}(B) \times_{\mathrm{h}_{Y,y}(B)} \mathrm{h}_{Y,y}(A)$ is a pair of morphisms $b : \operatorname{Spec} B \to X$, $a : \operatorname{Spec} A \to Y$, such that

1. the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "b"] \arrow[d, "\iota"] & X \arrow[d, "G"] \\ \operatorname{Spec} A \arrow[r, "a"] & Y; \end{tikzcd}$$

2. the set theoretic image of $b : \operatorname{Spec} B \to X$ is the point x.

An element $a' \in \mathrm{h}_{X,x}(A)$ is a morphism $a' : \operatorname{Spec} A \to X$ such that the set theoretic image of a' is x. The pair (b, a) is induced by a' if and only if $b = a' \circ \iota$ and $a = G \circ a'$. A morphism $a' \in \mathrm{h}_{X,x}(A)$ which induces (b, a) is called a *lifting* of (b, a).
Let TAN and OB be abelian groups; a sequence of groups and sets

$$0 \to TAN \to \mathrm{h}_{X,x}(A) \to \mathrm{h}_{X,x}(B) \times_{\mathrm{h}_{Y,y}(B)} \mathrm{h}_{Y,y}(A) \to OB$$

is exact if and only if

1. a pair $(b, a) \in \mathrm{h}_{X,x}(B) \times_{\mathrm{h}_{Y,y}(B)} \mathrm{h}_{Y,y}(A)$ goes to zero in OB if and only if it has a lifting $a' \in \mathrm{h}_{X,x}(A)$;

2. the action of TAN on $\mathrm{h}_{X,x}(A)$ is free;

3. two morphisms in $\mathrm{h}_{X,x}(A)$ induce the same pair in $\mathrm{h}_{X,x}(B) \times_{\mathrm{h}_{Y,y}(B)} \mathrm{h}_{Y,y}(A)$ if and only if they are in the same orbit. $\Diamond$

Definition 1.3.10. Let $\mathcal{X}$ be an algebraic stack and let $x \in \mathcal{X}(\mathbb{C})$. We can define a pseudo functor

$$\mathrm{h}_{\mathcal{X},x} : (Art) \to (Grpds)$$

by $\mathrm{h}_{\mathcal{X},x}(A) = \{(a, \alpha) \mid a : \operatorname{Spec} A \to \mathcal{X}; \alpha : a \circ \rho_A \Rightarrow x \text{ is a 2-isomorphism}\}$. An isomorphism $(a, \alpha) \to (a', \alpha')$ in $\mathrm{h}_{\mathcal{X},x}(A)$ is an isomorphism $\gamma : a \Rightarrow a'$ in $\mathcal{X}(\operatorname{Spec} A)$ such that $\alpha' \circ (\gamma \star \mathrm{id}_{\rho_A}) = \alpha$.

Remark 1.3.11. Let $G : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and let $x \in \mathcal{X}(\mathbb{C})$, $y := G(x)$. Let $0 \to I \to A \to B \to 0$ be a semismall extension in (Art) and let $\iota : \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced closed embedding. Let $(b, \beta) \in \mathrm{h}_{\mathcal{X},x}(B)$ and $(a, \alpha) \in \mathrm{h}_{\mathcal{Y},y}(A)$, and let $\gamma : a \circ \iota \Rightarrow G \circ b$ be a 2-isomorphism such that $(\mathrm{id}_G \star \beta) \circ (\gamma \star \mathrm{id}_{\rho_B}) = \alpha$.
A *lifting of* (b, β) *and* (a, α) is a triple (l, δ, ϵ) such that

1. $l : \operatorname{Spec} A \to \mathcal{X}$ is a morphism;

2. $\delta : G \circ l \Rightarrow a$ is a 2-isomorphism;

3. $\epsilon : l \circ \iota \Rightarrow b$ is a 2-isomorphism;

4. $(\mathrm{id}_G \star \epsilon) = \gamma \circ (\delta \star \mathrm{id}_\iota)$.

Notice that a lifting (l, δ, ϵ) of (b, β) and (a, α) naturally induces an object $(l, \lambda) \in \mathrm{h}_{\mathcal{X},x}(A)$ by defining

$$\lambda := \beta \circ (\epsilon \star \rho_B)\,.$$

An isomorphism between two liftings (l, δ, ϵ) and (l', δ', ϵ') is an isomorphism $l \Rightarrow l'$ in $\mathcal{X}(\operatorname{Spec} A)$ with the natural compatibility conditions.
Hence, we have defined a groupoid $\mathcal{L} := \mathcal{L}(b, \beta, a, \alpha, \gamma)$ that we call the *groupoid of liftings of* (b, β) *and* (a, α). $\diamondsuit$

If $G : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks and $x \in \mathcal{X}(\mathbb{C})$, it is still possible to define tangent and obstruction spaces for the induced morphism of pseudo functors $G_* : \mathrm{h}_{\mathcal{X},x} \to \mathrm{h}_{\mathcal{Y},G(x)}$. However the language to use is slightly more complicated. In the next result we prove that if the morphism G is strongly representable, then tangent and obstruction spaces for G_* can be computed assuming that $\mathcal{X}$ and $\mathcal{Y}$ are schemes.

Proposition 1.3.12. *Let $G : \mathcal{X} \to \mathcal{Y}$ be a strongly representable morphism of algebraic stacks, let $x \in \mathcal{X}(\mathbb{C})$ and let $y := G(x)$. Let $0 \to I \to A \to B \to 0$ be a semismall extension in (Art) and let $\iota : \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced closed embedding. Let $(b, \beta) \in \mathrm{h}_{\mathcal{X},x}(B)$ and $(a, \alpha) \in \mathrm{h}_{\mathcal{Y},y}(A)$, and let $\gamma : a \circ \iota \Rightarrow G \circ b$ be a 2-isomorphism such that $(\mathrm{id}_G \star \beta) \circ (\gamma \star \mathrm{id}_{\rho_B}) = \alpha$. Let X be the 2-fiber product $\operatorname{Spec} A \times_{\mathcal{Y}} \mathcal{X}$, and let $\bar{G} : X \to \operatorname{Spec} A$ be the canonical projection and $\bar{b} : \operatorname{Spec} B \to X$ be the morphism defined by $(\iota, b, \gamma) \in X(\operatorname{Spec} B)$. Then the groupoid of liftings of (b, β) and (a, α), $\mathcal{L} = \mathcal{L}(b, \beta, a, \alpha, \gamma)$, is canonically equivalent to the set*

$$L := \{ f : \operatorname{Spec} A \to X \,|\, \bar{G} \circ f = \mathrm{id}\,;\ f \circ \iota = \bar{b} \}\,.$$

Proof. The following pictures help understanding the definitions of $\mathcal{L}$ and L:

$$
\begin{tikzcd}
\operatorname{Spec} B \arrow[r, "b"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\Uparrow\epsilon", near start] & \mathcal{X} \arrow[d, "G"] \\
\operatorname{Spec} A \arrow[r, "a"] \arrow[ur, "l", color=red] \arrow[ur, phantom, "\delta\Downarrow"', near end] & \mathcal{Y};
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Spec} B \arrow[dr, "\bar{b}"] \arrow[drr, bend left, "b"] \arrow[dd, hook, "\iota"'] & & \\
& X \arrow[r] \arrow[dd, "\bar{G}"] \arrow[ddr, phantom, "\square"] & \mathcal{X} \arrow[dd, "G"] \\
\operatorname{Spec} A \arrow[ur, "f", color=red] \arrow[dr, "\mathrm{id}"'] & & \\
& \operatorname{Spec} A \arrow[r, "a"] & \mathcal{Y}.
\end{tikzcd}
$$

We claim that $\mathcal{L}$ is canonically equivalent to the following groupoid

$$\mathcal{L}' := \{(l, \delta) \,|\, l : \operatorname{Spec} A \to \mathcal{X}, (\delta : G \circ l \Rightarrow a) \in \mathcal{Y}(\operatorname{Spec} A)_1\}\,;$$

i.e. for any $(l, \delta, \epsilon) \in \mathcal{L}$, ϵ is induced by l and δ. Indeed, by Remark 1.3.11(4) we have that

$$(\mathrm{id}_G \star \epsilon) = \gamma \circ (\delta \star \mathrm{id}_\iota)\,.$$

But G is representable and therefore $G_{\operatorname{Spec} A} : \mathcal{X}(\operatorname{Spec} A) \to \mathcal{Y}(\operatorname{Spec} A)$ is faithful. So ϵ is induced by $(\mathrm{id}_G \star \epsilon)$. That proves the claim.
There is a canonical faithful morphism of groupoids

$$\begin{aligned} \mathcal{L}' &\to X(\operatorname{Spec} A)\,, \\ (l, \delta) &\mapsto (\mathrm{id}_{\operatorname{Spec} A}, l, \delta)\,. \end{aligned}$$

Hence $\mathcal{L}'$ is a rigid groupoid. To conclude the proof it's enough to observe that an object $(\mathrm{id}_{\operatorname{Spec} A}, l, \delta) \in X(\operatorname{Spec} A)$ induces an object $f \in L$ by definition of 2-fiber product. □

1.4 Obstruction theories

In this section we introduce the notion of obstruction theory for a Deligne Mumford stack and the notion of relative obstruction theory for a Deligne Mumford morphism of algebraic stacks. A reference for the material introduced in this section is the original paper by Behrend and Fantechi [BF97].

Definition 1.4.1. Let $\mathcal{X}$ be an algebraic stack and let $E^\bullet \in \operatorname{obj} D^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X})$. We say that $E^\bullet$ is perfect of perfect amplitude contained in $[-1, 0]$ if it is locally isomorphic to a complex of locally free sheaves.

Definition 1.4.2. Let $\mathcal{X} \to \mathcal{Y}$ be a Deligne Mumford morphism of algebraic stacks. Let $E^\bullet \in \operatorname{obj} D^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X})$. Let $\mathrm{L}_{\mathcal{X}/\mathcal{Y}}$ be the cotangent complex of $\mathcal{X}$ over $\mathcal{Y}$. A morphism

$$\xi : E^\bullet \longrightarrow \tau_{\geq -1} \mathrm{L}_{\mathcal{X}/\mathcal{Y}}$$

in $D^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X})$ is called an *obstruction theory for* $\mathcal{X}/\mathcal{Y}$ (or equivalently an obstruction theory for the morphism $\mathcal{X} \to \mathcal{Y}$) if

1. $h^0(\xi)$ is an isomorphism;

2. $h^{-1}(\xi)$ is surjective.

If $\mathcal{Y} = \operatorname{Spec}\mathbb{C}$, hence $\mathcal{X}$ is a Deligne Mumford stack, the morphism $\xi : E^\bullet \to \tau_{\geq -1}\,\mathrm{L}_{\mathcal{X}}$ is callad an *(absolute) obstruction theory for the stack* $\mathcal{X}$.
We say that an obstruction theory $E^\bullet \to \tau_{\geq -1}\,\mathrm{L}_{\mathcal{X}/\mathcal{Y}}$ is *perfect*, if the complex $E^\bullet$ is perfect of perfect amplitude contained in $[-1,0]$.

The following Lemma is an operative tool to check if a morphism $E^\bullet \to \tau_{\geq -1}\,\mathrm{L}_{\mathcal{X}/\mathcal{Y}}$ in the derived category is an obstruction theory.

Lemma 1.4.3. *Let* $\xi : E^\bullet \longrightarrow \tau_{\geq -1}\,\mathrm{L}_{\mathcal{X}/\mathcal{Y}}$ *be a morphism in* $D^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X})$*; the following conditions are equivalent.*

(a) ξ *is an obstruction theory for* $\mathcal{X}/\mathcal{Y}$*.*

(b) For any semismall extension $0 \to I \to A \to B \to 0$ *in* $(Art/\mathbb{C})$ *and for any pair of morphisms* $g_0 : \operatorname{Spec} B \to \mathcal{X}$ *and* $h : \operatorname{Spec} A \to \mathcal{Y}$ *making the following diagram commute:*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "g_0"] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} A \arrow[r, "h"] & \mathcal{Y} \end{tikzcd},$$

there exists a unique element $ob(g_0) \in \operatorname{Ext}^1(\mathrm{L}\, g_0^*\, E^\bullet, I)$*, called the obstruction class of* g_0*, such that*

1. g_0 *can be extended to a morphism* $g : \operatorname{Spec} A \to \mathcal{X}$ *over* $\mathcal{Y}$ *if and only if* $ob(g_0) = 0$*;*
2. *if* $ob(g_0) = 0$*, then the set of extension classes of* g_0 *is a torsor over the group* $\operatorname{Ext}^0(\mathrm{L}\, g_0^*\, E^\bullet, I)$*.*

Proof. That is a consequence of [BF97, Section 4]. □

1.5 Coherent Systems

In this secion we recall the definition of coherent systems, as it is given in [BGPMN03]. We provide a short summary of the results concerning coherent systems moduli spaces. In particular, we recall the definition of α-stability for a coherent system and we recall that moduli spaces of α-stable coherent systems have expected dimension equal to the Brill-Noether number $\beta(n,d,k)$. References for the material introduced in this section are [BGPMN03] and [HE98].

Definition 1.5.1. A *coherent system* of type (n,d,k) on a smooth projective curve C is a pair (E,V) where E is an algebraic vector bundle over C of rank n and degree d, and V is a linear subspace of dimension k of the space of sections $\mathrm{H}^0(C,E)$.

A *subsystem of* (E,V) is a pair (E',V') where E' is a subbundle of E and V' is a subspace of $V \cap \mathrm{H}^0(E')$. A subsystem (E',V') is called a *proper subsystem* if E' is non-zero and $(E',V') \neq (E,V)$

Definition 1.5.2. Fix $\alpha \in \mathbb{R}$. Let (E,V) be a coherent system of type (n,d,k) with $n>0$. The α-slope $\mu_\alpha(E,V)$ is defined by

$$\mu_\alpha(E,V) := \frac{d}{n} + \alpha\frac{k}{n}\,.$$

We say that (E,V) *is α-stable* if

$$\mu_\alpha(E',V') < \mu_\alpha(E,V)$$

for all proper subsystem (E',V') of (E,V).
We define *α-semistability* by replacing the above strict inequality with a weak inequality.

We denote the moduli space of α-stable coherent systems of type (n,d,k) by $G(\alpha) = G(\alpha;n,d,k)$. The GIT construction of these moduli spaces has been given in [Le 93] and [KN95].

Definition 1.5.3 ([KN95, 2.5])**.** Let $\mathcal{A}$ be the abelian category whose objects are arbitrary (sheaf) maps $\varphi : V \otimes \mathcal{O}_C \to E$ where V is a finite dimensional vector space and E is any coherent sheaf. A morphism in $\mathcal{A}$ from $\varphi : V \otimes \mathcal{O}_C \to E$ to $\xi : W \otimes \mathcal{O}_C \to F$ is given by a linear map $f : V \to W$ and a sheaf map $g : E \to F$ such that the following diagram commutes:

$$\begin{tikzcd} V \otimes \mathcal{O}_C \arrow[r, "\varphi"] \arrow[d, "f\otimes \mathrm{id}"] & E \arrow[d, "g"] \\ W \otimes \mathcal{O}_C \arrow[r, "\xi"] & F\,. \end{tikzcd}$$

Remark 1.5.4. By replacing a coherent system (E,V) by its "evaluation map" $V \otimes \mathcal{O}_C \to E$ we get an equivalence between the category of coherent systems and a full subcategory of $\mathcal{A}$. An object $\varphi : V \otimes \mathcal{O}_X \to E$ of $\mathcal{A}$ is induced by a coherent system if and only if E is locally free and $\mathrm{H}^0(\varphi) : V \to \mathrm{H}^0(E)$ is injective. Note that the subcategory of $\mathcal{A}$ induced by the category of coherent systems is not abelian. $\Diamond$

We may extend the notions of α-semistability and α-stability to any object of the category $\mathcal{A}$.

Definition 1.5.5. Fix $\alpha \in \mathbb{R}$. Let $\varphi : V \otimes \mathcal{O}_C \to E$ be an object of $\mathcal{A}$ of type (n, d, k) with $n > 0$, let $\xi : W \otimes \mathcal{O}_C \to F$ be a subobject of φ and let $\zeta : H \otimes \mathcal{O}_C \to G$ be a quotient of φ. Define

$$\mu_\alpha(\xi \hookrightarrow \varphi) := (d + \alpha k) \operatorname{rk} F - n(\deg F + \alpha \dim W)\,,$$

and

$$\mu_\alpha(\varphi \twoheadrightarrow \zeta) := (d + \alpha k) \operatorname{rk} G - n(\deg G + \alpha \dim H)\,.$$

We say that $\varphi : V \otimes \mathcal{O}_C \to E$ is *α-semistable* if for all subobjects $\xi : W \otimes \mathcal{O}_C \to F$ we have that $\mu_\alpha(\xi \hookrightarrow \varphi) \geq 0$, and that φ is *α-stable* if the inequality is strict for all proper subobjects.

Remark 1.5.6. Let $\varphi : V \otimes \mathcal{O}_C \to E$ be an object of $\mathcal{A}$ of type (n, d, k) with $n > 0$, and let

$$0 \to \xi \to \varphi \to \zeta \to 0$$

be an exact sequence in $\mathcal{A}$. Then

$$\mu_\alpha(\xi \hookrightarrow \varphi) > 0 \;\Leftrightarrow\; \mu_\alpha(\varphi \twoheadrightarrow \zeta) < 0\,. \qquad \Diamond$$

Proposition 1.5.7 ([KN95, Lemma 2.5])**.** *An object $\varphi : V \otimes \mathcal{O}_C \to E$ of $\mathcal{A}$ of type (n, d, k) is α-semistable (resp. α-stable) if and only if it is (the evaluation map of) an α-semistable (resp. α-stable) coherent system.* $\square$

Definition 1.5.8. A coherent system $\varphi : V \otimes \mathcal{O}_C \to E$ is *simple* if $\operatorname{End}(\varphi) = \mathbb{C}$.

