Scuola Internazionale Superiore di Studi Avanzati - Trieste

Secret Symmetries And Supersymmetry: Investigations

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Abstract
This is a report of the author's research in the fields of Non-linearly Realized Space-time Symmetries and Phenomenology of Supersymmetry.

In the first part, the equivalence of two theories that non-linearly realize a space-time symmetry, and which are related by a well motivated mapping, is discussed (with focus on the so-called Galileon group). This is done by studying how their coupling with gravity changes under the mapping.

The second part treats some aspects of the phenomenology of the supersymmetric partner of the goldstino (the sgoldstino) in the context of Gauge Mediation.

The work includes introductory sections on the two subjects.

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Chapter 1

Introduction

The present thesis is an extract of the work done during my PhD at SISSA. Even if somehow inhomogeneous, its two main lines find a curious common point: they can be seen as different evasions from the Coleman-Mandula theorem on the symmetry structure of four-dimensional relativistic particle theories. Let me then begin the introduction with the statement of the theorem, directly quoted from the paper that first presented it [1]:

> *"Let G be a connected symmetry group of the S matrix, which contains the Poincaré group and which puts a finite number of particles in a supermultiplet.*[1] *Let the S matrix be non trivial and let elastic scattering amplitudes be analytic functions of s and t in some neighbourhood of the physical region. Finally, let the generators of G be representable as integral operators in momentum space, with kernels that are distributions. Then G is locally isomorphic to the direct product of the Poincaré group and an internal symmetry group."*

It's necessary to stress that the hypotheses which are required for the theorem to hold are rather weak, at least from a physical point of view.

The theorem is very powerful, not just because of its generality, but also thanks to the simplicity of its result: any symmetry generator apart from P_μ and $J_{\mu\nu}$ – which are always there to implement relativistic invariance of the theory – must commute with them. Physically, this implies that the additional generators are Lorentz scalars, and in particular that they don't change the spin of the particle they act on. From the algebraic point of view, this is of course the simplest possibility, and in fact a trivial one in the authors' words. In formulas:

$$Lie(G) = Lie(ISO(3,1)) \oplus \mathfrak{h}. \tag{1.1}$$

Here $\mathfrak{h}$ is the Lie algebra of the additional symmetry generators, and $\oplus$ indicates the direct sum of Lie algebras (or sum of ideals). The Lie algebra $\mathfrak{h}$ can be further

[1] As we will see, the authors didn't know Supersymmetry, so here supermultiplet means just multiplet.

characterized: it is the direct sum of simple compact Lie algebras and U(1) Lie algebras.[2]

What about the symmetry operators that are considered? These are *"symmetries of the S matrix"*. Given U among them, it must respect the following properties [1]:

> *"(1) U turns one-particle states into one-particle states; (2) U acts on many-particle states as if they were tensor products of one-particle states; and (3) U commutes with S."*

A further property of this kind of operators that comes implicitly in the definition, is that they leave the vacuum state invariant; this is a crucial point, since not all the symmetries of a physical theory share this property.

Even if the Coleman-Mandula theorem doesn't require the theory to be defined in terms of a local action, it is simpler to consider now this restriction. A symmetry of the action $I[\Phi]$ is a transformation on the fields Φ that leaves the functional unchanged; let's call $\tilde{G}$ the group made up by these transformations. In general, symmetries of the S-matrix (and of the vacuum) are also symmetries of the action, but the converse is not true, and so $G < \tilde{G}$. When the inclusion is strict, the group $\tilde{G}$ is said to be spontaneously broken down to G. The theorem, therefore, deals with only a restricted set of symmetries; using a terminology which will be more clear in the following, with those that are *linearly realized*.

From a purely logical point of view (later, few words on the history), one could ask if there are physically interesting ways of violating one of the hypotheses that would lead to an algebraic structure of symmetry generators which is not of the form (1.1).

A first possibility, suggested by the previous discussion, is to include in the analysis the set $Lie(\widetilde{G})$ of all the symmetry generators of the action: is it of the same form as $Lie(G)$, or can it include also extra 'spacetime' symmetries?[3] The answer is that in general it can, and the first part of the present work deals with a study of spontaneously broken *spacetime* symmetries.

A second subtle possibility is to consider generators which are again symmetries of the S-matrix but that, instead of forming a Lie algebra, generate a *graded* Lie algebra: a more general algebraic structure containing also elements which are subject to anticommutation relations. Also in this context, the answer to the question is positive: there can be consistent scattering matrices which are invariant under extra operators that do not commute with the Poincaré algebra. Moreover, all the possibilities are classified, leading to the finite class of *Supersymmetry* algebras. The idea that Supersymmetry could be a key to understand particle physics led to tremendously many applications, particularly in the field of phenomenology; one of these applications will be the content of the second part of this work.

Let me now show a couple of examples, one for each of the above scenarios. Consider a six-dimensional flat spacetime in which a 3-brane lives; this configuration

[2] The property of compactness comes essentially from the fact that these symmetries act like unitary operators on finite-dimensional multiplets, as required by the first hypothesis and, as a matter of fact, non-compact groups don't admit finite-dimensional unitary representations.

[3] That is, symmetries which don't commute with the Poincaré algebra.

– which we can treat as the ground state of the theory – is not invariant under the full six-dimensional Poincaré $ISO(5,1)$, since for example any translation that is not parallel to the brane would change the configuration, and the same for boosts. What are then the symmetries that leave the vacuum invariant? They are:

$$Lie\left(ISO(3,1)_{\parallel}\right) \oplus Lie\left(SO(2)_{\perp}\right). \tag{1.2}$$

Consider now the excitations of the brane: these are described by two bending modes, each of which lives effectively in a four-dimensional spacetime. A consistent theory for them should be invariant not just under the unbroken generators (1.2), but under the full six-dimensional Poincaré. Later, we will see how to construct a class of actions that includes this case. For the time being, let's just stress two points: (1) $Lie(ISO(5,1))$, the algebra of the group $\widetilde{G}$ of the symmetries of the action, is not of the form $Lie(ISO(3,1)) \oplus \mathfrak{h}$;[4] therefore, a theory describing the brane bending modes would be an example of the first kind we were seeking; (2) the algebra of unbroken generators is instead of exactly that form – as it should, not to contradict the theorem –, being in particular $SO(2)$ a compact simple group which, from the four-dimensional point of view, has to be seen as an internal symmetry group, classifying multiplets and commuting with the S-matrix.

The next example is an instance of an algebra of symmetries of the S-matrix which is a graded one, probably the most paradigmatic: the '$N = 1$' Supersymmetry algebra. This is defined by the following commutators and anticommutators between the Poincaré generators and additional operators $Q_{\alpha=1,2}$ and $\bar{Q}_{\dot{\alpha}=1,2}$:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\,(\sigma^\mu)_{\alpha\dot{\alpha}}\, P_\mu, \tag{1.3}$$

$$[J_{\mu\nu}, Q_\alpha] = i\,(\sigma_{\mu\nu})_\alpha^{\ \beta}\, Q_\beta, \qquad \left[J_{\mu\nu}, \bar{Q}_{\dot{\alpha}}\right] = i\,(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\ \dot{\beta}}\, \bar{Q}_{\dot{\beta}}. \tag{1.4}$$

The anticommutators between the Q's and those between the $\bar{Q}$'s vanish, and similarly the commutators between them and the translations. The object called σ^μ is the four-vector of two by two matrices $(id, -\vec{\sigma})$;[5] $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are also two by two matrices (obtained by specific combinations of two σ^μ's) which, through relations (1.4), identify the new operators as *spinors* of the Lorentz group.

Interacting four-dimensional theories can be built that are symmetric under this algebra, a prominent example being the Minimal Supersymmetric Standard Model, MSSM in brief. Again, from the algebraic perspective, we were able to violate the triviality of formula (1.1), this time by including objects satisfying anticommutation relations.

On the physics side, formula (1.1) implies that there cannot be symmetry multiplets made by particles with different spin; by evading it, Supersymmetry gives its most beautiful and striking prediction: *supermultiplets*. A massive vector, a Dirac spinor and a real scalar can now be part of the same 'super-massive-vector', whose component particles are transformed into each other by Supersymmetry transformations.

[4] *Example.* Let the perpendicular directions be 4 and 5. The generator of boosts in the 5^{th} direction, which is broken by the brane, does not commute with the generator of boosts in the 1^{st} direction: $[J_{01}, J_{05}] = J_{15}$.

[5] $\sigma_{1,2,3}$ are the Pauli matrices

Historically, the idea of four-dimensional Supersymmetry [2] was not driven by the attempt of generalizing the Coleman-Mandula analysis to graded algebras: it was inspired by earlier speculations on a two-dimensional version of it, proposed in the context of String Theory [3]. Still, the authors Wess and Zumino became aware of the apparent contradiction with the theorem, correctly arguing that it was due to the fermionic[6] nature of the new generators. On the other side, the existence of *"secret symmetries"*[7] in particle physics was already well known at the time when the theorem was proposed: the paper by Coleman and Mandula does in fact explicitly state that the result is valid only for symmetries of the S-matrix. It is nice, by the way, to underline that is intrinsically necessary – due to the theorem – to realize non-linearly the spacetime symmetries. As we now know, this is of course true only *modulo* the peculiar (and finite) set of super-symmetries [4]: any other spacetime symmetry which is not of this kind, should be only looked for in its secret realization. To close the introduction without forgetting very important pieces of information, let me mention that Supersymmetry, even if could in principle be a symmetry of the spectrum of particles, is not so in the world we know: once more, to study realistic implementations of Supersymmetry, one has to understand its spontaneous breaking.

[6] This is a more directly physical way of talking about generators satisfying anticommutation relations.

[7] That is, symmetries which are spontaneously broken.

Chapter 2

Spontaneously Broken Spacetime Symmetries

One could ask what is the simplest object that carries with itself complete knowledge of the rotation group. This could be the set of vectors of three-dimensional real space R^3 with standard norm, or perhaps a rigid sphere S^2 with constant curvature. In both cases, the set of rotations finds a natural definition, being in the first case the group of three by three matrices (linear transformations) that preserve the norm of vectors, and in the second the set of isometries. Of course these two descriptions are related: a sphere of Gaussian curvature k is equivalent to the set of points in R^3 distant $1/\sqrt{k}$ from the origin; furthermore, any rotation of the ambient space brings the sphere into itself preserving all of its metric properties, being this true for any sphere about the origin. From this perspective, one could conclude that the linear space carries some redundancy, since it can be split into subsets that are transformed into themselves under the action of rotations. These subsets can be unambiguously characterized by their radius, and cannot be further split: from this last property they are called orbits of the group action. Another apparent gain in minimality can be read in the dimensionality, since the sphere has one dimension less. What about the mathematical simplicity? Points on a sphere are described by two coordinates, whose transformations under rotations are in general somewhat complicated. On the contrary, rotation matrices act *linearly* on vectors. Related to this, R^3 as a linear space has another important feature: it has a point which is left untouched by every rotation: the origin or zero vector. On a sphere, instead, every point is moved by some rotation. Finally, let me mention one last point. The choice of a sphere – a physical one, one that carries some metric information – implies necessarily a choice of scale: its curvature or radius; again, this is to be contrasted with the case of a linear space, where all scales are a priori included.

Far from having answered the initial question, we face now two peculiar scenarios, sharing distinct geometric features but being nonetheless strictly related. We will see later that, put in a physical context through the concept of field, the two become paradigm of field manifolds carrying respectively a linear or a non-linear realization of a group of symmetries, the group being spontaneously broken in the non-linear case.

2.1 Particles and Fields

The main conceptual tool used by particle physicists is that of *field*. I like therefore to spend a few words in showing the beautiful relation existing between a relativistic particle and its quantum field [5].

Particle physics experiments imply physical regimes in which both quantum-mechanical and relativistic effects are relevant. This observation leads to conclude that a proper description of these kind of systems has to involve a Hilbert space of states $\mathfrak{H}$ which carries a representation of the Poincaré group through unitary operators

$$U(\Lambda, a) : \mathfrak{H} \to \mathfrak{H}, \tag{2.1}$$

where Λ is a Lorentz transformation and a a spacetime translation; the generators of the U's are hermitian operators satisfying the Poincaré algebra.

Starting from an intuition of their basic properties, we seek those states in the Hilbert space which describe single particles. Since the generators of translations all commute between themselves, it is natural to diagonalize them simultaneously in the Hilbert space. Any state $\Psi_{\dots}$ will therefore be characterized by a label p such that

$$P^\mu \Psi_{p,\dots} = p^\mu \Psi_{p,\dots}. \tag{2.2}$$

A single particle state is a state that has just a single continuous label. All of its other degrees of freedom are then collected in the discrete label σ, so that the state will be denoted as

$$\Psi^{(1)}_{p,\sigma}.$$

By using the minimal equipment in the Hilbert space – that is the operators in (2.1) –, the objective is now to understand the nature of the transformations over the discrete index: this will bring to the most basic classification of the different kinds of relativistic particles. Since translations act trivially on these states, let's study the effect of Lorentz transformations. The fact that $U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\rho P^\rho$, implies that the state $U(\Lambda)\Psi^{(1)}_{p,\sigma}$ must be a linear combination of states with momentum Λp:

$$U(\Lambda)\Psi^{(1)}_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma} \Psi^{(1)}_{\Lambda p,\sigma'}. \tag{2.3}$$

Using this, and given that any momentum p can be transformed into any other p' through a Lorentz transformation – provided they lie on the same hyperboloid – we can carry on the following construction: take a reference momentum k lying in the hyperboloid and, for any p, take a standard Lorentz transformation $L(p)$ such that $L(p)k = p$; then *define*:

$$\Psi^{(1)}_{p,\sigma} \equiv U(L(p))\Psi^{(1)}_{k,\sigma}. \tag{2.4}$$

Acting now with a Lorentz transformation on $\Psi^{(1)}$, we find:

$$U(\Lambda)\Psi^{(1)}_{p,\sigma} = U(\Lambda)U(L(p))\Psi^{(1)}_{k,\sigma} = U(L(\Lambda p))U(L(\Lambda p))^{-1}U(\Lambda)U(L(p))\Psi^{(1)}_{k,\sigma}$$

$$= U(L(\Lambda p))U\left(L(\Lambda p)^{-1}\Lambda L(p)\right)\Psi^{(1)}_{k,\sigma} = U(L(\Lambda p))U\left(W(\Lambda, p)\right)\Psi^{(1)}_{k,\sigma}$$

$$= U(L(\Lambda p)) \sum_{\sigma'} \mathcal{D}_{\sigma'\sigma}(W(\Lambda, p)) \Psi^{(1)}_{k,\sigma'} = \sum_{\sigma'} \mathcal{D}_{\sigma'\sigma}(W(\Lambda, p)) \Psi^{(1)}_{\Lambda p,\sigma'}. \tag{2.5}$$

This is of the form we were seeking: some specific matrix (to be described) acting only on the discrete indices; that $U(W)$ acts just on those is because, by construction, $W(\Lambda, p)$ is a Lorentz transformation that keeps the reference four-momentum k fixed.[1] The matrices $\mathcal{D}$ do in fact furnish a representation of the subgroup of $ISO(3,1)$ whose elements respect $W^{\mu}{}_{\nu} k^{\nu} = k^{\mu}$, which is called little group. If k is timelike – and for massive particles this is the case –, this group is $SO(3)$, implying that massive particles are classified according to their *spin*, pretty much like in non-relativistic quantum mechanics. If k is null, the little group is $ISO(2)$, whose semisimple part is $SO(2)$, corresponding to rotations along the direction of the particle's motion; massless particles are therefore characterized by a single number, called *helicity*, which turns out to be a half-integer.

* * *

Having classified all possible kinds of particles, we face now the task of building a theory for them, a fundamental step being the construction of the Hamiltonian operator. Pretty much like in classical mechanics, where the Hamiltonian is a function of the dynamical variables p and q, here it is constructed in terms of fundamental operators which encode the dynamical degrees of freedom of the theory (that is our particles); these are the creation and annihilation operators. Given a state with N particles

$$\Psi^{(N)}_{\mathbf{p}_1\sigma_1 n_1 \ldots \mathbf{p}_N \sigma_N n_N}, \tag{2.6}$$

the creation operator $a^{\dagger}_{\mathbf{p}\sigma n}$ has the property of adding to the left of the list a particle with the specified quantum numbers[2]

$$a^{\dagger}_{\mathbf{p}\sigma n} \Psi^{(N)}_{\mathbf{p}_1\sigma_1 n_1 \ldots \mathbf{p}_N \sigma_N n_N} = \Psi^{(N+1)}_{\mathbf{p}\sigma n, \mathbf{p}_1\sigma_1 n_1 \ldots \mathbf{p}_N \sigma_N n_N}. \tag{2.7}$$

Together with its adjoint $a_{\mathbf{p}\sigma n}$, which has the effect of destroying a particle with the given quantum numbers, it is seen to respect the following relation:

$$a_{\mathbf{p}\sigma n} a^{\dagger}_{\mathbf{p}'\sigma' n'} \mp a^{\dagger}_{\mathbf{p}'\sigma' n'} a_{\mathbf{p}\sigma n} = E(\mathbf{p}, n) \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} \delta_{nn'}, \tag{2.8}$$

the sign being plus when both particles are fermions, minus if at least one is a boson. Two identical particles are bosons if the state vector which describes them remains the same after their exchange, fermions if it changes sign. It is easy with these ingredients to construct the *free* Hamiltonian, whose value on a state with given particles is just the sum of their energies:

$$H_0 = \sum_{n,\sigma} \int \frac{d^3 p}{E(\mathbf{p}, n)} a^{\dagger}_{\mathbf{p}\sigma n} a_{\mathbf{p}\sigma n} E(\mathbf{p}, n). \tag{2.9}$$

[1] Its definition is implicit in the second line.

[2] Here momentum $\mathbf{p}$, spin (or helicity) σ and particle type n are specified.

What about an interacting one? It can be shown that the interacting part V of the Hamiltonian will lead to a Poincaré covariant theory if written as the integral of a local function

$$V = \int d^3x \mathcal{H}_I(\mathbf{x}, t), \tag{2.10}$$

the integrand being a Lorentz scalar which commutes with itself at spacelike separations:

$$U(\Lambda, a)\mathcal{H}_I(x)U(\Lambda, a)^{-1} = \mathcal{H}_I(\Lambda x + a), \tag{2.11}$$

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \ \text{ if } \ (x - x')^2 < 0. \tag{2.12}$$

The problem we face now is of practical nature since, as can be deduced from (2.5), creation and annihilation operators transform under Lorentz transformations with a matrix depending on their four momentum:

$$U(\Lambda, a)a^\dagger_{\mathbf{p}\sigma n}U(\Lambda, a)^{-1} = e^{-i(\Lambda p)\cdot a}\sum_{\sigma'} \mathcal{D}^{(n)}_{\sigma'\sigma}(W(\Lambda, p))a^\dagger_{\mathbf{p}_\Lambda \sigma' n}, \tag{2.13}$$

making the problem of constructing the most general scalar a difficult task, especially for particles with spin (of course some of these can be built easily and are manifestly covariant, like for example the free Hamiltonian). This motivates the search for a more clever rearrangement of the degrees of freedom of the theory, and the introduction of fields, which are particular spacetime dependent linear combinations of the creation and annihilation operators that transform under a Poincaré transformation according to a representation of the homogeneous Lorentz group:

$$U(\Lambda, a)\psi_\ell(x)U(\Lambda, a)^{-1} = \sum_{\ell'} D_{\ell\ell'}(\Lambda^{-1})\psi_{\ell'}(\Lambda x + a). \tag{2.14}$$

By the use of fields, scalars are simply made by properly contracting the ℓ indices with constant tensors. The additional requirement of causality expressed in (2.12) asks for fields with trivial relations of commutation or anticommutation[3] at spacelike separations.

Let's then restate the problem: given a particle with certain mass and spin – the negatively charged electron with spin 1/2 and mass of 0.51 MeV, for example, or the massless photon with helicity 1 –, we should find a proper linear combination of creation and annihilation operators that makes a field – that is an object transforming like in (2.14) with *some* D – which is also causal. Many questions naturally arise: is this always possible? What is the relation between the spin of the particle and the representation D that rules the corresponding field? What governs the choice between commuting or anticommuting fields? The analysis of the field construction leads to many amazing answers, the most famous being the spin-statistics theorem, stating that causal bosonic and fermionic fields can only be made out of particles with respectively integer and half-integer spin or helicity.

The simplest field to construct is the scalar, which, it turns out, must be made of creation and annihilation operators for a spinless boson:

$$\varphi(x) = \int d^3p \frac{1}{\sqrt{2(2\pi)^3}} \left(a_{\mathbf{p}}e^{ip\cdot x} + a^\dagger_{\mathbf{p}}e^{-ip\cdot x}\right). \tag{2.15}$$

[3] Depending on whether the particles they are made of are bosons or fermions.

