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A mi madre y a la memoria de mi padre

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INTRODUCTION

1.1 Overview

The theoretical description of the Quantum Electromagnetic Field was developed in the thirties, shortly after the birth of quantum theory. In spite of some immediate succesful results such as the justification of the existence of photons, it is not until the construction of the full quantum theory of fermions interacting with the electromagnetic field, *i.e.* QED, that the full power of this theory became clear. Recently, a certain number of advances toward the construction of Quantum General Relativity (GR) have been achieved [1, 2, 3, 4, 5] (for an introduction see chapter 2 and [6, 7, 8]). However, we suspect that this theory too will express its full physical value only when a fully interacting matter + gravity theory is constructed, and, in particular, when the realistic fermions+gravity theory is constructed. By analogy with QED, we shall denote the quantum theory of fermions + gravity as Quantum Gravitational Dynamics, or QGD.

There are a number of reasons for suspecting that matter couplings are needed for clarifying the Quantum Gravity puzzle. The first of these, is that it is very difficult to write fully gauge invariant quantities on the phase space of General Relativity, due to diffeomorphism invariance [1, 9]. Equivalently, it is extremely difficult to imagine well-defined experiments to be performed on the gravitational field *alone.* Thus, in the pure-gravity case we are in the situation of constructing a theory, but not being sure of what precisely should we ask to the theory – a situation quite familiar, we believe, to anybody working on non-perturbative quantum gravity. On the other hand, diffeomorphism-invariant quantities, as well as realistic experiments described by those quantities, can be constructed in a relatively simple fashion in the presence of dynamically coupled matter [9, 10, 11]. For instance, the solar system, or a binary pulsar emitting gravitational radiation, are examples of matter + gravity systems that we understand well as far as measurements are concerned: we know with precision which are the observables that we can measure meaningfully. If these observables could be measured with Planck-scale sensitivity, then these systems could be seen as quantum-gravity laboratories. Clearly, we need a matter + gravity quantum theory in order to describe them theoretically.

A second reason for coupling matter to quantum gravity, is given by the peculiarity of the non-perturbative quantum theory of gravity. In the Loop Representation, the theory has a characteristic geometrical structure: Quantum states of gravity are classified by Knot Theory [1], and the dynamics can be represented in a fully combinatorial-algebraic fashion on Knot Space [5]. These features are not accidental, but rather are consequences of diffeomorphism invariance; equivalently, they are related to the fact that the Loop Representation is a genuine background-independent quantum field theory. Since the theory relies on these geometrical structure, it is mandatory to check whether these structures are lost when further fields are coupled. If so, doubts could be cast on the value of the Loop Representation of Quantum GR.

Motivated by the considerations above, we have studied the fermions + gravity system, or QGD. The choice of fermions is mainly motivated by realism, but also by the fact that they are very natural objects in the Ashtekar formalism. The study of the gravitational interaction of fermions in the light of General Relativity, has been carried out by Dirac and Sciama [12, 13], and, more recently, by Nelson and Teitelboim [14] using a canonical approach. The theory has been formulated in terms of Ashtekar variables in ref. [15]; we have developed an alternative equivalent form appearing in chapter 3. This is based on a first-order action for a self-dual connection, with non-vanishing torsion, and a soldering form. This alternative formulation enables one to interpret the equivalence of the self-dual theory to general relativity via the Bianchi symmetry (including torsion) of the curvature tensor. It makes also possible to identify an extra term in the Dirac equation as an effect of the non-vanishing torsion and, further, it reinforces Jacobson's proposal of picking gravity theories with or without torsion as different real sections of the phase space of complex general relativity. The role of boundary terms for gravity and their extension to include fermions is the subject of chapter 4. Supersymmetry-inspired local boundary conditions used in 1-loop quantum cosmology are studied for a model consisting of massless Fermionic fields on a four-dimensional Euclidean flat space bounded by a three-sphere.

We then construct, in chapter 5, the quantum theory following the lines along which the quantum theory of pure General Relativity was constructed. We define the natural extension of the loops observables to fermions (these are given by a parallel-transport operator associated to an open curve with fermions at its end points), study their Poisson algebra, and define the quantum theory as a linear representation of this algebra. In analogy with the pure-gravity case, we also show that the resulting representation can be heuristically obtained from a naive Schrödinger-like representation by means of a (ill defined) Loop Transform.

The loop representation of QGD turns out to be a very natural extension of the pure-gravity case, obtained by including *open curves* into Loop Space. This is certainly not surprising, since the kinematics of the loop representation can be seen as the continuum version of the Wilson-Kogut construction in lattice Yang-Mills theory [16], where fermions are represented by the end points of open lines of flux on the lattice. In the rest of the thesis, we will denote both open and closed curves as *loops* , disregarding consistency with the dictionary. It is not difficult to solve the diffeomorphism constraint on the resulting state space. The complete classification of the solutions is given by a generalization of the Knot Classes of the pure-gravity case – the new classes include graphs with an arbitrary number of intersections and open ends. Thus, quantum states of QGD admit the same kind of topological description as the states of pure Quantum GR, contrary to the fear that this aspect of the Loop Representation could be lost in presence of matter couplings. We view this as an encouraging result, though a result that could have been anticipated.

On the other hand, the results we obtain about the *dynamics* of the theory are unexpected and, we believe, rather surprising. The dynamics is given by the Hamiltonian constraint, including the fermion-gravity interaction. We construct in this thesis the corresponding quantum operator, and its action on the loops turns out to have an extremely simple geometrical interpretation: The Hamilonian-constraint operator essentially "shifts" loops along their tangent. This same simple geometrical action of the Hamiltonian constraint was recognized in the context of pure gravity in ref.[1]. The surprising result here is that the very same action, extended to *open* loops, codes the fermions-gravity interaction.

While suggestive, the above construction is not fully satisfactory for three reasons. First, there is a divergence in the action of the Hamiltonian operator which is difficult to control. Second, in spite of the simplicity of the action of the Hamiltonian operator, we have not been able to solve the corresponding quantum constraint equation. Third, the presence of fermions does not take us completely away from the difficulties of constructing gauge-invariant observables: it simplifies the task of finding three-dimensional diffeomorphism-invariant quantities, but it does not help with the problem of finding quantities that commute with the Hamiltonain constraint. Thus, the actual physical content of the theory is still quite unaccessible, as it is in pure gravity.

To face these problems, we take one further step. In chapter 6, we combine our results on fermions with the results obtained in ref.[5]. In that paper, the idea was proposed to unravel the dynamics of quantum gravity by coupling a scalar field, which could behave as a clock-field, following a long tradition [11] of ideas of using matter for simplifying the gravitational-theory analysis (see [9, 10, 17, 18, 19]). It was shown that by a suitable gauge fixing one can express the dynamics of gravity as intrinsic evolution with respect to the intrinsic time (or physical time) defined by the scalar field. This evolution is explicitly generated on the state space by a Hamiltonian operator $\hat{H}$. Here, we extend this construction to fermions. Namely, we consider the generally covariant gravity+fermions+scalar field system, we solve with respect to the scalar field, so that the Hamiltonian constraint is replaced by a genuine diffeomorphism-invariant Hamiltonian that evolves both gravity and fermions in the scalar-field-clock time. We explicitly construct the quantum Hamiltonian operator $\hat{H}$ (as opposed to Hamiltonian-constraint operator), by making use of the regularization techniques on manifolds that have recently been introduced [2] in quantum gravity. In the fermionic sector, we recover in this way the simple action described above, namely the shift of the loops along themselves. However, now the resulting operator is fully *diffeomorphism invariant*, and *finite*.

The resulting QGD is then given by a set of quantum states represented by graphs with a finite number of intersections and open ends, and by an Hamiltonian operator that acts in a simple geometrical and combinatorial fashion on these graphs. Matrix elements of this Hamiltonian can be interpreted as *first-order transition amplitudes between the graph states in a time-dependent perturbative expansion in the clock time.* The explicit computation is complicated by the need of extracting the square root of an infinite matrix, a task we expect could be solved order by order. Here, we only begin the explicit computation of matrix elements of the operator.

The picture of Quantum Gravitational Dynamics that begins to emerge from this construction has a simple and perhaps appealing general structure: A graph with two open ends, say, represents two fermions interacting gravitationally among themselves, and with the surrounding gravitational field. With the machinery developed in this thesis we could (at least in principle) follow the quantum evolution of this system in clock time.

Chapter 7 contains the conclusions reached in this thesis and the problems left over and how it may be possible to tackle them. The rest of this introduction consists of a review of the different ideas involved in the construction of a quantum theory of gravity.

The original contributions in the present thesis are all those concerning spin-$\frac{1}{2}$ Fermionic fields. Thus, they are the contents of chapter 3, section 4.2, chapter 5 and chapter 6. As research papers they are references [20, 21, 22].

Conventions are the same as in [7]. Unless otherwise stated, units are used in which $G = \hbar = c = 1$, un-hat-ed lower-case Latin letters, $a, b, c, \ldots = 0, 1, 2, 3$, are used as space-time indices whereas hat-ed ones, $\hat{a}, \hat{b}, \hat{c}, \ldots = \hat{0}, \hat{1}, \hat{2}, \hat{3}$, are local Lorentz indices. Upper-case Latin letters, $A, B, \ldots = 0, 1$ and $A', B', \ldots = 0', 1'$, are two-spinor indices. Symmetrization operations act only on two indices, those nearest to, and contained by, the respective symbols. Thus, $[a(MA' \ \ N)b]$ implies antisymmetrization in a, b and symmetrization in M, N.

1.2 Review of Quantum-Gravity approaches

Most conceptual problems in Quantum Gravity (QG) concern the status of our conventional ideas on space, time and matter. Such problems are usually linked with the technical problem of the compatibility of the standard geometrical ideas of General Relativity (GR) and Quantum Theory (QT). In fact, the analysis of this compatibility may provide a strong motivation for doing QG [23]. Thus, as remarked by Isham [24], the discussion of geometrical and conceptual issues in the same framework is justified. We closely follow him in this introduction.

The significance of the conceptual problems that stem from QG has no consensus. In constructing the theory we have in our grasp only minimal requirements for it: to reproduce i) classical General Relativity and ii) normal Quantum Theory. These should hold in the appropriate domains. Namely, for distances and times much bigger than the *Planck length* l_P *and time* t_P. Here $l_P = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33}cm \sim 10^{28}eV$ and $t_P = \frac{l_P}{c} \sim 10^{-42}sec$, where G is the Newton's constant, $\hbar$ is Planck's constant and c the speed of light in vacuum. On the other hand, near the Planck-length scale itself the views vary according to the extent to which the conceptual and structural frameworks of GR and QT are still applicable. There is the conservative view that nothing changes at such a scale and the revolutionary one that suggests a reassessment of the traditional ideas of spacetime and quantum matter e.g. i) Continuum concepts (Differential Geometry) are inapplicable in this domain and ii) Penrose's proposal [25] that QT becomes non-linear at the Planck length in the way needed to explain the problem of the reduction of the state vector of Quantum Mechanics.

The above minimal requirements on a QG theory are however not strong enough to single it out. By using a *covariant quantization* method, particle physicists have shown [26, 27, 28, 29, 30] that any Lorentz-invariant theory of a spin-2 massless quantum field coupled to a conserved energy-momentum tensor will necesarily yield the same low-energy scattering results as those obtained from the tree graphs of a weak-field perturbative expansion of the Einstein Lagrangian[1].

The existence of this set of "equivalent" theories comes from the fact that no precise a priori information is known about the requirements on the theory. Probably the main consequence is that no axiomatic formulation, Wightman-type or C^*-algebras [32], exists so far as opposite to Quantum Field Theory (see however [33] for recent developments).

The above ideas already show that the properties of a QG theory are not well-defined. This observation is supported by the many proposals that have appeared. However, it is through such proposals that a "feeling" can be got about what physicists understand as QG. A rough description of them is given below.

The different proposals can be divided into two groups relying on the feature of gravity that is emphasized: the *field properties* of the gravitational interaction is the viewpoint of particle physicists and the *structure of spacetime* as linked with gravity that is the approach of general relativists.

More precisely, the former group uses techniques drawn from conventional, Poincaré-group-based Quantum Field Theory. Here, the key concepts are special relativity and gravitons propagating in a fixed Minkowski spacetime. The minimal expectation is to produce scattering amplitudes for gravitons and other particles that are free of irremovable divergences, i.e. to have a perturbatively renormalizable theory or, maybe, a genuinely finite theory. The great goal is to have this theory as a part of a general Grand Unified Theory (GUT) in which the presence of the gravitational sector is essential.

In the early stages of this approach the expansion $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$, with $\kappa = \sqrt{16\pi G/c^2}$ and $\eta_{\mu\nu}$ the metric in Minkowski spacetime, was used; then the field $h_{\mu\nu}(x)$ was quantised using the standard techniques drawn from relativistic quantum field theory. The concept of gravitons as the quanta of the gravitational field came about. Background-field

[1] It has been shown [31] that there exists a consistent theory of a massless interacting spin-2 field that is not generally covariant, that is, it is not possible to change the dynamical field variable in such a way that the background flat metric disappears from the theory. This implies that the equivalence of all the theories mentioned above is not valid in every aspect.

methods were introduced afterwards, taking instead of the Minkowski spacetime another solution of Einstein equations: $\eta_{\mu\nu} \rightarrow g^{(0)}_{\mu\nu}(x)$ as the background and then quantising[2] $h_{\mu\nu}(x)$. More recently, attention has been paid to theories of superstrings and other extended objects where massless spin-2 fields (gravitons) also appear.

Among the most important problems in this approach is the lack of a meaningful causal structure of the theory: there is no reason why it should be the same as that of the background space. Also, it is difficult to handle relevant cosmological issues; for example, when studying the influence of QG on the initial spacetime singularity, the use of a classical cosmological model (e.g. Robertson-Walker) as the background is not sufficient. This is obvious if one thinks, as is done in this approach, that the spacetime structure is given by the background and gravitons – as defined on this background – are the entities that scatter between them and with the rest of the matter particles.

General relativists emphasise instead the geometrical structure of the theory and the role played by the spacetime structure. This can be considered closer to the essence of GR in which the gravitational field is replaced by the geometry of the spacetime. Quantisation adopting any special background spacetime is not accepted. If such a spacetime has a special role it should emerge naturally as part of the structure of the theory itself and not just put in by hand.

The minimal expectation here is to improve the understanding of the problems posed by spacetime singularities like those associated with black holes and similar situations in the classical theory. In particular, one expects to recover Hawking's results on the quantum production of particles by black holes and to extend them to tackle the problems of the back reaction of the created particles on the background spacetime as well as the final state of the evaporating hole. A more ambitious idea is to apply the theory to cosmological issues, especially to study the universe as a quantum entity.

The philosophy used here is one of "back to the basics", by relying on QT rather than on Quantum Field Theory. The main problems are related to i) the quantum status of

[2]Note that now we have a Quantum field theory for $h_{\mu\nu}(x)$ in a non-dynamical curved background $g^{(0)}_{\mu\nu}(x)$. This is because the back reaction of $h_{\mu\nu}(x)$ exerted on $g^{(0)}_{\mu\nu}(x)$ is not taken into account.

the spacetime concepts of classical GR and ii) the extent to which conventional QT ideas can be applied. Thus, in this approach one is more concerned with the conceptual issues that arise in QG.

As pointed out in [24], it is guaranteed that the uncertainty will be maintained about what one is trying to do in QG until the following question is unswered satisfactorily: *is the central problem of Quantum Gravity one of i) physics, ii) mathematics or iii) philosophy?* Moreover, how severe are the conceptual difficulties? and is it possible that one needs to get to grips with them before any serious technical development can be made? A brief account of some problems in QG is given below.

1.2.1 Basic problems in Quantum Gravity

Here we want to stress only the main difficulties one deals with when QG is investigated. One of the broadest of all the problems is the extent to which a quantum theory of gravity maintains: i) the picture of spacetime as afforded by GR and ii) the interpretative and structural frameworks of conventional QT. It is evident that this is a highly non-trivial question and hence it is worth mentioning it even if no satisfactory answer has been given so far. Instead, more specific problems involved in the construction of a QG theory are next touched on.

1. **Spacetime Diffeomorphism group and the definition of the observables**

   General Relativity equations are covariant with respect to the group Diff($\mathcal{M}$) of diffeomorphisms of the spacetime manifold $\mathcal{M}$. In a sense the role of the diffeomorphisms group in both classical and quantum GR is analogous to that of the gauge group in Yang-Mills theory. For instance, in both cases the "gauge fields" are non dynamical[3]. On the other hand, however, the two groups are quite different. Diffeomorphism group moves spacetime points around whereas the transformations

[3]Roughly, when one extends the partial derivatives to covariant derivatives in a theory of a free matter field say, in order to get the corresponding invariance, one arrives at a coupling between the matter fields and the gauge fields but does not get a kinetic term for the latter.

involved in Yang-Mills theory are made at a fixed spacetime point. A conclusion can be arrived at that invariance under Diff($\mathcal{M}$) means that individual mathematical points in $\mathcal{M}$ have no intrinsic physical significance. Certainly this is related to the question of what is an "observable" in GR [9, 10, 34]. As an example let us consider the Riemann scalar curvature $R(x) \equiv g^{\mu\nu}(x)R_{\mu\nu}(x)$. Even if it is a scalar function on $\mathcal{M}$ its value at any $x \in \mathcal{M}$, hence, cannot be regarded as an observable. At the quantum level the same result follows by considering a unitary representation of Diff($\mathcal{M}$). For instance, take $f \in$ Diff($\mathcal{M}$) and $U(f)$ in the chosen unitary representation. The action on the quantised metric of spacetime would lead to the transformation law:

$$U(f)R(x)[U(f)]^{-1} = R(f^{-1}(x)), \tag{1.1}$$

provided $R(x)$ can be defined as a proper operator function of the metric operator and its derivatives. Since a physical observable is defined as one that commutes with the action of the gauge group, $R(x)$ turns out not to be one. Alternatives can be tried for generating observables. One is to construct genuine invariants by integrating scalar functions of the metric of spacetime over the entire spacetime, e.g. $\int_{\mathcal{M}} R(x)(g(x))^{\frac{1}{2}} d^4x$; $\int_{\mathcal{M}} R_{\mu\nu}(x)R^{\mu\nu}(x)(g(x))^{\frac{1}{2}} d^4x$. It is worth noting that these are highly non-local and the corresponding quantisation will be very different from any conventional quantum field theory. Another possibility is an old idea about observables in GR. The basic point is that although $R(x)$ is not an observable, $R(X)$ is whenever X is a point on the spacetime manifold occupied by an actual physical particle. That is, we locate ourselves on the spacetime manifold with the help of a material reference system. This implies to some extent, that simple GR is an incomplete theory since the equations of motion do not involve the reference system, for example, its energy-momentum tensor. Adoption of this approach is related to the so-called "physical" coordinates which are used sometimes in the canonical quantisation of gravity, that we will deal with, as well as in the treatment of the problem of time in QG.

The diffeomorphism-invariance problem can be seen as arising through the insistence that QG should reflect the diffeomorphism-group invariance of classical GR [1]. Three related examples make this assertion clear. First, in normal quantum field theory the two-point function for a scalar field ϕ shows the behaviour:

$$W(x,y) \equiv \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \sim \frac{const.}{(x-y)^2} \tag{1.2}$$

with x, y approaching each other (the distance is measured with the Minkowski metric of the background). The dependence of $W(x,y)$ on its arguments is a direct consequence of the invariance of the vacuum state $|0\rangle$ under the action of the Poincaré group. If we require $|0\rangle$ to be Diff($\mathcal{M}$)-invariant, and if $\hat{\phi}(x)$ transforms as $R(x)$ in (1.1) then

$$W(x,y) \equiv W(f(x), f(y)) \quad \forall\, f\ \in \text{Diff}(\mathcal{M}). \tag{1.3}$$

However, for any two pairs of points (x, y) and (x', y') which are sufficiently close to each other –that lie in a single coordinate chart e.g.– there exists a diffeomorphism f such that $x' = f(x)$ and $y' = f(y)$. It follows that $W(x,y)$ is a constant for any y in a sufficiently small neighbourhood of x. When interpreting this result one has to note that the value $\phi(x)$ of the scalar field at $x \in \mathcal{M}$ is not an observable in a Diff($\mathcal{M}$)-invariant theory. The conclusion here is that the short-distance behaviour and ultraviolet divergences are likely to be different, in quantum gravity, from quantum field theory. In addition, the regularisation method for the operators will have to change since now there is not a background metric affording the measure of nearness of spacetime points.

Second, Diff($\mathcal{M}$) invariance also affects functional-integral quantisation methods. One might try to construct a theory of QG by using functional integrals, in analogy to standard quantum field theory, to produce vacuum expectation values of a time-ordered product of a set of fields say

$$G(x_1, x_2, ..., x_n) = \int \mathcal{D}[g] R(x_1) R(x_2) ... R(x_n) e^{i \int_{\mathcal{M}} R(g) \sqrt{g} d^4 x}. \qquad (1.4)$$

Here the difficulty is to recognise what "time-ordered" product means in the absence of any background metric providing the preferred notion of spacelike and timelike[4]. Even if such a background is provided there seems to be still inconsistency since the attribute of spacelike or timelike of any pair of points can be changed one into the other by the action of the Diff($\mathcal{M}$) group.

Third, we have the spacetime operator version of quantisation. In quantum field theory, a scalar field $\hat{\phi}$ obeys the microcausality condition

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0, \quad \forall \;\; x, y \;\; \textit{spacelike separated.} \qquad (1.5)$$

In the case of QG it has been shown [35] that for most pairs of points $x, y \;\; \in \mathcal{M}$ there will exist at least one Lorentzian metric with respect to which they are not spacelike separated, and hence, as far as all metrics are summed over in a functional integral (e.g. (1.4)), the r.h.s of (1.5) will not vanish!

2. **Background structure and the problem of time**

The background structure is a key feature of any approach to QG that can take different forms. It can consist of choosing a particular mathematical element of the theory or it can refer to the conceptual or interpretative framework assumed a priori. In the former case we have the examples of theories that take a fixed manifold representing spacetime or in which a particular spacetime metric is considered central. Concerning the latter case a remark is in order. It can be argued that the conventional Copenhagen interpretation of QT assumes as part of its background a fixed spacetime (in both topological and metric sense) and it is therefore intrinsically incompatible with the idea of QG. Also, the very existence of such a background is

[4]Apparently this problem can be circumvented by using an Euclidean-time formalism in which the functional integral is over metrics possessing a Riemannian rather than Lorentzian signature. The problem is transformed into one of interpreting physical amplitudes from Riemann n-point functions.

usually associated with a division of the universe into "system" and "observer". This split is in fact one of the current problems in Quantum Cosmology. The importance of the understanding of the precise background structure that is assumed should be clear by now. The issue on the compatibility of the ideas of QT and GR mentioned at the beginning of this chapter acquires a defined form in the background structure here discussed.

Time is not an observable in conventional quantum physics since there is not an operator associated to it. Instead, it is treated as a background parameter, as in classical physics, to express the evolution of a system. This applies to non-relativistic quantum theory, relativistic particle dynamics as well as to quantum field theory. Hence, time can be regarded as an element of the classical background that is essential in the Copenhagen interpretation of the theory. We can see now that, in any particular approach to QG, the nature of the problem of time is strongly related to the background structure assumed. As used by particle physicists, the background Minkowski metric provides the usual notion of time of special relativity and quantum field theory. However, it is not clear whether or not a measure of time, as given in such an approach, is physically correct. This is an aspect of the question about the extent to which the spacetime concepts of GR can be described adequately by a weak-field perturbation around a Minkowski background. For instance, the behaviour of the lightcones at the event horizon of a black hole cannot be readily reproduced in a graviton-based picture. This criticism holds also for the generalizations in which, instead of a Minkowskian background, another curved background is used.

In the relativists' approach the issue of time is different. The background is now the "three-manifold of space". If it is non-compact the asymptotic structure might be used to define an absolute time. In dealing with cosmological models, however, this three-manifold is taken to be compact and the notion of time has to be extracted from the variables involved in the description: canonical variables of gravity, matter fields or particles added to the system.

3. A minimal length in QG

A problem encountered when dealing with local quantum fields may be the existence of a minimal length (or time) related to the Planck units. It has been argued recently [36, 37, 38] that: i)geodesic distance is intrinsically bounded from below in QG and ii) the uncertainty relations and the existence of the Schwarzschild radius, impose lower bounds on measurements of both space and time. The latter being a result coming from an analysis of "quantum clocks" as related to time in the canonical quantization of gravity. Another possibility is that the minimal length may arise in the context of a lattice approach to QG [39]. Furthermore, there are indications that string theory may lead to a natural minimal length.

Concerning the meaning of the existence of such a minimal length one can interpret it as a length that can be "measured" only to an accuracy of the Planck value (in principle w.r.t. some background metric) if the underlying model of a continuum spacetime still holds. On the other hand, it can be interpreted as a signal of the breakdown of the continuum picture itself. Anyway, both views make more obscure the idea of quantising gravity by quantising the point fields of classical GR. For instance, if time cannot be measured to an accuracy greater than the Planck time, one needs to recast the equal-time commutation relations to make them meaningful as well as the general quantum-mechanical idea of a complete commuting algebra of "simultaneously" measurable observables.

A more difficult matter would be the breakdown of the continuum picture. This amounts to think of the Diff($\mathcal{M}$) invariance as only a coarse-grained feature [5]. This also causes problems to the Regge-calculus approach to QG in contrast to the case of the gauge group in Yang-Mills theories.

4. **Quantum Topology**

The framework of GR involves an equivalence class of pairs $(\mathcal{M}, g)$; the spacetime manifold and metrics on it. However, once the classical continuum picture of space-time has been entertained a number of possibilities may occur: in particular if ge-

[5]This idea is difficult to implement in practice, partly because of the absence of any finite-dimensional approximation to the Diff($\mathcal{M}$) group.

ometries are to be quantised (g or related objects) it may be possible to consider the quantisation of $\mathcal{M}$. This idea goes back to J.A. Wheeler [40]. It is far from clear what this would mean. The mathematical framework of GR has the following symbolic hierarchical form

$$\text{set} \rightarrow \text{topology} \rightarrow \text{differential structure} \rightarrow \text{Lorentzian metric}$$

in which each step represents a structure superimposed on the previous in the chain. A "priori" quantisation might be applied at any of these stages. To keep $\mathcal{M}$ (the manifold) fixed and just quantise the metric is not an adventurous approach. There are other possibilities, for instance, to keep the differential structure but let the manifold become part of the quantum structure. S. Hawking [41, 42] and collaborators developed this idea through the so-called "Euclidean" quantum-gravity program. This is based on the use of path integrals over Riemannian, rather than Lorentzian, metrics. A typical quantity is the functional

$$Z\Big[\text{boundary data}\Big] = \sum_{\mathcal{M}} \int_{\text{Riem}(\mathcal{M})} \mathcal{D}[g] e^{\int_{\mathcal{M}} R(g)\sqrt{g} d^4 x} \tag{1.6}$$

where the integral is over the set Riem($\mathcal{M}$) of all Riemannian metrics on $\mathcal{M}$ and the sum is over all four-manifolds[6] $\mathcal{M}$. The current theory of wormholes with their possible consequences in determining the constants of nature is an application of this idea [43, 44, 45]. The use of certain complicated manifolds $\mathcal{M}$ raises intriguing possibilities, e.g. losses of information may occur: when particles fall into the event horizon of a (virtual) black hole. An observer external to the black hole would interpret this as a transition from a pure state to one that is mixed, leading to what Hawking calls the "\$-matrix": the pure→mixed analogue of a normal S-matrix [46, 47, 48].

[6]It is not quite clear what is the meaning of this sum since it is not possible to classify four-manifolds in any (algorithmic) way.

There are other alternatives not less interesting based on the quantisation of sets, topologies and manifolds that have been developed in a number of directions [24]. However, as remarked by C. Isham himself , they are more speculative and difficult to relate to the conventional approaches to QG and we will not discuss them here.

Finally, we want to mention other important questions appearing in the context of QG. They are, for example, the interpretation of the role played by complex metrics (e.g. in Asthekar's formalism –Chapter 3), or those that are degenerate [49]. Both have arisen in recent work on QG posing non-trivial problems to the interpretation of the theory. The Quantum Cosmology issue contains several points which deserve discussion. Here we just quote them: essentially, shadows are cast when the interpretative framework of the quantum theory is applied to the entire universe. i)The conventional Copenhagen interpretation of quantum theory emphasizes the role of measurement and probability (often considered in a relative-frequency sense). However, an observer cannot be out of the universe to measure it and, also, we do not know what an ensemble of universes is. ii) Theories of the Quantum Creation of the Universe (QCU) recently rely on a unique quantum state based on some quantum boundary conditions "near the big-bang". It is not certainly kown if this is compatible with the standard notions of quantum theory. iii)The world around us is remarkably classical. It is a main question how to get this feature from a totally quantum mechanical description. iv) QCU theories involve the idea of a beginning of time. The compatibility of such an idea with both GR and conventional QT should be checked.

1.2.2 Approaches to Quantum Gravity

In this section we present a brief description of several of the different approaches to tackle the quantisation of gravity. We include only the particle physicists' schemes since a more complete discussion of the canonical framework, used by relativists, will be given in chapter 2. We give the schemes followed by a series of remarks concerning their conceptual and geometrical aspects.

1. **Quantisation of GR**

Detailed reviews in this respect are [50, 51, 52]. The analysis involves several points to be discussed.

Gravitons

Gravitons are the quanta of the gravitational field. The particle is conceived of as propagating on a background Minkowski spacetime and it is associated (like the other elementary particles) with a specific representation of the Poincaré group labelled by its mass m and spin s [32]. The especification of m and s for the case of the graviton is obtained as follows:

i) t-channel exchange of a particle of mass m can give rise to a static force of the form e^{-mr}/r^2 where r is the distance between two particles. Thus, the usual gravitational inverse-square law can be secured only if the graviton is massless.

ii) s cannot be half-odd since the Pauli exclusion principle makes it impossible to construct a classical-sized field from a coherent superposition of fermions.

iii) S. Weinberg showed [28, 29] that a particle whose spin is greater than two will not produce a static force. Furthermore, $s = 1$ gives a repulsive force between like particles (e.g. spin-1 photons play this role in Electrodynamics). Hence one arrives at the only two possibilities that $s = 0$ or $s = 2$.

Scalar fields, $\phi(x)$ are associated with zero spin while symmetric Lorentz tensor fields $h_{\mu\nu}(x)$ are associated with spin-2. They can be interpreted as corresponding to Newtonian gravity and General Relativity respectively. According to quantum field theory (based on special relativity) a free massless spin-2 field satisfies the field equation

$$h_{\mu\nu,\alpha}{}^{\alpha} - h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\nu}{}^{\alpha}{}_{,\alpha\mu} + h^{\rho}{}_{\rho,\mu\nu} + \eta_{\mu\nu}\left(h^{\alpha\beta}{}_{,\alpha\beta} - h^{\alpha}{}_{\alpha,}{}^{\beta}{}_{\beta}\right) = 0. \tag{1.7}$$

It is worth mentioning two properties of this equation: it is invariant under a) a redefinition of the field, $h_{\mu\nu} \rightarrow h_{\mu\nu} + \lambda\eta_{\mu\nu}h^{\alpha}{}_{\alpha}$, with $\lambda \neq \frac{1}{4}$ to avoid the new fields being traceless, and b) the gauge transformations $h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$, where $\xi_\nu(x)$ is an arbitrary Lorentz tensor field[7].

Gravitons from GR

The derivation of the graviton field from GR is through the lowest-order approximation of the Einstein-Hilbert action in the absence of boundaries

$$S[g] = \frac{1}{\kappa^2}\int R(g(x))[|detg|]^{\frac{1}{2}}d^4x \tag{1.8}$$

in the expansion $g_{\mu\nu}(x) = g^0_{\mu\nu}(x) + \kappa h_{\mu\nu}(x)$. The field eq. for $h_{\mu\nu}(x)$ is precisely (1.7) when the lowest order in κ is considered. The two properties mentioned above concerning the field equation have an interpretation here. The first corresponds to using $g_{\mu\nu}(|detg|)^\lambda$ as the field variable instead of $g_{\mu\nu}$. The second is just the effect induced by an infinitesimal diffeomorphism of Minkowski space generated by the vector field ξ.

Advantages

The advantages of adopting such a scheme can be summarised in:

1) Short-distance behaviour, operator-product expansions, regularisation and related topics are faced conventionally due to the existence of the background metric.

2) A fixed causal structure is afforded by the background metric that allows to define microcausal spacetime commutation relations for spacetime fields, equal-time commutation relations for canonical fields and a good notion of time ordering for use in a functional integral or other formalisms of coventional quantum field theories.

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[7]This invariance is a consequence of the graviton being massless and it turns out to be necessary to project out the lower-spin ghosts which are otherwise associated with the tensor field $h_{\mu\nu}(x)$.

3) Some of the difficulties with the Diff($\mathcal{M}$) invariance are translated into standard problems for gauge-invariant quantum-field systems and then they can be tackled with such methods.

4) The definition of observables can be approached by using the underlying Minkowski structure in analysing the asymptotic behaviour of the fields. The key point is that the gauge-group generators ξ have compact support and therefore do not affect the fields asymptotically[8].

Criticisms

Concerning the geometrical and conceptual criticisms we have:

1) There is no reason to adopt the causal structure of the Minkowski metric as the physically correct one. In fact, it has been suggested that a non-perturbative treatment could lead to light cones that do not coincide with those provided by the Minkowski structure. The status of the initial microcausal structure is uncertain.

2) The Minkowski background fixes also the topology of spacetime to coincide with that of a trivial vector space. In this way any feature of classical GR involving non-trivial topological structure is made difficult to discuss, e.g. cosmological problems, spacetimes singularities, black holes and event horizons.

3) The expansion the graviton field comes from is a poor one in the geometrical perspective of classical GR. For example, $g_{\mu\nu}(x)$ will be a genuine metric tensor (an invertible matrix with signature (-1,1,1,1)) only for small values of $h_{\mu\nu}(x)$. However, in some quantisation methods, one integrates over all values of $h_{\mu\nu}$. Indeed, rather than quantising on the space of pseudo-Riemannian metrics, we are quantising on the tangent space to the specific $\eta_{\mu\nu}$.

[8]Asymptotic observables played an important role in the seminal investigations of B.S. DeWitt [53, 54, 55] on the spacetime covariant approach to QG.

Non-Renormalizability

When the corresponding expansion, for the metric in terms of the graviton field, is inserted in the Einstein-Hilbert action (1.8) the resulting Lagrangian for $h_{\mu\nu}(x)$ contains terms that are non-polynomial, derivatively-coupled and with a dimensional coupling constant κ. Each one of the last features is an indication of the perturbative non-renormalisability of a quantum field theory in four spacetime dimensions. This can be considered the major disease of the approach to QG[9] we are discussing now. And that is the reason why several schemes have been proposed to avert it. This is the case, for instance, of the *"$R + R^2$" theories* in which to the Einstein-Hilbert action is added the square of the Riemann curvature [58, 59, 60] but which have not succeded because of problems of non-unitarity and negative-norm states [61]. *Supergravity* theory [62] was another major program aimed at removing ultraviolet divergences with the hope that the additional fermionic loops would cancel the infinities produced by the bosonic graviton loops. The appealing feature of this approach is that it yields a definite prediction for the fundamental matter Lagrangian to be coupled to GR. Unfortunately, it was found the idea does not work for more than 2 loops in the case of $N = 1$ supergravity and for more than 7 loops in the $N = 8$ case.

2. **Quantisation of a theory that gives GR as its low-energy limit**

The key step here is to find a system possessing a well-defined quantum theory and which yields classical GR as a low-energy limit, even though that is not the starting point.

Induced gravity

Here, the Einstein-Hilbert action is not fundamental, but rather an effective action induced by the quantum structure (for a detailed account see [63]). In constructing

[9] In the early stages, only power-counting estimates indicated the non-renormalisability of the theory. Due to kinematical reasons the pure-gravity one-loop graphs are finite on-shell. This is not the case when matter is included. Finally, two-loop calculations [56, 57] explicitely showed this failure.

the theory one starts with an action including only the usual coupling of matter fields to the metric tensor, and the pure-gravity term arises as a counterterm from quantised matter fields. Such term coincides with the weak-field expansion of the Einstein-Hilbert action. Hence, the criticisms to the weak-field scheme apply to the present scheme.

String theory

This is a more sophisticated scheme in which the graviton occurs as just one of an infinite number of particles associated with the quantised string.

The idea that a quantum theory of gravity can be constructed starting from closed strings comes from studies of zero-slope limit of the dual resonance model for non-hadrons [64, 65] (an extensive review on this link is [66]).

The main idea is to quantise certain fields appearing in the Polyakov action

$$S[q,X] \equiv \frac{1}{4\pi\alpha'}\int_W q^{ij}(\sigma)\partial_i X^\mu(\sigma)\partial_j X^\nu(\sigma) g_{\mu\nu}(X(\sigma))[det g]^{\frac{1}{2}} d^2\sigma \qquad (1.9)$$

where q_{ij} is a metric on the two-manifold W (the world-sheet), $X : W \to \mathcal{M}$ are the string fields which map W into the spacetime manifold $\mathcal{M}$, and $g_{\mu\nu}$ is a background metric on $\mathcal{M}$. The constant α' is related to the string tension and is assumed to be of the order of the Planck length.