Lemma 1.5.9. *Let $\varphi : V \otimes \mathcal{O}_C \to E$ be an α-stable coherent system, and let $f \in \operatorname{End}(\varphi)$ be such that $f \neq 0$. Then $f \in \operatorname{Aut}(\varphi)$.*

Proof. Since $\mathcal{A}$ is an abelian category, we can define the objects $\xi := \operatorname{Ker}(f)$ and $\zeta := \operatorname{Coker}(f)$. Then we have a long exact sequence

$$0 \to \xi \to \varphi \xrightarrow{f} \varphi \to \zeta \to 0\,,$$

which splits in two short exact sequences:

$$0 \to \xi \to \varphi \to \rho \to 0\,,$$
$$0 \to \rho \to \varphi \to \zeta \to 0\,.$$

Since $f \neq 0$, we have that $\xi \hookrightarrow \varphi$ is not an isomorphism, $\varphi \twoheadrightarrow \zeta$ is not an isomorphism, and $\rho \neq 0$.
If $\xi \neq 0$, then ξ is a proper subobject of φ and $\mu_\alpha(\xi \hookrightarrow \varphi) > 0$. Hence $\mu_\alpha(\varphi \twoheadrightarrow \rho) < 0$ and, in particular, $\rho \hookrightarrow \varphi$ is not an isomorphism. That is a contradiction, since it implies that ρ is a proper subobject of φ and that $\mu_\alpha(\rho \hookrightarrow \varphi) < 0$. So $\xi = 0$.
On the other hand, if $\zeta \neq 0$ then ρ is a proper subobject of φ and $\mu_\alpha(\rho \hookrightarrow \varphi) > 0$. Then $\varphi \twoheadrightarrow \rho$ is not an isomorphism, otherwise $\mu_\alpha(\varphi \twoheadrightarrow \rho) = \mu_\alpha(\rho \hookrightarrow \varphi) = 0$. That is a contradiction, since $\xi = 0$. So $\zeta = 0$, too.
Since $\operatorname{Ker} f = \operatorname{Coker} f = 0$, we deduce that f is an automorphism of φ. □

Proposition 1.5.10. *Every α-stable coherent system is simple.*

Proof. Let $\varphi : V \otimes \mathcal{O}_C \to E$ be an α-stable coherent system. We prove that if $f \in \operatorname{Aut}(\varphi)$ then $f \in \mathbb{C} \smallsetminus \{0\}$. By Lemma 1.5.9 that implies that $\operatorname{End}(\varphi) = \mathbb{C}$.
Let $f \in \operatorname{Aut}(\varphi)$ and consider the commutative $\mathbb{C}$-algebra $\mathbb{C}[f]$, where $f \cdot f := f \circ f$. We have injective morphisms of $\mathbb{C}$-vector spaces

$$\mathbb{C}[f] \hookrightarrow \operatorname{End}(\varphi) \hookrightarrow \operatorname{End}(V) \times \operatorname{End}(E)\,.$$

Being $\operatorname{End}(V) \times \operatorname{End}(E)$ a finite dimensional $\mathbb{C}$-vector space, also $\mathbb{C}[f]$ is finite dimensional. Hence $\mathbb{C}[f]$ is a field; indeed, if $g \in \mathbb{C}[f] \smallsetminus \mathbb{C}$ then $g \in \operatorname{Aut}(\varphi)$ by Lemma 1.5.9 and there exists a minimal integer $n \geq 1$ such that

$$\lambda_0 + \lambda_1 g + \cdots + \lambda_n g^n = 0,$$

where $\lambda_i \in \mathbb{C}$ and $\lambda_0 \neq 0$ (otherwise n is not minimal); therefore $g^{-1} = \lambda_0^{-1}(-\lambda_1 - \lambda_2 g - \cdots - \lambda_n g^{n-1}) \in \mathbb{C}[f]$.
Since $\mathbb{C}$ is algebraically close and $\mathbb{C}[f]$ is a finite extension of $\mathbb{C}$, we have that $\mathbb{C}[f] = \mathbb{C}$. Hence $f \in \mathbb{C} \smallsetminus \{0\}$. □

Definition 1.5.11 ([BGPMN03, 2.7]). For any (n, d, k), the Brill-Noether number $\beta(n, d, k)$ is defined by

$$\beta(n, d, k) = n^2(g-1) + 1 - k(k - d + n(g-1))\,.$$

Remark 1.5.12. Given two coherent systems (E, V) and (E', V'), one defines the extension groups

$$\operatorname{Ext}^i((E, V), (E', V'))\,,$$

as described in [HE98]. The following result is about the infinitesimal properties of the moduli spaces $G(\alpha)$. ♢

Theorem 1.5.13 ([HE98, Théorème 3.12]). *Let (E, V) be an α-stable coherent system.*

1. *If* $\mathrm{Ext}^2((E,V),(E,V)) = 0$*, then every moduli space of* α*-stable coherent systems is smooth in a neighborhood of the point defined by* (E,V)*.*

2. *The Zariski tangent space to the moduli space at the point defined by* (E,V) *is isomorphic to* $\mathrm{Ext}^1((E,V),(E,V))$*.*

Let (E,V) *be an* α*-stable coherent system of type* (n,d,k)*. Then*

$$\dim \mathrm{Ext}^1((E,V),(E,V)) = \beta(n,d,k) + \dim \mathrm{Ext}^2((E,V),(E,V))\,.$$ [2]

Corollary 1.5.14 ([BGPMN03, 3.6])**.** *Every irreducible component* G *of every moduli space* $G(\alpha; n,d,k)$ *has dimension*

$$\dim G \geq \beta(n,d,k)\,,$$

and the expected dimension of $G(\alpha; n,d,k)$ *is equal to* β*.*

1.6 Comodules

In this section we recall the definition of coalgebra and comodule and we provide a short review of some standard results for comodules. In particular, we recall that every affine group scheme naturally induces a coalgebra and every representation of an affine group scheme on a vector space induces a comodule. A reference for the material introduced in this section is [Mil12].

Definition 1.6.1. A *coalgebra* (over $\mathbb{C}$) is a $\mathbb{C}$-vector space A with a pair of $\mathbb{C}$-linear maps

$$\Delta : A \to A \otimes A\,, \qquad\qquad \epsilon : A \to \mathbb{C}\,,$$

satisfying the following conditions:

1. (co-associativity) $(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta\,;$
2. (co-identity) $(\mathrm{id}_A \otimes \epsilon) \circ \Delta = \mathrm{id}_A$, and $(\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A$.

A *morphism of coalgebras* is a $\mathbb{C}$-linear map $f : A \to B$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Delta_A"] & B \arrow[d, "\Delta_B"] \\ A \otimes A \arrow[r, "f\otimes f"] & B \otimes B\,; \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\epsilon_A"] & B \arrow[d, "\epsilon_B"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C}\,. \end{tikzcd}$$

[2]This relation has been proven in [BGPMN03, 3.5].

Definition 1.6.2. Let (A, Δ, ϵ) be a coalgebra over $\mathbb{C}$. An *A-comodule* is a $\mathbb{C}$-linear map $\rho : V \to V \otimes A$, where V is a $\mathbb{C}$-vector space, such that the following diagrams commute:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes A & \xrightarrow{\rho \otimes \mathrm{id}_A} & V \otimes A \otimes A\,; \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \| & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \epsilon} \\ V & \xrightarrow{\sim} & V \otimes \mathbb{C}\,. \end{array}$$

A morphism of comodules $(V, \rho) \to (W, r)$ is a morphism of $\mathbb{C}$-vector spaces $f : V \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle r} \\ V \otimes A & \xrightarrow{f \otimes \mathrm{id}_A} & W \otimes A\,. \end{array}$$

A comodule is said to be finite-dimensional if it is finite-dimensional as a vector space.

Let $\rho : V \to V \otimes A$ be an A-comodule. A $\mathbb{C}$-linear subspace W of V is said a *subcomodule of* V if $\rho(W) \subseteq W \otimes A$; in this case $\rho_{|W} : W \to W \otimes A$ is a comodule structure on W.

Definition 1.6.3. A *bi-algebra* A (over $\mathbb{C}$) is a $\mathbb{C}$-vector space with compatible structures of $\mathbb{C}$-algebra and of coalgebra over $\mathbb{C}$. In detail, a bi-algebra over $\mathbb{C}$ is a coalgebra (A, Δ, ϵ) such that

1. A is a $\mathbb{C}$-algebra;
2. $\Delta : A \to A \otimes A$ is a homomorphism of $\mathbb{C}$-algebras;
3. $\epsilon : A \to \mathbb{C}$ is a homomorphism of $\mathbb{C}$-algebras.

Remark 1.6.4. (a) Let A be a bi-algebra and let V be a $\mathbb{C}$-vector space. Then there is a canonical $\mathbb{C}$-linear map

$$1_V : V \to V \otimes A\,,$$

given by $v \mapsto v \otimes 1$. The map 1_V induces a structure of A-comodule on V; we say that $1_V : V \to V \otimes A$ is a *trivial comodule.*

(b) Let A be a bi-algebra and let $\rho : V \to V \otimes A$ be an A-comodule. The comodule structure of (V, ρ) canonically induces a $\mathbb{C}$-linear subspace of V:

$$V_t := \ker(\rho - 1_V).$$

V_t is is the biggest trivial comodule which is a subcomodule of V; we call V_t *the trivial subcomodule of* (V, ρ). ◊

Definition 1.6.5. Let V be a $\mathbb{C}$-vector space. We have a group valued functor

$$\mathrm{GL}(V) : (Alg/\mathbb{C}) \to (Grps)\,,$$

given by $R \mapsto \mathrm{Aut}_R(V \otimes R)$ where $\mathrm{Aut}_R(V \otimes R)$ is the group of R-linear automorphisms on $V \otimes R$.
A *linear representation* of a group scheme G on a vector space V is a morphism of group valued functors

$$r : G \to \mathrm{GL}(V)\,.$$

Proposition 1.6.6 ([Mil12, Proposition VIII.6.1])**.** *Let G be an affine group scheme and let $\mathcal{O}(G) := \Gamma(G, \mathcal{O}_G)$. Then $\mathcal{O}(G)$ is a bi-algebra.*
Let V be a $\mathbb{C}$-vector space.There is a canonical equivalence between the linear representations $r : G \to \mathrm{GL}(V)$ and the $\mathcal{O}(G)$-comodules $\rho : V \to V \otimes \mathcal{O}(G)$.

Sketch of proof. The structure of coalgebra on $\mathcal{O}(G)$ is naturally induced by the structure of group scheme on G. By Yoneda Lemma, a linear representation $r : G \to \mathrm{GL}(V)$ is equivalent to an R-linear automorphism $V \otimes \mathcal{O}(G) \to V \otimes \mathcal{O}(G)$, which by the universality of the tensor product is uniquely determined by a $\mathbb{C}$-linear morphism $\rho : V \to V \otimes \mathcal{O}(G)$. This is the associated comodule. □

Proposition 1.6.7. *Let (A, Δ, ϵ) be the bi-algebra given by $A := \mathbb{C}[t, t^{-1}]$, $\Delta(t) = t \otimes t$ and $\epsilon(t) = 1$; let $\rho : V \to V \otimes A$ be an A-comodule. Then V decomposes as a direct sum of $\mathbb{C}$-linear subspaces*

$$V = V_t \oplus W\,,$$

where V_t is the trivial subcomodule of (V, ρ); i. e.

1. *$\rho_{|V_t} : V_t \to V_t \otimes A$ is a trivial comodule;*

2. *$\rho(v) = v \otimes 1$ if and only if $v \in V_t$.*

Proof. A is the bi-algebra associated to the affine group scheme $\mathbb{G}_m = \mathrm{Spec}\,\mathbb{C}[t, t^{-1}]$. Let $Z : A \to \mathbb{C}$ be the $\mathbb{C}$-linear map given by $t^n \mapsto 0$ for any $n \in \mathbb{Z} \smallsetminus \{0\}$. Define a $\mathbb{C}$-linear map

$$f_0 : V \to V$$

as the composition $f_0 := (id_V \otimes Z) \circ \rho$. Then, for any $v \in V$ there exist $v_i \in V$ and $a_i \in A$ such that $Z(a_i) = 0$ and

$$\rho(v) = f_0(v) \otimes 1 + \sum v_i \otimes a_i\,.$$

By definition of A-comodule, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes A \arrow[d, "\mathrm{id}_V \otimes \Delta"] \\ V \otimes A \arrow[r, "\rho \otimes \mathrm{id}_A"] & V \otimes A \otimes A\,. \end{tikzcd}$$

Therefore, $f_0 \circ f_0 = f_0$. Define $V_t := \mathrm{Im}(f_0)$ and $W := \ker(f_0)$ to conclude the proof. $\square$

1.7 Noether normalization

In this section we recall the Noether normalization Lemma for projective varieties and some of its corollaries. Standard references for this material are the book [Har77] and the book [GW10].

Lemma 1.7.1 (Projective Noether normalization [GW10, 13.89])**.** *Let X be a projective scheme over $\mathbb{C}$, let $\mathcal{O}_X(1)$ be a very ample line bundle on X and let $n := \dim X$. Then there exists a finite surjective morphism*

$$\phi : X \to \mathbb{P}^n_{\mathbb{C}}\,,$$

such that $\phi^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_X(1)$.

Proof. Being X projective, there exists a closed embedding $\iota : X \to \mathbb{P}^m$ with $m \geq n$ and $\iota^*\mathcal{O}_{\mathbb{P}^m}(1) \cong \mathcal{O}_X(1)$. If $X = \mathbb{P}^m$, then $m = n$ and the proof is complete. Otherwise, let $p \in \mathbb{P}^m \smallsetminus X$ and let $U := \mathbb{P}^m \smallsetminus \{p\}$; then $\iota : X \to \mathbb{P}^m$ factors through U. Choosing suitable coordinates, we may assume that $\Gamma(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) = \mathrm{Span}_{\mathbb{C}}(x_0, \dots, x_m)$ and that $x_1, \dots, x_m$ generate $\mathcal{O}_{\mathbb{P}^m}(1)_{|U}$. Let $\pi : U \to \mathbb{P}^{m-1}$ be the projection morphism induced by $(\mathcal{O}_{\mathbb{P}^m}(1)_{|U}\,;\, x_1, \dots, x_m)$; then $\mathcal{O}_{\mathbb{P}^m}(1)_{|U} \cong \pi^*\mathcal{O}_{\mathbb{P}^{m-1}}(1)$. Define $\phi_1 : X \to \mathbb{P}^{m-1}$ as the composition $X \to U \to \mathbb{P}^{m-1}$; then ϕ_1 is proper and $\phi_1^*\mathcal{O}_{\mathbb{P}^{m-1}}(1) \cong \mathcal{O}_X(1)$. Let $\tilde{q} \in \mathbb{P}^{m-1}$ and assume that $\phi_1^{-1}(\tilde{q}) \neq \varnothing$. Let $q := [0 : \tilde{q}] \in \mathbb{P}^m$ and let l_{pq} be the line in $\mathbb{P}^m$ through p and q. Then $\phi_1^{-1}(\tilde{q})$ is a closed subset of l_{pq}, but $\phi_1^{-1}(\tilde{q}) \neq l_{pq}$ since $p \notin \phi_1^{-1}(\tilde{q})$. Hence $\phi_1^{-1}(\tilde{q})$ is a finite set. We deduce that $\phi_1 : X \to \mathbb{P}^{m-1}$ is quasi finite. Hence $\phi_1 : X \to \mathbb{P}^{m-1}$ is finite, as it is quasi finite and proper [GW10, 12.89]. Therefore $\phi_1(X)$ is closed in $\mathbb{P}^{m-1}$ and $\dim \phi_1(X) = \dim X = n$ [GW10, 12.12]. If ϕ_1 is surjective, then $n = m - 1$ and the proof is complete; otherwise we iterate the argument and define a finite morphism $\phi_2 : X \to \mathbb{P}^{m-2}$ such that $\phi_2^*\mathcal{O}_{\mathbb{P}^{m-2}}(1) \cong \mathcal{O}_X(1)$. The procedure stops after a finite number of steps and leads to the desired finite surjective morphism $\phi : X \to \mathbb{P}^n$. $\square$

Definition 1.7.2. Let X be a scheme and let $\mathcal{F} \in \mathrm{Qcoh}(X)$. We say that $\mathcal{F}$ is *pure of dimension* d if for any open subset U of X and for any $f \in \mathcal{F}(U)$ we have that $\dim \mathrm{Supp}(f) = d$ (recall that $\mathrm{Supp}(f) := \{x \in X \mid f_x \neq 0 \text{ in the stalk } \mathcal{F}_x\}$).

Remark 1.7.3. Let X be an integral scheme of dimension n. Then a sheaf $\mathcal{F} \in \mathrm{Qcoh}(X)$ is pure of dimension n if and only if $\mathcal{F}$ is torsion free. $\diamondsuit$

Lemma 1.7.4. *Let $\pi : X \to Y$ be a finite and surjective morphism of schemes and let $n := \dim X = \dim Y$. If $\mathcal{E} \in \mathrm{Qcoh}(X)$ is pure of dimension n, then also $\mathcal{F} := \pi_*\mathcal{E} \in \mathrm{Qcoh}(Y)$ is pure of dimension n.*

Proof. By contradiction, suppose that there exists an open subset V of Y and an element $0 \neq f \in \mathcal{F}(V)$ such that $\dim \mathrm{Supp}_V(f) < n$. By restricting to $\pi^{-1}(V)$ in X and to V in Y, we may assume that f is a global section of Y.
We claim that $\mathrm{Supp}_Y(f) = \pi(\mathrm{Supp}_X(f))$. Indeed, assume that $y \notin \mathrm{Supp}_Y(f)$. Then there exists an open neighborhood $y \in V_y \subseteq Y$ such that $f_{|V_y} = 0$. So $0 = f_{|V_y} = f_{|\pi^{-1}(V_y)}$, where $f_{|\pi^{-1}(V_y)}$ is the restriction of f considered as a global section of $\mathcal{E}$. Hence $\pi^{-1}(V_y) \subseteq X \setminus \mathrm{Supp}_X(f)$, that is $y \notin \pi(\mathrm{Supp}_X(f))$. Viceversa, assume that $y \notin \pi(\mathrm{Supp}_X(f))$. Since π is surjective, there is $x \in X \setminus \mathrm{Supp}_X(f)$ such that $y = \pi(x)$. We have that $0 = f_x = \pi^\sharp_x(f_y)$. Since π is finite (and surjective), $\pi^\sharp_x$ is injective and therefore $f_y = 0$, that is $y \notin \mathrm{Supp}_Y(f)$.
By assumption, there is a closed subset C of Y such that

$$\pi(\mathrm{Supp}_X(f)) = \mathrm{Supp}_Y(f) \subsetneq C \subsetneq Y.$$

Hence, we get the following chain of inclusions:

$$\mathrm{Supp}_X(f) \subsetneq \pi^{-1}(C) \subsetneq X,$$

which contradicts the hypothesis that $\mathcal{E}$ is pure of dimension n. $\square$

Proposition 1.7.5. *Let X be a a projective curve over $\mathbb{C}$, and let $\mathcal{E} \in \mathrm{Coh}(X)$ be pure of dimension 1. Then, for a sufficiently large nuber $N \gg 1$, we have that*

$$\Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)) = 0.$$

Proof. By Lemma 1.7.1 there exists a finite and surjective morphism $\phi : X \to \mathbb{P}^1$ such that $\phi^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_X(1)$. Since $\mathcal{E}$ is pure of dimension 1 and $\phi : Z \to \mathbb{P}^1$ is finite and surjective, $\phi_*\mathcal{E}$ is torsion free on $\mathbb{P}^1$ (by Lemma 1.7.4 and Remark 1.7.3). Then there exist $n \geq 1$ and $d_1, \ldots, d_n \in \mathbb{Z}$ such that $\phi_*\mathcal{E} = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(d_i)$. Let $N := \max d_i + 1$. By projection formula we have that

$$\begin{aligned}\Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N)) &\cong \Gamma(\mathbb{P}^1, \phi_*\mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-N)) \\ &\cong \bigoplus_{i=1}^n \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_i - N)) = 0.\end{aligned}$$

$\square$

Proposition 1.7.6 ([Har77, Theorem II.8.18]). *Let k be an algebraically closed field. Let X be a closed subscheme of $\mathbb{P}^n_k$ such that X has a finite number of singular points. Then there is an hyperplane $H \in \mathbb{P}^n_k$, not containing X, and such that the scheme $H \cap X$ is regular at every point. The set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$, considered as a projective space.* □

Corollary 1.7.7. *Let Z be a projective reduced curve over an algebraically closed field k, and let $\mathcal{E} \in \mathrm{Coh}(Z)$ be a locally free sheaf. Then, for a sufficiently large number $N \gg 1$, every generic divisor $D \in |\mathcal{O}_Z(N)|$ is such that*

- *for any $x \in \operatorname{Supp} D$, x is a regular point of Z;*
- *the scheme $D \cap Z$ is regular;*
- $\Gamma(Z, \mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(-D)) = 0$.