What about the fields describing the particles mentioned above? The electron, whose nature would make it suitable for the construction of a fermionic field with a spinorial representation of the Lorentz group, finds an interesting difficulty related to its property of carrying an electric charge $-e$, since creation and annihilation operators carry in this case opposite charge:

$$[Q, a_{\text{el}}^\dagger] = -e a_{\text{el}}^\dagger, \qquad [Q, a_{\text{el}}] = +e a_{\text{el}}. \tag{2.16}$$

To build an interaction density that commutes with the charge operator Q, fields are needed that have simple commutation relation with it and, since causality requires the electron field to be constructed with both creation and annihilation fields, the only possibility is to postulate the existence of a particle with the same mass and spin of the electron, but with opposite charge $+e$, such that $[Q, a_{\overline{\text{el}}}^\dagger] = +e a_{\overline{\text{el}}}^\dagger$; the *positron* and electron can then be combined to build the positively charged electron field, which is a causal fermionic field of Dirac type:

$$\Psi_\ell(x) = (2\pi)^{-3/2} \sum_\sigma \int d^3p \left(u_\ell(\mathbf{p}, \sigma) e^{ip\cdot x} a_{\mathbf{p}\sigma,\text{el}} + v_\ell(\mathbf{p}, \sigma) e^{-ip\cdot x} a_{\mathbf{p}\sigma,\overline{\text{el}}}^\dagger \right), \tag{2.17}$$

where the u's and v's, the form of which is not made explicit, are an example of a non-trivial solution to the initial problem of making a homogeneous field out of particles with some given spin, 1/2 in this case.

In the attempt of building a field for the photon, it turns out that it is impossible to construct a four-vector[4] out of the two helicity states ± 1 of the particle. It is possible, on the other hand, to construct an object $\mathcal{A}_\mu$ which transforms in the following way:

$$U(\Lambda)\mathcal{A}_\mu(x)U(\Lambda)^{-1} = \Lambda^\nu{}_\mu \mathcal{A}_\nu(\Lambda x) + \partial_\mu \Omega(x, \Lambda). \tag{2.18}$$

The second term makes it insufficient, for the purpose of constructing invariants, to simply contract $\mathcal{A}$ and other vectorial or tensorial fields with the invariant tensor $\eta_{\mu\nu}$. Still, it is possible to build scalars by contracting it with conserved vectorial currents, that is fields $j^\mu(x)$ such that $\partial_\mu j^\mu = 0$. Another well behaved object is

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \tag{2.19}$$

which is a tensor under Lorentz transformations, since the spurious term $\partial\Omega$ gets cancelled here. In other words, Poincaré covariance requires that the Hamiltonian density $\mathcal{H}_I(x)$ for a theory with photons should be invariant not only under Lorentz transformations of the fields, but also under transformations of the field $\mathcal{A}$ by a real function $\Omega(x)$:

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \Omega. \tag{2.20}$$

But this is the usual requirement of gauge invariance for the electromagnetic potential! In this logic, this is not a postulate, since it comes from more basic requirements.

[4]That is, an object which transforms according to the standard vector representation of the Lorentz group.

Once the proper field is assigned to each particle, we can begin the construction of the interaction density $\mathcal{H}_I$ as a polynomial in the fields and their derivatives. Using the methods of perturbation theory, this can be confronted with the rate of particle scatterings (and other observables related to the S-matrix) and in principle be fixed, at least order by order in, say, the number of fields.

It can be shown that this formulation of perturbation theory, based on the splitting $H_0 + V$ of a Hamiltonian H acting on a Fock space of particles, is equivalent to a path-integral formulation, where the fundamental object is the action, which is the integral over spacetime of a Lagrangian density:

$$I[\phi] = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x), ...). \tag{2.21}$$

The Lagrangian density is split into a free part $\mathcal{L}_0$, which is quadratic in the fields, and an interacting part $\mathcal{L}_I$.[5] In this formalism, physical amplitudes are computed as functional integrals over all field configurations of some product of operators, weighted by the exponential of the action:

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_N(x_N) \rangle = \mathcal{N} \int \left[\prod_x d\phi(x) \right] \mathcal{O}_1(x_1) \dots \mathcal{O}_N(x_N) e^{iI[\phi]}. \tag{2.22}$$

The perturbation theory is then obtained by expanding $\exp(i \int d^4x \mathcal{L}_I)$, leaving an infinite sum of calculable Gaussian path-integrals over the quadratic action $\int d^4x \mathcal{L}_0$.

More than just being able to reproduce the perturbative expansion for the S-matrix, the path-integral formulation gives a very intuitive way to handle non-perturbative phenomena. Moreover, the ‘particle-based’ formulation presented before relies somehow on the evidence of a unique vacuum state over which we perform experiments; the new formalism does instead make more clear that any theory defines many possible vacua, equivalent or not, each with its own S-matrix. Finally, Lorentz invariance is guaranteed simply by taking the Lagrangian density to be a scalar: in the next section, we will consider a Lorentz and gauge invariant Lagrangian density and take it as a starting point to discuss further properties of the theory.

2.2 The Geometric Principle

Suppose that a theory for some spinless particles is built, whose Lagrangian takes the form:

$$\mathcal{L} = -\frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi). \tag{2.23}$$

Following from a construction like the one presented in the previous section, $\mathcal{L}$ automatically satisfies some properties, a fundamental one being the positivity of the symmetric matrix g.[6] Suppose now that we do a field redefinition $\phi = \phi(\phi')$;

[5] In the simplest cases, this is just $-\mathcal{H}_I$.

[6] It is the necessity of equation (2.9) that implies this property at the Lagrangian level.

since any properly defined observable is independent of this choice, we state that

$$\mathcal{L}' = -\frac{1}{2}\left(g_{ij}(\phi(\phi'))\frac{\partial\phi^i}{\partial\phi'^k}\frac{\partial\phi^j}{\partial\phi'^l}\right)\partial_\mu\phi'^k\partial^\mu\phi'^l - V(\phi(\phi')) \tag{2.24}$$

describes the same physics as $\mathcal{L}$. This simple observation leads naturally to the description of fields as coordinates on a manifold $\mathcal{M}$, the field redefinition as a change of coordinates, and finally g as a Riemannian metric on $\mathcal{M}$ (as can be seen from its transformation). A field configuration is a mapping $\Phi : \Sigma \to \mathcal{M}$ from the spacetime to the field manifold, and the Lagrangian can be written as a pullback

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}[\Phi^* g]_{\mu\nu} - V \circ \Phi. \tag{2.25}$$

Extending these observations, we take as a principle that any local field theory can be cast in the language of differential geometry [6]. This is quite powerful, because it leads naturally to the investigation of which is the *global* structure of the manifold.

This is a remarkable step, especially from the point of view according to which the Lagrangian is built from particle physics experiments: from the global point of view, these experiments do in fact probe just a neighborhood of a special field configuration, that is the vacuum. Not any field configuration can be a particle physics vacuum: Poincaré invariance requires that it must be uniform and isotropic, so that only scalars can take non-zero values; furthermore, their value has to be constant in spacetime. In the geometric language, the vacuum identifies a point ϕ_∞ on the scalar manifold $\mathcal{M}$. Any physical state with a finite number of particles is just a small perturbation on top of this configuration; technically, the expectation value of any field is the same as the vacuum one:

$$\left(\Psi^{(N)}, \phi(x)\Psi^{(N)}\right) = \left(\Psi^{(0)}, \phi(x)\Psi^{(0)}\right) = \phi_\infty. \tag{2.26}$$

In making particle experiments, we are doing something very similar to the Taylor expansion, the reconstruction of a function $f(x)$ from its derivatives at a point x_0. Under this similarity, the function corresponds to the Lagrangian, and the terms of the Taylor expansion to the various pieces of its interacting part. Once a global meaning is given to the fields of the Lagrangian, other striking properties of the theory can be deduced like, for example, the fact that the vacuum we collide particles on is just a metastable state, and that in field space there exists a less energetic – and therefore more "true" – vacuum.

* * *

An interesting possibility is that the Lagrangian (or some specific term in it) is invariant under the action on the fields of a symmetry group G; at the infinitesimal level, this corresponds to the existence of vector fields $K^i_A\partial_i$ on $\mathcal{M}$ acting on coordinates like

$$\phi^i \to \phi^i + \epsilon^A K^i_A. \tag{2.27}$$

For this transformation to be a symmetry of $\mathcal{L}$, it must be a symmetry of the metric, meaning that G is a subgroup of the isometries of $\mathcal{M}$; the above fields are then called Killing fields. The existence of symmetries has fundamental consequences both at the local and global level: on one side, it implies the existence of conserved currents, on the other it provides indication on the overall structure of the manifold it acts on.

Consider for example the following two theories, for respectively two and three scalar fields $\varphi^{i=1,2}$ and $\phi^{I=1,2,3}$:

$$\mathcal{L}_\varphi = -\frac{\partial_\mu \varphi^i \partial^\mu \varphi^i}{2(1+\varphi^i\varphi^i/F^2)^2}, \tag{2.28}$$

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu \phi^I \partial^\mu \phi^I - \frac{m^2}{2}\phi^I\phi^I - \lambda\left(\phi^I\phi^I\right)^2. \tag{2.29}$$

These are manifestly invariant under the groups of rotations $SO(2)$ and $SO(3)$, acting on the fields like $\rho^{ij}\varphi^j$ and $R^{IJ}\phi^J$, where ρ is a two by two matrix satisfying $\rho^{-1} = \rho^t$, and similarly R. But this is not the end of the story, at least for $\mathcal{L}_\varphi$, which is in fact invariant under a larger group of symmetries. Consider the vector fields $\mathcal{K}_{A=1,2,3} = \mathcal{K}^i_A(\varphi)\partial/\partial\varphi^i$, whose components are

$$\mathcal{K}^i_3 = \varepsilon^{ij}\varphi^j,$$
$$\mathcal{K}^i_{1,2} = \delta^i_{1,2}(F^2 - \varphi^j\varphi^j) + 2\varphi^i\delta^j_{1,2}\varphi^j, \tag{2.30}$$

where ε is the antisymmetric matrix with $\varepsilon^{12} = 1$. The vector field $\mathcal{K}_3$ generates, according to formula (2.27), a coordinate change which is the infinitesimal form of an $SO(2)$ rotation, and it is trivially a symmetry of the Lagrangian. Less straightforwardly, it can be deduced that also $\mathcal{K}_{1,2}$ generate symmetries; in particular, calling

$$\mathcal{D}_\mu\varphi = \frac{\partial_\mu\varphi}{1+\varphi^2/F^2}, \tag{2.31}$$

its transformation under $\varphi^i \to \varphi^i + \epsilon^1\mathcal{K}^i_1 + \epsilon^2\mathcal{K}^i_2$ is given by the following field dependent infinitesimal $SO(2)$ rotation:

$$\mathcal{D}_\mu\varphi \to \begin{pmatrix} 0 & \epsilon^2\varphi^1 - \epsilon^1\varphi^2 \\ \epsilon^1\varphi^2 - \epsilon^2\varphi^1 & 0 \end{pmatrix} \mathcal{D}_\mu\varphi, \tag{2.32}$$

implying that the Lagrangian is invariant. What is the structure of the group of symmetries generated by the $\mathcal{K}_A$? This can be understood by studying the commutators of the Killing fields. It can be shown that the set of three generators is closed under brackets, and that it has the following Lie algebraic structure:

$$[\mathcal{K}_A, \mathcal{K}_B] = \varepsilon_{ABC}\mathcal{K}_C, \tag{2.33}$$

which is nothing but the Lie algebra of $SO(3)$. Even though they are both $SO(3)$ invariant, the two theories show remarkably different properties. First of all, the φ theory is intrinsically non-renormalizable: invariance under $\mathcal{K}_{1,2}$ requires terms with arbitrarily many fields. Second, and for the same reason, we cannot allow for a potential $V(\varphi)$. What about the vacuum structure? According to the previous discussion, we must study constant field configurations, but this is not the only requirement: for the vacuum to be perturbatively stable,[7] we must look for minima of the quantum effective potential, whose tree level approximation is just the classical potential V. For $m^2, g > 0$, the ϕ vacuum is simply $\phi^I = 0$. In the φ theory, things look more complicated, but it can be shown (e.g. by computing the Ricci scalar of the manifold metric, which turns out to be a constant proportional to F^2) that the fields are actually parameters on a sphere S^2, so that all points are equivalent, and we can take $\varphi^i = 0$ as the special vacuum point. It is immediate then to see that the quantization of $\mathcal{L}_\varphi$ leads to a theory of massless scalar particles; since the absence of a potential is due to symmetry reasons, this is a non-perturbative result. On the contrary, the ϕ theory allows for a mass term. Another important feature is that the quanta of the two theories fall in representations of the $SO(3)$ and $SO(2)$ groups, respectively.

The potential $V(\phi)$ we chose is not the only one which is $SO(3)$-invariant, stable and renormalizable: another possibility is to choose $m^2 < 0$. In this case, one expects the effective potential to have minima outside zero. Due to the $SO(3)$ symmetry, these must lie on a sphere of radius $v = \sqrt{-m^2/2g} + \ldots$. If we now quantize the theory on top of one of these equivalent vacua, we find that it describes a massive particle with $\mu^2 = -2m^2 + \ldots$, corresponding to the direction perpendicular to the sphere, and two massless which are tangent to the sphere. The $SO(3)$ symmetry is what dictates their masslessness and protects it from radiative corrections. Actually, in the limit in which $\mu^2 \to \infty$ at fixed v, the leading term of the effective theory for the massless excitations is exactly the φ theory.

Even if these examples are chosen somehow ad hoc, it is a remarkable fact that the Lagrangian density term $\mathcal{L}_\varphi$ is entirely fixed by symmetries: any theory which is invariant under $SO(3)$, but whose vacuum is only invariant under an $SO(2)$ subgroup, will have two massless scalars ξ^1 and ξ^2, with a Lagrangian which is given, up to reparametrizations, by

$$\mathcal{L}(\xi^1, \xi^2) = \mathcal{L}_\varphi(\xi^1, \xi^2) + \ldots, \tag{2.34}$$

where the additional terms are invariant under the vector fields of equation (2.30) and contain more that two derivatives. It is immediate to see that both the φ vacuum and, when $m^2 < 0$, the ϕ one are left invariant by just an $SO(2)$ subgroup: respectively, the one generated by $\mathcal{K}_3$ and the one made by rotations along the direction of the vacuum.

* * *

More generally, any theory

1. that is invariant under the action of an internal Lie group of symmetries G,
2. whose vacuum is invariant under a subgroup H,

will have in its spectrum as many massless scalars as $\dim G - \dim H$; moreover, their geometry is captured by the Riemannian manifold G/H with natural metric.[8]

[7] That is, letting aside tunnelling phenomena.

[8] For what concerns the example just discussed, notice that $SO(3)/SO(2)$ is isomorphic to S^2.

These particles are the Goldstone bosons. In the next section, it will be presented the standard procedure to build the most general Lagrangian for them.

Relativistic fields have been presented in section (2.1) as a powerful tool to construct Poincaré invariant actions out of creation and annihilation operators. The examples shown in this section suggest an independent motivation (which is also more tightly related to the subject of this work): while symmetries that leave the vacuum invariant are easily dealt with even in a Fock space formulation of the theory – they are in fact described by unitary operators –, symmetries that are broken by the vacuum are, in that context, practically intractable. On the contrary, they find a natural description in terms of actions on fields.

2.3 The Coset Construction

We have shown in the previous section that a scalar manifold $\mathcal{M}$ that describes physical spinless particles must possess a Riemannian[9] metric g. Furthermore, we argued that the Goldstone bosons of spontaneous symmetry breaking of a group G down to a subgroup H naturally lie on the manifold G/H, which is endowed with a canonical action of the group G:

$$g'(gH) = g'gH. \tag{2.35}$$

At the infinitesimal level, this corresponds to the existence of vector fields with commutators given by $Lie(G)$. Since we are looking for a theory which is invariant under the action of G, the metric in particular must be G-invariant. The existence of such a metric on G/H in not guaranteed for a generic choice of Lie groups, and one should in principle study conditions that allow for it, a prominent one being reductivity, that is the possibility to decompose the Lie algebra of G as

$$Lie(G) = Lie(H) \oplus X, \quad \text{with} \quad [Lie(H), X] \subseteq X. \tag{2.36}$$

A requirement on G that is physically relevant and guarantees reductivity is compactness: if the effective theory emerges as the low energy limit of a theory where G is a symmetry of the S-matrix, then we can infer from the Coleman-Mandula theorem that in fact G is compact. When this is the case, its Lie algebra has a negative definite Killing form $B(-,-)$: X can than be taken to be the orthogonal complement of $Lie(H)$ in $Lie(G)$ with respect to B.

While a geometric perspective on the problem is useful to understand its generalities and possible obstructions, the following construction in coordinates [7, 8] allows to capture why in practice some requirements are needed or helpful (like reductivity or compactness).

Consider a basis of $Lie(G)$ of the form

$$\{e_I\}_{I=1\dots n} = \{t_1 \dots t_m, x_1 \dots x_{n-m}\}\,, \tag{2.37}$$

where $\{t_i\}_{i=1\dots m}$ is a basis of $Lie(H)$. If we consider G compact, we can take the above basis to be orthonormal with respect to some negative multiple of the Killing

[9] That is, positive definite.

form $\tilde{B} = -cB$; in this case, thanks to the associative property of the Killing form, the structure constants are completely antisymmetric under exchange of the indices:

$$f_{IJK} = \tilde{B}([e_I, e_J], e_K) = \tilde{B}(e_I, [e_J, e_K]) = f_{JKI}. \tag{2.38}$$

Together with the obvious antisymmetry, this implies complete antisymmetry in all the indices. Thanks to this, reductivity follows from the closure of $Lie(H)$ which, in terms of structure constants, is encoded in the vanishing of $f_{ija} = \tilde{B}([t_i, t_j], x_a)$, implying that also $f_{iaj} = 0$ and therefore

$$[t_i, x_a] = f_{iab} x_b. \tag{2.39}$$

The existence of such a basis can be taken as the definition of a reductive coset G/H; compactness of G is a natural requirement that implies this key property.

Consider now the following group element, defined in terms of the exponential:

$$e^{\xi^a x_a}. \tag{2.40}$$

With this choice, the ξ^a parameters define a good set of coordinates on G/H, meaning that any element of the form gH has a unique representative of the form above; these coordinates will correspond to the Goldstone fields we want to write a theory for.

Let's then describe how the group G acts on these fields:

$$g e^{\xi x} = e^{\xi'(\xi;g)x} h(\xi; g) \cong e^{\xi'(\xi;g)x}. \tag{2.41}$$

We want, eventually, to write a theory which is invariant under this action. Strikingly enough, we will never need to know what is the precise form of the functions $\xi'(\xi; g)$; nicely transforming combinations are in fact provided by the following object, which is a $Lie(G)$-valued 1-form, called Maurer-Cartan form:

$$e^{-\xi x} d e^{\xi x} = D^a(\xi) x_a + A^i(\xi) t_i = D(\xi) + A(\xi). \tag{2.42}$$

Let's perform now the transformation (2.41). The first step follows simply from the fact that g is a fixed transformation, so it is not acted upon by the differential:[10]

$$\begin{aligned} e^{-\xi x} d e^{\xi x} &= e^{-\xi x} g^{-1} d(g e^{\xi x}) = h^{-1}(\xi; g) e^{-\xi' x} d(e^{\xi' x} h(\xi; g)) \\ &= h^{-1}(e^{-\xi' x} d e^{\xi' x}) h + h^{-1} dh = h^{-1} D'(\xi') h + h^{-1} A'(\xi') h + h^{-1} dh. \end{aligned} \tag{2.43}$$

Let's compare this with the previous equation; the property of being reductive crucially enters here, implying that $h^{-1} D' h$ belongs to X and can be expanded in terms of its basis $\{x_a\}_{a=1\ldots,n-m}$. Moreover, $Lie(H)$ is closed under the adjoint action by elements of H, and $h^{-1} dh$ also lies in the same subalgebra. Therefore one has the two projections:

$$D'^a x_a = D' = h D h^{-1} = D^a h x_a h^{-1} = D^a \rho_a{}^b x_b = \left(D^b \rho_b{}^a(h)\right) x_a; \tag{2.44}$$

[10] If the Lie group is a matrix group, g can be thought as some fixed numerical matrix belonging to it.

$$A'^i t_i = A' = hA^i t_i h^{-1} - (dh)h^{-1} = \left(A^j \sigma_j{}^i(h) - (dhh^{-1})^i\right) t_i. \tag{2.45}$$

The magic of the above result is that, under a general non-linear G transformation, the 1-forms D and A transform according to linear representations of H.[11] In particular, A transforms universally as an H-connection, being σ the adjoint representation of H on $Lie(H)$, and D transforms linearly according to ρ, so that any ρ-invariant positive contraction of two D's can be used to define a metric on G/H which is G-invariant, like for example:

$$\mu = D^a D^a, \tag{2.46}$$

which is unique up to a positive constant if ρ is irreducible.

