SISSA
Scuola Internazionale Superiore di Studi Avanzati

Area of Mathematics

# Untangling of trajectories for non-smooth vector fields and Bressan's Compactness Conjecture

**Ph.D. Thesis**

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Academic Year 2016/2017

Il presente lavoro costituisce la tesi presentata da Paolo Bonicatto, sotto la direzione del Prof. Stefano Bianchini, al fine di ottenere l'attestato di ricerca post-universitaria Doctor Philosophiæ presso la SISSA, Curriculum in Analisi Matematica, Modelli e Applicazioni. Ai sensi dell'art. 1, comma 4, dello Statuto della SISSA pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di Dottore di Ricerca in Matematica.

Trieste, Anno Accademico 2016-2017

Jeder kann zaubern, jeder kann seine Ziele erreichen, wenn er denken kann, wenn er warten kann, wenn er fasten kann.

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H. Hesse, *Siddharta*

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# Introduction

This thesis aims to present some recent advances in the study of two partial differential equations of the first order: we will consider the *continuity equation*

$$\begin{cases} \partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0, & \text{in } [0,T]\times\mathbb{R}^d \\ u(0,\cdot) = \overline{u}(\cdot) \end{cases} \tag{1}$$

and the *transport equation*

$$\begin{cases} \partial_t u + \boldsymbol{b}\cdot\nabla u = 0, & \text{in } [0,T]\times\mathbb{R}^d \\ u(0,\cdot) = \overline{u}(\cdot) \end{cases} \tag{2}$$

where $\boldsymbol{b}\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is a given vector field, $u\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a scalar function and $\overline{u}\colon \mathbb{R}^d \to \mathbb{R}$ is a given initial datum.

The continuity and the transport equations are among the *cardinal equations* of Mathematical Physics: for instance, the conservation of mass in Euler's equations of fluid-mechanics has the form of (1). In that case, a solution $u$ to (1) can be thought as the density of a continuous distribution of particles moving according to the velocity field $\boldsymbol{b}$; in other terms, the quantity $u(t,x)$ represents the number of particles per unit volume at time $t\in[0,T]$ and position $x\in\mathbb{R}^d$. Notice, moreover, that (1) and (2) are equivalent when $\operatorname{div}\boldsymbol{b} = 0$.

When $\boldsymbol{b}$ is sufficiently regular, existence and uniqueness results for (classical) solutions to Problems (1) and (2) are well known. They rely on the so called *method of characteristics* which establishes a deep connection between the "Eulerian" problems (1), (2) and their "Lagrangian" counterpart, given by the ordinary differential equation driven by $\boldsymbol{b}$:

$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t,\boldsymbol{X}(t,x)), & (t,x)\in[0,T]\times\mathbb{R}^d \\ \boldsymbol{X}(0,x) = x. \end{cases} \tag{3}$$

Under suitable regularity assumptions on $\boldsymbol{b}$, it is well known (and goes under the name of Cauchy-Lipschitz theory) that a *flow* exists, i.e. there is a smooth map $\boldsymbol{X}$ solving (3). A simple observation yields that, if $u$ is a solution to (2), then the function $t\mapsto u(t,\boldsymbol{X}(t,x))$ has to be constant: this allows to conclude that the unique solution $u$ of (2) is the *transport* of the initial data $\overline{u}$ along the *characteristics* of (3), i.e. along the curves $[0,T]\ni t\mapsto \boldsymbol{X}(t,x)$. Thus we end up with an explicit formula for the solution $u$ to (2):

$$u(t,x) = \overline{u}\left(\boldsymbol{X}(t,\cdot)^{-1}(x)\right).$$

Similarly one can obtain an explicit formula for solutions to (1).

However, in view of the applications to fluid-mechanics, one would like to deal with velocity fields or densities which are not necessarily smooth. For instance, continuity equation and transport equation with non-smooth vector fields are related to Boltzmann [**DL89b**, **DL91**] and Vlasov-Poisson equations [**DL89a**], and also to *hyperbolic conservation laws*. In particular the *Keyfitz and Kranzer system* (introduced in [**KK80**]) is a system of conservation laws that reads as

$$\partial_t u + \operatorname{div}\left(\boldsymbol{f}(|u|)u\right) = 0 \qquad \text{in } [0,T]\times\mathbb{R}^d, \tag{4}$$

where the map $\boldsymbol{f}\colon \mathbb{R}^+ \to \mathbb{R}^d$ is assumed to be smooth. It has been shown in [ABDL04] that (4) can be formally decoupled in a scalar conservation law for the modulus $r = |u|$ and a transport equation (with field $\boldsymbol{f}(r)$) for the angular part $\vartheta = u/|u|$:

$$\begin{cases} \partial_t r + \operatorname{div}(\boldsymbol{f}(r)r) = 0, \\ \partial_t \vartheta + \boldsymbol{f}(r) \cdot \nabla \vartheta = 0. \end{cases}$$

As it is well known, solutions to systems of conservation laws are in general non-smooth, hence the vector field $\boldsymbol{f}(r)$ appearing in the transport equation is not regular enough to apply the method of characteristics: we thus have to go beyond the Cauchy-Lipschitz setting.

**The classical approach: renormalized solutions.** The exploration of the non-smooth framework started with the paper of DiPerna and Lions [DL89c]. They realized that an interplay between Eulerian and Lagrangian coordinates could be exploited to deduce well-posedness results for the ODE (3) from analogous results on PDEs (1) and (2).
On the one hand, due to the linearity of the PDEs, the *existence* of weak solutions to (1), (2) is always guaranteed under reasonable summability assumptions on the vector field $\boldsymbol{b}$ and its spatial divergence; on the other hand, the problem of uniqueness turns out to be much more delicate. A possible strategy, introduced by [DL89c], to recover uniqueness, is based on the notions of *renormalized solution* and of *renormalization property*.
Roughly speaking, a bounded function $u \in L^\infty([0,T] \times \mathbb{R}^d)$ is said to be a *renormalized solution* to (2) if for all $\beta \in C^1(\mathbb{R})$ the function $\beta(u)$ is a solution to the corresponding Cauchy problem:

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0, \\ u(0,\cdot) = \bar{u} \end{cases} \implies \begin{cases} \partial_t(\beta(u)) + \boldsymbol{b} \cdot \nabla(\beta(u)) = 0 \\ \beta(u(0,\cdot)) = \beta(\bar{u}(\cdot)) \end{cases} \quad \text{for every } \beta \in C^1(\mathbb{R}).$$

This can be interpreted as a sort of weak "Chain Rule" for the function $u$, saying that $u$ is differentiable along the flow generated by $\boldsymbol{b}$. In [DL89c] it is shown that the validity of this property for every $\beta \in C^1(\mathbb{R})$ implies, under general assumptions, uniqueness of weak solutions for (2). Moreover, when this property is satisfied by all solutions, this can be transferred into a property of the vector field itself, which will be said to have the *renormalization property*.

The problem of uniqueness of solutions is thus shifted to prove the renormalization property for $\boldsymbol{b}$: this seems to require some regularity of vector field (tipically in terms of spatial weak differentiability), as counterexamples by Depauw [Dep03a] and Bressan [Bre03] show. With an approximation scheme, in [DL89c] the authors proved that renormalization property holds under Sobolev regularity assumptions on the vector field; some years later, Ambrosio [Amb04] improved this result, showing that renormalization holds for vector fields which are of class BV (locally in space) with absolutely continuous divergence.

From the Lagrangian point of view, the uniqueness of the solution to the transport equation (2) translates into well-posedness results of the so-called *Regular Lagrangian Flow* of $\boldsymbol{b}$, which is the by-now standard notion of flow in the non-smooth setting. This concept was introduced by Ambrosio in [Amb04]: in a sense, among all possible integral curves of $\boldsymbol{b}$ passing through a point, the Regular Lagrangian Flow selects the ones that do not allow for concentration, in a quantitative way with respect to some reference measure (usually the Lebesgue measure $\mathscr{L}^d$ in $\mathbb{R}^d$). It is worth pointing out that a number of recent papers are devoted to the study of its properties, in particular we mention [ACF15] where a purely *local* theory of Regular Lagrangian Flows has been proposed, thus establishing a complete analogy with the Cauchy-Lipschitz theory.

**Bressan's Compactness Conjecture.** As we have seen, the theory developed by DiPerna-Lions-Ambrosio settles the Sobolev and the BV case, when the divergence of $\boldsymbol{b}$ does not contain singular terms (with respect to $\mathscr{L}^d$). However, in connections with applications to conservation laws, it would be interesting to cover also the case in which $\boldsymbol{b}$ is of bounded variation in the space, but its divergence may contain non-trivial singular terms: indeed, the natural assumption at the level of the divergence of $\boldsymbol{b}$ seems to be not really absolute continuity with bounded density, as considered in Ambrosio [Amb04], but rather the existence of a nonnegative density $\rho$ transported by $\boldsymbol{b}$, with $\rho$ uniformly bounded from above and from below away from zero. Such vector fields are called *nearly incompressible*, according to the following definition.

Definition 1. A locally integrable vector field $\boldsymbol{b}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ is called *nearly incompressible* if there exists a function $\rho\colon (0,T)\times\mathbb{R}^d\to\mathbb{R}$ (called *density* of $\boldsymbol{b}$) and a constant $C>0$ such that $0<C^{-1}\le\rho(t,x)\le C$ for Lebesgue almost every $(t,x)\in(0,T)\times\mathbb{R}^d$ and

$$\partial_t\rho + \mathrm{div}_x(\rho\boldsymbol{b}) = 0 \qquad \text{in the sense of distributions on } (0,T)\times\mathbb{R}^d.$$

Notice that no assumption is made on the divergence of $\boldsymbol{b}$; on the other hand, it is rather easy to see (for instance, by mollifications) that if $\mathrm{div}\,\boldsymbol{b}$ is bounded then $\boldsymbol{b}$ is nearly incompressible.

Nearly incompressible vector fields are strictly related to a conjecture, raised by A. Bressan (studying the well-posedness of the Keyfitz and Kranzer system (4)), predicting the strong compactness of a family of flows associated to smooth vector fields:

Conjecture 1 (Bressan's Compactness Conjecture - Lagrangian formulation). *Let $\boldsymbol{b}_k\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, $k\in\mathbb{N}$, be a sequence of smooth vector fields and denote by $\boldsymbol{X}_k$ the associated flows, i.e. the solutions of*

$$\begin{cases}\partial_t\boldsymbol{X}_k(t,x) = \boldsymbol{b}_k(t,\boldsymbol{X}_k(t,x))\\ \boldsymbol{X}_k(0,x) = x.\end{cases}$$

*Assume that the quantity $\|\boldsymbol{b}_k\|_\infty + \|\nabla\boldsymbol{b}_k\|_{L^1}$ is uniformly bounded and that the flows $\boldsymbol{X}$ are* uniformly nearly incompressible, *in the sense that there exists $C>0$ such that*

$$\frac{1}{C}\le\det\left(\nabla_x\boldsymbol{X}_k(t,x)\right)\le C.$$

*Then the sequence $\{\boldsymbol{X}_k\}_{k\in\mathbb{N}}$ is strongly precompact in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$.*

By standard compactness arguments, it is readily seen that Conjecture 1 deals essentially with an ordinary differential equation, driven by a nearly incompressible, BV vector field. From the Eulerian point of view, one can thus expect that Conjecture 1 is proved as soon as one can show well posedness at the PDE level for a vector field of class BV and nearly incompressible, extending the well-posedness result of Ambrosio [Amb04]. This is indeed the case: as it has been proved in [ABDL04], Conjecture 1 would follow from the following one:

Conjecture 2 (Bressan's Compactness Conjecture - Eulerian formulation). *Any nearly incompressible vector field $\boldsymbol{b}\in L^1([0,T];\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$ has the renormalization property.*

The main result contained in this thesis is the following Theorem, which answers affirmatively to the conjectures above.

Main Theorem. *Bressan's Compactness Conjecture holds true.*

More precisely, we will prove Conjecture 2. It is important to mention various approaches that have been tried in the recent years, also at a purely Lagrangian level: for instance, explicit compactness estimates have been proposed in [ALM05, CDL08] (and further developed in [BC13]; see also [Jab10, CJ10]).

Before presenting the techniques we use to prove the Main Theorem we briefly discuss a particular setting, namely the two-dimensional one, where finer results are availble in view of the Hamiltonian structure.

**The two-dimensional case.** The problem of uniqueness of weak solutions to the transport equation (2) in the two dimensional (autonomous) case is addressed in the papers [ABC14], [ABC13] and [BG16]. In two dimensions and for divergence-free autonomous vector fields, renormalization theorems are available under quite mild assumptions, because of the underlying Hamiltonian structure. Indeed, if $\operatorname{div} \boldsymbol{b} = 0$ in $\mathbb{R}^2$, then there exists a Lipschitz Hamiltonian $H\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\boldsymbol{b} = \nabla^\perp H$, where $\nabla^\perp = (-\partial_2, \partial_1)$. Heuristically it is readily seens that level sets of $H$ are invariant under the flow of $\boldsymbol{b}$, since

$$\frac{d}{dt} H(\gamma(t)) = \nabla H(\gamma(t)) \cdot \dot{\gamma}(t) = \nabla H(\gamma(t)) \cdot \boldsymbol{b}(\gamma(t)) = 0$$

as $\boldsymbol{b}$ and $\nabla H$ are orthogonal. This suggests the possibility of decomposing the two-dimensional transport equation into a family of one-dimensional equations, along the level sets of $H$. By means of this strategy, and building on a fine description of the structure of level sets of Lipschitz maps (obtained in the paper [ABC13]), in [ABC14], the authors characterize the autonomous, divergence-free vector fields $\boldsymbol{b}$ on the plane for which uniqueness holds, within the class of bounded (or even merely integrable) solutions. The characterization they present relies on the so called *Weak Sard Property*, which is a (weaker) measure theoretic version of Sard's Lemma and is used to separate the dynamic where $\boldsymbol{b} \neq 0$ from the regions in which $\boldsymbol{b} = 0$.

The first contribution (in the chronological order) of the author to this topic is an extension of these Hamiltonian techniques to the two-dimensional nearly incompressible case:

Theorem A ([BBG16]). *Every bounded, autonomous, compactly supported, nearly incompressible* BV *vector field on* $\mathbb{R}^2$ *has the renormalization property.*

The main lines of the proof of Theorem A were present in the author's Master Thesis [Bon14] but, since then, some simplifications (at a technical level) have occurred. In addition, the techniques developed in [BBG16] allow to transfer properties from vector fields of the form $(1, \boldsymbol{b}(x))$ to the vector field $\boldsymbol{b}(x)$. The argument of [BBG16] is based on the paper [BG16], where the authors established, always by splitting techniques, uniqueness of weak solutions for a BV vector field, which is nearly incompressible with a time independent density.
Notice, however, that in the general $d$-dimensional case, with $d > 2$, the Hamiltonian approach cannot be applied, as there are not enough first integrals of the ODE (which is to say, bounded divergence-free vector fields in $\mathbb{R}^d$ do not admit in general a Lipschitz potential).

**The chain rule approach.** We now come back to the general $d$-dimensional setting and we briefly discuss an approach towards Bressan's Conjecture 2 that has been tried.

In [ADLM07], the authors proposed to face the conjecture by establishing a *Chain rule formula* for the divergence operator. Given a bounded, Borel vector field $\boldsymbol{b}\colon \mathbb{R}^d \to \mathbb{R}^d$, a bounded, scalar function $\rho\colon \mathbb{R}^d \to \mathbb{R}$, one would like to characterize (compute) the distribution $\operatorname{div}(\beta(\rho)\boldsymbol{b})$, for $\beta \in C^1(\mathbb{R};\mathbb{R})$, in terms of the quantities $\operatorname{div} \boldsymbol{b}$ and $\operatorname{div}(\rho\boldsymbol{b})$. In the smooth setting one can use the standard chain rule formula to get

$$\operatorname{div}(\beta(\rho)\boldsymbol{b}) = \beta'(\rho) \operatorname{div}(\rho\boldsymbol{b}) + (\beta(\rho) - \rho\beta'(\rho)) \operatorname{div} \boldsymbol{b} \tag{5}$$

In the general case, however, the r.h.s. of (5) cannot be written in that form, being only a distribution. In the case the vector field $\boldsymbol{b} \in \mathrm{BV}(\mathbb{R}^d)$, it can be shown that $\operatorname{div}(\beta(\rho)\boldsymbol{b})$ is a measure, controlled by $\operatorname{div} \boldsymbol{b}$ but, as noted in [ADLM07], the main problem is to give a meaning to the r.h.s. of (5) when the measure $\operatorname{div} \boldsymbol{b}$ is singular and $\rho$ is only defined almost everywhere with respect to Lebesgue measure. To overcome this difficulty, in the BV setting, the authors split the measure $\operatorname{div} \boldsymbol{b}$ into its absolutely continuous part, jump part and Cantor part and treat the cases separately.

*The absolutely continuous part.* Their first result ([ADLM07, Thm. 3]) is that in all Lebesgue points of $\rho$ the formula (5) holds (possibly being $\operatorname{div} \boldsymbol{b}$ singular), where $\rho$ is replaced by its Lebsgue value $\widetilde{\rho}$. This is achieved along the same techniques of [Amb04], which are in turn a (non-trivial) extension of the ones employed in [DL89c]: essentially, an approximation argument via convolution is performed (leading to the study of the so called *commutators*). One can control the singular terms by taking suitable convolution kernels which look more elongated in some directions.

*The jump part.* By exploiting properties of Anzellotti's weak *normal traces* for measure divergence vector fields (see [Anz83]), Ambrosio, De Lellis and Malý managed to settle also the jump part: they obtain an explicit formula (in the spirit of (5)), involving the traces of $\boldsymbol{b}$ and $\rho\boldsymbol{b}$ along a $\mathscr{H}^{d-1}$-rectifiable set (see also [ACM05] for an extension of these results to the BD case).

*The Cantor part.* In order to tackle the Cantor part, a "transversality condition" between the vector field and its derivative is assumed in [ADLM07]: it is shown that, if in a point $(\bar{t}, \bar{x})$ one has $(D\boldsymbol{b} \cdot \boldsymbol{b})(\bar{t}, \bar{x}) \neq 0$ (where $\boldsymbol{b}(\bar{t}, \bar{x})$ is the Lebesgue value of $\boldsymbol{b}$ in $(\bar{t}, \bar{x})$) then the point $(\bar{t}, \bar{x})$ is a Lebesgue point for $\rho$.

From the analysis of [ADLM07], it thus remains open the case of tangential points, i.e. the set of points at which $D\boldsymbol{b} \cdot \boldsymbol{b}$ vanishes, which make up the so called *tangential set*. This is actually relevant, as shown in [BG16]: answering negatively to one of the questions in [ADLM07], in [BG16] the authors exhibited an example of BV, nearly incompressible vector field with non empty tangential set. Even worse, the tangential set is a Cantor-like set of non integer dimension but, at level of the density $\rho$, one sees a pure jump. This severe pathology is depicted in Figure 1 and we refer the reader to [BG16] for a detailed construction.

## Overview of our approach

We now want to present in more details our main contribution, discussing briefly the theorems we obtain and the strategy leading to their proofs.

Our analysis starts from the following observation: the two techniques presented above (Hamiltonian in two-dimensional setting and Chain Rule) are not suited for the general case for two different reasons. On the one hand, as already noticed, in the general $d$-dimensional case with $d > 2$, the Hamiltonian approach cannot be applied, as divergence free vector fields in $\mathbb{R}^d$ do not admit in general a Lipschitz potential. On the other hand, in the Chain Rule approach the problem is more subtle: clearly, it seems arduous to construct suitable convolution kernels, which adapt to the irregularity of the vector field, controlling the errors, once the main term is exhibited. The subtle problem is however to determine *which* are the main terms: one has to compute some sort of trace on sets which are not rectifiable, i.e. Cantor-like sets. Lacking a suitable notion of trace, this task seems quite difficult. Such a notion could be given by means of a *Lagrangian representation* $\eta$ of the $\mathbb{R}^{d+1}$-valued vector field $\rho(1, \boldsymbol{b})$, and this is the starting point of our approach.

**Lagrangian representations.** In the general non-smooth setting, one could recover a link between the continuity equation (1) and the ODE (3) thanks to the so called *Superposition Principle*, which has been established by Ambrosio in [Amb04] (see also [Smi94]). Roughly speaking, it asserts that, if the vector field is globally bounded, every

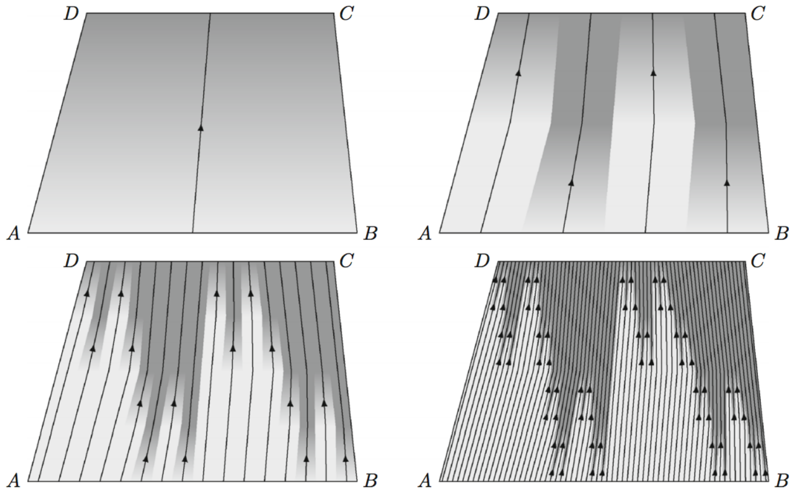



**Figure 1.** Example of [BG16]: the tangential set of the vector field $\boldsymbol{b}$ (only the integral curves have been drawn here) is a Cantor like set of dimension $3/2$. Notice that each trajectory $\gamma$ meets the tangential set in exactly one point, at time $t_\gamma$: the density $\rho$, computed along the curve, is piecewise constant, having a unique jump of size 1 in $t_\gamma$.

non-negative (possibly measure-valued) solution to the PDE (1) can be written as a superposition of solutions obtained via propagation along integral curves of $\boldsymbol{b}$, i.e. solutions to the ODE (3).

More generally, let us consider a locally integrable vector field $\boldsymbol{b} \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^d)$ and let $\rho$ be a non-negative solution to the balance law

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{b}) = \mu, \qquad \mu \in \mathscr{M}((0,T) \times \mathbb{R}^d). \tag{6}$$

with $\rho \in L^1_{\text{loc}}((1+|\boldsymbol{b}|)\mathscr{L}^{d+1})$ (so that a distributional meaning can be given). For simplicity, we will often write (6) in the shorter form

$$\operatorname{div}_{t,x}\big(\rho(1,\boldsymbol{b})\big) = \mu. \tag{7}$$

Let us denote the space of continuous curves by

$$\Upsilon := \Big\{(t_1,t_2,\gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times C(\mathbb{R}^+,\mathbb{R}^d),\, t_1 < t_2\Big\}$$

and let us tacitly identify the triplet $(t^-_\gamma, t^+_\gamma, \gamma) \in \Upsilon$ with $\gamma$, so that we will simply write $\gamma \in \Gamma$. We say that a finite, non negative measure $\eta$ over the set $\Upsilon$ is a *Lagrangian representation* of the vector field $\rho(1,\boldsymbol{b})$ if the following conditions hold:

(1) $\eta$ is concentrated on the set of characteristics $\Gamma$, defined as

$$\Gamma := \{(t_1,t_2,\gamma) \in \Upsilon : \gamma \text{ characteristic of } \boldsymbol{b} \text{ in } (t_1,t_2)\};$$

we explicitly recall that a curve $\gamma$ is said to be a characteristic of the vector field $\boldsymbol{b}$ in the interval $I_\gamma$ if it is an absolutely continuous solutions to the ODE

$$\dot\gamma(t) = \boldsymbol{b}(t,\gamma(t)),$$

in $I_\gamma$, which means that for every $(s,t) \subset I_\gamma$ we have

$$\int_\Gamma \left| \gamma(t) - \gamma(s) - \int_s^t \boldsymbol{b}(\tau,\gamma(\tau))\, d\tau \right| \eta(d\gamma) = 0.$$

(2) The solution $\rho$ can be seen as a superposition of the curves selected by $\eta$, i.e. if $(\mathbb{I},\gamma)\colon I_\gamma \to I_\gamma \times \mathbb{R}^d$ denotes the map defined by $t \mapsto (t,\gamma(t))$, we ask that

$$\rho \mathscr{L}^{d+1} = \int_\Gamma (\mathbb{I},\gamma)_\sharp \mathscr{L}^1 \, \eta(d\gamma);$$

(3) we can decompose $\mu$, the divergence of $\rho(1,\boldsymbol{b})$, as a local superposition of Dirac masses without cancellation, i.e.

$$\mu = \int_\Gamma \Big[\delta_{t_\gamma^-,\gamma(t_\gamma^-)}(dt\,dx) - \delta_{t_\gamma^+,\gamma(t_\gamma^+)}(dt\,dx)\Big]\, \eta(d\gamma),$$
$$|\mu| = \int_\Gamma \Big[\delta_{t_\gamma^-,\gamma(t_\gamma^-)}(dt\,dx) + \delta_{t_\gamma^+,\gamma(t_\gamma^+)}(dt\,dx)\Big]\, \eta(d\gamma).$$

The existence of such a decomposition into curves is a consequence of general structural results of 1-dimensional normal currents (see [Smi94] and, for the case $\mu = 0$, [AC08, Thm. 12]). The non-negativity assumption on $\rho \geq 0$ (i.e. the *a-cyclicity* of $\rho(1,\boldsymbol{b})$ in the language of currents) plays here a role, allowing to reparametrize the curves in such a way they become characteristic of $\boldsymbol{b}$, i.e. they satisfy Point (1).

**Restriction of Lagrangian representations and proper sets.** One problem we face immediately lies in the fact that $\eta$ is a *global* object, thus it is not immediate to relate suitable *local estimates* with $\eta$: in other words, in general, $\eta$ cannot be restricted to a set, without losing the property of being a Lagrangian representation. If we are given an open set $\Omega \subset \mathbb{R}^{d+1}$ and a curve $\gamma$, we can write

$$\gamma^{-1}(\Omega) = \bigcup_{i=1}^{\infty} (t_\gamma^{i,-}, t_\gamma^{i,+})$$

and then consider the family of curves

$$\mathtt{R}_\Omega^i \gamma := \gamma \llcorner_{(t_\gamma^{i,-}, t_\gamma^{i,+})}.$$

We can now define

$$\eta_\Omega := \sum_{i=1}^{\infty} (\mathtt{R}_\Omega^i)_\sharp \eta. \tag{8}$$

In general, the series in (8) does not converge. Moreover, even if the quantity in (8) is well defined as a measure, since $\eta$ satisfies Points (1) and (2) of the definition of Lagrangian representation 3.6, it certainly holds

$$\rho(1,\boldsymbol{b})\, \mathscr{L}^{d+1} \llcorner_\Omega = \int_\Gamma (\mathbb{I},\gamma)_\sharp \big((1,\dot\gamma)\mathscr{L}^1\big)\, \eta_\Omega(d\gamma).$$

but, in general, Point (3) is not satisfied by $\eta_\Omega$ (more precisely the second formula): in other words, $\eta_\Omega$ might not be a Lagrangian representation of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}\llcorner_\Omega$: the key point is that the sets of $\gamma$ which are exiting from or entering in $\Omega$ are not disjoint.