The classical system is invariant under the conformal transformations $q_{ij}(\sigma) \mapsto F(\sigma)q_{ij}(\sigma), \quad F(\sigma) > 0$, which can hold at the quantum level only when the metric $g_{\mu\nu}$ satisfies an equation that is effectively the vanishing of the trace of the energy-momentum tensor of the two-dimensional quantum field theory. Such an equation has the form

$$0 = R_{\mu\nu} + \frac{1}{2}\alpha' R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + \cdots \qquad (1.10)$$

where higher-order powers and derivatives of the Riemann curvature are not written down explicitely. Other background spacetime fields may also be introduced (e.g. massless dilatons and 2-form fields) producing similar equations.

Conformal invariance also constrains the dimension of $\mathcal{M}$ to be equal to some critical value depending on the background fields that are present or any two-dimensional spinor fields added to the system (like in superconformal field theories).

The above field equations are the string-theory substitute for the classical equations of GR. The metric $g_{\mu\nu}$ is not considered as background structure since it comes about as a solution of the dynamical equations. However, it works like that once calculations of quantum fluctuations around it are performed.

Several exact solutions to (1.10) have been found but probably among the most interesting ones are those which contain spacetime singularities [49, 67, 68]. The existence of these singular solutions may be taken as disappointing if one thinks that quantum gravity is supposed to remove the singularities coming from the classical framework of GR. Also, it is often claimed that the existence of a minimal length in the theory implies that quantum amplitudes should be free of ultraviolet divergences that plague conventional quantum field theory. However, this idea does not clarify whether the strings can only probe to a minimal length (keeping the spacetime continuum) or this fact shows the breakdown of the whole continuum picture. It cannot be decided adopting the Polyakov approach since the presence of a continuum manifold $\mathcal{M}$ is part of the background structure.

The main problem arises when an assessment of the singular solutions of the effective field equations is made. The next step following this line of work involves the calculation of the quantum fluctuations around the classical background solutions; we are back to the weak-field scenario of the old approaches to QG. Even when many of these high-energy calculations involve non-perturbative methods for summing the various contributions, any complete, non-perturbative alternative to the Polyakov approach is lacking. This might help to investigate the main issues in QG: spacetime singularities, quantum topology, quantum cosmology, etc.

3. **General-Relativise Quantum Theory**

Rather than starting with classical GR which is then quantised, one begins instead with conventional quantum theory and studies the extent to which it can be made compatible with the ideas of GR. This is very intriguing but it has not been developed much as the quantum field theory has. Remarkable in this approach is the work of K. Fredenhagen and R. Haag [35] who study the problem of making a quantum theory invariant under spacetime diffeomorphisms (see also [69]). This seems to be the reason why C.J. Isham used the term "General-Relativise".

4. **The semi-classical option**

An idea of Moller [70] in the opposite extreme to the one here exposed is that perhaps it is not necessary to quantise the gravitational field but only the matter to which it couples. This is in general what is meant by semi-classical approach. The induced-gravity approach mentioned above is an example.

Originally, the idea was to study the system

$$G_{\mu\nu}(g) = {}_t\langle\psi|T_{\mu\nu}(\widehat{matter}, g)|\psi\rangle_t, \tag{1.11}$$

$$i\hbar\frac{d}{dt}|\psi\rangle_t = H(\widehat{matter}, g)|\psi\rangle_t \tag{1.12}$$

where the source of the gravitational field is the expectation value of the energy-momentum tensor in some special state $|\psi\rangle_t$. Several remarks are in order.

1) higher powers of the Riemann curvature appear when regularisation and renormalisation of the energy-momentum tensor are made. These are needed because of the quantum matter fields considered as source in the first equation [71].

2) the system seems to be intrinsically unstable [72, 73, 74, 75]; the two equations are strongly coupled and the effective equations for the metric tensor are far more non-linear than those of GR. The calculations used were, however, mainly of perturbative nature and have been recently challenged [76, 77].

3) the effective equation for $|\psi\rangle_t$ is non-linear, hence, the superposition principle is lost. Whether or not this is a problem is related to one's attitude to conventional quantum theory.

It is not at all clear that quantising everything but the gravitational field is something inconsistent. For instance, the Bohr-Rosenfeld argument [78] showing that the electromagnetic field should be quantised if it couples consistenly to the current generated by the quantised matter does not apply to the gravitational case. This can be seen as follows. The proof for electromagnetism involves taking to infinity the ratio e/m of the electric charge e to the inertial mass m of a test particle. This is forbidden in the gravitational case by the equivalence principle since the analogue of e is the gravitational mass [79]. Several attempts have appeared [80, 81, 82] but no-one has succeeded in clarifying the situation completely (see also [23]).

The semiclassical approach has recently reappeared in the form of a Born-Oppenheimer (-WKB)approximation to QG. Again, quantum matter effectively couples to a classical gravitational field. Now, however, this is considered as an approximation to the unknown full theory of Quantum Gravity (see [83] for a comprehensive analysis).

There is no hope the lack of any experimental evidence on the would-be quantum aspects of gravity can be soon remediated. This makes it rather difficult to decide what approach to follow in investigating the theory. Self-consistency and indirect consequences on experimentally accesible phenomena will be the only guide to study it in the immediate future. Nevertheless, the formalisms put forward by Ashtekar and Rovelli-Smolin seem to be highly promising in the sense that they have managed to cope with some long-standing problems of quantum gravity. It remains to be seen, however, how much we can push the standard ideas on quantum theory and general relativity so that they can succeed at least in some respects towards the understanding of the quantum gravity issue. Thus, the following chapter reviews the basics of the general relativity based theory of quantum gravity, *i.e. Non-perturbative canonical gravity.* The rest of the present thesis will present results concerning the inclusion of Fermionic matter along these lines.

TWO

NON-PERTURBATIVE CANONICAL GRAVITY

This chapter is devoted to summarize the basis which are at the root of the results reported in this work. These basis are the canonical program for classical and quantum gravity introduced by Bergmann, Dirac, DeWitt, Wheeler and others, and the recent variations of it coming in the form of the Ashtekar and Rovelli-Smolin Loop approaches. They appear under the generic name of *Non-perturbative canonical gravity.*

In the introductory chapter the advantages and disadvantages of the different quantum-gravity approaches were elucidated. Now the general relativistic view is adopted. From here on the guideline will be Non-Perturbative Canonical gravity as described below. First, by discussing the initial-value problem for general relativity, as in [84], the constrained character of the theory is established. That is to say, initial data are not enough to get the "evolution in time" of gravity; the initial data must satisfy certain constraints. The explicit form of these constraints is given together with the interpretation of the Cauchy problem in this case, contrasting it with the standard field-theories analogue. Then, following [84], the Hamiltonian formulation is set. Hereby *geometrodynamical* variables are introduced to express the contents of general relativity. Using these variables the two constraints of the theory: vector and scalar, are interpreted as generating spatial diffeomorphisms and dynamics, respectively. This set of constraints are consistent, *i.e.* they form a closed algebra in Dirac's terminology. The quantisation of the theory is described following the Dirac quantisation method for constrained systems as given in [24, 85]. The quantum version of the constraints admits the original classical interpretation as generating three-dimensional diffeomorphisms and dynamics, respectively, of the quantum states for the gravitational field. Indeed, physical states are defined to be those which are anihilated by such quantum generators. Also, the main problems coming out by using this whole framework are remarked. Among them, the lack of exact solutions to the above equations involving the quantum constraints is taken as a strong motivation for the next two approaches to be discussed. This problem can be traced back to the one of making sense of non-polynomial functionals of the basic canonical operators.

Having introduced the gemetrodynamical framework makes it easier to understand the aim of the Ashtekar formalism. The idea is to define new canonical variables to describe gravity so that quantum dynamics becomes more manageable.

To settle the formalism we start with the action functional of Samuel and Jacobson and Smolin [86] containing a (complex) self-dual connection and a soldering form. It is shown this action is equivalent modulo the equation of motion for the connection to the standard Einstein-Hilbert action for pure gravity by virtue of Bianchi symmetry of the curvature tensor. It is worth stressing the connection here is assumed to be torsion-free. (This is not automatic, *e.g.* when spin-$\frac{1}{2}$ fields are coupled to gravity the connection develops a non-zero torsion, as it is shown in chapter 3). The constraints of the theory are read off from the action just introduced. It turns out now there are four types of constraints: Gauss, vector, scalar and *reality conditions*. The first three types can be interpreted as generators of *internal* rotations, three-dimensional diffeomorphisms and dynamics, respectively. The reality conditions are necessary to pick out the real section of the phase space of the complex theory introduced hereby corresponding to classical general relativity. With this *connection-dynamics* description one gets polynomial structure for the constraints, including the reality conditions, in the canonical variables. The transition to quantum theory becomes slightly easier because of this. The quantization program put forward by Ashtekar [7] is then described. In particular, the quantum Gauss constraint is automatically solved by looking at Wilson-loop-like objects [87]. The solutions to the quantum scalar constraint (wave functionals with support on smooth loops or containing certain kind of kinks and intersections) found by Jacobson and Smolin [87] are commented upon. The remaining problems once the geometrodynamical approach is replaced by the connection-dynamical one are stressed.

The search for solutions to the vector, or rather diffeomorphism quantum constraint, together with the character of connection of one of the Ashtekar canonical variables, led Rovelli and Smolin to put forward the Loop quantization framework. We briefly describe their approach at the end of this chapter.

2.1 Initial-value formulation for General Relativity

Every theory considered to have predictive power ought to have an initial-value formulation. Normally one considers spacetime as a given background and poses the problem of determining the evolution of quantities starting with their initial values and derivatives. The difficulty with general relativity is that spacetime itself is the quantity looked for; the notions of "initial data" and "time evolution" acquire their familiar significance only after one has obtained a solution of the field equations. Thus, the problem here becomes to find what quantities should be prescribed initially so that spacetime can be determinated from them.

In addressing the above problem we follow [84, 89]. Their approach is based on work of Lichnerowicz [90]. The idea is to define induced fields on a spacelike hypersurface using a four-dimensional metric and then to translate Einstein's equations in terms of these fields. This yields a theory of fields on a three-manifold with no reference to any four-geometry. Four-manifold and four-metric are the end products here.

To begin with we take a Lorentzian four-manifold M, with topology $\Sigma \times \mathbb{R}$ and a metric g_{ab} defined on it with signature $(-,+,+,+)$. M is assumed to admit foliations into three-manifolds Σ_t, spacelike w.r.t. g_{ab}, each of which is diffeomporphic to Σ. As in [84], they represent instants of time. The fields on Σ_t induced by g_{ab} which are useful for our aim are:

- the future-directed timelike unit normal to Σ_t: n^a,
- the positive-definite three-metric (first fundamental form) $h_{ij} = q_{ij}$ induced by $q_{ab} := g_{ab} + n_a n_b$. Such a q_{ab} clearly serves as a projector for tensor fields on M to Σ,

- the extrinsic curvature (second fundamental form) containing information about how Σ_t is embedded in (M, g_{ab}): $K_{ab} := q_a^m q_b^n \nabla_m n_n$.

Since notation may be confusing it is worth noticing that, given that M and Σ are two different manifolds, tensor fields associated with them should appear with different indices. Nevertheless, q_{ab}, which is orthogonal in each index to n^a, and K_{ab}, although defined on (M, g), reduce on Σ to the induced three-metric h_{ij} and extrinsic curvature K_{ij} respectively since they are automatically projected down. Thus, these tensor fields on (M, g) and (Σ, h) are (naturally) isomorphic. Whenever appropriate, however, we shall still use $h_{ij} = q_{ij}$ rather than q_{ab}, to avoid any residual confusion between the projector, the four-metric and the three-metric.

The vacuum Einstein equations

$$G_{ab} := {}^4R_{ab} - \frac{1}{2} g_{ab} {}^4R = 0 \tag{2.1}$$

having 10 components get decomposed when using the above three-dimensional quantities into 4 constraint equations and 6 dynamical equations [84, 89]. It is convenient to introduce first the unique connection on Σ, D, which is metric-compatible $(D_k h_{ij} = 0)$ and torsion-free $(D_{[i} D_{j]} f = 0)$. With it, four- and three-dimensional curvatures can be related [88, 89, 84]. The constraint equations are $(m = 1, 2, 3)$

$$2G_{ab}\, n^a n^b = R + (K_m^m)^2 - K_{ab}K^{ab} = 0 \,, \tag{2.2}$$

$$G_{ab}\, n^a q_m^b = q^{an} D_n \left(K_{am} - q_{am} K_r^r\right) = 0 \,. \tag{2.3}$$

Where R stands for the Ricci scalar formed with the three-dimensional curvature tensor of D on Σ, and hence it is a function of $q_{ij} = h_{ij}$ and its derivatives. Hence, q_{ab} and K_{ab} are restricted to satisfy (2.2) and (2.3).

To settle the evolution equations two more quantities are required. A function t on M with each of its level surfaces diffeomorphic to Σ and the timelike future-directed vector field t^a so that $t^a \nabla_a\, t = -1$. In this way t^a can be identified with $\frac{\partial}{\partial t}$ and its integral curves can be interpreted as connecting a spatial point at different times ($t =$ *constant* hypersurfaces). At every hypersurface the vector field t^a can be decomposed into normal and tangential components:

$$t^a := Nn^a + N^a \ , \tag{2.4}$$

with N measuring the rate at which time elapses (*lapse*) and N^a representing the necessary spatial shift to remain perpendicular to the hypersurface (*shift*). Finally, the equations of motion can be written as [84]

$$\dot{q}_{ab} := (\mathcal{L}_t\, q)_{ab} = 2NK_{ab} + \left(\mathcal{L}_{\vec{N}}\, q\right)_{ab} \ , \tag{2.5}$$

$$\dot{K}_{ab} := (\mathcal{L}_t\, K)_{ab} = -NR_{ab} + 2K_a^{\ m}K_{bm} - NKK_{ab} + D_aD_bN + \left(\mathcal{L}_{\vec{N}}K\right)_{ab} \ . \tag{2.6}$$

The first of them gives the sense in which K_{ab} is interpreted as the "velocity" of q_{ab}. In the second use was made of (2.3). It yields the evolution of K_{ab}. It is worth stressing they refer only to three-dimensional fields and contain all the information of the four-dimensional Einstein equations, for giving t^a (*i.e.* N and N^a) one solves (2.5)-(2.6) to get q_{ab} and hence g_{ab}.

We end this section with some further remarks [84]. The constraints (2.2) and (2.3) are preserved in time, as can be seen by taking their Lie derivatives and making use of the equations of motion (2.5) and (2.6). The inverse problem, however, is more difficult: a set of initial data satisfying the constraints can be evolved for a finite time so as to obtain a solution to Einstein field equations (well-posed Cauchy problem). This problem is solved by a theorem [91] asserting that given the pair (q_{ij},K_{ij}) of positive-definite metric q_{ij} and symmetric second-rank tensor field K_{ij} on Σ, satisfying the constraints (2.2)-(2.3), there exists a metric g_{ab}, unique up to diffeomorphisms, with signature $(-,+,+,+)$ on the four-manifold $\Sigma \times \mathbb{R}$ and an embedding $i : \Sigma \to M$ such that the induced metric and extrinsic curvature of $i(\Sigma)$ are identified with the images under i of the original q_{ij} and K_{ij}.

Based on the ideas just described concerning the structure of the initial-value problem for general relativity, in the next section the canonical quantisation program applied to general relativity is reviewed.

2.2 Geometrodynamical variables

As summarized by K. Kuchař [85] the canonical quantisation program applied to any classical theory can be set as follows: 1) Translate the classical theory into the Hamiltonian formalism and identify the corresponding canonically conjugate variables. 2) Turn such variables into operators satisfying the Dirac commutation relations, and substitute these operators into the Hamiltonian in order to get the Schrödinger equation. 3) An inner product should be defined that is conserved by this equation along the dynamical evolution of the state. The existence of this product makes the space of solutions into a Hilbert space where the probabilistic interpretation of the theory comes from.

In the case of gravity, however, there is no Hamiltonian in the usual sense but Hamiltonian constraints as described in the previous section. The implementation of the canonical quantisation approach is not straightforward and, in particular, a completely satisfactory Hilbert space has not been constructed so far. Among the most important consequences is that a clear probabilistic interpretation of the theory is lacking. Nevertheless, it worth understanding the sources of the difficulties for, eventually, circumventing them.

2.2.1 The Canonical Structure of Classical GR

This section can be considered as the first step, mentioned above, of the canonical quantisation program. Early studies aiming to cast classical GR in a canonical form, that is, adapting it to a Hamiltonian structure, were developed by using a specific coordinate system for spacetime. Thus, the involved global aspects were not emphasized [92, 93, 94]. These aspects of global character turn out to be of vital importance for the analysis of topological structure in QG, *e.g.* changes of topology, wormholes. Also, a clear understanding of them is convenient in carrying out the 3+1 foliation of spacetime. Here we do not consider such a kind of problems and hence we use a rather standard approach as adopted by Ashtekar [84]. Details of the geometrical, global view are given in [95, 96, 97, 98, 99] (see also [24]).

Starting with the structure provided by the initial-value formulation discussed in section 3.1 one readily realizes an adequate configuration variable for the canonical description is the three-metric h_{ij}. The next step is of course to determine its canonically conjugate momentum p^{ij}. The standard procedure is to use the corresponding Lagrangian L and define it in the form $p^{ij} := \frac{\delta L}{\delta \dot{h}_{ij}}$. Since the Lagrangian, $\int d^3x\, \sqrt{-g}\; {}^4R$, associated to the Einstein-Hilbert action dependes on second time derivatives of h_{ij} this task is not straightforward. This can be settled by substracting a total divergence that removes the undesired second-order time derivatives[10]. Indeed, the Gauss-Codazzi equations yield the identity

$$ {}^4R = R + K_{ij}K^{ij} + K^2 - \frac{2}{N}\frac{\partial K}{\partial t} - \frac{2}{N}\left(h^{ij}\, D_jD_iN - N^iD_iK\right) \quad . \qquad (2.7) $$

Thus, using $\sqrt{-g} = N\sqrt{h}$, and adding suitable boundary terms (see chapter 4), the Langrangian we look for is found to be

$$ L = \int_\Sigma d^3x\, \sqrt{h}\, N\left(R + K_{ij}K^{ij} - K^2\right) . \qquad (2.8) $$

The canonical momentum conjugate to h_{ij} is hence the weight-one tensor density

$$ p^{ij} := \frac{\delta L}{\delta \dot{h}_{ij}} = \sqrt{h}\left(K^{ij} - Kh^{ij}\right) . \qquad (2.9) $$

As usual, to obtain the Hamiltonian of the theory a Legendre transform has to be carried out. In our case, bearing in mind that $h_{ij} = q_{ij}$, one looks for

$$ H[q,p] := \int_\Sigma d^3x \left[\left(p^{ij}\dot{q}_{ij}\right) - L\right] . \qquad (2.10) $$

Neglecting surface terms the result of using the above definitions is

$$ H[q,p] = \int_\Sigma d^3x\; N\left[-h^{\frac{1}{2}}R + h^{-\frac{1}{2}}\left(p^{ij}p_{ij} - \frac{1}{2}p^2\right)\right] + \int_\Sigma d^3x\; N_j\left(-2D_ip^{ij}\right) , \qquad (2.11) $$

which is actually a combination of the constraints (2.2)-(2.3) weighted by the lapse and shift as can be shown using the definition of the momentum p^{ij}. This is not exactly the same in the case when Σ is non-compact; the Hamiltonian then turns out to contain a boundary term [84]. It should be clear at this point the relevance of the constraints for general relativity: *they are linked to the dynamics of gravity.*

We go on with the analysis of the contraints. In section 2 we saw that not all pairs (q_{ij}, K^{ij}) are allowed as initial data because they must fulfil the constraints (2.2)-(2.3).

[10] This problem and its solution are made clear in chapter 4, where the variational problem is studied.

Thus, also the points (q_{ij}, p^{ij}) of the phase space $\widetilde{\Gamma}$ should be restricted so that only physical states of the gravitational field are picked up. To do so one first has to recast the constraints (2.2)-(2.3) in terms of the canonical variables q_{ij}, p^{ij}. They become $(l = 1, 2, 3)$

$$C_l(q,p) = -2q_{lm}D_n p^{mn} , \tag{2.12}$$

$$C(q,p) = h^{\frac{1}{2}}R + h^{-\frac{1}{2}}\left(p^{ij}p_{ij} - \frac{1}{2}p^2\right) . \tag{2.13}$$

Several technical details can be faced more easily if these constraint maps from phase space into covector and scalar densities are smeared out with test fields on Σ. For this purpose, take

$$C_{\vec{v}}(q,p) := \int_\Sigma d^3x \, v^l C_l(q,p) , \tag{2.14}$$

$$C_N := \int_\Sigma d^3x \, N C(q,p) . \tag{2.15}$$

These constraint functions generate motions in phase space. This was expected since the Hamiltonian of the theory (*cf* (2.11)) is a combination of these functions (Although for the non-compact case the situation is more complicated the results hold the same [84]). Indeed, calculating the transformations they generate by using the canonical pair q_{ij}, p^{ij}, one gets [84] the vector field on the phase space $\widetilde{\Gamma}$ produced by $C_{\vec{v}}$ as

$$X_{C_{\vec{v}}} = \int_\Sigma d^3x \left[(\mathcal{L}_{\vec{v}}q)_{ij} \frac{\delta}{\delta q_{ij}} + (\mathcal{L}_{\vec{v}}p)^{ij} \frac{\delta}{\delta p^{ij}} \right] , \tag{2.16}$$

whereas the one associated to C_N, *when restricted* to the constraint surface $\bar{\Gamma}$ (*i.e.* the surface defined by $C_l(q,p) = 0$ and $C(q,p) = 0$)), becomes

$$X_{C_N} = \int_\Sigma d^3x \left[(\mathcal{L}_{N\vec{n}}q)_{ij} \frac{\delta}{\delta q_{ij}} + (\mathcal{L}_{N\vec{n}}p)^{ij} \frac{\delta}{\delta p^{ij}} \right] . \tag{2.17}$$

The geometrical interpretation of this construction is that, in light of (2.16), every vector field v defined on Σ generates a one-parameter family of diffeomorphisms on Σ. An infinitesimal diffeomorphism acting on the fields q_{ij}, p^{ij} on Σ, maps them into $q_{ij} + \epsilon(\mathcal{L}_{\vec{v}}q)_{ij}$ and $p^{ij} + \epsilon(\mathcal{L}_{\vec{v}}p)^{ij}$, respectively. This is why C_l is called the *diffeomorphism* constraint. On the other hand the insight one gains through (2.17) is that, on the constraint surface $\bar{\Gamma}$, the initial data (q_{ij}, K^{ij}) (or equivalently (q_{ij}, p^{ij})) "evolve in time" along the integral curves of the timelike vector field Nn^a. Thus, $C(q,p)$ is called the *Hamiltonian* constraint.

These constraints turn out to be first-class in that the Poisson bracket, with canonical variables q_{ij}, p^{ij}, between any two of them gives a combination of the constraints themselves. Explicitly

$$\{C_{\vec{v}}, C_{\vec{w}}\} = -C_{[\vec{v},\vec{w}]} \tag{2.18}$$

$$\{C_{\vec{v}}, C_N\} = -C_{\mathcal{L}_{\vec{v}}N} \tag{2.19}$$

$$\{C_N, C_M\} = -C_{\vec{K}} \tag{2.20}$$

where $K^i := ND^iM - MD^iN$. This algebra is sometimes *Dirac algebra.* It is not a Lie algebra since (2.18)-(2.20) involve structure *functions* rather than structure constants. In particular, they are not a realization of the Lie algebra of vector fields on spacetime. This can be traced back to the necessity of using the spacetime metric to decompose the vector field $t^a = Nn^a + N^a$ into lapse and shift and does not depend *only* on the manifold structure of spacetime.

Physical degrees of freedom. It is possible to construct a reduced phase space, $\hat{\Gamma}$ by dividing out the constraint surface $(C(q,p) = 0,\ \ C_l(q,p) = 0)$ $\bar{\Gamma}$ by orbits of the fields generated by the constraints (2.16) and (2.17). To every point of $\hat{\Gamma}$ there are associated two true degrees of freedom of the gravitational field. This is readily seen by first noticing that a point in $\tilde{\Gamma}$ represents twelve functions $(q_{ij},\ p^{ij})$ per point of Σ. Beacuse of the four constraints, each point of $\bar{\Gamma}$ has associated eight functions and, by divinding out by the four constraint vector fields, $\hat{\Gamma}$ represents four functions or, equivalently, two true degrees of freedom.

Having studied the canonical structure of GR we face in the following section its quantisation.

2.2.2 Canonical Quantisation

Whenever constraints are involved in a canonical theory the quantisation becomes difficult. Several possibilities may be tried. The most natural perhaps is to reduce the theory to a true canonical form by eliminating the constraints and the corresponding Lagrange multipliers before the quantisation is carried out. We have already faced the constraint equations of GR and, in fact, they cannot be solved in a closed form, the only known alternative being the Ashtekar formalism (where the corresponding constraints have a simpler structure, see next section) and the perturbative weak-field methods we critiziced in chapter 1. Further unappealing reasons, given in [24, 100], make this reduction to the true canonical form a program not easy to implement.

The possibility that has received more attention is the one in which the complete set of variables $(q_{ij}(x), p^{kl}(x))$ are given a quantum status and only at the quantum level the problem of the constraints and other (like gauge fixing) are tackled. The first step consists in setting the form of the canonical commutation relations:

$$\begin{aligned} \left[\hat{q}_{ij}(x), \hat{q}_{kl}(x')\right] &= 0 \\ \left[\hat{p}^{ij}(x), \hat{p}^{kl}(x')\right] &= 0 \\ \left[\hat{q}_{ij}(x), \hat{p}^{kl}(x')\right] &= i\hbar\delta_{(i}{}^{k}\delta_{j)}{}^{l}\delta(x, x') \end{aligned} \tag{2.21}$$

Remarks in order are:

1. The Schrödinger representation is adopted since the canonical variables do not carry any time dependence (however, see the analysis of the quantum constraints below).

2. Smeared operators should be introduced in order to avoid eventual use of the components of q and p in a specific coordinate system. That is:

$$\begin{aligned} \left[\hat{q}(h), \hat{q}(h')\right] &= 0 \\ \left[\hat{p}(k), \hat{p}(k')\right] &= 0 \\ \left[\hat{q}(h), \hat{p}(k)\right] &= i\hbar \int_{\Sigma} h^{ab}(x) k_{ab}(x) d^3x \end{aligned} \tag{2.22}$$

   where h and k are, respectively, a tensor density and a tensor [11].

[11] The third of these equations is coordinate independent since the r.h.s. is integrated over all Σ

3. Microcausality. The first equation in (2.21) might be interpreted as a mean to guarantee that the points on Σ can be taken as spacelike separated, independently of the spacetime structure that could be adopted.

4. Affine commutation relations. By insisting in the metric character of $q_{ij}(x)$ it should be investigated whether there exists the corresponding quantum operator and if it is compatible with the above canonical commutation relations. It has been argued that a positive-definite smeared operator may correspond to the notion of the classical metric q_{ij} and even more there are suggestions that certain degeneracy should be allowed [49, 101, 102, 103], *i.e.* the action of the smeared operator on non-zero vectors can give zero. Furthermore, it turns out to be case that there is no compatibility of the semi-definite smeared operator and the canonical commutation relation. Affine commutation relations should replace the canonical ones[12] [105].

2.2.3 Treatment of the constraints a la Dirac

What Dirac demands concerning the quantisation of theories with first-class constraints is that such constraints should be imposed on the physical states [106]. That is to say, at the quantum level the constraints should pick out only those states which are physically meaningful, *i.e.* those annihilated by the quantum constraint operators. In the present case we have:

$$C_l(\hat{q},\hat{p})\Psi = 0 \tag{2.23}$$

$$C(\hat{q},\hat{p})\Psi = 0 \tag{2.24}$$

They are the quantum analogue of the classical result about the equivalence of the constraints and dynamical equation, i.e., they are the whole technical content of the theory of QG. This can be related also to the following result. The canonical Hamiltonian (2.11)

[12] The analogue is a particle restricted to move in $\mathbb{R}^+$. Canonical commutation relations, $[\hat{x},\hat{p}] = i\hbar$, imply the spectrum of $\hat{x}$ is $\mathbb{R}$. However, compatibility is obtained by replacing the conventional commutation relation by the affine algebra: $[\hat{x},\hat{p}] = i\hbar\hat{x}$ [104].

is taken with N and $\vec{N}$ regarded as c-number functions in constructing the "Schrödinger equation":

$$i\frac{d}{dt}\Psi_t = \hat{H}[N,\vec{N}]\Psi_t \tag{2.25}$$

which by virtue of (2.24) implies tha Ψ_t is time independent. Also, it seems to be meaningless to talk about "Schrödinger" or "Heisenberg" picture since matrix elements between physical states in both pictures will coincide. These aspects of QG can be traced back to the absence of any intrinsic definition of "time" in GR. We have not gauge-fixed the theory and hence no such coordinate has been selected.

The problems arising when the implementation of Dirac scheme is attempted may be very severe. Probably among the most difficult ones is whether and to what extent the classical Poisson-algebra structure (2.20) can be translated into the quantum theory. Other important issues are:

1. Regularisation and renormalisation of the operators constraints (2.12)-(2.13). The origin of these problem is the non-linearity of these equations in the field operators expressed also as products evaluated at the same point. This is the analogue of the ultraviolet divergence problem in the quantum field theory.

2. Operator ordering. Its origin is the appearance of product operators in the constraints and, more precisely,

   (a) *e.g.* it has to be decided where to place $\hat{q}_{ij}(x)$ on the r.h.s. of the third equation in (2.20). A restriction here can be set in that no further constraints on the physical state vectors are desired to be generated by the commutators of the given first-class constraints.

   (b) the Hermiticity or non-Hermiticity of the constraints [107, 108]. The non-hermiticity option comes about because of the absence of a well-established relation between the Hilbert-space structure carrying the representation of the canonical (or affine) algebra and the one that should be imposed on the physical states (i.e. those satisfying tha constraints).

(c) singular operator products in the constraints seem to imply that any ordering is likely to be ambiguous [109].

(d) an anomaly should, possibly, be present in the theory as a genuine Planck-length effect. The mathematical problem here is that, for example, not much is known about central extension of the Dirac algebra.

3. The lack of a clear definition of the inner product on the physical states to be constructed from the Hilbert-space structure on the original space $\mathcal{H}$ that carries the representation of the canonical, or affine, commutation relations.

4. The recovering of the Diff($\mathcal{M}$) group. This is the converse problem of the translation of the Diff($\mathcal{M}$) invariance into the Dirac algebra in the canonical decomposition [101].

2.2.4 The meaning of the quantum constraints

Our aim here is to extract information from the quantum constraints since, as we have seen, they provide the whole dynamical content of QG. We will see that the quantum version of the vector constraints imposes a structure on the domain of the state vectors of QG leading to the notion of superspace, while the classical Hamiltonian constraint becomes the so-called Wheeler-DeWitt equation which provides us with the, more properly said, dynamics of gravity.

To achieve the above goals, however, the introduction of a given representation for the canonical algebra is needed. It is natural to adopt the analogue of the quantum operators associated to the canonically conjugate variables in standard quantum mechanics, i.e.

$$\begin{aligned} (\hat{h}_{ij}(x)\Psi)[h] &\equiv h_{ij}(x)\Psi[h] \text{ and} \\ (\hat{p}^{kl}(x)\Psi)[h] &\equiv -i\hbar\frac{\delta\Psi}{\delta h_{kl}(x)}[h], \end{aligned} \tag{2.26}$$

for the quantum operators $\hat{q}$ and $\hat{p}$. They are commonly used in spite of the following:

1. the incompatibility of the positiveness (or semi-positiveness) of the classical Riemannian metric with the above canonical algebra implies that the states functionals do

not have domain Riem(Σ). This is possible only if the commutation relations are the affine ones.

2. the measure problem. There do not exist Lebesgue measures on the space Riem(Σ) of Riemannian metrics with which an Hermitian inner product between state vectors can be defined [13].

3. distributional metrics are expected to be the objects around which any measure can be concentrated. However, Riem(Σ) is not a vector space and hence its dual space cannot be defined (at least conventionally). The above distributions could live in such a dual space. On the other hand, affine commutation relations do allow an appropriate distribution of a distributional metric and, furthermore, they admit representations in which state vectors are concentrated on distributional analogues of degenerate metrics, as well as some in which the state vector has an internal index analogously to the spin of a relativistic particle [101, 105].

Now we face the interpretation of the quantum momentum-constraint. We expect, as in the classical case, to have the $\hat{C}(\vec{\xi})$ as the generator of Diff(Σ). While insisting in keeping at quantum level the structure of the classical algebra of the $\hat{C}(\vec{\xi})'s$ (*i.e.* Diff(Σ) algebra) operator-ordering problems come about.

One way to avoid these problems is to force the $\hat{C}(\vec{\xi})'s$ generators to form a Hermitian representation of the algebra of Diff(Σ). Thus it is assumed that the quantum theory carry a unitary representation of Diff(Σ) and the quantum momentum constraint is translated into [14]:

$$(D(f)\Psi)[h] = \Psi[h] \tag{2.27}$$

[13] There is the possibility of introducing an infinite-dimensional weighted measure which requires $(\hat{p}^{ij}(x)\Psi)[h] \equiv -i\hbar\frac{\delta\Psi}{\delta h_{ij}(x)}[h] + i\rho(h)\Psi[h]$ where $\rho(h)$ is a function that compensates for the weight factor in the measure [24].

[14] Roughly, $D(f)$ is an element of the group Diff(Σ) and then it can be obtained by exponentiating the generators $\hat{C}(\vec{\xi})$ belonging to the Lie algebra of Diff(Σ). Since $(\hat{C}(\vec{\xi})\Psi)[h] = 0$ we get $(e^{\hat{C}(\vec{\xi})}\Psi)[h] = \Psi[h]$, the desired result.

where $D(f)$ is a unitary operator representing $f \in \text{Diff}(\Sigma)$. On the other hand, the natural representation of the operators $\hat{q}$ and $\hat{p}$ (2.26) suggests the action of $D(f)$ on Ψ as:

$$(D(f)\Psi)[h] \equiv \Psi[f^*h] \tag{2.28}$$

where f^*h is the usual pull-back of h by the diffeomorphism $f : \Sigma \to \Sigma$. From the above two aspects of the action of $D(f)$ on Ψ one is led to the conclusion that

$$\Psi[f^*h] = \Psi[h] \;\; \forall f \in \text{Diff}(\Sigma),\; h \in \text{Riem}(\Sigma). \tag{2.29}$$

We see the group Diff(Σ) acts on the space Riem(Σ) by sending h to f^*h, both elements of Riem(Σ), through $f \in \text{Diff}(\Sigma)$. Modulo metrics with isometry groups [15], one can think of Riem(Σ) as a fibre bundle with base space Riem(Σ)/Diff(Σ) and fibres the orbits of Diff(Σ).

The base space Riem(Σ)/Diff(Σ) of "inequivalent Riemannian metrics" under diffeomorphisms was called superspace by J.A.Wheeler [40, 110]. The quantum vector constraint in its version (2.29) says that the state functional Ψ is constant on the orbits of Diff(Σ) and hence it is a function on superspace, i.e. superspace is the true domain space of the QG state vectors.

Among the cautionary remarks once the above view is adopted are:

i) θ-states may be present due to the possible existence of non-trivial transformations under large diffeomorphism which cannot be continuously connected to the identity. We have discussed here only infinitesimal transformations [111, 112].

ii) (2.28) contains a unitary action only if the Hilbert-space measure on the domain space is itself a Diff(Σ) invariant, which probably does not exist. The only alternatives are to modify the structure of (2.28) and hence the idea that the state functional is constant on the Diff(Σ) orbits does not hold anymore.

[15] This problem can be circumvented by considering only those diffeomorphisms that leave fixed some particular frame at a base point in Σ.

iii) If the domain space of the state vectors is a space of distributional metrics, the action of Diff(Σ) on it would change. The bundle picture is no longer correct.

The analysis of the quantum Hamiltonian constraint is not as straightforward as in the case of the quantum vector constraint. Essentially, the implicit h factor in the r.h.s. of (2.20) spoils the Lie-algebra structure and as a consequence the operator-ordering problem is far more severe. By choosing the simple ordering in which the p^{kl}'s are always to the right of the $q_{ij} = h_{ij}$ the quantum Hamiltonian constraint becomes

$$-\hbar^2\kappa^2 \mathcal{G}_{ijkl}(h)\frac{\delta^2\Psi}{\delta h_{ij}\delta h_{kl}}[h] - \frac{(h)^{1/2}}{\kappa^2}R(h)[h] = 0 \tag{2.30}$$

where $\mathcal{G}_{ijkl}(h) \equiv \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$ They are the Wheeler-DeWitt equation (WDW) and the "metric" on Riem(Σ), respectively [54].

We now briefly describe the main problems concerning the WDW equation. Their relevance coming from the fact that the whole canonical-quantisation program has been reduced to this equation.

Factor ordering The WDW equation was obtained by using a specific ordering inspired by simplicity. Other orderings are possible, for example one commonly used is to express the "kinetic term" (the one containing functional derivatives) as a covariant functional Laplacian taking the DeWitt metric as the underlying structure. The appealing property of this ordering is the invariance under "coordinate transformations" on Riem(Σ). From our point of view the most relevant aspect to be taken into account when deciding the operator ordering, is whether the $\hat{C}(x)$ are expected to be Hermitian.