Notice that in this derivation the explicit form of the coset representative has not been used. One of its virtue lies in the fact that it implies a linear transformation on the coordinates ξ when $g \in H$:

$$he^{\xi x} = he^{\xi x}h^{-1}h \cong he^{\xi x}h^{-1} = e^{\xi^a h x_a h^{-1}} = e^{\xi^a \rho_a{}^b x_b}, \tag{2.47}$$

where ρ now doesn't depend on the point of the coset, and it acts like a standard linear representation on the fields. Even though this property doesn't fix uniquely the exponential parametrization, simplicity in the computation of the Maurer-Cartan form usually does.

We see that equation (2.46) is an explicit solution to the problem of building a G-invariant metric on the coset G/H. Therefore, once the Goldstone field configuration $\Xi : \Sigma \to G/H$ is specified, we can use the general formula (2.25) to write the covariant kinetic term for the Goldtone fields:

$$\mathcal{L}_\xi = \eta^{\mu\nu}[\Xi^*\mu]_{\mu\nu} = \eta^{\mu\nu}[\Xi^*(D^a{}_b d\xi^b D^a{}_c d\xi^c)]_{\mu\nu} = \eta^{\mu\nu}\frac{\partial \xi^b}{\partial x^\mu}\frac{\partial \xi^c}{\partial x^\nu} D^a{}_b(\xi) D^a{}_c(\xi). \tag{2.48}$$

While the above term is necessary to describe dynamical particles, since it contains the free kinetic term $\partial\xi^2$, more invariants can be built with the ingredients provided by the Maurer-Cartan form: the pullback through Ξ of the forms A^i can in fact be used to define derivatives which are well behaved under G-transformations:

$$\mathcal{D}_\mu \psi^n = \partial_\mu \psi^n + A^i_\mu \, (T_i)^n{}_m \, \psi^m, \tag{2.49}$$

where $\psi(x)$ is any field which transforms under G according to a ξ-dependent representation of the subgroup H:

$$g \cdot \psi(x) = \mathcal{R}(h(\xi(x); g))\psi(x), \tag{2.50}$$

with $h(\xi(x); g)$ defined in (2.41). The matrices T_i of the previous formula are the generators of $\mathcal{R}$.[12] Thanks to the transformation properties of A, this "covariant derivative" behaves under the G action in the same way as the field it acts on.

[11]Notice that the matrices ρ and σ depend, through $h(\xi; g)$, on the point of the coset they act on.

[12]For example, it can be deduced from expression (2.44) that the generators that govern the transformations of the fields D^a_μ – the components of the pullback of D^a – are expressible in terms of the structure constants of $Lie(G)$:

$$(T_i)^a{}_b = f_{iba}.$$

By taking higher and higher covariant derivatives on the fields D^a and ψ^m, more and more objects are constructed which transform under the action of $g \in G$ according to ξ-dependent representations of H; with these, by making H and Lorentz-invariant contractions, we are actually building a Lorentz scalar which is fully G-invariant. Actually, it turns out that

> any Lagrangian density with linear H symmetry and secret G symmetry can be constructed like this.

The reach if this result is somehow weakened by the fact that not all of the symmetries of a field theory are encrypted in an invariant Lagrangian density, since also terms in $\mathcal{L}$ which change by a total derivative lead to an invariant action. Even though the Galileon theory – which will be the prominent example in the core of the following discussion – is made of terms which are not strictly invariant at the level of the Lagrangian density, a full discussion on their constructibility in a generic theory is not necessary for this work.[13] These non-invariant operators are called Wess-Zumino terms, since the first term of this kind was proposed by the two authors in the context of $SU(3) \times SU(3)/SU(3)_{\text{diag}}$ [9].

What instead will be of central importance, is the simple observation that the choice of coset element (2.40) is not unique, since for example any other basis in X can be taken to define an exponential $\exp(\tilde{\xi}\tilde{x})$ with the property of representing unambiguously the points on G/H. While different choices lead in general to different Lagrangian densities, the theories they describe are fully equivalent, and this because ξ and $\tilde{\xi}$ are just different coordinates on the same manifold. More precisely, given a theory

$$\mathcal{L}(\xi) = \mathcal{L}^{(0)}(\xi) + c_1 \mathcal{L}^{(1)}(\xi) + \ldots, \tag{2.51}$$

where $\mathcal{L}^{(0)}$ is the universal "dressed" kinetic term, and $\mathcal{L}^{(k \geq 1)}$ are other non-universal invariants or Wess-Zumino terms, this is equivalent to the theory specified by

$$\tilde{\mathcal{L}}(\tilde{\xi}) = \mathcal{L}^{(0)}(\xi(\tilde{\xi})) + c_1 \mathcal{L}^{(1)}(\xi(\tilde{\xi})) + \ldots. \tag{2.52}$$

Notice that, due to the stated completeness of the construction, this expression must be equivalent to a sum of invariants or Wess-Zumino terms made out of $\tilde{\xi}$.

We conclude that two theories which, like $\mathcal{L}$ and $\tilde{\mathcal{L}}$, are related by a "twist" of coset element, are physically equivalent. An analogous claim, in the case of theories that implement *spacetime* symmetry breaking, will be exposed to critique in the following.

2.4 Spacetime Symmetries

The Coleman-Mandula theorem implies that any bosonic symmetry generator in a relativistic particle theory that

1. is not in the Poincaré algebra $\{P_\rho, J_{\mu\nu}\}$

[13] This analysis is carried out in [10]. See also [11] for a beautiful interpretation of the 'original' Wess-Zumino interaction term.

2. does not commute with it

must be broken on the vacuum. This in itself leads to some interest in a general analysis of non-linear realizations of spacetime symmetry groups $\mathcal{G}$. In this perspective, it is remarkable that a slight modification of the construction presented in the previous section is able to capture every invariant of a theory which is symmetric under $\mathcal{G}$ broken down to $ISO(3,1) \times H$ [12, 13].

Because we consider relativistic theories, the Poincaré group is always by definition a symmetry of the vacuum; still, since the translations act non-linearly on the coordinates $y \to y + a$, and given the interplay between the translations and the other generators in this context, the coset element is taken to be

$$g(y(\xi), \pi(\xi)) = e^{y^a P_a} e^{\pi^i X_i}, \tag{2.53}$$

where $\{X_i\}$ are the broken generators, and both y^a and π^i are taken to depend on the four spacetime coordinates ξ^μ. The components of the Maurer-Cartan form, as a 1-form on the spacetime manifold, can be expanded as before on a basis of $Lie(\mathcal{G})$:

$$g^{-1}\frac{\partial g}{\partial \xi^\mu} = E^a_\mu(\xi)\left(P_a + \nabla_a \pi^i(\xi) X_i + A^I_a(\xi) T_I\right). \tag{2.54}$$

Once again, the coefficients of this expansion are all is needed to build any invariant:

- $\nabla_a \pi^i$ transform according to field and y-dependent representations of the unbroken group $SO(3,1) \times H$;
- A^I_a serve to define covariant derivatives since, pretty much like A^i_μ in the previous section, they transform, under a general $\mathcal{G}$ transformation, as H-connections;
- E^a_μ is a novel object: it transforms as a Lorentz vector in the a index, and as a 1-form under ξ^μ diffeomorphisms. In this respect, notice that μ indices are curved, so no a priori $\eta^{\mu\nu}$ is available to contract them and, moreover, any well defined action has to be also diff-invariant. On the other side, a indices are flat, and can be contracted with η^{ab}. E^a_μ is called 'vierbein'.

These observations imply that any invariant action has to be constructed with H and Lorentz-invariant contractions of the $\nabla_a \pi^i$ and their higher covariant derivatives, integrated over the volume form $d^4\xi \ \det E$, like for example:

$$S = \int d^4\xi \ \det(E^a_\mu)\left(\eta^{ab}\nabla_a \pi^i \nabla_b \pi^j \sigma_{ij} + \ldots\right), \tag{2.55}$$

where σ is an invariant (constant) tensor of the H representation carried by the π's.

The non-trivial Lie algebraic relations existing between the broken generators and the Poincaré group – peculiar to spacetime symmetry groups like $\mathcal{G}$ – lead to the possibility that commutation relations hold which take this schematic form:

$$[P, X'] = cX + \ldots. \tag{2.56}$$

When this happens, the expansion of the Maurer-Cartan form contains a term with the field π' multiplying the generator X

$$g^{-1}\partial_\mu g = e^{-\pi X-\pi' X'}\partial_\mu e^{\pi X+\pi' X'} + e^{-\pi X-\pi' X'}\left(e^{-yP}\partial_\mu e^{yP}\right)e^{\pi X+\pi' X'}$$

$$= \partial_\mu \pi X + \ldots + \partial_\mu y \left(P - \pi'[X', P] + \ldots\right) = \left(\partial_\mu \pi + \partial_\mu y \pi' c + \ldots\right) X + \ldots, \quad (2.57)$$

and the covariant condition $\nabla\pi = 0$, called an 'inverse-Higgs constraint', can be imposed and used to express π' in terms of the derivatives of π [13]. When this is possible, it means, on the physics side, that not every independent broken symmetry has a counterpart in a dynamical massless degree of freedom: *in general these are less.*

This general statement has many famous instances: the Dilaton of broken dilations, for example, is enough to carry a non-linear realization of the full conformal symmetry group broken to the Poincaré group (one degree of freedom for five broken generators); similarly, the physical excitations of a flat membrane are as many as the broken translations, while implementing also the broken Lorentz transformations. Algebraically, one finds respectively:

$$[P_a, K_b] = -2\eta_{ab}D + 2J_{ab}, \quad (2.58)$$

$$[P_a, J_{bi}] = -\eta_{ab}P_i. \quad (2.59)$$

Let's focus the attention on the case of the conformal group. Taking the coset element appropriate for its breaking to the Poincaré group, we can explicitly compute the Maurer-Cartan form. We get:

$$e^{-\Pi^b K_b}e^{-\pi D}e^{-y^a P_a}\partial_\mu e^{y^a P_a}e^{\pi D}e^{\Pi^b K_b} = e^{-\pi}\delta^a_\mu P_a + \left(\partial_\mu \pi - 2e^{-\pi}\Pi_\mu\right)D + \ldots, \quad (2.60)$$

where the dots stand for terms proportional to K_b and J_{ab}. The inverse-Higgs constraint, which is obtained by putting to zero the coefficient of D, reads:

$$\Pi_\mu = \frac{1}{2}e^{\pi}\partial_\mu \pi. \quad (2.61)$$

This condition is compatible with all the symmetries, and makes it possible to eliminate the fields Π_a from the effective theory.

In the following section, the machinery presented here will be used to introduce, from a coset perspective, the Galileon theories, whose pathologies triggered the curiosity on whether different parametrization of the coset lead always to equivalent theories.

2.5 Galileons from the Coset

Galileon theories were defined in [14] as four-dimensional relativistic theories for a single scalar, whose equations of motion take a specific form, each field being acted upon by exactly two derivatives:

$$\mathcal{E}_\pi = F(\partial_\mu\partial_\nu\pi), \quad (2.62)$$

which implies immediately that, for any solution π, also $\pi + c + b_\mu \xi^\mu$ is a solution. Notice that the converse is not true, since not every theory which is invariant under this transformation of π takes the above form (for example, one can consider equations of motion with more than two derivatives per field).

This symmetry operation, together with the standard action of the Poincaré group on a scalar field, defines an algebra of symmetries with the following commutation relations:

$$[C, B_a] = [C, P_a] = [B_a, B_b] = 0, \tag{2.63}$$

$$[P_a, B_b] = \eta_{ab} C, \tag{2.64}$$

$$[J_{ab}, B_c] = \eta_{ac} B_b - \eta_{ab} B_c. \tag{2.65}$$

Supplemented with the usual relations between the Poincaré generators, they define the Lie algebra of the Galileon group $Gal(3+1,1)$.[14] Since the galileon π transforms with shifts, its realization of the Galileon group is non-linear, and we could hope to be able to build its most general Lagrangian from a coset construction, and in particular find those terms that give equations of motion of the form (2.62). Let's then take, as coset element of $Gal(3+1,1)/SO(3,1)$,

$$g(\xi) = e^{y^a(\xi) P_a} e^{\pi(\xi) C} e^{\Omega^b(\xi) B_b}. \tag{2.66}$$

To check whether we get a consistent picture, we study the action of the group on the coordinates of the coset. By the use of the commutators above, we find:

$$e^{cC} g(y, \pi, \Omega) = e^{y\cdot P} e^{(\pi + c)C} e^{\Omega \cdot B} = g(y, \pi + c, \Omega) \tag{2.67}$$

$$\begin{aligned} e^{b\cdot B} g(y, \pi, \Omega) &= \left(e^{b\cdot B} e^{y\cdot P} e^{-b\cdot B}\right) e^{\pi C} e^{(b+\Omega)\cdot B} = \exp\left(e^{b\cdot B} y^a P_a e^{-b\cdot B}\right) e^{\pi C} e^{(b+\Omega)\cdot B}, \\ &= e^{y\cdot P} e^{(\pi - y\cdot b)C} e^{(b+\Omega)\cdot B} = g(y, \pi - y\cdot b, \Omega + b). \end{aligned} \tag{2.68}$$

We see that the fields transform as expected.

From the above coset element, the Maurer-Cartan form is easily computed and, thanks to relation (2.64), the term proportional to C can be put to zero – implementing therefore an inverse-Higgs constraint –, so that the Ω field is eliminated and the Lagrangian can be expressed in terms of the single field π:

$$g^{-1}\partial_\mu g = \partial_\mu y^a P_a + (\partial_\mu \pi + \partial_\mu y^a \Omega_a) C + \partial_\mu \Omega^b B_b, \tag{2.69}$$

$$\Omega_\mu = -\partial_\mu \pi. \tag{2.70}$$

The last equation comes after identification of y^a with ξ^μ, which is always possible in flat space.

Unfortunately, the coset construction fails in producing Lagrangian terms of the kind we were seeking. As a clear signal that something is missing, it should be noticed that, while the free kinetic term gives equations of motion of the proper kind:

$$\mathcal{L}_{\pi 2} = -\frac{1}{2}(\partial_\mu \pi)^2 \;\rightarrow\; \mathcal{E}_{\pi 2} = -\partial^2 \pi, \tag{2.71}$$

[14] The slots refer respectively to the number of spacetime dimensions, and to the number of π's; see reference [15].

which is reassuring (since we want to define a meaningful theory), it is not possible to construct it in terms of the only object that is produced by the coset construction, that is $\partial_\mu \Omega_\nu = -\partial_\mu \partial_\nu \pi$. Actually, the free Lagrangian is invariant only up to a total derivative, and it turns out that this is true for any Lagrangian which would produce equations of the desired form [14]. As it was pointed out before, any such object, called a Wess-Zumino term, is not captured by the coset procedure.

The coset language will be used, later on, to motivate a specific mapping between Galileon theories. For what concerns the solution to equation (2.62), let me, for the time being, exhibit two more between the five 'Galileons'.[15] The simplest one is the tadpole $\mathcal{L}_{\pi 1} = \Lambda^3 \pi$, whose variation is $\delta \mathcal{L}_{\pi 1} = \Lambda^3 \partial_\mu (c\xi^\mu/4 + b^\mu \xi_\nu \xi^\nu/2)$. After the kinetic term we find, with energy dimension equal to 7, the following operator:

$$\mathcal{L}_{\pi 3} = -\frac{1}{2\Lambda^3}(\partial_\mu \pi)^2 \partial^2 \pi \;\rightarrow\; \mathcal{E}_{\pi 3} = \frac{1}{\Lambda^3}\left((\partial^2 \pi)^2 - (\partial_\mu \partial_\nu \pi)^2\right). \tag{2.72}$$

This term shows a worrisome aspect. Consider the simplest Galileon theory that includes it, that is $\mathcal{L}_\pi = \mathcal{L}_{\pi 2} + c\mathcal{L}_{\pi 3}$. Pretty much like studying, in the context of QED, how do electrons travel on a classical background field $A_{0\mu}$, we can study how perturbations of the galileon field travel on their own background π_0.

Far from sources, a weak $(\partial\partial\pi_0/\Lambda^3 \ll 1)$ and stationary background satisfies $\nabla^2 \pi_0 \simeq 0$, so that the linearized equation of motion for the perturbation ϖ reads:

$$\left(\eta^{\mu\nu} - \frac{2c}{\Lambda^3}\partial^\mu \partial^\nu \pi_0\right)\partial_\mu \partial_\nu \varpi = (g_{\text{eff}}^{-1})^{\mu\nu}\partial_\mu \partial_\nu \varpi \simeq 0. \tag{2.73}$$

This is equivalent to state that perturbations follow null geodesics of the metric $(g_{\text{eff}})_{\mu\nu}$, which is given by:

$$(g_{\text{eff}})_{\mu\nu} \simeq \eta_{\mu\nu} + \frac{2c}{\Lambda^3}\delta^i_\mu \delta^j_\nu \partial_i \partial_j \pi_0. \tag{2.74}$$

The problem here is that, as a consequence of the Poisson equation, the eigenvalues of $\partial_i \partial_j \pi_0$ have different signs, so that perturbations are induced to move superluminally at least in one direction [16, 17]. As pointed out in [16], this pathology is related to the impossibility of UV completing the effective theory to a standard Lorentz-invariant and local theory.

This fact will be considered again in the following, where we study the effect of a twist of coset parametrization in a Galileon theory.

* * *

Before going into the peculiarities of the Galileon case, let me spend a few words to systematize the question in the broader context of theories for Goldstone bosons.

First, let me recall that one of the main achievements of [8] is the clarification of whether a symmetry can be realized *linearly* on the Goldstone fields. This is indeed encrypted in a sharp physical criterion: linear realization is possible whenever the

[15] Many things are called *Galileon*: a group of symmetries, a particle and a few Wess-Zumino operators.

vacuum is invariant under the symmetry operation. Due to the obvious advantage of a linear, with respect to a generic realization of a symmetry, it will always be assumed that a choice of fields is made that implements in this way any unbroken symmetry. Let's limit even more the arena of possibilities, by considering just the exponential parametrizations; this amounts to a choice of an independent set in the Lie algebra of the full group $\mathcal{G}$, which (1) completes the Lie algebra of the restricted unbroken subgroup $H \times SO(3,1)$,[16] and (2) is such that the decomposition so implied is a reductive one.

Consider then two choices of the above kind, called $\{X\}$ and $\{X'\}$. For any Goldstone field configuration ξ, there is a unique $h \in H \times SO(3,1)$ such that:

$$e^{\xi' X'} = e^{\xi X} h, \tag{2.75}$$

and an identification is induced $\xi(\xi')$. Through the methods of the coset construction, each parametrization produces a set of invariants, which can be represented by infinite columns of Lagrangians, $\mathfrak{L}_n$ and $\mathfrak{L}'_n$. A theory is a contraction with an infinite row of real couplings:

$$\mathcal{L} = c_n \mathfrak{L}_n. \tag{2.76}$$

A natural question is whether the set of theories is the same; that is, given $\mathcal{L}$ like in the above expression, if there is *any* c'_n such that $c'_n \mathfrak{L}'_n$ is equivalent to $\mathcal{L}$. A negative answer to this question would put strong doubts on the completeness of the coset construction.

What we ask is instead more specific. The mapping defined through equation (2.75) must give a one to one correspondence between invariants:

$$\mathfrak{L}'_n \cong \mathcal{U}_{nm} \mathfrak{L}_m. \tag{2.77}$$

This, in turn, induces an identification between theories in the two parametrizations

$$c_n \mathfrak{L}_n \cong c_n \left(\mathcal{U}^{-1}\right)_{nm} \mathfrak{L}'_m = c'_m(c)\ \mathfrak{L}'_m, \tag{2.78}$$

with a fixed relation between the couplings. Now, the question is whether *this* specific theory is equivalent to the one we started with. Of course, a mathematical equivalence can be *defined* in terms of these relations, but we want to go beyond this. The example, soon to be presented, of a mapping between Galileon theories – induced by a change in the parametrization of the exponential –, will clarify that a mathematical correspondence is not always enough to define theories which are equivalent from a physical prospect.

* * *

Consider the following, alternative choice of parameters on the Galileon coset:

$$g_\alpha(x^a, q, \hat{\Omega}^b) = e^{x^a P_a} e^{qC} e^{\hat{\Omega}^b (B_b + \alpha P_b)}. \tag{2.79}$$

According to the previous discussion, there is a mapping between these parameters and those of equation (2.66). This is given by the following relations:

$$y^a = x^a + \alpha \hat{\Omega}^a, \tag{2.80}$$

[16] That is, translations are treated like broken generators.

$$\Omega^b = \hat{\Omega}^b, \tag{2.81}$$

$$\pi = q - \frac{\alpha}{2}\hat{\Omega}^2. \tag{2.82}$$

Supplemented with the inverse-Higgs constraint for the new fields, which is $\hat{\Omega}_a = -\partial q/\partial x^a$, any theory written with π can be translated to a theory for q. Notice that the second relation, together with the inverse-Higgs constraint, implies that the field derivatives transform trivially: $\partial\pi(y) = \partial q(x)$.