Thus the first question we have to answer to is to characterize the open sets $\Omega \subset \mathbb{R}^{d+1}$ for which $\eta_\Omega$ is a Lagrangian representation of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}\llcorner_\Omega$. It turns out that there are sufficiently many open sets $\Omega$ with this property: apart from having a piecewise $C^1$-regular boundary and assuming that $\mathscr{H}^d\llcorner_{\partial\Omega}$-a.e. point is a Lebesgue point for $\rho(1,\boldsymbol{b})$, the fundamental fact is that there are two Lipschitz functions $\phi^{\delta,\pm}$ such that

$$\mathbb{1}_\Omega \leq \phi^{\delta,+} \leq \mathbb{1}_{\Omega + B_\delta^{d+1}(0)}, \qquad \mathbb{1}_{\mathbb{R}^{d+1}\setminus\Omega} \leq \phi^{\delta,-} \leq \mathbb{1}_{\mathbb{R}^{d+1}\setminus\Omega + B_\delta^{d+1}(0)}$$

and

$$\lim_{\delta\to 0} \rho|(1,\boldsymbol{b})\cdot\nabla\phi^{\delta,\pm}|\, \mathscr{L}^{d+1} = \rho|(1,\boldsymbol{b})\cdot\mathbf{n}|\, \mathscr{H}^d\llcorner_{\partial\Omega} \qquad \text{in the sense of measures on } \mathbb{R}^{d+1},$$

which essentially mean that $\rho(1,\boldsymbol{b})\mathscr{H}^d\llcorner_{\partial\Omega}$ is measuring the flux of $\rho(1,\boldsymbol{b})$ across $\partial\Omega$. We call these set $\rho(1,\boldsymbol{b})$-*proper* (or just *proper* for shortness) and we study carefully their properties: we show that there are sufficiently many proper sets and that they can be perturbed in order to adapt to the vector field under study.

**Cylinders of approximate flow.** Once we are able to localize the problem in a proper set, we can start studying which are the pieces of information on the local behavior of the vector field that one needs in order to deduce global uniqueness results. To begin with, we consider again the case of the jump part of $\boldsymbol{b}$ in the $L^\infty \cap \mathrm{BV}$ (or $L^\infty \cap \mathrm{BD}$) case: in this framework, in [**ACM05**, Thm. 6.2] it has been proved the existence of a *strong* trace for $\rho$ over the jump set of $\boldsymbol{b}$ by taking suitable cylinders, so that on both sides of the discontinuity the later flux becomes negligible w.r.t. their base (see Figure 2a). By *strong trace* we mean that the trace operator, defined in the Anzellotti's distributional sense, agrees with the (approximate) pointwise limits defined with integral averages on balls. One could be tempted at this point to reproduce the proof in the tangential points: ignoring the fact that we do not have a suitable notion of (strong) trace on these Cantor sets, the main difference lies in the fact that, since the vector field is not transversal to the measure theoretic normal of the set, the cylinders should be much more elongated (see Figure 2b).

Thus we have to look for a different approach. Given a proper set $\Omega \subset \mathbb{R}^{d+1}$, we assume we can construct locally *cylinders of approximate flow* as follows:

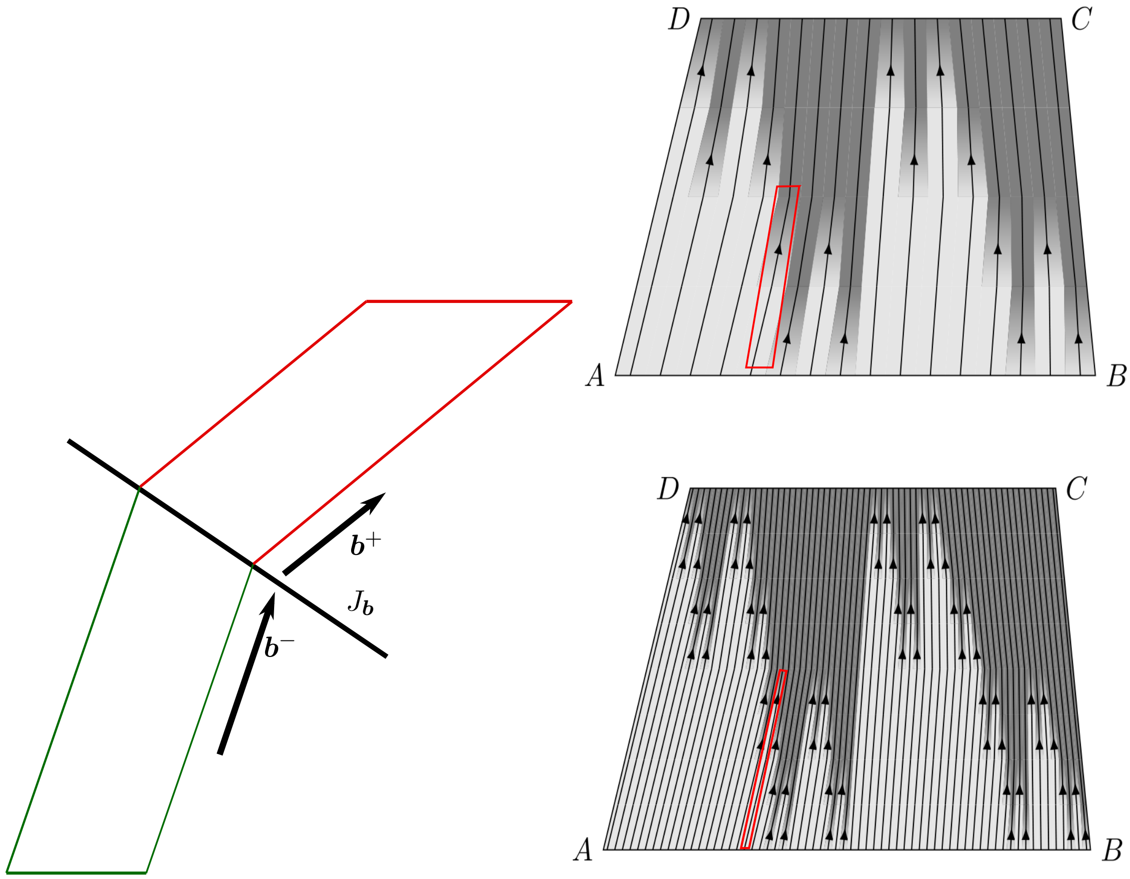
Assumption 1. There are constants $\mathtt{M}, \varpi > 0$ and a family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ such that:

(1) for every $\gamma \in \Gamma, \ell \in \mathbb{R}^+$, the function $\phi^\ell_\gamma \colon [t^-_\gamma, t^+_\gamma] \times \mathbb{R}^d \to [0,1]$ is Lipschitz, so that it can be used as a test function;

(2) the *shrinking ratio* of the cylinder $\phi^\ell_\gamma$ is controlled in time, preventing it collapses to a point: more precisely, for $t \in [t^-_\gamma, t^+_\gamma]$ and $x \in \mathbb{R}^d$,
$$\mathbb{1}_{\gamma(t)+B^d_{\ell/\mathtt{M}}(0)}(x) \le \phi^\ell_\gamma(t,x) \le \mathbb{1}_{\gamma(t)+B^d_{\mathtt{M}\ell}(0)}(x);$$

(3) we control in a quantitative way the flux through the "lateral boundary of the cylinder" (compared to the total amount of curves starting from the "base of the cylinder") with the quantity $\varpi$: more precisely, denoting by
$$\begin{aligned}\mathsf{Flux}^\ell(\gamma) :=\ & \begin{array}{c}\text{flux of the the vector field } \rho(1,\boldsymbol{b}) \\ \text{across the "boundary of the cylinder" } \phi^\ell_\gamma\end{array} \\ =\ & \iint_{(t^-_\gamma,t^+_\gamma)\times\mathbb{R}^d} \rho(t,x)\big|(1,\boldsymbol{b})\cdot\nabla\phi^\ell_\gamma(t,x)\big|\,\mathscr{L}^{d+1}(dx\,dt),\end{aligned}$$
$$\sigma^\ell(\gamma) := \text{amount of curves starting from the base of the cylinder } \phi^\ell_\gamma$$
and
$$\eta^{\mathrm{in}}_\Omega := \eta_\Omega \llcorner \{\text{curves entering in } \Omega\}$$
we ask that
$$\int_\Gamma \frac{1}{\sigma^\ell(\gamma)} \mathsf{Flux}^\ell(\gamma)\,\eta^{\mathrm{in}}_\Omega(d\gamma) \le \varpi. \tag{9}$$

We decided to call *cylinders of approximate flow* the family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$: indeed, if $\gamma$ is a characteristic of the vector field $\boldsymbol{b}$, the function $\phi^\ell_\gamma$ can be thought as generalized, smoothed cylinder centered at $\gamma$. Notice that the measure $\eta^{\mathrm{in}}_\Omega$ makes sense if $\Omega$ is a proper set, in view of the above analysis. Thus the ultimate meaning of the assumption is that one controls the ratio between the flux of $\rho(1,\boldsymbol{b})$ across the lateral boundary of the cylinders and the total amount of curves entering through its base in a uniform way (w.r.t. $\ell$), as the cylinder shrinks to a trajectory $\gamma$. A completely similar computation can be performed backward in time, by considering $\eta_\Omega$ restricted to the exiting trajectories and adopting suitable modifications.




**(a)** Visual proof of Thm. 6.2 in [**ACM05**]: the trace of $\rho$ on the jump set $J_{\boldsymbol{b}}$ is strong, as a consequence of the fact that the flux of $\boldsymbol{b}$ through the lateral boundaries of the cylinders is asymptotically negligible w.r.t. the base of the cylinders.

**(b)** Back to the example of [**BG16**]: in the tangential points, the cylinders should be much more elongated than in the jump case, as the vector field is not transversal.

**Figure 2.** Strong traces via cylinders: the jump case and the Cantor case.

**Passing to the limit via transport plans.** At this point, one would like to determine what the cylinder estimate (9) yields in the limit $\ell \to 0$. In order to perform this passage to the limit, we borrow some tools from the Optimal Transportation Theory. The language of transference plans is particularly suited for our purposes: we define

$$\Gamma^{\mathrm{cr}}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t_\gamma^\pm) \in \partial\Omega\big\}, \qquad \Gamma^{\mathrm{in}}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t_\gamma^-) \in \partial\Omega\big\}$$

and we consider plans between $\eta_\Omega^{\mathrm{cr}} := \eta_{\Omega \llcorner \Gamma^{\mathrm{cr}}(\Omega)}$ and the entering trajectory measure $\eta_\Omega^{\mathrm{in}}$. Notice that $\eta_\Omega^{\mathrm{cr}}$ is concentrated, by definition, on the set of trajectories entering in and exiting from $\Omega$ (*crossing* trajectories). In the correct estimate one has to take into account also of trajectories which end inside the set $\Omega$ and this, in view of Point 3 of the definition of Lagrangian representation, is estimated by the negative part $\mu^-$ of the divergence $\mu$, defined in (7). Thus one obtains the following

Proposition 1. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and $\eta$ be a Lagrangian representation of $\rho(1, \boldsymbol{b})$. If Assumption 1 holds then there exist $N_1 \subset \Gamma^{\mathrm{cr}}(\Omega), N_2 \subset \Gamma^{\mathrm{in}}(\Omega)$ such that*

$$\eta_\Omega^{\mathrm{cr}}(N_1) + \eta_\Omega^{\mathrm{in}}(N_2) \leq \inf_{C>1} \left\{ 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1} \right\}$$

*and for every* $(\gamma,\gamma')\in(\mathit{\Gamma}^{\mathrm{cr}}\setminus N_1)\times(\mathit{\Gamma}^{\mathrm{in}}\setminus N_2)$

$$\textit{either }\ \operatorname{clos}\operatorname{Graph}\gamma'\subset\operatorname{clos}\operatorname{Graph}\gamma\ \textit{ or }\ \operatorname{clos}\operatorname{Graph}\gamma,\operatorname{clos}\operatorname{Graph}\gamma'\ \textit{are disjoint.}\qquad(\star)$$

Proposition 1 gives essentially a uniqueness result (from the Lagrangian point of view) at a *local* level, namely inside a proper set $\Omega$: it says that, under Assumption 1, up to removing a set of trajectories whose measure is controlled, one gets a family of essentially disjoint trajectories (meaning that are either disjoint or one contained in the other).

***Untangling* of trajectories.** It seems at this point natural to try to perform some "local-to-global" argument, seeking a global analog of Proposition 1. In order to do this, we introduce the following *untangling functional for* $\eta^{\mathrm{in}}$, defined on the class of proper sets as

$$\mathfrak{f}^{\mathrm{in}}(\Omega):=\inf\Big\{\eta^{\mathrm{cr}}_{\Omega}(N_1)+\eta^{\mathrm{in}}_{\Omega}(N_2):\forall(\gamma,\gamma')\in(\mathit{\Gamma}\setminus N_1)\times(\mathit{\Gamma}\setminus N_2)\text{ the condition }(\star)\text{ holds}\Big\}$$

and, in a similar fashion, one can define an untangling functional for the trajectories that are exiting from the domain $\Omega$. In a sense, these functionals are measuring the minimum amount of curves one has to remove so that the remaning ones are essentially disjoint, i.e. they satisfy condition ($\star$). The main property of these functionals is that they are subadditive with respect to the domain $\Omega$, meaning that

$$\mathfrak{f}^{\mathrm{in}}(\Omega)\le\mathfrak{f}^{\mathrm{in}}(\mathrm{U})+\mathfrak{f}^{\mathrm{in}}(\mathrm{V}),$$

whenever $\mathrm{U},\mathrm{V}\subset\mathbb{R}^{d+1}$ are proper sets whose union $\Omega:=\mathrm{U}\cup\mathrm{V}$ is proper. The subadditivity suggests the possibility of having a local control in terms of a measure $\varpi^\tau$, whose mass is $\tau>0$, replacing the constant $\varpi$ in Proposition 1 with $\varpi^\tau(\Omega)$. In view of Proposition 1 one has to combine $\varpi^\tau$ with the divergence and this can be done by introducing a suitable measure $\zeta^\tau_C\approx C\varpi^\tau+\frac{|\mu|}{C}$ on $\mathbb{R}^{d+1}$. If Assumption 1 is satisfied locally by a suitable family of balls, then one can show, by means of a non-trivial covering argument, the following fundamental proposition, which is the global analog of Proposition 1.

PROPOSITION 2. *There exists a set of trajectories* $N\subset\mathit{\Gamma}$ *such that*

$$\eta(N)\le C_d\zeta^\tau_C(\mathbb{R}^{d+1})$$

*and for every* $(\gamma,\gamma')\in(\mathit{\Gamma}\setminus N)^2$ *it holds*

$$\begin{aligned}&\textit{either }\ \operatorname{Graph}\gamma\subset\operatorname{Graph}\gamma'\ \textit{ or }\ \operatorname{Graph}\gamma'\subset\operatorname{Graph}\gamma\\&\textit{or }\ \operatorname{Graph}\gamma,\operatorname{Graph}\gamma'\ \textit{are disjoint (up to the end points).}\end{aligned}\qquad(\star\star)$$

The interesting situation is when the measure $\zeta^C_\tau$ can be taken arbitrarily small, i.e. when $\tau\to0$: in that case $\eta$ is said to be *untangled*, i.e. it is concentrated on a set $\mathit{\Delta}$ such that for every $(\gamma,\gamma')\in\mathit{\Delta}\times\mathit{\Delta}$ the condition ($\star\star$) holds.

**Partition via characteristics and Lagrangian uniqueness.** The *untangling* of trajectories is the core of our approach and it encodes, in our language, the uniqueness issues and the computation of the chain rule. Indeed, once the untangled set $\mathit{\Delta}$ is selected, we can construct an equivalence relation on it, identifying trajectories whenever they coincide in some time interval: this gives a partition of $\mathit{\Delta}$ into equivalence classes $E_{\mathfrak{a}}:=\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}}$, being $\mathfrak{A}$ a suitable set of indexes. This, in turn, induces a partition of $\mathbb{R}^{d+1}$ (up to a set $\rho\mathscr{L}^{d+1}$-negligible) into disjoint trajectories (that we still denote by $\wp_{\mathfrak{a}}$): both partitions admit a Borel section (i.e. there exist Borel functions $\mathtt{f}\colon\mathbb{R}^{d+1}\to\mathfrak{A}$ and $\widehat{\mathtt{f}}\colon\mathit{\Delta}\to\mathfrak{A}$ such that $\wp_{\mathfrak{a}}=\mathtt{f}^{-1}(\mathfrak{a})$ and $\widehat{\mathtt{f}}^{-1}(\mathfrak{a})=E_{\mathfrak{a}}$ for every $\mathfrak{a}\in\mathfrak{A}$): hence a disintegration approach can be adopted, like in the two-dimensional setting. One reduces the PDE (7) into a family of one-dimensional ODEs along the trajectories $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$: we are thus recovering a sort of method of the characteristic in the weak setting.

To formalize this disintegration issue, we propose to call a Borel map $\mathtt{g}\colon\mathbb{R}^{d+1}\to\mathfrak{A}$ a *partition via characteristics* of the vector field $\rho(1,\boldsymbol{b})$ if:

- for every $\mathfrak{a} \in \mathfrak{A}$, $\mathtt{g}^{-1}(\mathfrak{a})$ coincides with $\operatorname{Graph}\gamma_{\mathfrak{a}}$, where $\gamma_{\mathfrak{a}}\colon I_{\mathfrak{a}} \to \mathbb{R}^{d+1}$ is a characteristic of $\boldsymbol{b}$ in some open domain $I_{\mathfrak{a}} \subset \mathbb{R}$;
- if $\widehat{\mathtt{g}}$ denotes the corresponding map $\widehat{\mathtt{g}}\colon \varDelta \to \mathfrak{A}$, $\widehat{\mathtt{g}}(\gamma) := \mathtt{g}(\operatorname{Graph}\gamma)$, setting $m := \widehat{\mathtt{g}}_\sharp \eta$ and letting $w_{\mathfrak{a}}$ be the disintegration
$$\rho \mathscr{L}^{d+1} = \int_{\mathfrak{A}} (\mathbb{I}, \gamma_{\mathfrak{a}})_\sharp (w_{\mathfrak{a}} \mathscr{L}^1)\, m(d\mathfrak{a})$$
then
$$\frac{d}{dt} w_{\mathfrak{a}} = \mu_{\mathfrak{a}} \in \mathscr{M}(\mathbb{R}), \tag{10}$$
where $w_{\mathfrak{a}}$ is considered extended to 0 outside the domain of $\gamma_{\mathfrak{a}}$;
- it holds
$$\mu = \int (\mathbb{I}, \gamma_{\mathfrak{a}})_\sharp \mu_{\mathfrak{a}}\, m(d\mathfrak{a}) \qquad \text{and} \qquad |\mu| = \int (\mathbb{I}, \gamma_{\mathfrak{a}})_\sharp |\mu_{\mathfrak{a}}|\, m(d\mathfrak{a}).$$

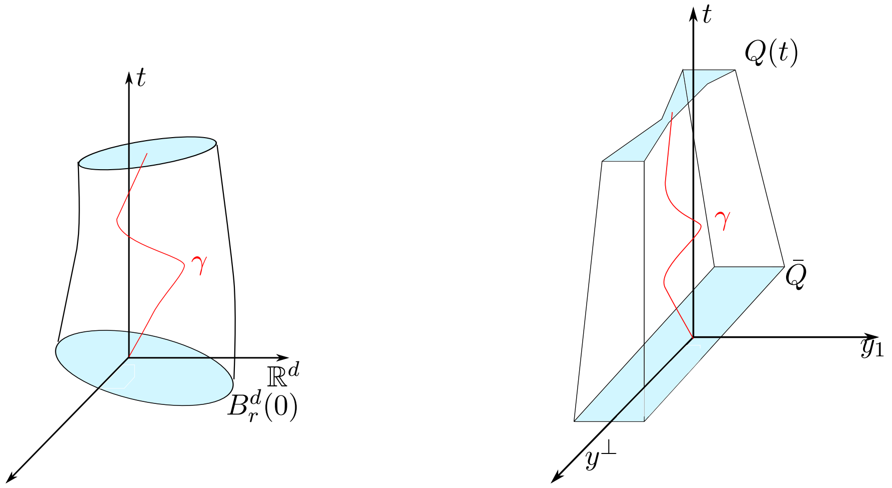
We will say the partition is *minimal* if moreover
$$\lim_{t\to\bar t\pm} w_{\mathfrak{a}}(t) > 0 \qquad \forall \bar t \in I_{\mathfrak{a}}.$$

In view of the discussion above, the family of equivalence classes $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$ arising from the untangled set $\varDelta$ constitutes a partition via characteristics. Since the function $w_{\mathfrak{a}}$ is a BV function on $\mathbb{R}$, in view of (10), we can further split the equivalence classes so that it becomes a minimal partition via characteristics of $\rho(1, \boldsymbol{b})$. Furthermore, if we take $u \in L^\infty((0,T) \times \mathbb{R}^d)$ such that $\operatorname{div}(u\rho(1, \boldsymbol{b})) = \mu'$ is a measure, we can repeat the computations for the vector field $(2\|u\|_\infty + u)\rho(1, \boldsymbol{b})$ obtaining that the same partition via characteristics works also for $u\rho(1, \boldsymbol{b})$. This yields the following uniqueness result, which is the core of the thesis:

Theorem B ([BB17b]). *If $\eta$ is untangled, then there exists a minimal partition via characteristics $\mathtt{f}$ of $\rho(1, \boldsymbol{b})$. Furthermore, if $u \in L^\infty((0,T) \times \mathbb{R}^d)$ is a solution to $\operatorname{div}(u\rho(1, \boldsymbol{b})) = \mu'$, then map $\mathtt{f}$ is a partition via characteristics of $u\rho(1, \boldsymbol{b})$ as well.*

In particular, by disintegrating the PDE $\operatorname{div}(u\rho(1, \boldsymbol{b})) = \mu'$ along the characteristics $\wp_{\mathfrak{a}} = \mathtt{f}^{-1}(\mathfrak{a})$, we obtain the one-dimensional equation
$$\frac{d}{dt}\Big(u\big(t, \wp_{\mathfrak{a}}(t)\big) w_{\mathfrak{a}}(t)\Big) = \mu'_{\mathfrak{a}}.$$
At this point, an application of Volpert's formula for one-dimensional BV functions allows an explicit computation of $\frac{d}{dt}(\beta(u \circ \wp_a) w_{\mathfrak{a}})$, i.e. of $\operatorname{div}(\beta(u)\rho(1, \boldsymbol{b}))$ thus establishing the Chain rule in the general setting.

**The BV nearly incompressible case and Bressan's Compactness Conjecture.** To conclude the proof of the Main Theorem, establishing Bressan's Compactness Conjecture, it remains to show how we can construct cylinders of approximate flow satisfying Assumption 1, for a vector field of the form $\rho(1, \boldsymbol{b})$, with $\rho \in (C^{-1}, C)$ and $\boldsymbol{b} \in L^1((0,T); \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$. In view of Theorem B, without loss of generality, we can assume $\rho = 1$ so that the vector field under consideration is exactly $(1, \boldsymbol{b})$: as usual, we denote by $D\boldsymbol{b}$ the derivative of $\boldsymbol{b}$ and we split it into the absolutely continuous part and the singular part.
In a Lebesgue point $(\bar t, \bar x)$ of the absolutely continuous part, the construction of the cylinders is rather easy: essentially, one replaces the real evolution under the flow of $\boldsymbol{b}$ of a ball $B^d_\ell(0)$ with an ellipsoid, obtained by letting everything evolve under a fixed matrix $A$ (compare with Figure 3a). Some standard computations show that the difference between the two evolutions can be made arbitrarily small, when compared to the volume of $B^d_\ell(0)$, by taking $A$ to be the Lebesgue value of $D\boldsymbol{b}$ in the point $(\bar t, \bar x)$.




**(a)** In the absolutely continuous part of $D\boldsymbol{b}$ the cylinders evolve under a constant matrix $A$, which will be taken close to $D^a\boldsymbol{b}$.

**(b)** The singular case: the cylinders shrink (if $\operatorname{div}\boldsymbol{b} < 0$) in a controlled way, their sides being graph of monotone Lipschitz functions which solve suitable differential equations.

**Figure 3.** Approximate cylinders of flow in the BV (nearly incompressible) case.

The estimates for the singular part are more delicate and depend heavily on the shape of the approximate cylinders of flow. Here the geometric structure of BV functions (Alberti's Rank-One Theorem) plays a role, as in the original proof of [Amb04]. The main idea is to choose properly the (non-transversal) sides' lenghts of the cylinders, in such a way to cancel the effect of the divergence. Indeed, by Rank One Theorem, we can find a suitable (local) coordinate system $\boldsymbol{y} = (y_1, y^\perp) \in \mathbb{R}^d$ in which the derivative $D\boldsymbol{b}$ is essentially directed toward a fixed direction (without loss of generality, the one given by $\boldsymbol{e}_1$). Accordingly, we define the (section at time $t$ of the) cylinder

$$Q = Q_{\ell^\pm_{1,\gamma},\ell}(t) := \gamma(t) + \Big\{\boldsymbol{y} = (y_1, y^\perp) : -\ell_1^-(t,y^\perp) \le y_1 \le \ell_1^+(t,y^\perp),\ |y^\perp| \le \ell\Big\}, \tag{11}$$

where $\ell > 0$ is a real number and $\ell^\pm_{1,\gamma}$ are suitable functions to be chosen, Lipschitz in $y^\perp$ and monotone in $t$. This is indeed a crucial step: we show it is possible to adapt locally the cylinders of approximate flows, by imposing that the sides' lengths $\ell^\pm_{1,\gamma}(t)$ are monotone functions satisfying suitable differential equations (see Figure 3b). In a simplified setting, i.e. if the level set of $b_1(t)$ were exactly of the form $y_1 =$ constant, then we would impose

$$\frac{d}{dt}\ell^+_{1,\gamma}(t) = b_1(t,\gamma(t)) + (Db_1)\big(\gamma(t), \gamma(t) + \ell^+_{1,\gamma}(t)\big)\ell^+_{1,\gamma}(t) \tag{12}$$

(and an analogous relation for $\ell^-_{1,\gamma}$). Plugging the solution of (12) into the definition of the cylinder (11), we can show that the flux of $\boldsymbol{b}$ through the lateral boundary of $Q$ is under control. Actually, a technical variation of this is needed in order to take into account the fact that the level sets are not of the form $y_1 =$ constant: to do this we exploit Coarea Formula and a classical decomposition of finite perimeter sets into rectifiable parts (relying ultimately on De Giorgi's Rectifiability Theorem). We show that, up to a $|D^s\boldsymbol{b}|$-small set, one can find Lipschitz functions $y_1 = L_{t,h}(y^\perp)$ in a fixed set of coordinates $(y_1, y^\perp) \in \mathbb{R} \times \mathbb{R}^{d+1}$, whose graphs cover a large fraction of the singular part $D^s\boldsymbol{b} \llcorner_{B_r^{d+1}(\bar{t},\bar{x})}$. We can at this point reverse the procedure, i.e. we construct a vector field starting from the level sets: this yields a BV vector field $\mathcal{U}(t)$ whose component $\mathcal{U}_1$ can be put into the right hand side of (12) and we can now perform the precise estimate of the flux of $\boldsymbol{b}$ through the lateral boundary of $Q$. By an application of the Radon-Nikodym Theorem, it follows that on large compact set it holds that the flow integral (9) is controlled by $\tau|D^{\mathrm{s}}\boldsymbol{b}|(B_r^{d+1}(\bar{t},\bar{x}))$. Finally a covering argument implies that the measure $\zeta_\tau^C$ can be taken, in the BV case, to be $\tau|D\boldsymbol{b}|$: in view of the discussion above this is enough to conclude finally the proof of the Main Theorem.