Regularization The second-order functional derivative taken at the same spatial point when acting on some state functionals is likely to produce $\delta^3(0)$ singularities. A regularization procedure will be eventually required.

Time and time-evolution These concepts should be introduced in some way. The key idea is to consider time as an internal property of the gravitational system, probably including matter, instead of taking it as an external parameter of the universe.

Solutions of WDW The immediate plan of attack may be to handle it by using the notions on functional differential equations. The validity of this view depends on the interpretation of the constraint equation $\hat{C}(x)\Psi = 0$. If it is considered to be a self-adjoint functional differential equation with eigenvector Ψ corresponding to the zero eigenvalue, some sort of boundary conditions should be imposed on Ψ, as in conventional eigenfunction problems. The theory, however, does not give much information about it. In addition, problems due to the zero eigenvalue lead to the suggestion that a renormalisation of the Wheeler-DeWitt operator is required [24]. Other ways to look for solutions of WDW are the expansion in $1/G$ corresponding to perturbation theory where the coupling constant is the coefficient in front of $R(h)$ in WDW and the WKB approximation, which has been used in tackling the problem of time. Another possibility exists that has become the most popular way of studying the WDW. This is the minisuperspace technique, it involves freezing all but a finite number of the infinite degrees of freedom in $\mathrm{Riem}(\Sigma)$ and quantising the small number that remain. WDW becomes a second-order partial differential equation that can be studied using the conventional methods of differential equations. Such an approach is mainly used in the studies of quantum cosmology since the finite number of degrees of freedom can be chosen in a way that is adapted to the classical models of cosmology. Note that there is no way of estimating the effect of the infinite degrees of freedom that are dropped and thus any conclusion coming from this approach should be handled carefully. Remarkable, however, is the claimed utility of the minisuperspace models in the discussion of the problem of time, interpretation of the state vector and related issues.

We end this review of methods for solving WDW by talking about the use of functional integrals. The motivation is traced back to the result of ordinary quantum mechanics that a solution of the time-dependent Schrödinger equation $i\hbar\partial\Psi/\partial t = \hat{H}\Psi$ can be given as

$$\Psi(x,t) = \int \mathcal{D}x[s] e^{\frac{i}{\hbar}\int_{t_0}^{t} L(s)ds} \tag{2.31}$$

where the path integral is over paths that end at the point x of the path at the initial time t_0. The case of gravity has been studied in detail by C. Teitelboim [113, 114, 115], and has been recently used in the Hartle-Hawking approach to quantum cosmology [116] in which the Euclidian-time version of the Einstein action is adopted. It was formally shown that the state function constructed in this form is in fact a solution of WDW [117, 118]. It is worth mentioning that all the above proposals provide approximate solutions in that the full WDW is not solved. The situation seems to be improved by the introduction of the Ashtekar variables (*cf* next section).

We end this section by noting the advantages of a canonical approach to QG. Since canonical QG is discussed in an operator-based framework, the problems involved appear more explicitly than in the conventional methods of quantisation of gravity mentioned in chapter 1. In the canonical framework several techniques are background-metric independent. This leads to the possibility of developing a non-perturbative analysis of QG and therefore the problems of quantum cosmology, spacetime singularities and related issues may be faced in a more suitable way. Also, as we have seen, in this approach a strong emphasis is made on the geometrical structure of the spacetime as viewed in GR, and thus, the extent to which it holds in the quantum theory can be addressed as well as other deep conceptual problems like the one of "time".

2.3 Ashtekar variables

Apart from the severe conceptual problems mentioned in chapter 1, progress in the canonical approach to gravity has been so far slowed down by the highly non-trivial structure of the field equations when expressed in the canonical variables (q_{ij}, p^{kl}). Recently, A. Ashtekar introduced a set of new variables which seem to have improved the situation

[102, 119]. In terms of the Ashtekar variables all the equations of the theory become polynomial (at most quartic). Also, when these variables are used, a relation between Yang-Mills theory and GR is revealed. This relation allows an exchange of techniques between them. These two features, polynomiality of the constraints and Yang-Mills-like structure, hold even in the cases where one adds to the gravity system a non-zero cosmological constant, matter fields (scalars, spinors or Yang-Mills fields) or considers the supersymmetric extension of the theory [15, 120, 121].

There exist several equivalent approaches to arrive at the Ashtekar variables each one stressing different aspects of the construction (see e.g. [7] and references therein). Since we will be using the same kind of ideas in the next chapter to couple Fermionic fields to gravity, within this framework we here describe the work of Jacobson and Smolin [86].

2.3.1 Canonical variables

The aim here is to see how Ashtekar variables come out from a four-dimensional action rather than by a canonical transformation in the phase space of GR.

One of the Ashtekar variables is a connection, thus, it is natural to start with the first-order form of the Palatini-like tetrad action

$$S[e,\omega] = \int d^4x \; e \, e^{a\hat{a}} \; e^{b\hat{b}} R_{ab\hat{a}\hat{b}}(\omega) \, , \tag{2.32}$$

where e is the determinant of the tetrad. It is known that every rank-two antisymmmetric tensor can be decomposed into self-dual and anti-self-dual parts [122]. Decomposing the connection ω in this way leads to the curvature decomposition

$$R_{ab\hat{a}\hat{b}}(^+\omega + {}^-\omega) = R_{ab\hat{a}\hat{b}}(^+\omega) + R_{ab\hat{a}\hat{b}}(^-\omega) \tag{2.33}$$

with $^{\pm}\omega$ the self-dual and anti-self-dual parts of ω, defined by (*cf* Appendix A)

$$^{\pm}\omega = \frac{1}{2}(\omega \mp i * \omega) \tag{2.34}$$

$*$ being the Hodge duality operator acting upon the local Lorentz indices $\hat{a}, \hat{b}$. It is worth noticing this decomposition of the curvature is only possible in four dimensions because only there the Hodge dual of a two-form is a two-form. Consider the action

$$S[e,{}^{+}\omega] = \int d^4x\; e\; e^{a\hat{a}}\, e^{b\hat{b}} R_{ab\hat{a}\hat{b}}({}^{+}\omega)\,. \tag{2.35}$$

Such an action is complex with real part (2.32), and, at first, one may think such an action produces too many equations of motion for the fields. This is not the case as it is shown next. The variation w.r.t. ${}^{+}\omega$ yields the result that ${}^{+}\omega = {}^{+}\omega(e)$, *i.e.* it is the self-dual part of the tetrad connection $\omega(e)$. Using this result one gets

$${}^{+}R_{ab\hat{a}\hat{b}}({}^{+}\omega) = \frac{1}{2}R_{ab\hat{a}\hat{b}}(\omega) - \frac{i}{2} * R_{ab\hat{a}\hat{b}}(\omega)\,. \tag{2.36}$$

Now the product $e \cdot *R$ appearing in (2.35) becomes, by virtue of the Bianchi symmetry of the curvature tensor,

$$e^{b\hat{b}} * R_{ab\hat{a}\hat{b}}(\omega(e)) = \frac{1}{2}e^{b\hat{b}}\epsilon_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}R_{ab\hat{c}\hat{d}} = \frac{1}{2}\epsilon_{\hat{a}}{}^{\hat{b}\hat{c}\hat{d}}R_{a\hat{b}\hat{c}\hat{d}} = 0\,. \tag{2.37}$$

Hence, modulo the equation of motion for ${}^{+}\omega$, (2.35) is exactly half the standard Einstein-Hilbert action for GR (up to boundary terms, *cf* chapter 4):

$$S[e,{}^{+}\omega] = \frac{1}{2}S[e,\omega(e)]\,. \tag{2.38}$$

It will be shown in chapter 3 this result also holds, with suitable generalizations, when Fermionic fields are coupled to gravity. In this argument no use was made of complex conjugation. Thus, the same reasoning applies to complex relativity where both e, ω are complex.

We present here the results of [86] using two-component spinor notation. There are at least two reasons to work with spinors when Ashtekar variables are studied. First, since one deals only with the self-dual part of the connection spinors are naturally adapted to do so (*cf* [122]). Second, when coupling Fermionic fields to gravity spinors become a necessary ingredient.

Local Lorentz indices $\hat{a}, \hat{b}, \ldots$ are replaced by pairs of unprimed and primed indices $AA', BB', \ldots$ associated to an internal two-complex-dimensional vector space W and its complex conjugate $\overline{W}$. These internal indices become spinor indices via one of the dynamical variables: $\sigma^{aAA'}$, the soldering-form. A non-degenerate antisymmetric two-form ϵ_{AB}

is fixed and together with its inverse ϵ^{AB} and complex conjugates $\epsilon_{A'B'}, \epsilon^{A'B'}$ are used to lower and raise spinor indices (*cf* Appendix A and [122]).

An action will be formed with the quantities $\mathcal{D}$ and $\sigma^{aAA'}$. The former is a $SL(2,\mathbb{C})$ connection defined only to act on unprimed indices[16] by

$$\mathcal{D}_a \epsilon_{AB} = 0 \,. \tag{2.39}$$

It is convenient to introduce a connection one-form A_{aBC} through

$$\mathcal{D}_a \lambda_A \equiv \partial_a \lambda_A + A_{aM}{}^{N} \lambda_N \,, \tag{2.40}$$

which by virtue of (2.39) leads to the result that A_{aBC} is traceless (or, equivalently, symmetric in BC), having then associated twelve complex degrees of freedom. Such an A_{aBC} contains the same information as self-dual connection ${}^{+}\omega$ due to the correspondence (*cf* [122]) ${}^{+}\omega_{a\hat{b}\hat{c}} \leftrightarrow A_{aBC}\epsilon_{B'C'}$, since the identification of BB', CC' with $\hat{b}, \hat{c}$ can be made with a fixed map known as Infeld-Van der Waerden symbols (*cf* Appendix A): $\mathcal{I}_{\hat{a}}^{AA'}$.

The latter variable $\sigma^{aAA'}$ is an invertible linear map between the four-dimensional space of (1,1) spinors (*i.e.* those belonging to $W \times \bar{W}$) and the (complexified) tangent space. It is the spinor version of the tetrads: $\sigma^{aAA'} = e^{a\hat{a}}\mathcal{I}_{\hat{a}}^{AA'}$; *e.g.* the metric is given by $g^{ab} = \sigma^{aAA'}\sigma^{b}{}_{AA'}$. For real relativity it is required to be imaginary [17] $\overline{\sigma}^{aAA'} = -\sigma^{aAA'}$. It yields 16 complex or real degrees of freedom respectively. The determinant of $\sigma^{aAA'}$ is a density of weight minus one whose *inverse* is denoted σ. Finally, according to the analysis at the beginning of this section and the language just settled down, a convenient action to adopt is [86]

$$S[\sigma^{aAA'}, A_{aBC}] = \int d^4x \; \sigma \; \sigma^{aM}{}_{M'} \sigma^{bNM'} \, F_{abMN} \tag{2.41}$$

where $\frac{1}{2}F_{abMN} := \partial_{[a}A_{b]MN} + A_{[a|M|}{}^{P}\, A_{b]PN}$.

This is indeed an action for general relativity in light of the correspondence with the Palatini-like tetrad action (2.32) and the correspondence $F_{abMN}\epsilon_{M'N'} \leftrightarrow {}^{+}R_{ab\hat{m}\hat{n}}$. In fact this is done explicitly in [86], here, however, we just mention that the variation of the

[16] Although it is possible to extend it to act on spacetime indices it turns out not to be necessary [86].

[17] This is so due to the signature we chose to work with: $(-,+,+,+)$; *cf* Appendix A of [7].

action (2.41) w.r.t. independent variables $\sigma^{aAA'}, A_{bBC}$ yields two equations of motion, the first of which tells us $\mathcal{D}$ is the self-dual part of the unique torsion-free "σ-compatible" connection (say $\nabla_a \sigma^{bBB'} = 0$) whereas the second is the spinorial form of the Einstein equations for pure gravity [86].

Similarly to the case of geometrodynamics above one should develop a 3+1 decomposition of the action and identify convenient canonical variables: the Ashtekar variables. Since this is carried out in detail in chapter 3 for the Einstein-Dirac system we limit ourselves to quote the main results for pure gravity.

Following the lines of the geometrodynamical approach it is assumed that the spacetime manifold M admits a foliation through three-dimensional hypersurfaces. A vector field $t^a = Nn^a + N^a$ is introduced besides the time t defined by $t^a \nabla_a t = -1$ and so on. The action (2.41) becomes

$$\begin{aligned} S_{SD} &= \int dt \int_{\Sigma_t} d^3x \left\{ i\sqrt{2}\,\mathrm{tr}\left({}^3\tilde{\sigma}^b\, {}^3\dot{A}_b \right) + (t\cdot A)^{BC} G_{BC} + N^a \tilde{V}_a + \underset{\sim}{N} \tilde{\tilde{S}} \right\} && (2.42) \\ G_{AB} &:= -i\sqrt{2}\, {}^3\mathcal{D}_b {}^3\tilde{\sigma}^b{}_{AB} && (2.43) \\ \tilde{V}_a &:= -i\sqrt{2}\,\mathrm{tr}\left({}^3\tilde{\sigma}^b\, {}^3F_{ab} \right) && (2.44) \\ \tilde{\tilde{S}} &:= -\mathrm{tr}\left({}^3\tilde{\sigma}^a\, {}^3\tilde{\sigma}^b\, {}^3F_{ab} \right) && (2.45) \end{aligned}$$

Because of their structure $G_{AB}, \tilde{V}_a, \tilde{\tilde{S}}$ are called the *Gauss*, *vector* and *scalar* constraints of the theory respectively. The factors in front of them in (2.42) are Lagrange multipliers due to the absence of their time derivatives in the action. The canonical variables can be read off from the structure $\int dt\,[p\dot{q} - H(p,q)]$ of (2.42): the set ${}^3A_{aBC}, {}^3\tilde{\sigma}^a{}_{AB}$ is a canonical pair. They are the Ashtekar variables in spinor form [86, 102]; Ashtekar used another way to arrive to them though. The symplectic structure for the phase space Γ formed by the fields ${}^3A, {}^3\tilde{\sigma}$ is given thus by

$$\left\{ {}^3\tilde{\sigma}^a{}_{AB}(x), {}^3A_b{}^{CD}(y) \right\} = -\frac{i}{\sqrt{2}}\, \delta^3(x,y)\, \delta_a{}^b \delta_{(A}{}^C \delta_{B)}{}^D \,, \qquad (2.46)$$

besides the trivial result that $\{ {}^3A_{aBC}, {}^3A_{bDE} \} = \left\{ {}^3\tilde{\sigma}^{mMN}, {}^3\tilde{\sigma}^{nRS} \right\} = 0$. Using this symplectic structure and smearing the constraints

$$G_T \equiv \int_\Sigma d^3x\, T^{BA}\, {}^3\mathcal{D}_a\, {}^3\tilde{\sigma}^a{}_{BA} \qquad (2.47)$$

$$D_{\vec{v}} \equiv \int_\Sigma d^3x\, v^a \mathrm{tr}\left({}^3\tilde{\sigma}^b F_{ab} - A_a {}^3\mathcal{D}_b {}^3\tilde{\sigma}^b\right) \tag{2.48}$$

$$H_{\underset{\sim}{N}} \equiv i\sqrt{2} \int_\Sigma d^3x\, \underset{\sim}{N} \mathrm{tr}\left({}^3\tilde{\sigma}^a\, {}^3\tilde{\sigma}^b\, F_{ab}\right). \tag{2.49}$$

With $T^{AB}, v^a, \underset{\sim}{N}$ test fields with compact support, $D_{\vec{v}}$ has been chosen as a combination of G_{AB} and V_a since in this way it acquires a direct significance as the generator of small diffeomorphisms on Σ. The action of the constraints on the phase-space variables is given by:

$$\left\{G_T, {}^3A_{aB}{}^C\right\} = -\mathcal{D}_a T_B{}^C, \tag{2.50}$$

$$\left\{G_T, {}^3\tilde{\sigma}^a{}_A{}^B\right\} = [T, \sigma^a]_A{}^B, \tag{2.51}$$

$$\left\{D_{\vec{v}}, {}^3A_{aB}{}^C\right\} = \mathcal{L}_{\vec{v}} {}^3A_{aB}{}^C \tag{2.52}$$

$$\left\{D_{\vec{v}}, {}^3\tilde{\sigma}^a{}_A{}^B\right\} = \mathcal{L}_{\vec{v}} {}^3\tilde{\sigma}^a{}_A{}^B \tag{2.53}$$

$$\left\{H_{\underset{\sim}{N}}, {}^3A_{aB}{}^C\right\} = -\frac{1}{2}\left[\underset{\sim}{N} {}^3\tilde{\sigma}^b, F_{ab}\right]_B{}^C \tag{2.54}$$

$$\left\{H_{\underset{\sim}{T}}, {}^3\tilde{\sigma}^a{}_A{}^B\right\} = -{}^3\mathcal{D}_b\left(\underset{\sim}{N} {}^3\tilde{\sigma}^{[a} {}^3\tilde{\sigma}^{b]}\right)_A{}^B \tag{2.55}$$

That is to say, G_T generates infinitesimal canonical transformations which are infinitesimal gauge rotations on spinor indices, and $D_{\vec{v}}$ generates diffeomorphisms. The scalar constraint generates "dynamics" on the constraint surface $\bar{\Gamma}$ (*cf* [7]).

To conclude this section we just stress the following [7]

- The lapse appearing above is a density of weight minus one. This is natural since the scalar constraint is a density of weight two. Integrals of densities of weight one can be carried out with no reference to a specific volume element.

- The constraints are polynomial in the canonical variables and, furthermore, the inverse of ${}^3\tilde{\sigma}^a{}_{AB}$ does not appear anywhere; degeneracies of the three-metric $\approx{q}^{ij} = -\mathrm{tr}\left({}^3\tilde{\sigma}^i {}^3\tilde{\sigma}^j\right)$ can be allowed.

- The constraints turn out to form a first-class set in that the Poisson bracket between any two of them is a linear combination of constraints. However, because the coefficients are functions of the canonical variables they do not form a proper Lie algebra (*cf* next chapter and [7] for more details).

- The phase space of general relativity in terms of the Ashtekar variables is identical to that of the complex Yang-Mills theory with internal group $SU(2)$: the Gauss-law constraint appears to be the same in both theories. General relativity has the additional vector and scalar constraints. This is why it is often stated the constraint surface of general relativity is embedded in the Yang-Mills constraint surface.

- Reality Conditions. Since the original action (2.41) is complex the Hamiltonian coming from it is also complex. In other words, even though ${}^3\tilde{\sigma}$ can be Hermitian initially so that a three-metric built out of it is real, under evolution it will not be so necessarily. This can be solved by imposing reality conditions on the canonical variables so that the Hermitian character of ${}^3\tilde{\sigma}$ is preserved and Lorentzian general relativity is projected out of the complexified phase space of general relativity. Such conditions are [7]

$$\left({}^3\tilde{\sigma}\right)^{\dagger\, a} = {}^3\tilde{\sigma}^a \qquad \left(A_{aB}{}^{C} - \Gamma_{aB}{}^{C}\right)^{\dagger} = -\left(A_{aB}{}^{C} - \Gamma_{aB}{}^{C}\right) . \qquad (2.56)$$

Here $\Gamma_{aA}{}^{B}$ is the unique torsion-free intrinsic three-dimensional connection compatible with ${}^3\sigma^a{}_{AB}$ on Σ. These conditions are more easily understood taking into account that $A_j = \Gamma(\sigma)_j - \frac{i}{\sqrt{2}} K_{jl}\sigma^l$, where K_{jl} is the extrinsic curvature of Σ. Roughly, the reality of the extrinsic curvature ensures the reality of the "time" derivative of the three-metric, thus, (2.56) implements how to pick out the real-relativity section of the phase space of complex GR. Obviusly (2.56) is not polynomial when translated, as they are, into the Ashtekar variables, nevertheless, Ashtekar *et al* have found an equivalent form of reality conditions which are indeed polynomial in the basic canonical variables ${}^3A_j, {}^3\tilde{\sigma}$ [7, 15] (see also [123]).

To discuss the quantisation based on the Ashtekar variables we have found it easier to set first the program to be applied and then summarize the advances and problems left in performing it. This is done next.

2.3.2 Ashtekar Quantisation Program

We outline the quantisation program given by A. Ashtekar [124]. The scheme will be given by insisting on the canonical algebra and the connection representation even when there exist alternatives. For instance, one of them is the $\mathcal{T}$-algebra and its loop representation that will be considered in the next section. For the sake of simplicity we use the isomorphism between $SU(2)$ spinors and triads [7]. In the discussion of the loop formulation we shall come back to spinor notation. Take the Pauli matrices divided by $\sqrt{2}$: $\tau_i{}^A_B$, $A, B = 1, 2$ and let

$$A_a{}^B{}_C = A_a{}^i \tau_i{}^B{}_C, \tag{2.57}$$

$$\tilde{E}^{aA}{}_B = \tilde{E}^{ai} \tau_i{}^A{}_B. \tag{2.58}$$

Let us set the program in the following steps.

1. Introduce operator-valued distributions, $\tilde{E}^a{}_i(x)$ and $A_b{}^j(x)$, subject to the canonical commutation relations

$$\begin{aligned}\left[\hat{A}_a{}^i(x), \hat{A}_b{}^j(y)\right] &= 0,\\ \left[\hat{\tilde{E}}{}^a{}_i(x), \hat{\tilde{E}}{}^b{}_j(y)\right] &= 0,\\ \left[\hat{\tilde{E}}{}^a{}_i(x), \hat{A}_b{}^j(y)\right] &= \hbar\, \delta^a{}_b\, \delta_i{}^j\, \delta^3(x, y).\end{aligned} \tag{2.59}$$

2. On the algebra generated by these operators, introduce a $*$-operation by requiring that $\tilde{E}^a{}_i$ be its own $*$-adjoint as well as its "time derivative", yielded by the commutator with the Hamiltonian, be its own $*$-adjoint. Thus, the reality conditions are incorporated at the algebraic level.

3. Choose a representation for the algebra. The most convenient choice is to use for

states holomorphic functions[18] of the complex connection ${A_b}^j$, represent $\hat{A_b}^j$ as a multiplication operator and $\hat{\tilde{E}}^a{}_i$ as a differential operator, $\hbar\frac{\delta}{\delta {A_a}^i}$. At this stage the $*$-relations are ignored. This is so because to incorporate such relations requires the availability of a Hermitian inner product. An unambiguous inner product is expected to exist only on *physical states* and thus it is appropriate to postpone the incorporation of these " quantum reality" conditions until after the *physical states* have been extracted.

4. Solve the quantum constraints. Since at the classical level the constraints involve only ${A_b}^j$ and $\tilde{E}^a{}_i$ and not their complex conjugates, we can continue avoiding to use the $*$-relations in the algebra. The space of solutions is the complex vector space of physical states. The operators of interest will be those in our algebra that map this space to itself.

5. On this space of physical states, introduce a Hermitian inner product that now incorporates the $*$-relations. The operators $\tilde{E}^a{}_i$ and its "time derivative" themselves will not be observables. Nevertheless, the $*$-relations of the initial algebra induce $*$-relations on the space of observables, which maps the space of physical states onto itself, and these relations are to be faithfully reflected in the Hermitian adjointness relations by the appropriate choice of the inner product. Thus, in the quantum theory, the secondary constraints (Gauss,vector and scalar) and the reality conditions are not on the same footing; the former determine the space of physical states (step 4) while the latter constrains the inner product on this space. In practice, the introduction of an inner product may require that we isolate "time" from among the various components of ${A_b}^j$ and interpret the scalar constraint as a Schrödinger equation.

6. Select physically interesting observables and make predictions.

[18]In simple examples as the harmonic oscillator, studied with complex variables, this requirement is equivalent to choosing the Hilbert space $L^2(\mathbb{R}, dx)$ for the conventional variables. In more general cases the situation is not clear.

Several remarks are in order.

1. The variable which is diagonalised is ${A_a}^i$, the analogue of the canonical momentum in the conventional theory, whereas the triad variable acts through a functional derivative. The A-representation is analogous to the functional Fourier transform of the representation used in the conventional canonical program, and thus the geometrical interpretation of the functionals on which the above operators act is very different from the familiar functionals $\Psi[h]$.

2. When imposing the constraints a' la Dirac it is necessary first, to choose an ordering for the operator version of the constraints. Jacobson and Smolin [87] chose the expressions

$$\hat{C}_i(x) = D_l \frac{\delta}{\delta {A_l}^i(x)} , \tag{2.60}$$

$$\hat{C}_l(x) = {F_{lm}}^i(x) \frac{\delta}{\delta A_m^i(x)} , \tag{2.61}$$

$$\hat{C}(x) = {\epsilon^{ij}}_k {F_{lm}}^k \frac{\delta}{\delta {A_l}^i(x) \delta A_m^j(x)} , \tag{2.62}$$

which have the virtue that $\hat{C}_i$ and $\hat{C}_l$ correctly generate the Lie algebras of the gauge groups $C^\infty(\Sigma, SO(3))$ and Diff(Σ), respectively. Remarkably, Jacobson and Smolin were able to find a number of exact solutions to the WDW equation in the present case: $\hat{C}\Psi = 0$. Among them there is the functional $\Psi[A] = 1$ that satisfies all the constraints. The lack of knowledge about measures in the space of connections does not allow any physical interpretation of this result. Another, formal, solution to the WDW equation was found to be [87]

$$\Psi_{\{\eta_a\}}[A] \equiv \prod_{a \in I} H_{\eta_a}[A] \tag{2.63}$$

where the product extends over the finite set I of indices s.t. $\{\eta_a | a \in I\}$ is a set of smooth non-intersecting loops, and

$$H_\eta[A] = Tr(Pexp \oint_\eta A) \tag{2.64}$$

is an element of a class of holomorphic $C^\infty(\Sigma, SO(3))$ gauge-invariant functionals. The P means that the line integral is a path-ordered one and η is supposed to be smooth in Σ. Further solutions exist which include intersecting curves; they involve linear combinations of states corresponding to the different ways in which such curves can be splitted.

3. The main problems raised by this line of work are:

   (a) The loop-based solutions given above come from exploiting the antisymmetry properties of F_{ab}. It is surprising they capture the full content of the WDW equation.

   (b) The operator products in the quantum constraints are ill-defined since they contain factors of $\delta^3(0)$. The *regularisation* method is still subject of debate (a point-splitting regularization needs a background metric and curve; the ultimate effect of this undesirable background is unclear).

   (c) The dependence of the functional solutions on loops in Σ amounts to their non-Diff(Σ)-invariant character. The Diff(Σ) constraints seem to be intractable here whereas in the conventional approach they are considered as innocuous. A way to get round this difficulty is given below in the loop formalism of QG.

The problem of finding physical states, and hence also of implementing the $*$-relations, remains open in the connection representation. Consequently, as far as full quantum gravity is concerned, so far, the connection representation has not led to qualitatively new insight into the dynamics of the gravitational field in the Planck regime. At present, its importance lies mainly in the fact that it provides a suitable general framework to address certain conceptual issues of QG in a concrete way. Among these are the issue of time and the large gauge transformations as related to θ-vacua and CP-violation. They are discussed in [7]. Also, using the connection representation some solutions to the quantum minisuperspace models have been found [125, 126]. It is worth mentioning that for some cosmological models [127] and spherically symmetric model [128] the above quantisation program has been fully completed.

2.4 Loop formalism

The motivations leading to this construction were the Ashtekar reformulation of GR and the Jacobson and Smolin's discovery of a class of solutions to the WDW equation, in terms of the Ashtekar variables, related to loops in three dimensions. No solution was found to the spatial diffeomorphism constraint. The loop representation was invented to solve this problem by introducing a representation space on which the spatial diffeomorphism group acts naturally, whereas the simplicity of the action of the Hamiltonian constraint in the self-dual representation is preserved. Recent reviews are [6, 7, 8].

Once the loop representation is introduced, the complete set of solutions to the constraints that generate diffeomorphisms of Σ are readily found. They can be related to a countable basis, whose elements are in one-to-one correspondence with the knot and link classes of Σ (More properly said, the elements are in one-to-one correspondence with the generalized link classes, which allows the loops to intersect and be kinked). The basic tool to handle the structure of the space of physical states of nonperturbative QG will be knot theory [129, 130, 131, 132]. The action of the Hamiltonian constraint on elements of the loop representation gets simplified. Thus, a large class of solutions to the Hamiltonian constraint is obtained which contains, in turn, a set of states that are also annihilated by the diffeomorphism constraints. It may happen that they are not the most general solutions to the combined set of constraints but they are exact physical states of the gravitational field.

2.4.1 Dirac quantisation to include loop variables

The strategy followed in the case of the loop approach is the Dirac method, whose steps we give now. We follow Smolin in [124].

1. Choice of a preferred subalgebra $\mathcal{A}$ of the classical observables to be the elementary observables of the quantum theory.

2. Choice of a linear space $\mathcal{S}$, on which there exists a completely regularized algebra of linear operators $\tilde{\mathcal{A}}$ that is a deformation of the classical algebra of the elementary observables, $\mathcal{A}$.

3. Definitions of the constraints and the Hamiltonian of GR in terms of the elements of $\mathcal{A}$.

4. Solution of the quantum constraints by finding the subspace $\mathcal{S}_{\mathrm{Phys}}$ of $\mathcal{S}$ that is in the kernel of the regularized constraints in the limit that the regularization is removed.

5. Definition of the physical observables that constitute the operator algebra on the space $\mathcal{S}_{\mathrm{Phys}}$. At this stage, we are required to do two things; first, find the algebra, and second, give its elements a physical interpretation.

6. Definition of the physical inner product on $\mathcal{S}_{Phys}$. This choice must implement both the reality conditions of the classical theory and the physical interpretation of physical observables in that operators which correspond to classical physical observables that are real must be Hermitian w.r.t. the physical inner product.

The first three steps have been completed in the loop representation, whereas 4, 5 and 6 are still under study. Indeed, points 4 and 5 are the matter of chapters 5 and 6 for the Einstein-Dirac system, while about point 6 some encouraging results have been recently obtained by Ashtekar and Lewandowski [33]. We sketch the progress that has been made, in this approach, on points 1-3.

2.4.2 Classical loop algebra

Quantisation of any classical theory consists of the association of classical observables, defined as functions on the phase space of the theory, with linear operators on some representation space such that the commutator algebra of the latter goes over into the Poisson algebra of the former in the limit $\hbar \to 0$. Due to the operator ordering and regularisation problems in any quantisation of a field theory, most of the classical observables will not have an unambiguous representation in terms of the operator algebra of the quantum theory. What we can do is to choose a subalgebra of classical observables that will be represented unambiguously in terms of the operator algebra of the quantum theory. These are called the elementary observables. We say that the rest of the quantisation procedure is constrained by the choice of these elementary observables in the following sense. The set of elementary observables should form a closed algebra under the Poisson brackets. This set must be small enough so that every element in its algebra can be represented in terms of a well-defined linear operator on the representation space (regularisation). Also, the set must be large enough so that the constraints, Hamiltonian, and a large enough set of physical observables must be expressible at the classical level through limits of sequences of elementary observables. When this happens it is said that the algebra of the elementary observables is complete.

The uppercase indices of the connection $A_a{}^B{}_C$ take values in the spin-one-half representation of the $so(3)$ Lie algebra.

To construct the loop representation we choose a set of elementary observables based on loops in the three-manifold Σ. The phase space of GR will be coordinatized by the Ashtekar variables (A, E). The loops are assumed to be piecewise smooth and parametrized, with non-vanishing tangent vectors.

Given a loop γ, and two points on it given by the parameter values, s and t, one defines the parallel transport to be

$$[U_\gamma(s,t)]^A{}_B \equiv \left[Pe^{\int_s^t du A_a(\gamma(u))\dot\gamma^a(u)}\right]^A{}_B , \tag{2.65}$$

where P means path ordered. The trace of this parallel transport all around the loop is known as the Wilson loop of the Ashtekar connection. In fact another symbol is used for it

$$T^0[\gamma] \equiv \mathrm{Tr} U_\gamma = \mathrm{Tr} P e^{\oint_\gamma A}, \tag{2.66}$$

and it is one of the loop variables we will define that form a closed algebra under Poisson brackets which is called the classical $\mathcal{T}$-algebra. It is found necessary to introduce observables corresponding to unordered sets of loops in Σ. Such a set is called a multiloop, and is denoted by $\{\gamma\} \equiv \{\gamma_1, \gamma_2, \ldots\}$. Corresponding to each multiloop $\{\gamma\}$, we have also a T^0 observable

$$T^0[\{\gamma\}] \equiv \prod_i \mathrm{Tr} U_{\gamma_i}. \tag{2.67}$$

Under Poisson bracket, the T^0's form an overcomplete set of commuting $SU(2)$ gauge-invariant observables. This result arises from certain relations that hold because of the involved $SL(2,C)$ matrices and their definitions in terms of parallel transport. They include

i) Invariance under reparametrisation of the loop parameter s.

ii) Invariance under inversion

$$T^0[\gamma^{-1}] = T^0[\gamma]. \tag{2.68}$$

iii) The spinor identity:

$$T^0[\alpha]T^0[\beta] = T^0[\alpha\#\beta] + T^0[\alpha\#\beta^{-1}], \tag{2.69}$$

where the loop $\alpha\#\beta$ is defined as follows. If α and β intersect at a point P, it is the loop obtained starting from P, going through α, then through β, and finally closing at P. This equation only holds if α and β intersect.

iv) The "retracing" identity:

$$T^0[\alpha] = T^0[\alpha \cdot l \cdot l^{-1}] \tag{2.70}$$

where l is a line with one end on α and $\alpha \cdot l \cdot l^{-1}$ is the loop obtained by going around α, then along the line, and then back along the line to α.

To have a complete algebra of observables we need some observables that also depend on the conjugate E fields. Looking ahead to the problem of regularization, at quantum level, we should require that the elementary observables do not include any that involve more than one E at any point of Σ. A T^1 observable is defined by inserting a conjugate E field into the trace of the parallel transport around the loop at some given point s:

$$T^1[\gamma]^a(s) \equiv \mathrm{Tr}[U_\gamma(s)\tilde{E}^a(\gamma(s))]. \tag{2.71}$$

This definition can be extended to the T^n observable, and to multiloops also. Such insertion of the E variable is called a "hand". The important consequences relevant for the quantization program are

1. The T^n's form a closed algebra under Poisson brackets.

$$\{T^n[\alpha], T^m[\beta]\} = i \sum_{\text{grasps}} \Delta[\alpha,\beta] T^{n+m-1}[\text{result of the grasp}] \tag{2.72}$$

   the grasps are the resulting loops by considering the possible combinations of the initial loops at the "hands".

2. Completeness on the gauge-invariant observables. Any gauge-invariant and local functions of F_{ab}, E^c may be constructed in terms of limits of sequences of T observables. (e.g. the constraints)

3. The loop algebra is closed under the action of the spatial diffeomorphisms. The easiest example is the T^0. Given $\phi \in \mathrm{Diff}(\Sigma)$

$$\phi \circ T^0[\alpha] = T^0[\phi \circ \alpha]. \tag{2.73}$$

4. The distributional singularities appearing in the loop algebra may be removed by an appropriate regularization procedure.

2.4.3 Loop-representation construction

The key idea in the program is to use the above algebra as the basic algebra whose representations determine the quantum theory. In particular, it is possible to construct a type of Fock-space quantization in which the analog of a "n-particle" state is a function $\psi(\eta_1, \eta_2, \ldots \eta_n)$ of n loops. The $\tilde{T}^0$ acts like a creation operator, for example

$$(\tilde{T}^0_\eta \psi)[\xi] \equiv \psi[\eta, \xi], \tag{2.74}$$

while the $\tilde{T}^1$ operators map each n-loop sector into itself. By this means a deformation of the classical $\mathcal{T}$-algebra is successfully constructed:

$$\left[\tilde{T}^n, \tilde{T}^m\right] = \hbar\Delta\tilde{T}^{n+m-1} + \hbar^2\Delta\Delta\tilde{T}^n + \cdots + \hbar^n\Delta\ldots\Delta\tilde{T}^m. \tag{2.75}$$

The next major step is to construct the quantum constraints as a limit of sequences of these $\tilde{T}$ variables. Rovelli and Smolin showed that the WDW equation, $\hat{C}\Psi = 0$, can be satisfied provided the Fock-space functions $\psi(\eta_1, \eta_2, \ldots \eta_n)$ are concentrated on smooth, non-intersecting loops [1].

Evidently, these states are not Diff(Σ)-invariant since the diffeomorphism group moves the loops around. Nevertheless, Diff(Σ)-invariant states can be found by requiring the n-loop functions $\psi(\eta_1, \eta_2, \ldots \eta_n)$ to be constant on the Diff(Σ) orbits, which are the *link* classes of the manifold Σ.

2.4.4 Advances and perspectives

So far, it is possible to summarize the advances in the loop representation as follows.

1. The loop representation can be considered as a complete quantisation of the phase space of GR. It is completely regularized and diffeomorphism-covariant since the operators involved, once regularized, carry a representation of the spatial diffeomorphism group.

2. The diffeomorphism and Hamiltonian constraints may be expressed in the loop representation, the former by their natural geometrical action on the loop space, the latter in regularized form.

3. The general solution to the diffeomorphism constraint is found in the loop representation, and expressed in terms of a countable basis. This countable basis is in one-to-one correspondence with the generalized link classes of the manifold.