It is important to stress that this is not a standard field redefinition: due to the interplay between translation generators and internal ones, also a change of spacetime coordinates is induced, depending in general on the fields themselves. As shown in [18], the above transformation can be rewritten as a redefinition of π at fixed spacetime coordinates, crucially involving an infinite series of terms with arbitrarily many derivatives:

$$\pi(y) = q(y + \alpha\partial\pi(y)) - \frac{\alpha}{2}(\partial\pi(y))^2 = q(y) + \frac{\alpha}{2}(\partial q(y))^2 + \dots \tag{2.83}$$

To see the mapping at work, we consider now the simplest theory for π, that is the free theory. It turns out that it is mapped to a non-trivial interacting Galileon theory:

$$\int \mathcal{L}_{\pi 2} = -\frac{1}{2}\int d^4y(\partial\pi)^2 \cong -\frac{1}{2}\int d^4x \det(\delta_a^b - \alpha\partial_a\partial^b q)(\partial q)^2$$
$$= \int (\mathcal{L}_{q2} - \mathcal{L}_{q3} + \frac{1}{2}\mathcal{L}_{q4} - \frac{1}{6}\mathcal{L}_{q5}). \tag{2.84}$$

Since the field derivatives transform trivially, the new terms in q are entirely generated by the (finite) expansion of the determinant. $\mathcal{L}_{q2}$ is the standard kinetic term for q, while $\mathcal{L}_{q3}$ is like in formula (2.72), with the substitutions $\pi \to q$ and $\Lambda^{-3} \to \alpha$. The remaining Galileons are given by:

$$\mathcal{L}_{q4} = -\frac{\alpha^2}{2}(\partial q)^2\left([Q]^2 - [Q^2]\right),$$

$$\mathcal{L}_{q5} = -\frac{\alpha^3}{2}(\partial q)^2\left([Q]^3 - 3[Q^2][Q] + 2[Q^3]\right), \tag{2.85}$$

where Q is the matrix of second derivatives of q, and $[\cdot]$ denotes the operation of taking the trace.

Given that also $\mathcal{L}_{q3}$ is present, we start wondering how the symbol $\cong$ can have a physical meaning, since it seems implausible that a theory which features superluminal propagation is equivalent to one that, being free, propagates luminal signals on any background.

It should be stressed that for a theory, like $\mathcal{L}_q$, of a scalar alone in the Universe, superluminality is not necessarily a problem: the propagation cone of excitations is simply not $\eta_{\mu\nu}$, the "coordinate one".

As was pointed out in reference [18], physical problems start to arise when the theory is coupled recklessly to other fields: our scope here is to extend that analysis, and get some insight on the relation between the two Galileon theories by coupling them to other dynamical fields.

First, it should be stressed that, if the possibility of having a q_0 dependent null cone is tolerated, then there is a simple procedure to map a theory from π to q coordinates in such a way that the new theory

1. is invariant under the Galileon transformations;

2. respects the causal structure defined by the background q_0.

To this end, one needs to transform the other fields as under the diffeomorphism defined by equation (2.80). Let's see, as a trivial example, how the theory of a free scalar ϕ is mapped to an interacting theory of q and ϕ'. Under diffeomorphisms, the scalar transforms as $\phi(y) = \phi'(x)$:

$$-\frac{1}{2}\int d^4y\ \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\frac{1}{2}\int d^4x\ \det\left(\frac{\partial y}{\partial x}\right)\eta^{\mu\nu}\frac{\partial x^\rho}{\partial y^\mu}\partial_\rho\phi'\frac{\partial x^\sigma}{\partial y^\nu}\partial_\sigma\phi'$$
$$= -\frac{1}{2}\int d^4x\ \sqrt{-\tilde{g}_{\rm eff}}(\tilde{g}^{-1}_{\rm eff})^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' \tag{2.86}$$

In the last expression, the Lagrangian for ϕ' has been rewritten in terms of the effective metric

$$(\tilde{g}_{\rm eff})_{\mu\nu} = \eta_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} = \eta_{\mu\nu} - 2\alpha\partial_\mu\partial_\nu q + \dots, \tag{2.87}$$

dictating what is the null cone along which the ϕ' excitations move. By comparison with equation (2.74), it is clear that actually $g_{\rm eff}$ and $\tilde{g}_{\rm eff}$ coincide, at least up to the first non trivial order. Upon realizing that the new Lagrangian is also *Gal*-invariant – since, besides the Galileons $\mathcal{L}_{qi}$, all the rest depends on just the second derivatives of q –, we see that the transformed Lagrangian possesses the two requirements listed above.

In general, by similar manipulations, it is possible to map a healthy theory for the π galileons, to a healthy theory for the q galileons which features a deformed propagation cone, depicted according to the background q_0.

Let me now go through these last steps with a more critical eye. A formal procedure has just been presented that tells how a field, which is inert under the Galileon transformations, has to transform under a twist of coset parametrization. Since the prescription is that it should transform under the *diffeomorphism* $y(x)$, it is useful to compare this with what happens in gravitational theories.

Let's then consider the minimally coupled kinetic term of a massless scalar ϕ. Calling $J^\mu_\nu = (\partial y^\mu/\partial x^\nu)$, field and metric transform as:

$$\partial_\mu\phi = (J^{-1})^\nu_\mu\partial_\nu\phi', \quad g^{\mu\nu} = J^\mu_\rho J^\nu_\sigma g'^{\rho\sigma}. \tag{2.88}$$

These transformations are such that the action integral is invariant in form:

$$-\frac{1}{2}\int d^4y\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\frac{1}{2}\int d^4xJ\sqrt{-g'}\frac{1}{J}J^\mu_\rho J^\nu_\sigma g'^{\rho\sigma}(J^{-1})^\tau_\mu\partial_\tau\phi'(J^{-1})^\lambda_\nu\partial_\lambda\phi'$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g'}g'^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi'. \qquad (2.89)$$

This is not the same procedure as that of equation (2.86), since in that case there is no analogue of the transformation of the metric. In fact, the scalar kinetic term transforms as follows:

$$-\frac{1}{2}\int d^4y\ \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\frac{1}{2}\int d^4xJ\ \eta^{\mu\nu}(J^{-1})^\rho_\mu\partial_\rho\phi'(J^{-1})^\sigma_\nu\partial_\sigma\phi'. \qquad (2.90)$$

One should not get confused by the fact that, here, the metric is flat and it is just a numerical matrix: indeed, we can think of it as the flat limit of a dynamical metric. Let's then be guided by the intuition that, for the coset transformation, *the metric doesn't change* (this corresponds in fact to the Minkowski metric, both before and after). We stress that this is not an option: if we transform η, then we don't recover the correct transformation of $\mathcal{L}_\pi$ into $\mathcal{L}_q$. This means that, in general, both the q galileons and the other primed particles move on a lightcone which is different from the one defined by the metric.

While on one side we are just repeating what we said before, and it seems that one has simply to abandon the prejudice that it is η that defines causality, thinking in terms of a dynamical metric leads to a puzzle. Suppose that one starts with a theory as simple as the free galileon one, minimally coupled with gravity; this has the feature that both gravitons and galileons move on the cone defined by the metric g.[17] After the mapping induced by the coset twist is performed, one has galileons moving 'outside' the metric cone; what about gravitons? Since we are talking about physical excitations, they must move on the same propagation cone as that of galileons, like they do on the other side of the mapping.

But this is *not* the normal behaviour of gravitons! Gravitational interactions, as we know them, are characterized by the very large scale M_{Pl}. In the limit $M_{Pl} \to \infty$ gravity is decoupled from matter, and its dynamics is led by the standard kinetic operator $\sqrt{-g}R$ which, in particular, makes the graviton move on the metric cone.

From this observation, we are forced to conclude that, since in the q theory gravitons do not move, in general, on the metric cone, the action of gravity must be a non-standard one.

The point of view that we took in our work [19] is that, for the matter Lagrangians $\mathcal{L}_\pi$ and $\mathcal{L}_q$ to be called *equivalent*, they have to "see" gravity in an analogous way: at least by sharing the same notion of the limit $M_{Pl} \to \infty$.

Through the computations that we are reporting in the next sections, the scope is to

1. present a rigorous way to couple gravity in a 'coset invariant' way, so to properly define the Galileon twist in the presence of a dynamical metric;

2. analyse the properties of the new action, with emphasis on its scales and on the regime in which the galileons become superluminal;

3. corroborate the statement that the two theories are not equivalent in the above sense.

[17] The Minkowski light cone, in the flat space limit.

2.6 Coupling to Gravity

Before going into the details of the Coset Construction in the presence of gravity, let me spend a few words on a formulation of pure gravity which, due to its accent on algebraic properties, can be considered as its starting point [20].

Even though, as a first principle mathematical formulation, it ultimately finds legitimation in its capability of describing physics correctly, it is worth to try and list what are the physical intuitions that guide it.

On one side, there is the evidence that any event can be identified uniquely thanks to a set of four coordinates, to be called ξ^μ. There are infinitely many choices of good coordinates, any such two being related by an invertible mapping: if ξ'^μ is another set of coordinates, then it must be that

$$J = \det\left(\frac{\partial \xi'^\mu}{\partial \xi^\nu}\right) \neq 0, \tag{2.91}$$

where the functions $\xi'^\mu(\xi)$ are defined by identifying labels that label the same event. This suggests a formulation in terms of a spacetime manifold $\mathcal{M}$.

On the other side, there are all the group theoretic properties of the Minkowski spacetime, that one wants to mimic and transpose in this context. First of all: its group of symmetries, which is the Poincaré group $ISO(3,1)$ and is taken here as fundamental.

One is therefore led to consider, as the basic mathematical object, a 'principal bundle' P over the manifold $\mathcal{M}$, with $G = ISO(3,1)$ as its structure group.

The fibers of the bundle are glued together smoothly, but there is no preferred way to move, so to say, from one to the other. The identification of such a preferred direction corresponds to the introduction of a connection on the bundle. Once a local trivialization of P is specified, the connection can be defined in terms of a set of gauge fields, as many as the generators of the group:

$$\mathcal{A}_\mu(\xi) = \tilde{e}^a_\mu(\xi) P_a + \frac{1}{2}\tilde{\omega}_\mu{}^{ab}(\xi) J_{ab}. \tag{2.92}$$

For a change of local trivialization – which, in coordinates, corresponds to the specification of a function $g(\xi)$ with elements in the structure group –, the gauge fields transform as follows; for $g(\xi) = e^{ia^a(\xi)P_a}$:

$$\tilde{e}'^a_\mu = \tilde{e}^a_\mu - \tilde{\omega}_\mu{}^{ab} a_b - \partial_\mu a^a, \qquad \tilde{\omega}'_\mu{}^{ab} = \tilde{\omega}_\mu{}^{ab}, \tag{2.93}$$

while, for $g(\xi) = e^{i\alpha^{ab}(\xi)J_{ab}}$:

$$\tilde{e}'^a_\mu = \Lambda(\alpha)^a{}_b \tilde{e}^b_\mu, \qquad \tilde{\omega}'_\mu{}^{ab} = \Lambda^a{}_c \Lambda^b{}_d \tilde{\omega}_\mu{}^{cd} + (\Lambda \partial_\mu \Lambda^{-1})^{ab}. \tag{2.94}$$

If a theory is to be constructed out of these ingredients – the gauge fields –, invariance under the above transformations must be guaranteed. Along with this, one has to follow the usual rules of general covariance under coordinate transformations on the base manifold $\mathcal{M}$.

The picture, so far, looks a bit too abstract. To make contact with a space that locally resembles Minkowski spacetime, an additional algebraic property must be taken into account: that, as explained before, the translations are realized non-linearly on the Minkowski coordinates y^a. To properly take this into account, a 'spontaneous breaking' is introduced through the function

$$e^{y^a(\xi)P_a}. \tag{2.95}$$

In the previous sections, a procedure has been developed to construct simply behaved objects out of non-linearly transforming fields, centred in the computation of the Maurer-Cartan form for the proper coset element. This involves crucially a derivation with respect to the spacetime coordinates.

In the context of gauge theories, derivatives of fields which transform non-trivially under the gauge symmetry are ill-behaved: gauge-covariant derivatives are therefore constructed with the help of gauge fields.

From the above observations, we conclude that to make the Maurer-Cartan form compatible with a gauge symmetry, the derivative must be substituted with the gauge-covariant one.

Let's then come back to the 'algebraic' formulation of gravity: from a purely formal point of view, we know that a nicely behaving object will be produced out of the non-linearly transforming fields y^a and of the gauge fields $\tilde{e}^a_\mu$ and $\tilde{\omega}_\mu{}^{ab}$, if we combine them as follows:

$$e^{-y^a(\xi)P_a}\left(\partial_\mu + \tilde{e}^a_\mu(\xi)P_a + \frac{1}{2}\tilde{\omega}_\mu{}^{ab}(\xi)J_{ab}\right)e^{y^a(\xi)P_a} \equiv e^a_\mu P_a + \frac{1}{2}\omega_\mu{}^{ab}J_{ab}. \tag{2.96}$$

As a matter of fact, it turns out that: (1) ω (which is simply equal to $\tilde{\omega}$), serves to define covariant derivatives for fields that transform non-trivially under the Lorentz group, and is called spin connection, while (2) $e^a_\mu = \tilde{e}^a_\mu + \partial_\mu y^a + \omega_\mu{}^{ab}y_b$ is a vierbein. Like in Yang-Mills theories, tensors can be constructed from the non-tensorial connections by proper antisymmetrized derivation. Once this is done, invariant actions take the form

$$\int d^4\xi \det e\ \mathcal{L}(\omega, e). \tag{2.97}$$

Before going into the details of this construction, let me briefly introduce a notation which is more economic, based on the suppression of spacetime indices and the use of form-theoretic formalism. In particular, ∂_μ is translated into the exterior derivative d, while the vierbein and spin connection become simply e^a and ω^{ab}. The wedge symbol $\wedge$ is used to produce higher order forms, and it has to be taken into account that a minus sign is picked up whenever d encounters a $\wedge$ (the first chapters of [21] are a very detailed reference for this formalism).

The following tensors can then be built: the curvature tensor, which is an $so(3,1)$-valued two-form, and the torsion tensor, a vector-valued two-form:

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \tag{2.98}$$

$$T^a = de^a + \omega^a{}_b \wedge e^b. \tag{2.99}$$

The Einstein-Hilbert action is then recovered by firstly putting to zero the torsion tensor, in such a way that ω becomes a function of the vierbein:

$$\omega_\mu{}^{ab} = \frac{1}{2}e^{\nu a}\left(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu\right) + e_{\mu c}e^{\nu a}e^{\lambda b}\partial_\lambda e^c_\nu - (a \leftrightarrow b)\big), \tag{2.100}$$

and then by making the simplest invariant contraction of the curvature tensor, which is nothing but the Ricci scalar $R = e^{\mu}_a e^{\nu}_b R_{\mu\nu}{}^{ab}$, so that:

$$S_{E\text{-}H} = \frac{1}{16\pi G}\int d^4\xi \,\det e\; R. \tag{2.101}$$

To solve the torsion-free condition for ω and to contract the inhomogeneous indices of the curvature tensor, we used the inverse of the vierbein: e^{μ}_a.

* * *

At this point, the inclusion of gravity in the Coset Construction is nothing but a generalization of what was explained before. Given the coset element (2.53), one simply needs to substitute in the Maurer-Cartan form the normal derivative with the Poincaré covariant one [22]. In this way, the various coset covariant objects are now also covariant under the gauge group.

Let's then consider the Galileon case, with focus on the two parametrizations (the π and the q) of the coset element, based respectively on the following choices of broken generators (see equations (2.66) and (2.79)):

$$\mathcal{B}_\pi = \{P_a, C, B_a\}\,, \quad \mathcal{B}_q = \left\{P_a, C, B_{(\alpha)a} \equiv B_a + \alpha P_a\right\}. \tag{2.102}$$

For each parametrization, the gauge covariant Maurer-Cartan form can be computed. After equality is imposed, we find relations between vierbeins and gauge connections, old and new. Even if distant in form, these have to be recognized as a promotion of equations (2.80-2.82), valid as well in a regime in which gravity is taken into account:

$$e^a = \hat{e}^a + \alpha(d\hat{\Omega}^a + \hat{\omega}^a{}_b\hat{\Omega}^b), \tag{2.103}$$

$$\Omega^a = \hat{\Omega}^a, \quad \omega^{ab} = \hat{\omega}^{ab}. \tag{2.104}$$

In the two parametrizations, the inverse Higgs constraint reads:

$$d\pi + e^a\Omega_a = 0 = dq + \hat{e}^a\hat{\Omega}_a. \tag{2.105}$$

Notice that, in the above expressions, all the fields are evaluated at the same point ξ: this makes it non-trivial to match them with the flat space ones, since those were involving also a spacetime coordinate transformation. Let's then study explicitly how to recover the correspondence.

First of all, observe that, in the old transformations, the metric is $\eta^{\mu\nu}$ on both sides. Consider now the new transformation rules in the absence of gravity; in this case one can take $\omega = \hat{\omega} = 0$ and $e = \delta$. The rules then impose that

$$\hat{e}^a_\mu(\xi) = \delta^a_\mu - \alpha\partial_\mu\hat{\Omega}^a(\xi), \tag{2.106}$$

which does not correspond to the η metric.[18] To match with the old transformations, one has therefore to perform a diffeomorphism $\xi = \xi(\xi')$ such that $\hat{e}'(\xi') = \delta$. This can be found as follows:

$$\hat{e}'^a_\nu(\xi') = \frac{\partial\xi^\mu}{\partial\xi'^\nu}\hat{e}^a_\mu(\xi) = \frac{\partial\xi^\mu}{\partial\xi'^\nu}\frac{\partial}{\partial\xi^\mu}\left(\xi^a - \alpha\hat{\Omega}^a(\xi)\right) = \frac{\partial}{\partial\xi'^\nu}\left(\xi^a + \alpha\hat{e}'^{\rho a}(\xi')\frac{\partial q}{\partial\xi'^\rho}\right), \tag{2.107}$$

[18] The metric is given, in terms of the vierbeins, as $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$.

where in the last equality, besides using the chain rule of differentiation, we use (1) the inverse-Higgs constraint to express $\hat{\Omega}^a$ in terms of derivatives of q and (2) the invariance under diffeomorphisms of the same $\hat{\Omega}^a$ to write it in ξ' coordinates.

Imposing that $\hat{e}'^a_\nu = \delta^a_\nu$ gives finally the following condition on ξ:

$$\xi^a = \xi'^a - \alpha\eta^{a\mu}\frac{\partial q}{\partial \xi'^\mu}. \tag{2.108}$$

Upon identifying $\xi \equiv y$ and $\xi' \equiv x$, the old relations are recovered and the equivalence – in the flat limit – of the two formulations established.

2.7 Mapping of the Action

By the use of relations (2.103) and (2.104), we can now transform any action in the π frame to the corresponding action in the q frame. Our primary interest is to study how a 'standard' gravitational action transforms under the mapping.

Before going into that, we check for consistency how the simplest π Lagrangian is mapped in the q frame: when the metric is η (or equivalently the vierbein is δ), we must recover the corresponding Galileon theory $\mathcal{L}_q$. We consider, of course, the standard kinetic term for π, minimally coupled to gravity:

$$\begin{aligned}-\frac{1}{2}\int d^4\xi\sqrt{-g}g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi &= -\frac{1}{2}\int d^4\xi\,\det e\,\eta^{ab}e^\mu_a\partial_\mu\pi e^\nu_b\partial_\nu\pi = -\frac{1}{2}\int d^4\xi\,\det e\,\eta^{ab}\Omega_a\Omega_b\\ = -\frac{1}{2}\int d^4\xi\,\det(\hat{e}^a_\mu + \alpha\hat{D}_\mu\hat{\Omega}^a)\eta^{ab}\hat{\Omega}_a\hat{\Omega}_b &= -\frac{1}{2}\int d^4\xi\,\det(\hat{e}^a_\mu + \alpha\hat{D}_\mu\hat{\Omega}^a)\eta^{ab}\hat{e}^\mu_a\partial_\mu q\hat{e}^\nu_b\partial_\nu q,\end{aligned} \tag{2.109}$$

where we have used, respectively: the definition of the metric in terms of the vierbein, the inverse Higgs constraint and the mapping. D stands for the covariant derivative with respect to the spin connection ω.

If the spacetime is flat, one can choose coordinates such that $\hat{e} = \delta$. When this is done, the action becomes simply

$$-\frac{1}{2}\int d^4\xi\,\det(\delta^a_\mu - \alpha\partial_\mu\partial^a q)\eta^{ab}\partial_a q\partial_b q. \tag{2.110}$$

The determinant that appears in this formula should be recognized as the Jacobian of the old Galileon transformation, showing once again that the new procedure is consistent with the old.