**Further developments of the *untangling*.** In a work in progress with S. Bianchini (that will appear in a forthcoming paper [BB17c]) we study some possible refinements of the concept of *untangling*. In particular, by imposing a control on the intersection of the curves only *forward in time* some estimates and propositions of the approach presented above simplify. More precisely, we define a Lagrangian representation $\eta$ of $\rho(1,\boldsymbol{b})$, with $\mathrm{div}(\rho(1,\boldsymbol{b})) = \mu \in \mathscr{M}([0,T]\times\mathbb{R}^d)$, to be *forward untangled* when it is concentrated on a set $\Delta^{\mathrm{for}}$ of curves which may intersect, but if they do then they remain the same curve in the future. In a sense, this means that trajectories can bifurcate only in the past.

This formulation arises naturally when one translates well-posedness of the ODEs in terms of Lagrangian representations: restricting for simplicity to the case in which $\mu = 0$ one would like to replace Assumption 1 with the following one:

ASSUMPTION 2. Let $\eta$ be a Lagrangian representation of $\rho(1,\boldsymbol{b})$ in $(0,T)\times\mathbb{R}^d$. Let $\varpi > 0$ and assume that for all $R > 0$ there exists $r = r(R) > 0$ such that

$$\int_\Gamma \frac{1}{\sigma^r(\gamma)}\eta\Big(\Big\{\gamma' \in \Gamma : |\gamma(0)-\gamma'(0)| \le r, |\gamma(T)-\gamma'(T)| \ge R\Big\}\Big)\eta(d\gamma) \le \varpi.$$

where now

$$\sigma^r(\gamma) := \text{amount of curves starting from the ball or radius } r > 0 \text{ around } \gamma(0).$$

Assumption 2 has the advantage of making more transparent and easier some of the proofs used in the approach presented above. One can repeat the general scheme presented above: first one formulates Assumption 2 locally, in a proper set and shows that - up to a set of curves whose measure is controlled - the (restricted) Lagrangian representation $\eta$ is forward untangled. In this way, one obtains a simpler proof of Theorem 1, avoiding the introduction of the crossing trajectories. Then one introduces the *forward untangling functional*, which turns out to be subadditive as well, exactly as in the setting above, allowing the usual "local-to-global" argument.

Using this formulation of the untangling, we are able to recover in our setting the results of [BC13], where the authors considered vector fields whose derivative can be written as convolution between a singular kernel and a $L^1$ function and we also derive a quantitative stability estimate for a class of vector fields satisfying a suitable weak $L^p$ bound on the gradient.

### A Lagrangian approach to scalar multidimensional conservation laws

The final part of the thesis is devoted to present a result obtained in collaboration with S. Bianchini and E. Marconi (in an ongoing project [BBM]), where the Lagrangian techniques discussed above are adapted and applied to the context of *scalar multi-dimensional conservation laws*. These are first order partial differential equations of the form

$$\partial_t u + \mathrm{div}_x(\boldsymbol{f}(u)) = 0 \qquad \text{for every } (t,x) \in [0,+\infty)\times\mathbb{R}^d, \tag{13}$$

where $u\colon [0,+\infty)\times\mathbb{R}^d \to \mathbb{R}$ is a scalar, unknown function and $\boldsymbol{f}\colon \mathbb{R}\to\mathbb{R}^d$ is a smooth map, called the *flux function*. In a sense, (13) is the non-linear counterpart to (1)-(2): indeed, the main difference lies in the non-linearity of the flux $\boldsymbol{f}$, which prevents equation (13) to have smooth solutions. To see this, one can use the chain rule to rewrite the equation in the form

$$\partial_t u + \boldsymbol{f}'(u)\cdot\nabla u = 0 \qquad \text{for every } (t,x) \in [0,+\infty)\times\mathbb{R}^d,$$

which means that $u$ is constant along *characteristic lines*, i.e. solutions to the trajectories of the ordinary differential equation $\dot{\gamma}(t) = \boldsymbol{f}(u(t,\gamma(t)))$ for $t \in (0,+\infty)$. Notice that, in this setting, the characteristics are straight lines, hence assuming we couple (13) with an initial condition $u(0,\cdot) = \bar{u}(\cdot)$ we obtain the implicit relation

$$u(t,x) = \bar{u}(x - t\boldsymbol{f}'(u(t,x))) \qquad \text{for every } (t,x) \in [0,+\infty) \times \mathbb{R}^d.$$

Due to the non-linearity of the flux, characteristic lines typically intersect somewhere and thus classical solutions do not exist globally in time.

Hence we are forced to consider distributional solutions, but this in turn implies a loss of uniqueness, as it is easily seen by well-known counterexamples. To restore the uniqueness we are forced to add some conditions, obtaining the notion of *entropy solution*: we ask that some non-linear functions of the solution $u$ are dissipated along the flow. More precisely, in the case of equation (13) we ask that it holds

$$\mu_\eta := \partial_t(\eta(u)) + \operatorname{div}(\boldsymbol{q}(u)) \leq 0 \qquad \text{in the sense of distributions over } (0,+\infty) \times \mathbb{R}^d, \quad (14)$$

for all convex entropy-entropy flux pairs $(\eta, \boldsymbol{q})$, where $\eta$ is a convex function and $\boldsymbol{q}$ is defined by $\boldsymbol{q}' = \eta' \boldsymbol{f}'$. Notice that, in view of the sign condition (14), the distribution $\mu_\eta$ is induced by a measure, usually called *entropy-dissipation measure*. A celebrated theorem by Kružkov [**Kru70**] ensures existence, uniqueness and stability for entropy solutions of scalar conservation laws for every initial datum $u_0 \in L^\infty(\mathbb{R}^d)$.

Kružkov theory yields also regularity result for the solution: if $\bar{u} \in \mathrm{BV}(\mathbb{R}^d)$, then also $u(t,\cdot) \in \mathrm{BV}(\mathbb{R}^d)$ for all $t > 0$. It remains thus open the question of the structure of the solution in the general case, i.e. when $\bar{u} \in L^\infty(\mathbb{R}^d)$. In [**DLOW03**] it is proved that the solution $u$ has a BV-like structure in the more general situation when $\mu_\eta$ is a (possibily signed) measure (this may be relevant in views of the connections with [**AG87, DKMO02**]: see also [**DLR03**] and the references therein). The authors show that, under suitable non-linearity assumptions on the flux, the solution $u$ is of vanishing mean oscillation, up to a $\mathscr{H}^{d-1}$-rectifiable set on which they conjecture the measure $\mu_\eta$ is concentrated.

The fine description of the entropy solution in one space dimension for a generic initial datum $\bar{u} \in L^\infty$ has been recently obtained in [**BM17**], where the authors, exploiting a Lagrangian approach, proved the desidered BV-like structure and, as a consequence, the concentration of the entropy-dissipation measure.

In [**BBM**] we introduce a suitable notion of Lagrangian representation for the multidimensional scalar equation (13). In the spirit of the aforementioned papers, our construction is based on an a-priori compactness estimate and an approximating scheme which exploits it: in this situation, we use the transport-collapse method introduced by Brenier [**Bre84**]. After showing the existence of a Lagrangian representation, we use it to prove a result on continuous solutions to (13) (see also [**Daf06**] for the one-dimensional case): we show that, in this case, the entropy-dissipation measure (14) vanishes.

Theorem C ([**BBM**]). *Let $u$ be a continuous, bounded entropy solution in $[0,T) \times \mathbb{R}^d$ to (13). Then for every $(t,x) \in [0,T) \times \mathbb{R}^d$, it holds*

$$u(t,x) = u_0\big(x - \boldsymbol{f}'(u(t,x))t\big).$$

*Moreover for every $\eta\colon \mathbb{R} \to \mathbb{R}$, $\boldsymbol{q}\colon \mathbb{R} \to \mathbb{R}^d$ Lipschitz such that $\boldsymbol{q}' = \eta' \boldsymbol{f}'$, a.e. with respect to $\mathscr{L}^1$, it holds*

$$\partial_t \eta(u) + \operatorname{div}_x \boldsymbol{q}(u) = 0$$

*in the sense of distributions. In the particular, the solution $u$ does not dissipate.*

## Structure of the thesis

The thesis is divided into three parts.

In the first part we collect some known results from the classical theory and from the approach of renormalized solutions; then we turn to consider the two-dimensional case, presenting first the proof of Theorem A and then discussing some counterexamples related to the Hamiltonian structure.

Part two is the core of the thesis and presents the proof of Bressan's Compactness Conjecture in the general $d$-dimensional case. We discuss first the localization issues related to the theory of proper sets (Chapter 6) and then we present the concept of untangling via cylinders of approximate flow and how it is related to uniqueness (Chapter 7), thus establishing Theorem B. Chapter 8 contains the computations needed to construct the cylinders in the BV case, concluding thus the proof of the Main Theorem. Finally, in Chapter 9 we present the theory of *forward untangling* and its applications.

Part three is devoted to present the content of the note [**BBM**], where a Lagrangian approach to scalar, multidimensional convervation laws is proposed. In particular, in Chapter 10 we give a suitable notion of Lagrangian representation for conservation laws and we exploit it to deduce that, for continuous solutions to conservation laws, the entropy dissipation measure vanishes.

We now describe more in details the structure of the single chapters.

An unnumbered chapter collects the basic notation used throughout the thesis. We present also a survey of the results we will need, mainly from Geometric Measure Theory and from the theory of BV functions. Two short paragraphs are devoted to the theory of weak traces for $L^\infty$-vector fields whose distributional divergence is a measure and to some tools borrowed from Optimal Transportation theory. No proof is given in this chapter, whose aim is to be merely a useful notational reference for the reader.

Chapter 1 is devoted to a brief exposition of the classical theory of flows. In Section 1.1 we collect known results of the Cauchy-Lipschitz theory for ordinary differential equations (Theorem 1.1) and we present the method of characteristics. Then we introduce the weak formulations of the partial differential equations we want to study: in Section 1.2 we clarify our notion of weak solution for the continuity equation (discussing, in particular, the regularity in time of the solution) and in Section 1.3 we introduce the weak formulation of the transport equation. We show first a quick and general result of existence of solutions (Proposition 1.13) and, to conclude the chapter, we begin investigating the problem of uniqueness of solutions to the transport equation in the non-smooth setting: we present the notions of *renormalized solution* and of *renormalization property* and show their links with the well-posedness problem for the PDE.

In Chapter 2, we present an approach to obtain the renormalization property based on *commutators estimate*, Section 2.1. We then illustrate two important results along these lines, which go back respectively to DiPerna-Lions [**DL89c**] and Ambrosio [**Amb04**] and are studied in Section 2.2 and Section 2.3 respectively. Concerning Section 2.3, we do not give a detailed account of the proof due to Ambrosio, but rather a slightly different argument, which however builds on the very same ingredients of the original proof.

Chapter 3 contains the definition of Lagrangian representation, which is a central tool in the thesis. We begin, in Section 3.1, by presenting a well-known result, due to Ambrosio, usually known as *Superposition Principle*. We give a proof of this theorem, building on a decomposition result formulated in language of 1-dimensional normal currents by Smirnov [**Smi94**]: this leads us to introduce the notion of Lagrangian representation, see Definition

3.6. In Section 3.2 we present in a rigorous way how Lagrangian representations can be used as a bridge between Lagrangian and Eulerian points of view, transferring well-posedness results in terms of the PDEs into corresponding statements for the Regular Lagrangian Flow of the ODE. In particular, we present an example of how they can be used to deduce in a quick way well-posedness results for vector fields having special structure (recovering, in particular, results going back to [**LBL04**] and [**Ler04**]). In the final part of the chapter, in Section 3.3 we introduce the class of nearly incompressible vector fields and present the statement of Bressan's Compacntess Conjecture.

In Chapter 4 and Chapter 5 we restrict our attention to the two dimensional case. We start by discussing, in Section 4.1, how the equation can be disintegrated onto the level sets of a suitable Lipschitz Hamiltonian and we explain the relevance, within this context, of the *Weak Sard Property*. Roughly speaking, if $H\colon \mathbb{R}^2 \to \mathbb{R}$ is a compactly supported, Lipschitz map, this amounts to ask that

$$H_\sharp \left(\mathscr{L}^2 \llcorner_S\right) \perp \mathscr{L}^1, \tag{15}$$

where $S$ is the set of critical points of $H$, i.e. $S := \{\nabla H = 0\}$: it turns out that this is a necessary and sufficient condition for uniqueness of weak solutions to the transport equation driven by $\boldsymbol{b} = \nabla^\perp H$. The rest of Chapter 4 is entirely devoted to the proof of Theorem A, which is based on [**ABC13, ABC14, BG16**]: we summarize here the main points of the argument, for the reader's convenience.
To begin, let us observe that, given $\boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of bounded variation and nearly incompressible case, it is not possible to construct the Hamiltonian $H\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla^\perp H = \boldsymbol{b}$, directly as in the case divergence-free case. However, locally, we can reduce the problem to the steady case using a Lagrangian representation $\eta \in \mathscr{M}_+(C([0,T];\mathbb{R}^d))$, whose existence is granted by the nearly incompressibility assumption and by the discussion of Chapter 3. Using the measure $\eta$, in Section 4.2, we construct a suitable partition of the plane and we reduce our problem *locally* (inside a ball $B$) to the case when the density is steady, which has been studied in [**BG16**]. Thus, inside a ball $B$, we can construct a local Hamiltonian $H_B$ and then we show how we can split an equation of the form

$$\operatorname{div}(u\boldsymbol{b}) = \mu, \qquad u\colon \mathbb{R}^2 \to \mathbb{R} \tag{16}$$

where $\mu$ is a measure on $\mathbb{R}^2$, into an equivalent family of equations along the level sets of $H_B$ (see Subsection 4.2.2). In Section 4.3 we establish the Weak Sard Property for the local Hamiltonians $H_B$ and then we turn to study in detail the relationship between level sets of the local Hamiltonian $H_B$ and the trajectories of the vector field $\boldsymbol{b}$: in Section 4.4, we present some lemmata which show that (up to a $\eta$ negligible set) all non constant integral curves of $\boldsymbol{b}$ are contained in "good" level sets of $H_B$.
In Section 4.5 we prove that the divergence operator is *local*, in the sense that the measure $\mu$ in (16) vanishes on the set $M := \{\boldsymbol{b} = 0\}$ (Proposition 4.28). We stress that this result is true for every space dimension and it is crucial to obtain a better description of the link between the level sets and the trajectories. This is achieved in Section 4.6, where in particular, we prove that "good" level sets of $H$ cover almost all the set $M^c = \{\boldsymbol{b} \neq 0\}$. Finally, in Section 4.7 we first show how the time-dependent problem

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2). \tag{17}$$

can be reduced to a family of one-dimensional problems on level sets of the Hamiltonians, which can be solved explicitly. This allows to construct a $\eta$-negligible set $R$ of trajectories with the following property, which is reminiscent of the standard method of characteristics: if $u$ is a solution of 17, then for all $\gamma \notin R$ the function $t \mapsto u(t,\gamma(t))$ is constant. This crucial result (Lemma 4.44) combined with an elementary observation (Lemma 4.45) concludes in Section 4.7.3 the proof of the Theorem A.

Chapter 5 is devoted to the presentation of some counterexamples, still in the two-dimensional setting. We begin in Section 5.1 discussing more in details the Weak Sard Property; in particular, we present an example, borrowed from [**ABC13**] of a Lipschitz function which does not satisfy (15). Then we consider the Chain Rule problem, presenting in particular the results obtained in [**ADLM07**]. This leads, in Section 5.2, to the introduction of the so called *tangential set* of a BV vector field: we discuss its role in the framework of the chain rule, in view of the results of [**ADLM07**]. In Section 5.3 we propose a two-dimensional counterexample, which answers in the negative to a question raised in [**ACM05**] about the size of the tangential set. The construction is inspired by the work [**BG16**]. The final part of Chapter 5 contains a variation on the theme of the Chain Rule, which is related to the recent paper [**CGSW17**]: using the Lipschitz function constructed in Section 5.1, we give an example of an autonomous vector field $\boldsymbol{b} \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and a bounded function $u\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{div} \boldsymbol{b} = 0$, $\partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0$ in the sense of distributions but the *renormalization defect* $\partial_t(u^2) + \operatorname{div}(u^2\boldsymbol{b})$ is not representable by a measure (let alone it is zero).

The second part of the thesis begins with Chapter 6, where we study localization issues and the theory of *proper sets*. In Section 6.1 we explain their definition (see Definition 6.1) and we study their main properties: there are sufficiently many sets which have a simple geometry and are proper (Lemma 6.7). Furthermore, we show how one can construct perturbations $\Omega^\varepsilon$ of proper sets $\Omega$, which are still proper, arbitrarily close to the original $\Omega$ and such that the entering/exiting fluxes mainly occur across finitely many time-constant planes, Theorem 6.15.
Section 6.2 is devoted to the study of the behaviour, at a Lagrangian level, of the restriction operation $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1} \mapsto \rho(1,\boldsymbol{b})\mathscr{L}^{d+1} \llcorner \Omega$. The first result we propose is Lemma 6.20, where we give a representation of the distributional trace as a countable sum of measures $(\mathtt{T}_\Omega^{i,\pm})_\sharp \eta$. As we already mentioned in the introduction, it may happen that the restriction operator induced by cutting the curves inside the domain $\Omega$ does not generate a Lagrangian representation of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1} \llcorner \Omega$: Example 6.21 shows a situation in which this occurs. However, as soon as one has some regularity of the vector field (e.g. $\mathrm{BV} - \mathrm{BD}$), one can obtain an absolutely convergent sequence of measures representing the trace for every Lipschitz set: this is proved in Proposition 6.22, relying on the chain rule for traces of [**ACM05**]. In this case, one can find two disjoint subsets $A^\pm$ of $\partial\Omega$ such that $(\mathtt{T}_\Omega^{i,\pm})_\sharp \eta$ is concentrated on $A^\pm$.
The same can be done (without regularity assumptions on the vector field) for proper sets (and not for a generic Lipschitz set), and it is studied in Section 6.3. The key fact, used several times in the section, is that the trace controls the flux of trajectories across $\partial\Omega$: using the perturbations $\Omega^\varepsilon$ of Section 6.1 one can finally show that, if $\Omega$ is proper, the measure $\eta_\Omega$ is a Lagrangian representation of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1} \llcorner \Omega$, Theorem 6.30.

The starting point of Chapter 7 is to give a set of assumptions in a proper set $\Omega$ which implies uniqueness up to a set of trajectories whose $\eta$-measure is controlled: this is contained in Assumption 7.1. We derive the uniqueness result in two steps: in the first, Proposition 7.2, we control the amount of trajectories starting from the same point of the boundary and subsequently bifurcating. In Proposition 7.3 instead we use transference plans to control the amount of trajectories starting from two different points of $\partial\Omega$ and intersecting at a later time. The final result is Theorem 7.14, which follows from the two steps above together with the deep duality results of [**Kel84**] (recalled in the Preliminaries): after removing a set of trajectories whose $\eta$-measure is explicitly controlled, the remaining curves are either disjoint or one a subset of the other. For convenience, the analysis is performed first on perturbation of propers sets, and then passed to the limit as shown in Theorem 7.15.
The next section, Section 7.2, addresses the problem of passing the local results obtained in Section 7.1 to global ones. From the estimates of Theorem 7.15, it is natural to introduce

the untangling functionals $\mathfrak{f}^{\rm in}$ and $\mathfrak{f}^{\rm out}$, Definitions 7.16 and 7.17. Their main property is subadditivity w.r.t. the domain $\Omega$, as explained in Proposition 7.18, and this makes natural the comparison of the untangling functionals with a measure: this is exactly Assumption 7.20 and a covering argument yields Theorem 7.24, which is the global version of Theorem 7.15. In the case the comparison measure can be made arbitrarily small (which will be the case if Assumption 7.27 is satisfied) then Corollary 7.26 shows that the Lagrangian representation $\eta$ is *untangled*, in the sense of Definition 7.25.
Finally, in the last part of the chapter, we show that if $\eta$ is untangled then there exists a partition of $\mathbb{R}^{d+1}$ into trajectories of $\boldsymbol{b}$ (Proposition 7.29) such that the PDE $\operatorname{div}(\rho(1,\boldsymbol{b})) = \mu$ can be decomposed into a family of ODEs on the characteristics (Proposition 7.31). The final result is the existence of a partition via characteristics for $\rho(1,\boldsymbol{b})$, according to Theorem 7.33, and its uniqueness in the class $\rho' \in L^\infty(\rho)$, Theorem 7.34. This allows the explicit computation of the chain rule, performed in Proposition 7.36.

Chapter 8 contains the relevant case for Bressan's Conjecture: we show that $\boldsymbol{b} \in L^1((0,T);\mathrm{BV}_{\rm loc}(\mathbb{R}^d))$ satisfies Assumption 7.27 from which untangling of trajectories is derived and uniqueness follows. In Section 8.1, we exploit Coarea formula and Rank-One Theorem to show that it is possible to approximate locally the singular part of $D\boldsymbol{b}$ with a measure concentrated on uniformly Lipschitz graphs, Corollary 8.5. The proof of this fact is split into several steps (Propositions 8.2-8.3 and Corollary 8.4). Using the Rank one property, the vector valued case is reduced to the previous analysis and we obtain the desired decomposition in Corollary 8.5. Then, in Section 8.2, the explicit form of the local cylinders is exhibited: in the absolutely continuous part of $D\boldsymbol{b}$, Section 8.2.1, one compares the Lagrangian flow $\eta$ with the linear flow generated by a constant matrix $A$. In this case, the analysis is pretty much similar to the standard renormalization estimates, we give in Proposition 8.6 the correct bounds.
The singular part (Section 8.2.2) is definitely more involved: the cylinders are constructed by solving a PDE (equations (8.11)) using the approximate vector field constructed in the previous section. Lemma 8.7 guarantees that the Lipschitz regularity of sets is preserved in time, so that they can be used as approximate cylinders of flows. Lemmata 8.8, 8.9 estimate the lateral flux through these cylinders and yield Proposition 8.10, which give the correct bounds for the singular case. We collect in the last section, Section 8.3, the technical proofs of Lemmata 8.8, 8.9.

Finally, in Chapter 9, we present the concept of *forward untangling*, which simplify some of the propositions shown in Chapter 7: it is discussed in Definition 9.1 the notion of *forward untangled Lagrangian representation* and we explain, in Section 9.1, the local theory (i.e. in a given proper set: see Proposition 9.2 and Proposition 9.3). Then we perform the usual "local-to-global" argument in Proposition 9.9, relying again on the subadditivity of the forward untangling functional (Proposition 9.7).
Section 9.2 aims to give an interesting example in which the forward untangling method applies: after recalling some preliminary results on weak Lebesgue spaces (in particular, the useful Lemma 9.12) we show that every Lagrangian representation of a vector field $\rho(1,\boldsymbol{b})$ is forward untangled, whenever $\boldsymbol{b}$ satisfy a suitable estimate on its difference quotients (stated precisely in 9.13). We remark that this case is relevant for at least two reasons: on the one hand, it allows to recover, in the untangling formulation, the results contained in [**BC13**] about vector fields whose gradient is given by a singular integral (of an $L^1$ function). On the other hand, this gives an example of a Regular Lagrangian Flow which has only a Hölder-Lusin property (and not a Lipschitz-Lusin property: see Remark 9.19).
Finally, Section 9.3, contains a quantitative stability estimate (Proposition 9.20). We compare the Regular Lagrangian Flow of a vector field satisfying Assumption 9.13 with the generalized flow of a given vector field: we estimate how much the two flows differ in terms of the $L^1$ norm of the difference of the vector fields.

The last part of the thesis deals with scalar, multidimensional conservation laws. In Chapter 10, after claryfing the notation and recalling some preliminary results (Section 10.1.2), we give the precise definition of Lagrangian representation of a solution (Definition 10.3). The first fact we show is that, whenever a solution has a Lagrangian representation, then it is an entropy solution (Proposition 10.5). Next, in Section 10.2.2 we prove the existence of a Lagrangian representation first for a BV entropy solution (Proposition 10.13), and then the general case (Theorem 10.14). The chapter is concluded with a first application of the above construction (Section 10.3), where we give the proof of Theorem C.

# Notation and mathematical preliminaries

Abstract. In this chapter we collect some mathematical preliminaries which will be used throughout the thesis. In particular, for the usefulness of the reader, in Section I we fix some notations we will use in the following. Section II collects the tools from Geometric Measure Theory we will use and Section III contains a synopsis of the theory of functions of bounded variation. Section IV contains a glimpse of the theory of traces for bounded, measure-divergence vector fields. Finally, in Section V we will present some topics from the Optimal Transportation Theory we will use later on in the Thesis.

## I. General notation

We fix in this section, for the usefulness of the reader, some notations which will be used throughout the thesis.