Proof. Since Z is projective, there is an embedding $\iota : Z \hookrightarrow \mathbb{P}^r_k$. Consider a Veronese embedding of degree N, $\Phi_N : \mathbb{P}^r_k \hookrightarrow \mathbb{P}^{\tilde{r}}_k$. Then $(\Phi_N \circ \iota)^* \mathcal{O}_{\mathbb{P}^{\tilde{r}}}(1) = \mathcal{O}_Z(N)$. The statement follows from Proposition 1.7.6 and Lemma 1.7.5. □

Lemma 1.7.8. *Let $f : X \to S$ be a morphism of schemes and let $p \in S(\mathbb{C})$. Let $x_1, \ldots, x_n \in X(\mathbb{C})$ be such that f is smooth at every x_i and $f(x_i) = p$. Then there exist an étale map $u : \tilde{S} \to S$, a point $\tilde{p} \in \tilde{S}(\mathbb{C})$ such that $u(\tilde{p}) = p$ and n morphisms $s_i : \tilde{S} \to X$ such that $f \circ s_i = u$ and $x_i = s_i(\tilde{p})$.*

Proof. That is a consequence of [DG67, IV, 17.16.3 (ii)]. □

Remark 1.7.9. After the base change $\tilde{f} : \tilde{X} := X \times_S \tilde{S} \to \tilde{S}$, Lemma 1.7.8 implies the existence of n sections $\tilde{s}_i : \tilde{S} \to \tilde{X}$ such that $\tilde{f} \circ \tilde{s}_i = \mathrm{Id}_{\tilde{S}}$ (indeed, $\tilde{s}_i := (s_i, \mathrm{Id}_{\tilde{S}})$). ♢

1.8 Cohomology and base change

This section collects a group of results which are consequences of "cohomology and base change" for projective morphisms. The standard reference for cohomology and base change is [DG67, III.3.2.1]; a further reference is the book Abelian Varieties by Mumford [Mum74, II.5].

Theorem 1.8.1 ([DG67, III.3.2.1]). *Let $p : X \to S$ be a proper morphism of schemes and let $\mathcal{F} \in \mathrm{Coh}(X)$ be flat over S. Let s be a point of S and denote by X_s the fiber $X \times \operatorname{Spec} \kappa(s)$. For any $i \geq 0$ there is a canonical morphism*

$$\varphi^i_s : s^* \mathbb{R}^i p_*(\mathcal{F}) \longrightarrow \mathrm{H}^i(X_s, \mathcal{F}_{|X_s}) .$$

If φ_s^i is surjective, then it is an isomorphism and there exists an open neighborhood U_s of s such that for any $t \in U_s$ the morphism $\varphi_t^i : t^*\mathbb{R}^i p_*(\mathcal{F}) \to \mathrm{H}^i(X_t, \mathcal{F}_{|X_t})$ is an isomorphism.
Furthermore, if φ_s^i is surjective the following conditions are equivalent:

1. *φ_s^{i-1} is surjective;*

2. *$\mathbb{R}^i p_*(\mathcal{F})$ is locally free near s.* □

Corollary 1.8.2. *Let $p : X \to S$ be a proper morphism of schemes and let $\mathcal{F} \in \mathrm{Coh}(X)$ be flat over S. Let s be a point of S, let $i \geq 0$ and let $\varphi_s^i : s^*\mathbb{R}^i p_*(\mathcal{F}) \to \mathrm{H}^i(X_s, \mathcal{F}_{|X_s})$ be the canonical morphism defined in Theorem 1.8.1. If $\mathrm{H}^i(X_s, \mathcal{F}_{|X_s}) = 0$ then there exists an open neighborhood U_s of s such that*

1. *$\mathrm{H}^i(X_t, \mathcal{F}_{|X_t}) = 0$ for any $t \in U_s$;*

2. *$\mathbb{R}^i p_*(\mathcal{F}) = 0$ near s;*

3. *φ_s^{i-1} is an isomorphism near s.*

Proof. That is a consequence of Theorem 1.8.1 and of Nakayama Lemma [Ati94, 2.6]. □

Definition 1.8.3. Let $p : X \to S$ be a proper morphism of schemes and let $\mathcal{F} \in \mathrm{Coh}(X)$ be flat over S. Let $i \geq 0$; we say that $\mathbb{R}^i p_*\mathcal{F}$ *commutes with base change* if for any scheme T and for any morphism of schemes $f : T \to S$ there is a canonical base change isomorphism

$$f^*\mathbb{R}^i p_*\mathcal{F} \longrightarrow \mathbb{R}^i \bar{p}_* \bar{f}^*\mathcal{F}\,,$$

where $\bar{p} := (p)_f$ and $\bar{f} := (f)_p$.

Remark 1.8.4. By Theorem 1.8.1 we have that $\mathcal{F}$ commutes with base change if and only if for any $s \in S(\mathbb{C})$ the canonical morphism $\varphi_s^i : s^*\mathbb{R}^i p_*(\mathcal{F}) \to \mathrm{H}^i(X_s, \mathcal{F}_{|X_s})$ is surjective. ◊

Lemma 1.8.5. *Let $p : X \to S$ be a projective morphism of schemes and let $\mathcal{E} \in \mathrm{Coh}(X)$ be flat sheaf over S such that $\mathrm{H}^1(X_s, \mathcal{E}_{|X_s}) = 0$ for any $s \in S(\mathbb{C})$. Then the set*

$$\{s \in S \mid \mathcal{E}_{|X_s} \textit{ is generated by global sections}\} \subseteq S$$

is an open subset of $\mathcal{S}$.

Proof. Let $\mathcal{F} := p_*\mathcal{E}$. By Theorem 1.8.1 the fact that $\mathrm{H}^1(X_s, \mathcal{E}_{|X_s}) = 0$ for any $s \in S(\mathbb{C})$ implies that $\mathcal{F}$ is locally free of finite rank and that $p_*\mathcal{E}$ commutes with base change. Cosider the canonical map

$$\varphi : p^*\mathcal{F} \to \mathcal{E}\,,$$

which is the adjoint map of $\mathrm{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$, and define

$$Z := \{x \in X \mid \varphi_x \text{ is not surjective}\}\,.$$

We claim that Z is a closed subset of X. Indeed, let $\mathcal{G} := \operatorname{Coker}\varphi \in \mathrm{Coh}(X)$; then

$$Z = \{x \in X \mid \mathcal{G}_x \neq 0\} = \operatorname{Supp}\mathcal{G}\,,$$

and the support of a coherent sheaf is closed. Now consider the set

$$H := \{s \in S \mid \mathcal{E}_{|X_s} \text{ is not generated by global sections}\}\,.$$

First of all, notice that if $s \in H$ then $X_s \neq \varnothing$. We claim that $H = p(Z)$; checking this claim is sufficient to colclude the proof since p is closed, being projective. Fix $s \in S$. The sheaf $\mathcal{E}_{|X_s}$ is not generated by global sections if and only if the canonical map

$$\mathcal{O}_{X_s} \otimes_{\mathbb{C}} \mathrm{H}^0(X_s, \mathcal{E}_{|X_s}) \to \mathcal{E}_{|X_s} \tag{$*$}$$

is not surjective. The map $(*)$ is exactly the map

$$\bar{s}^*\varphi : \bar{s}^*(p^*\mathcal{F}) \to \bar{s}^*\mathcal{E}\,,$$

where $\bar{s} := (s)_p$. The map $\bar{s}^*\varphi$ is not surjective if and only if there exists $x \in X_s$ such that $(\bar{s}^*\varphi)_x$ is not surjective, if and only if (Nakayama) $\exists x \in X_s : (x^*\bar{s}^*\varphi =)\, x^*\varphi$ is not surjective, if and only if (Nakayama) $\exists x \in X_s : \varphi_x$ is not surjective, if and only if $\exists x \in X_s : x \in Z$, if and only if $\exists x \in Z : p(x) = s$. Therefore $H = p(Z)$. □

Proposition 1.8.6. *Let $p : X \to S$ be a projective morphism of schemes and let $\mathcal{E} \in \mathrm{Coh}(X)$ be flat over S. Then for any sufficiently large integer $N \gg 0$ we have that*

- *the sheaf $\mathcal{A} := p_*(\mathcal{E} \otimes_X \mathcal{O}_X(N))$ is locally free of finite rank on S;*
- *the natural morphism of sheaves*

$$p^*\mathcal{A} \to \mathcal{E} \otimes_X \mathcal{O}_X(N) \to 0$$

is surjective.

Proof. Fix $s \in S(\mathbb{C})$ and denote $\bar{s} := (s)_p$. By [Har77, III.5.2] we deduce that

$$\exists N_1^s \gg 0 : \forall n \geq N_1^s \;\; \mathrm{H}^1(X_s, \bar{s}^*\mathcal{E}(n)) = 0\,.$$

By Theorem 1.8.1 we have that there exists an open neighborhood U_s of s such that for any $q \in U_s$, for any $n \geq N_1^s$, $\mathrm{H}^1(X_q, \bar{q}^*\mathcal{E}(n)) = 0$. Varying $s \in S$ we obtain an open cover $\{U_s\}$ of S. Since S is of finite type over $\mathbb{C}$, it is quasicompact. Therefore we can extract a finite subcover $\{U_{s_i}\}_{i=1}^k$. Define $N_1 := \max_i N_1^{s_i}$. For every $s \in S$ we have that $\mathrm{H}^1(X_s, \bar{s}^*\mathcal{E}(N_1)) = 0$. Let $\mathcal{F} := \mathcal{E}(N_1)$; then $\mathcal{F}$ satisfies the hypotheses of Lemma 1.8.5. Moreover, by [Har77, II.5.17] we obtain that for any $s \in S$

$$\exists N_2^s \gg 0 : \forall n \geq N_2^s \;\; \bar{s}^*\mathcal{F}(n) \text{ is generated by global sections.}$$

As above we deduce that there exists $N_2 \gg 0$ such that, for any $s \in S$, $\bar{s}^*\mathcal{F}(N_2)$ is generated by global sections. Define $N := N_1 + N_2$; we have proved that $\bar{s}^*\mathcal{E}(N)$ is generated by global sections, for any $s \in S$. Equivalently, for any $s \in S$ the canonical map

$$\mathcal{O}_{X_s} \otimes \mathrm{H}^0(X_s, \bar{s}^*\mathcal{E}(N)) \to \bar{s}^*\mathcal{E}(N) \to 0$$

is surjective. But, for any $s \in S$, this map is exactly

$$\bar{s}^* p^* p_*(\mathcal{E}(N)) \to \bar{s}^*\mathcal{E}(N) \to 0.$$

Hence, by Nakayama, the map

$$p^* p_*(\mathcal{E}(N)) \to \mathcal{E}(N) \to 0.$$

is surjective. Finally, since $\mathrm{H}^1(X_s, \bar{s}^*\mathcal{E}(N)) = 0$ for any $s \in S$, $\mathcal{A} := p_*(\mathcal{E}(N))$ is locally free of finite rank by Theorem 1.8.1 and that concludes the proof. □

Lemma 1.8.7. *Let $p : X \to S$ be a morphism of schemes and let*

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence of coherent sheaves on X which are flat over S. Let $f : T \to S$ an affine morphism of schemes, let $X_T := X \times_S T$ and let $\bar{f} := (f)_p : X_T \to X$. Then the sequence

$$0 \to \bar{f}^*\mathcal{K} \to \bar{f}^*\mathcal{G} \to \bar{f}^*\mathcal{H} \to 0$$

is still exact.

Proof. By restricting to an affine open subset U of S and to an affine open subset V of $p^{-1}(U)$ we may assume that X and S are affine schemes; i.e. $X = \operatorname{Spec} R$ and $S = \operatorname{Spec} A$ where R and A are $\mathbb{C}$-algebras. By hypothesis T is affine, too;

i.e. $T = \operatorname{Spec} B$ where B is a $\mathbb{C}$-algebra. Then $\bar{f}^*\mathcal{K} = \mathcal{K} \otimes_A B$, $\bar{f}^*\mathcal{G} = \mathcal{G} \otimes_A B$, $\bar{f}^*\mathcal{H} = \mathcal{H} \otimes_A B$ and we have an exact sequence

$$\operatorname{Tor}_1^A(\mathcal{H}, B) \to \mathcal{K} \otimes_A B \to \mathcal{G} \otimes_A B \to \mathcal{H} \otimes_A B \to 0\,.$$

But $\mathcal{H}$ is a flat A-module, hence $\operatorname{Tor}_1^A(\mathcal{H}, B) = 0$ [Sta17, Tag 00M5]. That concludes the proof. □

Remark 1.8.8. Lemma 1.8.7 is still valid if we consider a point $s : \operatorname{Spec} \mathbb{C} \to S$ instead of an affine morphism $f : T \to S$. More precisely if $s \in S(\mathbb{C})$ is a point of S, $X_s := X \times_S \operatorname{Spec} \kappa(s)$ is the fiber of $p : X \to S$ w.r.t. s, and $\bar{s} := (s)_p : X_s \to X$, the sequence

$$0 \to \bar{s}^*\mathcal{K} \to \bar{s}^*\mathcal{G} \to \bar{s}^*\mathcal{H} \to 0$$

is still exact. ◊

Proposition 1.8.9. *Let $p : X \to S$ be a projective morphism of schemes of relative dimension 1 and let $\mathcal{H}$ be a coherent sheaf on X which is flat over S. Then there is an exact sequence in* $\operatorname{Coh}(X)$*:*

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

such that

1. $\mathcal{K}, \mathcal{G} \in \operatorname{Coh}(X)$*,* $\mathcal{G}$ *is locally free and* $\mathcal{K}$ *is flat over* S*;*
2. $p_*\mathcal{K} = p_*\mathcal{G} = 0$*;*
3. $\mathbb{R}^1 p_*\mathcal{K}$ *and* $\mathbb{R}^1 p_*\mathcal{G}$ *are locally free sheaves of finite rank on* S *and commute with base change.*

Proof. According to Proposition 1.8.6, if $N \gg 0$ there exists a locally free sheaf $\mathcal{A} \in \operatorname{Coh}(S)$ and a canonical surjection $p^*\mathcal{A} \twoheadrightarrow \mathcal{H} \otimes \mathcal{O}_X(N)$. Let $\mathcal{G} := p^*\mathcal{A} \otimes \mathcal{O}_X(-N)$ and $\mathcal{K} := \operatorname{Ker}(\mathcal{G} \twoheadrightarrow \mathcal{H})$. The sequence

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

is exact in $\operatorname{Coh}(X)$; $\mathcal{K}$ is flat over S, since both $\mathcal{G}$ and $\mathcal{H}$ are flat over S. Fix a point $s \in S(\mathbb{C})$ and consider the following diagram:

$$\begin{tikzcd} X_s \arrow[r, "\bar{s}"] \arrow[d, "\bar{p}"] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ \operatorname{Spec} \mathbb{C} \arrow[r, "s"] & S \end{tikzcd} .$$

Since X_s is a 1-dimensional scheme, we have that the morphism

$$s^*\mathbb{R}^1 p_*\mathcal{E} \to \mathrm{H}^1(X_s, \bar{s}^*\mathcal{E}) \tag{$*$}$$

is an isomorphism for any sheaf $\mathcal{E} \in \mathrm{Coh}(X)$.
We claim that $\mathrm{H}^0(X_s, \bar{s}^*\mathcal{G}) = 0$. Indeed, $\bar{s}^*\mathcal{G} = \bar{s}^* p^*\mathcal{A} \otimes \mathcal{O}_{X_s}(-N)$ and the claim is a consequence of Theorem 1.7.5. Hence, also $\mathrm{H}^0(X_s, \bar{s}^*\mathcal{K}) = 0$ (since $\bar{s}^*\mathcal{K} \to \bar{s}^*\mathcal{G}$ is injective by Lemma 1.8.7). Therefore, the two natural morphisms

$$\varphi_s^0 : s^* p_*\mathcal{K} \to \mathrm{H}^0(X_s, \bar{s}^*\mathcal{K})\,,$$
$$\psi_s^0 : s^* p_*\mathcal{G} \to \mathrm{H}^0(X_s, \bar{s}^*\mathcal{G})$$

are surjecrtive. By 1.8.1 we get that

- $p_*\mathcal{K} = 0 = p_*\mathcal{G}$ near s;
- $\mathbb{R}^1 p_*\mathcal{K}$ and $\mathbb{R}^1 p_*\mathcal{G}$ are locally free near s.

Finally, by Nakayama lemma we deduce that $p_*\mathcal{K} = 0 = p_*\mathcal{G}$ and that $\mathbb{R}^1 p_*\mathcal{K}$ and $\mathbb{R}^1 p_*\mathcal{G}$ are locally free. □

Remark 1.8.10. Let $p : X \to S$ be a projective morphism of schemes of relative dimension 1 and let $\mathcal{H}$ be a coherent sheaf on X which is flat over S. Let

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence in $\mathrm{Coh}(X)$ such that

1. $\mathcal{K}, \mathcal{G} \in \mathrm{Coh}(X)$, $\mathcal{G}$ is locally free and $\mathcal{K}$ is flat over S;
2. $p_*\mathcal{K} = p_*\mathcal{G} = 0$;
3. $\mathbb{R}^1 p_*\mathcal{K}$ and $\mathbb{R}^1 p_*\mathcal{G}$ are locally free sheaves of finite rank on S and commute with base change.

Let $f : T \to S$ be an affine morphism of schemes, let $X_T := X \times_S T$, let $\bar{f} := (f)_p : X_T \to X$ and $\bar{p} := (p)_f : X_T \to T$. By Lemma 1.8.7

$$0 \to \bar{f}^*\mathcal{K} \to \bar{f}^*\mathcal{G} \to \bar{f}^*\mathcal{H} \to 0$$

is still exact and by Theorem 1.8.1

1. $\bar{f}^*\mathcal{K}, \bar{f}^*\mathcal{G} \in \mathrm{Coh}(X_T)$, $\bar{f}^*(\mathcal{G})$ is locally free and $\bar{f}^*(\mathcal{K})$ is flat over T;
2. $\bar{p}_*(f^*\mathcal{K}) = \bar{p}_*(f^*\mathcal{G}) = 0$;
3. $\mathbb{R}^1\bar{p}_*(f^*\mathcal{K})$ and $\mathbb{R}^1\bar{p}_*(f^*\mathcal{G})$ are locally free sheaves of finite rank on T and commute with base change. ♢

Chapter 2

Generalized Coherent Systems

In this chapter we define the notion of generalized coherent system, we introduce a moduli stack $\mathcal{S}$ which classifies families of generalized coherent systems (see Definitions 2.1.1 and 2.1.5), and we investigate the geometric properties of $\mathcal{S}$. In particular, we prove that $\mathcal{S}$ is an abelian cone over a smooth Artin stack $\mathcal{N}$ which classifies families of pairs of vector bundles (see Proposition 2.1.12). We prove also that the canonical forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ is strongly representable (see Proposition 2.1.13) and that it factors as the composition of a closed embedding followed by a smooth morphism (see Proposition 2.1.14). In the second section we prove that the locus of coherent systems inside $\mathcal{S}$ is open (see Proposition 2.2.3).