When gravity is turned on, derivatives of fields which carry Lorentz or spacetime indices are promoted to covariant derivatives. In this formalism, covariantization is obtained through the spin connection $\omega = \hat{\omega}$ which, in Einstein gravity, has the standard relation to the vierbein given in formula (2.100). From a geometrical point of view, this constraint between metric and connection is encoded in the vanishing of the torsion tensor.

Surprisingly enough, the Galileon mapping implies that, in some readily specified circumstances, a non-vanishing torsion is induced in the q frame, even if absent in the π one. This can be deduced as follows, by using the mapping to rewrite the torsion-free condition for e in terms of the 'hatted' fields:

$$T^a = 0 = de^a + \omega^a_{\ b} \wedge e^b = d\left(\hat{e}^a + \alpha(d\hat{\Omega}^a + \hat{\omega}^a_{\ b}\hat{\Omega}^b)\right) + \hat{\omega}^a_{\ b} \wedge \left(\hat{e}^b + \alpha(d\hat{\Omega}^b + \hat{\omega}^b_{\ c}\hat{\Omega}^c)\right)$$

$$= d\hat{e}^a + \hat{\omega}^a_{\ b} \wedge \hat{e}^b + \alpha\left(d\hat{\omega}^a_{\ c} + \hat{\omega}^a_{\ b} \wedge \hat{\omega}^b_{\ c}\right)\hat{\Omega}^c = \hat{T}^a + \alpha\hat{R}^a_{\ b}\hat{\Omega}^b. \tag{2.111}$$

The torsion is non-zero whenever both curvature and a background for Ω are present: $\hat{T}^a = -\alpha\hat{R}^a_{\ b}\hat{\Omega}^b$. Because of this, the spin connection is non-standard, and it includes a *contorsion* term as well:

$$\hat{\omega}_\mu^{\ ab} = \overline{\omega}_\mu^{\ ab}(\hat{e}) + \hat{K}_\mu^{\ ab}. \tag{2.112}$$

Since the tensor K is linear in T,[19] we can see that the above formula for the torsion in the q frame is recursive. In fact, due to the presence of the contorsion in the connection, the curvature tensor contains, besides its standard contribution – function of the metric alone –, also additional pieces with K:

$$\hat{R}^a_{\ b}(\hat{\omega}) = \overline{R}^a_{\ b}(\overline{\omega}(\hat{e})) + \overline{D}\hat{K}^a_{\ b} + \hat{K}^a_{\ c} \wedge \hat{K}^c_{\ b}, \tag{2.113}$$

where $\overline{D}$ is the covariant derivative with respect to the torsion free connection. In other words, the torsion contains implicitly an infinite sum of higher derivative operators, constructed out of the dynamical fields $\hat{e}$ and $\hat{\Omega} \approx \partial q$.

Let's finally study how the standard gravity action looks like in the q frame. To do this, we write it using the form-theoretic language, which greatly simplifies computations.

In the π frame, the action reads:

$$S_{E\text{-}H} = \frac{1}{32\pi G}\int \epsilon_{abcd}\; e^a \wedge e^b \wedge R^{cd}(\omega). \tag{2.114}$$

Once the condition of absence of torsion is imposed, R^{ab} becomes a function of e alone, and standard Einstein gravity is recovered.

It is straightforward, using the mapping, to rewrite the above action in the q frame:

$$\hat{S}_{E\text{-}H} = \frac{1}{32\pi G}\int \epsilon_{abcd}\;\left(\hat{e}^a \wedge \hat{e}^b + 2\alpha\hat{D}\hat{\Omega}^a \wedge e^b + \alpha^2\hat{D}\hat{\Omega}^a \wedge \hat{D}\hat{\Omega}^b\right) \wedge R^{cd}(\hat{\omega}). \tag{2.115}$$

The purpose is now to confront it with the standard gravity action.

By some manipulations, we can show that the first deviation from it comes at second order in α. These require the following identities:

- for any p-form β which is an $SO(3,1)$ singlet, trivially $D\beta = d\beta$. If β is a 3-form, then

$$\int d\beta = 0 \tag{2.116}$$

  for the Stokes theorem.

[19] Their relation is given by $K_{\mu\ \rho}^{\ \nu} = -\frac{1}{2}\left(T_{\mu\ \rho}^{\ \nu} - T^\nu_{\ \rho\mu} + T_{\rho\mu}^{\ \ \nu}\right)$.

- The definition of the Riemann tensor implies that

$$D^2\lambda^i = R^i{}_j\lambda^j. \tag{2.117}$$

- The Bianchi identity reads

$$DR^{ij} = 0. \tag{2.118}$$

- Finally, for the very definition of torsion, we have

$$De^a = T^a. \tag{2.119}$$

Moreover, we will make extensive use of the integration by parts for the covariant derivative D. From here on, to make the notation lighter, hats will be omitted.

Let's first consider the term proportional to α:

$$2\alpha \int \epsilon_{abcd}\, D\Omega^a\wedge e^b\wedge R^{cd} = 2\alpha \int \left(d(\epsilon_{abcd}\Omega^a e^b \wedge R^{cd}) - \epsilon_{abcd}(\Omega^a De^b \wedge R^{cd} - \Omega^a e^b \wedge DR^{cd})\right)$$

$$= -2\alpha \int \epsilon_{abcd}\, \Omega^a T^b \wedge R^{cd}. \tag{2.120}$$

The term in α^2 can be similarly manipulated:

$$\alpha^2 \int \epsilon_{abcd} D\Omega^a\wedge D\Omega^b\wedge R^{cd} = -\alpha^2 \int \Omega^2 D^2\Omega^b\wedge R^{cd} = -\alpha^2 \int \Omega^a R^b{}_i \Omega^i\wedge R^{cd}. \tag{2.121}$$

As explained before, if we want the connection to be the standard (torsion-free) one in the π frame, we must deal with a non-zero torsion in the q frame. Using its expression found in formula (2.111), and remembering that $R(\omega)$ contains terms with the contorsion tensor, we find that

$$\hat{S}_{E\text{-}H} = \frac{1}{32\pi G} \int \epsilon_{abcd}\, (e^a \wedge e^b + \alpha^2\Omega^a R^b{}_i(\omega)\Omega^i) \wedge (\overline{R}^{cd}(e) + \alpha\overline{D}\mathcal{K}^{cd} + \alpha^2\mathcal{K}^c{}_j \wedge \mathcal{K}^{jd}), \tag{2.122}$$

where we redefined $K^{ab} = \alpha\mathcal{K}^{ab}$, to make explicit that it comes at first order in α. The only term which is linear in α is given by

$$\alpha \int \epsilon_{abcd}\, e^a \wedge e^b \wedge \overline{D}\mathcal{K}^{cd}, \tag{2.123}$$

but this is a total derivative, since the barred covariant derivative is by definition torsion-free, and therefore $\overline{D}e^a = 0$. As anticipated, the standard gravity action in the q coordinates is of the form $\hat{S}_{E\text{-}H} = S_{E\text{-}H} + O(\alpha^2)$:

$$\hat{S}_{E\text{-}H} = S_{E\text{-}H} + \frac{\alpha^2}{32\pi G} \int \epsilon_{abcd}(e^a \wedge e^b \wedge \mathcal{K}^c{}_j \wedge \mathcal{K}^{jd} + \Omega^a R^b{}_i(\omega)\Omega^i \wedge R^{cd}(\omega)) \tag{2.124}$$

Notice that no truncation was made, so the result is exact. While the new action is written as the sum of a finite number of terms, it secretly contains infinitely many higher derivative operators, due to the appearance of the 'full' Riemann tensor $R(\omega)$ and of the contorsion tensor $\mathcal{K}$.

2.8 Conclusions

Let's now qualitatively analyse the gravity action in the dual frame.

On a slowly varying background q_0, that is such that $\alpha\partial\partial q_0 \ll 1$, the recursion (2.111) contains terms of the form

$$\alpha^n \partial^{n-1} R (\partial q_0)^n. \tag{2.125}$$

For processes with typical frequency ω, this series is governed by the expansion parameter $\alpha\omega\partial q_0$. Whenever it happens that

$$\alpha\omega\partial q_0 \gtrsim 1, \tag{2.126}$$

the series cannot be truncated, since each operator contributes at the same level. As shown in [18], this regime is the one in which 'local' superluminality becomes measurable in the $\mathcal{L}_q$ Galileon theory (even in flat space).

Due precisely to the recursion contained in the torsion, the gravity action contains, in the same regime, infinitely many equally relevant higher dimension operators: these must be responsible for the deviation of the gravitons' propagation cone with respect to the metric null cone. Notice that, consistently with the fact that this deviation must happen independently of the strength of the gravity coupling, the suppression scale of the higher dimension operators is unrelated to the Planck scale, and in fact it is much lower

$$\alpha^{-1/3} \ll M_{Pl}. \tag{2.127}$$

It is for this reason that we claim that, in the q frame, we feature a non-standard theory of gravity.

Independently from questions related to superluminality, we can consider this as a criterion for equivalence of two theories that, like $\mathcal{L}_\pi$ and $\mathcal{L}_q$, are related by a 'coset twist': that they must preserve a standard coupling with gravity, once mapped into each other. We see therefore that the two Galileon theories are not equivalent in this physical sense.

Although we chose to focus on the example of the Galileon group, due primarily to the simplicity of its algebra, we believe that our conclusions apply also to other spacetime cosets. Pretty much like for the case we considered, the spontaneous breaking of the conformal group admits at least two interesting descriptions, which can be put in correspondence to two different parametrizations of the coset [23]. The coset generators are related by a twist with exactly the same form as (2.102):

$$\hat{K}_\mu = K_\mu + \alpha' P_\mu, \tag{2.128}$$

where K_μ are the generators of special conformal transformations. Also in this case, we expect that a standard coupling with gravity in one representation is mapped to a non-standard coupling in the other.

Let me conclude by stressing that this is not just an academical problem: as a matter of fact, gravity is always coupled to any object that carries energy and momentum. The same Galileon theories, in both the π and q representations (more precisely, their conformal versions) have been used to discuss cosmological scenarios alternative to inflation [24]. Since in each case the galileons were coupled to gravity in a standard way, we see now that the two scenarios are not equivalent from a physical point of view or, in other words, that we can make no use of the mapping (2.75) to relate the two. We have in fact shown that standard couplings with gravity are not mapped into each other.

Chapter 3

The Sgoldstino at Colliders

The origin of masses is understood, within the Standard Model of particles' interactions, in accordance with two distinct mechanisms, each one associated with a physical scale. On one side, particles which are made of quarks and anti-quarks, that is hadrons, owe their masses mainly to confinement imposed by the strong color interactions: constituents are forced to move in a region of order $L_{QCD} = \Lambda_{QCD}^{-1}$, so that their energy, and therefore the mass of the hadron, is of order Λ_{QCD}. Any other mass can be traced back to the mechanism of spontaneous breaking of the electroweak gauge group by the vacuum expectation value of the Higgs field; more precisely, the mass of a given particle is given by

$$m_i = c_i v, \tag{3.1}$$

where v is the modulus of the Higgs vev, and c_i measures the strength of the interaction of particle i with the Higgs boson. Altogether, the two mechanisms outline a remarkably economic picture, since any mass can be in principle understood thanks to just two scales, that is $\Lambda_{QCD} \sim 1\ GeV$ and $v = 246\ GeV$. Moreover, the structure of the Model is rigid enough to forbid any explicit mass term, because particles are either gauge bosons of the group $SU(3) \times SU(2) \times U(1)$, or fermions that carry a chiral representation of it.

Notwithstanding this simplicity, the general picture leaves somehow unsatisfied. The reason is that there are other, much higher physical scales: the scale of gravity and the scale of approximate unification of the three forces. These are respectively $M_{Pl} \approx 10^{19}\ GeV$ and $\Lambda_{GUT} \approx 10^{16}\ GeV$.[1] What explains such large hierarchies between the scales of the Standard Model and these more fundamental scales? This puzzle, the hierarchy problem, applies in fact only to the electroweak scale. This is because the strong scale admits a natural explanation for its smallness; as a matter of fact, the only known natural explanation for hierarchies in particle physics.

Suppose that, at very high energy $E \gg \Lambda_p$, physics is described by a perturbative unified theory, which is broken to the Standard Model group G_{SM}, times perhaps some other factor, at the scale Λ_p. Below that scale, the couplings associated to the different factors run differently. If some of them, like what happens for g_s, the coupling of QCD, gets stronger and stronger for smaller and smaller energies, it

[1] We stress that these scales are not on the same footing, and that the first is considered to be more fundamental.

eventually becomes non-perturbative at a scale $\Lambda_{n\text{-}p}$, approximately given by

$$\Lambda_{n\text{-}p} = \Lambda_p e^{-\frac{8\pi^2 b}{\mathcal{G}^2(\Lambda_p)}}, \tag{3.2}$$

where b is the order one factor that governs the running of $\mathcal{G}$. The relation between the high and low scales is naturally hierarchical.

For what concerns the electroweak sector, an explanation along these lines is not available within the Standard Model, and the hierarchy between v and M_{Pl} is considered to be one of the major problems of fundamental particle physics. Even more dramatically, the Higgs field mass squared (which can be equivalently considered as the fundamental mass parameter of the electroweak sector), receives radiative corrections which tend to push it towards the highest scale of the theory. In this respect, the electroweak scale is seen to be highly unnatural.

Such unnaturalness would be alleviated if there was a symmetry forbidding the Higgs mass term, whose spontaneous breaking at a proper scale Λ_{sb} would set in turn the electroweak scale. In this picture, radiative corrections are kept under control, but it would remain to explain what sets Λ_{sb} hierarchically smaller than the Planck scale. This can in principle be obtained, thanks to a mechanism like the one that sets the scale of strong interactions.

A theory that embeds the Standard Model and that: (1) protects the Higgs mass from getting large radiative corrections and (2) explains the hierarchy between the electroweak and gravity scales, can be considered to be the main goal of particle physics.

As explained in the introduction, Supersymmetry puts in the same multiplet fermions and bosons, implying that, if the fermions are kept massless by some symmetry (like the chiral one), also bosons are forced to be so, and viceversa. The initial hope in embedding the Standard Model in a supersymmetric theory, was to control in such a way the mass of the Higgs. As we will see, this is not straightforward,[2] since cancellation of gauge anomalies requires the Higgs superfield to come in pairs carrying conjugated representations of G_{SM}, so that supersymmetric masses can be assigned to the Higgs scalars and their superpartners. Still, supersymmetric theories enjoy non-renormalization theorems, whose consequence is that radiative corrections are kept under control by the scale of Supersymmetry breaking, and large radiative corrections are avoided. From the above discussion, any supersymmetric extension of the Standard Model has to confront with at least three questions:

- What sets the scale of Supersymmetry breaking Λ_{sb}?
- What sets the supersymmetric Higgs mass μ?
- Do the parameters make the theory natural?

Unfortunately, there is no implementation of Supersymmetry that, being compatible with experiments, cleanly respects all of these requirements. Still, many ideas have been developed that deserve full exploration and, moreover, there is a single piece of prediction that makes the field very attractive, that is gauge coupling unification.

[2] At least without enlarging the internal symmetry group.

In the following section, the Minimal Supersymmetric Standard Model will be presented, along with a clarification of the above statements.

3.1 The Minimal Supersymmetric Standard Model

This and the following introductory sections are based on [25] and [26]. Knowledge of the basics of superfield formalism is assumed here.

A minimal implementation of Supersymmetry requires, at least, that to any boson there corresponds a fermion with the same transformation properties under internal symmetries. At the Lagrangian level, this requirement is efficiently satisfied by constructing the interactions in terms of superfields, the field analogue of particle supermultiplets.

Due to the fact that they carry inequivalent representations of the gauge group, it can be concluded that no Standard Model fermion can be the superpartner of any gauge boson. These last are to be embedded in 'vector superfields', whose components include new fermionic degrees of freedom: the gauginos. For any Standard Model fermion, instead, there must be a complex scalar field,[3] so that together they make a 'chiral superfield'. Then, for any quark there is a squark, and for any lepton a slepton. The chiral superfields are then:

$$Q(3,2)_{1/6}, U^c(\bar{3},1)_{-2/3}, D^c(\bar{3},1)_{1/3}, L(1,2)_{-1/2}, E^c(1,1)_1. \tag{3.3}$$

It remains to discuss what is the proper supersymmetric embedding of the Higgs field. From the point of view of the transformation properties under G_{SM}, the Higgs field could be a slepton, but this possibility is usually excluded, and the Higgs sector is taken to have zero lepton number. To give masses to the *up* and *down* quarks in the Standard Model, Yukawa couplings are needed that involve both the Higgs field and its complex conjugate; schematically:

$$\mathcal{L}_{SM} \supseteq y_u h q u + y_d h^* q d. \tag{3.4}$$

A supersymmetric theory is more constrained, so that Yukawa couplings must be derived from a superpotential, which is a holomorphic function of the chiral superfields. This excludes the possibility that h and h^* appear on the same footing, and at least two Higgs superfields in conjugated representations are needed to implement fermion mass generation in a supersymmetric context. These are called

$$H_u(1,2)_{1/2}, H_d(1,2)_{-1/2}. \tag{3.5}$$

This minimal choice actually squares with two fundamental facts.

The first is the cancellation of gauge anomalies: while the fermionic partners of gauge bosons carry a real representation of the gauge group, and therefore do not introduce any new anomaly, a single supersymmetric Higgs, by enlarging the content of chiral fermions of the theory through its superpartner the higgsino, in fact does. An even number of Higgs fields, on the other hand, does not introduce

[3] This is the proper choice for the spin of the superpartner: pairing a fermion with a vector would require the first to carry a *vectorial* representation of the gauge group.

any new anomaly. It should be stressed that this is a fundamental point, since a gauge anomaly makes the theory devoid of meaning.

The second is gauge couplings unification. One of the most striking features of the Standard Model is the prediction that the couplings related to its gauge group factors, that is g_s, g and g', take approximately the same value if renormalized at the scale Λ_{GUT}. This, together with the following group theoretical findings:

1. that G_{SM} can be naturally embedded in simple Lie groups, like for example $SU(5)$ or $SO(10)$;

2. that the chiral fermions can be properly "summed" to make them representations of these unified groups;

led to the spread belief that the Standard Model is the low energy appearance of a theory that is actually unified at Λ_{GUT}. The aesthetic appeal that a unified theory produces is so strong that, neglecting the possibility of a coincidence, many physicists tend to consider this as a fundamental fact, and the GUT scale as a fundamental physical scale, characterizing energies at which some novel dynamics unveils. This is enforced by the relative closeness of Λ_{GUT} to M_{Pl}, the scale at which gravity is expected to become strong.

On top of this, Supersymmetry gave what is maybe its most beautiful result: that the version of the supersymmetric Standard Model with the above listed fields, and with *exactly two* Higgs doublets, leads to perfect unification (within the experimental errors on the value of the gauge couplings at the electroweak scale). Due to the importance of this as a motivation for exploring Supersymmetry, we will always take as a basic requirement that:

> any addition of extra ingredients to the minimal version of the supersymmetric Standard Model has to preserve gauge coupling unification.

We will see in the next section that, to implement Supersymmetry breaking, such additions are in fact necessary. Before going to that, let me say a few words on the structure of interactions that emerges in this context, focusing in particular on the superpotential, which is the only term that is not unambiguously fixed after the specification of the "matter" field content and its gauge structure. It turns out that, besides the Yukawas

$$\mathcal{W}_Y = (y_u)_{ij} H_u Q_i U^c_j - (y_d)_{ij} H_d Q_i D^c_j - (y_e)_{ij} H_d L_i E^c_j, \tag{3.6}$$

and thanks also to the coincidence of the gauge quantum numbers of L and H_d, also the following renormalizable terms are allowed:

$$\mathcal{W}_{\text{extra}} = (c_d)_{ijk} L_i Q_j D^c_k + (c_e)_{ijk} L_i L_j E^c_k + d_{ijk} D^c_i D^c_j U^c_k. \tag{3.7}$$

Not to spoil the clean pattern of Baryon and Lepton number conservation in the Standard Model, a symmetry is assumed that has the virtue of excluding these last terms: R-parity. This has value $+1$ on all the Standard Model fields, and -1 on their superpartners.

With this additional symmetry requirement, a unique mass term is allowed to be part of the superpotential, the μ-term:

$$\mu H_u H_d. \tag{3.8}$$

As mentioned earlier, this comes together with the particularly compelling question of what mechanism sets it around the electroweak scale, and in this sense it can be seen as a drawback of the necessity of having two Higgs doublets.

The above comment does not do justice to the μ term, since it turns out that it is actually needed from a phenomenological point of view: it gives masses to higgsinos and it takes part in the mechanism of electroweak symmetry breaking.