**Euclidean spaces and topology.** For an integer $d \geq 1$, the $d$-dimensional euclidean real vector space will be written as $\mathbb{R}^d$, and its norm by $|\cdot|$. In the following we will often consider the space $\mathbb{R}^{d+1}$ or the space $\mathbb{R}^+ \times \mathbb{R}^d$, whose coordinates will be denoted by $t$ (time) and $x$ (space), with $t \in \mathbb{R}, x \in \mathbb{R}^d$. The open ball in $\mathbb{R}^d$ centered at a point $x \in \mathbb{R}^d$ with radius $r$ is

$$B_r^d(x) := \{y \in \mathbb{R}^d : |y - x| < r\}.$$

When $x = 0$ and there is no risk of confusion we will simply write $B_r^d$ to denote $B_r^d(0)$. The unit sphere in $\mathbb{R}^{d+1}$ of center 0 will be denoted by $\mathbb{S}^d := \partial B_1^{d+1}(0) \subset \mathbb{R}^{d+1}$ .

If $X$ is a metric space, the ball centered in $x \in X$ with radius $r$ will denoted $B_r^X(x)$, and $B_r(x)$ when no confusion occurs about $X$. If $E \subset X$ then $\mathrm{dist}(x, E)$ is the distance of $x$ from the set $E$, defined as the infimum of $d(x, y)$ as $y$ varies in $E$.

The norm in a generic Banach space will be denoted by $\|\cdot\|$, with an index referring to the space whenever some confusion may occur. If not otherwise stated, $\Omega$ will stand for a generic open set in $\mathbb{R}^d$.

The closure of a set $A$ is denoted by $\mathrm{clos}\, A$, usually being clear the ambient topological space. The relative closure of $A$ in the topological space $B$ is $\mathrm{clos}(A, B)$. Similarly, the interior will be written as $\mathrm{int}\, A$ or $\mathrm{int}(A, B)$. The frontier/boundary will be written as $\mathrm{Fr}\, A$ or $\mathrm{Fr}(A, B)$ and, in some cases (mainly for $\Omega \subset \mathbb{R}^d$), we will use the more conventional notation $\partial\Omega$. We will say that $A \Subset B$ if $\mathrm{clos}\, A$ is a compact set contained in $B$. A neighbourhood of $x \in X$ will be written as $U_x$. The power set of set $X$ will be denoted by $\mathcal{P}(X)$. Given a product space $X \times Y$, we denote the projection on the space $X$ by $\mathtt{p}_X$: sometimes we will also write $\mathtt{p}_j : \prod_i X_i \to X_j$ to denote the projection on the $j$-component $X_j$. In the product space $X \times Y$, for all sets $A$ we will denote its sections as

$$A(x) = \{y : (x, y) \in A\}, \quad A(y) = \{x : (x, y) \in A\}.$$

We say that the family $\{A_\alpha\}_{\alpha \in I}$ ($I$ some set of index) is a *covering* of $A$ if

$$A \subset \bigcup_\alpha A_\alpha,$$

and, if the elements of the family are disjoint, i.e. $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$, we say it is a *partition*.

**Functions and vectors.** If $A$ is a set, we will denote by $\mathbb{1}_A$ the characteristic function

$$\mathbb{1}_A(x) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

The notation $\mathbb{I}$ is for the identity function $\mathbb{I}(x) = x$. The symbol $g \circ f$ denotes the usual composition of the two functions $f\colon X \to Y$ and $g\colon Y \to Z$. When the function is defined in a subset of the ambient space, the domain will be written as $\mathcal{D}(f)$ and its range (or image) as $\mathcal{R}(f)$. The graph of a function $f$ is denoted as $\operatorname{Graph} f$, and the support by $\operatorname{supp} f$. In the case of vector valued functions (vector fields), we will use bold letters, e.g. $\boldsymbol{b} = (b_i)_{i=1}^d$. We will write $x_{\mathbf{n}} = (x \cdot \mathbf{n})\mathbf{n}$ and $x_{\mathbf{n}}^{\perp} = x - x_{\mathbf{n}}$: often we will identify each of these vectors with their subspace vectors, e.g. $x_{\mathbf{n}} \simeq x \cdot \mathbf{n}$. A generic vector in $\mathbb{R}^{d+1}$ will be written as $B$: we will sometime use this notation when the particular structure of $B$ is not important. If the vector field $\boldsymbol{b} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is time dependent, then we will use either the notation $\boldsymbol{b}(t)\colon \mathbb{R}^d \to \mathbb{R}^d$ or $\boldsymbol{b}_t$. In general, given $X, Y$ topological spaces, $C(X, Y)$ will stand for the space of continuous functions $f\colon X \to Y$ . If $X = \mathbb{R}^d$, $C^k(\mathbb{R}^d)$ is the space of real valued functions with continuous partial derivatives up to order $k$. The space of compactly supported smooth functions in an open set $\Omega \subset \mathbb{R}^d$ will be denoted by $C_c^\infty(\Omega)$. The space of distributions over $\Omega$ will be $\mathscr{D}'(\Omega)$. The duality pairing between a distribution $f \in \mathscr{D}'(\Omega)$ and a function $\psi \in C_c^\infty(\Omega)$ will be written as $\langle f, \psi\rangle$.

A smooth, non negative function $\varphi$ with compact support and integral equal to 1 will be called *convolution kernel*. We will use the notation $\varphi^\varepsilon = \varepsilon^{-d}\varphi_0^{1/\varepsilon}$. We will denote the convolution in $\mathbb{R}^d$ by $*$.

**Differential operators.** The distributional partial derivatives of a real-valued function $f$ defined on a subset of $\mathbb{R} \times \mathbb{R}^d$ will be written as $\partial_t f_t$, $\partial_{x_i} f$. The differential of a smooth function $f$ will be written as $Df$, and the divergence of a vector field $\boldsymbol{b}$ by $\operatorname{div} \boldsymbol{b}$. Finally, if $\boldsymbol{e}$ is a unit vector, the derivative of $\boldsymbol{b}$ along $\boldsymbol{e} = (e_i)_{i=1}^d$ will be denoted by $D_{\boldsymbol{e}} f = \sum_{i=1}^d e_i \partial_i f$.

**About the value of the constants.** To conclude this section, we will use $L$ for a scale constant, $C_d$ for a dimensional constant and $C$ for a generic constant which may change from line to line. If $f$ is some function, we will write $\mathcal{O}(f)$ for a quantity equivalent to $f$ or $o(f)$ for an infinitesimal quantity w.r.t. $f$: usually the point about where the limit is taken is clear from the context.

## II. Tools from Geometric Measure Theory

In this section we collect some results in measure theory which will be used in the next chapters. The main reference is [**AFP00**].

**Abstract measure theory.** If $X$ is a set and $\mathscr{A}$ is a $\sigma$-algebra on $X$, we will call $(X, \mathscr{A})$ a *measure space*. A *positive measure* on $(X, \mathscr{A})$ is a function $\mu\colon \mathscr{A} \to [0, +\infty]$ such that $\mu(\emptyset) = 0$ and which is $\sigma$-additive, i.e. for any sequence $\{A_n\}_n \subset \mathscr{A}$ of pairwise disjoint sets it holds $\mu(\bigcup_n A_n) = \sum_{n=0}^\infty \mu(A_n)$, A positive measure is said to be *finite* if $\mu(X) < \infty$ and is a *probability measure* if $\mu(X) = 1$; We say that a measure is $\sigma$-*finite* if there exists a countable family $(X_n)_{n\in\mathbb{N}} \subset \mathscr{A}$ such that $X = \bigcup_i X_i$ and $\mu(X_n) < +\infty$ for every $n \in \mathbb{N}$. We will write $\mathscr{P}(X)$ to denote the set of *probability measures* on $X$.
More generally, if $(X, \mathscr{A})$ is a measure space and $m \in \mathbb{N}$, $m \geq 1$, any $\sigma$-additive function $\mu\colon \mathscr{A} \to \mathbb{R}^m$ such that $\mu(\emptyset) = 0$ will be called *real measure* (if $m = 1$) or *vector valued measure* (if $m > 1$). Notice that this means that, for any sequence $\{A_n\}_n \subset \mathscr{A}$ of pairwise disjoint sets, the series $\sum_{n=0}^\infty \mu(A_n)$ is absolutely convergent and it holds $\mu(\bigcup_n A_n) = \sum_{n=0}^\infty \mu(A_n)$.

When $X$ is the $d$-dimensional Euclidean space $\mathbb{R}^d$, we will denote the Lebesgue measure by $\mathscr{L}^d$ and the $(d-1)$-dimensional Hausdorff measure by $\mathscr{H}^{d-1}$. The *Dirac mass at* $x$

will be written as $\delta_x$. The $d$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^d$ will be denote by $\omega_d$, so that

$$\mathscr{L}^d(B^d_r(0)) = \omega_d r^d.$$

Consequently, the $\mathscr{H}^{d-1}$-measure of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ is $d\omega_d$.

We say that a set $E \in \mathscr{A}$ is $\mu$ negligible if $\mu(E) = 0$. A negligible set (w.r.t. some given measure) will be often denoted by $N$. We say that a property $P(x)$ depending on the point $x \in X$ holds $\mu$-*almost everywhere* in $X$ if the set where $P$ fails is $\mu$-negligible. If $\mu = \mathscr{L}^d$ is the Lebesgue measure on (a subset of) $\mathbb{R}^d$ , we will often say the the property $P(x)$ holds *almost everywhere*.

*Measures induced by functions.* If $\mu$ is a positive measure on $(X, \mathscr{A})$ and $f \in L^1(X, \mu)$ we define $f\mu$ as the measure on $X$ given by $f\mu(E) := \int_E f\, d\mu$, for every $E \in \mathscr{A}$.

*Total variation of a measure.* For every measure $\mu$ on $(X, \mathscr{A})$ we define its *total variation* as the measure $|\mu|$ defined on $\mathscr{A}$ by

$$|\mu|(E) := \sup \left\{ \sum_{n=0}^{\infty} |\mu(E_n)| : E_n \subset \mathscr{A} \text{ disjoint, } E = \bigcup_{n=0}^{\infty} E_n \right\}.$$

The measure $|\mu|$ is a positive finite measure.

*Concentration.* We will say that a measure $\mu$ on $(X, \mathscr{A})$ is *concentrated* on a set $C \subset X$ if $|\mu|(X \setminus C) = 0$.

*Restriction of a measure.* Let $\mu$ be a measure on $(X, \mathscr{A})$ and $A \in \mathscr{A}$. We denote by $\mu \llcorner_A$ the measure on $\mathscr{A}$ defined by $\mu\llcorner_A(E) := \mu(A \cap E)$ for any $E \in \mathscr{A}$ which will be called *restriction* of $\mu$ to $A$. We will use a similar notation also for functions, $f\llcorner_A$.

*Absolute continuity and singularity.* Let $\mu$ be a positive measure and $\nu$ a real or vector-valued measure on $(X, \mathscr{A})$. We say that $\nu$ is *absolutely continuous* with respect to $\mu$ if

$$\forall E \in \mathscr{A} : \ \mu(E) = 0 \quad \Longrightarrow \quad |\nu|(E) = 0.$$

In this case, we will write $\nu \ll \mu$. Furthermore, if $\nu$ is now a positive measure on $(X, \mathscr{A})$, we say that $\mu$ and $\nu$ are *mutually singular* (and we write $\mu \perp \nu$) if there exists a set $E \in \mathscr{A}$ such that $\mu(E) = 0$ and $\nu(X \setminus E) = 0$. In the case where $\mu$ or $\nu$ are real or vector valued measures we say they are *mutually singular* if $|\mu|$ and $|\nu|$ are singular.

*Borel $\sigma$-algebra and Borel maps.* Let now $(X, \tau)$ be a topological space. The $\sigma$-algebra generated by open sets is called *Borel* $\sigma$-algebra and will be denoted by $\mathscr{B}(X)$. If $X, Y$ are two metric spaces, we say that a function $f \colon X \to Y$ is *Borel* if $f^{-1}(A) \in \mathscr{B}(X)$ for every open subset $A \subset Y$.

*Support of a measure.* Let $\mu$ be a positive Borel measure on $X$. The support of $\mu$ is defined as the closed set

$$\operatorname{supp} \mu := \{x \in X : \mu(U) > 0 \text{ for every neighbourhood } U \text{ of } x\}.$$

*Integration theory.* If $\mu$ is a measure on $X$ and $f \colon X \to [-\infty, +\infty]$ is a $\mu$-measurable map, we denote the integral of $f$ over $X$ w.r.t. the measure $\mu$ (whenever it exists) by

$$\int_X f(x)\, \mu(dx) \text{ or by } \int_X f(x)\, d\mu(x).$$

When $\mu = \mathscr{L}^d$ is the Lebesgue measure on (a subset of) $\mathbb{R}^d$ we write

$$\int_{\mathbb{R}^d} f(x)\, dx.$$

Sometime we will not write the set of integration, being implicitly characterized by the measure w.r.t. we are integrating. The Lebesgue spaces $L^p(X, \mu)$ are defined in the usual way and their natural norm is denoted by $\|\cdot\|_p$ or by $\|\cdot\|_{L^p}$. The notation $L^p_+$ will be used for the space of non-negative functions with integrable $p$-power. We will add the index loc to denote properties which holds locally, e.g. local integrability, local boundedness.

*Average.* The average integral on a set will be written as

$$\fint_A f(x)\,\mu(dx) := \frac{1}{\mu(A)}\int_A f(x)\,\mu(dx).$$

*Push-forward.* Let $X, Y$ be two metric spaces, $\mu$ a measure on $(X, \mathscr{B}(X))$ and $f\colon X \to Y$ a Borel function. We define the *push-forward* of $\mu$ with respect to $f$ as the measure on $(Y, \mathscr{B}(Y))$ given by $f_\sharp\mu(B) := \mu(f^{-1}(B))$ for all $B \in \mathscr{B}(Y)$. Notice that for a Borel map $g\colon Y \to \mathbb{R}$ it holds

$$\int_Y g(y)\,(f_\sharp\mu)(dy) = \int_X (g\circ f)(x)\,\mu(dx).$$

*Scaling of functions and measures.* Given a point $x \in \mathbb{R}^d$ and $r > 0$, we define the rescaling of $f$ about $x \in \mathbb{R}^d$ as

$$f^r_x(y) := f(x + ry).$$

For a measure $\mu$, similarly we define $\mu^r_x$ as

$$\int_Y f(y)\,\mu^r_x(dy) = \int_Y f\left(x + \frac{y-x}{r}\right)\mu(dy).$$

**Measures in metric spaces.** Let now $X$ be a locally compact separable (l.c.s.) metric space. A positive measure on $(X, \mathscr{B}(X))$ is called a *Borel measure*. If a Borel measure is finite on compact sets, it is called *positive Radon measure*. A *Radon measure* on $X$ is a function defined on relatively compact Borel subsets of $X$ such that it is a measure on $(K, \mathscr{B}(K))$ for any $K \subset X$ compact. If $\mu\colon \mathscr{B}(X) \to \mathbb{R}^m$ is a measure then it is called *finite Radon measure*. We will denote by $[\mathscr{M}_{\mathrm{loc}}(X)]^m$ and by $[\mathscr{M}(X)]^m$ respectively the space of $\mathbb{R}^m$-valued Radon measures and the space of $\mathbb{R}^m$-valued finite Radon measures. We remark that the space $[\mathscr{M}(X)]^m$ is a Banach space with the norm $\|\mu\|_{\mathscr{M}} := |\mu|(X)$. In the case $m = 1$, the set of signed Radon measures over $X$ is denoted by $\mathscr{M}(X)$, the positive Radon measures with $\mathscr{M}^+(X)$ and the finite Radon measures by $\mathscr{M}_b(X)$.

**Weak-star convergence.** Let $X$ be a l.c.s. metric space, equipped with the Borel $\sigma$-algebra. Let $\mu \in \mathscr{M}(X)$ and $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathscr{M}(X)$. We say that $(\mu_n)_n$ *weakly-star converges to* $\mu$ if

$$\int_X f(x)\mu_n\,(dx) \to \int_X f(x)\mu(dx),$$

for any $f \in C_0(X)$. Taking into account Riesz's Theorem [**AFP00**, Thm. 1.54], we see that the notion of weak-star convergence of measures coincides with the standard notion of weak-star convergence in the dual of a Banach space. Since the results we propose are often local in space-time, sometimes we will not distinguish between weak-star and narrow convergence, and sometime we will just write weak (or weak-star) convergence of measures to denote both of them.

**Radon-Nikodým theorem and polar decomposition.** We will use several times the following

THEOREM I (Radon-Nikodým). *Let $\mu$ be a positive measure and $\nu$ a $\mathbb{R}^m$-valued measure on $(X, \mathscr{A})$. Moreover, let $\mu$ be $\sigma$-finite. Then there exist two vector measures $\nu^a, \nu^\perp$ and a function $f \in [L^1(X,\mu)]^m$ such that*

$$\nu^a \ll \mu, \qquad \nu^\perp \perp \mu, \qquad \nu = \nu^a + \nu^\perp \ \text{ and } \ \nu^a = f\mu.$$

*Moreover, the measures $\nu^a, \nu^s$ are unique and the function $f$ is unique up to modification in $\mu$-negligible sets.*

The function $f$ is also called *density* of $\nu$ with respect to $\mu$ or *Radon-Nikodým derivative* and it also denoted by $\frac{d\nu}{d\mu}$. When the measure $\mu = \mathscr{L}^d$, then the first term in the decomposition above will be denoted by $\nu^{\text{a.c.}}$ (either for the function $f$ or the measure $f\mu$).

**Disintegration theorem.** We recall here a version of Disintegration Theorem we will widely use in the thesis. Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be open sets, $\mu$ a positive Borel measure on $E$ and $x \mapsto \nu_x$ a function which assigns to each $x \in X$ a $\mathbb{R}^m$-valued Borel measure $\nu_x$ on $Y$: we say that this map is *Borel* (or, equivalently, that the family of measures $\{\nu_x\}_{x\in X}$ is Borel) if $x \mapsto \nu_x(B)$ is Borel for any $B \in \mathscr{B}(Y)$.

THEOREM II (Disintegration). *Let $f\colon X \to Y$ be a Borel map, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$; let $\mu$ be a finite Radon measure on $X$ and $\nu$ a finite Radon measure on $Y$ such that $f_\#|\mu| \ll \nu$. Then there exists a Borel family of probability measures $\{\mu_y\}_{y\in Y}$ on $X$ such that for a.e. $y \in Y$ the measure $\mu_y$ is concentrated on $f^{-1}(y)$ and*

$$\mu = \int_Y \mu_y \, \nu(dy),$$

*which means that for any bounded Borel measurable function $\phi\colon X \to \mathbb{R}$*

$$\int_X \phi(x)\,\mu(dx) = \int_Y \left( \int_X \phi(x)\,\mu_y(dx) \right) \nu(dy). \tag{II.1}$$

*Moreover, if $\{\mu'_y\}_{y\in Y}$ is another family for which* (II.1) *holds, then $\mu_y = \mu'_y$ for $\nu$-a.e. $y \in Y$.*

**Geometric Measure Theory.** We now present the main results needed in the sequel about Lipschitz functions, rectifiable sets, Area and Coarea formulæ.

*Lipschitz functions.* Let $E \subset \mathbb{R}^n$ be an arbitrary subset and $f\colon E \to \mathbb{R}^m$. The function $f$ is said to be *Lipschitz* if there exists a constant $K$ such that

$$|f(x) - f(y)| \le K|x - y|, \qquad \forall x, y \in E. \tag{II.2}$$

The smallest number $K$ for which (II.2) holds is called the Lipschitz constant of $f$ and denoted by $\mathrm{Lip}(f)$. The linear space of Lipschitz functions on $E$ will be denoted by $\mathrm{Lip}(E)$. We recall that the classical Rademacher's Theorem guarantees that Lipschitz functions are differentiable $\mathscr{L}^d$-a.e.

*Lipschitz domains.* An open set $\Omega \subset \mathbb{R}^d$ is said to be *Lipschitz* or *Lipschitz regular* if $\partial\Omega$ is *Lipschitz*: this means that for every point $x \in \partial\Omega$ there exists a neighbourhood $U_x$ of $x$ and a Lipschitz function $\varsigma_x : \mathbb{R}^{d-1} \supset U_x \to \mathbb{R}$ and $r > 0$ such that in a local coordinate system

$$\partial\Omega \cap B^d_r(x) = \mathrm{Graph}(\mathbb{I}, \varsigma_x).$$

*Rectifiable sets.* Let $E \subset \mathbb{R}^n$ be an $\mathscr{H}^k$ measurable set. We say that $E$ is *countably $\mathscr{H}^k$-rectifiable* if there exist countably many Lipschitz functions $f_i\colon \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathscr{H}^k \left( E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k) \right) = 0.$$

We say that $E$ is *$\mathscr{H}^k$-rectifiable* if $E$ is countably *$\mathscr{H}^k$-rectifiable* and $\mathscr{H}^k(E) < \infty$.

*Area formula for Lipschitz curves.* For the general Area Formula we refer the reader to [**AFP00**, §2.10]. We present here the formulation we will use. Let $\gamma\colon I \to \mathbb{R}^2$ (where $I \subset \mathbb{R}$ is an open interval) be an injective, Lipschitz map and let us consider the curve $C := \gamma(I)$: then it holds

$$\int_{\mathbb{R}^2} \mathscr{H}^0\left(I \cap f^{-1}(y)\right) d\mathscr{H}^1(y) = \int_I |\gamma'(s)|\, ds.$$

By injectivity, $\mathscr{H}^0\left(I \cap f^{-1}(y)\right) = 1$ for every $y \in C$ and hence

$$\mathscr{H}^1\llcorner_C = \int_I |\gamma'(s)|\, ds$$

which can be written shortly as

$$\mathscr{H}^1\llcorner_C = \gamma_\sharp(|\gamma'|\mathscr{L}^1). \tag{II.3}$$

*Coarea formula for Lipschitz maps.* For the general Coarea Formula for Lipschitz functions we refer the reader to [**AFP00**, §2.12]. We present here the version of the formula in which we will use it.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function: according to Coarea Formula we have

$$|\nabla f|\mathscr{L}^2 = \int_{\mathbb{R}} \mathscr{H}^1\llcorner_{E_y}\, dy. \tag{II.4}$$

where $E_y$ denotes the level set $f^{-1}(y)$. Note that this provides a characterization of the disintegration $\{\mu_y\}_{y\in\mathbb{R}}$ of the measure $\mu := |\nabla f|\mathscr{L}^2$.

*Structure of level sets of Lipschitz functions.* Since we will need some results on the structure of level sets of Lipschitz functions defined in the plane, we recall them here. First, for any $A \subseteq \mathbb{R}^2$, we denote by

$$\mathrm{Conn}(A) := \Big\{C \subset A : C \text{ is a connected component of } A\Big\},$$
$$\mathrm{Conn}^\star(A) := \Big\{C \in \mathrm{Conn}(A) : \mathscr{H}^1(C) > 0\Big\},$$

and

$$A^\star := \bigcup_{C\in\mathrm{Conn}^\star(A)} C.$$

Suppose now that $\Omega \subset \mathbb{R}^2$ is an open, simply connected domain and $H\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a compactly supported Lipschitz function. For any $h \in \mathbb{R}$, let $E_h := H^{-1}(h)$ and define the *critical set* of $H$ as

$$S := \{x \in \mathbb{R}^2 : \nabla H(x) = 0 \text{ or } H \text{ is not differentiable at } x\}.$$

We recall the following deep

Theorem III ([**ABC13**, Theorem 2.5]). *The following statements hold for $\mathscr{L}^1$-a.e. $h \in H(\Omega)$:*

(1) *$\mathscr{H}^1(E_h) < \infty$ and $E_h$ is countably $\mathscr{H}^1$-rectifiable (in what follows, we will say $E_h$ is* regular*);*
(2) *for $\mathscr{H}^1$-a.e. $x \in E_h$ the function $H$ is differentiable at $x$ with $\nabla H(x) \neq 0$;*
(3) *$\mathrm{Conn}^\star(E_h)$ is countable and every $C \in \mathrm{Conn}^\star(E_h)$ is a closed simple curve with an injective, Lipschitz parametrization $\gamma\colon \mathbb{R}/\ell\mathbb{Z} \to C$ (for some $\ell > 0$ )which satisfies $\dot\gamma(t) \in \mathbb{R}^2 \setminus S$ and $\dot\gamma(t) = \nabla^\perp H(\gamma(t))$ for a.e. $t$;*
(4) *$\mathscr{H}^1(E_h \setminus E_h^\star) = 0$.*

## III. Functions of bounded variation and of bounded deformation

In this section, we recall the main facts about functions of bounded variation. For a complete presentation of the topic, see e.g. [**AFP00**, Chapter 3] or [**Zie89**]. For BD we refer the reader to [**ACDM97**].

Throughout the section, $\Omega \subset \mathbb{R}^d$ will denote a generic open set.

**Definition and main properties.** As we already said, if $\boldsymbol{b} \in L^1(\Omega;\mathbb{R}^m)$ we denote by $D\boldsymbol{b} = (D_i b^j)_{i,j}$ the derivative in the sense of distributions of $\boldsymbol{b}$, i.e. the $\mathbb{R}^{m\times d}$-valued distribution defined by

$$\langle D_i b^j, \varphi\rangle = \int_{\mathbb{R}^d} b^j \frac{\partial \varphi}{\partial x_i}\, dx \qquad \forall \varphi \in C_c^\infty(\Omega), \qquad 1 \le i \le d,\, 1 \le j \le m.$$

*Bounded variation.* We say that $\mathbf{b} \in L^1(\Omega;\mathbb{R}^m)$ has *bounded variation* in $\Omega$, and we write $\boldsymbol{b} \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ if $D\boldsymbol{b}$ is representable by a $\mathbb{R}^{m\times d}$-valued measure, still denoted with $D\boldsymbol{b}$, with finite total variation in $\Omega$. The space of functions defined on the whole $\mathbb{R}^d$ which are of bounded variation in $B^d_r(0))$ for every $r > 0$ are said to be of *locally bounded variation* and they make up the space $\mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$. The space $\mathrm{BV}(\Omega;\mathbb{R}^m)$ is a Banach space with the norm

$$\|\boldsymbol{b}\|_{\mathrm{BV}(\Omega)} := \|\boldsymbol{b}\|_{L^1(\Omega)} + |D\boldsymbol{b}|(\Omega).$$

*Bounded deformation.* When $m = d$, given a vector field $\boldsymbol{b} \in L^1(\Omega;\mathbb{R}^d)$ we can define the symmetric part of the distributional derivative of $\boldsymbol{b}$, which will be written as $E\boldsymbol{b} = (E_{ij}\boldsymbol{b})_{ij}$, by setting

$$E_{ij}\boldsymbol{b} := \frac{1}{2}(D_i b^j + D_j b^i), \qquad 1 \le i, j \le d.$$

Accordingly, we say that $\boldsymbol{b} \in L^1(\Omega;\mathbb{R}^d)$ has *bounded deformation* in $\Omega$, and we write $\boldsymbol{b} \in \mathrm{BD}(\Omega;\mathbb{R}^d)$, if $E_{ij}\boldsymbol{b}$ is a Radon measure with finite total variation in $\Omega$ for any $i, j = 1,\dots,d$.