2.1 The moduli stack of GCS

Throughout this chapter we fix a base algebraic stack $\mathcal{M}$ together with a flat projective relatively Gorestein morphism $\mathcal{M}' \to \mathcal{M}$ or relative dimension 1. A concrete example of such a stack $\mathcal{M}$ is the algebraic stack of genus g smooth curves $\mathcal{M}_g$, or its Deligne Mumford compactification $\overline{\mathcal{M}}_g$. However our intention is to work in the greatest possible generality. For this purpose we do not assume that $\mathcal{M}$ is Deligne Mumford.

Definition 2.1.1. A *generalized coherent system* (GCS) on a curve $C \in \mathcal{M}(\mathbb{C})$ is defined by the following data:

1. two locally free sheaves F, E on C ;
2. a morphism $\varphi \in \mathrm{Hom}_{\mathcal{O}_C}(F, E)$.

Remark 2.1.2. Any coherent system on a smooth curve C (see Definition 1.5.1) induces a generalized coherent system on C. Indeed, let $F := V \otimes \mathcal{O}_C$: the morphism $\varphi : F \to E$ is induced by the injection $V \hookrightarrow \mathrm{H}^0(E)$. $\Diamond$

Definition 2.1.3. Let $\mathcal{N} : (Sch)^{op} \to (Grpds)$ be the prestack defined by

$$\mathcal{N}(T)_0 := \left\{ (C \to T, E, F) \;\middle|\; \begin{array}{l} T \to \mathcal{M}\,;\; C := T \times_{\mathcal{M}} \mathcal{M}'\,; \\ E, F \in \mathrm{Coh}(C) \text{ are locally free.} \end{array} \right\},$$

where an isomorphism

$$(C \to T, E, F) \to (C' \to T, E', F')$$

is a triple (α, β, γ) such that $\alpha : C \to C'$ is induced by an isomorphism in $\mathcal{M}(T)$, and $\beta : E \to \alpha^* E'$ and $\gamma : F \to \alpha^* F'$ are isomorphisms of sheaves.

Proposition 2.1.4. *The prestack $\mathcal{N}$ is an algebraic stack.*

Proof. The stack $\mathcal{V}$ defined by

$$\mathcal{V}(T) := \left\{ (C \to T, E) \;\middle|\; \begin{array}{l} T \to \mathcal{M}\,;\; C := T \times_{\mathcal{M}} \mathcal{M}'\,; \\ E \in \mathrm{Coh}(C) \text{ is locally free.} \end{array} \right\},$$

is an algebraic stack, as described in [Ols16, Exercise 8.J]. Moreover $\mathcal{N} = \mathcal{V} \times_{\mathcal{M}} \mathcal{V}$; hence $\mathcal{N}$ is algebraic, too. □

Definition 2.1.5. Let $\mathcal{S} : (Sch)^{op} \to (Grpds)$ be the prestack defined by

$$\mathcal{S}(T) := \left\{ (C \to T, E, F, \varphi) \;\middle|\; \begin{array}{l} (C \to T, E, F) \in \mathcal{N}(T)\,; \\ \varphi \in \mathrm{Hom}_{\mathcal{O}_C}(F, E)\,. \end{array} \right\},$$

where an isomorphism

$$(C \to T, E, F, \varphi) \to (C' \to T, E', F', \varphi')$$

is a triple $(\alpha, \beta, \gamma) \in \mathcal{N}(T)_1$ such that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\gamma} & \alpha^* F' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha^*\varphi'} \\ E & \xrightarrow{\beta} & \alpha^* E' \end{array} .$$

We call $\mathcal{S}$ the *moduli stack of generalized coherent systems.* We will prove that $\mathcal{S}$ is actually an algebraic stack in Proposition 2.1.12.

Remark 2.1.6. There is a natural forgetful morphism $G : \mathcal{S} \to \mathcal{N}$, defined by $G_T(C \to T, E, F, \varphi) = (C \to T, E, F)$ for any scheme T. ◊

Remark 2.1.7. (1) There is a natural forgetful morphism $\mathcal{N} \to \mathcal{M}$. The universal curve $\mathcal{M}' \to \mathcal{M}$ pulls back to a universal curve $\mathcal{N}' \to \mathcal{N}$ and to a universal curve $\mathcal{S}' \to \mathcal{S}$.

(2) The stack $\mathcal{N}$ has a universal family $(\pi : \mathcal{N}' \to \mathcal{N}, \mathcal{E}, \mathcal{F})$ and the stack $\mathcal{S}$ has a universal family $(\bar{\pi} : \mathcal{S}' \to \mathcal{S}, \bar{\mathcal{E}}, \bar{\mathcal{F}}, \phi)$. Here $\mathcal{S}' = \mathcal{N}' \times_{\mathcal{N}} \mathcal{S}$ and the sheaves $\bar{\mathcal{E}}$ and $\bar{\mathcal{F}}$ are respectively the pullback of $\mathcal{E}$ and of $\mathcal{F}$. ♢

(3) Since the morphism $\mathcal{M}' \to \mathcal{M}$ is relatively Gorestein, its dualizing sheaf is a line bundle. It follows that both $\mathcal{N}' \to \mathcal{N}$ and $\mathcal{S}' \to \mathcal{S}$ are relatively Gorestein morphisms. We call ω the dualizing bundle of $\mathcal{N}' \to \mathcal{N}$ and $\bar{\omega}$ the dualizing bundle of $\mathcal{S}' \to \mathcal{S}$.

Remark 2.1.8. Let T be a scheme and let $(\pi : C \to T, E, F, \varphi) \in \mathcal{S}(T)$. Let $f : T' \to T$ be a morphism of schemes. If there is no ambiguity, we use the following notation:

- $f^*E := (f)_\pi^*\, E\,;$
- $f^*F := (f)_\pi^*\, F\,;$
- $f^*\varphi := (f)_\pi^*\, \varphi\,,$

where the morphism $(f)_\pi : C \times_T T' \to C$ is the pullback of f, as defined in 1.1. The notation concerning the stack $\mathcal{N}$ is analogous. ♢

The following lemmas are basic results concerning derived categories. We need them in order to prove that $\mathcal{S}$ is an abelian cone stack over $\mathcal{N}$.

Lemma 2.1.9. *Let $\mathcal{A}$ be an abelian category and let $E, F \in \mathrm{D}(\mathcal{A})$. Assume that*

1. $\mathrm{H}^i(E) = 0$ *for any* $i \neq 0$*;*
2. $\mathrm{H}^i(F) = 0$ *for any* $i > 0$.

Then the map

$$\mathrm{H}^0 : \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(F, E) \to \mathrm{Hom}_{\mathcal{A}}(H^0(F), H^0(E))$$

is a natural bijection.

Proof. By [Huy06, Exercise 2.31] we may assume that $E^i = 0$ if $i \neq 0$ and $F^i = 0$ if $i > 0$.
We check that a morphism $\mathrm{H}^0(F) \to E$ in $\mathcal{A}$ naturally induces a morphism of complexes $F \to E$. Let $f : \mathrm{H}^0(F) \to E$ be a morphism in $\mathcal{A}$. We have an exact sequence

$$F^{-1} \to F^0 \to \mathrm{H}^0(F) \to 0\,.$$

Hence f induces a morphism $f' : F^0 \to E$ by composition with $F^0 \to \mathrm{H}^0(F)$, and f' makes the following diagram commute:

$$\begin{tikzcd} F^{-1} \arrow[r] \arrow[d] & F^0 \arrow[d, "f'"] \\ 0 \arrow[r] & E\,. \end{tikzcd}$$

Therefore f naturally induces a morphism of complexes $F \to E$. $\square$

Lemma 2.1.10. *Let $f : X \to Y$ be a proper morphism of relative dimension d and let $E \in \mathrm{D}^{[a,b]}_{\mathrm{Coh}}(X)$. Then $\mathbb{R}f_*E \in \mathrm{D}^{[a,b+d]}_{\mathrm{Coh}}(Y)$.*

Proof. That is a consequence of [Huy06, Theorem 3.20]. $\square$

Lemma 2.1.11. *Let $p : C \to T$ be a flat morphism of schemes of relative dimension 1 and let $\omega_p \in \mathrm{Pic}(C)$ be the dualizing sheaf of p; let $H \in \mathrm{Coh}(C)$ be a sheaf which is flat on T. Then there is a canonical map*

$$\mathrm{Hom}_{\mathcal{O}_C}(H, \mathcal{O}_C) \longrightarrow \mathrm{Hom}_{\mathcal{O}_T}(\mathbb{R}^1 p_*(H \otimes \omega_p), \mathcal{O}_T)\,.$$

Such a map is a canonical bijection if $p : C \to T$ is Gorestein.

Proof. By tensoring with ω_p we get a canonical map $\mathrm{Hom}(H, \mathcal{O}_C) \to \mathrm{Hom}(H \otimes \omega_p, \omega_p)$. Such a map is a bijection if p is Gorestein. Lemma 2.1.9 implies that $\mathrm{Hom}_{\mathcal{O}_C}(H \otimes \omega_p, \omega_p) \cong \mathrm{Hom}_{\mathrm{D}(C)}(H \otimes \omega_p, \omega_p)$. By Grothendieck duality [Huy06, Section 3.4] we have that $\mathrm{Hom}_{\mathrm{D}(C)}(H \otimes \omega_p, \omega_p) \cong \mathrm{Hom}_{\mathrm{D}(T)}(\mathbb{R}p_*(H \otimes \omega_p[1]), \mathcal{O}_T)$. By Lemma 2.1.9 and Lemma 2.1.10 we deduce that $\mathrm{Hom}_{\mathrm{D}(T)}(\mathbb{R}p_*(H \otimes \omega_p[1]), \mathcal{O}_T) \cong \mathrm{Hom}_{\mathcal{O}_T}(\mathbb{R}^0 p_*(H \otimes \omega_p[1]), \mathcal{O}_T)$. Finally $\mathbb{R}^0 p_*(H \otimes \omega_p[1]) \cong \mathbb{R}^1 p_*(H \otimes \omega_p)$. $\square$

We are now ready to prove that $\mathcal{S}$ is an abelian cone stack over $\mathcal{N}$. We will use the notation $\mathbb{V}(\mathcal{H})$ meaning $\mathrm{Spec}\,\mathrm{Sym}(\mathcal{H})$, as explained in 1.1.

Proposition 2.1.12. *Let $(\pi : \mathcal{N}' \to \mathcal{N}, \mathcal{E}, \mathcal{F})$ be the universal family of the stack $\mathcal{N}$ and let ω be the dualizing sheaf of the morphism $\pi : \mathcal{N}' \to \mathcal{N}$. There is a natural isomorhism*

$$\mathcal{S} \xrightarrow{\sim} \mathbb{V}[\mathbb{R}^1\pi_*(\mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega)]\,.$$

Proof. Let $\mathcal{H} := \mathcal{F} \otimes \mathcal{E}^\vee$ and denote $\mathbb{V} := \mathbb{V}[\mathbb{R}^1\pi_*(\mathcal{H} \otimes \omega)]$. Let T be any scheme. By definition, we have that

$$\mathcal{S}(T) = \{t : T \to \mathcal{N}, \varphi : t^*\mathcal{F} \to t^*\mathcal{E}\}\,,$$

where the notation $t^*\mathcal{F}$, $t^*\mathcal{E}$ has been introduced in Remark 2.1.8. On the other hand, we have that

$$\mathbb{V}(T) = \{t : T \to \mathcal{N}, \gamma : t^*\mathbb{R}^1\pi_*(\mathcal{H} \otimes \omega) \to \mathcal{O}_T\}\,.$$

Fix a morphism $t : T \to \mathcal{N}$ and let $(p : C \to T, E, F)$ be the object of $\mathcal{N}(T)$ induced by t; then $F = t^*\mathcal{F}$, $E = t^*\mathcal{E}$. Denote $H := t^*\mathcal{H} = F \otimes E^\vee$. Being $\pi : \mathcal{N}' \to \mathcal{N}$ a morphism of relative dimension 1, the sheaf $\mathbb{R}^1\pi_*(\mathcal{H} \otimes \omega)$ commutes with base change. Hence $t^*\mathbb{R}^1\pi_*(\mathcal{H}\otimes\omega) \cong \mathbb{R}^1 p_*(H \otimes \omega_p)$, where ω_p is the dualizing sheaf of p. By Lemma 2.1.11 there is a canonical bijection $\mathrm{Hom}_{\mathcal{O}_C}(F, E) \cong \mathrm{Hom}_{\mathcal{O}_C}(H, \mathcal{O}_C) \cong \mathrm{Hom}_{\mathcal{O}_T}(\mathbb{R}^1 p_*(H \otimes \omega_p), \mathcal{O}_T)$. Therefore every object (t, φ) of $\mathcal{S}(T)$ induces an object (t, γ) of $\mathbb{V}(T)$.
That is compatible with isomorphisms. Indeed, let (t, φ) and (t', φ') be two objects of $\mathcal{S}(T)$, let $(p : C \to T, E, F)$ be the object of $\mathcal{N}(T)$ induced by t and let $(p' : C' \to T, E', F')$ be the object of $\mathcal{N}(T)$ induced by t'. Denote $H := F \otimes E^\vee$ and $H' := F' \otimes (E')^\vee$. Then φ induces a morphism $H \to \mathcal{O}_C$ and φ' induces a morphism $H' \to \mathcal{O}_{C'}$. An isomorphism $t \Rightarrow t'$ in $\mathcal{N}(T)$ is induced by an isomorphism of T-schemes $\alpha : C \to C'$ and an isomorphism of sheaves $\beta : H \to \alpha^* H'$. The isomorphism $t \Rightarrow t'$ induces an isomorphism in $\mathcal{S}(T)$ if and only if the following diagram commutes:

$$\begin{tikzcd}
H \arrow[dd] \arrow[dr] & \\
& \mathcal{O}_C \\
\alpha^* H' \arrow[ur] &
\end{tikzcd},$$

where we have identified $\mathcal{O}_C$ and $\alpha^*\mathcal{O}_{C'}$. By Lemma 2.1.11 that is a commutative diagram if and only if the following is a commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^1 p_*(H \otimes \omega_p) \arrow[dd] \arrow[dr] & \\
& \mathcal{O}_T \\
\mathbb{R}^1 p_*(\alpha^* H \otimes \omega_p) \arrow[ur] &
\end{tikzcd}.$$

Since $\mathbb{R}^1 p'_*(H' \otimes \omega_{p'})$ commutes with base change, we have that $\mathbb{R}^1 p'_*(H' \otimes \omega_{p'}) \cong \mathrm{id}_T^* \mathbb{R}^1 p'_*(H' \otimes \omega_{p'}) \cong \mathbb{R}^1 p_*(\alpha^* H' \otimes \omega_p)$. Hence $t \Rightarrow t'$ induces an isomorphism in $\mathcal{S}(T)$ if and only if it induces an isomorphism in $\mathbb{V}(T)$.
We have defined a functor $\mathcal{S}(T) \to \mathbb{V}(T)$. By construction such a functor is essentially surjective and fully faithful, hence it is an equivalence. It can be checked that it is also compatible with pullbacks, so that it induces an isomorphism of stacks $\mathcal{S} \to \mathbb{V}$. □

Corollary 2.1.13. *The prestack $\mathcal{S}$ is an algebraic stack and the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ is strongly representable.*

Proof. That is a consequence of 2.1.12 and of [Ols16, Proposition 10.2.2]. □

Proposition 2.1.14. *Let $G : \mathcal{S} \to \mathcal{N}$ be the forgetful morphism. The morphism G factors as the composition of a closed embedding followed by a smooth morphism.*

Proof. Define $\mathcal{H} := \mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega \in \mathrm{Coh}(\mathcal{N}')$; $\mathcal{H}$ is a locally free sheaf. By Proposition 2.1.12 we have a natural isomorphism

$$\mathcal{S} \cong \mathbb{V}(\mathbb{R}^1\pi_*\mathcal{H}) = \mathbb{V}(\mathbb{R}^1\pi_*(\mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega)) .$$

Furthermore, by Proposition 1.8.9 there is an exact sequence in $\mathrm{Coh}(\mathcal{N}')$,

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0 , \tag{$*$}$$

such that

- $\mathcal{K}, \mathcal{G} \in \mathrm{Coh}(\mathcal{N}')$, $\mathcal{G}$ is locally free and $\mathcal{K}$ is flat over $\mathcal{N}$;
- $\pi_*\mathcal{K} = \pi_*\mathcal{G} = 0$;
- $\mathbb{R}^1\pi_*\mathcal{K}$ and $\mathbb{R}^1\pi_*\mathcal{G}$ are locally free sheaves of finite rank on $\mathcal{N}$.

If we apply the functor $\pi_*(-)$ to the sequence $(*)$, we get the following long exact sequence:

$$0 \longrightarrow \pi_*\mathcal{H} \longrightarrow \mathbb{R}^1\pi_*\mathcal{K} \longrightarrow \mathbb{R}^1\pi_*\mathcal{G} \longrightarrow \mathbb{R}^1\pi_*\mathcal{H} \longrightarrow 0 .$$

Let $p : \mathbb{V}(\mathbb{R}^1\pi_*\mathcal{G}) \to \mathcal{N}$ be the structure morphism of the geometrical vector bundle $\mathbb{V}(\mathbb{R}^1\pi_*\mathcal{G})$; p is a smooth morphism, since $\mathbb{R}^1\pi_*\mathcal{G}$ is locally free of finite rank. The surjection $\mathbb{R}^1\pi_*\mathcal{G} \to \mathbb{R}^1\pi_*\mathcal{H}$ gives rise to a closed embedding $\iota : \mathbb{V}(\mathbb{R}^1\pi_*\mathcal{H}) \to \mathbb{V}(\mathbb{R}^1\pi_*\mathcal{G})$ such that $p \circ \iota = G$. $\square$

Corollary 2.1.15. *Let $G : \mathcal{S} \to \mathcal{N}$ be the forgetful morphism and let*

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "G"] \arrow[dr, "\iota"'] & & \mathcal{N} \\ & \mathcal{X} \arrow[ur, "p"'] & \end{tikzcd}$$

be a factorization of G as the composition of a smooth morphism p and a closed embedding ι. Let $\mathcal{I}$ be the ideal sheaf of ι and Ω be the cotangent bundle of p. Then

$$\tau_{\geq -1}\,\mathrm{L}_G \cong [\iota^*\mathcal{I} \to \iota^*\Omega_p] .$$

Proof. That is a consequence of [Sta17, Tag 08SH]. $\square$

2.2 The open locus of coherent systems

Let $\mathcal{V}$ be the stack given by

$$\mathcal{V}(T) := \left\{ (\pi : C \to T, E) \;\middle|\; \begin{array}{l} [\pi : C \to T] \in \mathcal{M}(T)_0\,; \\ E \in \mathrm{Coh}(C) \text{ locally free.} \end{array} \right\}.$$

The stack $\mathcal{V}$ is algebraic [Ols16, Exercise 8.J]. There is a canonical morphism

$$\mathcal{V} \times \mathrm{BGL} \to \mathcal{N}\,,$$

given by $((\pi : C \to T, E), V) \mapsto (\pi : C \to T, E, p^*V)$ for any $T \in (Sch)$, for any $(p : C \to T, E) \in \mathcal{V}(T)$ and for any $V \in \mathrm{BGL}(T)$. Let

$$\tilde{\mathcal{S}} := \mathcal{S} \times_{\mathcal{N}} (\mathcal{V} \times \mathrm{BGL})\,.$$

A $\mathbb{C}$-point $(C, E, V, \varphi) \in \tilde{\mathcal{S}}(\mathbb{C})$ is a coherent system on C if the induced map $\mathrm{H}^0(\varphi) : V \to \mathrm{H}^0(E)$ is injective (see Definition 1.5.1).
In this section we prove that the locus of coherent systems inside $\tilde{\mathcal{S}}$ is open.