4. An infinite, but not complete, set of states that are in the kernel of the Hamiltonian constraint is also found. These states consist of all loop functionals with support on loops that are smooth and non-intersecting.

5. The Hamiltonian and diffeomorphism constraints are compatible, in that an infinite set of physical states that are in the simultaneous kernel may be constructed. This space has a countable basis, which is in one-to-one correspondence with the ordinary link classes.

6. A functional transform taking states in the self-dual representation to states in the loop representation may be constructed formally.

7. For free-field theories this transform may be explicitly constructed, and gives a construction of a loop representation for the Fock space of free photons and free gravitons [133, 134].

The perspectives, on the other hand, can be set in the following manner.

- Completeness of the solution space of the constraints. The set of solutions to the Hamiltonian constraint mentioned above is almost certainly not a complete set. There are two reasons for that. First, a large set of additional solutions has been found in the self-dual representation associated with intersecting loops [87, 135]. It is expected that those solutions will exist in the loop representation as well. The second is that the mentioned solutions are constructed using only the antisymmetry of the indices of the operator. Thus, it is not impossible that there exist other operators whose continuum limit classically is not the Hamiltonian constraint, but that also annihilate the above states.

- Physical interpretation of the physical operator algebra. Given a description of the solution space to the constraints in terms of a countable basis implies that one knows how to construct the general operator acting on that space. Thus, given the results about states, we have the general diffeomorphism-invariant operator, and a large class of completely physical operators. What we do not have is any correspondence between these operators and diffeomorphism-invariant or physical observables in classical GR.

- The Physical inner product. We already mentioned that the choice of an inner product is related to the reality conditions. This means that, if anyone proposes an inner product on the space of physical states, one must be able to check that any operator on the physical states whose classical limit is real when the reality condition are imposed is Hermitian. However, this condition requires that we have a correspondence between classical and quantum physical observables, which, as we have just mentioned, we do not have. Thus, at present, while there are candidates for the physical inner product (e.g. an L^2 norm, if any, for the countable basis of link classes) it has not been possible to check whether any of them correctly expresses the reality condition.

THREE

SELF-DUAL FRAMEWORK FOR FERMIONIC FIELDS AND GRAVITATION

To extend the non-perturbative canonical formulation in coupling matter to gravity using Ashtekar and Loop variables it is very convenient to start with an analysis of the corresponding action. This yields the important result of identifying the canonical variables of the theory and makes it easier to relate them to the standard geometrodynamical variables, for the same system, we are used to. The action functional for spin-$\frac{1}{2}$ fields and gravity within the framework of Ashtekar's variables was put forward by Jacobson [120] and thoroughly studied including other matter fields by Ashtekar *et al* [15]. In this chapter an alternative construction of this action is presented and its canonical analysis carried out.

Since the gravitational part of the action using these variables is first-order it is natural to consider a gravitational connection admitting torsion[19] [20]. By extending the self-dual connection to admit torsion and using the Bianchi symmetry of the curvature tensor for non-vanishing torsion, a self-dual action is obtained and shown to be equivalent to the ECSK-Dirac one. Results by Ashtekar *et al* are recovered when splitting out the above self-dual connection into its torsion-free part and its torsion contribution. Hereby the modification to the Dirac equation they found is shown to be a torsion effect. Then the

[19]There are several kinds of matter which can support a non-vanishing torsion when coupled to gravity [13, 136]. Attention is here focused on the spin-$\frac{1}{2}$ Dirac field minimally coupled to gravity.

reality conditions are briefly discussed. The possibility is considered of adopting the reality of the antisymmetric part of the extrinsic curvature as part of the reality conditions to keep or eliminate torsion, so that two different real sectors of the phase space of complex general relativity and spinor fields can be associated to ECSK-Dirac and Einstein-Dirac theories respectively.

The results given here can be regarded as complementary to those of Jacobson and Ashtekar *et al*, for spin-$\frac{1}{2}$ fields, by providing a specific link between two first-order actions: the ECSK-Dirac (Einstein-Cartan-Sciama-Kibble-Dirac) [136, 13] and the self-dual Einstein-Dirac action [120, 15], hereafter referred to as ECSKD and sd-ED respectively; ED will stand for torsion-free Einstein-Dirac theory in either its first- or second-order forms. Furthermore, since the key of the equivalence is based on the Bianchi symmetry of the curvature tensor for non-vanishing torsion, this is an extension of the Jacobson and Smolin [86] idea of using the Bianchi identity (with vanishing torsion) for pure gravity to establish the equivalence between its chiral and Palatini-like actions. The equations of motion, Einstein and Dirac, have the standard structure, the difference coming from the existence of a non-vanishing torsion supported by the fermionic fields. By expressing the above results in terms of a torsion-free connection, the results of Ashtekar *et al* [15] are recovered. In particular, the interpretation of an extra term in the Dirac equation in [15] naturally emerges from the present analysis as a torsion effect, considered previously in the literature [137].

The present analysis also enables one to revive and idea put forward by Jacobson [120] of identifying ECSKD and ED theories with two different sectors of the phase space of complex general relativity via the reality conditions, to be imposed on the canonical variables, necessary to get real gravity.

In section 1 the standard second-order action for gravity coupled to four-component Dirac fields is given as well as its first-order analogue, *i.e.* the ECSKD one using the language of two-component spinors. This provides the origin of the chiral action to be studied. Further details concerning four- and two-spinors are given in Appendix A. Section 2 establishes the equivalence between the sd-ED and ECSKD actions. The equations of motion coming from the self-dual action are the contents of section 3 together with their relation to the results of [15]. The canonical decomposition of the theory is studied in section 4. Section 5 deals with the issue of the reality conditions. Finally, in section 6, some conluding remarks are given.

3.1 . Einstein-Dirac and ECSKD actions

In coupling fermionic fields to gravity the introduction of orthonormal tetrads is natural because spinors are defined w.r.t. orthonormal frames [122]. Furthermore, whenever tetrads are adopted, connections also enter the description of gravity. In building up an action from which to obtain the equations of motion, one has the possibility of considering tetrads and connections as independent fields or not. If they are not one gets the Einstein-Dirac (second-order) action. Instead, by taking them as independent, one gets the ECSKD (first-order) action. The corresponding variational problems differ on what should be fixed on the boundary. One finds it is necessary to add boundary terms to get a well-posed problem only in the second-order case [138]. In the first-order case, on the other hand, one is left with an equation of motion associating a non-vanishing torsion to the connection [13, 136].

Alluding to both minimal coupling and equivalence principles one is led to the following second-order Einstein-Dirac action [14] (see also [12]), written here with all of its constants[20],

$$\begin{aligned} S_{ED} = \int_M d^4x \, \frac{e}{c} \Big\{ & \frac{1}{16\pi G/c^2} R[e, \omega(e)] \\ & + \frac{i\hbar c}{2} e^a_{\hat{a}} \Big[\left(\nabla_a \overline{\Psi}\right) \gamma^{\hat{a}} \Psi - \overline{\Psi} \gamma^{\hat{a}} \left(\nabla_a \Psi\right) \Big] + mc^2 \overline{\Psi} \Psi + \text{Boundary Terms} \Big\} \end{aligned} \tag{3.1}$$

where Ψ is a four-component Dirac spinor, $\gamma^{\hat{a}}$ are standard Dirac 4×4-matrices defined in flat space-time and $\overline{\Psi} := \Psi^{\ddagger} \gamma_{\hat{0}}$ is the Dirac-conjugate spinor; the ‡ stands for the standard hermitian-conjugation operation acting on complex matrices including order reversal in the case of Grassmann-valued fields. The tetrads are such that $e_{a\hat{a}} e_b^{\hat{a}} = g_{ab}$, $e_{a\hat{a}} e^a_{\hat{b}} =$

[20] We assume the dimension of the spinors Ψ to be length$^{-\frac{3}{2}}$; $\overline{\Psi}\Psi$ is associated to a probability density in first-quantized theories.

$\eta_{\hat{a}\hat{b}} = \mathrm{diag}\{-1,+1,+1,+1\}$ and $e = \det(e_{a\hat{a}})$. $R[e,\omega(e)]$, as usual, is the scalar of curvature formed with the tetrads and the curvature of the connection $\omega_{a\hat{a}\hat{b}}(e) = e^{b}{}_{\hat{a}}\left(\nabla_a e_{b\hat{b}}\right)$. Namely, $R[e,\omega(e)] := e^{a\hat{c}}e^{b\hat{d}}R_{ab\hat{c}\hat{d}}[\omega](e)$. On the other hand, $\nabla_a\Psi = [\partial_a\Psi - B_a\Psi]$ together with $\nabla_a\overline{\Psi} = [\partial_a\overline{\Psi} + \overline{\Psi}B_a]$ are the extension of the unique torsion-free connection ∇ to act on four-component spinors, with $B_a := \frac{1}{8}\omega_{a\hat{a}\hat{b}}[\gamma^{\hat{a}},\gamma^{\hat{b}}]$ [14].

Then a first-order action is formed which is compatible with the action introduced by Ashtekar *et al* [7, 15]. This is achieved by translating four- into two-spinors through the chiral representation for the Dirac γ's (*cf* Appendix) and using the two-spinor decomposition of the curvature (*cf* [7]). In it $G = c = \hbar = 1$ and the 16π factor is dropped so it yields a different numerical factor in the Einstein equations. Furthermore, an extra factor of 2 is included in the fermionic contribution to the action. Both numerical factors can be compensated through the definition of the stress-energy tensor. The resulting action functional is

$$
\begin{aligned}
S_{ECSKD} \;=\; \int_M d^4x \Big\{ & \sigma\left[\sigma^{aMA'}\sigma^{b}{}_{AA'}R_{abM}{}^{A}[{}^{+}\omega] + \sigma^{aAM'}\sigma^{b}{}_{AA'}\overline{R}_{abM'}{}^{A'}[{}^{-}\omega]\right] \\
& - \sqrt{2}\,\sigma\,\sigma^{a}{}_{AA'}\left[\overline{\kappa}^{A'}\left(\nabla_a\kappa^{A}\right) - \left(\nabla_a\mu^{A}\right)\overline{\mu}^{A'}\right] \\
& + \sqrt{2}\,\sigma\,\sigma^{a}{}_{AA'}\left[\left(\nabla_a\overline{\kappa}^{A'}\right)\kappa^{A} - \mu^{A}\left(\nabla_a\overline{\mu}^{A'}\right)\right] \\
& - 2im\sigma\left[\mu_A\kappa^{A} - \overline{\kappa}^{A'}\overline{\mu}_{A'}\right]\Big\}
\end{aligned}
\tag{3.2}
$$

where $\sigma := \det(\sigma_a{}^{AA'})$ and $\sigma_a{}^{AA'}$ is the soldering form (i.e. the two-spinor version of the tetrad in curved space-time). Note that the connection, here splitted into self-dual and anti-self-dual parts (*cf* Appendix), develops a torsion contribution supported by the fermionic fields as we use a first-order formalism with connection and soldering forms taken to be independent fields instead of adopting *a priori* a relation between them [14]. Furthermore, should Grassmann variables for fermions be used this action would remain even and manifestly real. The even character holds by adopting a chiral action but not its reality.

3.2. Self-dual action for Einstein-Dirac theory

Ashtekar *et al* studied the coupling of fermions to gravity in terms of a *chiral* action containing the self-dual part of a connection only [15], disregarding the anti-self-dual part. They based their analysis on the unique torsion-free connection defined by the metric $g_{ab} = \sigma_{aAA'}\sigma_b^{AA'}$. In this chapter the analysis is carried out by extending the connection to admit torsion. Let the chiral action [15] be

$$\begin{aligned} S_{SD} \;=\; & \int_M d^4x \Big\{ - \sigma\, \sigma^{aMA'}\, \sigma^{bJ}{}_{A'}\, F_{abMJ} \\ & - \sqrt{2}\, \sigma\, \sigma^a{}_{AA'} \left[\overline{\kappa}^{A'} \left(\mathcal{D}_a \kappa^A\right) - \left(\mathcal{D}_a \mu^A\right) \overline{\mu}^{A'}\right] \\ & - im\sigma \left[\mu_A \kappa^A - \overline{\kappa}^{A'} \overline{\mu}_{A'}\right] \Big\} \end{aligned} \tag{3.3}$$

where F_{abMN} is the curvature of the connection $\mathcal{D}$ defined at this stage to act only on unprimed spinor indices. Clearly, (3.3) is obtained from (3.2) by taking only the contribution of the *self-dual* piece of the connection, ${}^{+}\omega$ $(= \omega_{aAB}$ *cf* Appendix A), and half of the mass term. The name *chiral* hence accounts for this. Thus (3.3) is manifestly not real. Nevertheless, it will be shown below it reproduces (3.2) modulo the equations of motion for $\mathcal{D}$ and via the Bianchi symmetry of the curvature for a "metric-compatible" connection having torsion so that no spurious equations of motion are picked up.

The goal here is to determine $\mathcal{D}$ dynamically. The variation of (3.3) with respect to $\mathcal{D}$ can be carried out by introducing the auxiliary forms $Q_a{}^M{}_N$ and $P^c{}_{ab}$ so as to define $\mathcal{D}$ with respect to ∇, the connection compatible with the soldering form

$$\nabla_a \sigma_b^{AA'} = 0 \tag{3.4}$$

and having associated a non-vanishing torsion $T_{ab}{}^c$

$$2\nabla_{[a}\nabla_{b]} f := T_{ab}{}^c\, \nabla_c f\,,\; f \text{ a zero} - \text{form.} \tag{3.5}$$

Namely,

$$\mathcal{D}_a \lambda_b^A = \nabla_a \lambda_b^A + Q_a{}^A{}_B\, \lambda_b^B + P^c{}_{ab}\, \lambda_c^A \tag{3.6}$$

with associated torsion $\mathcal{T}_{ab}{}^c$

$$\begin{aligned} 2\mathcal{D}_{[a}\mathcal{D}_{b]} f \;&:=\; \mathcal{T}_{ab}{}^c\, \nabla_c f\,,\; f \text{ a zero} - \text{form} & (3.7) \\ \mathcal{T}_{ab}^{\;c} \;&=\; T_{ab}^{\;c} - 2P^c{}_{[ab]}\,. & (3.8) \end{aligned}$$

By requiring the annihilation of the symplectic form ϵ_{AB} one gets a restriction on Q_{aAB} above:

$$\mathcal{D}_a\epsilon_{AB} = \nabla_a\epsilon_{AB} = 0 \quad \Rightarrow Q_{aAB} = Q_{a(AB)} \text{ (trace} - \text{free)} . \tag{3.9}$$

Concerning the action on space-time indices, and thus $P^c{}_{ab}$, it is known that to control both metricity (compatibility condition (3.4)) and torsion it is necessary to include kinetic terms for them in the corresponding action [139]; otherwise one should impose either of them and get the other as an equation of motion [139]. We follow the latter possibility by imposing

$$P^c{}_{ab} = 0 . \tag{3.10}$$

This amounts to specify that the torsion of ∇, $T_{ab}{}^c$, is exactly that of $\mathcal{D}$, $\mathcal{T}_{ab}{}^c$ (*cf* (3.8)), to be determined dynamically. Also, from (3.6), the action on space-time indices of both $\mathcal{D}$ and ∇ is identified.

Varying $\mathcal{D}$ is equivalent to varying Q so the action (3.3) should be re-expressed in terms of ∇, Q_{aMN} and $T_{ab}{}^c$. The curvatures, F_{abMN} and R_{abMN} of $\mathcal{D}$ and ∇, respectively, fulfill [122]

$$F_{abMN} = R_{abMN} - 2\nabla_{[a}Q_{b]MN} + 2Q_{[aM}{}^P\, Q_{b]PN} + T_{ab}{}^c\, Q_{cMN}. \tag{3.11}$$

Plugging (3.9) in (3.3) to carry out its variation makes it necessary a previous integration by parts of the ∇Q term. This gives a total divergence and a term containing the derivative of products of soldering forms

$$\begin{aligned} &\int_M d^4x \left\{-2\sigma\, \sigma^{aMA'}\sigma^{bN}{}_{A'}\nabla_{[a}Q_{b]MN}\right\} \\ &= \int_M d^4x \left\{-2\nabla_a\left[\sigma\, \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\, Q_{bMN}\right] + 2\left[\nabla_a\left(\sigma\, \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\right)\right] Q_{bMN}\right\} \\ &= -2\int_{\partial M} dS_a\, \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\, Q_{bMN} - 2\int_M d^4x\, T_{am}{}^m\, \sigma\, \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\, Q_{bMN} . \end{aligned} \tag{3.12}$$

The second term on the second line above drops by virtue of the compatibility condition (3.4) whereas the total divergence turns into a sum of a boundary and a volume term, the latter containing torsion[21].

[21] Note that whenever a connection ∇ has torsion T, a tensor density of weight $+1$, $\tilde{\phi}^a$, has divergence $\nabla_a\tilde{\phi}^a = \partial_a\tilde{\phi}^a - T_{an}{}^n\tilde{\phi}^a$.

Using the above results in varying (3.3) w.r.t. Q_{gMN} yields the equation

$$\sigma^{[a(MA'}\sigma^{b]N)}_{A'}\left(2T_{am}{}^{m}\,\delta_b{}^g - T_{ab}{}^g\right) + 4\sigma^{[g(MA'}\sigma^{a]}{}_{AA'}\,Q_a{}^{N)A} + i\sigma^{g(M}{}_{A'}\,k^{N)A'} = 0. \qquad (3.13)$$

Here $[a(MA'\ b]N)$ means antisymmetrization in a, b and symmetrization in M, N and similarly for the other terms. $k^{AA'}$ and k^m are defined by

$$\begin{aligned} k^{AA'} &:= -i\sqrt{2}\left(\overline{\kappa}^{A'}\kappa^A - \mu^A\overline{\mu}^{A'}\right) \\ k^m &:= \sigma^m{}_{AA'}\,k^{AA'}\,. \end{aligned} \qquad (3.14)$$

One readily solves (3.13) for Q_{aMN} observing that $i\sigma^{g(M}{}_{A'}\,k^{N)A'} = i\sigma^{g(M}{}_{A'}\,\epsilon_R{}^{N)}k^{RA'}$, whose r.h.s., in turn, obeys the identity[22]

$$2i\sigma^{g(M}{}_{A'}\,\epsilon_R{}^{N)} = \sigma^{p(MB'}\sigma^{qN)}{}_{A'}\,\sigma^m{}_{RB'}\,\epsilon_{pqm}{}^g\,. \qquad (3.15)$$

Note that there is an implicit antisymmetrization in p, q in the r.h.s. of this identity due to the contraction with the volume four-form. It is possible now to factor out the soldering-form factors in (3.13). This leads to

$$\sigma^{pRA'}\sigma^{qS}{}_{A'}\left\{\left(2T_{[pm}{}^{m}\,\delta_{q]}{}^g - T_{pq}{}^g\right)\epsilon_R{}^{(M}\epsilon_S{}^{N)} + 4\epsilon_{SA}\delta_{[p}{}^gQ_{q]}{}^{A(N}\epsilon_R{}^{M)} + \frac{1}{2}\epsilon_{pqm}{}^g\,k^m\epsilon_R{}^{(M}\epsilon_S{}^{N)}\right\} = 0\,. \qquad (3.16)$$

Assuming $\sigma^{aAB'}$ is non-degenerate enables one to set to zero the factor in braces. Tracing over R, M of such a factor then yields

$$\epsilon_S{}^N\left(2T_{[pm}{}^{m}\,\delta_{q]}{}^g - T_{pq}{}^g\right) - 4Q_{[qS}{}^N\,\delta_{p]}{}^g + \frac{1}{2}\epsilon_{pqm}{}^g\,k^m\,\epsilon_S{}^N = 0\,. \qquad (3.17)$$

Since Q is traceless, taking the traces over S, N and q, g, one arrives at

$$T_{am}{}^m = 0 \qquad (3.18)$$

so that torsion takes the value

$$T_{pq}{}^g = \frac{1}{2}\epsilon_{mpq}{}^g\,k^m\,. \qquad (3.19)$$

[22] Given the real, totally antisymmetric (in pairs of indices AA', ...) four-form [7] $e_{AA'BB'CC'DD'} := -i\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} + i\epsilon_{A'B'}\epsilon_{C'D'}\epsilon_{AC}\epsilon_{BD}$, related to the volume four-form through $\epsilon_{abcd} = e_{AA'BB'CC'DD'}\sigma_a{}^{AA'}\sigma_b{}^{BB'}\sigma_c{}^{CC'}\sigma_d{}^{DD'}$, (3.15) is a lengthy but otherwise straightforward result.

Furthermore, this value of torsion substituted back in (3.17) yields

$$Q_{aSN} = 0 \ . \tag{3.20}$$

Hence, according to (3.11), $\mathcal{D}$ is the self-dual part of the connection ∇ and it has associated torsion (3.19) by virtue of (3.8) and (3.10).

Reproducing (3.2) from (3.3) is easy at this stage. Recall that the Bianchi symmetry of the curvature of a connection ∇ having torsion T (see *e.g.* [122])

$$R_{[abc]}{}^{d} - T_{[ab}{}^{e}\, T_{c]e}{}^{d} - \nabla_{[a} T_{bc]}{}^{d} = 0 \tag{3.21}$$

where antisymmetrization is understood on all three indices a, b, c, can be related to the self-dual Riemann tensor [7]

$${}^{+}R_{abc}{}^{d} = \frac{1}{2}\left(\delta_c^m \delta_n^d - \frac{i}{2}\epsilon_c{}^{dm}{}_n\right) R_{abm}{}^{n} = R_{abA}{}^{B}\, \sigma_c{}^{AM'} \sigma^d{}_{BM'} . \tag{3.22}$$

Such a relation is as follows. The self-dual scalar curvature providing the total pure-gravity contribution to the self-dual action can be written as

$${}^{+}R := \delta_d{}^b g^{ac}\, {}^{+}R_{abc}{}^{d} = \frac{1}{2}R - \frac{i}{4}\epsilon^{abm}{}_n\, R_{abm}{}^{n}. \tag{3.23}$$

The second term of the last equality can be obtained by means of the Bianchi symmetry (3.21) and of the torsion (3.19) as

$$\epsilon_d{}^{abc}\, R_{abc}{}^{d} = \epsilon_d{}^{abc}\, \nabla_{[a} T_{bc]}{}^{d} = 3 \nabla_a k^a \quad . \tag{3.24}$$

The term quadratic in torsion drops out in view of the form of the torsion (3.19). Finally, one arrives at

$${}^{+}R = \frac{1}{2}R + \frac{3i}{4}\nabla_a k^a \, . \tag{3.25}$$

Correspondingly, the terms containing derivatives of the fermionic fields in the self-dual action (3.3) can be re-written in terms of ∇ and k^m as follows:

$$\begin{aligned} -\sqrt{2}\,\sigma\,\sigma^a{}_{AA'}\left[\overline{\kappa}^{A'}\left(\mathcal{D}_a\kappa^A\right) - \left(\mathcal{D}_a\mu^A\right)\overline{\mu}^{A'}\right] \quad = \quad & -\frac{\sigma}{\sqrt{2}}\sigma^a{}_{AA'}\left[\overline{\kappa}^{A'}\left(\nabla_a\kappa^A\right) - \left(\nabla_a\mu^A\right)\overline{\mu}^{A'}\right] \\ & + \frac{\sigma}{\sqrt{2}}\sigma^a{}_{AA'}\left[\left(\nabla_a\overline{\kappa}^{A'}\right)\kappa^A - \mu^A\left(\nabla_a\overline{\mu}^{A'}\right)\right] \\ & - \frac{i}{2}\sigma\nabla_a k^a \, . \end{aligned} \tag{3.26}$$

In light of (3.25) and (3.26) we have shown that the ECSK-Dirac action (3.2) and the chiral action (3.3) are equivalent modulo total divergences and the equation of motion for $\mathcal{D}$ (*i.e.* $\mathcal{D}$ is the self-dual part of ∇). Note that the mass terms in the actions differ by a factor of 2. Because of the non-vanishing torsion of ∇ these divergences give, apart from the boundary terms, volume terms involving the trace of the torsion. However, for the Einstein-Dirac system, torsion is traceless (*cf* (3.18)) and hence we get, indeed, a complete dynamical equivalence. Explicitly,

$$S_{SD}[{}^{+}\omega, \sigma, \kappa, \mu] = \frac{1}{2} S_{ECSKD}[\omega, \sigma, \kappa, \mu] + \frac{i}{4} \int_{\partial M} dS^a k_a \tag{3.27}$$

${}^{+}\omega$ being the self-dual part of the connection ω. For real GR, it is then evident that, although S_{SD} is not real, its imaginary part is a boundary term. This is a non-trivial generalization to spin-$\frac{1}{2}$ fields coupled to gravity of the results given in [86] for pure gravity.

3.3. Equations of motion

The structure of the action (3.3) makes it easy to get the equations of motion for the rest of the fields. By varying w.r.t. $\mu^A, \overline{\mu}^{A'}, \kappa^A, \overline{\kappa}^{A'}$ and using $\widetilde{\sigma}^a{}_{AA'} := \sigma\sigma^a{}_{AA'}$, the equations of motion for the Dirac field are

$$\begin{aligned} \widetilde{\sigma}^a{}_{AA'}\mathcal{D}_a\kappa^A &= \frac{im}{\sqrt{2}}\sigma\, \overline{\mu}_{A'} & \qquad \mathcal{D}_a\left(\widetilde{\sigma}^a{}_{AA'}\overline{\kappa}^{A'}\right) &= \frac{im}{\sqrt{2}}\sigma\, \mu_A \ \text{ and} \\ \mathcal{D}_a\left(\widetilde{\sigma}^a{}_{AA'}\overline{\mu}^{A'}\right) &= \frac{im}{\sqrt{2}}\sigma\, \kappa_A & \qquad \widetilde{\sigma}^a_{AA'}\mathcal{D}_a\mu^A &= \frac{im}{\sqrt{2}}\sigma\, \overline{\kappa}_{A'}\ . \end{aligned} \tag{3.28}$$

Note that $\mathcal{D}$ does not act on primed indices (hence its compatibility with the soldering form is undefined) but one needs to know its action on space-time indices, whereas in the pure-gravity case it is independent of its extension to act on space-time indices because of the torsion-free condition [7]. Here, however, it develops a non-vanishing torsion. This problem is solved by introducing the connection ∇ to coincide with $\mathcal{D}$ when acting on space-time indices and thus having identical torsion (*cf* (3.6), (3.10)).

To compare with the standard Dirac equations of motion we simply replace $\mathcal{D}$ with ∇.

One gets

$$\sigma^a{}_{AA'}\nabla_a\kappa^A = \frac{im}{\sqrt{2}}\overline{\mu}_{A'} \qquad \sigma^a{}_{AA'}\nabla_a\overline{\kappa}^{A'} = \frac{im}{\sqrt{2}}\mu_A \quad \text{and}$$
$$\sigma^a{}_{AA'}\nabla_a\overline{\mu}^{A'} = \frac{im}{\sqrt{2}}\kappa_A \qquad \sigma^a{}_{AA'}\nabla_a\mu^A = \frac{im}{\sqrt{2}}\overline{\kappa}_{A'}\,. \tag{3.29}$$

With our conventions $\sigma^a{}_{AA'}$ are taken to be antihermitian (*cf* Appendix). Although (4.2) resemble ordinary Dirac equations in curved space-time one should bear in mind ∇ is *not* torsion-free. These are the equations for a Dirac field minimally coupled to gravity with torsion (see *e.g.* [137]).

The field equation for $\sigma^a{}_{AA'}$ can be more easily understood by adopting the conventions of Ashtekar *et al* [7]. Set

$$H_{ab} := \frac{1}{\sigma}\sigma_{bAA'}\frac{\delta S^{\text{Gravity}}_{SD}}{\delta\sigma^a{}_{AA'}} \quad \text{and} \quad E_{ab} := -\frac{1}{8\pi\sigma}\sigma_{bAA'}\frac{\delta S^{\text{Dirac}}_{SD}}{\delta\sigma^a{}_{AA'}} \tag{3.30}$$

so that

$$H_{ab} = 8\pi E_{ab}$$

are the Einstein equations we are looking for. Using the connection ∇, due to its simple relation with $\mathcal{D}$, one easily gets

$$H_{ab} = \sigma_{bAA'}\left[2\sigma^c{}_B{}^{A'}R_{ac}{}^{AB} + \sigma^d{}_D{}^{D'}\sigma^c{}_{BD'}R_{dc}{}^{DB}\sigma_a{}^{AA'}\right] \tag{3.31}$$

$$\begin{aligned} 8\pi E_{ab} &= \sqrt{2}\,\sigma_{bAA'}\left[\overline{\kappa}^{A'}\left(\nabla_a\kappa^A\right) - \left(\nabla_a\mu^A\right)\overline{\mu}^{A'}\right] \\ &\quad - \sqrt{2}\,g_{ab}\,\sigma^c{}_{CC'}\left[\overline{\kappa}^{C'}\left(\nabla_c\kappa^C\right) - \left(\nabla_c\mu^C\right)\overline{\mu}^{C'}\right] \\ &\quad - im\,g_{ab}\left(\mu_D\kappa^D - \overline{\kappa}^{D'}\overline{\mu}_{D'}\right) \end{aligned} \tag{3.32}$$

which can be reduced to[23]

$$H_{ab} = G_{ab} + \frac{i}{2}\nabla_a k_b - \frac{i}{2}g_{ab}\nabla_c k^c \tag{3.33}$$

$$8\pi E_{ab} = \sqrt{2}\,\sigma_{bAA'}\left[\overline{\kappa}^{A'}\left(\nabla_a\kappa^A\right) - \left(\nabla_a\mu^A\right)\overline{\mu}^{A'}\right] - \frac{i}{2}g_{ab}\nabla_c k^c \tag{3.34}$$

[23] Use has been made of (3.21) modified to be antisymmetric in the last three indices
$\epsilon_m{}^{bc}{}_d\, R_{abc}{}^d = \epsilon_m{}^{bc}{}_d\left(\nabla_{[a}T_{b]c}{}^d - \frac{1}{2}T_{s[a}{}^d\, T_{b]c}{}^s - \frac{1}{2}T_{ab}{}^t\, T_{tc}{}^d\right)$.
This can be verified by relating the curvature tensors, of a torsion-free connection, $\tilde{R}_{abcd}$ ($\epsilon_m{}^{bcd}\tilde{R}_{abcd} = 0$) and that of a connection with non-vanishing torsion, R_{abcd} (*cf* [122]).

where G_{ab} is the Einstein tensor of the curvature of ∇. The resulting Einstein equation can be further simplified by developing the term $\frac{i}{2}\nabla_a k_b$. One thus finds

$$G_{ab} = \frac{1}{\sqrt{2}}\sigma_{bAA'}\left[\overline{\kappa}^{A'}\left(\nabla_a \kappa^A\right) - \left(\nabla_a \overline{\kappa}^{A'}\right)\kappa^A + \mu^A\left(\nabla_a \overline{\mu}^{A'}\right) - \left(\nabla_a \mu^A\right)\overline{\mu}^{A'}\right] \quad . \tag{3.35}$$

Apart from a factor of $\frac{1}{2}$, due to our conventions (*cf* (3.2)), the r.h.s. of this equation is the stress-energy tensor of a Dirac field in a curved space-time with torsion, *i.e.* in a U_4-theory (see *e.g.* [13, 136, 137]).

It is now possible to make contact with the results of Ashtekar *et al* [15]. They found a cubic term in fermionic fields in their Dirac equation and stressed it has its origin in the kind of theory they started with, *i.e.* torsion in the ECSK-Dirac theory. This is explicitly shown below by splitting out the torsion contribution from the connection ∇ introduced above.

Let $\overset{T=0}{\nabla}$ be the unique torsion-free connection compatible with the metric $g_{ab} = \sigma_{aAA'}\sigma_b^{AA'}$. Hence, there exists a tensor $Q_{ab}{}^c$, and its spinor version Θ_{aBC}, $\overline{\Theta}_{aB'C'}$, relating both $\overset{T=0}{\nabla}$ and ∇ through [122]

$$\begin{aligned}
\left(\overset{T=0}{\nabla}_a - \nabla_a\right) v^b &= Q_{ac}{}^b\, v^c && (3.36)\\
\left(\overset{T=0}{\nabla}_a - \nabla_a\right) \kappa^A &= \Theta_a{}^A{}_B\, \kappa^B && (3.37)\\
\left(\overset{T=0}{\nabla}_a - \nabla_a\right) \lambda^{A'} &= \overline{\Theta}_a{}^{A'}{}_{B'}\, \lambda^{B'} && (3.38)
\end{aligned}$$

where the spinor decomposition

$$\begin{aligned}
Q_{ab}{}^c &= \left[\Theta_a{}^C{}_B\, \epsilon_{B'}^{C'} + \overline{\Theta}_a{}^{C'}{}_{B'}\, \epsilon_B^C\right] \sigma_b^{BB'}\, \sigma^c{}_{CC'} && (3.39)\\
\Theta_{aBC} &= \frac{1}{2}\sigma^b_{BB'}\, \sigma^c{}_C{}^{B'} Q_{abc} && (3.40)
\end{aligned}$$

is implied. With our notation, Θ_{aBD} corresponds to C_{aBD} appearing in [15]. Furthermore

$$\begin{aligned}
T_{ab}{}^c &= 2Q_{[ab]}^{\ \ \ c} && (3.41)\\
Q_{abc} &= T_{a[bc]} - \frac{1}{2}T_{bca} && (3.42)
\end{aligned}$$

the first of which states $\overset{T=0}{\nabla}$ is torsion-free and the second is a result of the metricity condition [122]. The torsion information is so thrown into Θ_{aBC}. In the case of Dirac fields one gets, plugging (3.19) into the above relations,

$$\Theta_{aBC} = \frac{i}{4} k_{(BA'} \, \sigma_{aC)}{}^{A'} . \tag{3.43}$$

Hereby the modification to the Dirac equation found in [15] is explicitly determined, its origin being the non-vanishing torsion; by virtue of (3.37) and (3.43), the first of the Dirac equations (3.29) takes the form

$$\sigma^a{}_{AA'} \nabla_a \kappa^A = \sigma^a{}_{AA'} \left(\overset{T=0}{\nabla}{}_a - \frac{3i}{8} k_a \right) \kappa^A = \frac{im}{\sqrt{2}} \overline{\mu}_{A'} \, . \tag{3.44}$$

This result extends to the primed-indices spinor equations (3.29) through $\overline{\Theta}_{aB'C'}$, and, similarly, to the Einstein equations (3.35). Such a modification was discussed previously in a U_4-theory [137]. The reduced action of Ashtekar *et al* [15] is thereby obtained. In particular, the four-Fermi interaction term, $\sigma k_m k^m$, is brought into the action; in other words, using the space-time-indices version of the identity (3.11) (*cf* [122]) for the curvatures of ∇ and $\overset{T=0}{\nabla}$, one gets the following relation between the corresponding scalars: $\frac{1}{2} \overset{T\neq 0}{R} = \frac{1}{2} \overset{T=0}{R} - \frac{3}{16} k_m k^m$.

3.4 Canonical sd-ED theory

The strength of Ashtekar's framework shows up in studying the canonical form of GR. Polynomiality of the constraints, among other appealing features, holds the same when matter fields are minimally coupled to the gravitational field [15, 120]. The novel feature now is the explicit non-vanishing torsion of the connection $\mathcal{D}$. From now on ${}^+\omega_{aBC}$ will be denoted by A_{aBC}. A non-vanishing torsion when one Weyl Fermionic field is considered was obtained by Jacobson [120]. The case of a Dirac field turns out to coincide, structurally, with the case studied by Ashtekar *et al* , because of the totally antisymmetric torsion.

Setting the 3+1 decomposition of the chiral action (3.3) is based on two main ingredients [7, 15] as shown in chapter 2. First it is assumed that the spacetime manifold M admits a foliation through three-dimensional hypersurfaces. Thus, four-dimensional quantities are related to the three-dimensional ones so as to express dynamics in their terms. Second, since spinors are here the building blocks of the theory and it is known that for a Riemannian three-dimensional hypersurface $SU(2)$-spinors are natural whereas those associated to a Lorentzian four-dimensional manifold are $SL(2,\mathbb{C})$, one requires a translation between them (*cf* Appendix A). The spacetime foliation is obtained as follows. Let t be the "time" defining the foliation such that Σ_t, the hypersurfaces of constant t, are all diffeomorphic to an initial one Σ_0. Take $\sigma^a{}_{AA'}$ to be anti-hermitian and such that Σ_t are all spacelike w.r.t. it. For Σ_t the unique future-directed timelike normal is denoted by n^a and the tensor field inducing the positive-definite three-metric h_{ij} by $q_{ab} := g_{ab} + n_a n_b$. Introduce the vector field t^a having affine parameter t: $t^a \nabla_a t = 1$. It is decomposed into lapse, N, and shift, N^a, functions: $t^a := N n^a + N^a$. The second ingredient is implemented by providing an hermitian metric $G_{AA'} \equiv -i\sqrt{2} n^a \sigma_{aAA'}$. It picks the unprimed $SL(2,\mathbb{C})$ spinors on M as $SU(2)$ spinors on Σ_t. Hereby the operation † defined as $\left(\chi^\dagger\right)_A = G_A{}^{A'} \overline{\chi}_{A'}$ yields the hermitian conjugate of χ_A. Finally, a soldering form on $SU(2)$ spinors is introduced through ${}^3\sigma^a_{AB} := i\sqrt{2}\sigma^a{}_{(A}{}^{A'} n_{B)A'}$. It is hermitian –w.r.t †–, trace-free and it is, by definition, automatically projected into Σ_t.