**Approximate limits.** We recall also that if $\boldsymbol{b} \in L^1(\Omega;\mathbb{R}^m)$, we say that $\boldsymbol{b}$ has an *approximate limit at* $x \in \Omega$ if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{r\downarrow 0} \fint_{B_r(x)} |\boldsymbol{b}(y) - z|\, dy = 0. \tag{III.5}$$

The set $S_{\boldsymbol{b}}$ of points where (III.5) does not hold is called the *approximate discontinuity set*; for any $x \in \Omega\setminus S_{\boldsymbol{b}}$, the vector $z$ is uniquely determined by (III.5) is called *approximate limit* of $\boldsymbol{b}$ at $x$ and is denoted by $\widetilde{\boldsymbol{b}}(x)$. We say that $\boldsymbol{b}$ is approximately continuous at $x$ if $x \notin S_{\boldsymbol{b}}$ and $\widetilde{\boldsymbol{b}}(x) = \boldsymbol{b}(x)$, i.e. $x$ is a Lebesgue point of $\boldsymbol{b}$. It is possible to check that $S_{\boldsymbol{b}}$ is a Borel set and the function $\widetilde{\boldsymbol{b}}\colon \Omega\setminus S_{\boldsymbol{b}} \to \mathbb{R}^m$ is a Borel function, coinciding $\mathscr{L}^d$-a.e. in $\Omega \setminus S_{\boldsymbol{b}}$ with $\boldsymbol{b}$. We also introduce the set of *approximate jump* $J_{\boldsymbol{b}} \subset S_{\boldsymbol{b}}$, to be the set of points $x \in S_{\boldsymbol{b}}$ such that there exist $z^\pm \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$ such that

$$\lim_{r\downarrow 0} \fint_{B_r^+(x,\nu)} |\boldsymbol{b}(y) - z^+|\, dy = 0, \quad \lim_{r\downarrow 0} \fint_{B_r^-(x,\nu)} |\boldsymbol{b}(y) - z^-|\, dy = 0, \tag{III.6}$$

where $B_r^\pm(x,\nu) := \{y \in B_r(x) : \langle y - x, \nu\rangle \gtrless 0\}$. The triplet $(z^+, z^-, \nu)$ is uniquely determined by (III.6) up to a permutation of $(z^-, z^+)$ and a change of sign of $\nu$ and is denoted by $(\boldsymbol{b}^+(x), \boldsymbol{b}^-(x), \nu(x))$. The set of approximate jump points is a Borel set denoted by $J_{\boldsymbol{b}}$; moreover, the map $J_{\boldsymbol{b}} \ni x \mapsto (\boldsymbol{b}^+(x), \boldsymbol{b}^-(x), \nu(x)) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{d-1}$ can be chosen to be Borel in its domain. In the case of 1-dimensional BV functions $f$ (or, in general, whenever the limits exist), we will often write

$$f(\bar{x}\pm) = \lim_{x\to\bar{x}} f(x)$$

for the right/left limit.

**Decomposition of the derivative in** BV**.** For $\boldsymbol{b} \in \mathrm{BV}(\Omega;\mathbb{R}^m)$, by Radon-Nikodým Theorem I, we can write

$$D\boldsymbol{b} = D^{\mathrm{a.c.}}\boldsymbol{b} + D^{\mathrm{s}}\boldsymbol{b}$$

where $D^{\mathrm{a.c.}}\boldsymbol{b}$ is the absolutely continuous part of $D\boldsymbol{b}$ w.r.t. $\mathscr{L}^d$ and $D^{\mathrm{s}}\boldsymbol{b}$ is the singular part of $D\boldsymbol{b}$ w.r.t. $\mathscr{L}^d$. Moreover, $D^{\mathrm{s}}\boldsymbol{b}$ can be further split in two parts, so that we obtain the decomposition

$$D\boldsymbol{b} = D^{\mathrm{a.c.}}\boldsymbol{b} + D^{\mathrm{c}}\boldsymbol{b} + D^{\mathrm{j}}\boldsymbol{b},$$

where

(1) $D^{\mathrm{j}}\boldsymbol{b}$ is the *jump part*: it is absolutely continuous w.r.t. to the $\mathscr{H}^{d-1}$-measure restricted to the $(d-1)$-countably rectifiable set $J$;
(2) $D^{\mathrm{c}}\boldsymbol{b}$ is the *Cantor part*, i.e. the residual part, orthogonal to the Lebesgue measure: each set with finite $\mathscr{H}^{d-1}$-measure is $D^{\mathrm{c}}\boldsymbol{b}$-negligible.

The following important result relative to the structure of BV functions holds.

Proposition IV (Structure of BV function). *Let $\boldsymbol{b} \in \mathrm{BV}(\Omega;\mathbb{R}^m)$. Then the approximate jump set $J_{\boldsymbol{b}}$ is countably $\mathscr{H}^{d-1}$-rectifiable, $\mathscr{H}^{d-1}(S_{\boldsymbol{b}} \setminus J_{\boldsymbol{b}}) = 0$ and*

$$D^j\boldsymbol{b} = (\boldsymbol{b}^+ - \boldsymbol{b}^-) \otimes \nu \mathscr{H}^{d-1} \llcorner J_{\boldsymbol{b}}.$$

*Furthermore, the blow-up $\boldsymbol{b}_x^r$ converges in $L^1$ to*

$$\bar{\boldsymbol{b}} = \begin{cases} \boldsymbol{b}^- & x\cdot\nu < 0, \\ \boldsymbol{b}^+ & x\cdot\nu > 0. \end{cases}$$

In the case of scalar functions $f \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ the following Coarea formula holds.

Theorem V (Coarea). *It holds*

$$|Df| = \int_{\mathbb{R}} |D\mathbb{1}_{\{f>h\}}|\, \mathscr{L}^1(dh), \qquad Df = \int_{\mathbb{R}} D\mathbb{1}_{\{f>h\}}\, \mathscr{L}^1(dh).$$

In the case of BV vector field $\boldsymbol{b}$, we recall the following deep result, due to Alberti:

Theorem VI (Alberti's Rank-one, [Alb93]). *It holds*

$$D\boldsymbol{b} = M(x)|D^{\mathrm{a.c.}}\boldsymbol{b}| + \mathsf{n}(x) \otimes \mathsf{m}(x)|D^{\mathrm{s}}\boldsymbol{b}|.$$

In the following we will use the notation $\mathsf{n}$ and $\mathsf{m}$ to denote the two unit vectors in the rank-one property. The matrix $M(x)$ will denote the Radon-Nikodým derivative of the absolutely continuous part. Note that from the orthogonality of the decomposition

$$|D^{\mathrm{a.c.}}\boldsymbol{b}| = |D\boldsymbol{b}|^{\mathrm{a.c.}}, \quad |D^{\mathrm{c}}\boldsymbol{b}| = |D\boldsymbol{b}|^{\mathrm{c}}, \quad |D^{\mathrm{j}}\boldsymbol{b}| = |D\boldsymbol{b}|^{\mathrm{j}}.$$

In case of time-dependent vector fields, i.e. $\boldsymbol{b}\colon I\times\mathbb{R}^d \to \mathbb{R}^d$, where $I = (0,T)$ for some fixed $T > 0$, we will say $\boldsymbol{b} \in L^1(I;\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$ if

$$\boldsymbol{b}_t(\cdot) = b(t,\cdot) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d) \text{ for } \mathscr{L}^1\text{-a.e. } t \in I$$

and for all $R > 0$

$$\int_{I\times B_R} |\boldsymbol{b}_t|\, dx\, dt + \int_I |D\boldsymbol{b}_t|(B_R)\, dt < +\infty$$

For a.e. $t \in I$ we can perform the usual decomposition of the spatial derivative of $\boldsymbol{b}_t$ into the absolutely continuous and the singular part, i.e.

$$D\boldsymbol{b}_t = D^{\mathrm{a.c.}}\boldsymbol{b}_t + D^s\boldsymbol{b}_t = D^{\mathrm{a.c.}}\boldsymbol{b}_t + D^{\mathrm{j}}\boldsymbol{b}_t + D^{\mathrm{c}}\boldsymbol{b}_t$$

where the measures in the decomposition are defined as above in the autonomous case. We denote by $|D\boldsymbol{b}|$, $|D^a\boldsymbol{b}|$, $|D^s\boldsymbol{b}|$ the measures obtained by integration of $|D\boldsymbol{b}_t|$, $|D^a\boldsymbol{b}_t|$, $|D^s\boldsymbol{b}_t|$ with respect to the time variable, that is

$$|D\boldsymbol{b}| = \int_0^T |D\boldsymbol{b}_t|\, dt.$$

This explicitly means that

$$\int_{I\times\mathbb{R}^d} \varphi(t,x)d|D\boldsymbol{b}|(t,x) = \int_{I\times\mathbb{R}^d} \varphi(t,x)d|D\boldsymbol{b}_t|(x)\,dt,$$

for every function $\varphi \in C_c(I\times\mathbb{R}^d)$ (similar formulæ hold for the other parts as well).

**Sets of finite perimeter.** Let $F\subset\mathbb{R}^d$: we will say it is of (locally) *finite perimeter* is the characteristic function $\mathbb{1}_F \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ (resp. locally of bounded variation). We recall that the *reduced boundary* $\partial^\star F$ of $F$ is the set of points such that

$$\lim_{r\searrow 0}\frac{|D\mathbb{1}_F|(B^d_r(x))}{r^{d-1}} = \omega_{d-1}, \quad \lim_{r\searrow 0}\frac{D\mathbb{1}_F(B^d_r(x))}{|D\mathbb{1}_F|(B^d_r(x)} = \mathsf{n}(x),$$

where $\mathsf{n}(x)$ is the *measure theoretical inner unit normal.* Furthermore, it holds

$$\lim_{r\searrow 0}\frac{1}{\omega_d r^d}\mathscr{L}^d\big(F\cap B^d_r(x)\cap\{x\cdot\mathsf{n}(x)>0\}\big)=1, \quad \lim_{r\searrow 0}\frac{1}{\omega_d r^d}\mathscr{L}^d\big(F\cap B^d_r(x)\cap\{x\cdot\mathsf{n}(x)<0\}\big)=0.$$

We finally recall the following (see, for instance, [**Zie89**, Thm. 5.7.3]):

Theorem VII (De Giorgi). *If $F\subset\mathbb{R}^d$ is of locally finite perimeter, then $\partial^\star F$ is countably $\mathscr{H}^{d-1}$ rectifiable and it holds $|D\mathbb{1}_F| = \mathscr{H}^{d-1}\llcorner_{\partial^\star F}$.*

## IV. Traces for measure divergence $L^\infty$-vector fields

We need to recall some basic facts about the trace properties of $L^\infty$ vector fields whose divergence is a measure. The main references are [**DL07, ACM05, Anz83, CF99**].

**Anzellotti's weak traces.** We consider a bounded vector field $\boldsymbol{V}\in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and we assume that its distributional divergence $\operatorname{div}\boldsymbol{V}$ is represented by some Radon measure: in this case, we will say that $\boldsymbol{V}$ is a *measure divergence vector field.* There are well known results that allows to give a meaning and to characterize the *trace* of such vector fields over rectifiable sets. To begin, we recall the following

Definition VIII. Given a bounded, open domain with $C^1$ boundary $\Omega\subset\mathbb{R}^d$, the *(Anzellotti) normal trace* of $\boldsymbol{V}$ over $\partial\Omega$ is the distribution defined by

$$\big\langle \mathsf{Tr}(\boldsymbol{V},\Omega)\cdot\mathbf{n},\psi\big\rangle := \int_\Omega \psi(x)\,d(\operatorname{div}\boldsymbol{V})(x) + \int_\Omega \boldsymbol{V}\cdot\nabla\psi(x)\,d\mathscr{L}^d(x)$$

for every compactly supported smooth test function $\psi\in C^\infty_c(\mathbb{R}^d)$.

The following Proposition says that the trace of a measure divergence vector field is not an arbitrary distribution, but it is induced by integration of a bounded function defined on $\partial\Omega$.

Proposition IX. *There exists a unique $g\in L^\infty_{\mathrm{loc}}(\partial\Omega;\mathscr{H}^{d-1}\llcorner_{\partial\Omega})$ such that*

$$\big\langle \mathsf{Tr}(\boldsymbol{V},\Omega)\cdot\mathbf{n},\phi\big\rangle = \int_{\partial\Omega} g\phi\,\mathscr{H}^{d-1}, \qquad \forall\phi\in C^\infty_c(\mathbb{R}^d).$$

It is not clear yet the relationship between the function $g$ and the vector field $\boldsymbol{V}$. We have thus the following theorem (see [**DL07**, Prop 7.10]):

Theorem X (Fubini's Theorem for traces). *Let $F\in C^1(\Omega)$. Then for a.e. $t\in\mathbb{R}$ we have*

$$\mathsf{Tr}(\boldsymbol{V},\{F>t\})\cdot\mathbf{n} = \boldsymbol{V}\cdot\nu_t \qquad \mathscr{H}^{d-1}\text{-a.e. on } \Omega\cap\{F>t\}, \tag{IV.7}$$

*where $\nu_t$ denotes the exterior unit normal to $\{F>t\}$.*

One can also define the traces of $\boldsymbol{V}$ on a *oriented* hypersurface of class $C^1$, say $\Sigma$. Indeed, choosing an open $C^1$ domain $\Omega' \Subset \mathbb{R}^d$ such that $\Sigma \subset \partial\Omega$ and the unit outer normals agree $\nu_\Omega = \nu_\Sigma$ we can define

$$\mathsf{Tr}^-(\boldsymbol{V}, \Sigma) \cdot \mathbf{n} := \mathsf{Tr}(\mathbf{V}, \Omega).$$

Analogously, choosing an open $C^1$ domain $\Omega''$ such that $\Sigma \subset \partial\Omega''$ and $\nu_{\Omega''} = -\nu_\Sigma$ we define

$$\mathsf{Tr}^+(\boldsymbol{V}, \Sigma) \cdot \mathbf{n} := -\,\mathsf{Tr}(\mathbf{V}, \Omega'') \cdot \mathbf{n}.$$

We remark that one can replace $C^1$ regularity with Lipschitz, so that it is possible to give the definition of normal trace of a measure divergence vector field on countable $\mathscr{H}^{d-1}$-rectifiable sets. We collect here other important results on Anzellotti's weak traces that will be used in the Thesis:

Proposition XI. *If $\boldsymbol{V}$ is a bounded, measure divergence vector field, then:*

- $\operatorname{div} \boldsymbol{V} \ll \mathscr{H}^{d-1}$ *as measures in* $\mathbb{R}^d$*;*
- *for any oriented, $C^1$ hypersurface $\Sigma$ it holds*

$$\operatorname{div} \boldsymbol{V} \llcorner_\Sigma = \big(\mathsf{Tr}^+(\boldsymbol{V}, \Sigma) \cdot \mathbf{n} - \mathsf{Tr}^-((\boldsymbol{V}, \Sigma) \cdot \mathbf{n}\big) \mathscr{H}^{d-1} \llcorner_\Sigma.$$

Finally, an interesting case is when the vector field $\boldsymbol{V}$ is of the form $\boldsymbol{V} := \rho \boldsymbol{v}$, where $\rho \in L^\infty(\mathbb{R}^d)$ and, for instance, $\boldsymbol{v} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$. In this situation, one has the usual definition of the trace of $\boldsymbol{v}$ over $\partial\Omega$ as BV function. We recall that the trace of BV functions $\mathbf{v}$ for open sets $\Omega \subset \mathbb{R}^d$ of class $C^1$ is a measure which is absolutely continuous w.r.t. $\mathscr{H}^{d-1} \llcorner_{\partial U}$. We conclude this section by recalling the following chain rule for traces, proved when $\boldsymbol{v} \in \mathrm{BV}$ in [**ADLM07**] (see also [**ACM05**, Theorem 4.2] for the case of vector fields of *bounded deformation*).

Theorem XII (Change of variables for traces). *Let $U \subset \mathbb{R}^d$ be an open domain of class $C^1$ and let $\boldsymbol{v} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ and $\beta \in \mathrm{Lip}(\mathbb{R}; \mathbb{R})$. Then if $\boldsymbol{V} = \rho \boldsymbol{v}$ is a measure divergence vector field, then also $\beta(\rho)\boldsymbol{v}$ is a measure divergence vector field and, moreover, it holds*

$$\mathsf{Tr}^\pm(\beta(\rho)\boldsymbol{v}, U) \cdot \mathbf{n} = \beta\left(\frac{\mathsf{Tr}^\pm(\rho\boldsymbol{v}, U) \cdot \mathbf{n}}{\mathsf{Tr}^\pm(\boldsymbol{v}, U) \cdot \mathbf{n}}\right) \mathsf{Tr}^\pm(\boldsymbol{v}, U) \cdot \mathbf{n}, \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial U,$$

*where the ratio is arbitrarily defined at points where the trace $\mathsf{Tr}(\boldsymbol{v}, U)$ vanishes.*

Other results concerning the trace of BD vector fields can be found in [**Bab15**]. For further use, we point out that minor modifications of [**ACM05**, Theorem 4.2] yield the following slight extension of Theorem XII to the case of time-dependent, measure-divergence vector fields (it is now intended that the divergence operator acts also on the time variable, i.e. $\operatorname{div} = \operatorname{div}_{t,x}$).

Proposition XIII. *Let $U \subset \mathbb{R}^d$ be a Lipschitz domain and let $\boldsymbol{v} \in L^1(\mathbb{R}; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$ and $\beta \in \mathrm{Lip}(\mathbb{R}; \mathbb{R})$. Then if $\boldsymbol{V} = \rho\boldsymbol{v}$ is a measure divergence vector field, then also $\beta(\rho)\boldsymbol{v}$ is a measure divergence vector field and, moreover, it holds*

$$\mathsf{Tr}^\pm(\beta(\rho)\boldsymbol{v}, U) \cdot \mathbf{n} = \beta\left(\frac{\mathsf{Tr}^\pm(\rho\boldsymbol{v}, U) \cdot \mathbf{n}}{\mathsf{Tr}^\pm(\boldsymbol{v}, U) \cdot \mathbf{n}}\right) \mathsf{Tr}^\pm(\boldsymbol{v}, U) \cdot \mathbf{n}, \qquad \mathscr{H}^{d-1}\text{-a.e. on } \partial U,$$

*where the ratio is arbitrarily defined at points where the trace $\mathsf{Tr}(\boldsymbol{v}, U)$ vanishes.*

## V. Tools from Optimal Transportation Theory and duality

In this section, we recall some results borrowed from the theory of Optimal Transport, which will be used in the Thesis. The main references are [**AG13, Kel84**].

Let $X, Y$ be Polish spaces (i.e. complete and separable metric spaces) and recall that $\mathscr{P}(X)$ (resp. $\mathscr{P}(Y)$) is the set of Borel probability measures on $X$ (resp. $Y$). Let $c\colon X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a fixed Borel *cost function*, and $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$ be given measures, called *marginal measures*. Kantorovich formulation of Optimal Transportation Problem is the following:

$$\min\left\{ \int_{X\times Y} c(x,y)\,\pi(dxdy) \,:\, \pi \in \mathsf{Adm}(\mu,\nu)\right\} \tag{V.8}$$

where the set of *admissible test plans* is defined as

$$\mathsf{Adm}(\mu,\nu) := \left\{\pi \in \mathscr{P}(X\times Y) \,:\, (\mathsf{p}_X)_{\#}\pi = \mu,\ (\mathsf{p}_Y)_{\#}\pi = \nu\right\}.$$

The set $\mathsf{Adm}(\mu,\nu)$ is always not empty, convex and compact w.r.t. the narrow topology on $\mathscr{P}(X\times Y)$; furthermore, we recall that the minimum of Problem V.8 exists, under mild assumptions on the cost function (e.g., $c$ is lower semicontinuous and bounded from below is sufficient). It is possible to formulate the *dual problem* to Problem V.8:

$$\begin{aligned}\max\Bigg\{ \int_X \phi(x)\,\mu(dx) + \int_Y \psi(y)\nu(dy) : \phi \in L^1(\mu), \psi \in L^1(\nu) \text{ such that}\\ \phi(x)+\psi(y) \le c(x,y), \forall (x,y)\in X\times Y\Bigg\}.\end{aligned} \tag{V.9}$$

The relation between Problem V.8 and Problem V.9 lies in the following classical

THEOREM XIV ([**AG13**, Thm.1.17]). *Assume that the cost function $c\colon X\times Y \to \mathbb{R}$ is* continuous, bounded from below *and satisfies*

$$c(x,y) \le a(x) + b(y)$$

*for some $a \in L^1(X,\mu)$, $b \in L^1(Y,\nu)$. Then the infimum of Problem* (V.8) *is equal to the supremum of Problem* (V.9)*.*

In [**Kel84**], general duality theorems for functionals similar to the ones considered in Problem V.8 and Problem V.9 are established under minimal assumptions on the ambient space in the multi-marginal case. Given finitely many finite measures $\mu_i \ge 0$ over Polish spaces $X_i$, $i \in I$ being $I$ some finite set of indeces, the set of *admissible transference plans* $\mathsf{Adm}(\mu_i)$ is now defined as

$$\mathsf{Adm}(\mu_i) = \left\{\pi \ge 0 : (\mathsf{p}_i)_\sharp \pi \le \mu_i\right\} \subset \mathscr{M}^+\left(\prod_i X_i\right).$$

The next theorem will be used in the thesis:

THEOREM XV ([**Kel84**, Thm. 2.14]). *The following duality result holds for any Borel, non-negative cost function $h\colon \prod_i X_i \to \mathbb{R}^+$:*

$$\begin{aligned}&\sup\left\{ \int_{\prod_i X_i} h(x)\,\pi(dx) \,:\, \pi \in \mathsf{Adm}(\mu_i)\right\}\\ &= \inf\left\{ \sum_i \int_{X_i} h_i(x)\,\mu_i(dx),\ h_i\colon X_i \to \mathbb{R}\ \textit{ Borel, such that } \sum_i h_i \ge h\right\}.\end{aligned} \tag{V.10}$$

Moreover, we will need the following Proposition, which considers the case $I = \{1,2\}$, i.e. when we have two Polish spaces $X, Y$ under the assumption that the cost function $c$ is the indicator function of some set $C \subset X\times Y$: in this case the special structure of the cost function can be transferred to the maximizing pair $(h_1,h_2)$ in (V.10), as the next proposition claims:

PROPOSITION XVI ([**Kel84**, Prop. 3.3]). *If $I = \{1,2\}$ and $c = \mathbb{1}_C$ for some Borel set $C \subset X\times Y$, then the r.h.s. of* (V.10) *can be replaced by*

$$\inf\left\{ \sum_{i=1,2} \mu_i(B_i), B_i \textit{ Borel}, \sum_{i=1,2} \mathbb{1}_{B_i} \ge \mathbb{1}_B\right\},$$

*and the minimum is attained.*

# Part 1

# Classical theory, renormalization and the two-dimensional case

CHAPTER 1

# An overview on the classical theory

ABSTRACT. The aim of this chapter is to give a rather quick and useful overview on the classical theory of flows. In Section 1.1 we recall the Cauchy-Lipschitz Theorem (giving existence, uniqueness and stability of solutions for *ordinary differential equations*). Then we introduce the formulations of the partial differential equation we want to study, namely the *continuity equation* and the *transport equation* and we show that, in the smooth framework, they are easily solved by the method of characteristics. As it is well-known, uniqueness of solutions to the ODE is readily lost, as soon as one drops the regularity assumptions on the velocity field: however, even for rough vector fields, one can still hope to show uniqueness of a suitable notion flow, the *Regular Lagrangian Flow*. It turns out that the theory of such flows (existence, uniqueness and stability) can be constructed from well-posedness results of the PDEs associated to the vector field: we are thus led to consider the continuity equation and the transport equation with non-smooth coefficients. In Section 1.2 we clarify our notion of weak solution for the continuity equation (discussing, in particular, the regularity in time). In Section 1.3, instead, we introduce the weak formulation of the transport equation and we prove a very general result of existence of bounded solutions. Finally, we begin investigating the problem of uniqueness of solutions to the transport equation, by presenting the notions of *renormalized solution* and of *renormalization property*, showing their links with the well-posedness problem for the PDE.

## 1.1. Cauchy-Lipschitz theory and the method of characteristics

**1.1.1. The ordinary differential equation.** Let $\Omega \subset [0,+\infty) \times \mathbb{R}^d$ be an open set and let $\boldsymbol{b}\colon \Omega \to \mathbb{R}^d$ be a given vector field. We want to study the ordinary differential equation (ODE)

$$\gamma'(t) = \boldsymbol{b}(t,\gamma(t)). \tag{1.1}$$

A (classical) solution of (1.1) consists of an interval $I \subset \mathbb{R}$ and a function $\gamma\colon I \to \mathbb{R}^d$ such that $(t,\gamma(t)) \in \Omega$ for every $t \in I$ and which satisfies (1.1) for every $t \in I$. We also say that $\gamma$ is an *integral curve* or a *characteristic curve* of the vector field $\boldsymbol{b}$. If we fix $(t_0,x_0) \in \Omega$, we can couple the equation (1.1) with the *initial condition* $\gamma(t_0) = x_0$. Usually, we will consider $t_0 = 0$ and we drop the index on $x_0$. We will thus consider the *Cauchy problem*

$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t,\boldsymbol{X}(t,x)), \\ \boldsymbol{X}(0,x) = x. \end{cases} \tag{1.2}$$

A map $\boldsymbol{X}$ solving (1.2), whenever it exists, will be called the *flow* of $\boldsymbol{b}$. When $\boldsymbol{b}$ enjoys suitable regularIty assumptions, mainly in the space variable, existence and uniqueness of the flow of $\boldsymbol{b}$ are ensured by the following, well-known result.

THEOREM 1.1 (Cauchy-Lipschitz). *Let $\boldsymbol{b}\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a continuous vector field and assume it is Lipschitz continuous with respect to the spatial variable, uniformly with respect to the time,*

$$|\boldsymbol{b}(t,x) - \boldsymbol{b}(t,y)| \le K|x-y|, \qquad \forall t \in [0,T],\ x,y \in \mathbb{R}^d.$$

*Then for every $x \in \mathbb{R}^d$ there exists $\delta > 0$ and a unique maximal solution $\boldsymbol{X}(\cdot,x)$ to (1.2) defined in a nonempty maximal time interval $[0,\delta)$. Moreover, the map $\boldsymbol{X}(t,\cdot)$ is locally Lipschitz on its domain.*

The proof of Theorem 1.1 is classical and it is based on the Banach-Caccioppoli fixed point Theorem.