Regarding this last point, we can start to see that the model is so far incomplete; consider in fact the scalar Higgs sector, its potential being the sum of F-terms coming from μ and D-terms coming from gauge interactions:

$$V(h_u, h_d) = |\mu|^2 \left(|h_u|^2 + |h_d|^2\right) + \frac{g^2}{2} \left|h_u^\dagger h_d\right|^2 + \frac{g^2 + g'^2}{8} \left(|h_u|^2 - |h_d|^2\right)^2. \tag{3.9}$$

This encounters many difficulties. If $\mu \neq 0$, then there is a single vacuum which preserves $SU(2) \times U(1)$. If μ is instead zero, then there is a flat direction, and the points along it correctly break the electroweak symmetry; still, there are too many massless particles (the physical 'Higgses' and the higgsinos), and moreover Supersymmetry is unbroken.

The solution to both problems of correctly implementing electroweak *and* Supersymmetry breaking is the content of the next section.

3.2 Supersymmetry Breaking

Unbroken Supersymmetry is highly constraining: it enforces the equality of the masses of particles within the same supermultiplets, and imposes relations among couplings, such that, qualitatively, particles and their superpartners enjoy interactions of the same strength.

This tells immediately that Supersymmetry must be broken: no particle, between those that it predicts, has been seen so far. Experiments, instead, suggest that both gauginos and sfermions are heavier than their superpartners.

There are arguments that seem to deny the possibility that such a structure can be obtained, at tree-level, within the theory constructed in the previous section. This is to be traced to a sum rule relating the masses of bosons on one side, to those of fermions on the other, valid at tree-level in renormalizable theories, even when Supersymmetry is spontaneously broken. The formula reads:

$$\sum_{j=0} m^2 + 3 \sum_{j=1} m^2 = 2 \sum_{j=1/2} m^2, \tag{3.10}$$

and it is valid for sums restricted to particles with a given choice of conserved quantum numbers. This implies that, for example, the sum over all of the *down* type squark masses is equal to two times the corresponding sum for quarks:

$$\sum_{d \text{ squarks}} m^2 = 2(m_d^2 + m_s^2 + m_b^2) \approx (7\ GeV)^2, \tag{3.11}$$

definitely a too small value. This result has been obtained by considering the minimal field content discussed in the previous section, and the argument can be evaded by assuming that there is a fourth generation of quarks with a large mass, let's say of order TeV.

In general, a realistic implementation of Supersymmetry breaking requires to enlarge the field content of the theory. More specifically, it is usually assumed that Supersymmetry is broken spontaneously in a separate sector of the theory. This means that, if the interactions of the new sector with the so called 'observable sector'[4] are controlled by a coupling $\mathcal{G}$, the breaking happens also when it is set to zero. When, on the other side, the coupling is switched on, Supersymmetry breaking is communicated to the observable sector.

In the class of models that we are going to consider, $\mathcal{G}$ stands for some gauge coupling: either between those of the Standard Model group, like in 'standard' Gauge Mediation, or characterizing additional factors, like for the direct generation of sfermion masses in 'tree-level' Gauge Mediation.

Another common feature of these models of Supersymmetry breaking, is the fact that the new, 'hidden' dynamics, lies at energies which are much higher than the scale of the supersymmetric Standard Model, so that its degrees of freedom can be integrated out. This procedure is expected to generate effective interactions between the observable fields that are negligibly small, with one exception: super-renormalizable terms. These are Lagrangian operators of dimension ≤ 3 that are constrained by G_{SM} gauge invariance and R-parity, and include masses for gauginos and sfermions:

$$\mathcal{L}_{\text{soft}} = -\frac{1}{2}\left(M_1\tilde{b}\tilde{b} + M_2\tilde{w}\tilde{w} + M_3\tilde{g}\tilde{g} + c.c.\right) - \sum_{\tilde{f}} m_{\tilde{f}}|\tilde{f}|^2 + \ldots, \tag{3.12}$$

trilinear scalar interaction allowed by the aforementioned symmetries, and additional mass terms for the Higgs doublets:

$$\ldots - m_{h_u}^2|h_u|^2 - m_{h_d}^2|h_d|^2 - (B\mu h_u h_d + c.c.)\,, \tag{3.13}$$

generating additional contributions to the supersymmetric Higgs potential (3.9) that, for proper choices of the parameters, successfully produce electroweak symmetry breaking.

When these terms are added to the supersymmetric ones, the renormalizable theory that one obtains features explicit Supersymmetry breaking. Still, all the powerful renormalization properties of supersymmetric theories are preserved, implying that, for example, the mass of the Higgs boson can in principle be kept naturally light.[5]

Once these 'soft' parameters are fixed, the theory makes sense on its own and it can be used to study phenomenological implications of Supersymmetry.

[4]That is quarks, squarks, gauge bosons and gauginos, etc.

[5]The amount of unnaturalness roughly depends, in this context, on the mass parameter that sets the Supersymmetry breaking masses in the Minimal Supersymmetric Standard Model. If this is too high, naturalness is lost but, still, the theory is not sensitive to what happens at the Planck scale.

Before going into the explicit implementations of gauge-mediated models, let's discuss an exception to the above argument that the hidden dynamics can be integrated out.

If the theory is invariant under a global continuous symmetry, and this is broken spontaneously by the fields of the hidden sector, then the same fields automatically provide the degrees of freedom necessary to respect the Goldstone theorem, that is the massless 'Goldstone particles'. Due to this very property, they cannot be integrated out (though their interactions turn out to be suppressed by the scale of symmetry breaking), and it is therefore interesting, in the spirit of the previous chapter, to study what are the leading interactions – universally fixed by symmetry principles – between them and the other phenomenologically relevant fields, the 'observable' ones.

A spontaneously broken continuous symmetry is of course always present in these theories: Supersymmetry. As a consequence, there exists a massless 'goldstino'. Due to the fermionic nature of this symmetry transformation, it follows that the goldstino is a fermion.

Let me review first the consequences of the spontaneous breaking of an internal symmetry group G, assumed to be linearly realized on the Lagrangian fields and broken by the *vev* of some G-charged scalar.

Consider first the action, which is a functional of some fields which are collectively called Φ:

$$S[\Phi] = \int d^4x \mathcal{L}\left(\Phi, \partial_\mu \Phi\right). \tag{3.14}$$

If the fields are acted upon by an infinitesimal x-dependent symmetry transformation

$$\Phi(x) \to \left(1 + i\epsilon^I(x) t_I\right) \Phi(x), \tag{3.15}$$

due to the invariance of S for constant ϵ, the action changes as follows

$$S \to S - \int d^4x \partial_\mu \epsilon^I(x) J_I^\mu \left(\Phi(x), \partial_\mu \Phi(x)\right). \tag{3.16}$$

Integrating by parts the last piece, and using the fact that – when the fields respect the equations of motion – $\delta S = 0$ for an arbitrary variation $\delta\Phi$, it is readily deduced that $\partial_\mu J_I^\mu = 0$: the J's are the conserved currents related to invariance under G. In the above expression, it is stressed that the currents are some given expressions of the field configuration.

Consider now the interesting case in which the fields take some non-zero expectation on the vacuum $\langle\Phi\rangle_{\mathrm{VAC}}$. It can be shown that any field configuration can be written as follows:

$$\Phi(x) = e^{i\xi^\alpha(x) t_\alpha} \tilde{\Phi}(x), \tag{3.17}$$

with the new fields linearly constrained by the condition that

$$\tilde{\Phi}(x) t_\alpha \langle\Phi\rangle_{\mathrm{VAC}} = 0. \tag{3.18}$$

With the notation used in the second chapter, the generators t_α are the broken ones, and the ξ^α are the Goldstone fields. Now, using the definition of current given through (3.16), we can figure out the linear coupling between the Goldstone fields and the other fields $\tilde{\Phi}$:

$$S[\Phi] = S[e^{i\xi t}\tilde{\Phi}] = S[\tilde{\Phi} + i\xi t\tilde{\Phi} + \ldots] = S[\tilde{\Phi}] - \int d^4x \partial_\mu \xi^\alpha J^\mu_\alpha(\tilde{\Phi}, \partial_\mu \tilde{\Phi}) + \ldots, \quad (3.19)$$

where the current is now to be evaluated on the 'Goldstone-free' field configuration; this is *not* the whole conserved current: other contributions, due to the Goldstone fields, are not included. It is important to realize that the fields ξ^α are not canonically normalized. To find out their normalization, we inspect the canonical kinetic term for the scalars. Calling γ the exponential of the Goldstone fields, one finds:

$$\frac{1}{2}\partial\phi \cdot \partial\phi = \frac{1}{2}\partial(\gamma\tilde{\phi}) \cdot \partial(\gamma\tilde{\phi}) = \frac{1}{2}\partial\tilde{\phi} \cdot \partial\tilde{\phi} + \partial\tilde{\phi} \cdot (\gamma^T \partial\gamma)\tilde{\phi} + \frac{1}{2}\tilde{\phi} \cdot (\partial\gamma^T \partial\gamma)\tilde{\phi}. \quad (3.20)$$

The kinetic term for the ξ's is obtained from the last term, by taking the *vev* of the $\tilde{\phi}$'s (which is precisely $\langle\Phi\rangle_{\text{VAC}}$), and expanding $\partial\gamma$ at linear order in the Golstone fields. We get:

$$\frac{1}{2}\partial_\mu \xi^\alpha (it_\alpha \langle\Phi\rangle_{\text{VAC}}) \cdot (it_\beta \langle\Phi\rangle_{\text{VAC}}) \partial^\mu \xi^\beta = \frac{1}{2}\partial_\mu \xi^\alpha f^2_{\alpha\beta} \partial^\mu \xi^\beta. \quad (3.21)$$

When the broken generators furnish an irreducible representation of the unbroken subgroup, the matrix $f^2_{\alpha\beta}$ reduces to a single number f^2, and the canonically normalized expressions are obtained through the substitution $\xi \to \xi/f$. In particular, we get the interaction term

$$-\frac{1}{f}\int d^4x \partial_\mu \xi^\alpha J^\mu_\alpha(\tilde{\Phi}, \partial_\mu \tilde{\Phi}). \quad (3.22)$$

The case of Supersymmetry breaking can be treated with similar manipulations, by considering the auxiliary fields (which are collectively called F) on the same footing as the scalars ϕ and the fermions ψ. We write any field configuration as

$$\Phi(x) = \Gamma(x)\tilde{\Phi}(x) = \exp(iG(x) \cdot \mathcal{Q})\tilde{\Phi}(x) \quad (3.23)$$

where Γ is an object which is bosonic in nature, but is made from the the contraction of a fermionic field $G(x)$, which is the goldstino,[6] and the fermionic operator $\mathcal{Q}$, the one that implements Supersymmetry transformations. Schematically (all the spinor and Lorentz contractions are left implicit):

$$\epsilon \cdot \mathcal{Q}\phi = \delta\phi = \epsilon \cdot \psi, \qquad \epsilon \cdot \mathcal{Q}\psi = \delta\psi = \epsilon\partial\phi + \epsilon F,$$
$$\epsilon \cdot \mathcal{Q}F = \delta F = \epsilon \cdot \partial\psi. \quad (3.24)$$

When rewritten in terms of G and $\tilde{\Phi}$, and up to linear terms in the goldstino field, the action reads:

$$S[\Phi] = S[\tilde{\Phi}] - \int d^4x\ \partial_\mu G(x) \cdot \mathcal{J}^\mu(\tilde{\Phi}(x), \partial_\mu \tilde{\Phi}(x)) + \ldots, \quad (3.25)$$

[6] At this level, its mass dimension is $-1/2$, and it is not canonically normalized.

where $\mathcal{J}^\mu$ is the fermionic Supersymmetry current, evaluated here on the goldstino-free configuration. To find the correct normalization of G, we have to inspect the fermions' kinetic term, since the goldstino degrees of freedom are to be found there. Again, the interesting piece is found after expanding Γ and taking the *vev* of $\tilde{\Phi}$:

$$\psi^\dagger i\sigma^\mu \partial_\mu \psi = (\Gamma\tilde{\psi})^\dagger i\sigma^\mu \partial_\mu (\Gamma\tilde{\psi}) = (\tilde{\psi} + iG\mathcal{Q}\tilde{\psi} + \ldots)^\dagger i\sigma^\mu i\partial_\mu G\mathcal{Q}\tilde{\psi} + \ldots$$
$$= \langle \mathcal{Q}\tilde{\psi}\rangle^\dagger_{\mathrm{VAC}} G^\dagger i\sigma^\mu \partial_\mu G \langle \mathcal{Q}\tilde{\psi}\rangle_{\mathrm{VAC}} + \ldots = |\langle F\rangle_{\mathrm{VAC}}|^2 \, G^\dagger i\sigma^\mu \partial_\mu G + \ldots \tag{3.26}$$

For Supersymmetry breaking, it is crucial that not the scalars, but the auxiliary fields take a vacuum expectation value. From the last expression, we see that the goldstino field has to be rescaled by $|\langle F\rangle_{\mathrm{VAC}}|$, which is a dimension two parameter, making the newly defined goldstino field correctly of dimension 3/2. Let me finally mention what is the analogue of constraint (3.18), which in this case practically projects out all but the goldstino direction:

$$\tilde{\Phi}^\dagger \langle \mathcal{Q}\Phi\rangle_{\mathrm{VAC}} = 0 = \tilde{\psi}^\dagger \langle F\rangle_{\mathrm{VAC}}. \tag{3.27}$$

This derivation assumes that all the breaking occurs through F-terms, but the generalization to the case where the D-terms are different from zero is trivial. When this happens, the goldstino has also gaugino components along the directions corresponding to the non-zero D-terms.

To consistently take into account gravitational effects in a supersymmetric theory, Supersymmetry has to be promoted to a local symmetry. Like for the spontaneous breaking of 'standard' gauge symmetries, where the Goldstone boson is "eaten" by the gauge boson, the goldstino becomes here the longitudinal component of the massive gravitino, which is a spin 3/2 field ψ_μ. Pushing further the analogy, we understand that there is a regime in which the gravitino interactions are governed by the goldstino ones.

Let's then briefly analyse the consequences of the spontaneous breaking of a $U(1)$ gauge symmetry, to understand what is the limit in which the Goldstone boson interactions dominate over the gauge boson ones. We consider a gauge invariant theory, and relabel the matter fields in a way analogous to eq. (3.17):

$$\mathcal{L}_M(\Phi, \nabla^{(gA)}\Phi) = \mathcal{L}_M(e^{i\xi Q}\tilde{\Phi}, \nabla^{(gA)}(e^{i\xi Q}\tilde{\Phi})) = \mathcal{L}_M(\tilde{\Phi}, \nabla^{(gA-\partial\xi)}\tilde{\Phi})$$
$$= \mathcal{L}_M(\tilde{\Phi}, \partial\tilde{\Phi}) + (gA_\mu - \partial_\mu\xi)J^\mu(\tilde{\Phi}, \partial\tilde{\Phi}) + \frac{1}{2}f^2(gA_\mu - \partial_\mu\xi)^2 + \ldots. \tag{3.28}$$

The second piece shows that, as required by symmetry, the vector and the Goldstone boson are coupled to the same 'matter' current. Once ξ is properly normalized by the relabelling $\xi \to \xi/f$, we can compare the strength of their couplings:

$$g \quad \text{vs} \quad E/f. \tag{3.29}$$

The last term in the above expansion of the Lagrangian dictates that the mass of the vector boson is $m_A = gf$, and we can express the limit in which the Goldstone boson interactions dominate as

$$E \gg m_A. \tag{3.30}$$

In a similar way, and using the knowledge that, roughly, the gravitino couples with M_{Pl}^{-1} strength:

$$\mathcal{L} \sim \left(\frac{\psi_\mu}{M_{Pl}} - \frac{\partial_\mu G}{F} \right) \cdot \mathcal{J}^\mu, \tag{3.31}$$

we conclude that the interesting regime is when

$$E \gg F/M_{Pl} \sim m_g. \tag{3.32}$$

After the presentation of Gauge Mediation models in the next section, these parameters will be translated into rough numerical estimates for the mass of the gravitino.

3.3 Gauge Mediation

Let's explore the possibility that Supersymmetry is broken in a 'hidden' sector of the theory – characterised by a mass scale much larger than the electroweak scale – and transmitted to the observable sector through gauge interactions shared by the two (see [27] and references therein).

The first requirement is that the hidden sector must contain fields which are charged under G_{SM}, called messengers. Moreover, these are taken to lie in vectorial representations of the gauge group, so to allow group invariant mass terms:

$$\mathcal{L} = \int d^2\theta M_\Phi \Phi \overline{\Phi}. \tag{3.33}$$

It is important to stress that the addition of charged fields can, in general, spoil the prediction of gauge coupling unification of the supersymmetric Standard Model. Since unification is taken here as a guiding principle, let's discuss straightaway what are the conditions for it to hold.

To study the *1-loop* running of gauge couplings, it is convenient to consider the functions $1/g_r^2$, which vary linearly with respect to the logarithm of the energy scale μ:

$$\frac{1}{g_r^2(\mu)} = \frac{1}{g_r^2(\Lambda_{\text{ref}})} - \frac{1}{8\pi^2} b_r \log\left(\frac{\mu}{\Lambda_{\text{ref}}}\right). \tag{3.34}$$

The coefficient b_r is fixed by the particle content of the theory, including only those particles with masses below the running scale μ. This means that the above formula cannot be valid at all energies, but only beneath the thresholds where new particles become effective. In each of these energy intervals, the coefficient has in principle a different magnitude, so that $1/g_r^2$ is approximately a piecewise linear continuous function.

It turns out that a good approximation consists indeed in attaching linear pieces, the junction points being specified by the various mass thresholds. Moreover, it is not numerically important if the points are not perfectly known, the "error" being logarithmic in $m_{\text{threshold}}$ and loop-suppressed.

Above the scale of soft terms, the running of the gauge couplings is dictated by the coefficients:

$$b_1 = 11, \quad b_2 = 1, \quad b_3 = -3. \tag{3.35}$$

As mentioned before, the evolution of the couplings for smaller and smaller distances, is such that they meet at $\Lambda_{GUT} \approx 2.2 \times 10^{16} GeV$.

Due to the peculiar form of the *1-loop* evolution expressed in (3.34), it is easy to give, within this approximation, a simple criterion to preserve unification with additional charged matter.

Consider the interesting example of Gauge Mediation, and its basic requirement that messenger fields Φ and $\overline{\Phi}$ exist, with masses roughly specified by M_Φ and transforming according to conjugate representations of G_{SM}:

$$\Phi \to \rho_r \Phi, \qquad \overline{\Phi} \to \overline{\rho}_r \overline{\Phi}. \tag{3.36}$$

With this specification, the messenger contribution to b_r can be computed, and it is equal to the Dynkin index of the representation $\rho_r \oplus \overline{\rho}_r$. If the following two conditions are met:

1. That all the additional particles have masses around M_Φ;
2. That $b_r|_{\rho\oplus\overline{\rho}}$ does not depend on the gauge factor r;

then the couplings, evolved according to formula (3.34), still meet each other at Λ_{GUT}, the change in their common value being given by:

$$\delta g^{-2}(\Lambda_{GUT}) = -2b|_{\rho\oplus\overline{\rho}} \log\left(\frac{\Lambda_{GUT}}{M_\Phi}\right). \tag{3.37}$$

Because this result is inherently perturbative, it is necessary for its validity that nowhere the couplings become strong, and this translates into a limitation on the amount of additional charged matter in the theory.

Consistently with the picture that a unified gauge group becomes effective at Λ_{GUT}, a natural way for the second condition to satisfy is to consider messengers in full representations of a group of this kind, like $SU(5)$ for example. The messenger contribution to b, called the 'messenger index', will be denoted by N in the following.

For the messengers to be capable of transmitting Supersymmetry breaking, they must feature it. A non-supersymmetric mass spectrum can be obtained in a minimal way by coupling them at tree level to a source of Supersymmetry breaking: a neutral chiral superfield X, for simplicity identified with the goldstino superfield, that takes both scalar and auxiliary *vev*'s

$$\langle X \rangle = M + \theta^2 F. \tag{3.38}$$

More general scenarios, like for example the inclusion of other sources of breaking, are not considered here, since they are not necessary to spell out the main consequences of Gauge Mediation.