The pure-gravity piece of the integrand of (3.3) becomes, after use of the definition of the $SU(2)$ soldering form inverted for $\sigma^a{}_{AA'}$ and the decomposition of t^a in lapse and shift,

$$\sigma \ \mathrm{tr} \left(-i\sqrt{2} N^{-1} t^a F_{ab} + {}^3\sigma^a \ {}^3\sigma^b F_{ab} + i\sqrt{2} N^{-1} N^a \ {}^3\sigma F_{ab} \right) \ . \tag{3.45}$$

Here the first subtle difference w.r.t. [7] comes about, because of the torsion. The Lie derivative of the self-dual connection A_{bBC} is given by $\mathcal{L}_t A_b = t^a \mathcal{D}_b A_a + A_a \mathcal{D}_b t^a - T_{ab}{}^c \ t^a A_c$, containing an additional torsion term. It can be re-expressed as $\mathcal{L}_t A_b = \mathcal{D}_b(A_a t^a) + t^a \left[2\mathcal{D}_{[a} A_{b]} - T_{ab}{}^c A_c \right]$. The last term being just the one necessary to cancel out the torsion contribution in $\mathcal{D}A$ and to yield the curvature F_{ba}. Hence,

$$\mathcal{L}_t A_b = \mathcal{D}_b(A_a t^a) + t^a F_{ab} \ . \tag{3.46}$$

This equation can be inverted for the projection of the curvature $t^a F_{ab}$. The result further simplifies by introducing

$$ {}^3\tilde{\sigma}^a{}_A{}^B := {}^3\sigma \ {}^3\sigma^a{}_A{}^B \qquad \text{and} \qquad \underset{\sim}{N} := {}^3\sigma^{-1} N \tag{3.47}$$

where ${}^3\sigma$ is the inverse of the determinant $\det({}^3\sigma^a{}_A{}^B)$. The following equality hence holds $\sigma = \sqrt{-g} = N\sqrt{h} = -N \ {}^3\sigma$, with g and h the determinants of g_{ab} and h_{ij}, respectively.

At the end one is left with

$$-\,\sigma\,\sigma^{a}{}_{A}{}^{A'}\,\sigma^{b}{}_{BA'}\,F_{ab}^{AB}=\mathrm{tr}\left[i\sqrt{2}\,{}^{3}\tilde{\sigma}^{b}\mathcal{L}_{t}A_{b}-i\sqrt{2}N^{a}\,{}^{3}\tilde{\sigma}^{b}F_{ab}-\underset{\sim}{N}{}^{3}\tilde{\sigma}^{a}{}^{3}\tilde{\sigma}^{b}F_{ab}\right]\,. \tag{3.48}$$

Next, using ${q_a}^b$ to project on Σ_t every quantity originally defined w.r.t. M gives us ${}^3A_{aBC}, {}^3F_{abBC}, {}^3\mathcal{D}f$ besides the identity (*cf* Appendix B) ${q_a}^b\mathcal{L}_tA_b = {q_a}^b\mathcal{L}_t{}^3A_a$ thus producing for the pure-gravity part of (3.3)

$$S_{SD}^{\mathrm{Einstein}}=\int dt\int_{\Sigma_t}d^3x\quad\mathrm{tr}\quad\left[i\sqrt{2}\,{}^{3}\tilde{\sigma}^{b}\mathcal{L}_{t}^{3}A_{b}-i\sqrt{2}\,{}^{3}\tilde{\sigma}^{b}\,{}^{3}\mathcal{D}_{b}(A_{a}t^{a})\right.$$
$$\left.-i\sqrt{2}N^{a}\,{}^{3}\tilde{\sigma}^{b}\,{}^{3}F_{ab}-\underset{\sim}{N}\,{}^{3}\tilde{\sigma}^{a}\,{}^{3}\tilde{\sigma}^{b}\,{}^{3}F_{ab}\right]\,. \tag{3.49}$$

It is already apparent here one can adopt ${}^3A_{aAB}$ and ${}^3\tilde{\sigma}^b{}_{AB}$ as a canonical pair. A further integration by parts of the second term above in order to identify the Lagrange multipliers in full of the theory reveals the existence of a volume term containing the trace of the torsion apart from the boundary term $-i\sqrt{2}\ \mathrm{tr}\left[\sigma^b n_b^{\partial\Sigma_t}(A_at^a)\right]$. Because of the antisymmetry of the torsion in our case such a volume term vanishes whereas the boundary term drops out as soon as one considers Σ to be compact. No time derivatives of A_at^a, N^a and $\underset{\sim}{N}$ appear at all so they are the Lagrange multipliers. This still holds, as shown below, when a Dirac field is coupled to gravity.

Concerning the Dirac contribution to the self-dual action (3.3) the treatment goes through similarly as for the pure-gravity case. Substituting the primed-indices spinors by the corresponding hermitian conjugate, *e.g.* $\left(\kappa^\dagger\right)^A = -\overline{\kappa}^{A'}G^A{}_{A'}$, using the relation between $SL(2,\mathbb{C})$ and $SU(2)$ soldering forms as well as $n^a = N^{-1}(t^a - N^a)$ it is found that

$$\sigma\,\sigma^{a}{}_{AA'}\overline{\kappa}^{A'}\mathcal{D}_{a}\kappa^{A}\;=\;\underset{\sim}{N}\,{}^{3}\sigma\,{}^{3}\tilde{\sigma}^{a}{}_{A}{}^{B}\left(\kappa^{\dagger}\right)_{B}{}^{3}\mathcal{D}_{a}\kappa^{A}+\frac{i}{\sqrt{2}}{}^{3}\sigma\left(\kappa^{\dagger}\right)_{A}\mathcal{L}_{t}\kappa^{A}$$
$$-\frac{i}{\sqrt{2}}{}^{3}\sigma\left(A_{b}t^{b}\right)_{B}{}^{A}\left(\kappa^{\dagger}\right)_{A}\kappa^{B}-\frac{i}{\sqrt{2}}{}^{3}\sigma N^{a}\left(\kappa^{\dagger}\right){}^{3}\mathcal{D}_{a}\kappa^{A}\,. \tag{3.50}$$

Here $\mathcal{L}_t\kappa^A = t^a\mathcal{D}_a\kappa^A$ and there is no explicit torsion contribution. As before, every quantity has been projected on Σ_t. Similarly the results apply to $\sigma\,\sigma^a{}_{AA'}\overline{\mu}^{A'}\mathcal{D}_a\mu^A$. Also, the factor $\kappa^{A'}\mu_{A'}$ becomes $\left(\kappa^\dagger\right)^A\left(\mu^\dagger\right)_A$. Finally, the Dirac-field contribution to the self-dual action (3.3) is

$$S_{SD}^{\mathrm{Dirac}}\;=\;\int dt\int_{\Sigma_t}d^3x\;\left\{-\sqrt{2}\,\underset{\sim}{N}\,{}^{3}\sigma\,{}^{3}\tilde{\sigma}^{a}{}_{A}{}^{B}\left[\left(\kappa^{\dagger}\right)_{B}{}^{3}\mathcal{D}_{a}\kappa^{A}-\left(\mu^{\dagger}\right)_{B}{}^{3}\mathcal{D}_{a}\mu^{A}\right]\right.$$

$$+ i\,{}^3\sigma\Big[\left(\kappa^\dagger\right)_A \mathcal{L}_t\kappa^A - \left(\mu^\dagger\right)_A \mathcal{L}_t\mu^A\Big] - i\,{}^3\sigma\left(A_b t^b\right)_B{}^A\Big[\left(\kappa^\dagger\right)_A\kappa^A - \left(\mu^\dagger\right)_A\mu^A\Big]$$
$$- i\,{}^3\sigma N^a\Big[\left(\kappa^\dagger\right)_B {}^3\mathcal{D}_a\kappa^A - \left(\mu^\dagger\right)_B {}^3\mathcal{D}_a\mu^A\Big]$$
$$+ im\,\underset{\sim}{N}\,\left({}^3\sigma\right)^2\left[\kappa^A\mu_A - \left(\kappa^\dagger\right)^A\left(\mu^\dagger\right)_A\right]\Big\}\ . \tag{3.51}$$

Now it is clear one can adopt $\kappa^A, {}^3\sigma\left(\kappa^\dagger\right)_A$ and $\mu^A, {}^3\sigma\left(\mu^\dagger\right)_A$ as the fermionic canonical couples. Note that there is no explicit torsion contribution and the Lagrange multipliers are the same as in S_{SD}^{Einstein} above. Indeed, one can rewrite the action (3.3) in the more transparent form [15, 120]

$$S_{SD} = \int dt\int_{\Sigma_t} d^3x\,\Big\{\Big[i\sqrt{2}\,\mathrm{tr}\left({}^3\tilde{\sigma}^b\,{}^3\dot{A}_b\right) + \tilde{\pi}_A\dot{\kappa}^A + \tilde{\omega}_A\dot{\mu}^A\Big] + (t\cdot A)^{BC}G_{BC} + N^a\tilde{V}_a + \underset{\sim}{N}\tilde{\tilde{S}}\Big\} \tag{3.52}$$

$$G_{AB} := -i\sqrt{2}\,{}^3\mathcal{D}_b{}^3\tilde{\sigma}^b{}_{AB} + \tilde{\pi}_{(A}\,\kappa_{B)} + \tilde{\omega}_{(A}\,\mu_{B)} \tag{3.53}$$

$$\tilde{V}_a := -i\sqrt{2}\,\mathrm{tr}\left({}^3\tilde{\sigma}^b\,{}^3F_{ab}\right) - \tilde{\pi}_A{}^3\mathcal{D}_a\kappa^A - \tilde{\omega}_A{}^3\mathcal{D}_a\mu^A \tag{3.54}$$

$$\tilde{\tilde{S}} := -\mathrm{tr}\left({}^3\tilde{\sigma}^a\,{}^3\tilde{\sigma}^b\,{}^3F_{ab}\right) + i\sqrt{2}\,{}^3\tilde{\sigma}^a{}_A{}^B\left(\tilde{\pi}_B{}^3\mathcal{D}_a\kappa^A + \tilde{\omega}_B{}^3\mathcal{D}_a\mu^A\right) + im\left(\left({}^3\sigma\right)^2\,\kappa^A\mu_A - \tilde{\pi}^A\tilde{\omega}_A\right) \tag{3.55}$$

The factors in front of $G_{AB}, \tilde{V}_a, \tilde{\tilde{S}}$ are Lagrange multipliers since no time derivative of them appears anywhere in the action. $G_{AB}, \tilde{V}_a, \tilde{\tilde{S}}$ are called, respectively, the *Gauss*, *vector* and *scalar* constraints of the theory. The canonical momenta are readily recognized from the structure $\int dt[p\dot{q} - H(q,p)]$ [15]. They are: $-i\sqrt{2}\,{}^3\tilde{\sigma}^a{}_{AB}$, $\tilde{\pi}_A := i\,\sigma\,\left(\kappa^\dagger_A\right)$ and $\tilde{\omega}_A := -i\,\sigma\,\left(\mu^\dagger\right)$.

As in the study of the equations of motion of the previous section the link with previous results can be established by expressing the self-dual connections here and in [15] in terms of the torsion-free connection compatible with ${}^3\sigma_{aAB}$ (the soldering form defined on Σ_t). In other words, we are identifying the torsion contribution of our self-dual connection $\mathcal{D}$ via Θ_{aAB}, Eq. (3.43), with the C_{aAB} of Ashtekar *et al* [7, 15] defining their self-dual connection.

The constraints of the theory do form an algebra (such a term is actually loose, since one deals with structure functions rather than structure constants, as emphasized below) which admits an interpretation along the same lines of the pure-gravity case. We will be dealing only with the case in which Σ is compact so that boundary conditions for the different fields do not play any role. The non-compact case is discussed in [15]. The symplectic structure one ends up with for the phase space Γ consisting of the fields $A, {}^3\tilde{\sigma}, \kappa, \tilde{\pi}, \mu, \tilde{\omega}$ is thus

$$\begin{aligned} \left\{{}^3\tilde{\sigma}^a{}_{AB}(x), {}^3A_b{}^{CD}(y)\right\} &= -\frac{i}{\sqrt{2}}\,\delta^3(x,y)\,\delta_a{}^b\delta_{(A}{}^C\delta_{B)}{}^D \\ \left\{\tilde{\pi}_A(x), \kappa^B(y)\right\} &= -\delta^3(x,y)\,\delta_A{}^B \qquad (3.56) \\ \left\{\tilde{\omega}_A(x), \mu^B(y)\right\} &= -\delta^3(x,y)\,\delta_A{}^B\,. \qquad (3.57) \end{aligned}$$

The analysis of the constraints is highly simplified by smearing them with suitable functions as usual (*cf* previous chapter). Hence, let us set

$$G_T \equiv \int_\Sigma d^3x\; T^{BA}\tilde{G}_{AB} \qquad (3.58)$$

$$V_{\vec{v}} \equiv \int_\Sigma d^3x\; v^a\tilde{V}_a \qquad (3.59)$$

$$H_{\underset{\sim}{N}} \equiv i\sqrt{2}\int_\Sigma d^3x\; \underset{\sim}{N}\,\tilde{\tilde{S}}. \qquad (3.60)$$

With $T^{AB}, v^a, \underset{\sim}{N}$ test fields with compact support. The action of the smeared Gauss constraint on the phase-space variables is given by:

$$\begin{aligned} \left\{G_T, A_{aA}{}^B(x)\right\} &= -\mathcal{D}_a T_A{}^B(x), \qquad (3.61) \\ \left\{G_T, \tilde{\sigma}^a{}_A{}^B(x)\right\} &= [T,\sigma^a]_A{}^B(x), \qquad (3.62) \\ \left\{G_T, \kappa^A(x)\right\} &= -\kappa^B(x)T_B{}^A(x), \qquad (3.63) \\ \{G_T, \tilde{\pi}_B(x)\} &= T_B{}^A(x)\tilde{\pi}_A(x) \qquad (3.64) \\ \left\{G_T, \mu^A(x)\right\} &= -\mu^B(x)T_B{}^A(x), \qquad (3.65) \\ \{G_T, \tilde{\omega}_B(x)\} &= T_B{}^A(x)\tilde{\omega}_A(x). \qquad (3.66) \end{aligned}$$

That is to say, G_T generates infinitesimal canonical transformations which are infinitesimal $SU(2)$ rotations on spinor indices. Obviously then the Poisson bracket with the other two constraints vanishes (no free spinor indices!), whereas one gets

$$\{G_T, G_R\} = -G_{[T,R]} \qquad (3.67)$$

for $R_{AB}(x)$ a test function with compact support. Going over to the vector constraint it turns out that it is rather a combination of the vector and Gauss constraint which has a nice and direct meaning. This is

$$D_{\vec{v}} := V_{\vec{v}} - \int_\Sigma d^3x \ \mathrm{Tr}\ (A_a G) \ , \tag{3.68}$$

whose action on the canonical variables becomes, for f any of the set $A, {}^3\tilde{\sigma}, \kappa, \tilde{\pi}, \mu, \tilde{\omega}$,

$$\{D_{\vec{v}}, f\} = \mathcal{L}_{\vec{v}} f \ . \tag{3.69}$$

It yields infinitesimal three-dimensional diffeomorphism transformations. According to the lines of chaper 2, the constraints of the theory should by implemented either classically or quantum mechanically by selecting quantities that are invariant under the corresponding action. We shall be looking at objects which are invariant under the action of the Gauss-law constraint in chapter 5 and hence it will be $D_{\vec{v}}$ rather than $V_{\vec{v}}$ the form of the constraint of interest for us. It is called the *Diffeomorphism* constraint.

The rest of the algebra satisfied by the constraints involves some lengthy calculations but still simpler w.r.t. the analogue using geometrodynamical variables. This is one of the aspects that is simplified in Ashtekar's framework. We just give the results here (*cf* [15]),

$$\{D_{\vec{v}}, G_T\} = -G_{\mathcal{L}_{\vec{v}} T} \tag{3.70}$$

$$\{D_{\vec{v}}, D_{\vec{w}}\} = -D_{\mathcal{L}_{\vec{v}} \vec{w}} \tag{3.71}$$

$$\left\{D_{\vec{v}}, H_{\underset{\sim}{M}}\right\} = -H_{\mathcal{L}_{\vec{v}} \underset{\sim}{M}} \tag{3.72}$$

$$\left\{H_{\underset{\sim}{N}}, H_{\underset{\sim}{M}}\right\} = D_{\vec{K}} + G_{K \cdot A} \qquad (= V_{\vec{v}}) \tag{3.73}$$

where $K^a \equiv \left(\underset{\sim}{N} \partial_b \underset{\sim}{M} - \underset{\sim}{M} \partial_b \underset{\sim}{N}\right) \mathrm{Tr}\ \left({}^3\tilde{\sigma}^a \ {}^3\tilde{\sigma}^b\right)$. Further remarks in order here are as follows.

- The constraints are polynomial in the canonical variables. As stressed in chapter 2 this is relevant for the quantisation and is by itself one of the motivations of the whole framework.

- Note that it is the connection with non-vanishing torsion the one that appears in the constraints. Although in [15] the connection is torsion-free the authors have an extra term which can be eliminated by the addition of yet another term to the action quartic in Fermionic fields. In our case such additional terms analogously do cancel torsion out of the theory. Nevertheless it is obviously simpler to work with the non-vanishing-torsion connection and hence sd-ECSKD theory. Furthermore, as shown below, it may be possible to eliminate the torsion via the reality conditions. In this way the problem of facing the quartic term in Fermionic fields is not faced at the starting point of the quantization, or more precisely, it is postponed until the implementation of the quantum reality conditions.

- The structure of the algebra of the constraints is the same as that in the pure-gravity case. Hojman *et al* [140] have emphasized this is so because the algebra itself has its roots in geometrodynamics. On the other hand although they are *first-class* in the language of Bergmann and Dirac, since they involve structure functions rather than structure constants they do not generate a proper Lie group [141]. A consequence of this is that the above algebra is not isomorphic to the obvious group in tetrad-gravity, *i.e.* the semi-direct product of tetrad rotations with the four-dimensional-diffeomorphisms group acting on $\Sigma \times \mathbb{R}$.
- At last, it is worth stressing that, by construction, exactly as in the pure-gravity case of chapter 2, there is no reference to the anti-self-dual part of the connection and hence at this point the two parts are independent one of the other. This means that we have a description of complex GR [142]. That is to say, within Ashtekar's framework further constraints are needed to obtain real GR or, rather, real ECSKD as here: the *reality conditions.* They are discussed in the next section.

3.5. Reality Conditions

The role of the reality conditions in Ashtekar's formalism has been studied by several authors (see *e.g.* [123] and ref. there). Originally it was found that in spite of having polynomial structure at the level of constraints the quantisation program put forward by Ashtekar was spoiled through the non-polynomiality of the reality conditions in the canonical variables. More recently Ashtekar *et al* [15, 7] have introduced another equivalent form of the reality conditions which does not have this problem. Still, a clear solution to the issue is missing. Since one is more used to think in terms of geometrodynamical variables the non-polynomial form is straightforward to interpret. According to chapter 2, these conditions reduce to the reality of the three-metric and its time derivative. This is the form we shall bear in mind in this section although it may be possible to translate our proposal to the polynomial form following the lines of [7, 15].

Although it has been shown that the constraints of the sd-ED theory keep their structure, independently of whether the connection is taken to develop torsion [120] or not [15], a definite answer to the question put forward by Jacobson [120] of identifying real ED and ECSKD theories with two different real sectors of the phase space of complex GR via the reality conditions has not been given. Ashtekar *et al* [15] gave a partial answer in the negative to it by looking at the effect on the reality conditions of adding the quartic combination of fermionic fields required to translate the ECSKD action into the ED one; this term decouples torsion in ECSKD theory from the rest of the dynamics and it does not affect the reality conditions as obtained by Ashtekar *et al*. Though, in their analysis, as remarked above, torsion was dealt with in an implicit fashion. Here, however, we adopt a self-dual connection developing a non-vanishing torsion. As given in [143] torsion vanishes for ECSK theory in vacuum (i.e. no source for torsion), using a Hamiltonian framework, by virtue of a second-class constraint, in contrast with the Lagrangian framework in which its vanishing can be traced back to the Lagrangian equations of motion. In [143], however, splitting torsion out of the connection is required to get this result. Though different, both strategies lead to a vanishing torsion. It may be possible to replace the way torsion is set to zero by imposing an extra constraint at the Hamiltonian level, rather than adding a quartic term in fermionic fields to the Lagrangian.

The reality conditions, for instance, to pick out real ECSKD from sd-ED, which is a complex theory, can be written in terms of the extrinsic curvature

$$\begin{aligned} K_a^{\ b} &:= K_{aAB}\, {}^3\sigma^{bAB} \\ K_{aBC} &= i\sqrt{2}\left({}^3A_{aBC} - {}^3\Gamma_{aBC}\right) \end{aligned} \tag{3.74}$$

and fermionic canonical variables $\kappa, \tilde{\pi}, \mu, \tilde{\omega}$ (*cf* [7]). ${}^3A_{aBC}$ and ${}^3\Gamma_{aBC}$ being associated, respectively, to the projection of $\mathcal{D}$, and the intrinsic connection, D, on Σ_t, compatible with the soldering form ${}^3\sigma_{aAB}$. They can be expressed as [7, 84]

$$\begin{aligned} \sigma^a{}_{AB} &= (\sigma^a{}_{AB})^\dagger \quad (\Rightarrow q_{ab} \text{ real}) \\ K_{(ab)} &= \left(K_{(ab)}\right)^\dagger \quad (\Rightarrow \dot{q}_{ab} \text{ real})\, K_{[ab]} = \left(K_{[ab]}\right)^\dagger (\Rightarrow \text{Torsion real}) \ . \end{aligned} \tag{3.75}$$

The conjugation † [144, 145] is that of Ashtekar [7]; it affects the matrix character only for spinor indices whereas on space indices it yields ordinary complex conjugation. In general the second condition would involve a symmetric piece of the torsion $2T_{(amb)}n^m$ (*cf* [143]) which however vanishes in the present case.

The antisymmetric part of the extrinsic curvature is different from zero because of the non-vanishing torsion. In fact, using the definitions of extrinsic curvature in terms of the normal and that of torsion, one gets $K_{[ab]} = \frac{1}{2} q_a{}^m q_b{}^n \; T_{mn}{}^p \; n_p$, $q_a{}^b \equiv g_a{}^b + n_a n^b$ being the projector to translate four-dimensional space-time indices to spatial three-dimensional ones, whereas n_a is the normal to Σ_t [7]. Note that no splitting of the torsion contribution out of the connection has been required. These conditions should be supplemented by the reality conditions for fermionic fields [7]. It is worth emphasizing that the above reality conditions do involve explicitly torsion. They are reasonable since initial reality of q_{ab}, and of its time-derivative, $\dot{q}_{ab}$, ensure it will keep being real as it evolves. Torsion is fully specified with a single (initial-) reality condition, since it does not propagate [143]. It is then clear that one can actually set torsion to zero through a suitable selection of the reality conditions, rather than decoupling it from the rest of the dynamics, to get ED. A similar approach was considered by Maluf [146] in the pure-gravity case. For instance one can alternatively demand

$$K_{[ab]} = -\left(K_{[ab]}\right)^\dagger \ . \tag{3.76}$$

$K_{[ab]}$ being purely imaginary would lead to a real ED-theory with vanishing torsion because the combination of soldering form and connection forming $K_{[ab]}$ is no longer equal to the combination of soldering form and fermionic fields forming $q_a{}^m q_b{}^n \; T_{mn}{}^p \; n_p$ (which is real). This, we think, amounts to a vanishing torsion, and, because torsion was not splitted out of the connection anywhere, the constraints (3.53)-(3.55) would simply remain the same.

Should it work, we expect this form of reality conditions has to be cast in a polynomial form so that it matches the important requirement in the Ashtekar quantization program.

Yet, another possibility may be considered. Penrose [147] has shown that complex-GR admitting complex conformal transformations, $\tilde{g}_{ab} = \Omega\overline{\Omega}g_{ab}$, leads to torsion related to the complex function Ω. It remains to be seen whether this may provide more insight into the issue of identifying theories with and without torsion as different sectors of the phase space of complex-GR by restricting the properties of the conformal factor Ω.

3.6. Final Remarks

The equivalence between the self-dual and the ECSK forms of the action coupling Dirac fields to gravity has been shown by introducing a connection with non-vanishing torsion. The key steps of the proof are the use of the Bianchi symmetry of the curvature of such a connection, here including torsion, and the result that the torsion for this system is totally antisymmetric. Thus, the actions differ by total divergences. They lead to boundary terms only because the volume terms they involve are proportional to the trace of the torsion, and hence vanish. This can be considered the explicit version of an observation first made by Dolan [148]. He studied the canonical transformation, in pure gravity, from tetrad and connection variables to Ashtekar variables. According to [148], the generating function, when torsion is present, remains the same structurally, whenever torsion is totally antisymmetric; this is the case for the ED and supergravity. On the other hand, Jacobson [120] used another approach to prove the above equivalence of the actions. The boundary terms he finds, can thus be traced back to the Bianchi identity using the present results.

For real GR, the above-mentioned boundary terms are the imaginary part of the self-dual action. This explains why there are no spurious equations of motion (*cf* (3.27)). Moreover, these imaginary terms, having origin in the Bianchi symmetry for the curvature of the connection with torsion (and certain combinations of the fermionic fields (3.25), (3.26)), provide a generalization of the analogous pure-gravity case [86], where equivalence between the Palatini-like and self-dual forms of the action follows from the Bianchi symmetry of the curvature for a torsion-free connection.

We have also shown explicitly that the extra term entering the Dirac equations obtained by Ashtekar *et al* [7, 15] from the self-dual action is essentially a torsion term. By splitting the self-dual connection into its torsion-free and torsion parts, the standard four-Fermi interaction in the action comes about [15, 120]. These results completely agree with Hehl and Datta [137]. They investigated the four-Fermi interaction using certain anholonomic basis and related connection; a U_4-theory [136, 137] with fermions admiting a $V - A$ (vector-axial) structure for the fermionic interaction. It is tempting to further analyze the problem within Ashtekar's framework so as to elucidate any implication concerning other non-gravitational interactions, where connections play an essential role.

To conclude this chapter, we find it helpful for the reader to emphasize again the main differences between our analysis and the work appearing in [15, 120], at the risk of repeating ourselves. They are as follows.

(1) We find that the imaginary part of the chiral action is a boundary term using the Bianchi identity. This point of view is a direct generalization to fermionic matter of the pure-gravity result. This is completely different from the analysis in [120]. Moreover, torsion effects in the Dirac equation were not considered by the author of [120], and we have tried to formulate our proposal for reality conditions in a more precise way.

(2) We work with the full connection, without splitting out torsion contributions (relevant applications of this point of view to a different, cosmological framework may be found in [145]) as done implicitly in [15]. This makes it possible for us to propose a set of reality conditions *different* from the ones appearing in [15], yielding real ED-theory.

FOUR

VARIATIONAL PROBLEMS FOR GRAVITY AND FERMIONIC FIELDS

Variational problems are relevant in approaching the problem of quantum gravity. As shown in the previous chapter, defining canonically conjugate variables for the theory is straightforward by starting with an action principle. Its availability also makes it easier to elucidate the coupling of matter fields to gravity for instance when using Ashtekar variables. On the other hand, in the path-integral approach, knowing what the action for gravity and matter fields is enables one to calculate the semiclassical value of transition amplitudes. In the present chapter the results by York [149] concerning boundary terms for pure gravity are described so that well-posedness of the corresponding variational problem is attained. Then, after touching on the relevance for quantum cosmology of the boundary conditions for the wave function of the universe as a motivation for our approach, the analogous classical variational problem for Fermionic fields is studied for a model consisting of a massless spin-$\frac{1}{2}$ field in flat Euclidean 4-space bounded by a three-sphere, S^3 [21]. At last, the local boundary conditions used in the one-loop quantum cosmological analysis in [150] for the massless Fermionic fields on S^3 are implemented through a boundary term to be included in the classical action.

4.1 Gravitational boundary terms

In second-order theories of gravitation in the presence of boundaries, boundary terms are essential to obtain a well-posed variational problem, i.e. a differentiable action functional which is stationary under arbitrary variations of the four-metric. Moreover, boundary terms lead to a Lagrangian density such that the corresponding Hamiltonian is well-defined, they yield a non-vanishing action for vacuum general relativity, and the fundamental formula relating entropy of black holes to the area of their event horizon is recovered using Wick-rotated path-integral techniques.

There are a number of different forms of the action integral from which the vacuum Einstein field equations can be derived. York has performed a comprehensive analysis of this issue [149] by focusing on the question of what is fixed on the spacetime boundary. Consider spacetime (M, g), with g having diagonal form $(-,+,+,+)$ on each tangent space, to be smoothly sliced and let one of such slices be the spacetime boundary ∂M. Take n^a to be the unit normal vector field associated to the slicing, such that $n^a n_a = \epsilon$, $\epsilon = +1$ if n^a is spacelike (i.e. ∂M is timelike), $\epsilon = -1$ if n_a is timelike (i.e. ∂M is spacelike). The tensor field on (M, g) whose restriction on ∂M yields the positive-definite three-metric induced on every slice is denoted by h $(h_{ij} = q_{ij})$ and the extrinsic-curvature tensor K of the slice is here defined as $K_{ab} \equiv q_a^{\ m} q_b^{\ n} \nabla_m n_n$, where ∇ is the unique torsion-free four-dimensional connection of (M, g), compatible with g. York's results are expressed in terms of the gravitational action, in $c = 1$ units, as

$$\begin{aligned} S \equiv & \frac{1}{16\pi G} \int_M {}^{(4)}R \sqrt{-g}\, d^4x - \sum_{j=1}^{P} \frac{b\, \epsilon_j}{16\pi G} \int_{B_j} K^l_{\ l} \sqrt{q}\, d^3x \\ & + \sum_{k=1}^{P} \frac{\epsilon_k}{8\pi G} \int dt \int_{B_k} \partial_i \left[\sqrt{h} \left(K^l_{\ l} N^i - q^{ij} N_{,j} \right) \right] d^3x + C\left(K^l_{\ l}, \widetilde{M} \right) \end{aligned} \tag{4.1}$$

where all contributions are given in terms of q, K, the lapse N and shift vector N^a [149]. Morover, $K^l_{\ l} = q^{ab} K_{ab}$, with q^{ab} the inverse of q_{ab}, and the B_j are the components of the spacetime boundary which add up to ∂M. Finally, the values of the dimensionless parameter b are determined according to what is fixed on ∂M. $b = 0$ is consistent with fixing π^{cd}, the canonically conjugate variable to q_{ab}, on the boundary, $b = \frac{2}{3}$ when the conformal three-metric, $\hat{q}_{cd} \equiv (\det q)^{-\frac{1}{3}} q_{cd}$, and the trace $K^l_{\ l}$ are both fixed on ∂M under the variation, whereas $b = 2$ corresponds to the case when q_{ab} is fixed (i.e. $\delta q_{ab}|_{\partial M} = 0$). The constant $C = C(K^l{}_l, \widetilde{M})$ is a term which, without affecting the equations of motion enables one to have a finite action in the asymptotically-flat case, and depends on $K^l{}_l$ and a suitable four-manifold $\widetilde{M}$ different from M [151]. Note also that the second sum appearing in equation (1.1) is here included to cancel an opposite-sign contribution arising from the Einstein-Hilbert part of the full action [54], so as to obtain the familiar ADM form of the action integral [84, 149]. Of course, $\epsilon_j = -1$ if B_j is spacelike (the case we are interested in), and $\epsilon_j = +1$ if B_j is timelike.

To ilustrate how the analysis goes through an example is next given. Consider the Einstein-Hilbert action

$$I_{\rm EH} \equiv \frac{1}{16\pi G} \int_M d^4x \sqrt{-g}\ {}^{(4)}R \ . \tag{4.2}$$

Varying it w.r.t. the metric g_{ab} yields [89]

$$16\pi G\ \delta I_{\rm EH} = -\int_M d^4x \sqrt{-g}\ G^{ab}\ \delta g_{ab} + \int_{\partial M} d^3x \sqrt{h}\ n^c \delta v_c \ , \tag{4.3}$$

$$\delta v_c \equiv \nabla^b \left(\delta g_{cb}\right) - g^{bd} \nabla_c \left(\delta g_{bd}\right) \ . \tag{4.4}$$

Hence, $I_{\rm EH}$ is not stationary under arbitrary variations of g_{ab}; due to the presence of the boundary contribution the normal derivatives of the variations of the metric should vanish on ∂M. As prescribed above, to build up an action which is stationary one can add suitable terms to $I_{\rm EH}$ such that they cancel out the corresponding contribution. Writing down δv_c explicitly in terms of the variations of the metric one gets [89]

$$[n^c \delta v_c]_{\partial M} = -\left[n^c q^{bd} \nabla_c \left(\delta g_{bd}\right)\right]_{\partial M} = -2\left[\delta K^m{}_m\right]_{\partial M} \ . \tag{4.5}$$

In the first equality use is made of the facts that *i)* $q_{ab} = g_{ab} + n_a n_b$ and *ii)* $\left[q^{bc} \nabla_c \left(\delta g_{ab}\right)\right]_{\partial M} = 0$ because one is taking $[\delta g_{ab}]_{\partial M} = 0$. The second equality can be seen to hold by directly working out the variation of the trace of the extrinsic curvature of the boundary $K^m_m|_{\partial M}$[24]. Hence, the action

$$S \equiv \frac{1}{16\pi G} \int_M d^4x \sqrt{-g}{}^{(4)}R + \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{q}\ K^l{}_l \tag{4.6}$$

[24] $K \equiv K^m_m = q^a{}_b \nabla_a n^b$ becomes, upon variation, $\delta K = q^a{}_b \left(\delta C^b_{ac}\right) n^c$, where C^b_{ac} is defined by $\nabla_a n^b \equiv \widetilde{\nabla}_a n^b + C^b_{ac} n^c$. Thus, using again condition *ii)* above, $[\delta K]_{\partial M} = \frac{1}{2}\left[n^c q^{ad} \nabla_c \left(\delta g_{ad}\right)\right]_{\partial M}$; the result one was looking for.

is stationary under arbitrary variations of the metric, δg_{ab}, with the condition $[\delta g_{ab}]_{\partial M} ==$ 0. Morover, it can be verified that it is also stationary when only the three-metric induced on the boudary is fixed, *i.e.* $\delta h|_{\partial M} = 0$ [89]. The boundary term in (4.6) accounting for the extra boundary term coming from $I_{\rm EH}$.

4.2 Boundary terms for massless Fermionic fields

4.2.1 Quantum Cosmology and the Hartle-Hawking proposal

The motivation for studying the boundary terms and boundary conditions discussed below can be traced back to the Hartle-Hawking proposal for the wave function of the universe in quantum cosmology [116]. The specific form, however, is taken from supersymmetry [152].

When fermionic fields are incorporated with gravity in a cosmological model using the path-integral approach [116, 138, 145] one looks for a solution of the Wheeler-DeWitt equation defined by the path integral over the class C of Riemannian four-metrics and matter fields matching the boundary data

$$\Psi = \int_C d[e_a^{\hat{b}}] d[\phi_A] d[\widetilde{\phi}_{A'}] d[\chi_A] d[\widetilde{\chi}_{A'}] \, e^{-\widetilde{I}} \, , \tag{4.7}$$

where $e_a^{\hat{b}}$ are the tetrads and $\phi_A, \widetilde{\phi}_{A'}, \chi_A, \widetilde{\chi}_{A'}$ the (two-component) spinor matter fields. Moreover, $\widetilde{I}$ is the Euclidean action for the coupled system. The Hartle-Hawking proposal [116] demands that four-metrics summed over in (3.7) should be compact, and that there is only a "final" boundary S_F for the Riemannian four-manifold. Since the fermionic action is quadratic in the corresponding Grassmann fields, and Berezin integration rules hold, the evaluation of the path integral is essentially semiclassical. Thus, boundary conditions suitable for investigating the Hartle-Hawking quantum state may be found by studying the classical version of the Hartle-Hawking path integral, *i.e.* , by asking for data on a three-sphere bounding a compact region with a Riemannian metric, such that the boundary-value problem for the Dirac equation is well-posed.

Local supersymmetry leads to boundary conditions for fermionic fields in one-loop quantum cosmology involving the Euclidean normal ${}_e n_A{}^{A'}$ to the boundary and a pair of independent spinor fields ψ^A and $\widetilde{\psi}^{A'}$ [145]. It is here found that if $\sqrt{2}\ {}_en_A^{\ A'}\ \psi^A \mp \widetilde{\psi}^{A'} \equiv \Phi^{A'}$ is set to zero on a 3-sphere bounding flat Euclidean 4-space, the modes of the massless spin-$\frac{1}{2}$ field multiplying harmonics having positive eigenvalues for the intrinsic 3-dimensional Dirac operator ${}_en_{AB'}\ e^{BB'j}\left({}^{(3)}D_j\right)$ on S^3 should vanish on S^3. Remarkably, this coincides with the property of the classical boundary-value problem when spectral boundary conditions are imposed on S^3 in the massless case. Moreover, the boundary term in the action functional turns out to be proportional to the integral on the boundary of $\Phi^{A'}\ {}_en_{AA'}\ \psi^A$.