REMARK 1.2. Theorem 1.1 remains valid if we assume that $\boldsymbol{b}$ is only summable with respect to the time variable and Lipschitz continuous in $x$ uniformly in $t$. Furthermore, we remind that it is possible to prove uniqueness of solutions under some milder assumptions on the vector field $\boldsymbol{b}$ than Lipschitz behaviour. For instance, the so called *one-sided Lipschitz condition* or the *Osgood condition* [Cri09, Prop. 1.2.6, 1.2.7] are sufficient to get uniqueness. ♠

**1.1.2. The transport equation and the continuity equation.** The ODE (1.2) is strictly related to the following linear partial differential equation, known as *transport equation*:

$$\begin{cases} \partial_t u(t,x) + \boldsymbol{b}(t,x) \cdot \nabla u(t,x) = 0, & \text{in } (0,T) \times \mathbb{R}^d \\ u(0,x) = \bar{u}(x) \end{cases} \tag{1.3}$$

where $\bar{u}\colon \mathbb{R}^d \to \mathbb{R}$ is a given initial datum. Indeed, if $u$ is a smooth solution of (1.3) and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} \frac{d}{dt} u(t, \boldsymbol{X}(t,x)) &= \partial_t u(t, \boldsymbol{X}(t,x)) + \partial_t \boldsymbol{X}(t,x) \cdot \nabla u(t, \boldsymbol{X}(t,x)) \\ &= \partial_t u(t, \boldsymbol{X}(t,x)) + \boldsymbol{b}(t, \boldsymbol{X}(t,x)) \cdot \nabla u(t, \boldsymbol{X}(t,x)) = 0, \end{aligned}$$

which means that $u$ is constant along the characteristics of $\boldsymbol{b}$. Hence, we have the following formula which yields the expression of the solution to (1.3) in terms of the flow of $\boldsymbol{b}$:

$$u(t,x) := \bar{u}\big(\boldsymbol{X}(t,\cdot)^{-1}(x)\big). \tag{1.4}$$

In the following, we will also consider the *continuity equation* which reads as

$$\partial_t \rho(t,x) + \operatorname{div}(\rho(t,x)\boldsymbol{b}(t,x)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d \tag{1.5}$$

and, in the case of a divergence-free vector field, is equivalent to the transport equation (1.3). It is possible to derive an explicit formula like (1.4) for the solution to the continuity equation (1.9), coupled with an initial datum of the form $\rho(0,\cdot) = \bar{\rho}(\cdot)$, for some given $\bar{\rho}\colon \mathbb{R}^d \to \mathbb{R}$: as one can easily check, the solution $\rho$ is given by

$$\rho_t(x) = \frac{\bar{\rho}}{\det(\nabla \boldsymbol{X}(t,\cdot))} \circ (\boldsymbol{X}(t,\cdot)^{-1})(x). \tag{1.6}$$

Formula (1.6) is usually known, within the language of fluid dynamics, as the *Transport Formula* (see, e.g. [MB02, Proposition 1.3]).

**1.1.3. Regular Lagrangian Flows.** When the vector field $\boldsymbol{b}$ is not assumed to be smooth (locally Lipschitz in space), but enjoys only Sobolev or BV bounds, there are well known examples that show that uniqueness at the ODE level, i.e. uniqueness of solutions to (1.1), is lost. For instance, one can consider the autonomous, continuous (but not Lipschitz) vector field $\boldsymbol{b}(x) := \sqrt{|x|}$ defined for $x \in \mathbb{R}$. Notice that $\boldsymbol{b} \in W^{1,p}_{\text{loc}}(\mathbb{R})$ for every $1 \le p < 2$. It is easy to check that the Cauchy problem

$$\begin{cases} x'(t) = \sqrt{|x(t)|}, \\ x(0) = 0 \end{cases} \tag{1.7}$$

has infinitely many solutions, given by

$$\gamma^c(t) = \begin{cases} 0 & \text{if } t \le c, \\ \frac{1}{4}(t-c)^2 & \text{if } t \ge c, \end{cases}$$

for every $c \in [0, +\infty]$. Heuristically, this means that the solution can "stay at rest" in the origin for an arbitrary long time.

However, one can still associate to the vector field $\boldsymbol{b}$ a notion of flow, made of a selection of trajectories of the ODE. Among all possible selections, we prefer the trajectories that do not allow for concentration, as presented in the following definition.

DEFINITION 1.3 (Regular Lagrangian Flow). Let $T > 0$ and $\boldsymbol{b}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a Borel, locally integrable vector field. We say that the Borel map $\boldsymbol{X} : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is a *Regular Lagrangian Flow* of $\boldsymbol{b}$ if the following two properties hold:

(1) for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$, $\boldsymbol{X}(\cdot,x) \in AC([0,T];\mathbb{R}^d)$ and solves the ODE (1.1) for $\mathscr{L}^1$-a.e. $t \in (0,T)$, with the initial condition $\boldsymbol{X}(0,x) = x$;
(2) there exists a constant $C = C(\boldsymbol{X})$ satisfying $X(t,\cdot)_\sharp \mathscr{L}^d \le C\mathscr{L}^d$ for every $t \in [0,T]$.

We notice that, thanks to the second condition in Definition 1.3, the notion of Regular Lagrangian Flow is invariant under modifications of $\boldsymbol{b}$ on negligible sets, i.e. if $\boldsymbol{b}(t,x) = \widetilde{\boldsymbol{b}}(t,x)$ for $\mathscr{L}^{d+1}$-a.e. $(t,x) \in [0,T]\times\mathbb{R}^d$ it is easy to check that $\boldsymbol{X}$ is a Regular Lagrangian Flow relative to $\boldsymbol{b}$ if and only if it is a Regular Lagrangian Flow relative to $\widetilde{\boldsymbol{b}}$.

It turns out that the theory of Regular Lagrangian Flows (namely existence, uniqueness and stability) can be constructed from corresponding results of the PDEs (continuity and transport equations) associated to the vector field, in the spirit of the following general principle: if one is able to prove uniqueness, at the PDE level, in the class of bounded (nonnegative) solutions, uniqueness at the Lagrangian level, i.e. of the Regular Lagrangian Flow, follows. For instance, this is the point of view assumed in [**Cri11**] to show the uniqueness of Regular Lagrangian Flow of (1.7) (see also the recent paper [**BDPRS15**] for generalizations). We will come back later with more details on the relationship between transport equation and ordinary differential equation. In view of what we have just said, it is thus natural to consider the continuity equation and the transport equation with non-smooth coefficients: in the next sections we precise the definitions of weak solutions to these problems we will adopt.

## 1.2. Weak formulation of the continuity equation

From a physical point of view, (1.5) is the equation satisfied by the density $u$ of a continuous distribution of particles moving according to the velocity field $\boldsymbol{b}$; in other words, the quantity $u(t,x)$ represents the number of particles per unit volume at time $t$ and position $x$. Therefore it is often convenient to consider the unknown function $u$ in (1.5) as the density of a measure and more generally we can consider the equation

$$\partial_t\mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0, \tag{1.8}$$

where $[0,T] \ni t \mapsto \mu_t$ is a Borel measure-valued function. The continuity equation (1.8) is intended in the distributional sense, according to the following definition.

DEFINITION 1.4. A Borel family $\mu = \{\mu_t\}_{t\in[0,T]}$ of locally finite signed measures on $\mathbb{R}^d$ such that $\boldsymbol{b}_t\mu_t$ is a locally finite measure is a solution to the continuity equation (1.8) if

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t\phi(t,x) + \boldsymbol{b}(t,x)\cdot\nabla\phi(t,x)]d\mu_t(x)dt = 0 \qquad \forall\phi \in C_c^\infty((0,T)\times\mathbb{R}^d).$$

For the Cauchy problem, we consider distributional solutions to

$$\begin{cases}\partial_t\mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0,\\ \mu_0 = \bar{\mu},\end{cases} \tag{1.9}$$

where $\bar{\mu}$ is a given measure on $\mathbb{R}^d$ by requiring that

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t\phi(t,x)+\boldsymbol{b}(t,x)\cdot\nabla\phi(t,x)]d\mu_t(x)dt+\int_{\mathbb{R}^d}\phi(0,x)\,d\bar{\mu}(x) = 0 \quad \forall\phi \in C_c^\infty((0,T)\times\mathbb{R}^d). \tag{1.10}$$

Remark 1.5. We remark, in passing, that the continuity equation (1.8) makes sense (in the distributional formulation) without any regularity requirement on $\boldsymbol{b}$ and/or on $\mu$, provided

$$\int_I \int_A |\boldsymbol{b}|\, d|\mu_t|\, dt < \infty \qquad \text{for every } A \Subset \mathbb{R}^d. \tag{1.11}$$

When we consider singular measures $\mu_t$, the vector field $\boldsymbol{b}$ has to be defined *pointwise* and not only $\mathscr{L}^d$-a.e., since the product $\boldsymbol{b}\mu_t$ is sensitive to modifications of $\boldsymbol{b}$ in $\mathscr{L}^d$-negligible sets. However, we will often consider only measures $\mu_t$ which are absolutely continuous with respect to $\mathscr{L}^d$, i.e. $\mu_t = \rho(t,\cdot)\mathscr{L}^d$, so that everything depends only on the equivalence class of $\boldsymbol{b}$ in $L^1((0,T)\times\mathbb{R}^d)$. ♠

We notice that it is easy to derive an explicit formula like (1.4) for the solution to the continuity equation (1.9) even in the measure-valued case. For instance, if $\boldsymbol{b} \in C^1([0,T]\times\mathbb{R}^d)$ then one can readily check that

$$\mu_t := \boldsymbol{X}(t,\cdot)_\# \bar{\mu} \tag{1.12}$$

is a solution to (1.9). Indeed, the condition $\mu_0 = \bar{\mu}$ is trivially satisfied, as $\boldsymbol{X}(\cdot,x) = x$. On the other hand, testing (1.8) against test functions of the form $\varphi(t,x) = \psi(t)\phi(x)$ we get

$$\int_{\mathbb{R}} \psi'(t)\langle \mu_t, \phi\rangle\, dt + \int_{\mathbb{R}} \psi(t) \int_{\mathbb{R}^d} \boldsymbol{b}(t,x)\cdot\nabla\phi(x)\, d\mu_t(x)\, dt = 0.$$

Since the flow $\boldsymbol{X}$ is $C^1$ with respect to the time variable, the map $t \mapsto \langle \mu_t, \phi\rangle$ belongs to $C^1([0,T])$ as well, so that we only have to compute the pointwise derivative. Since $\partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t,\boldsymbol{X}(t,x))$ we thus deduce that

$$\begin{aligned}
\frac{d}{dt}\langle \mu_t, \phi\rangle &= \frac{d}{dt}\int_{\mathbb{R}^d} \phi(\boldsymbol{X}(t,x))d\bar{\mu}(x) \\
&= \int_{\mathbb{R}^d} \nabla\phi(\boldsymbol{X}(t,x))\cdot \boldsymbol{b}(t,\boldsymbol{X}(t,x))d\bar{\mu}(x) \\
&= \int_{\mathbb{R}^d} \nabla\phi(y)\cdot\boldsymbol{b}(t,y)d\mu_t(y),
\end{aligned}$$

which gives that (1.12) is a solution to (1.8). The assumption $\boldsymbol{b} \in C^1$ is not necessary and a formal proof can be conducted along the lines above within the Cauchy-Lipschitz framework (see, for instance, [**AC08**, Prop. 4]). Notice that, by using Area formula, we can recover (1.6) from (1.12), in the particular case when $\bar{\mu} = \bar{\rho}\mathscr{L}^d$.

**1.2.1. Weak continuity in time of measure-valued solutions.** We now want to consider the problem of the time continuity of $t \mapsto \mu_t$, where $\{\mu_t\}_{t\in[0,T]}$ is a Borel family of measures which solves (1.8). Indeed, the fact that the family solves the continuity equation gives automatically some regularity conditions on the map $t \mapsto \mu_t$: first, we see that, after (possibly) a modification on a negligible set of times, the map $t \mapsto \mu_t$ is weakly-star continuous in the sense of distributions.

Let us call $j\colon \mathscr{M}(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\mathbb{R}^d)$ the canonical embedding of measures in the space of distributions and set $T_t := j(\mu_t)$ for $t \in [0,T]$.

Proposition 1.6 (Regularity in the sense of distributions). *Let $\{\mu_t\}_{t\in[0,T]}$ be a measure-valued solution of (1.8). Then there exists a negligible set $N \subset [0,T]$ such that after redefinition (if necessary) for $t \in N$ the family $\{T_t\}_{t\in[0,T]}$, where $T_t = j(\mu_t)$, becomes weak-star continuous in the sense of distributions, i.e. if $t \to \bar{t} \in [0,T]$ then*

$$\langle T_t, \phi\rangle \to \langle T_{\bar{t}}, \phi\rangle, \qquad \textit{for every} \quad \phi \in C^\infty_c(\mathbb{R}^d).$$

Proof. Let $D \subset \mathscr{D}$ be a countable dense set. By (1.8) and (1.11), the map

$$f_\phi\colon [0,T] \ni t \mapsto \langle T_t, \phi\rangle$$

belongs to $W^{1,1}(0,T)$ for any $\phi \in D$, since

$$\frac{d}{dt}\langle T_t, \phi\rangle = \int_{\mathbb{R}^d} \boldsymbol{b}_t(x) \cdot \nabla\phi(x)\, d\mu_t(x) \in L^1(0,T). \tag{1.13}$$

Hence there exists a negligible set $L_\phi \subset [0,T]$ such that $f_\phi$ is uniformly continuous on $[0,T]\setminus L_\phi$. Let $L := \bigcup_{\phi\in D} L_\phi$: then, for any $\tau \in [0,T]$, there exists a sequence $t_k \in [0,T]\setminus I$ such that $t_k \to \tau$. By uniform continuity of $f_\phi$ the sequence $f_\phi(t_k)$ is Cauchy hence there exists

$$f_\phi(\tau) := \lim_{k\to\infty} f_\phi(t_k).$$

We can thus define a functional $\widetilde{T}_\tau$ on $D$, for any $\tau \in [0,T]$, by imposing

$$\langle \widetilde{T}_\tau, \phi\rangle = f_\varphi(\tau).$$

Taking into account (1.13), we see easily that the functional $\widetilde{T}_t$ can be extended in a continuous way to the whole $\mathscr{D}$. To conclude the proof, it is enough to observe that $\widetilde{T}_t$ coincides with $T_t$ for $t \in [0,T] \setminus L$. □

We now observe that if, in addition, the family $\{\mu_t\}_{t\in[0,T]}$ solves (1.8) and $\mu_t \in L^\infty([0,T];\mathscr{M}(\mathbb{R}^d))$, then Proposition 1.6 can be improved, showing that there exists a representative of $t \mapsto \mu_t$ which is weakly-star continuous in the sense of *measures*, i.e. in duality with $C_c(\mathbb{R}^d)$.

Proposition 1.7 (Regularity in the sense of measures). *Let $\{\mu_t\}_{t\in[0,T]}$ be a measure-valued solution of (1.8). Suppose that, in addition, $\operatorname{ess\,sup}_{t\in[0,T]} |\mu_t|(\mathbb{R}^d) < \infty$. Then there exists a negligible set $N \subset [0,T]$ such that after redefinition (if necessary) for $t \in N$ the family $\mu_t$ becomes weak* continuous in the sense of measures, i.e. i.e. if $t \to \bar{t} \in [0,T]$ then*

$$\int_{\mathbb{R}^d} \phi(x) d\mu_t(x) \to \int_{\mathbb{R}^d} \phi(x)\, d\mu_{\bar{t}}(x), \qquad \textit{for every} \quad \phi \in C_c(\mathbb{R}^d).$$

Proof. The proof is just an application of Riesz Theorem. Indeed, we can repeat verbatim the proof of Proposition 1.6, writing directly $\mu_t$ instead of $T_t$. We just need to observe that, if we set for any $\phi \in D$ and for any $t \in [0,T] \setminus L$

$$\ell_t(\phi) := f_\phi(t) = \int_{\mathbb{R}^d} \phi(x)\, d\mu_t(x)$$

then we have the estimate

$$|\ell_t(\phi)| \le M\|\phi\|_\infty, \tag{1.14}$$

where $M := \operatorname{ess\,sup}_t |\mu_t|(\mathbb{R}^d)$. By the time continuity of $\ell_t(\phi)$ the estimate (1.14) holds for all $t \in [0,T]$ and all $\phi \in D$. Moreover, using (1.14) for any $t \in [0,T]$ we can extend $\ell_t(\cdot)$ from $D$ to $C_c(\mathbb{R}^d)$, so that (1.14) holds for all $t \in [0,T]$ and all $\phi \in C_c(\mathbb{R}^d)$. By Riesz Representation Theorem for any $t \in [0,T]$ we can represent the functional $\ell_t(\cdot)$ as

$$\ell_t(\varphi) = \int \varphi\, d\widetilde{\mu}_t$$

using some finite Borel measure $\widetilde{\mu}_t$ satisfying $|\widetilde{\mu}_t|(\mathbb{R}^d) \le M$. The resulting family $\widetilde{\mu}_t$ is weak-star continuous, and coincides with $\mu_t$ for $t \in [0,T] \setminus L$ and this completes again the proof. □

Finally, an easy adaptation of the above proofs (see, for instance, [DL07, Lemma 3.7]) leads to the following result, which deals with the case where $\mu_t = \rho_t \mathscr{L}^d$, with uniform bounded densities $\rho \in L^\infty((0,T); L^\infty(\mathbb{R}^d))$. In this case, which will be very relevant throughout the thesis, the convergence can still be improved, as one can find a representative of $\rho_t$ which is weakly-star continuous in $L^\infty$.

Proposition 1.8 (Regularity in the sense of $L^\infty$). *Let $\{\mu_t\}_{t\in[0,T]}$ be a measure-valued solution of (1.8) and assume that $\mu_t = \rho_t \mathscr{L}^d$, with $\|\rho\|_{L^\infty((0,T)\times\mathbb{R}^d)} < \infty$. Then there exists a negligible set $N \subset [0,T]$ such that after redefinition (if necessary) for $t \in N$ the map $t \mapsto \rho_t$ becomes weakly-star continuous in $L^\infty(\mathbb{R}^d)$, i.e. i.e. if $t \to \bar{t} \in [0,T]$ then*

$$\int_{\mathbb{R}^d} \phi(x)\rho_t(x)\,dx \to \int_{\mathbb{R}^d} \phi(x)\rho_{\bar{t}}(x)\,dx, \qquad \textit{for every} \quad \phi \in L^1(\mathbb{R}^d).$$

We conclude with a couple of remarks.

Remark 1.9. In the general measure-valued case, if $\mu_t \geq 0$ for a.e. $t \in \mathbb{R}$ and $\boldsymbol{b}$ satisfies suitable growth conditions (for instance, it is globally bounded), then integrating (1.8) over $\mathbb{R}^d$ one gets the *conservation of mass*, i.e. $\mu_t(\mathbb{R}^d) = \bar{\mu}(\mathbb{R}^d)$. In particular, $\mu_t \in L^\infty([0,T];\mathscr{M}^+(\mathbb{R}^d))$, hence all non-negative measure-valued solutions are weakly-star continuous in the sense of measures. ♠

Remark 1.10 (On the definition of the initial condition for (1.8)). When we have weak-star continuous representative (say, in the sense of measures) one can prescribe the initial condition for (1.8) either in the distributional way (1.10) or, equivalently by imposing $\widetilde{\mu}_0 = \bar{\mu}$, being $\widetilde{\mu}_t$ the time continuous representative. ♠

Remark 1.11 (Counterexample in $L^1([0,T];\mathscr{M}(\mathbb{R}^d))$). We want to explicitly remark that the assumption $\operatorname{ess\,sup}_{t\in[0,T]} |\mu_t|(\mathbb{R}^d) < \infty$ in Proposition 1.7 cannot be relaxed to $\mu_t \in L^1([0,T];\mathscr{M}(\mathbb{R}^d))$. Indeed, let $f\colon (0,2) \to \mathbb{R}$ be the Lipschitz function defined by

$$f(x) = \sum_{n=0}^{\infty} (-1)^n (2^{-n} - |x - y_n|)^+$$

where $a^+ = \max(a,0)$ and $y_n := \sum_{k=0}^{n-1} 2^{-k} + 2^{-n-1}$. Being Lipschitz, $f$ is differentiable a.e. w.r.t Lebesgue measure and let us set $N := \{x \in (0,2) : f \text{ is not differentiable at } x\}$. Let us define $\boldsymbol{b}(x) := \frac{1}{f'(x)}$ if $x \notin N$ and $\boldsymbol{b}(x) := 0$ otherwise. Notice that $\boldsymbol{b}(x) \in \{-1, 0, +1\}$ for every $x \in (0,2)$. For any $\varphi \in C^\infty_c(\mathbb{R}\times(0,2))$ with $\varphi = \varphi(t,x)$ we have

$$\begin{aligned}
0 &= \int_{(0,2)} \partial_x\Big(\varphi\big(f(x),x\big)\Big)\,dx \\
&= \int_{(0,2)} \varphi_t\big(f(x),x\big)f'(x) + \varphi_x\big(f(x),x\big)\,dx \\
&= \int_{(0,2)} \Big(\varphi_t\big(f(x),x\big) + \frac{1}{f'(x)}\varphi_x\big(f(x),x\big)\Big)f'(x)\,dx \\
&= \int_{(0,2)} \Big(\varphi_t\big(f(x),x\big) + \boldsymbol{b}(x)\varphi_x\big(f(x),x\big)\Big)f'(x)\,dx \\
&= \int_{(0,2)} \Big(\varphi_t\big(f(x),x\big) + \boldsymbol{b}(x)\varphi_x\big(f(x),x\big)\Big)\operatorname{sign}(f'(x))|f'(x)|\,dx \\
&= \int_{(0,2)} \Big(\varphi_t\big(f(x),x\big) + \boldsymbol{b}(x)\varphi_x\big(f(x),x\big)\Big)\operatorname{sign}(\boldsymbol{b}(x))|f'(x)|\,dx.
\end{aligned}$$

By Coarea formula, we continue

$$\begin{aligned}
0 &= \int_{(0,2)} \Big(\varphi_t\big(f(x),x\big) + \boldsymbol{b}(x)\varphi_x\big(f(x),x\big)\Big)\operatorname{sign}(\boldsymbol{b}(x))|f'(x)|\,dx \\
&= \int_{\mathbb{R}} \sum_{x\in f^{-1}(t)} \Big(\varphi_t\big(t,x\big) + \boldsymbol{b}(x)\varphi_x\big(t,x\big)\Big)\operatorname{sign}(\boldsymbol{b}(x))\,dt.
\end{aligned}$$

We now set, for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$,

$$T_t := \sum_{x \in f^{-1}(t)} \operatorname{sign}(\boldsymbol{b}(x))\delta_x. \tag{1.15}$$

One can easily check that, for every $t \neq 0$, the distribution $T_t$ is induced by a measure, i.e. $T_t = \mu_t$. Indeed, for every $t \neq 0$ the set $f^{-1}(t)$ is finite, hence the distribution $T_t$ reduces to a finite linear combination of Dirac deltas, hence it is a measure. The computation above hence shows that, for $t \neq 0$, $T_t = \mu_t$ defined by (1.15) is a measure-valued solution to the continuity equation associated to $\boldsymbol{b}$. Notice furthermore that, by Coarea formula, the map $t \mapsto \mu_t$ is in $L^1(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} |\mu_t|(\mathbb{R})\, dt < \infty$$

while, on the other hand, $|\mu_t|(\mathbb{R})$ is not uniformly bounded in $t$, as the number of preimages $\#\{x \in f^{-1}(t)\}$ becomes arbitrarily large as $t \to 0$. More precisely, the function $t \mapsto \#\{f^{-1}(t)\}$ is piecewise constant and has jumps (of size 2, in each $x \in N$); for future reference, we will denote the element of the set $f^{-1}(0)$ as $x_k$, for $k \in \mathbb{N}$ (and we suppose they are ordered, $0 = x_0 < x_1 < \ldots \leq 2$). Notice that $T_0$ is not a measure but a distribution of order 1, since

$$\begin{aligned}
\langle T_0, \phi \rangle = -\phi(1) + \phi\left(\frac{3}{2}\right) - \phi\left(\frac{7}{4}\right) + \ldots &= \sum_{k \in \mathbb{N}} (-1)^k \phi(x_k) \\
&\leq \sum_{k \in \mathbb{N}} \|\nabla \phi\|_\infty |x_{k+1} - x_k| \\
&= \|\nabla \phi\|_\infty \sum_{k \in \mathbb{N}} 2^{-k} < \infty
\end{aligned}$$

for any $\phi \in C_c^\infty(0,2)$ but, for a merely continuous function $\phi \in C_c(0,2)$, the sum defining $T_0$ may diverge, thus $T_0$ is not a measure.

Finally, one can show that $T_t$ is continuous at *every* $t \in \mathbb{R}$ in the sense of distributions. We prove continuity for $t = 0$: if $t_j \to 0$ as $j \to \infty$, denoting with $(x_k^j)_{k=1,\ldots C_j}$ the ordered elements of $f^{-1}(t_j)$, we see that $x_k^j \to x_k$ for every $k$, as $j \to +\infty$ by continuity of $f$. Hence

$$\langle T_t - T_{t_j}, \phi \rangle = \sum_{k=1}^{\infty} (-1)^k \phi(x_k) - \sum_{k=1}^{C_j} (-1)^k \phi(x_k^j) = \sum_{k \geq C_j} (-1)^k \phi(x_k) - \sum_{k=1}^{C_j} (-1)^k \big(\phi(x_k^j) - \phi(x_k)\big).$$

For $j$ large enough the second term is small enough and the first is the remainder of a converging series, hence tends to 0 as well. The continuity in the sense of distributions for $t \neq 0$ is easier and is left to the reader. ♠

## 1.3. Weak formulation of the transport equation

Let us assume now that $\boldsymbol{b} \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a locally integrable vector field and let $u \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ and consider the transport equation

$$\partial_t u(t,x) + \boldsymbol{b}(t,x) \cdot \nabla u(t,x) = 0. \tag{1.16}$$

We observe that, being $u$ merely bounded, the term $\boldsymbol{b} \cdot \nabla u$ is not well defined. Nevertheless, if $\operatorname{div} \boldsymbol{b} \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d)$ then we can define the product $\boldsymbol{b} \cdot \nabla u$ as a distribution via the equality

$$\langle \boldsymbol{b} \cdot \nabla u, \phi \rangle = -\langle \boldsymbol{b} u, \nabla \phi \rangle - \langle u \operatorname{div} \boldsymbol{b}, \phi \rangle \qquad \forall \phi \in C_c^\infty((0,T) \times \mathbb{R}^d).$$

This allows us to give directly a distributional meaning to the transport equation (1.16) and therefore we have the following

DEFINITION 1.12. Suppose $\boldsymbol{b} \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$ and let $\operatorname{div} \boldsymbol{b} \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$. Then we say that a locally bounded function $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a *weak solution* of (1.16) if

$$\partial_t u + \operatorname{div}(u\boldsymbol{b}) - u \operatorname{div} \boldsymbol{b} = 0 \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^d).$$

Concerning the initial condition, an analogous statement like 1.8 applies for solutions to the transport equation: thus, we can give couple (1.16) with $u(0,x) = \bar{u}(x)$ (for a given $\bar{u}\colon \mathbb{R}^d \to \mathbb{R}$), by simply requiring that $\widetilde{u}(0,x) = \bar{u}(x)$, being $\widetilde{u}(t,\cdot)$ the weak-star $L^\infty$ continuous representative.