Notation 2.2.1. Let $T \in (Sch)$ and let $(\pi : C \to T, E, V, \varphi) \in \tilde{\mathcal{S}}(T)$; let $p \in T(\mathbb{C})$. Throughout the section we will use the following notation:

1. C_p denotes the fiber $C \times_T \operatorname{Spec} \kappa(p)$;
2. p^*E denotes the pullback $E_{|C_p}$;
3. p^*V is the pullback $V_{|\operatorname{Spec}\kappa(p)}$, as usual;
4. $\bar{V}$ denotes the pullback π^*V and $p^*\bar{V}$ denotes the pullback $\bar{V}_{|C_p}$;
5. $\varphi_p : p^*\bar{V} \to p^*E$ denotes the pullback of the morphism $\varphi : \bar{V} \to E$ on C_p.

Notice that $p^*\bar{V} = p^*V \otimes \mathcal{O}_{C_p}$ and $\mathrm{H}^0(C_p, p^*\bar{V}) = p^*V$.

Proposition 2.2.2. *Let $T = \operatorname{Spec} A$ be an affine scheme and let $(\pi : C \to T, E, V, \varphi) \in \tilde{\mathcal{S}}(T)$. Then the set*

$$\{p \in T(\mathbb{C}) \mid \mathrm{H}^0(\varphi_p) : p^*V \to \mathrm{H}^0(C_p, p^*E) \textit{ is injective}\}$$

is an open subset of T.

Proof. By restricting to an open subset of T, we may assume that $V \cong \widetilde{A^{\oplus g}}$. We need to show that for any $p \in T$ such that

$$\mathrm{H}^0(\varphi_p) : p^*V \to \mathrm{H}^0(C_p, p^*E) \tag{$*$}$$

is injective, there exists an element $\xi \in A$ such that

1. $p \in \mathrm{D}(\xi)$;
2. for any $q \in \mathrm{D}(\xi)$ the map $\mathrm{H}^0(\varphi_q) : q^*V \to \mathrm{H}^0(C_q, q^*E)$ is injective.

Fix $p \in S$ and apply Lemma 1.7.7 to p^*E, in order to find a divisor D on C_p such that

1. for any $x \in \operatorname{Supp} D$, x is a regular point of C_p;
2. the scheme $D \cap C_p$ is regular;
3. $\Gamma(C_p, p^*E \otimes_{\mathcal{O}_{C_p}} \mathcal{O}_{C_p}(-D)) = 0$.

Define $\{x_1, \dots, x_n\} := \operatorname{Supp} D$. Recall that D is a closed subset of C, since it is closed in C_p and C_p is closed in C. Every x_i is a smooth point of C with respect to the morphism $\pi : C \to T$, since the morphism $D \cap C_p \to \operatorname{Spec} \kappa(p)$ is regular. For any i there exists an affine open neighborhood $U_i = \operatorname{Spec} R_i$ of x_i in C such that $E_{|U_i} = \widetilde{R_i^{\oplus r_i}}$. By Lemma 1.7.8 we may assume that there exist n sections $s_i : T \to C$ such that $s_i(p) = x_i$. Hence, we have n maps

$$s_i^* \varphi : V \to s_i^* E\,.$$

Now, $s_i^* E \cong \widetilde{A^{\oplus r_i}}$ for any i; so, if we write $N := \sum_{i=1}^n r_i$, we have that

$$\bigoplus_{i=1}^n s_i^* E \cong A^{\oplus N}.$$

Consider the morphism

$$\Phi := \oplus s_i^* \varphi : V \longrightarrow \bigoplus_{i=1}^n s_i^* E\,;$$

Φ is equivalent to a morphism of A-modules $A^{\oplus g} \to A^{\oplus N}$.
We claim that the two following conditions are equivalent:

1. the morphism $\mathrm{H}^0(\varphi_p) : p^*V \to \mathrm{H}^0(C_p, p^*E)$ is injective;
2. the morphism
$$p^*\Phi : p^*V \longrightarrow \bigoplus_{i=1}^n p^* s_i^* E = \bigoplus_{i=1}^n x_i^* E$$
is injective.

Notice that the morphism $p^*\Phi$ is equivalent to a morphism of $\mathbb{C}$-vector spaces $\kappa(p)^{\oplus g} \to \kappa(p)^{\oplus N}$.
We have an exact sequence

$$0 \to p^*E \otimes_{\mathcal{O}_{C_p}} \mathcal{O}_{C_p}(-D) \to p^*E \to p^*E \otimes_{\mathcal{O}_{C_p}} \mathcal{O}_D \to 0$$

Furthermore

1. $\mathrm{H}^0(C_p, p^*E \otimes_{\mathcal{O}_{C_p}} \mathcal{O}_{C_p}(-D)) = 0$;
2. $\mathrm{H}^0(C_p, p^*E \otimes \mathcal{O}_D) = \oplus_{i=1}^n x_i^* E$.

Therefore, applying the functor H^0 to the exact sequence above we get an injective morphism of $\mathbb{C}$-vector spaces:

$$\mathrm{H}^0(C_p, p^*E) \hookrightarrow \bigoplus_{i=1}^n x_i^* E\,.$$

That proves the claim.
Assume that $p \in T$ is such that $\mathrm{H}^0(\varphi_p)$ is injective. Then $p^*\Phi$ is injective, too. Let (a_{ij}) be the matrix in $\mathrm{M}_{g,N}(A)$ which induces the homomorphism of A-modules $A^{\oplus g} \to A^{\oplus N}$ associated to Φ. Let $\overline{a_{ij}}$ be the canonical image of a_{ij} in $\kappa(p)$. Then $(\overline{a_{ij}})$ is the matrix which induces the homomorphism of $\mathbb{C}$-vector spaces $\kappa(p)^{\oplus g} \to \kappa(p)^{\oplus N}$ associated to $p^*\Phi$. Since $p^*\Phi$ is injective, there exists a minor $\bar{M}$ of the matrix $(\overline{a_{ij}})$ such that $\det \bar{M} \neq 0$ in $\kappa(p)$. Therefore, there exists a minor M of the matrix (a_{ij}) such that if we define $\xi := \det M \in A$, ξ is not in the maximal ideal of p. Hence $p \in \mathrm{D}(\xi)$.
We still need to prove that for any $q \in \mathrm{D}(\xi)$ the map $\mathrm{H}^0(\varphi_q) : q^*V \to \mathrm{H}^0(C_q, q^*E)$ is injective. Consider the divisors on C given by $\Delta_i := s_i(S)$, where we recall that $s_i : S \to X$ is a section of $\pi : X \to S$ for any $i = 1, \dots n$. Notice that $(\Delta_i)_{|C_p} = x_i$. Define

$$F := E \otimes \bigotimes_{i=1}^n \mathcal{O}_C(-\Delta_i)\,.$$

Then $p^*F = p^*E \otimes \mathcal{O}_{C_p}(-D_p)$ and so we have that $\mathrm{H}^0(C_p, p^*F) = 0$. Hence by semicontinuity $\mathrm{H}^0(C_q, q^*F) = 0$ for any q in an open neighborhood U of p. By restricting to a principal open subset of $U \cap \mathrm{D}(\xi)$, we may assume that $\mathrm{H}^0(C_q, q^*F) = 0$ for any $q \in \mathrm{D}(\xi)$. Then for any $q \in \mathrm{D}(\xi)$ we have that $\mathrm{H}^0(C_q, q^*E \otimes \mathcal{O}_{C_q}(-D_q)) = 0$, where $D_q := (\Delta_1 + \cdots + \Delta_n)_{|X_q}$. As previously, we can check that $q^*\Phi$ is injective if and only if $\mathrm{H}^0(\varphi_q)$ is injective. But if $q \in \mathrm{D}(\xi)$, then $q^*\Phi$ is injective. Hence the proof is complete. $\square$

Corollary 2.2.3. *Let $\mathcal{S}_c$ be the stack defined by*

$$\mathcal{S}_c(T) := \left\{ (\pi : C \to T, E, V, \varphi) \;\middle|\; \begin{array}{l} (\pi : C \to T, E, V, \varphi) \in \tilde{\mathcal{S}}(T)\,; \\ \mathrm{H}^0(\varphi_p) : p^*V \to \mathrm{H}^0(C_p, p^*E) \text{ is injective} \\ \text{for any } p \in T(\mathbb{C}) \end{array} \right\}.$$

The stack $\mathcal{S}_c$ is the moduli stack of coherent systems. Then the canonical morphism $\mathcal{S}_c \to \tilde{\mathcal{S}}$ is an open embedding.

Proof. Let T be an affine scheme, let $T \to \tilde{\mathcal{S}}$ be a morphism of stacks and let $U := T \times_{\tilde{\mathcal{S}}} \mathcal{S}_c$. Proposition 2.2.2 implies that $U \to T$ is an open embedding. $\square$

Chapter 3

Obstruction Theory

In this chapter we construct a perfect obstruction theory $E^\bullet \to \tau_{\geq -1} \mathrm{L}_G$ for the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ (the definitions of $\mathcal{S}$, $\mathcal{N}$ and $G : \mathcal{S} \to \mathcal{N}$ can be found in 2.1). The existence of the complex $E^\bullet$ depends on the fact that $\mathcal{S}$ has a universal family; indeed

$$E^\bullet = \mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1]) ,$$

where $\bar{\pi} : \mathcal{S}' \to \mathcal{S}$ is the universal curve over $\mathcal{S}$ and $\bar{\mathcal{E}}$ and $\bar{\mathcal{F}}$ are the universal sheaves in $\mathrm{Coh}(\mathcal{S}')$.
The chapter is structured in two sections. In the first section we compute tangent and obstruction spaces for the morphism $G : \mathcal{S} \to \mathcal{N}$ (see Proposition 3.1.4). In the second section we actually construct the obstruction theory $E^\bullet \to \tau_{\geq -1} \mathrm{L}_G$ (see Proposition 3.2.3) and we prove that it comes from a canonical morphism $E^\bullet \to \mathrm{L}_G$ (see Proposition 3.2.5).

3.1 Tangent and obstruction spaces

In this section we investigate the infinitesimal properties of the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ (defined in 2.1). In particular, we observe that it induces a natural transformation $\mathrm{h}_{\mathcal{S},s} \to \mathrm{h}_{\mathcal{N},G(s)}$ at every point $s \in \mathcal{S}$ and we compute tangent and obstruction spaces for such natural transformation (compare Section 1.3).

The following Lemma is a result in Homological Algebra. We need it in the proof of Proposition 3.1.4 to describe the infinitesimal properties of the morphism $G : \mathcal{S} \to \mathcal{N}$.

Lemma 3.1.1. *Let X be a scheme; let $\underline{e} := [E' \xrightarrow{e'} E \xrightarrow{e''} E'' \to 0]$ and $\underline{f} := [0 \to F' \xrightarrow{f'} F \xrightarrow{f''} F'' \to 0]$ be two exact sequences in* $\mathrm{Coh}(X)$*; let $h : E'' \to F''$ be a* *morphism of sheaves:*

$$\begin{array}{ccccccccc} & & E' & \xrightarrow{e'} & E & \xrightarrow{e''} & E'' & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & F' & \xrightarrow{f'} & F & \xrightarrow{f''} & F'' & \longrightarrow & 0\,. \end{array}$$

Assume that there exists a morphism $\lambda : E \to F$ *such that* $h \circ e'' = f'' \circ \lambda$. *Define a morphism of abelian groups:*

$$\begin{aligned} p : \mathrm{Hom}(E, F') &\to \mathrm{Hom}(E', F') \\ g &\mapsto g \circ e'\,. \end{aligned}$$

Define a set

$$\Gamma := \{\alpha : E'' \to F \,|\, f'' \circ \alpha = h\}\,.$$

Then

1. *the triple* $(\underline{e}, \underline{f}, h)$ *induces an element* $ob \in \mathrm{Coker}(p)$ *such that* Γ *is non empty if and only if* $ob = 0$*;*

2. *if* Γ *is non empty, it is a principal homogeneous space on* $\mathrm{Ker}(p)$*.*

Proof. Define

$$\Lambda := \{\lambda : E \to F \,|\, h \circ e'' = f'' \circ \lambda\}\,.$$

By hypothesis the set Λ is non empty. For any $\lambda \in \Lambda$ define $\bar\lambda := \lambda \circ e' : E' \to F'$ and write $[\bar\lambda]$ for the class of $\bar\lambda$ in $\mathrm{Coker}(p)$. Define a subset $\Lambda_0 \subseteq \Lambda$ as $\Lambda_0 := \{\lambda \in \Lambda \,|\, \bar\lambda = 0\}$. There is a canonical bijection $\Lambda_0 \leftrightarrow \Gamma$.
The group $\mathrm{Hom}(E, F')$ acts transitively on the set Λ:

$$\begin{aligned} \mathrm{Hom}(E, F') \times \Lambda &\to \Lambda\,, \\ (g, \lambda) &\mapsto \lambda + f' \circ g\,. \end{aligned}$$

The group $\mathrm{Hom}(E, F')$ also acts on the set $\mathrm{Hom}(E', F')$:

$$\begin{aligned} \mathrm{Hom}(E, F') \times \mathrm{Hom}(E', F') &\to \mathrm{Hom}(E', F')\,, \\ (g, k) &\mapsto k + p(g)\,. \end{aligned}$$

Furthermore, the map $\Lambda \to \mathrm{Hom}(E', F')$ is $\mathrm{Hom}(E, F')$-equivariant:

$$\overline{\lambda + f' \circ g} = \bar\lambda + p(g)\,.$$

Hence, the map $\Lambda \to \mathrm{Coker}(p)$ is $\mathrm{Hom}(E, F')$-invariant:

$$[\overline{\lambda + f' \circ g}] = [\bar\lambda + p(g)] = [\bar\lambda]\,.$$

Since the action of $\mathrm{Hom}(E, F')$ on Λ is transitive, we deduce that the map $\Lambda \to \mathrm{Coker}(p)$ is constant and we define *ob* as the only element in the image of $\Lambda \to \mathrm{Coker}(p)$. By construction $ob \neq 0$ if and only if $\Lambda_0 = \varnothing$, if and only if $\Gamma = \varnothing$. The second statement is a consequence of the injectivity of $f' : F' \to F$ (hence the action of $\mathrm{Hom}(E, F')$ on Λ is simple). $\square$

Remark 3.1.2. Lemma 3.1.1 is still valid if we replace $\mathrm{Coh}(X)$ with an arbitrary abelian category. $\lozenge$

Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$. In the following results we use the notation $\mathbb{V}(\mathcal{F})$ meaning $\mathrm{Spec}\,\mathrm{Sym}(\mathcal{F})$, as explained in 1.1. Recall that if $g : T \to \mathcal{X}$ is morphism of stacks and T is a scheme, giving an $\mathcal{X}$-morphism $T \to \mathbb{V}(\mathcal{F})$ is equivalent to giving a morphism of sheaves $g^*\mathcal{F} \to \mathcal{O}_T$ [Ols16, Section 10.2].

Lemma 3.1.3. *Let Y be a scheme, let*

$$\mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence in $\mathrm{Coh}(Y)$ *and assume that $\mathcal{G}$ is locally free. Let $X := \mathbb{V}(\mathcal{H})$, and let $G : X \to Y$ be the canonical morphism. Let $x \in X$ and $y \in Y$ be closed points such that $G(x) = y$. Let $G_* : \mathrm{h}_{X,x} \Rightarrow \mathrm{h}_{Y,y}$ be the natural transformation induced by G at x (see Remark 1.3.9). Let $z :$ $\mathrm{Hom}(y^*\mathcal{G}, \mathbb{C}) \to \mathrm{Hom}(y^*\mathcal{K}, \mathbb{C})$ *be the morphism induced by* $\mathcal{K} \to \mathcal{G}$.*
Then $\mathrm{Ker}(z)$ *is the tangent space of* G_* *and* $\mathrm{Coker}(z)$ *is an obstruction space of* G_*.

Proof. Let $q : A \twoheadrightarrow B$ be a semismall extension in (Art), let $I := \mathrm{Ker}\, q$ and let $(b, a) \in \mathrm{h}_{X,x}(B) \times_{\mathrm{h}_{Y,y}(B)} \mathrm{h}_{Y,y}(A)$: we have a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[r, "b"] \arrow[d, "j"] & X \arrow[d, "G"] \\ \mathrm{Spec}\, A \arrow[r, "a"] & Y . \end{tikzcd}$$

Hence the morphism b is induced by a morphism of sheaves $\beta : a^*\mathcal{H} \to B$.
The diagram

$$\begin{tikzcd} & a^*\mathcal{K} \arrow[r] & a^*\mathcal{G} \arrow[r] & a^*\mathcal{H} \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & I \arrow[r] & A \arrow[r, "q"] & B \arrow[r] & 0 \end{tikzcd}$$

satisfies the hypotheses of Lemma 3.1.1. Indeed $\mathcal{G}$ is locally free, hence the morphism $\mathbb{V}(\mathcal{G}) \to Y$ is smooth and therefore there exists a lifting $g : \mathrm{Spec}\, A \to \mathbb{V}(\mathcal{G})$

of a; the morphism g induces a morphism of sheaves $\gamma : a^*\mathcal{G} \to A$ making the diagram commute.
Define a set

$$\Gamma := \{\alpha : a^*\mathcal{H} \to A \,|\, q \circ \alpha = \beta\} \subseteq \mathrm{Hom}_{A\text{-lin}}(a^*\mathcal{H}, A)\,.$$

A morphism $\bar{a} \in \mathrm{h}_{X,x}(A)$ induces the pair (b, a) if and only if $\bar{a}$ is induced by a morphism of sheaves $\alpha : a^*\mathcal{H} \to A$ such that $q \circ \alpha = \beta$, i.e. an element of Γ.
By adjunction we get the isomorphisms

$$\mathrm{Hom}_{A\text{-lin}}(a^*\mathcal{G}, I) \cong \mathrm{Hom}_{\mathbb{C}\text{-lin}}(y^*\mathcal{G}, I_{\mathbb{C}}) \cong \mathrm{Hom}_{\mathbb{C}\text{-lin}}(y^*\mathcal{G}, \mathbb{C}) \otimes I_{\mathbb{C}}\,,$$

and $\mathrm{Hom}_{A\text{-lin}}(a^*\mathcal{K}, I) \cong \mathrm{Hom}_{\mathbb{C}\text{-lin}}(y^*\mathcal{K}, \mathbb{C}) \otimes I_{\mathbb{C}}$. Moreover the morphism obtained from $a^*\mathcal{K} \to a^*\mathcal{G}$ by applying the functor $\mathrm{Hom}(-, I)$ is exactly $z \otimes \mathrm{id}_{I_{\mathbb{C}}}$. By Lemma 3.1.1 we deduce that

$$0 \to \mathrm{Ker}\, z \otimes I_{\mathbb{C}} \to \mathrm{h}_{X,x}(A) \to \mathrm{h}_{X,x}(B) \times_{\mathrm{h}_{Y,y}(B)} \mathrm{h}_{Y,y}(A) \to \mathrm{Coker}\, z \otimes I_{\mathbb{C}}$$

is an exact sequence of groups and sets (see Definition 1.3.6). Hence $\mathrm{Ker}(z)$ is the tangent space of G_* and $\mathrm{Coker}(z)$ is an obstruction space of G_*. □

We are now ready to compute tangent and obstruction spaces for the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ (defined in 2.1).