Take then the following superpotential term, with index I running over the different irreducible representations of G_{SM}:

$$\mathcal{W} = \sum_I \lambda_I X \Phi_I \overline{\Phi}_I. \tag{3.39}$$

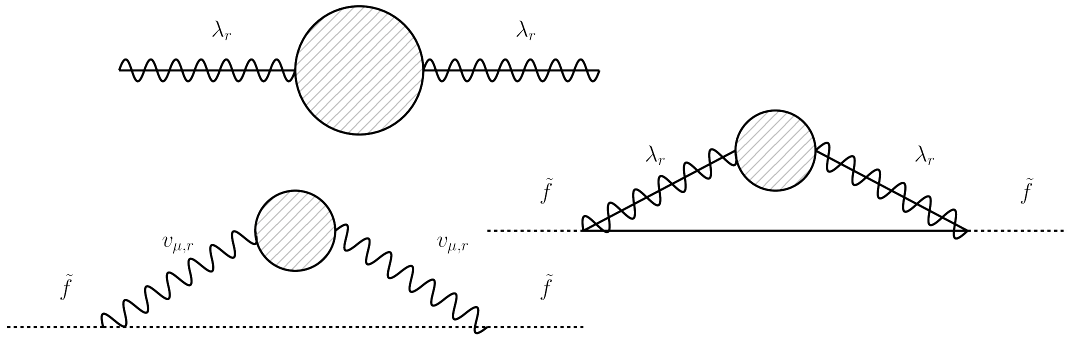

Figure 3.1: One-loop diagrams contributing to gaugino masses (first graph) and two-loop diagrams contributing to sfermion masses (second and third graphs). The discs represent loops of messengers.

The effect of the F-term on the mass spectrum of the particles contained in the messenger supermultiplets – that is, as many Dirac spinors and couples of complex scalars as the dimension of the representation ρ –, is to split the bosonic squared masses symmetrically with respect to the fermionic ones:

$$(m^2_{\text{scalar}})_I = \lambda_I^2 M^2 \pm \lambda_I F, \qquad (m^2_{\text{fermion}})_I = \lambda_I^2 M^2, \tag{3.40}$$

consistently with the constraints of the sum rule (3.10). It is remarkable that this spectrum is all is needed to produce Supersymmetry breaking soft masses for gauginos and for sfermions, respectively at one and two loops. On the other side, gauge bosons and chiral fermions are automatically preserved massless, since in no way G_{SM} is broken by the above dynamics. More precisely, this last point is to be taken as a requirement, which forces the scalar masses to be all positive definite. As a condition on the parameters, this reads:

$$F \leq \min\{\lambda_I\} M^2. \tag{3.41}$$

Explicit loop computations give, for gaugino masses and sfermion squared masses:

$$M_r = \frac{\alpha_r}{4\pi}\frac{F}{M}\sum_I N_{r,I}\, g(F/\lambda_I M^2) \approx \frac{\alpha_r}{4\pi}\frac{F}{M} N_r, \tag{3.42}$$

$$\tilde{m}_{\tilde{f}}^2 = 2\sum_r C_r(f)\frac{\alpha_r^2}{16\pi^2}\frac{F^2}{M^2}\sum_I N_{r,I}\, f(F/\lambda_I M^2) \approx 2\sum_r C_r(f)\left(\frac{\alpha_r}{4\pi}\frac{F}{M}\right)^2 N_r. \tag{3.43}$$

Here, the indices $N_{r,I}$ quantify the relative contribution to the messenger indices N_r of each irreducible component of ρ, and the functions g and f deviate significantly from one only when (3.41) is close to equality.[7] The coefficient $C_r(f)$ is the quadratic Casimir of the sfermion $\tilde{f}$, relative to the factor $G_{SM}|_r$.

The main features of gauge-mediated spectra are easily read from the above formulas: (1) due to the peculiar scaling existing between the mass dimension of the parameters and the number of loops at which they arise, gauginos and sfermions

[7]Still, the deviation is order 1. Later, at need, the form of g will be made explicit.

have physical masses of the same order; (2) more precisely, a hierarchical structure is present, and it is governed by the kind of gauge interaction that each particle feels, making squarks and gluinos much heavier than right selectrons and bino; (3) sfermion masses are generation independent, so that no additional flavor structure is introduced into the supersymmetric Standard Model (this is a virtue of gauge-mediated scenarios, since all the constraints coming from precision flavor experiments are automatically respected).

With these quantitative expressions, we can come back to a few points that were left open in the previous discussions. Coloured sparticles, and especially gluinos, have strong detector bounds, implying roughly that

$$\tilde{m}_{\text{strong}} \approx \frac{\alpha_3}{4\pi}\frac{F}{M} \gtrsim 1\ TeV. \tag{3.44}$$

Once the magnitude of the soft terms is fixed, one can vary independently one of the two dimensionful parameters M and F. A lower bound on M comes from expression (3.41) which, setting $\min\{\lambda_I\} = 1$, implies that

$$M \gtrsim \frac{\tilde{m}_{\text{strong}}}{\alpha_3/4\pi} \approx 100\ TeV \left(\frac{\tilde{m}_{\text{strong}}}{TeV}\right). \tag{3.45}$$

An upper bound is also to be imposed, coming after consistency considerations in the context of Supergravity. Pretty much like in Gauge Mediation, when a sector breaks Supersymmetry, gravitational interactions transmit the breaking to all the other sectors, including the observable one, so that soft masses are generated of order F/M_{Pl}. To make the Gauge Mediation scenario self-sufficient, gravity contributions should be suppressed with respect to gauge-mediated ones. Translated as a bound on M, this reads:

$$M \lesssim \frac{\alpha_1}{4\pi} M_{Pl} \sim 10^{15} GeV, \tag{3.46}$$

meaning that M can span over a range of 10 orders of magnitude.

One could ask where do the scales M and $\sqrt{F}$ come from. From this perspective, it seems that the boundary values allowed for M provide the most conservative scenarios. On one extreme, M is very close to Λ_{GUT}, leaving only F to be explained. On the other, $F \approx M^2$, and it could be argued that the two scales arise from the same dynamics. It is not our purpose, by the way, to give a complete description of gauge-mediated models, and the parameters will be taken as independent. On the contrary, it should be observed that the model is impressively predictive, since the spectrum depends basically on the combination F/M alone.

The other parameter that governs the spectrum is the messenger index, fixing a ratio of order $\sqrt{N}$ between the masses of gauginos and sfermions. Now that we have a picture of what could be the messenger scale, we can use formula (3.37) to set a useful bound on N, based on the requirement of perturbativity for the gauge couplings up to Λ_{GUT}:

$$N \lesssim \frac{150}{\log(\Lambda_{GUT}/M)}. \tag{3.47}$$

The bound is stronger for low scale mediation. In this case, it imposes $N \lesssim 5$.

Finally, we can check whether the gravitino interactions are dominated by its longitudinal components or, equivalently, whether its mass is smaller than the typical energy of interactions involving sparticles. The bounds on M can be translated to bounds on F or on the gravitino mass m_g (which is $\sim F/M_{Pl}$):

$$10^{-8} GeV \left(\frac{\tilde{m}_{\text{strong}}}{TeV}\right)^2 \lesssim m_g \lesssim 100\ GeV \left(\frac{\tilde{m}_{\text{strong}}}{TeV}\right). \tag{3.48}$$

For almost all the parameters range, we conclude that Gauge Mediation models imply a very light gravitino, which for many purposes can be thought as the goldstino of rigid Supersymmetry breaking, since m_{soft}, which sets the energy scale of processes that involve it (primarily, decay of sparticles), is at least 100 GeV.

Let's now discuss the possibility that, in a similar set-up, some of the sfermions get decoupled from the rest of the spectrum, by acquiring mass at tree-level.

In the simplest implementation of Tree-level Gauge Mediation [28, 29], it is assumed that the Standard Model fields are charged under an extra gauge symmetry $U(1)_{\text{x}}$. Once this, together with Supersymmetry, is broken by the vacuum expectations of some G_{SM}-neutral fields X'_i, a non-zero expectation value for the operator

$$J_{\text{x}} = \phi^\dagger Q_{\text{x}} \phi$$

can be in principle induced. In turn, $U(1)_{\text{x}}$ gauge interactions generate the contact term[8]

$$-\frac{1}{2} g_{\text{x}}^2 J_{\text{x}}^2 = -g_{\text{x}}^2 \langle J_{\text{x}} \rangle \sum_f q_{\text{x}}(f) |\tilde{f}|^2 + \dots, \tag{3.49}$$

which provides a mass for the sfermions that is proportional to their $U(1)_{\text{x}}$ charge. The requirement that the above mass terms do not destabilize the G_{SM} preserving vacuum, implies that their charges must have the same sign, which must be equal also to that of $\langle J_{\text{x}} \rangle$.

Charge assignment is not arbitrary in a quantum field theory: for gauge symmetries to be anomaly-free, all of the anomaly coefficients

$$d_{IJK} = tr(\{t_I, t_J\} t_K)$$

have to be zero. An important implication, which is dictated by the vanishing of coefficients $d_{\text{SM,SM,x}}$, is that there must exist some fields which carry a non-trivial representation of the Standard Model gauge group and a non-zero $U(1)_{\text{x}}$ charge, opposite in sign with respect to that of sfermions. To achieve this with the ingredients of Gauge Mediation models, one has to assign x-charges to some of the messenger fields, with the consequence that their masses get negative contributions from $\langle J_{\text{x}} \rangle$. This is not necessarily a problem for vacuum stability, since messengers have a Supersymmetric mass term as well, which is positive and can be made large enough.

The order of magnitude of this contribution to $\tilde{m}_f^2$ can be estimated to be

$$m^2 \sim \frac{F^2}{M^2}, \tag{3.50}$$

[8]Technically, this comes after integrating out the auxiliary field D_{x}.

as appropriate for a tree-level effect which goes to zero when Supersymmetry is unbroken as well as when the hidden sector decouples. In this way, x-charged sfermions are naturally heavier than gauginos or x-neutral sfermions, which still get masses from loops of messengers.

3.4 The Goldstino Supermultiplet

So far, the low energy implications of Supersymmetry breaking have been encrypted into two distinct types of informations: on one side, through soft terms for the observable fields and, on the other, through the introduction of a dynamical goldstino that couples universally to the Supersymmetry current of the observable fields.

Both the soft terms and the goldstino itself hide the underlying supersymmetric structure of the theory: the first by breaking it explicitly, the second by coming unpaired. In this respect, it is remarkable that we can, at the same time, restore linear Supersymmetry (that is, we can use the language of superfields) and unveil the intimate connection between soft terms and goldstino universal couplings.

Firstly, as a response to the above observation that the goldstino does not have a supersymmetric partner, an embedding in a chiral superfield is provided for this, having dynamical components x and G:

$$X = x + \sqrt{2}G\theta + f\theta^2. \tag{3.51}$$

As implied by equation (3.27), the goldstino direction in field space is given by $\langle F \rangle_i$. This means that, necessarily from first principles, f has expectation equal to

$$F = \sqrt{\sum_i \langle F \rangle_i^2}.$$

This is nothing but the square root of the energy density of the vacuum, and it measures the total amount of Supersymmetry breaking. This suggests that we can use X as a spurion, to build symmetry breaking objects out of manifestly invariant contractions [30]. Since we are especially interested in soft terms, let's consider the mass terms of gauginos, quoted in expression (3.12). As one can easily check, these are generated by the following supersymmetric contraction:

$$\frac{M_r}{2F} \int d^2\theta X \mathcal{W}_r^\alpha \mathcal{W}_{r\alpha}. \tag{3.52}$$

Besides reproducing the gaugino masses when two θ's are taken from X, this term gives additional couplings, including one which is linear in the goldstino field and bilinear in the dynamical components of the vector superfield (that is gauge bosons and gauginos). This term, given by

$$\frac{M_r}{F} \left(\lambda_r \sigma^{\mu\nu} G\right) F_{r\mu\nu}, \tag{3.53}$$

closely resembles the goldstino-current interaction, and it can in fact be shown that, upon use of the fields' equations of motion, they are actually equivalent. Given the universality, implied by equation (3.25), of terms linear in the goldstino field, this result has to be seen – for the consistency of the procedure – as a necessity.

With the above method, a non-obvious relation between soft terms and goldstino-current interactions is made manifest. When, moreover, a dynamical nature is given to x, the superpartner of the goldstino, this syntheses goes even further, by predicting new interactions between this – called the $s$goldstino – and the gauge fields.

To make clear whether these can be phenomenologically interesting, it should be first understood whether the sgoldstino is light enough[9] (let's say at most as heavy as $\tilde{m}_{\text{strong}}$). In the limit in which $F \to 0$, by continuity, one expects that a massless fermion is part of the spectrum of any supersymmetric theory since, for any $F > 0$, spontaneously broken Supersymmetry guarantees the existence of the goldstino. When $F = 0$, the symmetry is restored, implying that this fermion is accompanied by a massless scalar superpartner. When Supersymmetry breaking is switched back on, the sgoldstino mass is therefore expected to be controlled by F.

Other arguments impose that, in a broad context, the goldstino direction is associated to a whole tree-level flat direction in scalar fields' space. As shown in [31], this happens for any model of chiral fields with canonical Kähler potential and arbitrary renormalizable superpotential. Within these hypotheses, the sgoldstino mass arises at loop level.

Let's then consider the standard scenario of Gauge Meditation, to envisage what is the limit in which the above supersymmetric description is effective. We need to make sure that the only active degrees of freedom are X and the soft fields or, equivalently, that the messengers are decoupled. In terms of the dimensionful parameters of the theory, that

$$F \ll M^2. \tag{3.54}$$

In this limit, any scale that is controlled by Supersymmetry breaking is decoupled from the scale of the messengers.

A fundamental property of the Lagrangian that emerges after promoting, as we just described, the soft terms to supersymmetric interactions with X, is the following: that for fixed soft parameters, all the interactions depend just on the total amount of Supersymmetry breaking, in such a way that they grow in strength with the inverse power of the parameter F. Let's therefore estimate the value of F in the Gauge Mediation scenario, focusing in particular on its lower limit.

The predictivity of Gauge Mediation resides in the fact that the whole spectrum is dictated by basically just one ratio: F/M. This is especially true in scenarios where the scale of the messengers is relatively low, since in this case N must be of order one. This means that any bound on sparticle masses translates to a bound on F/M. Let's consider the constraint (3.44) on the masses of strongly interacting sparticles. Inequality (3.41) can then be turned to a lower bound on $\sqrt{F}$:

$$\sqrt{F} \gtrsim \frac{4\pi \tilde{m}_{\text{strong}}}{\alpha_3 \sqrt{\lambda_<}} \approx 100\ TeV \frac{1}{\sqrt{\lambda_<}} \left(\frac{\tilde{m}_{\text{strong}}}{TeV} \right), \tag{3.55}$$

[9] Unlike its fermionic partner, its mass is not protected by any symmetry, and it has to be considered as an additional parameter which depends on the specific model.

where $\lambda_< \equiv \min\{\lambda_I\}$. Notice that this lower bound is reached when $F \simeq \lambda_< M^2$, which is the opposite regime with respect to the effective theory one, in which all the messengers can be integrated out. For the effective theory to hold, even higher values of F are therefore needed (but not necessarily *much* higher, as experience suggests).

The important point is that, even for low-scale Gauge Mediation, the parameter F seems to be too high for the interactions of the goldstino supermultiplet to be of any relevance, since it is much higher than the typical energy scale of MSSM processes. This hierarchy is of course rooted in the loop factor that relates the soft masses to the Supersymmetry breaking scale.

With this warning in mind, the next section will be devoted to the analysis of the sgoldstino interactions induced by the operator (3.52), as it was born from the attempt of explaining the sadly "evaporated" 750 GeV excess in the diphoton channel, announced at CERN in December 2015 [32, 33].

3.5 The Sgoldstino

Let's now focus on the interactions that operator (3.52) dictates between sgoldstino and gauge bosons. These are given by:

$$\sum_r \frac{M_r}{2\sqrt{2}F}(sF_r^{\mu\nu}F_{r\mu\nu} - aF_r^{\mu\nu}\tilde{F}_{r\mu\nu}), \tag{3.56}$$

where the sgoldstino has been decomposed into its real and imaginary part as $x = (s + ia)/\sqrt{2}$.

From the point of view of LHC physics, these vertices look particularly appealing, since they directly provide the ingredients for the s-channel production and decay of the sgoldstino, through respectively the fusion of two gluons, which are abundant in protons, and the emission of two photons, which gives the cleanest signal to be detected. For invariant energies close to the mass of the sgoldstino, the diphoton signal would feature a characteristic resonant enhancement that, if large enough, could emerge from the background of Standard Model processes (the strength of the signal being rooted, at the fundamental level, in the coefficients of the vertices).

Let's then study the details of this process. In the resonant limit, the cross section for the production of two photons is expressible in terms of the resonance's partial widths into gluons and photons [34]:

$$\sigma(pp \to \text{res} \to \gamma\gamma) = \frac{\tilde{C}_{gg}}{m_{\text{res}}\Gamma_{\text{res}}s_{\text{c.m.}}}\Gamma(\text{res} \to gg)\Gamma(\text{res} \to \gamma\gamma) \tag{3.57}$$

where it is assumed that no other parton is significantly involved in the production beneath the gluon, whose participation in the process is quantified by $\tilde{C}_{gg}$.[10] With the assumption that the decay into two gluons dominates the total width Γ_{res}, the

[10]This coefficient is given by the product of the partonic integral C_{gg} of reference [34] and of the K factor K_{gg} of reference [35]. For $m_{\text{res}} = 750\ GeV$ and $\sqrt{s_{\text{c.m.}}} = 13\ TeV$, they are equal to respectively 2137 and 2.8.

analysis of the signal gets simplified, the width into photons being directly readable from the signal strength:

$$\Gamma_{\gamma\gamma} = \sigma m_{\rm res} s_{\rm c.m.}/\tilde{C}_{gg}. \tag{3.58}$$

This assumption, which is generically satisfied for the specific case of the sgoldstino (see, for example, reference [36] for early discussions on sgoldstino collider phenomenology), is also the one that selects the *minimum* width into photons that is compatible with the given signal. Using as central values the ones that were obtained by fitting the bump at 750 GeV, one gets:

$$\Gamma_{\gamma\gamma} \approx 0.4\ MeV \left(\frac{m_{\rm res}}{750\ GeV}\right)\left(\frac{\sigma}{8\ fb}\right)\left(\frac{\sqrt{s_{\rm c.m.}}}{13\ TeV}\right)^2\left(\frac{0.6\times 10^4}{\tilde{C}_{gg}}\right). \tag{3.59}$$

Even though many of the conclusions of our work [37] – whose analysis will be tracked in the following – are based on these specific numbers, let me try to argue why it is still meaningful to consider them as typical values for a realistic signal:

- $\sqrt{s_{\rm c.m.}}$ is just the $C.O.M.$ energy at which LHC operates, going from 8 to 14 TeV;
- $m_{\rm res}$, the mass of the new resonance virtually discovered in this experiment, should be around the TeV scale;
- the value of the cross section σ can also be considered prototypical of a process that, with tens of events, emerges from a clean background, the luminosity being of the order of few fb^{-1}.

Of course, these qualitative statements are compatible with quite large deviations from the central value. I would like therefore to anticipate what is the general trend that emerges from the analysis of the sgoldstino hypotheses: that it is more and more natural for smaller and smaller values of $\Gamma_{\gamma\gamma}$. Notwithstanding this, the fine-tuned regime where one is led with the quoted central number, still deserves special attention, due to its peculiar properties and the lack of previous thorough investigations (for other works on the sgoldstino interpretation of the 750 GeV excess, see [38]).

Not to make the formulas too long, whenever to a physical quantity $\mathcal{O}$ is assigned a central value $\mathcal{O}_c$, the notation $[\mathcal{O}]_*$ will be used to mean the adimensional ratio $\mathcal{O}/\mathcal{O}_c$.

Let's then consider the interaction (3.56), which gives a width into two photons equal to:

$$\Gamma_{\gamma\gamma} = \frac{m_{\rm res}^3 M_\gamma^2}{32\pi F^2}, \tag{3.60}$$

where the mass of the photino is given in terms of the Weinberg angle by $c_W^2 M_1 + s_W^2 M_2$. When combined with the previous expression in terms of experimental quantities, the scale of Supersymmetry is predicted to be:

$$\sqrt{F} \approx 4\ TeV \left(\frac{M_\gamma}{200\ GeV}\right)^{1/2} \times \left[\frac{m_{\rm res}^2 \tilde{C}_{gg}}{s_{\rm c.m.}\sigma}\right]_*^{1/4}. \tag{3.61}$$

Unfortunately, as expressed in formula (3.55), this value is too low to be compatible with Gauge Mediation: choosing a very large photino mass would not help since, due to the universality of gauge-mediated spectra, this would simply push all the other masses to higher and higher values, making the aforementioned bound on F more and more stringent.

The way out from this problem comes from a more careful analysis of the ultraviolet completion. The form of interactions (3.56) – with their magnitude fixed in terms of the gaugino masses – ultimately comes from the assumption that messengers are decoupled from the rest of the spectrum; it is by evading this assumption, and going in the limit where some scalar messenger becomes light, that such a large signal can be explained within Gauge Mediation.

3.6 Light Messengers

Let's then go back to the details of the ultraviolet completion, and consider the minimal Gauge Mediation setup, as it was described in section (3.3).