4.2.2 Variational problem for fermionic fields: a model

Locally supersymmetric boundary conditions have been recently studied in quantum cosmology to understand its one-loop properties (see references in [145]). They involve the normal to the boundary and the field for spin $\frac{1}{2}$, the normal to the boundary and the spin-$\frac{3}{2}$ potential for gravitinos, Dirichlet conditions for real scalar fields, magnetic or electric field for electromagnetism, mixed boundary conditions for the 4-metric of the gravitational field (and in particular Dirichlet conditions on the perturbed 3-metric). The aim here is to describe the corresponding classical properties in the case of massless spin-$\frac{1}{2}$ fields.

For this purpose, we consider flat Euclidean 4-space bounded by a 3-sphere of radius a. This is the limiting case of a more involved boundary-value problem, where flat Euclidean 4-space is bounded by two concentric 3-spheres of radii r_1 and r_2, say, and one finally takes the limit $r_1/r_2 \rightarrow 1$. The spin-$\frac{1}{2}$ field, represented by a pair of independent spinor fields ψ^A and $\widetilde{\psi}^{A'}$, is expanded on a family of 3-spheres centred on the origin as [138, 145, 150]

$$\psi^A = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\Big[m_{np}(\tau)\rho^{nqA} + \widetilde{r}_{np}(\tau)\overline{\sigma}^{nqA}\Big] \tag{4.8}$$

$$\widetilde{\psi}^{A'} = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\Big[\widetilde{m}_{np}(\tau)\overline{\rho}^{nqA'} + r_{np}(\tau)\sigma^{nqA'}\Big] \ . \tag{4.9}$$

With our notation, τ is the Euclidean-time coordinate, the α_n^{pq} are block-diagonal matrices with blocks $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, ρ and σ are spinor harmonics on the three-sphere, they obey the identities described in [138, 145]. Last but not least, the modes m_{np} and r_{np} are regular at

$\tau = 0$, whereas the modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$ are singular at $\tau = 0$ if the spin-$\frac{1}{2}$ field is massless. Bearing in mind that the harmonics ρ^{nqA} and $\sigma^{nqA'}$ have positive eigenvalues $\frac{1}{2}\left(n+\frac{3}{2}\right)$ for the 3-dimensional Dirac operator on the bounding S^3 [145], the decomposition (4.8-4.9) can be re-expressed as

$$\psi^A = \psi^A_{(+)} + \psi^A_{(-)} \qquad (4.10)$$

$$\widetilde{\psi}^{A'} = \widetilde{\psi}^{A'}_{(+)} + \widetilde{\psi}^{A'}_{(-)} \quad . \qquad (4.11)$$

In (4.10-4.11), the (+) parts correspond to the modes m_{np} and r_{np}, whereas the (−) parts correspond to the singular modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$, which multiply harmonics having negative eigenvalues $-\frac{1}{2}\left(n+\frac{3}{2}\right)$ for the 3-dimensional Dirac operator on S^3. If one wants to find a solution of the Weyl equation which is regular $\forall \tau \in [0, a]$, one is thus forced to set to zero the modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$ $\forall \tau \in [0, a]$ [138]. This is why, if one requires the local boundary conditions [145]

$$\sqrt{2}\ {}_e n_A^{\ A'}\ \psi^A \mp \widetilde{\psi}^{A'} = \Phi^{A'} \text{ on } S^3 \qquad (4.12)$$

such a condition can be expressed as [145]

$$\sqrt{2}\ {}_e n_A^{\ A'}\ \psi^A_{(+)} = \Phi_1^{A'} \text{ on } S^3 \qquad (4.13)$$

$$\mp \widetilde{\psi}^{A'}_{(+)} = \Phi_2^{A'} \text{ on } S^3 \qquad (4.14)$$

where $\Phi_1^{A'}$ and $\Phi_2^{A'}$ are the parts of the spinor field $\Phi^{A'}$ related to the $\overline{\rho}$- and σ-harmonics respectively. In particular, if $\Phi_1^{A'} = \Phi_2^{A'} = 0$ on S^3 as in [145, 150], one finds

$$\sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\ m_{np}(a)\ {}_e n_A^{\ A'}\ \rho^A_{nq} = 0 \qquad (4.15)$$

$$\sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\ r_{np}(a)\ \sigma^{A'}_{nq} = 0 \qquad (4.16)$$

where a is the 3-sphere radius. Since the harmonics appearing in (4.15-4.16) are linearly independent, these relations lead to $m_{np}(a) = r_{np}(a) = 0\ \forall n, p$. Remarkably, this simple calculation shows that the classical boundary-value problems for regular solutions of the Weyl equation subject to local or spectral conditions on S^3 share the same property provided $\Phi^{A'}$ is set to zero in (4.12): the regular modes m_{np} and r_{np} should vanish on the bounding S^3.

To study the corresponding variational problem for a massless fermionic field, we should now bear in mind that the spin-$\frac{1}{2}$ action functional in a Riemannian 4-geometry takes the form [145, 150]

$$I_E = \frac{i}{2} \int_M \Big[\widetilde{\psi}^{A'}\Big(\nabla_{AA'}\psi^A\Big) - \Big(\nabla_{AA'}\widetilde{\psi}^{A'}\Big)\psi^A\Big]\sqrt{g}\, d^4x + \widehat{I}_B \quad . \tag{4.17}$$

This action is *real*, and the factor i occurs by virtue of the convention for Infeld-van der Waerden symbols used in [145, 150]. In (4.17) $\widehat{I}_B$ is a suitable boundary term, to be added to ensure that I_E is stationary under the boundary conditions chosen at the various components of the boundary (e.g. initial and final surfaces, as in [138]). Of course, the variation δI_E of I_E is linear in the variations $\delta\psi^A$ and $\delta\widetilde{\psi}^{A'}$. Defining $\kappa \equiv \frac{2}{i}$ and $\kappa\widehat{I}_B \equiv I_B$, variational rules for anticommuting spinor fields lead to

$$\begin{aligned} \kappa\Big(\delta I_E\Big) \;=\; & \int_M \Big[2\delta\widetilde{\psi}^{A'}\Big(\nabla_{AA'}\psi^A\Big)\Big]\sqrt{g}\, d^4x - \int_M \Big[\Big(\nabla_{AA'}\widetilde{\psi}^{A'}\Big)2\delta\psi^A\Big]\sqrt{g}\, d^4x \\ & - \int_{\partial M} \Big[{}_e n_{AA'}\Big(\delta\widetilde{\psi}^{A'}\Big)\psi^A\Big]\sqrt{h}\, d^3x + \int_{\partial M} \Big[{}_e n_{AA'}\widetilde{\psi}^{A'}\Big(\delta\psi^A\Big)\Big]\sqrt{h}\, d^3x \\ & + \delta I_B \end{aligned} \tag{4.18}$$

where I_B should be chosen in such a way that its variation δI_B combines with the sum of the two terms on the second line of (4.18) so as to specify what is fixed on the boundary (see below). Indeed, setting $\epsilon \equiv \pm 1$ and using the boundary conditions (4.12) one finds

$${}_e n_{AA'}\widetilde{\psi}^{A'} = \frac{\epsilon}{\sqrt{2}}\psi_A - \epsilon\; {}_e n_{AA'}\Phi^{A'} \text{ on } S^3 \quad . \tag{4.19}$$

Thus, anticommutation rules for spinor fields [138] show that the second line of equation (4.18) reads

$$\begin{aligned} \delta I_{\partial M} \;\equiv\; & -\int_{\partial M} \Big[\Big(\delta\widetilde{\psi}^{A'}\Big){}_e n_{AA'}\psi^A\Big]\sqrt{h}\, d^3x + \int_{\partial M} \Big[{}_e n_{AA'}\widetilde{\psi}^{A'}\Big(\delta\psi^A\Big)\Big]\sqrt{h}\, d^3x \\ =\; & \epsilon \int_{\partial M} {}_e n_{AA'}\Big[\Big(\delta\Phi^{A'}\Big)\psi^A - \Phi^{A'}\Big(\delta\psi^A\Big)\Big]\sqrt{h}\, d^3x \quad . \end{aligned} \tag{4.20}$$

Now it is clear that setting

$$I_B \equiv \epsilon \int_{\partial M} \Phi^{A'}\, {}_e n_{AA'}\, \psi^A \sqrt{h}\, d^3x \quad , \tag{4.21}$$

enables one to specify $\Phi^{A'}$ on the boundary, since

$$\delta\Big[I_{\partial M} + I_B\Big] = 2\epsilon \int_{\partial M} {}_e n_{AA'} \Big(\delta\Phi^{A'}\Big) \psi^A \sqrt{h}\, d^3x \quad . \tag{4.22}$$

Hence the action integral (4.17) appropriate for our boundary-value problem is

$$\begin{aligned} I_E &= \frac{i}{2}\int_M \Big[\widetilde{\psi}^{A'}\Big(\nabla_{AA'}\psi^A\Big) - \Big(\nabla_{AA'}\widetilde{\psi}^{A'}\Big)\psi^A\Big]\sqrt{g}\, d^4x \\ &+ \frac{i\epsilon}{2}\int_{\partial M} \Phi^{A'}{}_e n_{AA'}\, \psi^A \sqrt{h}\, d^3x \quad . \end{aligned} \tag{4.23}$$

Note that, by virtue of (4.12), equation (4.20) may also be cast in the form

$$\delta I_{\partial M} = \frac{1}{\sqrt{2}}\int_{\partial M} \Big[\widetilde{\psi}^{A'}\Big(\delta\Phi_{A'}\Big) - \Big(\delta\widetilde{\psi}^{A'}\Big)\Phi_{A'}\Big]\sqrt{h}\, d^3x \quad , \tag{4.24}$$

which implies that an equivalent form of I_B is

$$I_B \equiv \frac{1}{\sqrt{2}}\int_{\partial M} \widetilde{\psi}^{A'}\ \Phi_{A'}\sqrt{\det h}\, d^3x \quad . \tag{4.25}$$

The local boundary conditions studied at the classical level here, have been applied to one-loop quantum cosmology in [145, 150, 153]. Interestingly, the present result seems to add evidence in favour of quantum amplitudes having to respect the properties of the classical boundary-value problem. In other words, if fermionic fields are massless, their one-loop properties in the presence of boundaries coincide in the case of spectral [138, 145, 152] or local boundary conditions [145, 150, 153] while we find that classical modes for a regular solution of the Weyl equation obey the same conditions on a 3-sphere boundary with spectral or local boundary conditions, provided the spinor field $\Phi^{A'}$ of (4.12) is set to zero on S^3. Furthermore, the analysis presented in Eqs. (4.17)-(4.25) may clarify the spin-$\frac{1}{2}$ variational problem in the case of local boundary conditions on a 3-sphere as a particular matter-field analogue of the analysis in [149] for pure gravity.

FIVE

LOOP VARIABLES FOR EC-WEYL THEORY

In this chapter, we study the quantum fermions+gravity system, that is, the gravitational counterpart of QED. Based on the results of chapter 3 we look at a theory of a Weyl field coupled to gravity: the Einstein-Cartan-Weyl theory in self-dual form. We construct the related non-perturbative quantum theory by extending the Loop Representation of General Relativity as described in [22].

To this aim, we introduce the fermion equivalent to the loop variables, and we define the quantum theory as a representation of their Poisson algebra. Not surprisingly, the fermions can be incorporated in the Loop Representation by simply including open curves into "Loop space", as expected from the experience in lattice Yang-Mills theory. We explicitely construct the diffeomorphism and hamiltonain operators. The first can be fully solved as in pure gravity. The hamiltonian constraint admits the same simple geometrical interpretation as its pure gravity counterpart: it is the operator that shifts curves along themselves ("shift operator"). An explicit divergence afflicts the results when adopting this approach. This is highlighted at the end of the chapter. To cope with it will be the matter of chapter 6.

5.1 Classical fermion paths

As described in chapter 2, Non-perturbative quantum General Relativity can be constructed in terms of the loop variables

$$T[\alpha] \;:=\; U_A{}^A[\alpha] \equiv H[\alpha] \tag{5.1}$$

$$T^a[\beta](s) \;:=\; U_A{}^B[\beta](s)\,\tilde{\sigma}^a{}_B{}^A(\beta(s)). \tag{5.2}$$

We indicate loops by greek letters. A loop is here a closed continuous piecewise differentiable curve in Σ, $\alpha : S_1 \to \Sigma$; and s is the parameter along the loop: $\alpha : s \mapsto \alpha^a(s)$. We indicate by $U_A{}^B[\alpha](s)$ the parallel transport $SL(2,\mathbb{C})$ matrix of the Ashtekar connection around the loop α, starting from the parameter value s; that is, the path order exponential of the line integral of the connection around the loop:

$$U_A{}^B[\alpha] = \left[\mathcal{P}\exp \int_\alpha ds\; \dot{\alpha}^a(s)\, A_a\right]_A{}^B . \tag{5.3}$$

The properties of these loop variables were also discussed in chapter 2. In order to construct the extension of the loop representation to fermions, we want to generalize these loop variables to the presence of the spinor field. We want to define objects invariant under the action of the Gauss constraint. *Loop*-like variables in theories with connections involving fermions, have been dealt with by Rovelli and Smolin [154] for lattice Yang-Mills, and by Kim *et al* [155], for 2+1 gravity. For earlier related ideas, see [16, 156]. We follow here the lines of [154]. Let us consider piecewise differentiable continuous *open* curves in Σ. Since it turns out to be convenient, we will also call these open lines loops, certainly with an abuse of terminology; and we denote them too by means of greek letters: $\alpha : (0,1) \to \Sigma$; and $\alpha : s \mapsto \alpha^a(s)$. We denote as α_i and α_f the initial and final point of the loop, namely:

$$\begin{aligned} \alpha_i \;&:=\; \alpha(0), \\ \alpha_f \;&:=\; \alpha(1). \end{aligned} \tag{5.4}$$

If $\alpha_f = \beta_i$, we denote the open line obtained joining α and β as $\alpha\cdot\beta$; that is

$$[\alpha\cdot\beta](s) := \alpha(2s), \quad \text{if } s \in [0, 1/2] \tag{5.5}$$

$$[\alpha\cdot\beta](s) := \beta(2s-1), \quad \text{if } s \in [1/2, 1]. \tag{5.6}$$

We then define

$$X[\alpha] \;:=\; \psi^A(\alpha_i)\, U_A{}^B[\alpha]\, \psi_B(\alpha_f) \tag{5.7}$$

$$Y[\alpha] \;:=\; \tilde{\pi}^A(\alpha_i)\, U_A{}^B[\alpha]\, \psi_B(\alpha_f). \tag{5.8}$$

X and Y, are parametrized by open curves. They are defined as path integral exponentials of the Ashtekar connection along these curves, with Grassmann-valued spinor variables attached to the end points. They are $SU(2)-$invariant[25] as can be shown in a straightforward manner by recalling the transformations induced by the Gauss law constraint (*cf* section 3.4) on the canonical variables of the theory. Other important properties of the X and Y variables are the following.

1. They are invariant under a positive derivative monotonic reparametrization of the open loops (as it is readily deduced from the definition (5.3)).

2. X is invariant under inversion of the open loop, $X[\alpha^{-1}] = X[\alpha]$. This important property follows from the fact the fermions are Grassmann variables. In fact:

$$\begin{aligned}
X[\alpha^{-1}] &= \psi^A(\alpha_i^{-1})U_A{}^B[\alpha^{-1}]\psi_B(\alpha_f^{-1}) \\
&= \psi^A(\alpha_f)U_A{}^B[\alpha^{-1}]\psi_B(\alpha_i) \\
&= -\,\psi^A(\alpha_f)U_{AB}[\alpha^{-1}]\psi^B(\alpha_i) \\
&= +\,\psi^A(\alpha_f)U_{BA}[\alpha]\psi^B(\alpha_i) \\
&= -\,\psi^B(\alpha_i)U_{BA}[\alpha]\psi^A(\alpha_f) \\
&= +\,\psi^B(\alpha_i)U_B{}^A[\alpha]\psi_A(\alpha_f) \\
&= +\,\psi^A(\alpha_i)U_A{}^B[\alpha]\psi_B(\alpha_f) \\
&= X[\alpha]
\end{aligned} \tag{5.9}$$

In the third and sixth line we have used the spinor index property $\xi^A\rho_A = -\xi_A\rho^A$. In the fourth line we have used the parallel propagator property $U_{AB}[\alpha] = -U_{BA}[\alpha^{-1}]$

[25]They become $SL(2,\mathbb{C})$-invariants by choosing accordingly an $SL(2,\mathbb{C})$-connection thus describing complex general relativity.

(recall that if the parallel propagator $U_A{}^B[\alpha]$ is the identity, then $U_{AB}[\alpha] = \epsilon_{AB}$). In the fifth line, we have switched the two fermions, gaining a minus sign due to their Grassmannian character.

3. Retracing identity. As their closed loops counterparts, the fermionic loop variables X and Y satisfies the retracing and spinor identities that follows from their being defined in terms of parallel propagators of an $SU(2)$ connection. For instance, if the open curve α is formed by the three segments β, γ, δ with $\beta_f = \gamma_i = \delta_i$, as $\alpha = \beta \cdot \gamma \cdot \gamma^{-1} \cdot \delta$, then $X[\alpha] = X[\beta \cdot \delta]$.

4. Spinor identity. The following notation is useful. Let α be an (oriented) open line. We define

$$\alpha^A \ := \ \psi^B(\alpha_i)\, U_B{}^A[\alpha]. \tag{5.10}$$

Then, if $\alpha_f = \beta_f$, we can write

$$X[\alpha \cdot \beta^{-1}] = \alpha^A \beta_A \tag{5.11}$$

Now, consider a point p where 4 lines terminate, that is $\alpha_f = \beta_f = \gamma_f = \delta_f = p$. There are three possible ways of connecting these four lines to form two gauge invariant X variables:

$$X[\alpha \cdot \gamma^{-1}]X[\delta \cdot \beta^{-1}] \ = \ \alpha_A \gamma^A \delta_B \beta^B = \alpha^A \beta^B \gamma^C \delta^D \epsilon_{AC} \epsilon_{DB} \tag{5.12}$$

$$X[\alpha \cdot \delta^{-1}]X[\gamma \cdot \beta^{-1}] \ = \ \alpha_A \delta^A \gamma_B \beta^B = \alpha^A \beta^B \gamma^C \delta^D \epsilon_{AD} \epsilon_{BC} \tag{5.13}$$

$$X[\alpha \cdot \beta^{-1}]X[\gamma \cdot \delta^{-1}] \ = \ \alpha_A \beta^A \gamma_B \delta^B = \alpha^A \beta^B \gamma^C \delta^D \epsilon_{AB} \epsilon_{CD}. \tag{5.14}$$

By using the fundamental spinor identity, which is at the root of the Mandelstam relations

$$\epsilon_{AB} \epsilon_{CD} + \epsilon_{AD} \epsilon_{BC} + \epsilon_{AC} \epsilon_{DB} = 0, \tag{5.15}$$

we have the fermion version of the spinor identity, namely

$$X[\alpha \cdot \gamma^{-1}]X[\delta \cdot \beta^{-1}] + X[\alpha \cdot \delta^{-1}]X[\gamma \cdot \beta^{-1}] + X[\alpha \cdot \beta^{-1}]X[\gamma \cdot \delta^{-1}] \ = \ 0\ . \tag{5.16}$$

Similarly, it is simple to derive the identitity that refers to the intersection between the open loop $\alpha \cdot \beta$ and the closed loop γ, where $\alpha_f = \beta_i = \gamma_i = \gamma_f$:

$$X[\alpha \cdot \beta]\, T[\gamma] = X[\alpha \cdot \gamma \cdot \beta] + X[\alpha \cdot \gamma^{-1} \cdot \beta] \tag{5.17}$$

5. Fermionic(Grassmann) identities. Since Grassmann variables anticommute, and since the fermion field has only two components, if we multiply three or more fields in the same point we obtain zero. It follows that we can have products of X variables with at most two coinciding ends with Fermions. Thus, for instance, if $\alpha_i = \beta_i = \gamma_i$, then $X[\alpha]X[\beta]X[\gamma] = 0$.

6. Gauge observables algebra. Finally, the most important property of the X, Y, T, T^a variables is that their Poisson algebra closes. A direct computation yields

$$\begin{aligned} \{X[\beta], X[\alpha]\} &= 0 \\ \{Y[\beta], X[\alpha]\} &= \delta^3(\alpha_f, \beta_i)\, X[\alpha \cdot \beta] + \delta^3(\beta_i, \alpha_i)\, X[\alpha^{-1} \cdot \beta] \\ \{Y[\alpha], Y[\beta]\} &= \delta^3(\alpha_f, \beta_i)\, Y[\alpha \cdot \beta] - \delta^3(\alpha_i, \beta_f)\, Y[\beta \cdot \alpha] \end{aligned} \tag{5.18}$$

Whereas the nonvanishing brackets with the T variables are

$$\begin{aligned} \{T^a[\gamma](s), X[\alpha]\} &= i\Delta^a[\gamma(s), \alpha] \sum_{\mu=\pm 1} \mu\, X[\alpha \#_s \gamma^\mu] \\ \{T^a[\gamma](s), Y[\alpha]\} &= i\Delta^a[\gamma(s), \alpha] \sum_{\mu=\pm 1} \mu\, \tilde{Y}[\alpha \#_s \gamma^\mu]. \end{aligned} \tag{5.19}$$

As for pure gravity, one may define also higher order observables, which will have simple quantum operators associated. For instance we can insert "hands" (*i.e.* gravitational momentum variable) into the X variable, and so on. In order to treat the dynamics, the following variable will be particularly useful.

$$Y^a[\alpha](s) \;:=\; \pi^A(\alpha_i)\, U_A{}^B[\alpha](0,s)\, \tilde{\sigma}^a{}_B{}^C(\alpha(s))\, U_C{}^D[\alpha](s,1)\, \psi_D(\alpha_f), \tag{5.20}$$

where the notation $U_A{}^B[\alpha](s,t)$ indicates the matrix of the parallel transport along α from s to t. This variable is quadratic in the momenta, $\tilde{\pi}^A, {}^3\tilde{\sigma}^a{}_A{}^B$, and will play a role analogous to the role of the T^2 variable in pure gravity, which is also quadratic in the momenta. Indeed, we will use it for defining the hamiltonian constraint.

Finally, we may introduce a further small simplification in the notation: if the context is clear, we may write $X[\alpha]$ as $T[\alpha]$. Namely we can use the same notation, $T[\alpha]$, to indicate the the trace of the holonomy of α if α is a closed loop, and to indicate the parallel propagator along α sandwiched between two fermion fields if α is open.

5.2 QGD: kinematics

Let us now begin the construction of the quantum theory. Following the philosophy described in refs. [1, 6, 105], we look for a quantum representation of the classical loop algebra. Rather than simply writing the representation, it is perhaps more instructive to use the Loop Transform, introduced in ref. [1], as an heuristic device to help us in this task. Thus we first consider a "Schrödinger-like", or perhaps "Bargmann-like", representation of the quantum EC-Weyl theory. We consider functionals $\Psi[A,\psi]$ on the configuration space, and we define the canonical coordinates operators A and ψ as multiplicative operators, and the corresponding momentum operators as functional derivative operators

$$\hat{\tilde{\sigma}}^{a}{}_{A}{}^{B}(x) = \frac{\delta}{\delta A_{a}{}^{A}{}_{B}(x)}, \tag{5.21}$$

$$\hat{\tilde{\pi}}_{A}(x) = i\frac{\delta}{\delta\psi^{A}(x)}. \tag{5.22}$$

Since we are using this construction only as an heuristic tool to find a possible form for the fermion loops operators, we are not particularly concerned with mathematical rigor here. We focus on loop states in this representation. These are defined as follows. Given a closed loop α, we write

$$\Psi_{\alpha}[A,\psi] = U[\alpha]_{A}{}^{A}. \tag{5.23}$$

In the case of an open loop α, we have

$$\Psi_{\alpha}[A,\psi] = \psi^{A}(\alpha_i)\, U[\alpha]_{A}{}^{B}\, \psi^{B}(\alpha_f). \tag{5.24}$$

Thus, for an arbitrary collection β of a finite number of open or closed loops $\alpha_1, \alpha_2, ..., \alpha_n$, we use

$$\Psi_{\beta} = \Psi_{\alpha_1}\Psi_{\alpha_2}...\Psi_{\alpha_n}. \tag{5.25}$$

Note that we follow here the usual convention of denoting loops (open or closed) and multiple loops (namely collections of a finite number of loops) by means of the same notation, that is greek letters from the begining of the alphabet. The idea of the loop representation is to take the Ψ_β states as an overcomplete basis of quantum states. We thus introduce the (ill defined) Loop Transform [1, 6] as follows.

$$\Psi[\beta] := \int \mathcal{D}A\mathcal{D}\psi \ \Psi_\beta[A,\psi] \ \Psi[A,\psi]. \tag{5.26}$$

Here $\Psi[A,\psi]$ represents a generic wave functional in the connection representation, and $\Psi[\beta]$ represents its Loop Transform. The novel features entering here come from the inclusion of fermions; states containing no fermions are described exactly as in the pure gravity case [1]. The transform (5.26) has a well defined meaning on the lattice, where the Ψ_β states are the Wilson-Susskind states. In the lattice case it is possible to show that the transform defines a unitary transformation to a new basis in the Hilbert space of the theory. The mathematics of the Loop representation in the continuum is under investigation, with extremely encouraging results by Ashtekar, Lewandowski and others [33].

Eq.(5.26) suggests that we may look for a representation of the full EC-Weyl loop algebra on a space of functionals of multiple loops, where the multiple loops are sets of closed as well as open loops. Then the transform gives us immediately the action of the X and Y operators in the loop representation

$$\begin{aligned}
\left(\widehat{X}[\alpha]\Psi\right)[\beta] &= \int \mathcal{D}A\mathcal{D}\psi \, \Psi_\beta[A,\psi] \left(\widehat{X}[\alpha]\Psi\right)[A,\psi] \\
&= \int \mathcal{D}A\mathcal{D}\psi \, \Psi_\beta[A,\psi]\Psi_\alpha[A,\psi]\Psi[A,\psi] \\
&= \int \mathcal{D}A\mathcal{D}\psi \, \Psi_{\beta\cup\alpha}[A,\psi]\Psi[A,\psi] \\
&= \Psi[\beta\cup\alpha]\,;
\end{aligned} \tag{5.27}$$

where the union operaton $\cup$ of set theory is well defined between the multiple loop β and the single open loop α. Thus we have

$$\widehat{X}[\alpha]\Psi[\beta] = \Psi[\beta\cup\alpha]. \tag{5.28}$$

Note that this is essentially the same action as the action of $T[\alpha]$ in pure gravity. Using the notation suggested at the end of the previous section, we can write

$$\widehat{T}[\alpha]\Psi[\beta] = \Psi[\beta \cup \alpha] \tag{5.29}$$

for open as well as for closed α's.

As far as Y is concerned, we have

$$\begin{aligned} \left(\widehat{Y}[\alpha]\Psi\right)[\beta] &= \int \mathcal{D}A\mathcal{D}\psi\ \Psi_\beta[A,\psi]\left(\widehat{Y}[\alpha]\Psi\right)[A,\psi] \\ &= \int \mathcal{D}A\mathcal{D}\psi\ \left(\widehat{Y}[\alpha]\Psi_\beta\right)[A,\psi]\ \Psi[A,\psi] \\ &= \int \mathcal{D}A\mathcal{D}\psi\ \left(\psi_B(\alpha_f)U_A{}^B[\alpha]\frac{\delta}{\delta\psi^A(\alpha_i)}\Psi_\beta\right)[A,\psi]\ \Psi[A,\psi] \\ &= \sum_{\beta_f}\delta^3(\alpha_i,\beta_f)\,\Psi[\alpha\cdot\beta] + \sum_{\beta_i}\delta^3(\alpha_i,\beta_i)\,\Psi[\beta\cdot\alpha^{-1}]. \end{aligned} \tag{5.30}$$

Here $\sum_{\beta_i}$ indicates the sum over all the initial points of the open loops in the multiple loop β, and $\sum_{\beta_f}$ indicates the sum over all the final points of the open loops in the multiple loop β. We introduce the following notation. We write β_e to indicate any end point of the multiple loop β. If $\beta_e = \alpha_i$, the notation $\beta\cdot_e\alpha$ indicates the multiple loops obtained by attaching α_i with β_e. Thus we have

$$\widehat{Y}[\alpha]\ \Psi[\beta] = \sum_{\beta_e}\delta^3(\alpha_i,\beta_e)\ \Psi[\beta\cdot_e\alpha]. \tag{5.31}$$

In words, the action of the operator $Y[\alpha]$ is simply to attach the open loop α to any open end that happens to be in the point α_i.

Next, we supplement (5.28), (5.31) with the usual quantum T-variables in the loop representation [1, 6]. The computation of the quantum commutation relations of the entire set is then straigthforward. The result is that the set of operators $\hat{X},\hat{Y},\hat{T},\hat{T}^a$ provides a representation of the classical algebra (5.18), (5.19) and the classical $\mathcal{T}$–algebra.

We can also naturally introduce quantum operators corresponding to higher order loop variables. As their pure gravity counterparts, these have a quantum algebra that reduces to the corresponding classical Poisson algebra in the limit in which the Planck constant goes to zero. We write here the quantum operator corresponding to the Y^a variable defined above, since it will be used in the construction of the Hamiltonian

$$
\begin{aligned}
Y^a[\alpha](s)\ \Psi[\beta] &= \sum_{\beta_e} \delta^3(\beta_e, \alpha_i) \int_\beta dt\ \dot{\beta}^a(t) \delta^3(\alpha(s), \beta(t)) \\
&\quad \left[\Psi(\alpha * *^{+}_{e,t}\beta) + \Psi(\alpha * *^{-}_{e,t}\beta)\right] = \\
&= \sum_{\beta_e} \delta^3(\beta_e, \alpha_i) \int_\beta dt\ \dot{\beta}^a(t) \delta^3(\alpha(s), \beta(t)) \\
&\quad \sum_{q=\pm} \Psi(\alpha * *^{q}_{e,t}\beta)
\end{aligned}
\tag{5.32}
$$

Here we have indicated by $\alpha**^{+}_{e,t}\beta$ and $\alpha**^{-}_{e,t}\beta$ the two loops obtained by joining the β_e end point of β with the point α_i, and rerouting the intersection $\alpha(t) = \beta(s)$ in the two possible ways. Note the plus sign between the two terms, which will play an important role in what follows. The expression (5.32) can be obtained for instance from the loop transform. To determine the correct overall coefficient and sign, an accurate computation with the SU(2) index algebra is needed. Note however, that the relative plus sign between the two terms in parenthesis is forced by symmetry, since neither of the two can be preferred.

5.3 Diffeomorphisms and diff-invariant states

The classical vector constraint generates spatial diffeomorphisms when acting on gauge invariant objects. This is also true in the quantum theory provided the correct ordering of the vector constraint quantum operator is chosen. Precisely as in the pure gravity theory, there are several fully equivalent ways for reaching this conclusion:

1. As suggested by Isham [105], the vector constraint can simply be defined as the generator of the natural action of the diffeomorphism group on the space of the open and closed loops. The commutator algebra of these generators among themselves and with all the other operators in the theory, then, reproduces the corresponding classical Poisson algebra. This is a sufficient condition to ensure that the classical limit of the quantum theory that we are constructing reproduces the classical theory we started from. Since the correct classical limit is the *sole* requirement we have on the theory, the quantum diffeomorphism constraint defined in this way represents a consistent quantization of its classical counterpart.

2. We can use the transform, and define the loop representation vector constraint operator as the transform of the vector constraint operator in the representation that diagonalizes A and ψ.

3. We can express the classical vector constraint in terms of loop variables, as a suitable limit of a sequence of these variables. The corresponding quantum constraint is then defined as the limit of the corresponding quantum loop operators.

As in pure gravity, it is not difficult to show that these different strategies yield the same quantum diffeomorphism constraint, and that this is can be expressed as follows. For every diffeomorphism $\phi \in \mathrm{Diff}[\Sigma]$

$$\Psi[\alpha] \;=\; \Psi[\phi\cdot\alpha] \tag{5.33}$$

where $[\phi\cdot\alpha](s) := \phi(\alpha(s))$. The general solution of the diffeomorphism constraint is given by the loop functionals constant along the orbits of the action of the Diff group on the loop space, namely, they are given by

$$\Psi[\alpha] \;=\; \Psi[K(\alpha)] \tag{5.34}$$

where $K(\alpha)$ is a generalized knot class, that is, an equivalent class under diffeomorphisms of sets of graphs formed by open and closed lines.

For every generalized knot class K, we can define a corresponding quantum state Ψ_K as the reprsentative function of the class. We will also use a Dirac notation $\Psi[\alpha] = \langle\alpha|\Psi\rangle$, and denote the state Ψ_K as $|K>$. Thus

$$\begin{aligned}\langle\alpha|K\rangle &= 1 \quad \textit{if} \quad K = K(\alpha),\\ \langle\alpha|K\rangle &= 0 \quad \textit{otherwise},\end{aligned} \tag{5.35}$$

Let us begin here some preliminary investigation of the structure of the ensemble of quantum states $|K>$.

Consider a fixed class K. Let α be one of the (diff-equivalent) multiple loops that belongs to K. Let α be composed by c closed loops and o open loops. There are $2o$ end points α_e in α. We distinguish the end points as simple or doubles. An end-point α_e is simple if there is no other end point α'_e in α such that $\alpha_e = \alpha'_e$. It is double otherwise. Note that there cannot be "triple" end points, because of the Pauli principle. Let S and D be the number of simple and double end points. And $N = S + 2D = 2o$ be the number of end points.

Consider the points i in the image of α such that one or more than one of the following is true:

1. α is non injective in i (*i.e.* it is an intersection point),
2. i is an endpoint,
3. α is non differentiable in i (*e.g.* a kink) ;

we denote these points i as *generalised intersections*, or, simply as *intersections*. We assume that the number of these intersections is finite, and we denote this number as I. Given an intersection i, we assume there is only a finite number of components of α coming out of it. We denote this number as m_i, and we call it the *order* of the intersection. Intersections of order 1 are single end points. Intersections of order 2 are either double end points or kinks along a loop. Intersections or order 3 are single end points that fall over a loop. Intersections of order 4 are either crossings of two loops, or a double endpoint that falls over a loop, and so on. We call the intersections of order one *free* end-points.

Consider an intersection i of order m_i in a loop α. Let $\alpha_j(s)$, with $j = 1...m_i$, and $\alpha_j(0) = i$, be the m_i lines (components of the loop α) that come out from i. Let $\vec{l}_j$, $j = 1...n(i)$ be the $n(i)$ tangents of $\alpha_j(s)$ in i. Let then $\vec{l}_j^{(k)}$, where k is a positive integer, the kth derivative of the jth component of the loop in i. For instance, if $m_i = 1$, and $i = \alpha(0)$, then $\vec{l}_1 = d\alpha(s)/ds$, and $\vec{l}_1^{(2)} = d^2\alpha(s)/ds^2$, and so on.

The vectors $\vec{l}_j^{(k)}$ transform among themselves under reparametrization of α and under diffeomorphisms. Considering any possible intersection of order m_i, we denote the space of the equivalence classes of the (full collection of) $\vec{l}_j^{(k)}$ vectors, under reparametrization of α and under diffeomorphisms, as the moduli space of the intersection of order m_i. The moduli space of an intersection of order $n < 5$ is discrete; this is not so, in general, for larger n. However, the moduli space is always finite dimensional. Let $a_i^{(m_i)}$ be a collection of parameters that coordinatizes this moduli space, as well as characterizing the rootings of α through the i intersection.

Finally, let $\mathcal{K}^c_P$ be a discrete index that labels the braids with P (ordered) open hands and c closed loops. We can (over-) characterise a quantum state as (see ref. [5]):

$$|N, I, D;\ \ a_1^{m_1}....a_I^{m_I};\ \ \mathcal{K}^o_{\sum_i m_i}\rangle \tag{5.36}$$

where, we recall, N is the number of end-points, I is the number of intersections, D is the number of double hands, $m_1....m_I$ are the orders of the I intersections, $a_1^{(m_1)}....a_I^{(m_I)}$ are the moduli space parameters of the intersections and $\mathcal{K}^o_{\sum_i m_i}$ is the discrete topological class of the braid obtained by taking away the intersection points from the loop.

5.4 The hamiltonian constraint has a simple action

Our last and main task is to deal with the dynamics, which is contained in the hamiltonian constraint. (The Hamiltonian constraint of the pure gravity Loop Representation was discussed in [1, 157, 158]. For a comprehensive and critical discussion of the various approaches see ref [159].) We shall perform this task in two stages. In the present section we introduce a simple and naive non- regularized definition of the quantum hamiltonian constraint. This is not really satisfactory because it does not allow us to control the divergences of the theory. However, we think it is usefull to present this non-regularized version of the dynamics first, because it allows one to appreciate the striking simplicity of the geometrical action of the EC-Weyl Hamiltonain constraint, which otherwise could improperly appear as an improbable product of the technicalities of the regularization procedure. In the next cahpter, we will transform the formal outcome we obtain here in a more solid result.