Proposition 3.1.4. *Let $n := (p : C \to \mathrm{Spec}\,\mathbb{C}, E, F) \in \mathcal{N}(\mathbb{C})$ be any point and let $s \in \mathcal{S}(\mathbb{C})$ be a point such that $G(s) = n$. Let $G : \mathrm{h}_{\mathcal{S},s} \to \mathrm{h}_{\mathcal{N},n}$ be the morphism of pseudo functor induced by the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$. Then*

- $\mathrm{Hom}(F, E)$ *is the tangent space for $G : \mathrm{h}_{\mathcal{S},s} \to \mathrm{h}_{\mathcal{N},n}$, and*
- $\mathrm{Ext}^1(F, E)$ *is an obstruction space for $G : \mathrm{h}_{\mathcal{S},s} \to \mathrm{h}_{\mathcal{N},n}$.*

Proof. By Proposition 1.3.12 it is enough to assume that $G : \mathcal{S} \to \mathcal{N}$ is a morphism of schemes.
Let $(\pi : \mathcal{N}' \to \mathcal{N}, \mathcal{E}, \mathcal{F})$ be the universal family of $\mathcal{N}$, let $\omega \in \mathrm{Pic}(\mathcal{N}')$ be the dualizing sheaf of π. Define $\mathcal{H} := \mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega \in \mathrm{Coh}(\mathcal{N}')$, so that $\mathcal{S} \cong \mathbb{V}(\mathbb{R}^1\pi_*\mathcal{H})$ (Proposition 2.1.12). By Proposition 1.8.9 we may choose an exact sequence

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

in $\mathrm{Coh}(\mathcal{N}')$ such that

1. $\mathcal{K}, \mathcal{G} \in \mathrm{Coh}(\mathcal{N}')$, $\mathcal{G}$ is locally free and $\mathcal{K}$ is flat over $\mathcal{N}$;
2. $\pi_*\mathcal{K} = \pi_*\mathcal{G} = 0$;

3. $\mathbb{R}^1\pi_*\mathcal{K}$ and $\mathbb{R}^1\pi_*\mathcal{G}$ are locally free sheaves of finite rank on $\mathcal{N}$ and commute with base change.

Hence
$$\mathbb{R}\pi_*\mathcal{H}[1] \cong [\mathbb{R}^1\pi_*\mathcal{K} \to \mathbb{R}^1\pi_*\mathcal{G}]$$
is perfect of perfect amplitude contained in $[-1,0]$. Therefore
$$\mathbb{L}n^*\mathbb{R}\pi_*\mathcal{H}[1] \cong [n^*\mathbb{R}^1\pi_*\mathcal{K} \to n^*\mathbb{R}^1\pi_*\mathcal{G}]$$
and $n^*\mathbb{R}^1\pi_*\mathcal{K} \to n^*\mathbb{R}^1\pi_*\mathcal{G} \to \mathbb{L}n^*\mathbb{R}\pi_*\mathcal{H}[1] \to n^*\mathbb{R}^1\pi_*\mathcal{K}[1]$ is a distinguished triangle (by definition of mapping cone). The functor $\mathrm{Hom}_{\mathrm{D}^b(\mathbb{C})}(-,\mathbb{C})$ is cohomological, so we get a long exact sequence
$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathbb{C}}(n^*\mathbb{R}^1\pi_*\mathcal{H},\mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(n^*\mathbb{R}^1\pi_*\mathcal{G},\mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(n^*\mathbb{R}^1\pi_*\mathcal{K},\mathbb{C}) \\ \to \mathrm{Hom}_{\mathrm{D}^b(\mathbb{C})}(n^*\mathbb{R}\pi_*\mathcal{H},\mathbb{C}) \to 0\,, \end{aligned}$$
where we have used Lemma 2.1.9 if possible.
By Lemma 3.1.3 we have that $\mathrm{Hom}_{\mathbb{C}}(n^*\mathbb{R}^1\pi_*\mathcal{H},\mathbb{C})$ is the tangent space of G and that $\mathrm{Hom}_{\mathrm{D}^b(\mathbb{C})}(n^*\mathbb{R}\pi_*\mathcal{H},\mathbb{C})$ is an obstruction space of G. Let $H := F \otimes E^\vee$ and let ω_p be the dualizing sheaf of C (i.e. the dualizing sheaf of $\pi : \mathcal{N}' \to \mathcal{N}$ at n). By cohomology and base change we have that $n^*\mathbb{R}\pi_*\mathcal{H} \cong \mathbb{R}p_*(H \otimes \omega_p)$, in particular $n^*\mathbb{R}^1\pi_*\mathcal{H} \cong \mathbb{R}^1p_*(H \otimes \omega_p)$. By Lemma 2.1.11 the tangent space of G is $\mathrm{Hom}(H,\mathcal{O}_C) \cong \mathrm{Hom}(F,E)$. By Grothendieck duality
$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}^b(\mathbb{C})}(\mathbb{R}p_*(H\otimes\omega_p),\mathbb{C}) &\cong \mathrm{Hom}_{\mathrm{D}^b(C)}(H\otimes\omega_p,\omega_p[1]) \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(C)}(H,\mathcal{O}_C[1]) \cong \mathrm{Ext}^1(H,\mathcal{O}_C) \\ &\cong \mathrm{Ext}^1(F,E)\,. \end{aligned}$$
□

3.2 Construction of the obstruction theory

In this section we construct a morphism $E^\bullet := \mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1]) \to \tau_{\geq -1}\,\mathrm{L}_G$ in the derived category of $\mathcal{S}$ and we prove that it is a perfect relative obstruction theory for the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ (defined in 2.1). We provide two different constructions of $E^\bullet \to \tau_{\geq -1}\,\mathrm{L}_G$. The first one (Proposition 3.2.3), elementary and very explicit, is suitable to prove that the morphism $E^\bullet \to \tau_{\geq -1}\,\mathrm{L}_G$ is indeed an obstruction theory; the second one (Proposition 3.2.5), more intrinsic and canonical, allow us to show that such an obstruction theory descends from a canonical morphism $E^\bullet \to \mathrm{L}_G$.

Lemma 3.2.1. *Let $\mathcal{X}$ be an algebraic stack, let*
$$\mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence in $\mathrm{Coh}(\mathcal{X})$*, let* $p : \mathbb{V}(\mathcal{G}) \to \mathcal{X}$ *be the canonical morphism and let* $\mathcal{I} \in \mathrm{Coh}(\mathbb{V}(\mathcal{G}))$ *be the ideal sheaf induced by the closed embedding* $\mathbb{V}(\mathcal{H}) \hookrightarrow \mathbb{V}(\mathcal{G})$*. Then the morphism* $\mathcal{K} \to \mathcal{G}$ *induces a surjective map*

$$p^*\mathcal{K} \twoheadrightarrow \mathcal{I}\,.$$

Proof. Denote the closed embedding of $\mathbb{V}(\mathcal{H})$ in $\mathbb{V}(\mathcal{G})$ by $\iota : \mathbb{V}(\mathcal{H}) \to \mathbb{V}(\mathcal{G})$. We have an exact sequence in $\mathrm{Coh}(\mathbb{V}(\mathcal{G}))$:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{V}(\mathcal{G})} \to \iota_*\mathcal{O}_{\mathbb{V}(\mathcal{H})} \to 0\,,$$

which induces an exact sequence in $\mathrm{Coh}(\mathcal{X})$:

$$0 \to p_*\mathcal{I} \to \mathrm{Sym}\,\mathcal{G} \to \mathrm{Sym}\,\mathcal{H}\,.$$

The morphism $\mathcal{K} \to \mathrm{Sym}\,\mathcal{G} \to \mathrm{Sym}\,\mathcal{H}$ is zero, hence $\mathcal{K} \to \mathrm{Sym}\,\mathcal{G}$ induces a morphism $\mathcal{K} \to p_*\mathcal{I}$. By adjunction we get a morphism $\psi : p^*\mathcal{K} \to \mathcal{I}$. To check that ψ is surjective we may assume that $\mathcal{X}$ is an affine scheme, i.e. $\mathcal{X} = \mathrm{Spec}\,A$ where A is a $\mathbb{C}$-algebra. The morphism $\mathcal{K} \to \mathcal{G}$ induces a morphism $\mathcal{K}\otimes_A \mathrm{Sym}\,\mathcal{G} \to \mathcal{G}\otimes_A \mathrm{Sym}\,\mathcal{G} \to \mathrm{Sym}\,\mathcal{G}$, which we denote by $f : \mathcal{K}\otimes_A \mathrm{Sym}\,\mathcal{G} \to \mathrm{Sym}\,\mathcal{G}$. But $\mathrm{Im}\, f = \mathcal{I}$ and $f : \mathcal{K} \otimes_A \mathrm{Sym}\,\mathcal{G} \twoheadrightarrow \mathcal{I}$ is precisely the morphism $\psi : p^*\mathcal{K} \to \mathcal{I}$ when $\mathcal{X} = \mathrm{Spec}\,A$ is affine. □

Lemma 3.2.2. *Let* $\mathcal{X}$ *be an algebraic stack, let*

$$\mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence in $\mathrm{Coh}(\mathcal{X})$*, let* $q : \mathbb{V}(\mathcal{H}) \to \mathcal{X}$ *be the canonical morphism, let* $\iota : \mathbb{V}(\mathcal{H}) \hookrightarrow \mathbb{V}(\mathcal{G})$ *be the closed embedding induced by the surjection* $\mathcal{G} \twoheadrightarrow \mathcal{H}$*, and let* $\mathcal{I} \in \mathrm{Coh}(\mathbb{V}(\mathcal{G}))$ *be the ideal sheaf induced by* ι*. Assume that* $\mathcal{G}$ *is a locally free sheaf and let* Ω *denote the relative cotangent sheaf of the morphism* $\mathbb{V}(\mathcal{G}) \to \mathcal{X}$*. Then the morphism* $\mathcal{K} \to \mathcal{G}$ *induces a commutative diagram*

$$\begin{tikzcd} q^*\mathcal{K} \arrow[r] \arrow[d, "f_{-1}"] & q^*\mathcal{G} \arrow[d, "f_0"] \\ \iota^*\mathcal{I} \arrow[r] & \iota^*\Omega\,, \end{tikzcd}$$

where f_{-1} *is a surjective morphism and* f_0 *is an isomorphism.*

Proof. Let $p : \mathbb{V}(\mathcal{G}) \to \mathcal{X}$ be the canonical morphism. By Lemma 3.2.1 the morphism $\mathcal{K} \to \mathcal{G}$ induces a surjective morphism $p^*\mathcal{K} \to \mathcal{I}$. Since $\mathcal{G}$ is locally free, there is a canonical isomorphism $p^*\mathcal{G} \to \Omega$. Applying the right-exact functor ι^* we get the morphisms f_{-1} and f_0 (notice that $p \circ \iota = q$). Checking that the diagram commutes is straightforward. □

Proposition 3.2.3. *Let $G : \mathcal{S} \to \mathcal{N}$ be the forgetful morphism, let $(\bar{\pi} : \mathcal{S}' \to \mathcal{S}, \bar{\mathcal{E}}, \bar{\mathcal{F}}, \phi)$ be the universal family of $\mathcal{S}$, let $\bar{\omega}$ be the dualizing sheaf of the morphism $\bar{\pi}$ and let L_G be the cotangent complex of G. Then there is a canonical morphism*

$$\mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1]) \longrightarrow \tau_{\geq -1} \mathrm{L}_G \ .$$

Proof. Let $(\pi : \mathcal{N}' \to \mathcal{N}, \mathcal{E}, \mathcal{F})$ be the universal family of $\mathcal{N}$ and let ω be the dualizing sheaf of $\pi : \mathcal{N}' \to \mathcal{N}$. Recall that we have a cartesian diagram

$$\begin{tikzcd} \mathcal{S}' \arrow[r, "\bar{G}"] \arrow[d, "\bar{\pi}"] \arrow[dr, phantom, "\square"] & \mathcal{N}' \arrow[d, "\pi"] \\ \mathcal{S} \arrow[r, "G"] & \mathcal{N}\,. \end{tikzcd}$$

Denote $\mathcal{H} := \mathcal{F} \otimes \mathcal{E}^\vee \otimes \omega \in \mathrm{Coh}(\mathcal{N}')$. By Proposition 1.8.9 we can choose a resolution of $\mathcal{H}$,

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0\,,$$

such that

1. $\mathcal{K}, \mathcal{G} \in \mathrm{Coh}(\mathcal{N}')$, $\mathcal{G}$ is locally free and $\mathcal{K}$ is flat over $\mathcal{N}$;
2. $\pi_*\mathcal{K} = \pi_*\mathcal{G} = 0$;
3. $\mathbb{R}^1\pi_*\mathcal{K}$ and $\mathbb{R}^1\pi_*\mathcal{G}$ are locally free sheaves in $\mathrm{Coh}(\mathcal{N})$ and they commute with base change.

Applying the functor π_* we get an exact sequence in $\mathrm{Coh}(\mathcal{N})$:

$$0 \to \pi_*\mathcal{H} \to \mathbb{R}^1\pi_*\mathcal{K} \to \mathbb{R}^1\pi_*\mathcal{G} \to \mathbb{R}^1\pi_*\mathcal{H} \to 0\,.$$

By Proposition 2.1.12 we have that $\mathcal{S} \cong \mathbb{V}(\mathbb{R}^1\pi_*\mathcal{H})$; hence we have a commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\iota"] \arrow[dr, "G"'] & \mathbb{V}(\mathbb{R}^1\pi_*\mathcal{G}) \arrow[d, "p"] \\ & \mathcal{N}\,, \end{tikzcd}$$

where ι is a closed embedding and p is smooth. By Lemma 3.2.2 we have a commutative diagram

$$\begin{tikzcd} G^*\mathbb{R}^1\pi_*\mathcal{K} \arrow[r] \arrow[d, "f_{-1}"] & G^*\mathbb{R}^1\pi_*\mathcal{G} \arrow[d, "f_0"] \\ \iota^*\mathcal{I} \arrow[r] & \iota^*\Omega\,, \end{tikzcd}$$

where $\mathcal{I}$ is the ideal sheaf induced by ι, Ω is the cotangent sheaf of p, f_{-1} is a surjective morphism and f_0 is an isomorphism. Notice that $\tau_{\geq -1} \mathrm{L}_G \cong [\iota^*\mathcal{I} \to \iota^*\Omega]$ by Corollary 2.1.15. Now, $\mathbb{R}^1\pi_*\mathcal{K}$ and $\mathbb{R}^1\pi_*\mathcal{G}$ commute with base change; hence $G^*\mathbb{R}^1\pi_*\mathcal{K} = \mathbb{R}^1\bar{\pi}_*\bar{G}^*\mathcal{K}$ and $G^*\mathbb{R}^1\pi_*\mathcal{G} = \mathbb{R}^1\bar{\pi}_*\bar{G}^*\mathcal{G}$.
Denote $\bar{\mathcal{H}} := \bar{G}^*\mathcal{H} = \bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega} \in \mathrm{Coh}(\mathcal{S}')$, and $\bar{\mathcal{K}} := \bar{G}^*\mathcal{K}$, $\bar{\mathcal{G}} := \bar{G}^*\mathcal{G}$. By Remark 1.8.10 we have that

$$0 \to \bar{\mathcal{K}} \to \bar{\mathcal{G}} \to \bar{\mathcal{H}} \to 0$$

is a resolution of $\bar{\mathcal{H}}$ such that

1. $\bar{\mathcal{K}}, \bar{\mathcal{G}} \in \mathrm{Coh}(\mathcal{S}')$, $\bar{\mathcal{G}}$ is locally free and $\bar{\mathcal{K}}$ is flat over $\mathcal{S}$;
2. $\bar{\pi}_*\bar{\mathcal{K}} = \bar{\pi}_*\bar{\mathcal{G}} = 0$;
3. $\mathbb{R}^1\bar{\pi}_*\bar{\mathcal{K}}$ and $\mathbb{R}^1\bar{\pi}_*\bar{\mathcal{G}}$ are locally free sheaves in $\mathrm{Coh}(\mathcal{S})$ and they commute with base change.