Each irreducible component of ρ, together with its conjugated representation, is characterized by (1) a coupling constant λ_I and (2) a set of indices $N_{r,I}$, quantifying respectively the strength of their interaction with the goldstino superfield X and their contribution to the running of the gauge couplings $g_{r=1,2,3}$.

The mass spectrum of the messenger sector is given by formula (3.40), and particular attention is to be devoted to the lightest scalar, whose mass is

$$m_<^2 = \lambda_<^2 M^2 - \lambda_< F. \tag{3.62}$$

The main upshot of the previous section was that, within Gauge Mediation, there is a fundamental obstacle in explaining a large cross section for sgoldstino production and, at the same time, keeping large enough gaugino masses. Where the effective coupling between sgoldstino and gauge bosons is given by expression (3.56), this conclusion is unavoidable. Still, this relies on the assumption that the messengers can be integrated out.

Here, more general formulas will be presented, that relax the assumptions on the messenger spectrum and suggest a way to tackle the obstacle.

At the root of the aforementioned effective coupling, there are the following trilinear interactions between the components of the sgoldstino and the messengers:

$$-\sum_I \lambda_I \left(\frac{s+ia}{\sqrt{2}} \psi_I \overline{\psi}_I + h.c. \right) + \sqrt{2}\lambda_I^2 M s (|\phi_{I+}|^2 + |\phi_{I-}|^2). \tag{3.63}$$

The leading contribution to the crucial amplitudes is given by diagrams where two photons or two gluons and a sgoldstino are attached to a loop of messengers (see figure (3.2)). Notice that, while s has trilinear interactions with the whole supermultiplet, a couples just to the fermions.

If the lightest messenger is heavier than the sgoldstino,[11] the decay widths are

[11]But we are not making, here, the hypotheses that the messengers are *much* heavier than the sgoldstino: for $m_< \gtrsim 2m_s$, it gives an error of just a few percent.

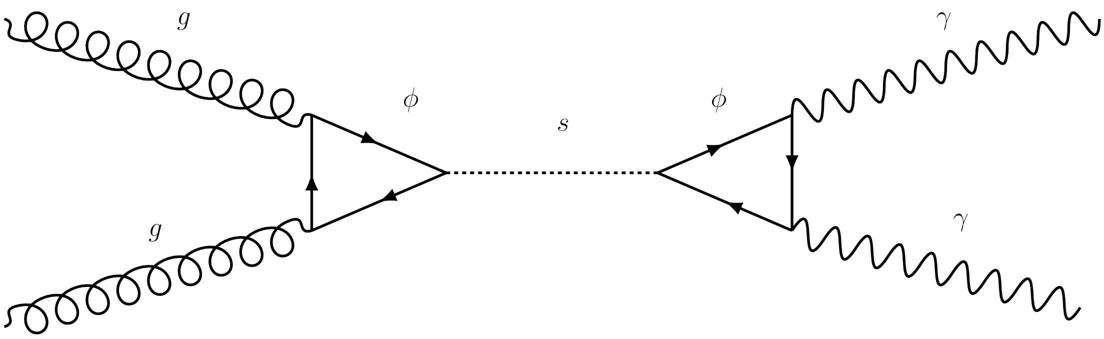

Figure 3.2: Feynman diagram corresponding to the resonant process. Both bosonic and fermionic messengers circulate in the loops.

well approximated by the following expressions:

$$\Gamma(s \to gg) = \frac{m_s^3}{M^2} \frac{4}{9} \frac{8\alpha_3^2}{(8\pi)^3} \left\{ \sum_I N_{3,I} \left[1 + \frac{1}{4} \left(\frac{\lambda_I^2 M^2}{m_{I-}^2} + \frac{\lambda_I^2 M^2}{m_{I+}^2} \right) \right] \right\}^2, \tag{3.64}$$

$$\Gamma(a \to gg) = \frac{m_a^3}{M^2} \frac{8\alpha_3^2}{(8\pi)^3} N_3^2, \tag{3.65}$$

$$\Gamma(s \to \gamma\gamma) = \frac{m_s^3}{M^2} \frac{4}{9} \frac{\alpha^2}{(8\pi)^3} \left\{ \sum_I N_{\gamma,I} \left[1 + \frac{1}{4} \left(\frac{\lambda_I^2 M^2}{m_{I-}^2} + \frac{\lambda_I^2 M^2}{m_{I+}^2} \right) \right] \right\}^2, \tag{3.66}$$

$$\Gamma(a \to \gamma\gamma) = \frac{m_a^3}{M^2} \frac{\alpha^2}{(8\pi)^3} N_\gamma^2, \tag{3.67}$$

where $N_\gamma = (5/3)N_1 + N_2$. In the effective field theory limit of the previous section, in which $F \ll M^2$, the messengers are not only decoupled, but are almost degenerate in mass, and the expression for $\Gamma_{\gamma\gamma}$ given there is readily recovered, upon use of the Gauge Mediation expression for the photino mass:

$$\Gamma(s \to \gamma\gamma) \approx \frac{m_s^3}{M^2} \frac{4}{9} \frac{\alpha^2}{(8\pi)^3} \left(\sum_I N_{\gamma,I} \left(1 + \frac{1}{4} + \frac{1}{4} \right) \right)^2 = \frac{m_s^3}{32\pi F^2} \left(\frac{\alpha N_\gamma F}{4\pi M} \right)^2. \tag{3.68}$$

Let's instead be guided by the previous formulas, and consider the limit in which the light messengers ϕ_- are much lighter than ϕ_+ and ψ. It is reasonable to assume that this can only happen for the scalar with the smallest coupling to X, to be called $\phi_<$: the reason is that the condition $m_{\phi_-} \ll m_\psi$ implies a fine tuning to 1^- of the ratio $F/(\lambda M^2)$, something that can be done just for one λ, unless these couplings are also tuned to be very close.[12]

In this limit, the widths for the decay of s, but not for that of a, get enhanced, and are approximately equal to

$$\Gamma(s \to gg) \approx \frac{m_s^3}{m_<^2} \frac{1}{36} \frac{8\alpha_3^2}{(8\pi)^3} N_{3<}^2 \left(\frac{\lambda_<^2 M}{m_<} \right)^2, \tag{3.69}$$

[12] This can naturally happen if there is an unbroken flavour symmetry relating messengers with the same gauge transformations.

$$\Gamma(s \to \gamma\gamma) \approx \frac{m_s^3}{m_<^2} \frac{1}{36} \frac{\alpha^2}{(8\pi)^3} N_{\gamma<}^2 \left(\frac{\lambda_<^2 M}{m_<} \right)^2 , \tag{3.70}$$

where $N_{3<}$ and $N_{\gamma<}$ are, respectively, the contributions of the near-critical messengers to N_3 and N_γ.

While the width of s changes drastically, the mass of the gauginos suffers only minor modifications: the only change is in the function g of formula (3.42),[13] which for near-critical messengers is approximately equal to log4. Then one has:

$$M_3 = \frac{\alpha_3}{4\pi} \frac{F}{M} \left(N_{3<} \log 4 + \sum_{I \neq <} N_{3,I}\ g(F/\lambda_I M^2) \right) = \frac{\alpha_3}{4\pi} \frac{F}{M} \overline{N}_3. \tag{3.71}$$

and it can be easily checked that

$$N_3 + N_{3<}(\log 4 - 1) < \overline{N}_3 < N_3 \log 4. \tag{3.72}$$

Since in any case $\overline{N}_3 / N_3$ is of order one, the bound on $\sqrt{F}$ given at the end of section (3.4) is only slightly modified in this limit.

Let's analyse closely expressions (3.69,3.70). Even though the parametric enhancement with respect to the non-critical regime is simply quantified by $(\lambda_< M/m_<)^4$, its physical origin is more clearly understood in terms of two distinct, cooperating mechanisms:

1. Due to the fact that $\phi_<$ is much lighter than the other messengers, s does preferably decay through its mediation; this implies that the suppression scale of the decay amplitude is far less severe: it is now $m_<$ instead of the messenger scale.

2. The previous consideration was dictated essentially by dimensional analysis. On top of this, the decay width shows a further enhancement of magnitude $(\lambda_<^2 M/m_<)^2$ which, if not explained out of the kinematics of the decay, must come from a coupling. As a matter of fact, the ratio in parentheses is the proper dimensionless measure of the interaction strength between s and the near-critical messengers:

$$-\sqrt{2}\lambda_<^2 M s |\phi_<|^2 = -\sqrt{2} \left(\frac{\lambda_<^2 M}{m_<} \right) m_< s |\phi_<|^2 = -\sqrt{2} g_{\text{eff}} m_< s |\phi_<|^2. \tag{3.73}$$

A way to understand this is to compute loop corrections within an effective theory for s and $\phi_<$ alone. In the limit in which $m_< \gg m_s$, it turns out that higher loop effects are indeed controlled by

$$\sim \frac{g_{\text{eff}}^2}{16\pi^2}, \tag{3.74}$$

modulo order one factors taking into account the multiplicity of messengers. In this sense, it is clear that the limit in which $\phi_<$ becomes anomalously light is the same as the regime in which its effective coupling to the sgoldstino becomes stronger and stronger.

[13] Which is given by: $g(x) = \frac{1}{2x^2} \left((1+x)\log(1+x) + (1-x)\log(1-x) \right)$

Because it is in this very regime that a large production rate can be explained, one should study, besides its virtues, what are the possible limitations. Of course, the closer to criticality the messengers are, the higher is the level of fine tuning, which is quantified in terms of the hierarchy within the near-critical supermultiplet

$$\Delta = \left(\frac{m_{\psi<}}{m_<}\right)^2 = \left(\frac{g_{\text{eff}}}{\lambda_<}\right)^2 . \tag{3.75}$$

Moreover, as a consequence of its role in governing loop corrections, g_{eff} is subjected to perturbativity constraints, implying roughly that

$$g_{\text{eff}} = \lambda_< \frac{m_{\psi<}}{m_<} \lesssim 4\pi. \tag{3.76}$$

When this is violated, perturbative computations no longer hold, invalidating the analysis.

Another constraint comes from the requirement that the light messengers, which for our scope must be charged and coloured, are heavy enough to escape detection. We take as an approximate constraint that $m_< \gtrsim 1\ TeV$.

With this limitations in mind, the next section will be devoted to a quantitative analysis of the problem, to see whether and in which conditions the signal can be fitted.

3.7 Quantitative Analysis

The following analysis uses approximations that are valid in the regime in which one messenger is anomalously light. Moreover, we assume as before that the width of the sgoldstino is dominated by the decay into gluons. A possible competitor is the tree-level decay into two goldstinos, which is induced by the same effective operator that gives mass to the sgoldstino: $(m_s^2 |X|^4/F^2)_D$. The decay width is given by

$$\Gamma(s \to GG) \approx \frac{m_s}{32\pi}\left(\frac{m_s^2}{F}\right)^2 , \tag{3.77}$$

and in the following it will be compared with the other widths.

For the numerical analysis, it turns sometimes useful to think in terms of adimensional quantities, like g_{eff} and

$$x \equiv \frac{4m_<^2}{m_s^2}. \tag{3.78}$$

Formula (3.58), valid in the gluon dominance hypotheses, states that the signal fixes the ratio $\Gamma(s \to \gamma\gamma)/m_s$, called Γ_{red} and equal to

$$\Gamma_{\text{red}}(s \to \gamma\gamma) \approx 0.53 \times 10^{-6} \left[\frac{\sigma s_{\text{c.m.}}}{\tilde{C}_{gg}}\right]_* . \tag{3.79}$$

Let's now take formula (3.70) and rewrite it in terms of the new variables. Before doing that, we modify it in order to take into account resonance effects that arise when the mass of the particles in the loop – that is the messengers – is close to half the mass of the sgoldstino; this modification makes the formula valid down to that threshold, at the price of introducing a dependence on the function F, defined as

$$F(x) = x \arctan^2 \frac{1}{\sqrt{x-1}} - 1 = \frac{1}{3x} + O(\frac{1}{x^2}). \tag{3.80}$$

The resonant behaviour corresponds to the limit $x \to 1^+$, while the simple behaviour of the approximate formula is recovered for large x by the use of the above expansion in powers of $1/x$. We have then:

$$\begin{aligned}\Gamma_{\text{red}}(s \to \gamma\gamma) &= \frac{m_s^2}{4m_<^2}\left(\frac{1}{8\pi}\right)^3 \frac{\alpha^2 N_{\gamma<}^2}{9}\left(\frac{\lambda_<^2 M}{m_<}\right)^2 \left[3\frac{4m_<^2}{m_s^2} F\left(\frac{4m_<^2}{m_s^2}\right)\right]^2 \\ &= \frac{1}{8\pi}\frac{\mathcal{F}(x)^2}{x}\frac{\alpha^2 N_{\gamma<}^2}{36}\left(\frac{g_{\text{eff}}}{4\pi}\right)^2 \approx 0.65 \times 10^{-7}\, \frac{\mathcal{F}(x)^2}{x} N_{\gamma<}^2 \left(\frac{g_{\text{eff}}}{4\pi}\right)^2, \end{aligned} \tag{3.81}$$

where $\mathcal{F} = 3xF$ rapidly approaches 1 for large x, and it's equal to $3((\pi/2)^2 - 1) \approx 4.4$ when $x = 1$.

Due to the constraints coming from the preservation of perturbative gauge coupling unification, $N_{\gamma<}$ cannot be too large. Moreover, depending on the particular value of m_s, the variable x is bounded from below by the non-observation of new coloured particles at colliders. Imposing that $m_< \gtrsim 1\ TeV$, we find:

$$x \gtrsim 7\left(\frac{750\ GeV}{m_s}\right)^2. \tag{3.82}$$

The goal is to fit signal (3.79) with formula (3.81), respecting the limits on x and g_{eff} and having first specified the representation furnished by the messengers. Remember that all three indices N_r have to be equal and $\lesssim 5$, and that only one irreducible component – carrying non-trivial electric charge and $SU(3)$ quantum numbers – can be chosen to become the near-critical one.

For example, we can choose the near-critical messengers to be the components $V(3,2)_{-5/6}$ and $\overline{V}(\overline{3},2)_{5/6}$ of the adjoint of $SU(5)$. Having $N = 5$, this choice is at the edge of gauge coupling unification. The photon index of the light messengers, $N_{\gamma<}$, takes the quite large value of $34/3$.

After imposing that the signal is reproduced, we obtain the following relation between x and g_{eff}:

$$\frac{x}{\mathcal{F}(x)^2} \approx 15.7\ \left(\frac{g_{\text{eff}}}{4\pi}\right)^2 \times \left[\frac{\tilde{C}_{gg}}{\sigma s_{\text{c.m.}}}\right]_* . \tag{3.83}$$

Qualitatively, one has in this case to choose between a very light scalar messenger and an effective coupling close to its non-perturbative limit. Of course, it can be, in general, that the window of allowed values closes, or instead that it is less tight. Notice that the range of the left hand side for $x > 1$ is $(0.052, +\infty)$;[14] if we instead put limits on the level of the fine tuning of x to 1^+, and consider for example $x > 1.5$, the range is reduced to $(0.54, +\infty)$.

[14]The condition $x > 1$ corresponds, on the physics side, to the requirement that the decay of the sgoldstino into two messengers is not kinematically allowed, or that $m_< > m_s/2$.

So far, the bounds on the mass of gluinos has not been taken into account: the following expression shows that, as a matter of fact, this mass can always be made large enough by choosing a small enough $\lambda_<$ (which is by now unconstrained)

$$M_3 = \frac{\alpha_3 \overline{N}_3}{4\pi} \frac{g_{\text{eff}}}{\lambda_<} m_<. \tag{3.84}$$

This, of course, at the price of worsening the fine tuning in the messenger sector.

Let's close the section with a comparison of those that are usually the largest partial widths in sgoldstino models. First, it is straightforward to see that decay into gluons dominates over that into photons, as a consequence of the larger coupling constant:

$$\frac{\Gamma(s \to gg)}{\Gamma(s \to \gamma\gamma)} = 8 \left(\frac{\alpha_3}{\alpha}\right)^2 \left(\frac{N_{3<}}{N_{\gamma<}}\right)^2 \approx 1.6 \times 10^3 \left(\frac{N_{3<}}{N_{\gamma<}}\right)^2. \tag{3.85}$$

Then, we can compare the photon width with the goldstino one. Using formula (3.77), and after some manipulation, we find that

$$\frac{\Gamma(s \to \gamma\gamma)}{\Gamma(s \to GG)} = \left(\frac{\alpha N_{\gamma<}}{96\pi}\right)^2 \left(\frac{g_{\text{eff}}}{\lambda_<}\right)^6 x \approx 2.56 \left(\frac{N_{\gamma<}}{5}\right)^2 \left(\frac{g_{\text{eff}}}{4\pi\lambda_<}\right)^6 \left(\frac{x}{10}\right). \tag{3.86}$$

The trend is not obvious, due especially to the large sensitivity on $g_{\text{eff}}/\lambda_<$. Using (3.84), we can express the above formula in terms of particles' masses and messenger indices:

$$\frac{\Gamma(s \to \gamma\gamma)}{\Gamma(s \to GG)} \approx 37 \left(\frac{N_{\gamma<}}{5}\right)^2 \left(\frac{5}{\overline{N}_3}\right)^6 \left(\frac{m_s}{m_<}\right)^4 \left(\frac{M_3}{m_s}\right)^6. \tag{3.87}$$

The width for the decay into goldstinos is maximized for M_3 and $m_<$ at respectively their lower and upper limits. In the example given before, with messengers in the adjoint representation, one finds

$$\frac{\Gamma(s \to \gamma\gamma)}{\Gamma(s \to GG)} \sim 500 \left(\frac{M_3}{1.7\ TeV}\right)^6 \left(\frac{1.5\ TeV}{m_<}\right)^4, \tag{3.88}$$

where we specified to the central values of (3.59), and took $\overline{N}_3 = 6$. As a reference value for M_3, we took its most conservative lower bound, while, to fix $m_<$, we extracted x from formula (3.83), with g_{eff} at its perturbative limit 4π. We see that, in this case, the goldstino width is always subdominant with respect to the photon width and, a fortiori, with respect to the gluino one.

3.8 Conclusions

For the present thesis, I preferred to develop what can be considered the core of reference [37]. What we found there is that, in order to make the sgoldstino interpretation of the 750 GeV excess compatible with Gauge Mediation, one needs to go in a peculiar regime, characterized by the presence of some anomalously light messenger. The choice of Gauge Mediation as a UV completion is in turn the most appropriate, since it allows for a relatively low value of the Supersymmetry breaking scale $\sqrt{F}$: the smaller this scale, the stronger the coupling of the sgoldstino to the the observable sector.

After the announcement – given in August 2016 – that the 750 GeV excess was just a statistical fluctuation, the interest in the specific numbers used in [37] of course decreased. Due to this, the formulas have been adapted to arbitrary experimental values. Still, all the approximations which are valid in the near-critical regime have been used. Our computations, in fact, give a general message: Gauge mediation, together with the bounds on sparticles' masses, imply a lower bound on the Supersymmetry breaking scale which forces it to be quite large (with respect to the scale of accelerator processes). Therefore, the interactions of the sgoldstino are expected to be generically suppressed. In the near-critical regime, with the light scalar messengers at the TeV scale, a window opens up in which the sgoldstino can take part in collider phenomenology. For this reason, therefore, the near-critical regime has an interest which goes beyond its original motivation.

Apart from the unavoidable increase of fine-tuning, we found that, in this same regime, light messengers and sgoldstino become strongly coupled. Notice that, with a superpotential for the hidden sector of the form

$$\mathcal{W} = \lambda X \Phi \overline{\Phi} - FX,$$

there is no tree-level dependence of the potential V on x, the scalar component of X. This means that, in this approximation, any value of $\langle x \rangle$ which makes the messengers non-tachyonic (that is $\langle x \rangle > \sqrt{F/\lambda} \equiv x_{\text{crit}}$) works equally well. For $\langle x \rangle \gg x_{\text{crit}}$, well-known loop corrections lift the flat direction, so that the potential increases for larger field values. What about the region close to the critical point? Here, strong interactions come into play, and one can speculate on the possibility that, as a consequence of these, a stable point is generated at $x_{\text{min}} = x_{\text{crit}}(1+\epsilon)$. If this was the case, the smallness of ϵ, that is the lightness of the scalar messengers with respect to their fermionic partners (or, in other words, the fine-tuning), would find a self-contained explanation. But this is just speculative thinking, and it would be interesting to support it with non-perturbative computations (carried out with lattice methods, for example).

Let me conclude by mentioning that, as a consequence of the gauge structure of the electroweak sector, decay into photons is always accompanied by decay into the other gauge bosons (at least ZZ and $Z\gamma$) at a comparable rate. The constraints coming from this are discussed in [37], along with a discussion on the possible role of the R-axion (the Goldtone boson of R-symmetry breaking) in faking the diphoton signal, and on the possibility that sfermion masses get decoupled thanks to the mechanism discussed at the end of section (3.3).

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