Thus, we begin by defining the hamiltonian constraint by using the simplest procedure, namely we define it in the connection representation and we transform it to the Loop Representation by using the Loop Transform. We choose the ordering in which the momenta are always to the right of the configuration variables. Taking into account the Grassmann character of the fermionic variables, the Hamiltonian constraint, smeared against a test (inverse density) scalar $\underset{\sim}{N}(x)$ is

$$\begin{aligned}\widehat{H}[\underset{\sim}{N}] &= -\int_\Sigma d^3x\ \underset{\sim}{N}\,\mathrm{Tr}\left(F_{ab}\widehat{\widetilde{\sigma}}^a\widehat{\widetilde{\sigma}}^b\right) - i\sqrt{2}\,\widehat{\widetilde{\sigma}}^a{}_A{}^B\,\mathcal{D}_a\psi^A\,\widehat{\widetilde{\pi}}_B \\ &= -\int_\Sigma d^3x\ \underset{\sim}{N}(x)\mathrm{Tr}F^A_{abB}(x)\frac{\delta}{\delta A^C_{aB}(x)}\frac{\delta}{\delta A^A_{bC}(x)} \\ &\quad -i\sqrt{2}\ \mathcal{D}_a\psi^A(x)\ \frac{\delta}{\delta A^A_{bB}(x)}\ \frac{\delta}{\delta\psi^B(x)}\end{aligned} \tag{5.37}$$

In order to compute the Loop transform of this operator, we have to compute its action on the kernel of the Loop Transform, that is, on the basis loop states $\Psi_\alpha[A,\psi]$. We need to compute

$$\widehat{H}[\underset{\sim}{N}]\ \Psi_\alpha[A,\psi]. \tag{5.38}$$

If α is formed by closed loops alone, then $\Psi_\alpha[A,\psi]$ is independent from ψ and therefore the second term in (6.4) does not act. It follows immediately that the action of $\widehat{H}[\underset{\sim}{N}]$ on the closed loop states is fully equivalent to the action of the pure gravity hamiltonian constraint. Let us then assume α is a single open loop. A straigthforward calculation gives

$$\begin{aligned}&\widehat{H}[\underset{\sim}{N}]\ \Psi_\alpha[A,\psi] = \\ &\quad -\int_0^1 dt\ \int_0^1 ds\,\underset{\sim}{N}(\alpha(t))\delta^3(\alpha(s),\alpha(t))\dot{\alpha}^a(s)\ \dot{\alpha}^b(t) \\ &\quad \psi^G(\alpha_i)\,U_{G[F}(0,t)F_{ab}{}^{FB}(\alpha(t))U_{B]H}(t,1)\,\psi^H(\alpha_f) \\ &\quad -\int_0^1 ds\,\underset{\sim}{N}(\alpha_i)\delta^3(\alpha(s),\alpha_i)\,\dot{\alpha}^a(0)\,\mathcal{D}_a\psi^A(\alpha_i)U_{AE}[\alpha]\,\psi^E(\alpha_f) \\ &\quad +\int_0^1 ds\,\underset{\sim}{N}(\alpha_f)\delta^3(\alpha(s),\alpha_f)\,\dot{\alpha}^a(1)\,\psi^D(\alpha_i)U_{DA}[\alpha]\,\mathcal{D}_a\psi^A(\alpha_f).\end{aligned} \tag{5.39}$$

We immediately see the difficulty in this equation: the three- dimensional delta functions are integrated only against two line integrals, leaving a divergent factor. As we said, let us disregard this infinity for the moment.

We now recall from ref.[1] that, if we disregard divergences, the action of the Hamiltonian operator on a pure loop state can be written as

$$\widehat{H}[\underset{\sim}{N}]\ \Psi_\alpha = \mathcal{S}[\underset{\sim}{N}]\Psi_\alpha \tag{5.40}$$

where $\mathcal{S}$, denoted as the "Shift operator", is a simple operator *acting on the loop argument*, as

$$\mathcal{S}[\underset{\sim}{N}]\Psi_\alpha := \int_0^1 ds \int_0^1 dt\ \underset{\sim}{N}(\alpha(s))\dot{\alpha}^a(s)\delta^a(\alpha(s),\alpha(t))\frac{\delta}{\delta\alpha^a(t)}\ \Psi_\alpha. \tag{5.41}$$

If α has no self-intersections, and up to a divergent factor k, the Shift operator becomes simply

$$\mathcal{S}[\underset{\sim}{N}]\Psi_\alpha = k \lim_{t\to 0} \Psi_{\alpha_{t\underset{\sim}{N}}} \tag{5.42}$$

where

$$\alpha^a_{t\underset{\sim}{N}}(s) := \alpha^a(s) + t\underset{\sim}{N}(\alpha(s))\dot{\alpha}^a(s) \tag{5.43}$$

The action of the Shift operator has thus a very simple geometrical interpretation: it shifts the loop ahead along its tangent. Clearly, it sends a smooth closed loop into itself, while it deforms a loop with kinks or intersections. This simple action is one of the ways in which one can (formally) understand the well known result that loop states corresponding to smooth non intersecting loops are in the kernel of the Hamiltonian constraint. The fact that the action of the hamiltonian constraint on the kernel of the Loop Transform can be expressed in terms of an operator acting on the loop variable enables us to interpret this operator as the operator that represents the Hamiltonian constraint in the Loop Representation [1]. Thus in pure gravity the Hamiltonian constraint can be simply expressed as the Shift operator.

How does the picture change if we include the fermions ? The striking result that we have mentioned in the introduction is that the picture *does not change at all*: Eq (5.40), which expresses the Hamiltonian as the Shift operator still holds. The shift operator being still given by equation (5.41), where now we also allow the loops to be open, or in the absence of intersections, by equation (5.42). Indeed, by computing the action of $\mathcal{S}[\underset{\sim}{N}]$, as defined in equation (5.41) on an *open* loop state, we get precisely the right hand side of equation (5.39). What happens is that the fermion term in the classical hamiltonian constraint give rise to the second and third term in equation (5.39) and these terms are precisely the terms that "move" the two fermions at the end of the loop in the correct direction !

Thus we have the following result:

- In pure gravity the action of the hamiltonian constraint on Loop Space can be expressed as the action of the Shift operator (5.41), which simply shifts non-intersecting loops along their own tangent.

- By applying this *same* geometrical operator on *open* loop states, we have the action of the EC-Weyl hamiltonian constraint.

We have been deeply puzled by this result, and we do not see any simple way of interpreting it. We have not been able to find any reason for which this result could be understood in terms of the "classical" Fermion dynamics. Its simplicity seems to us an indication of something, but we have not been able to decode the indication. We shall study in the next chapter how we can free the result from the divergences, *i.e.* the way to clean it up from the divergent factor k.

SIX

QUANTUM GRAVITATIONAL DYNAMICS

This chapter deals with the problem of making more rigourous in both physical and technical senses the analysis of the Hamiltonian constraint operator found in chapter 5 for the EC-Weyl theory using the Loop representation. This is based on ref. [22]. In particular, we aim to adopt a viewpoint in which there are no divergences present as it was the case in chapter 5.

To unravel the dynamics of the theory we study the evolution of the fermion-gravity system in the physical-time defined by an additional coupled ("clock"-) scalar field. We explicitely construct the Hamiltonian operator that evolves the system in this physical time. We show that this Hamiltonian is finite, diffeomorphism invariant, and has a simple geometrical action confined to the intersections and the end points of the "loops". The quantum theory of fermions+gravity evolving in the clock time is finally given by the combinatorial and geometrical action of this Hamiltonian on a set of graphs with a finite number of end points.

6.1 Coupling a clock field

To cope with the difficulties found in trying to define a sensible quantum operator for the Hamiltonian constraint of the EC-Weyl theory, we now modify our point of view and consider a richer theory: we couple a scalar field to EC-Weyl theory. We use a scalar field in order to define a physical internal time, or clock-time, in terms of which we can study physical evolution.

We have two independent motivations for choosing this road. First this procedure allows us to unravel the physics of the general covariant quantum theory, otherwise hidden in the frozen-time formalism, as discussed in detail for instance in refs.[9, 10]. Second, this is a way to overcome the divergence difficulties we had in the previous chapter. Indeed, we have learned from the experience in pure Quantum GR that non-diffeomorphism-invariant operators tend to be ill-defined in a generally covariant quantum field theory, while all the diffeomorphism invariant operators that we have been able to construct have good finiteness properties [2]. Thus, we wish to replace the Hamiltonian constraint with some diffeomorphism invariant operator [the Hamiltonian constraint, being a scalar (density) is diffeomorphism covariant, not diffeomorphism invariant]. As shown in ref.[5], the coupling of a scalar clock field and the replacement of the hamiltonian constraint with an hamiltonian is a way to achieve this result. The hamiltonian that generates the evolution in the clock time is a diffeomorphism invariant quantity and replaces the hamiltonian constraint.

We refer to [5] for the details of the scalar field construction and the gauge fixing that allows to define the hamiltonian. We simply recall here the main idea, so that this chapter can be independently read. Physical quantum states are represented (say in the connection representation) as functionals $\Psi[A, \psi]$ of the spatial fields $A(\vec{x}), \psi(\vec{x})$, satisfying the Wheler-DeWitt equation. As it is well known, the time coordinate t disappears from this frozen time formalism. In principle, the disappearence of the coordinate time is not a problem, since the observables of the theory must be 4-dimensional general covariant *anyway* , and thus, in particular, must be independent from t. Examples of these 4-dimensional general covariant observables are given by the invariant distance d_p of the solar system planets from the Earth, seen as a function dependent from, say, the invariant distance d of the Earth from the sun. In practice, however, it is notoriously too difficult to write the dynamical equations of GR *directly* in terms of coordinate invariant quantities: in classical GR we almost always work with coordinate dependent quantities, and extract coordinate invariant predictions only after the dynamics has been fully worked out in a particular gauge. For instance, we study the motion of the planets, including perhaps emitted gravitational radiation, in some arbitrary coordinate system (with some arbitrary coordinate time t: $d_p(t)$ and $d(t)$), and only when the dynamics has been solved we compute coordinate invariant quantities $d_p(d)$, which can be compared with astronomical observations. The quantum frozen time formalism, however, does not allow us to work with quantities dependent on the fictitious arbitrary coordinate time t, and thus makes the dynamical analysis particularly cumbersome: what one should do is to view the physical states $\Psi[A,\psi]$ as coding the quantum evolution of any variable in terms of any other variable. In general this is not easy.

The problem can be simplified by studying a version of the theory in which there is a simple quantity to be taken as the independent variable; that is, in which we know from the scratch which variable we want to use as the "clock variable". We thus introduce a scalar field $T(\vec{x},t)$, and we decide to study the evolution of the gravitational and fermions degrees of freedom, as they evolve in the value of T. If $A(\vec{x},t), \psi(\vec{x},t), T(\vec{x},t)$ is a solution of the field equations, we extract information invariant under a time coordinate transformation by solving t with respect to T, and substituting the resulting $t(\vec{x},T)$ in A and ψ: We get the two functions

$$\begin{aligned} A(\vec{x},T) &= A(\vec{x},t(\vec{x},T)), \\ \psi(\vec{x},T) &= \psi(\vec{x},t(\vec{x},T)), \end{aligned} \tag{6.1}$$

which are invariant under coordinate time reparametrization. Equivalently, we choose a (physical) coordinate system in which $T(\vec{x},t) = t$.

In the quantum theory, the frozen time formalism is defined in terms of the functionals $\Psi[A,\psi,T]$ of the spatial fields $A(\vec{x}), \psi(\vec{x}), T(\vec{x})$. We can interpret these states as giving the amplitude for a $A(\vec{x}), \psi(\vec{x})$ configuration *at the given configuration* $T(\vec{x})$ *of the clock* *field* , and to interpret the Wheeler-DeWitt equation as an evolution equation in the multifingered time $T(\vec{x})$. We can then further fix the multifingered $T(\vec{x})$ time as a constant function $T(\vec{x}) = T$, (we keep the same letter T to indicate also the real number T, besides the function $T(\vec{x})$), and restrict $\Psi[A, \psi, T]$ (*functional* of the field $T(\vec{x})$) to $\Psi[A, \psi](T) = \Psi[A, \psi, T]|_{T(\vec{x})=T}$ (*function* of the real numeber T), without any loss of information. The state $\Psi[A, \psi](T)$ expresses the quantum amplitude for the evolution in T of the fields $A(\vec{x}, T), \psi(\vec{x}, T)$ defined above. Moreover, if $\Psi[A, \psi, T]$ satsfies the Wheeler-DeWitt constraint, it is shown in [5] that $\Psi[A, \psi](T)$ satisfies the equation

$$i\hbar \frac{\partial}{\partial T}\ \Psi[A, \psi](T) = \widehat{H}\ \Psi[A, \psi](T), \tag{6.2}$$

where the operator $\widehat{H}$ will be defined in a moment. We can view equation (6.2) as a genuine Schrödinger equation which evolves in the time T.

More precisely, we can express the Einstein-Weyl-scalar-field theory in a canonical gauge-fixed form, in terms of the configuration variables $A_{aA}{}^{B}$ and ψ^{A}, the constraints

$$\begin{aligned} \widetilde{G}_{AB} &= -i\sqrt{2}\,\mathcal{D}_b \widetilde{\sigma}^b{}_{AB} + \widetilde{\pi}_{(A}\psi_{B)} \\ \widetilde{V}_a &= -i\sqrt{2}\,\mathrm{Tr}\left(\widetilde{\sigma}^b F_{ab}\right) - \widetilde{\pi}_A \mathcal{D}_a \psi^A \end{aligned} \tag{6.3}$$

and the *hamiltonian* (as opposed to hamiltonian constraint)

$$H = \int d^3x\ \sqrt{-\mathrm{Tr}\,(\widetilde{\sigma}^a \widetilde{\sigma}^b F_{ab}) + i\sqrt{2}\,\widetilde{\sigma}^a{}_A{}^B\, \widetilde{\pi}_B \mathcal{D}_a \psi^A}. \tag{6.4}$$

It is shown in [5] that the solutions of this theory, $A(\vec{x}, T), \psi(\vec{x}, T)$, are related to the solutions of the full Einstein- Cartan-Weyl-scalar-field theory via equation (6.1). We can define the kinematics of the quantum theory and treat the diffeomorphism constraint exactly as in the chapter 5, but now we do not have a quantum hamiltonian constraint, but rather a Schödinger equation (6.2) and a quantum hamiltonian $\hat{H}$, which is the quantum operator corresponding to the classical variable H given in (6.4).

The next section is devoted to the construction of the quantum operator $\widehat{H}$. This time we will not be content with formal manipulations of divergent expressions, and we require a somehow higher level of rigor.

6.2 Definition of the regularized hamiltonian

Following ref.[5], our first step is to define a regularised classical hamiltonian in terms of loop variables. We introduce a fictitious background flat metric and a preferred set of coordinates in which this metric is euclidean; in the following everything is written in those coordinates and we use the euclidean metric to define norms of vectors and raise and lower vector indices. It will be our task, later, to show that the operator we define is independent of the regularization background metric introduced here. We write

$$H = \lim_{\substack{L\to 0\ \delta\to 0 \\ A\to 0\ \tau\to 0}} H_{L\delta A\tau}, \tag{6.5}$$

$$H_{L\delta A\tau} = \sum_I L^3 \sqrt{-C^{A,L,\delta}_{\text{Einstein }I} - C^{L,\tau,\delta}_{\text{Weyl }I}}\ . \tag{6.6}$$

Here we have partitioned three-dimensional space into cubes of sides L, labelled by the index I. The quantity $C^{A,L,\delta}_{\text{Einstein }I}$ is the regularized form of the pure gravity hamiltonian constraint (integrated over the I-th cube), and has been defined in [5].

With the purpose of defining the fermion term $C^{L,\tau,\delta}_{\text{Weyl }I}$, we begin by defining the open loop $\gamma^{\tau}_{\vec{x},\vec{y}}$, where τ is a regularization parameter (to be taken small), $\vec{x}$ is a point in space and $\vec{y}$ is a vector in the tangent space around $\vec{x}$. Since we have a background flat metric we can write expressions as $\vec{x} + s\vec{y}$, for any real s, with obvious meaning. The open loop $\gamma^{\tau}_{\vec{x},\vec{y}}$ is defined as the (uniformly parametrized) straight line (in the background metric) that starts at $\vec{x}$ in the $\vec{y}$ direction and is long τ. That is

$$(\gamma^{\tau}_{\vec{x},\vec{y}})^a(0) = x^a, \quad (\dot{\gamma}^{\tau}_{\vec{x},\vec{y}})^a(s) = \frac{\tau}{|\vec{y}|}y^a. \tag{6.7}$$

Note that

$$(\gamma^{\tau}_{\vec{x},\vec{y}})^a(1) = x^a + \frac{\tau}{|\vec{y}|}y^a \tag{6.8}$$

and

$$(\gamma^{\tau}_{\vec{x},\vec{y}})^a(|\vec{y}|/\tau) = x^a + y^a. \tag{6.9}$$

By making use of this loop, we define the regularised fermion hamiltonian constraint by

$$C^{L,\tau,\delta}_{\text{Weyl }I} \;=\; \frac{1}{L^3}\int_I d^3x\ C^{\tau,\delta}_{\text{Weyl}}(\vec{x})\ , \tag{6.10}$$

$$C^{\tau,\delta}_{\mathrm{Weyl}}(\vec{x}) = c_{\tau\delta} \int d^3y \ \theta(\delta - |\vec{y}|) \ \frac{y^a}{|\vec{y}|} \ Y^a \left[\gamma^{\tau}_{\vec{x}\vec{y}}\right] (|\vec{y}|/\tau) \, . \tag{6.11}$$

$$c_{\tau\delta} = \frac{3}{\tau} \frac{1}{\frac{4}{3}\pi\delta^3} \, . \tag{6.12}$$

Here $\theta(x)$ is the conventional step function, that is the characteristic function of R^+. Note the three different roles of the three regularization parameters L, τ and δ: the parameter L fixes the size of the boxes. As we will see, the introduction of these boxes will allow us to deal with the square root. For every space point x, the fermion term of the Hamiltonian constraint $C_{\mathrm{Weyl}}(\vec{x})$ is approximated by means of the loop variable Y^a corresponding to the "small" loop $\gamma^{\tau}_{\vec{x}\vec{y}}$, that starts at x. The parameter τ gives the length of the "small" loop. The direction of this loop is integrated over (d^3y angular integration). Y^a has a special point where the gravitational "hand" is inserted. This point is chosen to be $\gamma^{\tau}_{\vec{x},\vec{y}}(|y|/\tau) = x^a + y^a$. Thus, also the position of this point (the hand) is integrated over (d^3y radial integration). This smearing of the position of the hand is what determines the point split of the functional derivatives operators in the quantum theory. Note that the d^3y integral is restricted (by the θ function) to a ball of size δ around x. Thus, the parameter δ gives the point splitting separation between the initial point of the loop and its hand. Note that in order this definition to make sense we must have $\tau > \delta$.

A straigthforward expansion in τ and δ shows that equation (6.5) is satisfied, namely that the quantity that we have defined represents a genuine regularization of the Hamiltonian. For completness, let us sketch this expansion. We begin by writing the regularized expression explicitely, by using the expression for Y^a, given in eq.(5.20)

$$\begin{aligned} C^{\tau,\delta}_{\mathrm{Weyl}}(\vec{x}) &= c_{\tau\delta} \int d^3y \ \theta(\delta - |\vec{y}|) \ \frac{y^a}{|\vec{y}|} \ \pi^A(\gamma^{\tau}_{\vec{x}\vec{y}i}) U_A{}^B \left[\gamma^{\tau}_{\vec{x}\vec{y}}\right] (0, |\vec{y}|/\tau) \\ &\quad \tilde{\sigma}^a{}_B{}^C (\gamma^{\tau}_{\vec{x}\vec{y}}(|\vec{y}|/\tau)) U_C{}^D \left[\gamma^{\tau}_{\vec{x}\vec{y}}\right] (|\vec{y}|/\tau, 1) \psi_D(\gamma^{\tau}_{\vec{x}\vec{y}f}). \end{aligned} \tag{6.13}$$

By using the explicit definition of the loop $\gamma^{\tau}_{\vec{x}\vec{y}}$, this becomes

$$\begin{aligned} C^{\tau,\delta}_{\mathrm{Weyl}}(\vec{x}) &= c_{\tau\delta} \int d^3y \ \theta(\delta - |y|) \ \frac{y^a}{|\vec{y}|} \ \pi^A(\vec{x}) U_A{}^B \left[\gamma^{\tau}_{\vec{x}\vec{y}}\right] (0, |y|/\tau) \\ &\quad \tilde{\sigma}^a{}_B{}^C (\vec{x} + \vec{y}) U_C{}^D \left[\gamma^{\tau}_{\vec{x}\vec{y}}\right] (|y|/\tau, 1) \psi_D(\vec{x} + \tau\vec{y}/|\vec{y}|). \end{aligned} \tag{6.14}$$

We may now expand everything in powers of δ and τ around the point $\vec{x}$. Before doing so, however, let us note what follows. Since we have a (d^3y) integration over a volume of order δ^3, and we divide by δ^3 (in $c_{\tau\delta}$) before taking the limit $\delta \to 0$, only the term in the integrand of zero order in δ may survive the limit. We also divide by τ and take the limit $\tau \to 0$, thus only terms of first order in τ survive in the limit. This means that to the relevant order we may replace quantities in $\vec{x} + \vec{y}$ by quantities in $\vec{x}$ (recall the d^3y integration is over a sphere of radius δ), and quantities in $\vec{x} + \tau\vec{y}/|\vec{y}|$, which is a distance τ from $\vec{x}$ by the first two terms in their Taylor expansion around $\vec{x}$. By doing so, the first of the two parallel propagators is just replaced by the identity, while the second can be replaced with the entire parallel propagator along the small loop. We therefore have, up to terms of order δ or τ,

$$\begin{aligned} C^{\tau,\delta}_{\text{Weyl}}(\vec{x}) &= c_{\tau\delta} \int d^3y\ \theta(\delta - |\vec{y}|)\ \frac{y^a}{|\vec{y}|}\ \ \pi^A(\vec{x})\delta_A{}^B \\ &\quad \tilde{\sigma}^a{}_B{}^C(\vec{x}) \left(\delta_C{}^D + \tau\hat{y}^c A^D_{cC}(\vec{x})\right) \left(\psi_D(\vec{x}) + \tau\hat{y}^c\partial_c\psi_D(\vec{x})\right) = \\ &= c_{\tau\delta} \int d^3y\ \theta(\delta - |\vec{y}|)\ \frac{y^a}{|\vec{y}|}\ \ \pi^A(\vec{x}) \\ &\quad \tilde{\sigma}^a{}_A{}^C(\vec{x}) \left[\psi_C(\vec{x}) + \tau hat y^c \left(A^D_{cC}(\vec{x})\psi_D(\vec{x}) + \partial_c\psi_C(\vec{x})\right)\right] \end{aligned} \tag{6.15}$$

with $\hat{y}^c$ the components, in the euclidean background, of the unit vector $\frac{\vec{y}}{|\vec{y}|}$. The first term in the square brackets vanishes because $\int d^3y y^a = 0$, and the second term is the covariant derivative. Thus

$$C^{\tau,\delta}_{\text{Weyl}}(\vec{x}) = c_{\tau\delta}\tau \left[\int d^3y\ \theta(\delta - |\vec{y}|)\ \frac{y^a\hat{y}^c}{|\vec{y}|}\right]\ \ \pi^A(\vec{x})\tilde{\sigma}^a{}_A{}^C(\vec{x})\mathcal{D}_c\psi_C(\vec{x}). \tag{6.16}$$

The integration is now immediate

$$\int d^3y\ \theta(\delta - |\vec{y}|)\ \frac{y^a\hat{y}^c}{|\vec{y}|} = \frac{1}{3}\ \delta^{ac}\ \frac{4}{3}\pi\delta^3 \tag{6.17}$$

Restoring the explicit form of $c_{\tau\delta}$ we thus obtain

$$C^{\tau,\delta}_{\text{Weyl}} = \pi^A\tilde{\sigma}^a{}_A{}^C\mathcal{D}_c\psi_C + O(\delta) + O(\tau), \tag{6.18}$$

namely the fermion term of the hamiltonian constraint, as we wanted.

We have shown that the quantity (6.11) provides indeed a regularized form of the fermion hamiltonian constraint. We now come to the quantum theory and define the corresponding regularized quantum operator

$$\widehat{H}_{L\delta A\tau} = \sum_I L^3 \sqrt{-\hat{C}^{A,L,\delta}_{\text{Einstein } I} - \hat{C}^{L,\tau,\delta}_{\text{Weyl } I}} \; . \tag{6.19}$$

$$\hat{C}^{L,\tau,\delta}_{\text{Weyl } I} = \frac{1}{L^3} \int_I d^3x \; \hat{C}^{\tau,\delta}_{\text{Weyl}}(x) \, , \tag{6.20}$$

$$\hat{C}^{\tau,\delta}_{\text{Weyl}}(x) = c_{\tau\delta} \int d^3y \; \theta(\delta - |y|) \, \frac{y^a}{|y|} \; \hat{Y}^a \left[\gamma^{\tau}_{\vec{x}\vec{y}}\right] (|y|/\tau) \, . \tag{6.21}$$

The operator $\hat{Y}^a$ is defined in equation (5.32). Note that in this regularized operator the two hands of the operator [1] do not overlap: they are point split.

Let us study the action of the operator that we have defined: in spite of it apparent complexity, this action will turn out to be relatively simple. Plugging the explicit definition of $\widehat{Y}^a$ (5.32) in (6.21) gives

$$\hat{C}^{\tau,\delta}_{\text{Weyl}}(\vec{x}) \; \Psi[\alpha] = c_{\tau\delta} \int d^3y \; \theta(\delta - |\vec{y}|) \, \frac{y^a}{|y|} \sum_{\alpha_e} \delta^3(\vec{x}, \alpha_e)$$
$$\int_\alpha ds \; \dot{\alpha}^a(s) \; \delta^3(\gamma^{\tau}_{\vec{x}\vec{y}}(|\vec{y}|/\tau), \alpha(s)) \; \sum_{q=\pm} \Psi \left[\alpha * *^q{}_{e,s} \gamma^{\tau}_{xy}\right] \; . \tag{6.22}$$

We now use the explicit form of the small loop, and we keep only terms small in τ and δ. The first δ^3 function in the last equation forces the loop α to have an end-point α_e at $\vec{x}$. The integration over the small ball (size δ) around this point, and the second δ^3 function, pick up a second point $\alpha(s) = \vec{x} + \vec{y}$ in α, close to α_e. If α_e is a free end point, we can write this second point (up to the relevant order) as $\vec{x} + |\vec{y}|\vec{l}_e$, where $\vec{l}_e$ was defined in the previous section as the tangent of α at the end-point α_e. In general, however, α_e needs not be a free end-point; in the general case there will be several components of α originating from α_e. Thus $\vec{l}_e$ takes a finite number of values, and the radial d^3y integration together with radial part of the second delta function pick up all these values; note that they turn out to be proportional to $\vec{y}$. The radial d^3y integration is then straightforward, and we obtain

$$\hat{C}^{\tau,\delta}_{\text{Weyl}}(\vec{x}) \; \Psi[\alpha] = \frac{c_{\tau\delta}}{L^3}\delta \sum_{\alpha_e} \delta^3(\alpha_e, \vec{x}) \sum_{\vec{l}_e} \sum_{q=\pm} \Psi \left[\alpha * *^q{}_{e,\delta} \gamma^{\tau}_{\vec{x}(\vec{x}+\delta\vec{l})}\right] \; . \tag{6.23}$$

We introduce the notation

$$\hat{C}^{\tau,\delta}_{\text{Weyl}}(x) \; \Psi[\alpha] = \frac{c_{\tau\delta}}{L^3}\delta \sum_{\alpha_e} \delta^3(\alpha_e, x) \; \widehat{\mathcal{F}}^{\tau\delta}_e \; \Psi[\alpha] \tag{6.24}$$

$$\hat{\mathcal{F}}_e^{\tau\delta}\ \Psi[\alpha] = \sum_{\vec{I}_e} \sum_{q=\pm} \Psi\left[\left[\alpha * *^q{}_{e,\delta}\gamma^\tau_{\vec{x}(\vec{x}+\delta\vec{I})}\right]\right] . \tag{6.25}$$

Finally, we may come to the Hamiltonian. Let us assume for a moment that the loop we are dealing with has no intersection nor kinks, so that we can set the Einstein term to zero. Inserting our last result into the Hamiltonian we have

$$\hat{H}\Psi[\alpha] = \lim_{\substack{L\to 0\ \delta\to 0\\ \tau\to 0}} \sum_I L^3 \sqrt{\frac{c_{\tau\delta}}{L^3}\delta \int_{\text{cube } I} d^3x \sum_{\alpha_e} \delta^3(\alpha_e, x)\ \hat{\mathcal{F}}_e^{\tau\delta}\ \Psi[\alpha]} . \tag{6.26}$$

For L small enough, and assuming that $\delta \ll L$, so that "boundary effects" of the box can be neglected, every cube I contains only one end-point, and since $\hat{C}_{\text{Weyl}}$ gives zero unless there is an end-point in I, we have

$$\hat{H}\Psi[\alpha] = \lim_{\substack{L\to 0\ \delta\to 0\\ \tau\to 0}} \sqrt{\frac{3}{\tau}\frac{1}{\frac{4}{3}\pi\delta^3}L^3\delta} \sum_{\alpha_e} \left(\hat{\mathcal{F}}_e^{\tau\delta}\right)^{\frac{1}{2}}\ \Psi[\alpha]. \tag{6.27}$$

where we have restored the explicit expression for $c_{\tau\delta}$.

It is now time to study the limits explicitely. Let us first focus on the crucial prefactor

$$C(L,\tau,\delta) = \sqrt{\frac{3}{\tau}\frac{1}{\frac{4}{3}\pi\delta^3}L^3\delta} = \frac{3}{2\sqrt{\pi}}\sqrt{\frac{L^3}{\tau\delta^2}}. \tag{6.28}$$

The question we have to address is the finitness of the above limit. Clearly the result depends on the order in which the limits are taken. This is precisely what we should have expected: different orders in which the limit is taken correspond to inequivalent definitions of the quantum operator. Since all these definitions correspond to the same classical limit, the choice amount to a choice of different orderings of the quantum hamiltonian. The question is whether there is *one* choice that gives us a finite quantum operator.

Of course we may not confine ourselves to the choice between taking one first or another one first of the three limits: we choose any combination. More precisely, we may consider the three dimensional L, τ, δ space, and study the limit of $C(L,\tau,\delta)$ as we approach the point $L = 0, \tau = 0, \delta = 0$: this point can be approached in a variety of alternative ways, not just along one of the coordinate axis. Let us introduce a parameter ϵ, and consider a curve $L(\epsilon), \tau(\epsilon), \delta(\epsilon)$ in this tree dimensional space, such that $L(0) = 0, \tau(0) = 0, \delta(0) = 0$. Our problem is to understand wether we can choose this curve in such a way that the limit

$$\lim_{\epsilon\to 0} C(L(\epsilon), \tau(\epsilon), \delta(\epsilon)) \tag{6.29}$$

is finite. We are not free to choose the curve $L(\epsilon), \tau(\epsilon), \delta(\epsilon)$ in a completly arbitrary way, because there is a certain number of conditions that we have imposed on the regularization parameters along the way. First, of course, we must have $L > 0, \tau > 0, \delta > 0$. Then, have required $\tau > \delta$, and, in order to avoid "boundary effects" in the box, $L > \delta$. Can all the conditions be satisfied and a finite limit be obtained ?

The first crucial point to be noted is that powers of lengths cancell exactly and the quantity $C(L, \tau, \delta)$ is dimensionless. This is a necessary condition for having a finite limit. (It was precisely the fact that we got a divergent quantity with the dimensions of a length, measured in the background metric, that prevented us from achieving a background independent renormalization in chapter 5). By itself, however, this fact does not suffice to guarantee a well defined limit. We now claim that this limit can indeed be chosen consistently with all the requirements, as follows

$$\begin{aligned} L(\epsilon) &= k\epsilon^3 a, \\ \tau(\epsilon) &= \epsilon\, a, \\ \delta(\epsilon) &= \epsilon^4 a, \end{aligned} \tag{6.30}$$

where a is an arbitrary length, and k is a arbitrary dimensionless positive number. It is easy to see that all the requested conditions are satisfied. In the limit, we have

$$\lim_{\epsilon\to 0} C(L(\epsilon), \tau(\epsilon), \delta(\epsilon)) = \frac{3}{2\sqrt{\pi}}\sqrt{\frac{L^3(\epsilon)}{\tau(\epsilon)\delta^2(\epsilon)}} = \frac{3k^{-3/2}}{2\sqrt{\pi}} := \lambda^2 \tag{6.31}$$

λ is a free constant that emerges from the regularization procedure. Thus, the prefactor is finite in the limit.

Thus, we write the action of the hamiltonian as

$$\hat{H}\Psi[\alpha] = \lambda^2 \sum_{\alpha_e} \left(\widehat{\mathcal{F}}_e\right)^{\frac{1}{2}} \Psi[\alpha] \tag{6.32}$$

where we have introduced the "end-point operator"

$$\widehat{\mathcal{F}}_e\Psi[\alpha] = \lim_{\epsilon\to 0} \widehat{\mathcal{F}}_e^{\tau(\epsilon)\delta(\epsilon)} \Psi[\alpha]. \tag{6.33}$$

We now examine this "end-point operator", its action, its finiteness and its transformation properties under diffeomorphisms.

Since δ (the point splitting distance of the two grasps) goes to zero much faster than τ (the length of the added loop, we can now simply take the $\delta \to 0$ limit first, and the $\tau \to 0$ limit second. Let us consider the $\delta \to 0$ limit of $\widehat{\mathcal{F}}_e^{\tau\delta}\Psi[\alpha]$ (with finite τ). It is easy to see that if the end- point is free, the action of the operator is simply to add a small straight line of length $\tau = \epsilon a$ to the end point of the loop, in the direction of the loop tangent. If the end-point is not free, the action of the operator produces one term for each component of α emerging from the end-point. The terms corresponding to the component of α that ends in α_e is again just an addition of a small straight line to the end-point; while the other terms imply the addition of the small loop and also a rerouting through the intersection. The rerouting pattern can be calculated in straight forward way from equation (6.25).

Before taking the limit $\tau \to 0$, let us now assume that $\Psi[\alpha]$ is a diffeomorphism invariant state. Thus $\Psi[\alpha]$ depends only on the diffeomorphism equivalence class of α. If α_e is a free end-point, we have then

$$\lim_{\tau\to 0} \widehat{\mathcal{F}}_e^{\tau 0}\Psi[\alpha] = \lim_{\tau\to 0} 2\Psi[\alpha * *\gamma^{\tau}_{\alpha_e,\dot{\alpha}|_e}] = 2\Psi[\alpha], \tag{6.34}$$

because for small enough τ the added loop will not intersect any other loop, and the addition of a small line at the end of a loop does not change the diffeomorphism equivalent class of the loop.

If, on the other way, α_e is not a free end-point, then the loop $\alpha * *\gamma^{\tau}_{\alpha_e,\vec{l}_e}$ *does* belong to a different knot class than α. For instance, if the end point α_e falls over a smooth component β of the loop α, then one of the terms in (6.25) will add to it a small straight line, so that the resulting loop contains an intersection and a free end-point.

The key point, now, is that in any case, since $\Psi[\alpha]$ is diffeomorphism invariant, for small enough τ we have that $\widehat{\mathcal{F}}_e^{\tau 0}\Psi[\alpha]$ becomes *independendent from* τ. Therefore the limit is the limit of a constant function, and therefore is *finite.*

Moreover, it is clear that the resulting action of $\widehat{\mathcal{F}}_e$ is well-defined on the diffeomorphism invariant states. Thus, the operator $\widehat{H}$ is finite and diffeomorphism invariant in the limit.

If we now reinstate $\hat{C}_{\text{Eintsein}} \neq 0$, we have

$$\hat{H} = \sum_{\substack{\text{intersections } i \\ \text{end-points } e}} \sqrt{\hat{M}_i + \lambda \hat{\mathcal{F}}_e}\,, \tag{6.35}$$

where $\hat{M}$ was constructed in [5]. $\hat{H}$ is a finite operator defined on knot states. Its action follows immediately from the construction above.

The matrix elements of the operators $\hat{M}_i$ and $\hat{F}_e$ can be directly computed between any two given knot states. The calculation amounts to a straightforward exercise in geometry and combinatorics. The next problem is to compute the square root of the resulting (infinite) matrix. We expect that the square root can be computed order by order as the complexity of the knots considered increase. Work is in progress to compute explicitely the matrix elements, and thus understand if the structure of the resulting matrix allows a simple argoritm for extracting the square root.

6.3 QGD: dynamics

We are now in the position of describing the general structure of Quantum Gravitational Dynamics, or QGD, the quantum theory of gravitationally interacting fermions, evolving in the clock time defined by a scalar field.

A physical quantum state $|K\rangle$ of the theory is specified by a generalized knot, namely an open braid K of order N (with N open end-points, N even), with an arbitrary finite number I of intersections. A more accurate notation for these states is given in equation (5.36). The quantum dynamics is given by the matrix $\hat{H}$ in braid space, given in equation (6.35), the matrix elements of which are computed, order by order, according to the geometrical and algebraic rules given by equations (6.25) and (6.33), and in ref.[5]. We can interpret the matrix elements of $\hat{H}$ as first order transitions amplitudes in a time dependent perturbation expansion in the clock time T. In principle, the exponentiation of the $\hat{H}$ action gives the full evolution.