Hence $\mathbb{R}\bar{\pi}_*\bar{\mathcal{H}} \cong [\mathbb{R}^1\bar{\pi}_*\bar{\mathcal{K}} \to \mathbb{R}^1\bar{\pi}_*\bar{\mathcal{G}}]$ and it is a perfect complex in degrees 0,1. It follows that $\mathbb{R}\bar{\pi}_*\bar{\mathcal{H}}[1] = \mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1])$ is a complex in degrees -1,0. Summing up, the commutative diagram

$$\begin{tikzcd} \mathbb{R}^1\bar{\pi}_*\bar{\mathcal{K}} \arrow[r] \arrow[d, "f_{-1}"] & \mathbb{R}^1\bar{\pi}_*\bar{\mathcal{G}} \arrow[d, "f_0"] \\ \iota^*\mathcal{I} \arrow[r] & \iota^*\Omega \end{tikzcd}$$

induces a morphism $\theta : \mathbb{R}\bar{\pi}_*\bar{\mathcal{H}}[1] \to \tau_{\geq -1} \mathrm{L}_G$ in $\mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(\mathcal{S})$ such that $\mathrm{h}^{-1}(\theta)$ is surjective and $\mathrm{h}^0(\theta)$ is an isomorphism. Therefore θ is an obstruction theory. □

Remark 3.2.4. Actually, the proof of Proposition 3.2.3 is incomplete: we should check that the morphism $E^\bullet := \mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1]) \to \tau_{\geq -1} \mathrm{L}_G$ does not depend on the choice of the resolution $0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$ of $\mathcal{H}$. However that is not necessary, as we are going to give a canonical intrinsic construction of $E^\bullet \to \tau_{\geq -1} \mathrm{L}_G$ in Proposition 3.2.5. We will also prove that the obstruction theory $E^\bullet \to \tau_{\geq -1} \mathrm{L}_G$ indeed comes from a canonical morphism $E^\bullet \to \mathrm{L}_G$. ♢

Proposition 3.2.5. *Let $G : \mathcal{S} \to \mathcal{N}$ be the forgetful morphism, let $(\bar{\pi} : \mathcal{S}' \to \mathcal{S}, \bar{\mathcal{E}}, \bar{\mathcal{F}}, \phi)$ be the universal family of $\mathcal{S}$, let $\bar{\omega}$ be the dualizing sheaf of the morphism $\bar{\pi}$ and let L_G be the cotangent complex of G. Then there is a canonical morphism*

$$\mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1]) \longrightarrow \mathrm{L}_G \ .$$

Proof. Let $(\pi : \mathcal{N}' \to \mathcal{N}, \mathcal{E}, \mathcal{F})$ be the universal family of $\mathcal{N}$. Let $p : \mathbb{V}(\mathcal{F}\otimes\mathcal{E}^\vee) \to \mathcal{N}'$ and $\bar{p} : \mathbb{V}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee) \to \mathcal{S}'$ be the structure morphisms respectively of $\mathbb{V}(\mathcal{F} \otimes \mathcal{E}^\vee)$ and $\mathbb{V}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee)$. Both p and $\bar{p}$ are smooth morphisms, since $\mathcal{F} \otimes \mathcal{E}^\vee$ and $\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee$ are locally free sheaves. The universal morphism $\phi : \bar{\mathcal{F}} \to \bar{\mathcal{E}}$ induces a section $f : \mathcal{S}' \to \mathbb{V}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee)$ of $\bar{p}$. Therefore we have a commutative diagram:

$$\begin{tikzcd}
\mathbb{V}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee) \arrow[r, "\bar{p}"] \arrow[d, "v"] \arrow[dr, phantom, "\square"] & \mathcal{S}' \arrow[r, "\bar{\pi}"] \arrow[d, "G'"] \arrow[l, bend right, dashed, "f"'] \arrow[dr, phantom, "\square"] & \mathcal{S} \arrow[d, "G"] \\
\mathbb{V}(\mathcal{F} \otimes \mathcal{E}^\vee) \arrow[r, "p"] & \mathcal{N}' \arrow[r, "\pi"] & \mathcal{N}
\end{tikzcd} .$$

Since both $p : \mathbb{V}(\mathcal{F} \otimes \mathcal{E}^\vee) \to \bar{\mathcal{C}}$ and $\pi : \mathcal{C} \to \mathcal{N}$ are flat morphism, we have the following natural isomorphisms:

$$\mathrm{L}_{G'} \cong \bar{\pi}^* \mathrm{L}_G ,$$
$$\mathrm{L}_{\bar{p}} \cong v^* \mathrm{L}_p .$$

Define $z := p \circ v$; then $G' = z \circ f$. Hence we have distinguished triangles

$$v^* \mathrm{L}_p \to \mathrm{L}_z \to \mathrm{L}_v \to v^* \mathrm{L}_p[1] ,$$
$$f^* \mathrm{L}_z \to \mathrm{L}_{G'} \to \mathrm{L}_f \to f^* \mathrm{L}_z[1] .$$

From the first triangle we get a map $\mathrm{L}_{\bar{p}} \to \mathrm{L}_z$ and hence a map $f^* \mathrm{L}_{\bar{p}} \to f^* \mathrm{L}_z$; from the second triangle we get a map $f^* \mathrm{L}_z \to \pi^* \mathrm{L}_G$. Composing them we obtain a map $f^* \mathrm{L}_{\bar{p}} \to \pi^* \mathrm{L}_G$. Since $\bar{p} : \mathbb{V}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee) \to \mathcal{S}'$ is smooth, we have that $\mathrm{L}_{\bar{p}} \cong \Omega_{\bar{p}} \cong \bar{p}^*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee)$. Therefore $f^* \mathrm{L}_{\bar{p}} \cong f^* p^*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee) \cong \bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee$ and we have a map

$$\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \to \bar{\pi}^* \mathrm{L}_G .$$

Now we use Grothendieck duality:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^b(\mathcal{S}')}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee, \bar{\pi}^* \mathrm{L}_G) &\cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{S}')}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1], \bar{\pi}^* \mathrm{L}_G \otimes \bar{\omega}[1]) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{S}')}(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1], \bar{\pi}^! \mathrm{L}_G) \\
&\cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{S})}(\mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1]), \mathrm{L}_G) .
\end{aligned}$$

Hence, the morphism $\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \to \bar{\pi}^* \mathrm{L}_G$ naturally induces a morphism

$$\mathbb{R}\bar{\pi}_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1]) \to \mathrm{L}_G .$$ $\square$

Chapter 4

Rigidification

In this chapter we recall the notion of rigidification of an algebraic stack with respect to the multiplication by scalars, as it is introduced in [ACV03]. We describe the rigidification of the stacks $\mathcal{S}$ and $\mathcal{N}$ (defined in 2.1) and we prove that the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ (see Proposition 2.1.6) is compatible with it. Then we prove that the obstruction theory $E^\bullet \to \tau_{\geq 1} \mathrm{L}_G$ (defined in 3.2) descends to the rigidification. Finally, we use all these results to produce a perfect obstruction theory for every moduli space of α-stable coherent systems (see Definition 1.5.2).

Notation 4.0.6. In this chapter we use the notation $\mathbb{G}$ to denote the multiplicative group scheme $\mathbb{G}_m = \operatorname{Spec} \mathbb{C}[t, t^{-1}]$ and $\mathcal{O}(\mathbb{G})$ to denote the space of global sections of $\mathcal{O}_{\mathbb{G}}$. In 1.6 we observed that $\mathcal{O}(\mathbb{G})$ is a bi-algebra over $\mathbb{C}$. If V is a $\mathbb{C}$-vector space, a representation $r : \mathbb{G} \to \mathrm{GL}(V)$ of $\mathbb{G}$ on V is equivalent to an $\mathcal{O}(\mathbb{G})$-comodule structure $\rho : V \to V \otimes \mathcal{O}(\mathbb{G})$ on V [Mil12, VIII.6].

Recall that $\mathrm{B}\mathbb{G}$ is the stack whose T-points are $\mathbb{G}$-torsors over the scheme T (compare [Ols16, Definition 8.1.14]).

4.1 The rigidification of the stack of GCS

In this section we give a short review of the rigidification of an algebraic stack [ACV03, Section 5.1] and we prove that the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ descends to the rigidification.

Theorem 4.1.1 ([ACV03, 5.1.5])**.** *Let $\mathcal{X}$ be an algebraic stack. Assume that for any scheme T and for any $x \in \mathcal{X}(T)$ there is an injective homomorphism of groups*

$$\iota_x : \mathbb{G}(T) \to \mathrm{Aut}(x)$$

compatible with pullbacks. Then there is a smooth surjective morphism of algebraic stacks

$$\mathrm{Rig} : \mathcal{X} \to \mathcal{X}^{\mathbb{G}}$$

such that

1. *for any scheme T, for any $x \in \mathcal{X}(T)$, the morphism $\iota_x : \mathbb{G}(T) \to \mathrm{Aut}(x)$ factors into the kernel of $\mathrm{Rig}_x : \mathrm{Aut}(x) \to \mathrm{Aut}(\mathrm{Rig}(x))$;*
2. *the morphism $\mathrm{Rig} : \mathcal{X} \to \mathcal{X}^{\mathbb{G}}$ is universal for morphisms of stacks $\mathcal{X} \to \mathcal{Y}$ satisfying property (1) above;*
3. *is $T = \mathrm{Spec}\,\mathbb{C}$, then in (1) above $\mathrm{Aut}(\mathrm{Rig}(x)) = \mathrm{Aut}(x)/\mathbb{G}(T)$.*

Proof. That is a reformulation of [ACV03, Theorem 5.1.5]. □

Definition 4.1.2. Let $\mathcal{X}$ be an algebraic stack which satisfies the hypotheses of Theorem 4.1.1. We call the morphism $\mathrm{Rig} : \mathcal{X} \to \mathcal{X}^{\mathbb{G}}$ the *rigidification of $\mathcal{X}$ along* $\mathbb{G}$.

Remark 4.1.3. The rigidification $\mathrm{Rig} : \mathcal{X} \to \mathcal{X}^{\mathbb{G}}$ makes $\mathcal{X}$ into a gerbe over $\mathcal{X}^{\mathbb{G}}$ banded by the group scheme $\mathbb{G}$ [Sta17, Tag 06QB]. In particular, locally $\mathcal{X} \cong \mathcal{X}^{\mathbb{G}} \times \mathrm{B}\mathbb{G}$ [Sta17, Tag 06QH]. ◊

We need the following algebraic lemma in order to prove the existence of a canonical injective morphism $\iota_s : \mathbb{G}(T) \to \mathrm{Aut}(s)$ for all families of GCS's, i.e. for every $s \in \mathcal{S}(T)$.

Lemma 4.1.4. *Let R be a noetherian ring, let M be a f.g. flat module on R such that $M \neq 0$ and let $r \in R \smallsetminus \{0\}$. Then the endomorphism $\mu_r^M : M \to M$ induced by r by multiplication is nonzero.*

Proof. Let $\mu_r : R \to R$ be the multiplication morphism induced by r. Then $\mu_r^M = \mu_r \otimes \mathrm{id}_M$. Since $r \neq 0$, we have that $\mu_r \neq 0$.
We may assume that R is a local ring. In case R is not local there exists $p \in \mathrm{Spec}\,R$ such that $(\mu_r)_p : R_p \to R_p$ is different from zero: it's enough to prove the statement on R_p.
Since R is local, every proper ideal of R is contained in the Jacobson radical of R. Let $I := \mathrm{Ker}\,\mu_r$; being $\mu_r \neq 0$, we have that $I \neq R$. Hence $IM \neq M$, by Nakayama Lemma. The morphism $\tilde{\mu}_r : R/I \to R$ naturally induced by μ_r is injective. So the morphism $\tilde{\mu}_r^M : M \otimes R/I \to M$ obtained by tensoring with M is still injective, because M is flat. But $M \otimes R/I \cong M/IM \neq 0$. Therefore $\tilde{\mu}_r^M \neq 0$. Finally $\mu_r^M = \tilde{\mu}_r^M \circ \pi$, where $\pi : M \to M \otimes R/I$ is the canonical surjection; hence we have that $\mu_r^M \neq 0$ if $\tilde{\mu}_r^M \neq 0$. □

Proposition 4.1.5. *Let $\mathcal{S}$ be the stack of GCS's (see Definition 2.1.5); let T be a scheme and let $s \in \mathcal{S}(T)$. There is a canonical injective morphism of groups $\iota_s : \mathbb{G}(T) \to \mathrm{Aut}(s)$ which is compatible with pullbacks.*

Proof. We may assume that $T = \operatorname{Spec} R$ is an affine scheme. The T-point s of $\mathcal{S}$ is induced by an object $(p : C \to T, \varphi : F \to E) \in \mathcal{S}(T)$. The sheaves $E, F \in \operatorname{Coh}(C)$ are flat R-modules. Hence any $r \in R^* = \mathbb{G}(T)$ induces an automorphism of s by multiplication: if $\mu_r^E : E \to E$ and $\mu_r^F : F \to F$ are the multiplication morphisms then $\varphi \circ \mu_r^F = \mu_r^E \circ \varphi$. By Lemma 4.1.4 the morphism of groups $\iota_s : \mathbb{G}(T) \to \operatorname{Aut}(s)$ so constructed is injective. □

Proposition 4.1.6. *Let $\mathcal{N}$ be the stack defined in 2.1.3; let T be a scheme and let $n \in \mathcal{N}(T)$. There exists a canonical injective morphism of groups $(\mathbb{G} \times \mathbb{G})(T) \to \operatorname{Aut}(n)$ which is compatible with pullbacks.*
By composing with the diagonal morphism $\mathbb{G}(T) \to (\mathbb{G} \times \mathbb{G})(T)$, that induces an injective morphism of groups $\mathbb{G}(T) \to \operatorname{Aut}(n)$.

Proof. The proof of this statement is analogous to the proof of Proposition 4.1.5. □

Remark 4.1.7. Notice that distinct elements $r, r' \in R^* = \mathbb{G}(T)$ in general do not induce an automorphism of $s \in \mathcal{S}(T)$. Anyway, that is true if s lies in the image of the zero section of the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ (recall that the morphism G makes $\mathcal{S}$ into an abelian cone over $\mathcal{N}$; compare Proposition 2.1.12). ♢

In the next result we check that the forgetful morphism $G : \mathcal{S} \to \mathcal{N}$ is compatible with the rigidification.

Corollary 4.1.8. *Let $G : \mathcal{S} \to \mathcal{N}$ be the forgetful morphism; let $\mathcal{S} \to \mathcal{S}^{\mathbb{G}}$ be the rigidification of $\mathcal{S}$ and let $\mathcal{N} \to \mathcal{N}^{\mathbb{G}}$ be the rigidification of $\mathcal{N}$. Then there exists a unique morphism (up to unique 2-isomorphism) $\tilde{G} : \mathcal{S}^{\mathbb{G}} \to \mathcal{N}^{\mathbb{G}}$ such that the following diagram is 2-cartesian:*

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d, "G"] & \mathcal{S}^{\mathbb{G}} \arrow[d, "\tilde{G}"] \\ \mathcal{N} \arrow[r] & \mathcal{N}^{\mathbb{G}} \end{tikzcd} .$$

Proof. The morphism $\mathcal{S} \to \mathcal{N} \to \mathcal{N}^{\mathbb{G}}$ satisfy property (1) in the theorem. Hence, the existence $\tilde{G} : \mathcal{S}^{\mathbb{G}} \to \mathcal{N}^{\mathbb{G}}$ is a consequence of the universal property of $\mathcal{S} \to \mathcal{S}^{\mathbb{G}}$. Checking that the diagram is cartesian is straightforward. □

4.2 Rigidification and obstruction theories

In this section we prove that the perfect obstruction theory $E^\bullet \to \tau_{\geq -1} \mathrm{L}_G$ defined in 3.2 descends to the rigidification $\tilde{G} : \mathcal{S}^{\mathbb{G}} \to \mathcal{N}^{\mathbb{G}}$.

Lemma 4.2.1. *Let $n := (C, E, F) \in \mathcal{N}(\mathbb{C})$; let $c \in \mathbb{G}(\mathbb{C})$ be a (nonzero) scalar. Then the automorphism induced on $\mathrm{Ext}^i(F, E)$ (for $i = 0, 1$) by acting simultaneously on E and on F with the scalar c is the identity.*

Proof. Since the functor Ext^i is contravariant in the first variable and covariant in the second, the scalar automorphism c applied to the first variable acts as c^{-1}, and applied to the second variable it acts as c (explicitly $\mathbb{C}^* \curvearrowright \mathrm{Hom}(F, E), (c, f(-)) \mapsto cf(c^{-1} \cdot -) = f(-))$. $\square$

In the next result we use the language of comodules (see Section 1.6) to describe the category of quasi coherent sheaves on $X \times \mathrm{B}\mathbb{G}$, in case X is an affine scheme.

Lemma 4.2.2. *Let $X := \operatorname{Spec} R$ be an affine scheme. Then there is an equivalence of categories*

$$\begin{aligned} \mathrm{Qcoh}(X \times \mathrm{B}\mathbb{G}) \cong \{(F, f) \mid F \textit{ is a } R\textit{-module and } f : F \to F \otimes \mathcal{O}(G) \\ \textit{is an } \mathcal{O}(\mathbb{G})\textit{-comodule structure on } F\}\,. \end{aligned}$$

Proof. As described in [Ols16, Example 8.1.12], we have that

1. $X \times \mathrm{B}\mathbb{G} = [X/\mathbb{G}]$, where $\mathbb{G}$ acts trivially on X;

2. there is a canonical smooth surjective morphism $X \to [X/\mathbb{G}]$;

3. the following diagram is 2-cartesian:

$$\begin{array}{ccc} \mathbb{G} \times X & \xrightarrow{a} & X \\ \downarrow{\scriptstyle p} & & \downarrow \\ X & \longrightarrow & [X/\mathbb{G}]\,, \end{array}$$

where $a : \mathbb{G} \times X \to X$ is the action of $\mathbb{G}$ on X, and $p : \mathbb{G} \times X \to X$ is the canonical projection.

Now, since $\mathbb{G}$ acts trivially on X we have that $p = a$. By Remark 1.2.30 we deduce that

$$\begin{aligned} \mathrm{Qcoh}([X/\mathbb{G}]) &\cong \{(\mathcal{F}, \phi) \mid \mathcal{F} \in \mathrm{Qcoh}(X)\,;\, \phi \in \mathrm{Aut}(a^*\mathcal{F}) \text{ satisfying cocycle}\} \\ &\cong \{(F, f) \mid F \text{ is a } R\text{-module and } f : F \to F \otimes \mathcal{O}(\mathbb{G}) \\ &\quad \text{is an } \mathcal{O}(\mathbb{G})\text{-comodule structure on } F\}\,, \end{aligned}$$

where the second equivalence is due to the fact that $X = \operatorname{Spec} R$ is affine. $\square$

Remark 4.2.3. Let X be an affine scheme, let $p : X \times \mathrm{B}\mathbb{G} \to X$ be the canonical surjection and let $F \in \mathrm{Qcoh}(X)$. It can be checked that p^*F induces the trivial comodule $(F, 1)$ (compare Remark 1.6.4(a)). $\diamondsuit$

Definition 4.2.4. Let X be an algebraic stack, let $p : \mathcal{X} \to X$ be a gerbe banded by $\mathbb{G}$ and let $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})$. We call

$$\mathcal{F}_t := p^* p_* \mathcal{F}$$

the *trivial subsheaf of* $\mathcal{F}$ *with respect to* $\mathbb{G}$ (in Lemma 4.2.8 we will prove that the canonical morphism $\mathcal{F}_t \to \mathcal{F}$ is injective).

Recall that if V is a vector space and $V \to V \otimes \mathcal{O}(\mathbb{G})$ is a comodule structure on V, the trivial subcomodule of V is the biggest subcomodule of V which is a trivial comodule (compare Remark 1.6.4(b)). In the following results we prove that the notion of trivial subsheaf (on $\mathbb{G}$-gerbes) is the global counterpart of the notion of trivial subcomodule.