**1.3.1. Existence of solutions.** Existence of weak solutions to (1.16) is rather easy to prove, thanks to the linearity of the transport equation. Indeed, it is sufficient to regularize the vector field and the initial data, obtaining a sequence of smooth solutions to the approximate problems, and then, passing to the limit, we get a solution. Namely, the following holds:

PROPOSITION 1.13 (Existence). *Let $\boldsymbol{b} \in L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ with $\operatorname{div} \boldsymbol{b} \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ and let $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a weak solution $u \in L^\infty([0,T] \times \mathbb{R}^d)$ to (1.16) with initial condition $u_0$.*

PROOF. By a regularization argument (for instance, via convolutions) there exists a sequence of smooth vector fields $\boldsymbol{b}^\varepsilon$ and smooth initial data $u_0^\varepsilon$ which converge (strongly in $L^1$) to $\boldsymbol{b}$ and $u_0$ respectively. Then we consider the regularized equation

$$\begin{cases} \partial_t u(t,x) + \boldsymbol{b}^\varepsilon(t,x) \cdot \nabla u(t,x) = 0, \\ u(0,x) = u_0^\varepsilon(x), \end{cases}$$

and let $u^\varepsilon$ be its solution (which exists and it is unique, since $\boldsymbol{b}^\varepsilon$ is smooth). From the explicit formula for the solution of the transport equation with a smooth vector field, we get that $\{u^\varepsilon\}_\varepsilon$ is equi-bounded in $L^\infty([0,T] \times \mathbb{R}^d)$. By compactness, we can find a subsequence $\{u^{\varepsilon_j}\}_j$ which converges weakly$^\star$ in $L^\infty([0,T] \times \mathbb{R}^d)$ to a function $u \in L^\infty([0,T] \times \mathbb{R}^d)$ which is clearly, by linearity, a solution to (1.16). □

**1.3.2. The problem of uniqueness.** If, on the one hand, the problem of existence of weak solutions to the transport equation is easily settled by Proposition 1.13, the problem of uniqueness is much harder. Even for bounded and divergence-free vector fields this is a nontrivial problem, since oscillations might lead to loss of uniqueness (see, e.g., the counterexample in [**Dep03a**]).

A possible strategy to prove uniqueness, which goes back to DiPerna-Lions, is to formally multiply both sides of (1.16) by $\beta'(u(t,x))$, being $\beta\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary $C^1$ function: we get

$$\beta'(u(t,x))\partial_t u(t,x) + \beta'(u(t,x))\boldsymbol{b}(t,x) \cdot \nabla u(t,x) = 0.$$

If all the functions were smooth, we could apply ordinary chain rule and we could rewrite the last equation as

$$\partial_t \beta(u(t,x)) + \boldsymbol{b}(t,x) \cdot \nabla \beta(u(t,x)) = 0. \tag{1.17}$$

This last passage is not justified (and is in general is false) without further regularity assumptions. However, the validity of this "chain rule" is enough to have uniqueness of solutions for the Cauchy problem associated to (1.16): heuristically, integrating the equation (1.17) on $\mathbb{R}^d$ we get

$$\int_{\mathbb{R}^d} \partial_t \beta(u(t,x))\,dx + \int_{\mathbb{R}^d} \boldsymbol{b}(t,x) \cdot \nabla \beta(u(t,x))\,dx = 0.$$

Assuming that the boundary term vanishes, we apply the divergence theorem and we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} \beta(u(t,x))\,dx = \int_{\mathbb{R}^d} \beta(u(t,x)) \operatorname{div} \boldsymbol{b}(t,x)\,dx.$$

Assuming that $\|\operatorname{div} \boldsymbol{b}\|_{L^\infty} \leq C$, for some $C \geq 0$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(u(t,x))\,dx \leq C\int_{\mathbb{R}^d} \beta(u(t,x))\,dx.$$

In particular, taking $\beta(s) = s^2$ and using Gronwall's Lemma, we get that

$$\int_{\mathbb{R}^d} u^2(t,x)\,dx \leq e^{Ct}\int_{\mathbb{R}^d} u^2(0,x)\,dx = 0,$$

if $\bar{u} = 0$ and hence $u \equiv 0$, which by linearity, is enough to conclude that uniqueness holds.

This informal discussion leads us to introduce to the following

DEFINITION 1.14 (Renormalized solutions). Let $I \subseteq \mathbb{R}$ be an interval and $\boldsymbol{b} \in L^1_{\mathrm{loc}}(I \times \mathbb{R}^d; \mathbb{R}^d)$ with $\operatorname{div} \boldsymbol{b} \in L^1_{\mathrm{loc}}(I \times \mathbb{R}^d)$. We say that a function $u \in L^\infty_{\mathrm{loc}}(I \times \mathbb{R}^d)$ is a *renormalized solution* if, for every function $\beta \in C^1(\mathbb{R}, \mathbb{R})$, the following implication holds:

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0, \\ u(0,\cdot) = \overline{u}(\cdot) \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \partial_t \beta(u) + \boldsymbol{b} \cdot \nabla \beta(u) = 0, \\ \beta(u(0,\cdot)) = \beta(\overline{u})(\cdot) \end{cases}$$

where the equations are understood in the sense of Definition 1.12.

REMARK 1.15. A similar definition (as well as the subsequent theorems) extends to transport equations with a linear right hand side of order zero in $u$ of the form

$$\partial_t u + \boldsymbol{b} \cdot \nabla u = cu, \qquad c \in L^1([0,T]; L^\infty_{\mathrm{loc}}(\mathbb{R}^d)).$$

In particular, choosing $c = -\operatorname{div} \boldsymbol{b}_t$, we are able to translate all the forthcoming statements for the transport equation into corresponding results for the continuity equation. ♠

When the renormalization property is satisfied by all bounded weak solutions, it can be transferred to a property of the vector field itself.

DEFINITION 1.16 (Renormalization property). Let $I \subseteq \mathbb{R}$ be an interval and $\boldsymbol{b} \in L^1_{\mathrm{loc}}(I \times \mathbb{R}^d; \mathbb{R}^d)$ and let $\operatorname{div} \boldsymbol{b} \in L^1_{\mathrm{loc}}(I \times \mathbb{R}^d)$. We say that $\boldsymbol{b}$ has the *renormalization property* if every bounded solution of the transport equation with vector field $\boldsymbol{b}$ is renormalized according to Definition 1.14.

The importance of the renormalization property lies in the following proposition, which states precisely the informal argument presented at the beginning of this section.

PROPOSITION 1.17 ([Cri09, Theorem 2.3.3]). *Let $\boldsymbol{b}\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded vector field with $\operatorname{div} \boldsymbol{b} \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^d))$. If $\boldsymbol{b}$ has the renormalization property, then bounded solutions to the Cauchy problem for (1.16) are unique.*

CHAPTER 2

# Renormalized solutions via commutator estimates

ABSTRACT. From the discussion of the previous chapter we deduce that the problem of uniqueness of solutions to the transport equation driven by some rough vector field $\boldsymbol{b}$ is a delicate issue and a possible way to tackle it is to show that $\boldsymbol{b}$ has the renormalization property. In this chapter, we want to present two important positive results in this direction, based on the common approach of *commutator estimates*, in a sense explained in Section 2.1. The two results we intend to discuss go back respectively to DiPerna-Lions [**DL89c**] and Ambrosio [**Amb04**] and are studied in Section 2.2 and Section 2.3. A comment is in order, concerning Section 2.3: we do not give a detailed account of the original proof due to Ambrosio (which can be found, beside [**Amb04**], also in [**Cri09, DL07**]), but rather a slightly different argument, which however builds on the very same ingredients of the original proof. In particular we will make use of the regularization scheme based on commutators, but instead of decoupling the difference quotients of $\boldsymbol{b}$ as in [**Amb04**], we will construct a suitable family of point-dependent convolution kernels and show strong convergence to zero of the commutator (w.r.t. this kernel) in $L^1$, in the spirit of DiPerna-Lions' argument for Sobolev vector fields [**DL89c**].

## 2.1. Commutators estimates

From Section 1.3.2 we thus deduce that to prove uniqueness of weak solutions to the transport equation is enough to prove the renormalization property for the vector field $\boldsymbol{b}$. In order to do this, following an idea due again to DiPerna-Lions [**DL89c**], we first write down the PDE satisfied by $u^\delta := u * \varphi^\delta$, being $(\varphi^\delta)_{\delta>0}$ a family of even convolution kernels in $\mathbb{R}^d$. More precisely, if $\boldsymbol{b}\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is a Borel vector field and $u \in L^\infty(\mathbb{R}\times\mathbb{R}^d)$ satisfies

$$\partial_t u + \boldsymbol{b}\cdot\nabla u = 0 \tag{2.1}$$

then, convolving both sides of (2.1) with $\varphi^\delta$ (and adding and subtracting the term $\boldsymbol{b}\cdot\nabla u^\delta$), we obtain

$$\partial_t u^\delta + \boldsymbol{b}\cdot\nabla u^\delta = \boldsymbol{b}\cdot\nabla u^\delta - (\boldsymbol{b}\cdot\nabla u) * \varphi^\delta. \tag{2.2}$$

We then define the *commutator* $r^{\varphi,\delta}$ as the error term in the right hand side of (2.2):

$$r^{\varphi,\delta} := \boldsymbol{b}\cdot\nabla u^\delta - (\boldsymbol{b}\cdot\nabla u) * \varphi^\delta. \tag{2.3}$$

The name *commutator* comes from the fact that this term measures the difference in exchanging the operations of convolution and differentiating in the direction of $\boldsymbol{b}$.

By multiplying both sides of (2.2) by $\beta'(u^\delta)$, and applying the chain-rule for Sobolev maps (notice that $u^\delta$ is smooth in the space and has the time derivative in $L^1$, as $\partial_t u^\delta = -(\boldsymbol{b}\cdot\nabla u)^\delta$) we deduce

$$\partial_t\beta(u^\delta) + \boldsymbol{b}\cdot\nabla\beta(u^\delta) = \beta(u^\delta) r^{\varphi,\delta}.$$

Now in order to recover the renormalization property, we would like to pass to the limit as $\delta\to 0$, showing the convergence to zero of the quantity $\beta(u^\delta)r^{\varphi,\delta}$. Notice that $r^{\varphi,\delta}$ always converges to zero weakly in the sense of distributions (without any regularity assumption on $\boldsymbol{b}$). However, in order to transfer the uniqueness from the approximate problems to the limit one, we need strong convergence to zero of the commutator $r^{\varphi,\delta}$. This is indeed the case, if $\boldsymbol{b}$ has Sobolev regularity, as shown in the next paragraph.

## 2.2. Renormalization in $W^{1,p}$: DiPerna-Lions' argument

By elementary facts in the theory of Sobolev spaces it is possible to show strong convergence to zero of the commutators, when $\boldsymbol{b}$ enjoys Sobolev bounds.

PROPOSITION 2.1 (Strong convergence of the commutator). *Let $\boldsymbol{b}$ be a bounded vector field, with $\boldsymbol{b} \in L^1_{\mathrm{loc}}(I; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$, where $I \subset \mathbb{R}$ is an interval and let $u \in L^q_{\mathrm{loc}}(I \times \mathbb{R}^d)$ with $p, q$ conjugates exponents. Then $r^{\varphi,\delta} \to 0$ strongly in $L^1_{\mathrm{loc}}(I;\mathbb{R}^d)$ as $\delta \to 0$.*

PROOF. By definition we have

$$\begin{aligned} r^{\varphi,\delta} &= \boldsymbol{b}\cdot\nabla u^\delta - (\boldsymbol{b}\cdot\nabla u) * \varphi^\delta \\ &= \boldsymbol{b}\cdot\nabla u^\delta - \big(\operatorname{div}(u\boldsymbol{b}) - u \operatorname{div}\boldsymbol{b}\big) * \varphi^\delta \\ &= \boldsymbol{b}\cdot\nabla u^\delta - \operatorname{div}(u\boldsymbol{b}) * \varphi^\delta + \big(u \operatorname{div}\boldsymbol{b}\big) * \varphi^\delta. \end{aligned}$$

Recalling the kernels $\varphi^\delta$ have compact support, we get by divergence theorem

$$\begin{aligned} r^{\varphi,\delta}(t,x) &= \boldsymbol{b}_t(x)\int_{\mathbb{R}^d} u_t(y)\nabla\varphi^\delta(x-y)\,dy - \int_{\mathbb{R}^d} \operatorname{div}\big(u_t\boldsymbol{b}_t\big)(y)\varphi^\delta(x-y)\,dy + (u_t \operatorname{div}\boldsymbol{b}_t) * \varphi^\delta \\ &= \int_{\mathbb{R}^d} u_t(y)\boldsymbol{b}_t(x)\cdot\nabla\varphi^\delta(x-y)\,dy - \int_{\mathbb{R}^d} u_t(y)\boldsymbol{b}_t(y)\cdot\nabla\varphi^\delta(x-y)\,dy + (u_t \operatorname{div}\boldsymbol{b}_t) * \varphi^\delta \\ &= \int_{\mathbb{R}^d} u_t(y)\big[\boldsymbol{b}_t(x) - \boldsymbol{b}_t(y)\big]\cdot\nabla\varphi^\delta(x-y)\,dy + (u_t \operatorname{div}\boldsymbol{b}_t) * \varphi^\delta \\ &= \frac{1}{\delta^d}\int_{\mathbb{R}^d} u_t(y)\big[\boldsymbol{b}_t(x) - \boldsymbol{b}_t(y)\big]\cdot\nabla\varphi\left(\frac{x-y}{\delta}\right)\frac{1}{\delta}\,dy + (u_t \operatorname{div}\boldsymbol{b}_t) * \varphi^\delta \\ &= \int_{\mathbb{R}^d} u_t(x+\delta z)\left[\frac{\boldsymbol{b}_t(x+\delta z) - \boldsymbol{b}_t(x)}{\delta}\right]\cdot\nabla\varphi(z)\,dz + (u_t \operatorname{div}\boldsymbol{b}_t) * \varphi^\delta, \end{aligned}$$

where in the last passage we have used the change of variables $y := x + \delta z$ and the fact that $\nabla\varphi$ is odd. To sum up, we have seen that the commutator can be written as

$$r^{\varphi,\delta}(t,x) = \int_{\mathbb{R}^d} u_t(x+\delta z)\left[\frac{\boldsymbol{b}_t(x+\delta z) - \boldsymbol{b}_t(x)}{\delta}\right]\cdot\nabla\varphi(z)\,dz + (u_t \operatorname{div}\boldsymbol{b}_t) * \varphi^\delta.$$

Now it is a standard fact in the theory of Sobolev spaces (see, e.g., [**Bre10**, Prop. 9.3] or [**Zie89**, Theorem 2.1.6]) that, as $\delta \to 0$,

$$\frac{\boldsymbol{b}_t(x+\delta z) - \boldsymbol{b}_t(x)}{\delta} \to \nabla\boldsymbol{b}_t(x)\cdot z \qquad \text{strongly in } L^p_{\mathrm{loc}}. \tag{2.4}$$

Moreover, we also have $u_t(x+\delta z) - u_t(x) \to 0$ strongly in $L^q$, as $\delta \to 0$, because the translation is a $L^q$-strongly continuous isomorphism; therefore we deduce that the commutator converges strongly in $L^1_{\mathrm{loc}}(I \times \mathbb{R}^d)$ to

$$u_t(x)\int_{\mathbb{R}^d} \big(\nabla\boldsymbol{b}_t(x)\cdot z\big)\cdot\nabla\varphi(z)\,dz + u_t \operatorname{div}\boldsymbol{b}_t.$$

Now we observe that

$$\begin{aligned} u_t(x)\int_{\mathbb{R}^d} \big(\nabla\boldsymbol{b}_t(x)\cdot z\big)\cdot\nabla\varphi(z)\,dz &= u_t(x)\int_{\mathbb{R}^d}\sum_{i,j=1}^d \frac{\partial b_i}{\partial x_j}(t,x) z_j \frac{\partial\varphi}{\partial z_i}(z)\,dz \\ &= u_t(x)\sum_{i,j=1}^d \frac{\partial b_i}{\partial x_j}(t,x)\int_{\mathbb{R}^d} z_j\frac{\partial\varphi}{\partial z_i}(z)\,dz \\ &= -u_t(x)\operatorname{div}\boldsymbol{b}_t(x) \end{aligned}$$

since, as it can be proved proved integrating by parts, it holds

$$\int_{\mathbb{R}^d} z_j\frac{\partial\varphi}{\partial z_i}(z)\,dz = -\delta_{ij}. \tag{2.5}$$

This concludes the proof. □

As a corollary, we obtain the following

THEOREM 2.2 (DiPerna-Lions). *Let $\boldsymbol{b} \in L^1_{\rm loc}((0,T); W^{1,p}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d))$. Then $\boldsymbol{b}$ has the renormalization property.*

## 2.3. Strong convergence of the commutators in the BV, divergence-free case

The main difficulty in extending the DiPerna-Lions theorem to the BV setting is the fact that strong convergence of the difference quotients (2.4) characterizes functions in Sobolev spaces, so in BV one cannot expect strong convergence of commutators. In 2004, in the paper [**Amb04**], Ambrosio extended the DiPerna-Lions theory and showed that the renormalization property holds for vector fields which are of class BV (locally in space) and whose divergence is absolutely continuous with respect to Lebesgue measure.

THEOREM 2.3 (Ambrosio). *Let $\boldsymbol{b} \in L^1_{\rm loc}((0,T); {\rm BV}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d))$ with $\operatorname{div} \boldsymbol{b}_t \ll \mathscr{L}^d$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$. Then $\boldsymbol{b}$ has the renormalization property.*

In this section we intend to present a detailed proof of Theorem 2.3, in the particular setting where $\operatorname{div} \boldsymbol{b}_t = 0$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$. The approach we present is slightly different from Ambrosio's original argument, although it involves the same basic ingredients. Indeed, the original proof in [**Amb04**] (see also [**Cri09, DL07**] for other presentations) passes through commutators estimates and by-passes the lack of strong convergence via a splitting of the difference quotients; then, an anisotropic regularization is performed, based on a local selection of a "bad" direction given by Alberti's Rank-one Theorem VI.

Instead of this decoupling of the difference quotients, we will construct a family of convolution kernels (depending on the point $(t,x)$) and show strong convergence to zero of the commutator (w.r.t. this family of kernels) in $L^1$, in the spirit of DiPerna-Lions' argument (see also [**Ler04**]). Indeed, the anisotropic regularization procedure performed in the original argument by Ambrosio is, in some sense, equivalent to let the convolution kernel vary from point to point. This is exactly the idea behind the alternative proof we present here: we optimize locally the so called *anisotropic energy* (see Definition 2.8). Then we use a Vitali type covering argument to glue toghether the locally optimizing kernels: in this way, we end up with a well defined function $\varphi = \varphi(t,x,z)$ which is globally smooth and, for every fixed $(t,x)$, it is a convolution kernel in the variable $z$. We then find an explicit expression of the commutator obtained by convolving a solution $u \in L^\infty$ with the kernel $\varphi$ and we finally show that it converges $L^1$-strongly to 0, thus concluding the argument.

As already said, in order to simplify the presentation, we will restrict our attention to the divergence-free case; minor adaptations of the proof presented below can lead to the proof of the general version stated in Theorem 2.3.

**2.3.1. Precise statement and preliminaries to the proof.** Let us set $I := (0,T)$ where $T > 0$ is fixed and consider a vector field $\boldsymbol{b}\colon I \times \mathbb{R}^d \to \mathbb{R}^d$. Assume that $\boldsymbol{b} \in L^1(I; {\rm BV}_{\rm loc}(\mathbb{R}^d))$, i.e.

$$\boldsymbol{b}_t(\cdot) = b(t,\cdot) \in {\rm BV}_{\rm loc}(\mathbb{R}^d) \text{ for } \mathscr{L}^1\text{-a.e. } t \in I$$

and for all $R > 0$

$$\int_{I\times B_R} |\boldsymbol{b}_t|\,dx\,dt + \int_I |D\boldsymbol{b}_t|(B_R)\,dt < +\infty$$

We will present the proof, for simplicity, in the divergence-free case, so from now onwards we assume $\operatorname{div} \boldsymbol{b}_t = 0$ in the sense of distributions on $\mathbb{R}^d$ for $\mathscr{L}^1$-a.e. $t \in I$. By polar decomposition we can write

$$D\boldsymbol{b}_t = M_t |D\boldsymbol{b}_t| \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in I. \tag{2.6}$$

Notice that, being $\operatorname{div} \boldsymbol{b}_t = 0$ for $\mathscr{L}^1$-a.e. $t \in I$ , the matrix $M_t$ satisfies

$$\operatorname{tr} M_t = 0 \qquad \text{for } \mathscr{L}^1\text{-a.e. } t \in I.$$

We will denote by $\varphi \in C_c^\infty\big(I \times \mathbb{R}^d \times \mathbb{R}^d\big)$ a non-negative, smooth function $\varphi = \varphi_t(x,z)$ which is a convolution kernel in the $z$ variable, i.e. for every $(t,x) \in I \times \mathbb{R}^d$ it holds

$$\varphi_t(x,\cdot) \text{ is radially simmetric;} \tag{2.7a}$$

$$\operatorname{supp} \varphi_t(x,\cdot) \subset B_1(0); \tag{2.7b}$$

$$\int_{\mathbb{R}^d} \varphi_t(x,z)\, dz = 1. \tag{2.7c}$$

Furthermore, we denote by $\frac{\partial}{\partial x_i}$ the operator of partial derivative w.r.t. $x_i$ and by $\frac{\partial}{\partial z_i}$ the operator of partial derivative w.r.t. $z_i$. The vector $\nabla_1 \varphi_t \in \mathbb{R}^d$ will be the gradient of $\varphi_t$ in the first set of variables $(x)$ and $\nabla_2 \varphi_t \in \mathbb{R}^d$ the gradient of $\varphi_t$ in the second set of variables $(z)$. As usual we write

$$\varphi_t^\delta(x,z) := \frac{1}{\delta^d} \varphi_t\left(x, \frac{z}{\delta}\right)$$

and, if $f_t \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a function of $x$

$$f_t^\delta(x) := \big(f_t * \varphi_t^\delta\big)(x) = \frac{1}{\delta^d} \int_{\mathbb{R}^d} f_t(y) \varphi_t\left(x, \frac{x-y}{\delta}\right) dy.$$

Recall that, by (2.3), the commutator is

$$r^{\varphi,\delta} := \boldsymbol{b} \cdot \nabla u^\delta - (\boldsymbol{b} \cdot \nabla u) * \varphi^\delta.$$

We are now ready to state the Theorem:

THEOREM 2.4. *Let $\boldsymbol{b} \in L^1_{\mathrm{loc}}(I; \mathrm{BV}(\mathbb{R}^d; \mathbb{R}^d))$ a compactly supported vector field with $\operatorname{div} \boldsymbol{b}_t = 0$ in the sense of distributions and let $u \in L^\infty(I \times \mathbb{R}^d)$. Then for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and a smooth kernel $\varphi \in C_c^\infty(I \times \mathbb{R}^d \times \mathbb{R}^d)$ satisfying (2.7) such that for any $\delta' \in (0,\delta)$ it holds*

$$\|r^{\delta',\varphi}\|_{L^1(I \times \mathbb{R}^d)} < \varepsilon$$

*where $r^{\delta',\varphi}$ is the commutator defined as in (2.3).*

Before entering into the proof of Theorem 2.4, we collect some preliminaries. To begin, we recall the following standard

LEMMA 2.5. *Let $\mu \in \mathscr{M}_b(\mathbb{R}; \mathbb{R}^m)$. Define the function*

$$\mu_\varepsilon(t) := \frac{\mu\big((t, t+\varepsilon]\big)}{\varepsilon}, \qquad t \in \mathbb{R}.$$

*Then it holds*

$$\int_K |\mu_\varepsilon(t)|\, dt \le |\mu|(K_\varepsilon) \qquad \text{for any compact set } K \subset \mathbb{R},$$

*where $K_\varepsilon$ is the open $\varepsilon$ neighbourhood of $K$. Furthermore, if $\mu \ll \mathscr{L}^1$ then $\mu_\varepsilon$ converges, as $\varepsilon \to 0$, strongly in $L^1_{\mathrm{loc}}$ to the density of $\mu$ w.r.t. $\mathscr{L}^1$.*

PROOF. For the first point, we have

$$\int_K |\mu_\varepsilon(t)|\, dt \le \frac{1}{\varepsilon} \int_K \big|\mu\big((t, t+\varepsilon]\big)\big|\, dt \le \frac{1}{\varepsilon} \int_{K+\varepsilon} d|\mu|(s) \varepsilon$$

and the conclusion follows. For the second point it is enough to notice that $\mu_\varepsilon = \mu * \mathbf{1}_{(0,\varepsilon]}$ and use standard results about convolutions. □

**2.3.2. Estimates and regularity of the anisotropic energy.** The following quantity will be involved in our computations.

Definition 2.6 (Anisotropic energy of a kernel). For any $d \times d$ matrix $M$ and any smooth function $\varphi \in C_c^\infty(\mathbb{R}^d)$ we define the *anisotropic* energy (in direction $M$) as

$$\Lambda(M,\varphi) := \int_{\mathbb{R}^d} \big|\langle Mz, \nabla\varphi(z)\rangle\big|\, dz. \tag{2.8}$$

We first show some additional properties enjoyed by the anisotropic energy defined in (2.8). The first result is the following estimate, due to Alberti. Set

$$\mathcal{K} := \left\{ \varphi \in C_c^\infty(\mathbb{R}^d) :\, \varphi \geq 0,\, \varphi \text{ even },\, \int_{\mathbb{R}^d} \varphi = 1 \right\}. \tag{2.9}$$

Then we have

Lemma 2.7 (Alberti). *It holds*

$$\inf_{\varphi\in\mathcal{K}} \Lambda(M,\varphi) = |\operatorname{tr} M|.$$

For the sake of completeness, we give also a proof of Lemma 2.7.