6.3.1 The simplest states

For instance, we can start from the simplest state formed by a single nonself intersecting open line. In terms of the notation (5.36), this can be denoted as

$$|2,2,0;1_2^0\rangle \tag{6.36}$$

where we have indicated the simplest value of $\mathcal{K}_2^0$, a single line, by 1_2^0. The moduli space of free open-ends is clearly formed by a single point, and thus we do not need a_i parameters.

There are two fermions in this quantum state. We have that $\hat{M}_i|2,2,0;1_2^0\rangle = 0$ because there are no intersections of kinks in $|2,2,0;1_2^0\rangle$. On the other side, we have from (6.34) that

$$\hat{F}_e\ |2,2,0;1_2^0\rangle = \lambda^2|2,2,0;1_2^0\rangle, \tag{6.37}$$

so that we get

$$\hat{H}\ |2,2,0;1_2^0\rangle = 2\lambda|2,2,0;1_2^0\rangle. \tag{6.38}$$

Therefore $|2,2,0;1_2^0\rangle$ is an eigenstate of the theory, or equivalently, the time dependent Schrödinger quantum state

$$|2,2,0;1_2^0,\ \ T\rangle = \exp\left\{i\lambda\sqrt{\frac{c^5}{\hbar G}}\ T\right\}\ \ |2,2,0;1_2^0\rangle \tag{6.39}$$

is a solution of the *exact* quantum interacting theory. (We have restored physical units, for clarity.) Perhaps this state corresponds to an extremely simple "universe" in which there are only two fermions gravitating around each other in the simplest of the quantum geometries. It is suggestive to think at this state as a kind of "atomic" "ground state" of a simple 2-fermions universe.

Next we can consider the generalized knot formed by n non intersecting copies of the above, and denote it as $|2n,2n,0;1_{2n}^0\rangle$. It is then straight forward to see that the time evolution of this state is given by

$$|2n,2n,0;1_{2n}^0,\ \ T\rangle = \exp\left\{i\ n\lambda\sqrt{\frac{c^5}{\hbar G}}\ T\right\}|2n,2n,0;1_{2n}^0\rangle \tag{6.40}$$

and that the corresponding energy eigenstates are

$$E_n = nE_1 = n\ \lambda \sqrt{\frac{c^5\hbar}{G}}. \tag{6.41}$$

As soon as we consider simple intersecting states, the full complexity of the operators $\hat{M}_i$ and $\hat{F}_e$ becomes relevant, and we have non-trivial time evolution.

6.4 Remarks on QGD

Before concluding, we list here a certain number of comments and considerations. In particular, we want to point out several important problems that remain open.

1. *Conservation of particle number.* The operator $\hat{F}_e$ defined above acts on end-points by displacing them, and possibly by changing the associated rootings at intersections, but never creates or destroys end-points. Since the operator $\hat{M}_i$ too, conserves the number of end-points, it follows that the hamiltonian that we have defined conserves the number of particles. This is at first surprising, given that in general there is particle creation from space-time dynamics. But a more close analysis shows that this conservation is to be expected. Unlike the Einstein-Dirac theory (in the standard semiclassical approach), indeed, the EC-Weyl theory does conserves particle number. This can be seen classically from the fact that the quantity

$$N\ :=\ \int_\Sigma d^3x\ \psi^A(x)\ \tilde{\pi}_A(x) \tag{6.42}$$

commutes with all the constraints, including the hamiltonain constraint [155]. One can immediately define the corresponding operator (say, using the Loop Transform), which turns out to be

$$\hat{N}||N,I,D;\ \ a_1^{m_1}....a_I^{m_I};\ \ \mathcal{K}^o_{\sum_i m_i}\rangle\ =\ N||N,I,D;\ \ a_1^{m_1}....a_I^{m_I};\ \ \mathcal{K}^o_{\sum_i m_i}\rangle. \tag{6.43}$$

This confirms the interpretation of the number N of end-points as the particle number. Since $[\hat{N},\hat{H}]=0$, the number of end-points N is a conserved quantum number in the theory.

2. *Particle anti-particle distinction.* The Weyl field theory describe a particle-antiparticle couple (say a neutrino and its anti- neutrino). In the Langrangian formulation the fermions are described by two complex fields. Since the action contains only first derivatives, the phase space has the same dimension as the space of the lagrangian fields, namely four real dimensions per point. These give two degrees of freedom, which describe, indeed, the particle and its antiparticle. Do the end-points of the loops represent particles or anti-particles ? The answer is that the distinction is not gauge invariant, thus the question is not well posed in the theory. In flat space one can globally distinguish particles from antiparticles; but when the Weyl system is coupled with gravitation, something curious happens: the particle antiparticle distinction becomes local. Consider two field excitations in two different space position, and assume the first is a particle; then, the fact that the second be a particle or an antiparticle depends on the parallel transport operator between the two. This is because the particle and the antiparticle are distinguished by different directions in the internal spin space, and we can only compare directions in spin space in different points by using the connection. Since the particle anti-particle distinction is gauge dependent, in a gauge-independent formulation there is not way to distinguish particles from anti-particles. This is why the end-points of the loop represent at the same time both kinds of excitations.

3. *Regime of validity of the formalism, and complex energy eigenvalues.* This is an important feature of the clock field formalism that we must be discussed in detail. The formalism cannot be used in *any* regime of the system. This is already obvious at the classical level: Consider an arbitrary solution of the field equations $A(\vec{x},t), \psi(\vec{x},t), T(\vec{x},t)$: in general it is *not* possible to invert $T(\vec{x},t) \rightarrow t(\vec{x},T)$ globally. Thus, we certainly cannot use T as a time variable for every solutions of the field equations and for every spacetime region. On the other side, *there are* solutions and spacetime reions where we can make the inversion. Consider an initial configuration of the $A(\vec{x}), \psi(\vec{x}), T(\vec{x})$ fields and their time derivatives $\dot{A}(\vec{x}), \dot{\psi}(\vec{x}), \dot{T}(\vec{x})$ on a given space like surface. This defines a point in phase space. Assume that $\dot{T}(\vec{x}) < 0$

in a spacial region R. There will be a time interval Δt for which $\dot{T}(\vec{x})$ will remain positive in R. More precisely, we can determine a region Γ in phase space, and a corresponding spacetime region S, such that for any initial condition in Γ, $\dot{T}(\vec{x})$ is positive in S. We shall say that the gravitational-fermion-scalar field system is in the *clock regime* in S if the initial conditions are in Γ, namely if $\dot{T}(\vec{x})$ is positive all over S. By definition, we can perform the inversion $T(\vec{x},t) \to t(\vec{x},T)$ in the region in which the system is in the clock regime. Thus, as far as the classical theory is concerned, the formalism makes sense only in this regime. The same is true in the quantum theory. The quantum formalism that we have constructed is meanigfull in the clock regime.

A paradigm for this construction can be found in the quantum system of two uncoupled simple harmonic oscillator variables $g(t), f(t)$, if we fix the total energy E and decide to never consider the evolution in the external clock time t, but rather use one of the two variables, say f as internal clock; namely if we decide to ask questions concerning what is the position g of the first oscillator, when the second is in f. We obtain the classical evolution $g(f)$ by inverting $f(t) \to t(f)$ and defining $g(f) = g(t(f))$. We can also do the same in quantum mechanics (see ref. [19], where the example is worked out in detail). However, along any orbit there is a point in which the internal time variable f "comes back" (in t), and therefore we obtain a non unitary evolution operator in f. The physical interpretation of this non-unitarity is clear: there is "no system" anymore for f arbitarry large.

The formalism reflect this fact, both classically and quantum mechanically, in the form of the hamiltonian. The hamiltonian that evolves the two oscillators in the external time t is $p_g^2 + p_f^2 + g^2 + f^2$. The hamiltonian that evolves the systemin the internal time f is easilly obtained solving for p_f (see ref.[19])

$$H(f) = \sqrt{E - p_g^2 - g^2 - f^2} \tag{6.44}$$

(where E is the total energy of the system in the t time). This (time dependent) hamiltonian generates the evolution equations for $g(f)$. The important point to note is that the hamiltonian becomes immaginary for large f. This simply signal the fact that it doesn't make sense anymore to evolve $g(f)$ in f. If we want to continue the evolution by using an internal time, we have to choose another, distinct, internal time, and "patch" the evolution. Note that this does not mean that the formalism that evolves in f is inconsistent or incorrect: it simply means that is has a certain domain of validity. The same holds in quantum mechanics. Indeed, it was shown in ref.[19] that the quantum hamiltonian corresponding to (6.44) is self- adjoint when suitably restricted to an ($f-$dependent) region of the Hilbert space, but develops immaginary eigenvalues if applied outside this region.

Similarly, we expect that the the hamiltonian that we have defined will also have immaginary eigenvalues. These simply signal that twe are going out from the domain of validity of the formalism, namely from the *clock* regime. We are trying to use evolution in T to describe the gravitational field in regions where the T fields fails to be monotonic. Explicitely, this possibility can be easilly traced back to the classical hamiltonian constraint. Roughly speaking, since the form of this constraint is $\Pi^2 + C_{\text{Einstein}} + C_{\text{Weyl}} = 0$ (Π being the scalar field momentum) and since in order $\dot{T}$ to change sign Π must vanish, it follows that the save region, is where $C_{\text{Einstein}} + C_{\text{Weyl}} > 0$, which is of course a sufficient condition for the Hamiltonian we have defininied, $H = \int \sqrt{C_{\text{Einstein}} + C_{\text{Weyl}}}$ to be real. Thus, imaginary eigenvalues of $\hat{H}$ signal that we are exiting the regime of validity of the formalism we have developed here: the object we have chosen as clock is running backward. We must therefore exclude from the state space of QGD, as formulated here, the graphs that are eigenvalues of $\hat{H}$ witha an eigenvalue that is not a real positive number.

In particular, all the vacuum solutions of pure quantum GR that where previously found lie outside the clock regime. They are eigenstate of $\hat{H}$ whith vanishing eigenvalue. Classically, the vacuum solutions of the theory are only compatible with $\Pi = 0$, namely $\dot{T} = 0$, which clearly indicates that we are outside the regime in which we can take the scalar field as a good clock.

We do not consider this necessary restriction of the formalism as a serious limitation. Our long term aim is to develop a usable theory that can be employed, at least in principle, to describe Planck scale measurements and the Planck scale evolution of quantum geometry. We would be very content of having a sensible general covariant field theory that correctly describe this physics in the regime in which whatever we are using as a clock keeps behaving as a clock.

4. *Scalar product.* One of the weak points of the Loop Representation is given by the fact that a complete and consistent definition of the scalar product is not yet available. The conventional wisdom is that once physical observables on the physical state space have been constructed, the scalar product is determined by the requirement these physical observables be self-adjoint. The present work is a step in this direction. The (real eigenvalue) eigenstates of the hamiltonian $\hat{H}$ must form, if the formalism is consistent, an orthogonal basis. Thus, working out explicitely the eigenstates of $\hat{H}$ in knot space should at the same time lead to a partial definition of the scalar product.

5. *Taking limits on knot space.* Finally, let us discuss a subtle point in the definition of $\hat{H}$, which we percieve as the most delicate and potentially problematic point in the construction above. We refer to the different way in which the $\delta \to 0$ and the $\tau \to 0$ limit have been dealt with, when dealing with knot states.

   To focus the point, let us consider a model example. Consider the space $C[R]$ of the continuous functions $f(x)$ on the real line. Consider the closure D of the space $C[R]$, say in the pointwise topology, such that D contains also piecewise continuous functions as the step function θ defined by

$$\theta(x) = 1 \text{ if } x > 0 \;\; ; \theta(x) = 0 \text{ otherwise.} \tag{6.45}$$

   Now define the linear functional k on $C[R]$ as follows: $(k, f) := lim_{x \to 0} f(x)$, and assume you want to extend k from $C[R]$ to D. There are two possible strategies: one is to keep the definition

$$(k, f) = \lim_{x \to 0} f(x). \tag{6.46}$$

The other is to note that an equivalent definition of k on $C[R]$ is $(k,f)=f(0)$, and thus to define

$$(k,f)=f(0). \tag{6.47}$$

According to the first definition

$$(k,\theta)=1, \tag{6.48}$$

according to the second

$$(k,\theta)=0. \tag{6.49}$$

We are in a similar situation when we need to study the action of the hamiltonian $\hat{H}$ on knot states. In quantum mechanics, operators are often defined on dense subspaces of the state space. For instance we begin by defining the momentum operator in Schrödinger mechanics not on the full L_2 state space but on the dense subspace of the differentiable functions; then we can extend it. The Hamiltonian $\hat{H}$ that we define in this paper contains a certain number of limiting procedures. We may first rigourously define it on a suitable restricition of the space of the loop functionals. For instance we may assume that the loop functionals are continuous in all the deformations that we consider. $\hat{H}$ is well defined on this space. Then, however, we want to consider the action of $\hat{H}$ on the knot states. These are not continuous in the deformations that we consider and thus we need to define a suitable extension of the operator. At this point we have a choice that essentially reflects the choice we described in the simple example above. As far as we understand, this choice, if not dictated by internal consistency, is again part of the quantization ambiguities as the ordering of the dynamical operators.

The important point we want to make here is not that a choice of the extension has been made in computing the action of $\hat{H}$ on the knot states, but that *two different* choices have been made for the two limits $\delta\rightarrow 0$ and $\tau\rightarrow 0$. In fact, as far as the $\delta\rightarrow 0$ is concerned, we have assumed that we should first take the limit, and then consider the extension of the action of the operator to diffeomorphism invariant states; while as far as the $\tau\rightarrow 0$ limit is concerned, we have assumed that we should first extend the the operator to diffeomorphism invariant states, and then take the limit.

This choice is not completely arbitrary: δ must go to zero faster that τ, and, if we take away the fake dimensions added by the integration, we see that the first significative term, which is the one that we are considering, is of order zero in δ and of first order in τ. Ths means that already at the classical level what we are doing is precisely considering a function $f(\delta,\tau)$ and picking up terms of the form $\frac{\partial}{\partial\tau}f(\delta,\tau)|_{\delta=\tau=0}$. Thus it is not completely unreasonable that this difference gets translated in the different ways in which the two limits are taken on loop space: roughly, we are "really" looking at the $\delta = 0$ point, and we are "really" looking at the limit in the first order expansion in τ. However, these are very wand-waving justifications of our choice. Untill a well-defined calculus on Loop Space is constructed [33], we do not see a way to transform these tentative explorations into solid mathematics. Our only real justification at this point, if any, is the hope that the (finite) structure we are constructing be internally consistent and, perhaps, related to Nature.

6. *Future developments*

   (a) The next step in the construction of the theory should be to compute explicitely matrix elements of $\hat{M}_i$ and $\hat{F}_e$ (see eq.(6.35)), and understand whether there is a direct algoritm for extracting the square root. If this can be done, the theory is essentially at the stage where the evolution of physical states can be described.

   (b) As noted above, the scalar product is partially fixed by the construction itself. The energy and the particle number are conserved observables. There are other observables in the theory that one may consider, and evolve, as the area observable discussed in references [2, 4]. A crucial test for the consistency of the scheme developed here is, as was noted in [5], whether the second order term of the time dependent perturbation expansion develops divergences.

SEVEN

CONCLUSIONS AND OUTLOOKS

In this thesis the coupled system formed by Fermionic fields and gravity has been studied along the lines of the recently introduced non-perturbative canonical gravity frameworks of Ashtekar and Rovelli-Smolin or Loop quantisation.

Coupling matter to gravity becomes mandatory when defining physically meaningful observables in the light of general relativity; purely gravitational experiments are rather difficult to implement. Furthermore, the Loop formalism applied to pure gravity yields an underlying genuinely background independent field theory, thus, one is entitled to prove whether this geometrical result holds the same after one has coupled matter to gravity. We chose Fermion fields as the matter in the present work mainly because of realism, but also because they are very natural objects when describing gravity in terms of the Ashtekar variables.

We can divide the results obtained here into classical and quantum aspects for the Fermionic fields + gravity system.

The classical aspects concern two problems. First, we have investigated the form of the action functional for our system a la Ashtekar and second, we looked at a model to study the variational problem for massless Fermionic fields.

For the first problem we have generalized the pure gravity result about the equivalence between the Einstein-Hilbert and the self-dual actions to the case in which Fermion fields are coupled to gravity. This generalization is made possible by using, in the case of the self-dual action, a self-dual connection developing torsion and is based on the Bianchi symmetry of curvature tensors. Thus the theory one deals with is actually Einstein-Cartan-Sciama-Kibble-Dirac; the result goes trough because of the property of the torsion for this theory of being totally antisymmetric. Our approach is appealing because it enables one to interpret the different modifications to the general relativistic equations of motion in Ashtekar framework directly to torsion and because it makes possible to propose a set of reality conditions, different from the standard ones, such that one can pick a real theory of gravity and Fermionic fields without torsion[26] out from the phase space of complex general relativity.

The second problem we dealt with at the classical level was a model consisting of a massless spin-$\frac{1}{2}$ field in flat Euclidean four-space bounded by a three-sphere. This could be considered as an an example of the matter field counterpart of the variational problem for gravity. We showed that this model has associated a well-defined variational problem which implements certain supersymmetric-inspired local boundary conditions used in one-loop quantum cosmology. This result adds evidence in favour of quantum amplitudes having to respect the properties of the classical boundary-value problem, *i.e.* the one-loop properties for masless Fermionic fields in the presence of boundaries coincide for both spectral and local boundary conditions.

Now we proceed with the quantum aspects of the gravity + Fermion fields system we analyzed.

Although we did not aim to investigate the role of torsion at the quantum level the loop formalism was indeed developed for the theory with torsion Einstein-Cartan-Weyl because it is simpler than having to deal with the quartic term in Fermion fields necessary to eliminate torsion. In any case, if our proposal concerning reality conditions worked we would have a way to deal with it at the quantum level. Besides, a Weyl field simplifies the work a great deal.

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[26]That this could happen was originally proposed by T. Jacobson [120].

Based on the classical canonical structure obtained for the Einstein-Cartan-Sciama-Kibble-Dirac theory, in its self-dual form, we performed the Loop quantization to get Quantum Gravitational Dynamics (QGD) in two stages.

First we constructed the loop representation for the EC-Weyl theory. As a generalization of the pure gravity case loop variables for the theory were defined to account for the Fermions. This was readily accomplished by admiting not only closed curves parametrizing parallel transport operators (defined with the Ashtekar connection) but also including open ones with the condition that Fermions sit at the end points. For convenience we referred to both, open and close curves, as loops the right meaning being clear from the context. They form an Poisson algebra compatible with the $\mathcal{T}$-algebra of loop variables for pure gravity. A linear representation of this algebra acting on wave functionals in the space of loops defines the quantum theory, which, we showed can be equivalently obtained through an heuristic Loop transform.

The diffeomorphism constraint in the resulting state space was solved. the solutions are classified by the generalization of the Knot classes that appeared in pure gravity. Now more classes are defined that include graphs with an arbitrary number of intersections and open ends. Hence, quantum states of QGD admit the same topological description as the states of pure Quantum General Relativity.

The action of the Hamiltonian constraint on the quantum states is, interestingly, rather simple: it *shifts* loops along themselves. This is again a generalization to the case of massless Fermions of the pure gravity results.

Though intuitively appealing various important problems made this first stage unsustainable as it is: i) a divergent quantity appears in the action of the Hamiltonian constraint on quantum states, ii) in spite of the simplicity of the action of the quantum constraint equation we did not find any non-trivial solution for it and, iii) the presence of Fermions by itself does not eliminate the difficulties of constructing physical observables, although makes it easier to find three-dimensional diffeomorphism invariant quantities, it does not help in providing quantities that commute with the Hamiltonian constraint.

To face part of the above difficulties in the second stage of the application of the Loop quantization we adopted the old idea of using matter as reference to define the gravitational analysis. A scalar field is coupled to the gravitational field and thus we have a compound system gravity+Fermions+scalar field, so that the scalar field plays the role of a clock-field. By a suitable gauge fixing the theory is solved for the scalar field and rather than a Hamiltonian constraint we end up with a genuine diffeomorphism invariant Hamiltonian that evolves gravity and Fermions in the scalar field clock time. We succeded in regularizing this quantum Hamiltonian so that it is diffeomorphism invariant and finite. Furthermore, it acts exactly the same way as the analogue constraint on loop states, *i.e.* shifting loops along themselves.

The picture is thus one in which quantum states are represented by graphs containing a finite number of intersections and open ends, the Hamiltonian being an operator that acts in a simple, geometrical, combinatorial way on these graphs. A time perturbation expansion may be considered in which the matrix elements of the operator Hamiltonian are the first order transition amplitudes between the graph states.

Further insight could be gained by obtaining the above mentioned matrix elements of the Hamiltonian operator. A rather difficult task that involves calculating the square root of an infinite matrix. It would give the first step in describing "time evolution". It remains to be seen whether the second order contribution to the time perturbation expansion does not diverge.

Among the most impressive results using the Loop representation for non-perturbative Quantum General Relativity is the physical interpretation of states [2]. Even though local operators such as the metric at a point may not be well-defined, there do exist non-local operators as for instance the area of a two-surface of which one can make perfect sense. Furthermore, there exists quantum states, *weaves*, which approximate a given flat or slightly curved geometries [160] at large scales having associated a discrete structure at the Planck scale. We believe, according to our results, this intepretation holds the same also when matter is coupled to gravitation. The precise form in which the matter fields enter this scheme started coming out from our analysis above as Fermions siting at the end points of open curves or at special points of closed curves. Obtaining the analogue of the weave states inlcuding matter has not been done so far.

The above conclusions and comments already outline some of the possible lines along which further work can be done. We end by mentioning some possible future developments which can be regarded as natural extensions to the results here reported.

We chose to work with a Weyl field because of simplicity since automatically, then, there is no mass term in the Hamiltonian constraint nor in the associated Hamiltonian, when a clock-field is used. It is very tempting to try to figure out what kind of interpretation can be given for instance to the mass of the Fermions in terms of quantum states labeled by graphs. Are they associated to the connectivity of the graphs? or some kind of topological defect of the graphs? There are results using other approaches in which a quantum-gravitational origin for masses is proposed [161]. As it could be expected, they get masses of order of the Planck scale, since it is this scale the only natural scale appearing in the Fermions+gravity system. They also figured out what kind of mechanisms could account for the suppresion of the Planck-size effective mass. An alternative solution along the lines of the Loop quantization may be tried by for instance alluding to the topological properties of the graphs associated to quantum states. This possibility requires more development.

After having studied spin-$\frac{1}{2}$ fields coupled to gravity if one thinks in incorporating further symmetries that may lead to a better understanding of the problem of quantum gravity the obvious option is to try a supergravity theory. Some advances have been done in this direction for some models in lower than four-dimensional spacetimes [162]. Indeed it is known supergravity plays a rather restricting role in the way gravity and matter can couple. Thus, it would not be surprising this can shed some further light on the structure of spacetime and matter at the Planck length.

APPENDIX A

SPINORS CONVENTIONS

A translation between two- and four-spinors is next given. Starting with the matrices $\tau_{\hat{a}}$ such that $\tau_{\hat{a}=\hat{0}}$ is the two-dimensional unit matrix, I_2, whereas $\tau_{\hat{j}=\hat{1},\hat{2},\hat{3}}$ are the Pauli matrices, introduce the *anti-hermitian* Infeld-van der Waerden symbols [122, 163]

$$\mathcal{I}_{\hat{a}}^{\ AA'} := \frac{i}{\sqrt{2}}\tau_{\hat{a}} \quad \Rightarrow \quad \mathcal{I}^{\hat{a}}_{\ AA'} := -\lambda_{\hat{a}}\frac{i}{\sqrt{2}}\rho^{\hat{a}} \quad \text{(no sum over } \hat{a}) \tag{A.1}$$

where $\lambda_{\hat{a}=\hat{2}} = -1$ and $\lambda_{\hat{a}}$ is $+1$ otherwise; $\rho^{\hat{a}} \equiv \tau_{\hat{a}}$ as matrices. Note that the primed index is always last thus assigning entries to the corresponding matrices; hat-ed and two-component spinor indices are raised and lowered with $\eta_{\hat{a}\hat{b}}$ and the antisymmetric forms [122, 163] ϵ_{AB}, $\epsilon_{A'B'}$, respectively (*e.g.* $\lambda^B = \epsilon^{BA}\lambda_A$ and $\chi_C = \chi^D\epsilon_{DC}$). The space $\mathcal{V}$ of spinors whose complex conjugate satisfies $\overline{\alpha}^{AA'} = -\alpha^{AA'}$ can be shown to be isomorphic to a *real* four-dimensional space admiting a Minkowski metric of signature $(-,+,+,+)$ [7]. The Infeld-van der Waerden symbols (A.1) provide such an isomorphism. A convenient two-component representation for the Dirac spinor field can be obtained as follows. Define

$$\Psi := \begin{pmatrix} \kappa^A \\ \overline{\mu}_{A'} \end{pmatrix} \tag{A.2}$$

as well as the action of the Dirac γ's

$$\gamma^{\hat{a}}\Psi := \sqrt{2}\begin{pmatrix} \mathcal{I}^{\hat{a}AA'}\overline{\mu}_{A'} \\ -\mathcal{I}^{\hat{a}}_{\ AA'}\kappa^A \end{pmatrix} \quad \Rightarrow \quad \gamma^{\hat{a}} = \begin{pmatrix} 0 & \sqrt{2}\,\eta^{\hat{a}\hat{b}}\,\mathcal{I}_{\hat{b}}^{\ AA'} \\ -\sqrt{2}\left[\mathcal{I}^{\hat{a}}_{\ AA'}\right]^{\text{Transposed}} & 0 \end{pmatrix}. \tag{A.3}$$

Since the Infeld-van der Waerden symbols fulfill

$$\epsilon_{AB}\,\epsilon_{A'B'} = \eta_{\hat{a}\hat{b}}\,\mathcal{I}^{\hat{a}}_{\ AA'}\,\mathcal{I}^{\hat{b}}_{\ BB'} \tag{A.4}$$

$$\frac{1}{2}\eta_{\hat{a}\hat{b}}\,\epsilon_A^{\ B} = \mathcal{I}_{(\hat{a}AA'}\,\mathcal{I}_{\hat{b})}^{\ BA'} \tag{A.5}$$

it is straightforward to show that the Clifford algebra $\gamma^{(\hat{a}}\gamma^{\hat{b})} = \eta^{\hat{a}\hat{b}}$ holds. In terms of the two-dimensional identity matrix and Pauli matrices one gets

$$\gamma^{\hat{0}} = \left(\gamma^{\hat{0}}\right)^{\ddagger} = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix} \qquad \gamma^{\hat{j}} = -\left(\gamma^{\hat{j}}\right)^{\ddagger} = \begin{pmatrix} 0 & i\tau_j \\ i\tau_j & 0 \end{pmatrix}$$
$$\gamma_{\hat{5}} := i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}} = (\gamma_{\hat{5}})^{\ddagger} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} . \tag{A.6}$$

This is a *chiral* representation for the γ's since it naturally enables one to split the four-spinor Ψ into left- and right-parts. Namely [163], $\Psi = \Psi_L + \Psi_R$,

$$\Psi_R := \tfrac{1}{2}(I_4 + \gamma_{\hat{5}})\Psi = \begin{pmatrix} \kappa^A \\ 0 \end{pmatrix} , \quad \Psi_L := \tfrac{1}{2}(I_4 - \gamma_{\hat{5}})\Psi = \begin{pmatrix} 0 \\ \overline{\mu}_{A'} \end{pmatrix} , \tag{A.7}$$

I_4 being the four-dimensional unit matrix. The Dirac-conjugate spinor $\overline{\Psi} = \Psi^{\ddagger}\gamma_{\hat{0}}$ thus becomes

$$\overline{\Psi} = -i\left(\mu_A \, , -\overline{\kappa}^{A'}\right) . \tag{A.8}$$

The extension of the connection ∇ to four-spinors can now be translated to the two-component language. One only has to re-express $B_a = \frac{1}{8}\omega_{a\hat{a}\hat{b}}\left[\gamma^{\hat{a}}, \gamma^{\hat{b}}\right]$ accordingly with the above results. It turns out that

$$\nabla_a\Psi = \begin{pmatrix} \nabla_a\kappa^A \\ \nabla_a\overline{\mu}_{A'} \end{pmatrix} = \begin{pmatrix} \partial_a\kappa^A - \omega_a{}^A{}_B\,\kappa^B \\ \partial_a\overline{\mu}_{A'} + \overline{\omega}_{aA'}{}^{B'}\,\overline{\mu}_{B'} \end{pmatrix} \tag{A.9}$$

and analogously for $\nabla_a\overline{\Psi}$. Here, $\omega_{aAB} := \frac{1}{2}\omega_{a\hat{a}\hat{b}}\,\mathcal{I}^{\hat{a}}{}_{AA'}\,\mathcal{I}^{\hat{b}}{}_B{}^{A'}$ and $\overline{\omega}_{aA'B'} := \frac{1}{2}\omega_{a\hat{a}\hat{b}}\,\mathcal{I}^{\hat{a}}{}_{AA'}\,\mathcal{I}^{\hat{b}A}{}_{B'}$ are the two-spinor version of the connection $\omega_{a\hat{a}\hat{b}}$. They correspond, on hat-ed indices, to the self-dual (${}^{+}\omega_{a\hat{a}\hat{b}}$) and anti-self-dual (${}^{-}\omega_{a\hat{a}\hat{b}}$) parts respectively, defined by [7, 122]

$${}^{\pm}\omega_{a\hat{a}\hat{b}} := \frac{1}{2}\left(\delta_{\hat{a}}{}^{\hat{c}}\delta_{\hat{b}}{}^{\hat{d}} \mp \frac{i}{2}\epsilon_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}\right)\omega_{a\hat{c}\hat{d}}$$

Namely, the (anti-) self-dual part of the connection couples to the (left-) right-part of the fermionic field.

Thus, the fermionic terms of (3.1) become, in two-component spinor language,

$$ie^a_{\hat{a}}\left[\left(\nabla_a\overline{\Psi}\right)\gamma^{\hat{a}}\Psi - \overline{\Psi}\gamma^{\hat{a}}\left(\nabla_a\Psi\right)\right] = -\sqrt{2}\sigma^a{}_{AA'}\left[\overline{\kappa}^{A'}\left(\nabla_a\kappa^A\right) - \left(\nabla_a\mu^A\right)\overline{\mu}^{A'}\right]$$

$$+\sqrt{2}\sigma^a{}_{AA'}\left[\left(\nabla_a\overline{\kappa}^{A'}\right)\kappa^A - \mu^A\left(\nabla_a\overline{\mu}^{A'}\right)\right] \quad \text{(A.10)}$$

$$2m\overline{\Psi}\Psi \quad = \quad -2im\left(\mu_A\kappa^A - \overline{\kappa}^{A'}\overline{\mu}_{A'}\right) \quad \text{(A.11)}$$

where the definition of the soldering forms $\sigma^a{}_{AA'} := e^a_{\hat{a}}\ \mathcal{I}^{\hat{a}}{}_{AA'}$ has been used. With our conventions, they are *anti-hermitian* for $e^a_{\hat{a}}$ real; *i.e.* for real general relativity.

APPENDIX B

THREE-DIMENSIONAL PROJECTION OF LIE DERIVATIVES

Proof of: $q_m{}^b \mathcal{L}_t A_b = q_m{}^b \mathcal{L}_t {}^3A_b \;\Leftrightarrow\; q_m{}^b \mathcal{L}_t q_b{}^d = 0 \;\Rightarrow \mathcal{L}_t q_a{}^b = 0$

From the definition of the Lie derivative,

$$\mathcal{L}_v w^a = [v, w]^a \ , \tag{B.1}$$

and torsion, $2\nabla_{[a}\nabla_{b]} f := T_{ab}{}^c \nabla_c f$, one gets

$$\mathcal{L}_v w^c = v^m (\nabla_m w^c) - w^n (\nabla_n v^c) + v^m T_{mn}{}^c w^n \ . \tag{B.2}$$

Since $\mathcal{L}_v(f) = v(f)$ then one also has

$$\mathcal{L}_v u_a = v^m (\nabla_m u_a) + u_n (\nabla_a v^n) - v^m T_{ma}{}^n u_n \ . \tag{B.3}$$

Using the above relations it is possible to show that the Lie derivative along the normal n^a of the projector q_{ab} becomes

$$\begin{aligned} \mathcal{L}_n q_{ab} &= n^m \nabla_m q_{ab} + q_{mb} \nabla_a n^m + q_{am} \nabla_b n^m - n^m \left(T_{ma}{}^n q_{nb} + T_{mb}{}^n q_{an} \right) \\ &= n^m \left[\nabla_m q_{ab} - \nabla_a q_{mb} - \nabla_b q_{am} - T_{ma}{}^n q_{nb} - T_{mb}{}^n q_{an} \right] \\ &= q_a{}^m \nabla_m n_b + q_b{}^m \nabla_m n_a - n^m \left[T_{ma}{}^n q_{nb} + T_{mb}{}^n q_{an} \right] \\ &= 2K_{(ab)} - 2n^m T_{m(a}{}^n q_{b)n} \ , \end{aligned} \tag{B.4}$$

where, in order to go from the first to the second equality, the orthogonality between q_{ab} and n^b and the Leibnitz rule were used, whereas to produce the third one use was made of the explicit form $q_{ab} = g_{ab} + n_a n_b$ together with the identity $n^m \nabla_a n_m = 0$. Finally the form of the extrinsic curvature, $K_{ab} = q_a{}^m \nabla_m n_b$, yielded the fourth equality.

The Lie derivative along t^a can be cast as

$$\mathcal{L}_t q_{ab} = N\mathcal{L}_n q_{ab} + \mathcal{L}_{\vec{N}} q_{ab} \ , \tag{B.5}$$

which holds because the factors in front of the derivatives of N are the contraction of $q_{am}n^m = 0$. By next using the above expression for $\mathcal{L}_n q_{ab}$, one arrives at

$$\mathcal{L}_t q_{ab} = 2NK_{(ab)} - 2Nn^m T_{m(a}{}^n q_{b)n} + \mathcal{L}_{\vec{N}} q_{ab} \ . \tag{B.6}$$

The combination of interest to us is $q_c{}^a \mathcal{L}_t q_a{}^b$. Taking into account $q_a{}^b = \delta_a{}^b + n_a n^b$ and using the Leibniz rule for the Lie derivative, one gets

$$\begin{aligned} q_c{}^a \mathcal{L}_t q_a{}^b &= n^b q_c{}^a \mathcal{L}_t n_a \\ &= n^b \left[q_c{}^a t^m \nabla_m n_a + q_c{}^a n_m \nabla_a t^m - t^m T_{ma}{}^n q_c{}^a n_n \right] \ . \end{aligned} \tag{B.7}$$

Decomposing t^m into shift and lapse the first term in square bracktes yields

$$q_c{}^a t^m \nabla_m n_a = Nn^m \nabla_m n_c + N^m K_{mc} \tag{B.8}$$

whereas the second term can be fiddled as

$$\begin{aligned} q_c{}^a n_m \nabla_a t^m &= q_c{}^a \left[\nabla_a (n_m t^m) - t^m \nabla_a n_m \right] \\ &= -q_c{}^a \nabla_a N - N^m K_{cm} \ . \end{aligned} \tag{B.9}$$

One is hence left with

$$q_c{}^a \mathcal{L}_t q_a{}^b = n^b \left[Nn^m \nabla_m n_c - q_c{}^a \nabla_a N + 2N^m K_{[mc]} - t^m T_{ma}{}^n q_c{}^a n_n \right] \ . \tag{B.10}$$

Recalling the relation $n_a = \alpha \nabla_a t$, which for our election of t ($t^a \nabla_a t = 1$) amounts to $n_a = -N \nabla_a t$, it is possible to prove

$$2\nabla_{[m} n_{a]} = \frac{2}{N} \left(\nabla_{[m} N \right) n_{a]} + T_{ma}{}^n n_n \ , \tag{B.11}$$

and thus that

$$Nn^m T_{ma}{}^n q_c{}^a n_n = Nn^m \nabla_m n_c - q_c{}^a \nabla_a N \ . \tag{B.12}$$

Putting all together yields

$$\begin{aligned} q_c{}^a \mathcal{L}_t q_a{}^b &= n^b \left[N n^m T_{ma}{}^n q_c{}^a n_n + 2N^m K_{[mc]} - t^m T_{ma}{}^n q_c{}^a n_n \right] \\ &= n^b \left[2N^m K_{[mc]} - N^m T_{ma}{}^n q_c{}^a n_n \right] \end{aligned} \tag{B.13}$$

which is zero for non-vanishing torsion by actual cancellation of the antisymmetric part of the extrinsic curvature with the torsion contribution (Recall $K_{[mc]} = \frac{1}{2} q_m{}^p T_{pa}{}^n q_c{}^a n_n$, using the definition of K_{mc}). For the vanishing-torsion case each term is zero separately. Furthermore, because of the identity $q_b{}^d q_c{}^a q_a{}^b = q_c{}^d$ taking Lie derivatives $\mathcal{L}_t$ on both sizes and using Leibnitz rule for it yields

$$(\mathcal{L}_t q_b{}^d) q_c{}^b + q_a{}^d (\mathcal{L}_t q_c{}^a) = \mathcal{L}_t q_c{}^d \tag{B.14}$$

with each of the terms on the L.H.S. being zero and then $\mathcal{L}_t q_c{}^d = 0$.

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