Lemma 4.2.5. *Let X be an algebraic stack, let $p : \mathcal{X} \to X$ be a gerbe banded by $\mathbb{G}$. Then the functor $p_* : \mathrm{Qcoh}(\mathcal{X}) \to \mathrm{Qcoh}(X)$ is exact.*

Proof. We need to prove that if $\mathcal{E} \to \mathcal{F}$ is a surjective morphism in $\mathrm{Qcoh}(\mathcal{X})$, then $p_*\mathcal{E} \to p_*\mathcal{F}$ is surjective, too. Surjectivity is a local property, so we may assume that $X = \mathrm{Spec}\, R$ is an affine scheme and that $\mathcal{X} = X \times \mathrm{B}\mathbb{G}$. Then $\mathcal{E}$ induces a comodule (E, e), $\mathcal{F}$ induces a a comodule (F, f) and $\mathcal{E} \to \mathcal{F}$ induces a surjective morphism of comodules $g : (E, e) \to (F, f)$. Now $p_*\mathcal{F} \cong \Gamma(X \times \mathrm{B}\mathbb{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{O}_{X \times \mathrm{B}\mathbb{G}}, \mathcal{F}) \cong \mathrm{Hom}((R, 1_R), (F, f))$. We have that $a \in \mathrm{Hom}((R, 1_R), (F, f))$ if and only if a is a morphism in $\mathrm{Hom}(R, F)$ which makes the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{a} & F \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f} \\ F & \xrightarrow{1_F} & F \otimes \mathcal{O}(\mathbb{G})\,. \end{array}$$

Therefore $p_*\mathcal{F} \cong F_t$, where $(F_t, 1)$ is the trivial subcomodule of (F, f) (see Remark 1.6.4). Analogously, $p_*\mathcal{E} \cong E_t$, where $(E_t, 1)$ is the trivial subcomodule of (E, e). By Proposition 1.6.7 we deduce that if $g : (E, e) \to (F, f)$ is surjective, then $E_t \to F_t$ is surjective; hence $p_*\mathcal{E} \to p_*\mathcal{F}$ is surjective if $\mathcal{E} \to \mathcal{F}$ is surjective. $\square$

Remark 4.2.6. Notice that in Lemma 4.2.5 we have proved the following statement: let X be an affine scheme and let $p : X \times \mathrm{B}\mathbb{G} \to X$ be the canonical surjection, let $\mathcal{F} \in \mathrm{Qcoh}(X \times \mathrm{B}\mathbb{G})$ and let (F, f) be the $\mathcal{O}(\mathbb{G})$-comodule induced by $\mathcal{F}$, then $p_*\mathcal{F} \cong F_t$, where $(F_t, 1)$ is the trivial subcomodule of (F, f). $\diamondsuit$

Corollary 4.2.7. *Let X be an algebraic stack, let $p : \mathcal{X} \to X$ be a gerbe banded by $\mathbb{G}$. Then the functor $p^*p_* : \mathrm{Qcoh}(\mathcal{X}) \to \mathrm{Qcoh}(\mathcal{X})$ is exact.*

Proof. The functor $p^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(\mathcal{X})$ is exact, because $p : \mathcal{X} \to X$ is flat (locally p is the canonical projection $X \times \mathrm{B}\mathbb{G} \to X$). The functor $p_* : \mathrm{Qcoh}(\mathcal{X}) \to \mathrm{Qcoh}(X)$ is exact by Lemma 4.2.5. Hence $p^*p_* : \mathrm{Qcoh}(\mathcal{X}) \to \mathrm{Qcoh}(\mathcal{X})$ is exact. □

Lemma 4.2.8. *Let X be an algebraic stack, let $p : \mathcal{X} \to X$ be a gerbe banded by $\mathbb{G}$, let $\mathcal{F} \in \mathrm{Qcoh}(\mathcal{X})$ and let $\mathcal{F}_t := p^*p_*\mathcal{F}$. Then the canonical morphism $\mathcal{F}_t = p^*p_*\mathcal{F} \to \mathcal{F}$ is injective.*

Proof. Injectivity is a local property, so we may assume that $\mathcal{X} \cong X \times \mathrm{B}\mathbb{G}$. Let (F, f) be the $\mathcal{O}(\mathbb{G})$-comodule induced by $\mathcal{F} \in \mathrm{Qcoh}(X \times \mathrm{B}\mathbb{G})$ and let $(F_t, 1)$ be the trivial subcomodule of (F, f). The comodule induced by $\mathcal{F}_t$ is $(F_t, 1)$; indeed $p_*\mathcal{F} \cong F_t$ by Remark 4.2.6, and p^*F_t induces the comodule $(F_t, 1)$ by Remark 4.2.3. But $(F_t, 1) \to (F, f)$ is an inclusion of comodules; hence $\mathcal{F}_t \to \mathcal{F}$ is an inclusion. □

Proposition 4.2.9. *Let X be an algebraic stack, let $p : \mathcal{X} \to X$ be a gerbe banded by $\mathbb{G}$; let $E \in \mathrm{Qcoh}(X)$. Then the canonical morphism $E \to p_*p^*E$ is an isomorphism.*

Proof. We need to check that the canonical map $E \to p_*p^*E$ is an isomorphism. That can be done locally, so we may assume that X is an affine scheme and that $\mathcal{X} = X \times \mathrm{B}\mathbb{G}$. By Remark 4.2.3 we have that p^*E induces the trivial comodule $(E, 1)$; hence, by Remark 4.2.6 we deduce that $p_*p^*E \cong E$. □

Proposition 4.2.10. *Let X be an algebraic stack and let $p : \mathcal{X} \to X$ be a gerbe banded by $\mathbb{G}$. Then*

1. $p^* : \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(X) \to \mathrm{D}^{[-1,0]}_{Coh}(\mathcal{X})$ *is fully faithful;*
2. *the essential image of p^* is*

$$\mathcal{T} := \left\{ \mathcal{E}^\bullet \in \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X}) \;\middle|\; \begin{array}{l} \textit{the } \mathcal{O}(\mathbb{G})\textit{-comodule induced by } x^*\,\mathrm{h}^i(\mathcal{E}^\bullet) \\ \textit{is a trivial comodule, for any } x \in X\,. \end{array} \right\}.$$

Proof. Let $[\mathcal{E}^{-1} \to \mathcal{E}^0] \in \mathcal{T}$. We have an exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules

$$0 \to \mathrm{h}^{-1}(\mathcal{E}^\bullet) \to \mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathrm{h}^0(\mathcal{E}^\bullet) \to 0\,.$$

By Corollary 4.2.7 if we pass to the trivial subsheaves, we still get an exact sequence:

$$0 \to \mathrm{h}^{-1}(\mathcal{E}^\bullet)_t \to \mathcal{E}^{-1}_t \to \mathcal{E}^0_t \to \mathrm{h}^0(\mathcal{E}^\bullet)_t \to 0\,.$$

Furthermore, by definition of $\mathcal{T}$ and by Nakayama lemma we deduce that $\mathrm{h}^i(\mathcal{E}^\bullet)_t = \mathrm{h}^i(\mathcal{E}^\bullet)$, for any $i = -1, 0$. Hence the following sequence is exact:

$$0 \to \mathrm{h}^{-1}(\mathcal{E}^\bullet) \to \mathcal{E}_t^{-1} \to \mathcal{E}_t^0 \to \mathrm{h}^0(\mathcal{E}^\bullet) \to 0\,.$$

Therefore the canonical morphism $p^* p_* \mathcal{E}^\bullet = [\mathcal{E}_t^{-1} \to \mathcal{E}_t^0] \to [\mathcal{E}^{-1} \to \mathcal{E}^0] = \mathcal{E}^\bullet$ is a quasi-isomorphism. That proves the second statement.
Let $E^\bullet, F^\bullet \in \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(X)$ and let $g : p^*E^\bullet \to p^*F^\bullet$ be a morphism in $\mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X})$. Then g is induced by an object $\mathcal{Q}^\bullet \in \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X})$, a quasi isomorphism $\mathcal{Q}^\bullet \to p^*E^\bullet$ and a morphism $\mathcal{Q}^\bullet \to p^*F^\bullet$. We have that $\mathcal{Q}^\bullet \in \mathcal{T}$, because $\mathrm{h}^i(\mathcal{Q}^\bullet) \cong \mathrm{h}^i(p^*E^\bullet)$. Let $Q^\bullet := p_*\mathcal{Q}^\bullet$; the canonical morphism $p^*Q^\bullet \to \mathcal{Q}^\bullet$ is a quasi isomorphism. Therefore $g : p^*E^\bullet \to p^*F^\bullet$ is induced by a diagram

$$p^*E^\bullet \xleftarrow{q.\,is} p^*Q^\bullet \longrightarrow p^*F^\bullet$$

By applying p_* and by Proposition 4.2.9 we get a diagram

$$E^\bullet \xleftarrow{q.\,is} Q^\bullet \longrightarrow F^\bullet\,.$$

That proves that $p^* : \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(X) \to \mathrm{D}^{[-1,0]}_{Coh}(\mathcal{X})$ is full.
Now let $E^\bullet, F^\bullet \in \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(X)$ and let $f, g : E^\bullet \to F^\bullet$ be two morphism in $\mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(X)$. Then we have a diagram of morphisms of complexes

$$\begin{tikzcd}
 & M^\bullet \arrow[dl, "q.\,is"'] \arrow[dr] & \\
E^\bullet & & F^\bullet\,. \\
 & N^\bullet \arrow[ul, "q.\,is"] \arrow[ur] &
\end{tikzcd}$$

Assume that there exists an object $\mathcal{Q}^\bullet \in \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(\mathcal{X})$ and a pair of quasi isomorphisms $\mathcal{Q}^\bullet \to p^*M^\bullet$ and $\mathcal{Q}^\bullet \to p^*N^\bullet$ such that the following diagram is commutative in the homotopy category:

$$\begin{tikzcd}
 & p^*M^\bullet \arrow[dl, "q.\,is"'] \arrow[dr] & \\
p^*E^\bullet & \mathcal{Q}^\bullet \arrow[u, "q.\,is"] \arrow[d, "q.\,is"'] & p^*F^\bullet\,. \\
 & p^*N^\bullet \arrow[ul, "q.\,is"] \arrow[ur] &
\end{tikzcd} \qquad (*)$$

Again, there exists $Q^\bullet \in \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(X)$ such that the canonical morphism $p^*Q^\bullet \to \mathcal{Q}^\bullet$ is a quasi isomorphism. By applying p_* we deduce that the following diagram is commutative in the homotopy category:

$$\begin{tikzcd} & M^\bullet \arrow[dl, "q.\,is"'] \arrow[dr] & \\ E^\bullet & \tilde{Q}^\bullet \arrow[u, "q.\,is"] \arrow[d, "q.\,is"] & F^\bullet . \\ & N^\bullet \arrow[ul, "q.\,is"] \arrow[ur] & \end{tikzcd}$$

□

Now we have the necessary techniques to prove that the obstruction theory defined in 3.2 descends to the rigidification.

Corollary 4.2.11. *Let $\tilde{G} : \mathcal{S}^{\mathbb{G}} \to \mathcal{N}^{\mathbb{G}}$ be the morphism defined in Corollary 4.1.8. The obstruction theory $E^\bullet \to \tau_{\geq -1} \mathrm{L}_G$ (defined in 3.2) induces a perfect obstruction theory for the morphism $\tilde{G}$:*

$$\tilde{E}^\bullet \to \tau_{\geq -1} \mathrm{L}_{\tilde{G}} .$$

Proof. Recall that $E^\bullet := \mathbb{R}\pi_*(\bar{\mathcal{F}} \otimes \bar{\mathcal{E}}^\vee \otimes \bar{\omega}[1])$. Denote by $p : \mathcal{S} \to \mathcal{S}^{\mathbb{G}}$ the rigidification of $\mathcal{S}$ along $\mathbb{G}$. By Proposition 4.2.10 and by Lemma 4.2.1 there exists $\tilde{E}^\bullet \in \mathrm{D}^{[-1,0]}_{\mathrm{Coh}}(\mathcal{S}^{\mathbb{G}})$ such that $E^\bullet \cong p^*\tilde{E}^\bullet$. Furthermore $\mathrm{L}_G \cong p^* \mathrm{L}_{\tilde{G}}$, since the morphism p is smooth. Hence, by Proposition 4.2.10 the morphism $E^\bullet \to \tau_{\geq -1} \mathrm{L}_G$ induces a morphism $\tilde{E}^\bullet \to \tau_{\geq -1} \mathrm{L}_{\tilde{G}}$.
The fact that $\tilde{E}^\bullet \to \tau_{\geq -1} \mathrm{L}_{\tilde{G}}$ is a perfect obstruction theory is a consequence of Lemma 4.2.5. □

4.3 Applications

In this section we prove that the relative obstruction theory defined in 3.2 induces a perfect obstruction theory for the moduli spaces of simple coherent systems (compare Definition 1.5.8). Since α-stable coherent systems are simple (by Proposition 1.5.10), such an obstruction theory is indeed an obstruction theory for every moduli space of α-stable coherent systems.

Definition 4.3.1. Let $s \in \mathcal{S}(T)$ and let $m \in \mathcal{M}(T)$ be the image of s via the forgetful morphism $F : \mathcal{S} \to \mathcal{M}$. Let

$$\mathrm{Aut}(s/\,\mathrm{id}_m) := \{\psi \in \mathrm{Aut}(s) \,|\, F(\psi) = \mathrm{id}_m\} .$$

We say that s is *simple* if $\iota_s : \mathbb{G}(T) \to \mathrm{Aut}(s/\,\mathrm{id}_m)$ is an isomorphism.

Notation 4.3.2. Throughout the chapter the symbol $\mathcal{S}_{smp}$ denotes the stack of simple GCS's. Note that the canonical morphism $\mathcal{S}_{smp} \to \mathcal{S}$ is an open embedding. With an abuse of notation, we still use the letter G to denote the morphism $G : \mathcal{S}_{smp} \to \mathcal{N}$, which is the restriction of the forgetful morphism $\mathcal{S} \to \mathcal{N}$ to $\mathcal{S}_{smp}$.

Remark 4.3.3. Let $G : \mathcal{S}_{smp} \to \mathcal{N}$ be the forgetful morphism. Let $\mathcal{S}_{smp} \to \mathcal{S}^{\mathbb{G}}_{smp}$ be the rigidification of $\mathcal{S}_{smp}$ and let $\mathcal{N} \to \mathcal{N}^{\mathbb{G}}$ be the rigidification of $\mathcal{N}$. It is still true that there exists a unique morphism $\tilde{G} : \mathcal{S}^{\mathbb{G}}_{smp} \to \mathcal{N}^{\mathbb{G}}$ such that the following diagram is cartesian (compare Corollary 4.1.8):

$$\begin{tikzcd} \mathcal{S}_{smp} \arrow[r] \arrow[d, "G"] & \mathcal{S}^{\mathbb{G}}_{smp} \arrow[d, "\tilde{G}"] \\ \mathcal{N} \arrow[r] & \mathcal{N}^{\mathbb{G}} . \end{tikzcd}$$ $\Diamond$

Remark 4.3.4. The forgetful morphism $\mathcal{N} \to \mathcal{M}$ (defined in Remark 2.1.7) factors through $\mathcal{N}^{\mathbb{G}} \to \mathcal{M}$. Indeed, $\mathcal{N} \to \mathcal{M}$ satisfies property (1) of Theorem 4.1.1. Notice that the morphism $\mathcal{S}^{\mathbb{G}}_{smp} \to \mathcal{M}$ is representable. Moreover the morphism $\mathcal{N}^{\mathbb{G}} \to \mathcal{M}$ is smooth, since $\mathcal{N} \to \mathcal{M}$ is smooth. That is true because we are working on curves: at any point $(C, E, F) \in \mathcal{N}(\operatorname{Spec}\mathbb{C})$ we have that $\mathrm{H}^2(\mathcal{H}\mathrm{om}(E, E)) = \mathrm{H}^2(\mathcal{H}\mathrm{om}(F, F)) = 0$. $\Diamond$

Corollary 4.3.5. *The obstruction theory we have defined indueces a perfect obstruction theory for the morphism*

$$\mathcal{S}^{\mathbb{G}}_{smp} \to \mathcal{M}\,.$$

Proof. Let $q : \mathcal{N}^{\mathbb{G}} \to \mathcal{M}$; since q is smooth, the complex L_q is perfect. Denote by $F : \mathcal{S}^{\mathbb{G}}_{smp} \to \mathcal{M}$ the composition $q \circ \tilde{G}$. By the properties of the cotangent complex we have a distinguished triangle

$$\tilde{G}^* \mathrm{L}_q \to \mathrm{L}_F \to \mathrm{L}_{\tilde{G}} \to \tilde{G}^* \mathrm{L}_q[1]\,.$$

Recall that the obstruction theory that we have constructed in 3.2 is induced by a morphism $\tilde{E}^\bullet \to \mathrm{L}_{\tilde{G}}$ (see Proposition 3.2.5). Hence, by composition we get a morphism $\tilde{E}^\bullet \to \tilde{G}^* \mathrm{L}_q[1]$. Let E' denote the mapping cone of such morphism, shifted by -1. By the axioms of the triangulated categories we obtain a morphism $E' \to \mathrm{L}_F$ and therefore a morphism $E' \to \tau_{\geq -1} \mathrm{L}_F$. That is the obstruction theory for the morphism F. $\square$

If we fix a smooth projective curve C, such obstruction theory induces a perfect obstruction theory for the moduli spaces of simple coherent systems on C of rank equal to the Brill-Noether number β (see Definition 1.5.11 and [BGPMN03, 2.7]).

Corollary 4.3.6. *For any smooth, projective, genus g curve C and for any triple (n,d,k) the moduli space of simple coherent systems of type (n,d,k) has a perfect obstruction theory of rank*

$$\beta := n^2(g-1) + 1 - k(k-d+n(g-1))\,.$$

Proof. To prove this result we may assume that $\mathcal{M} = \operatorname{Spec}\mathbb{C}$ and $\mathcal{M}' = C$ (recall that $\mathcal{M}'$ is a family of projective Gorestein curves over $\mathcal{M}$, as described in 2.1).
Let $\mathcal{S}_c$ be the stack of coherent systems and consider the morphism $F : (\mathcal{S}_c)^{\mathbb{G}}_{smp} \to (\mathcal{V}\times \mathrm{BGL})^{\mathbb{G}}$ (compare Section 2.2); denote by r the rank of the relative obstruction theory for the morphism F, and denote by δ the dimension of $(\mathcal{V}\times \mathrm{BGL})^{\mathbb{G}}$. We need to check that $\beta = r + \delta$.
The relative obstruction theory is perfect, so we can compute its rank at any point (C, E, V, φ). By hypothesis, the rank of E is n, the degree of E is d and the dimension of V is k. The rank of an obstruction theory at a point is given by the dimension of the tangent space minus the dimension of the obstruction space at that point, so we have that

$$r = \dim \operatorname{Hom}(V \otimes \mathcal{O}_C, E) - \dim \operatorname{Ext}^1(V\otimes \mathcal{O}_C, E)\,,$$

compare Proposition 3.1.4. Hence

$$r = \chi(E^{\oplus k}) = k(d + n(1-g))\,.$$

On the other hand

$$\delta := \dim(\mathcal{V}\times \mathrm{BGL})^{\mathbb{G}} = n^2(g-1) - k^2 + 1\,.$$

By comparison we deduce that $r + \delta = \beta$. □

Since α-stable coherent systems are simple (by Proposition 1.5.10), our computation provides a perfect obstruction theory of rank β for every moduli space of α-stable coherent systems.

Corollary 4.3.7. *Fix $\alpha \in \mathbb{R}$. Let C be a smooth, projective, genus g curve and let (n,d,k) be a suitable triple of positive integers. Let $\beta := \beta(n,d,k)$ be the Brill Noether number (see Definition 1.5.11 and [BGPMN03, 2.7]). Then the moduli space of α-stable coherent systems of type (n,d,k) has a perfect obstruction theory of rank β.* □

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