Proof. The geometric idea behind this Lemma is that the minimizing kernel $\varphi$ must be chosen in such a way that its gradient is as much orthogonal as possible to the vector field $Mz$ or, equivalently, its level sets are as much tangential as possible to $Mz$. Notice first that the lower bound is immediate

$$\begin{aligned}
\int_{\mathbb{R}^d} |\langle Mz, \nabla\varphi\rangle|\, dz &\geq \left| \int_{\mathbb{R}^d} \langle Mz, \nabla\varphi\rangle\, dz \right| \\
&= \left| \sum_{i,j} M_{ji} \int_{\mathbb{R}^d} z_j \frac{\partial\varphi}{\partial z_i} dz \right| \\
&= |\operatorname{tr} M|
\end{aligned}$$

where we have used once again (2.5). To show the upper bound, we take advantage of the identity

$$\langle Mz, \nabla\varphi\rangle = \operatorname{div}(Mz\varphi(z)) - \operatorname{tr} M\varphi(z)$$

so that it is enough to show that for any $T > 0$ there exists a kernel $\varphi$ such that

$$\int_{\mathbb{R}^d} |\operatorname{div}(Mz\varphi(z))|\, dz \leq \frac{1}{T}.$$

Given a smooth nonnegative convolution kernel $\psi$ with compact support, we consider the following function:

$$\varphi(z) := \fint_0^T \psi(e^{-tM}\cdot z)e^{-t\operatorname{tr} M}\, dt$$

where $e^{tM}$ is defined as usual by

$$\sum_{i=0}^\infty M^i \frac{t^i}{i!}$$

so that $e^{tM}\cdot z$ is the solution of the initial value problem

$$\begin{cases} \dot\gamma = M\cdot\gamma \\ \gamma(0) = z. \end{cases}$$

Thanks to the change of variable formula, we have for any integrable bounded function $\theta$ it holds

$$\begin{aligned}\int \varphi(z)\theta(z)\,dz &= \frac{1}{T}\int_0^T\int \theta(z)\psi(e^{-tM}\cdot z)e^{-t\,\mathrm{tr}\,M}\,dz\,dt\\ &= \frac{1}{T}\int_0^T\int \theta(e^{tM}w)\psi(w)\,dw\,dt.\end{aligned}$$

This says that the measure $\varphi\mathscr{L}^d$ coincides with the time average of the pushforward of the measure $\psi\mathscr{L}^d$ along the integral curves of $Mz$. It is therefore reasonable to hope that the gradient of $\varphi$ is very orthogonal to the field $Mz$. An easy computation shows indeed that

$$\mathrm{div}(M\cdot z\psi(e^{-tM}\cdot z))e^{-t\,\mathrm{tr}\,M} = -\frac{d}{dt}(\psi(e^{-tM}\cdot z)e^{-t\,\mathrm{tr}\,M})$$

from which we get

$$\begin{aligned}\int_{\mathbb{R}^d}|\,\mathrm{div}(Mz\varphi(z))|\,dz &= \int_{\mathbb{R}^d}\frac{1}{T}\left|\int_0^T \mathrm{div}(M\cdot z\psi(e^{-tM}\cdot z))e^{-t\,\mathrm{tr}\,M}\,dt\right|dz\\ &= \int_{\mathbb{R}^d}\frac{1}{T}\left|\int_0^T \frac{d}{dt}(\psi(e^{-tM}\cdot z)e^{-t\,\mathrm{tr}\,M})dt\right|dz\\ &= \int_{\mathbb{R}^d}\frac{1}{T}\left|\psi(e^{-TM}\cdot z)e^{-T\,\mathrm{tr}\,M}-\psi(z)\right|dz\\ &\le \frac{2}{T},\end{aligned}$$

where we have used again the change of variables and the fact the $\int_{\mathbb{R}^d}\psi\,dz = 1$. This concludes the proof. □

We now turn our attention to the regularity of the energy $\Lambda$ w.r.t. $M$ and $\varphi$: we will show that, on a suitably chosen compact set, the infimum of Alberti's Lemma is attained. Recall the set $\mathcal{K}$ defined in (2.9): if $\varphi_1,\varphi_2\in\mathcal{K}$ we have for any $M_1, M_2\in\mathbb{R}^{d\times d}$

$$\left|\Lambda(M_1,\varphi_1)-\Lambda(M_2,\varphi_2)\right| \le \omega_d|M_1-M_2|\|\nabla\varphi_1\| + \int_{\mathbb{R}^d}|z|\left|\nabla(\varphi_1-\varphi_2)\right|dz. \tag{2.10}$$

Now let $R>0$ be a fixed (large enough) real number and set

$$\mathcal{K}_R = \{\varphi\in\mathcal{K} : \|\nabla\varphi\|_\infty \le R\},$$

which is easily seen to be a compact set in the uniform topology thanks to Ascoli-Arzelà Theorem. From (2.10) it follows that for every fixed $\varphi\in\mathcal{K}_R$ the function $\Lambda(\cdot,\varphi)$ is uniformly Lipschitz. On the other hand, using standard facts in the Calculus of Variations (see, e.g., [Dac12, Thm. 3.22 and Remark 3.25ii]) it is easy to prove that for every fixed $M$, the map $\Lambda(M,\cdot)$ is lower semicontinuous with respect to the strong $C^0$ topology in $\mathcal{K}_R$. Indeed, let $\varphi_n\to\varphi$ uniformly with $\|\nabla\varphi_n\|_\infty\le R$ for every $n$: then, by Banach-Alaoglu, there exists a subsequence such that $\nabla\varphi_n \rightharpoonup^* \nabla\varphi$, in the $w^\star$ topology of $L^\infty$. Thus we have that $\varphi_n\rightharpoonup\varphi$ in $w^\star - W^{1,\infty}$ and we are in position now to apply Theorem 3.22 of [Dac12]. Hence, for fixed $M$, $\inf_{\mathcal{K}_R}\Lambda(M,\cdot)$ is attained and the function

$$M\mapsto \inf_{\mathcal{K}_R}\Lambda(M,\cdot)$$

is $R$-Lipschitz (being the infimum of Lipschitz functions). Notice that the family of functions $\inf_{\mathcal{K}_R}\Lambda(M,\cdot)$ is decreasing in $R$; furthermore, by Alberti's Lemma 2.7, for every fixed $M$ with $\mathrm{tr}\,M=0$ it holds

$$\inf_{\mathcal{K}_R}\Lambda(M,\cdot)\to 0, \qquad \text{as } R\to+\infty. \tag{2.11}$$

By Dini's Theorem, it follows that the convergence in (2.11) is uniform, so that

$$\sigma(R) := \sup_{\substack{\|M\|\le 1 \\ \operatorname{tr} M = 0}} \inf_{\varphi\in\mathcal{K}_R} \Lambda(M,\varphi) \to 0 \qquad \text{as } R \to +\infty. \tag{2.12}$$

**2.3.3. Explicit form of the commutator for point-depending convolution kernels.** We now show how the commutator can be written in a more explicit form when the convolution kernel depends also on the point, i.e. when we convolve with a function $\varphi$ satisfying (2.7). By the very definition of convolution, we have

$$\boldsymbol{b}\cdot\nabla u^\delta = \boldsymbol{b}\cdot\nabla(u*\varphi^\delta) = \sum_i b_i \frac{\partial}{\partial x_i}(u*\varphi^\delta)$$

hence we can write

$$\begin{aligned}
(\boldsymbol{b}\cdot\nabla u^\delta)(t,x) &= \frac{1}{\delta^d}\sum_{i=1}^d b_t^i(x)\frac{\partial}{\partial x_i}\int_{\mathbb{R}^d} u_t(y)\varphi\left(x,\frac{x-y}{\delta}\right)dy \\
&= \frac{1}{\delta^d}\sum_{i=1}^d b_t^i(x)\int_{\mathbb{R}^d} u_t(y)\frac{\partial\varphi}{\partial x_i}\left(x,\frac{x-y}{\delta}\right)dy \\
&= \frac{1}{\delta^d}\boldsymbol{b}_t(x)\int_{\mathbb{R}^d} u_t(y)\left[\nabla_1\varphi\left(x,\frac{x-y}{\delta}\right)+\frac{\nabla_2\varphi}{\delta}\left(x,\frac{x-y}{\delta}\right)\right]dy.
\end{aligned} \tag{2.13}$$

For the second term, instead we have

$$(\boldsymbol{b}\cdot\nabla u)*\varphi^\delta = \operatorname{div}(u\boldsymbol{b})*\varphi^\delta = \sum_{i=1}^d \frac{\partial}{\partial x_i}(ub^i)*\varphi^\delta = \sum_{i=1}^d ub^i * \frac{\partial\varphi^\delta}{\partial x_i}$$

from which it follows

$$\begin{aligned}
\big((\boldsymbol{b}\cdot\nabla u)*\varphi^\delta\big)(t,x) &= \frac{1}{\delta^d}\int_{\mathbb{R}^d}\sum_{i=1}^d u_t(y)b_t^i(y)\frac{\partial\varphi}{\partial x_i}\left(x,\frac{x-y}{\delta}\right)dy \\
&= \frac{1}{\delta^d}\sum_{i=1}^d\int_{\mathbb{R}^d} u_t(y)b_t^i(y)\left[\frac{\partial\varphi}{\partial x_i}\left(x,\frac{x-y}{\delta}\right)+\frac{1}{\delta}\frac{\partial\varphi}{\partial z_i}\left(x,\frac{x-y}{\delta}\right)\right]dy \\
&= \frac{1}{\delta^d}\int_{\mathbb{R}^d} u_t(y)\boldsymbol{b}_t(y)\cdot\left[\nabla_1\varphi\left(x,\frac{x-y}{\delta}\right)+\frac{1}{\delta}\nabla_2\varphi\left(x,\frac{x-y}{\delta}\right)\right]dy.
\end{aligned} \tag{2.14}$$

In conclusion, summing up all terms and using (2.13) and (2.14), we get:

$$
\begin{aligned}
r^{\varphi,\delta}(t,x) &= \frac{1}{\delta^d}\int_{\mathbb{R}^d} \boldsymbol{b}_t(x)u_t(y)\cdot\left[\nabla_1\varphi\left(x,\frac{x-y}{\delta}\right)+\frac{\nabla_2\varphi}{\delta}\left(x,\frac{x-y}{\delta}\right)\right]dy \\
&\quad -\frac{1}{\delta^d}\int_{\mathbb{R}^d} \boldsymbol{b}_t(y)u_t(y)\cdot\left[\nabla_1\varphi\left(x,\frac{x-y}{\delta}\right)+\frac{\nabla_2\varphi}{\delta}\left(x,\frac{x-y}{\delta}\right)\right]dy \\
&= \int_{\mathbb{R}^d} \boldsymbol{b}_t(x)u_t(x+\delta z)\cdot\left[\nabla_1\varphi\left(x,-z\right)+\frac{\nabla_2\varphi}{\delta}\left(x,-z\right)\right]dz \\
&\quad -\int_{\mathbb{R}^d} \boldsymbol{b}_t(x+\delta z)u_t(x+\delta z)\cdot\left[\nabla_1\varphi\left(x,-z\right)+\frac{1}{\delta}\nabla_2\varphi\left(x,-z\right)\right]dz \\
&= \int_{\mathbb{R}^d} \boldsymbol{b}_t(x)u_t(x+\delta z)\cdot\left[\nabla_1\varphi\left(x,z\right)-\frac{\nabla_2\varphi}{\delta}\left(x,z\right)\right]dz \\
&\quad -\int_{\mathbb{R}^d} u_t(x+\delta z)\boldsymbol{b}_t(x+\delta z)\cdot\left[\nabla_1\varphi\left(x,z\right)-\frac{1}{\delta}\nabla_2\varphi\left(x,z\right)\right]dz \\
&= \int_{\mathbb{R}^d} u_t(x+\delta z)\left[\boldsymbol{b}_t(x)-\boldsymbol{b}_t(x+\delta z)\right]\cdot\nabla_1\varphi\left(x,z\right)\,dz \\
&\quad +\int_{\mathbb{R}^d} u_t(x+\delta z)\left[\frac{\boldsymbol{b}_t(x+\delta z)-\boldsymbol{b}_t(x)}{\delta}\right]\cdot\nabla_2\varphi(x,z)\,dz.
\end{aligned}
$$

We have thus shown

$$
r^{\varphi,\delta} = r_1^{\varphi,\delta} + r_2^{\varphi,\delta} \tag{2.15}
$$

where we have set

$$
r_1^{\varphi,\delta}(t,x) := \int_{\mathbb{R}^d} u_t(x+\delta z)\big(\boldsymbol{b}_t(x)-\boldsymbol{b}_t(x+\delta z)\big)\cdot\nabla_1\varphi\left(x,z\right)\,dz \tag{2.16}
$$

and

$$
r_2^{\varphi,\delta}(t,x) := \int_{\mathbb{R}^d} u_t(x+\delta z)\left[\frac{\boldsymbol{b}_t(x+\delta z)-\boldsymbol{b}_t(x)}{\delta}\right]\cdot\nabla_2\varphi(x,z)\,dz. \tag{2.17}
$$

We are now going to estimate the $L^1$ norm of the quantities defined in (2.16) and (2.17).

**2.3.4. Estimate for the term (2.16).** We begin by estimating the term $r_1^{\varphi,\delta}$. Define the function $\omega\colon[0,1]\to\mathbb{R}$ as

$$
\omega(r) := \int_{B_r}\int_{I\times\mathbb{R}^d}\big|\boldsymbol{b}_t(x+y)-\boldsymbol{b}_t(x)\big|\,dt\,dx\,dy. \tag{2.18}
$$

The function $\omega$ allows to control the $L^1$ norm of $r_1^{\varphi,\delta}$, defined in (2.16). Applying Fubini and recalling the convolution kernel has support in the unit ball, we have

$$
\begin{aligned}
\|r_1^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} &= \int_{I\times\mathbb{R}^d}\left|\int_{\mathbb{R}^d} u_t(x+\delta z)\big(\boldsymbol{b}_t(x)-\boldsymbol{b}_t(x+\delta z)\big)\cdot\nabla_1\varphi\left(x,z\right)\,dz\right|\,dt\,dx \\
&\le \|u\|_\infty\int_{\mathbb{R}^d}\int_{I\times\mathbb{R}^d}|\boldsymbol{b}_t(x)-\boldsymbol{b}_t(x+\delta z)||\nabla_1\varphi\left(x,z\right)|\,dt\,dx\,dz \\
&\le \|u\|_\infty\int_{B_\delta}\int_{I\times\mathbb{R}^d}|\boldsymbol{b}_t(x)-\boldsymbol{b}_t(x+\delta z)||\nabla_1\varphi\left(x,z\right)|\,dt\,dx\,dz \\
&\le \|u\|_\infty\|\nabla_1\varphi\|_\infty\int_{B_\delta}\int_{I\times\mathbb{R}^d}|\boldsymbol{b}_t(x)-\boldsymbol{b}_t(x+\delta z)|\,dt\,dx\,dz \\
&\le \|u\|_\infty\|\nabla_1\varphi\|_\infty\omega(\delta).
\end{aligned} \tag{2.19}
$$

**2.3.5. Estimate for the term (2.17).** The term $r_2^{\varphi,\delta}$ is more delicate to estimate and it requires a preliminary study. Indeed, as already pointed out, for the proof of Theorem 2.4, we have to introduce a covering of the space $I \times \mathbb{R}^d$ made up of suitably chosen balls (which will allow to patch the local minimizers found thanks to Section 2.3.2). We first show how to find these balls: the idea is that the matrix $M_t$ is almost constant within them (in an $L^1$ sense, w.r.t. $D\boldsymbol{b}$). After that, we will present an estimate of the difference quotients of $\boldsymbol{b}_t$ in the spirit of Ambrosio's original proof.

*The cover and its properties.* Recall that the matrix $M$ comes from the polar decomposition of the measure $D\boldsymbol{b}_t$ and is defined in (2.6). By definition, for any Lebesgue point of $M$ w.r.t. $|D\boldsymbol{b}|$ we have that

$$\lim_{r\to 0} \frac{1}{|D\boldsymbol{b}|(B_r(t,x))} \int_{B_r(t,x)} \big|M_s(y) - M_t(x)\big|\, d|D\boldsymbol{b}|(s,y) = 0.$$

Let now $N \subset \mathbb{R}^+$ the set of radii $r$ such that $|D\boldsymbol{b}_t|(\partial B_r) > 0$: being $\partial B_s \cap \partial B_r = \emptyset$ for $s \neq r$, the set $N$ turns to be negligible (actually, it is at most countable). In particular, the map $r \mapsto |D\boldsymbol{b}|(B_r(t,x))$ is continuous at every $r \in \mathbb{R}^+ \setminus N$, hence for every $\varepsilon > 0$ there exists $\eta_r > 0$ such that

$$|D\boldsymbol{b}|\big(B_{(1-\eta_r)r}(t,x)\big) \geq (1-\varepsilon)|D\boldsymbol{b}|\big(B_{r+\eta_r}(t,x)\big).$$

Introduce now the following cover of $I \times \mathbb{R}^d$:

$$\mathscr{F}^\varepsilon := \left\{ \overline{B}_\rho(t,x) : (t,x) \in I \times \mathbb{R}^d, 0 < \rho \leq \varepsilon, \fint_{B_\rho(t,x)} \big|M_s(y) - M_t(x)\big|\, d|D\boldsymbol{b}|(s,y) < \varepsilon \right\},$$

which is easily seen to be a fine cover. By Vitali Covering Theorem [**AFP00**] (with $\mu = D\boldsymbol{b}$), there exists a (countable) disjoint subcover $\mathscr{F}' \subset \mathscr{F}$ such that

$$|D\boldsymbol{b}|\left( (I \times \mathbb{R}^d) \setminus \bigcup_{\mathscr{F}'} B \right) < \varepsilon. \tag{2.20}$$

Let us write the balls in $\mathscr{F}'$ as $B_i := B_{r_i}(t_i, x_i)$; up to restricting a bit $B_i$, we can assume $r_i \notin N$ for all $i \in \mathbb{N}$. In particular, by countable additivity, we have

$$\sum_i \int_{B_i} \big|M_t(x) - M_{t_i}(x_i)\big|\, d|D\boldsymbol{b}|(t,x) \leq \varepsilon |D\boldsymbol{b}|(I \times \mathbb{R}^d).$$

Being $r_i \notin N$ for every $i \in \mathbb{N}$, we can find $\eta_i > 0$ be such that

$$|D\boldsymbol{b}|\big(B_{(1-\eta_i)r_i}(t_i,x_i)\big) \geq (1-\varepsilon)|D\boldsymbol{b}|\big(B_{r_i+\eta_i}(t_i,x_i)\big)$$

so that

$$\sum_{i\in\mathbb{N}} |D\boldsymbol{b}|\big((B_{(1+\eta_i)r_i}(t_i,x_i)) \setminus B_{r_i+\eta_i}(t_i,x_i)\big) \leq \varepsilon |D\boldsymbol{b}|(I \times \mathbb{R}^d).$$

*Estimates inside $B_i$.* We now prove that up to a small term ($\|M - M_i\|_{L^1}$) the difference quotients of $\boldsymbol{b}$ in direction $z$ are close (inside the ball $B_i$) to a constant vector, whose direction is given by the matrix $M_i$. Without loss of generality we can assume $|z| = 1$ and, up to rotations, $z = \boldsymbol{e}_1$. Let also $t \in I$ be fixed. Let

$$\begin{aligned} \pi \colon \mathbb{R}^d &\longrightarrow \mathbb{R}^{d-1} \\ x = (x_1, x') &\mapsto x'. \end{aligned}$$

Consider now the measure $\mu_t := \pi_\sharp |D\boldsymbol{b}_t|$ and consider the disintegration

$$|D\boldsymbol{b}_t| = \int_{\mathbb{R}^{d-1}} \sigma_{x'}(ds)\, d\mu_t(x'). \tag{2.21}$$

Observe that clearly $D_1\boldsymbol{b}_t := D\boldsymbol{b}_t \cdot \boldsymbol{e}_1 = M \cdot \boldsymbol{e}_1 |D\boldsymbol{b}_t|$ and, by slicing theory of BV functions (see [**AFP00**, §3.11]) we have $D_1\boldsymbol{b}_t = \partial_1(\boldsymbol{b}_t)_{x'} \otimes \mathscr{L}^{d-1}(dx')$. Recall that if $(\zeta_h)_{h\in\mathbb{R}}$ is a (measurable) family of measures on the space $X$, we can define a new measure on the space $X$ by setting

$$\zeta := \int_{\mathbb{R}} \zeta_h \, dh \ \text{ which means } \ \zeta(A) = \int_{\mathbb{R}} \zeta_h(A) \, dh$$

for every measurable set $A \subset X$. Now let $K \subset \mathbb{R}$ be a compact set and let $A \subset \mathbb{R}^{d-1}$ be a measurable set. Then we can estimate

$$\begin{aligned}
&\left| \int_K \left[ \int_A \frac{\boldsymbol{b}_t(x+\delta \boldsymbol{e}_1) - \boldsymbol{b}_t(x)}{\delta} \, d\mathscr{L}^{d-1}(x') - \int_A \frac{M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big)}{\delta} d\mu_t(x') \right] dx_1 \right| \\
=& \frac{1}{\delta} \left| \int_K \left[ \int_A \partial_1(\boldsymbol{b}_t)_{x'}\big((x_1, x_1+\delta]\big) \, d\mathscr{L}^{d-1}(x') - \int_A M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big) d\mu_t(x') \right] dx_1 \right| \\
=& \frac{1}{\delta} \left| \int_K \left[ \int_A \int_{x_1}^{x_1+\delta} d\Big(\partial_1(\boldsymbol{b}_t)_{x'}\Big)(\ell) \, d\mathscr{L}^{d-1}(x') - \int_A M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big) d\mu_t(x') \right] dx_1 \right| \\
=& \frac{1}{\delta} \left| \int_K \left[ \int_{(x_1, x_1+\delta] \times A} d\Big(\partial_1(\boldsymbol{b}_t)_{x'} \otimes \mathscr{L}^{d-1}\Big)(\ell, x') - \int_A M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big) d\mu_t(x') \right] dx_1 \right| \\
=& \frac{1}{\delta} \left| \int_K \left[ \int_{(x_1, x_1+\delta] \times A} d\Big(D_1 \boldsymbol{b}_t\Big)(\ell, x') - \int_A M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big) d\mu_t(x') \right] dx_1 \right|
\end{aligned}$$

and, using Fubini and Lemma 2.5 (recall that $K^\delta$ denotes the $\delta$-neighbourhood of $K$), we can further estimate

$$\begin{aligned}
&\frac{1}{\delta} \left| \int_K \left[ \int_{(x_1, x_1+\delta] \times A} d\Big(M_t \cdot \boldsymbol{e}_1 |D\boldsymbol{b}_t|\Big)(\ell, x') - \int_A M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big) d\mu_t(x') \right] dx_1 \right| \\
=& \frac{1}{\delta} \left| \int_K \left[ \int_A M_t(x) \cdot \boldsymbol{e}_1 \sigma_{x'}\big((x_1, x_1+\delta]\big) \, d\mu_t(x') - \int_A M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big) \, d\mu_t(x') \right] dx_1 \right| \\
=& \left| \int_K \int_A \Big( M_t(x) - M_i \Big) \cdot \boldsymbol{e}_1 \frac{\sigma_{x'}\big((x_1, x_1+\delta]\big)}{\delta} \, d\mu_t(x') \, dx_1 \right| \\
\le& \int_A \big| (M_t(x) - M_i) \cdot \boldsymbol{e}_1 \big| \sigma_{x'}\big(K^\delta\big) \, d\mu_t(x')
\end{aligned}$$

By recalling the disintegration (2.21) we deduce

$$\begin{aligned}
&\left| \int_K \left[ \int_A \frac{\boldsymbol{b}_t(x+\delta \boldsymbol{e}_1) - \boldsymbol{b}_t(x)}{\delta} \, d\mathscr{L}^{d-1}(x') - \int_A \frac{M_i \cdot \boldsymbol{e}_1 \, \sigma_{x'}\big((x_1, x_1+\delta]\big)}{\delta} d\mu_t(x') \right] dx_1 \right| \\
&\le \int_{(K \times A)^\delta} |(M_t(x) - M_i) \cdot \boldsymbol{e}_1| \, d|D\boldsymbol{b}_t|(x).
\end{aligned} \tag{2.22}$$

*Estimate for the second term in* (2.15). We are now ready to estimate the $L^1$ norm of $r_2^{\varphi,\delta}$, as defined by (2.17). By splitting the integral into the sum of integrals over the balls

$B_i$ of the cover we have

$$
\begin{aligned}
\|r_2^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} &= \left\| \int_{\mathbb{R}^d} u_t(x+\delta z)\left[\frac{\boldsymbol{b}_t(x+\delta z)-\boldsymbol{b}_t(x)}{\delta}\right]\cdot\nabla_2\varphi(x,z)\,dz\right\|_{L^1(I\times\mathbb{R}^d)} \\
&\le \|u\|_\infty \int_{I\times\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\frac{\boldsymbol{b}_t(x+\delta z)-\boldsymbol{b}_t(x)}{\delta}\cdot\nabla_2\varphi(x,z)dz\right|dt\,dx \\
&\le \|u\|_\infty \sum_{i=1}^J\int_{B_i}\left|\int_{\mathbb{R}^d}\frac{\boldsymbol{b}_t(x+\delta z)-\boldsymbol{b}_t(x)}{\delta}\cdot\nabla_2\varphi(x,z)dz\right|dt\,dx \\
&\quad + \|u\|_\infty\|\nabla_2\varphi\|_\infty|D\boldsymbol{b}|\left(\bigcup_{i\ge J}B_i+B(0,\delta)\right)
\end{aligned}
$$

and now using Fubini and adding and subtracting we obtain

$$
\begin{aligned}
\|r_2^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} \le \|u\|_\infty \sum_{i=1}^I\int_{\mathbb{R}^d}\Bigg|\int_{B_i}&\frac{\boldsymbol{b}_t(x+\delta z)-\boldsymbol{b}_t(x)}{\delta}\cdot\nabla_2\varphi(x,z)\,dx\,dt \\
&-\int_{B_i}\langle M_i z,\nabla_2\varphi(x,z)\rangle\frac{\sigma_{x'}\big((x_z,x_z+\delta z]\big)}{\delta}\,d\mu_t(x')\,dx_1\,dt\Bigg|\,dz \\
+\|u\|_\infty\sum_{i=1}^J\int_{\mathbb{R}^d}&\left|\int_{B_i}\langle M_i z,\nabla_2\varphi(x,z)\rangle\frac{\sigma_{x'}\big((x_z,x_z+\delta z]\big)}{\delta}\,d\mu_t(x')\,dx_1\,dt\right|dz \\
+\|u\|_\infty\|\nabla_2\varphi\|_\infty&|D\boldsymbol{b}|\left(\bigcup_{i\ge J}B_i+B(0,\delta)\right).
\end{aligned}
$$

Thanks to the estimate (2.22) (integrated in time) and applying again Fubini and Lemma 2.5 for the term with $\sigma_{x'}$ we can write

$$
\begin{aligned}
&\|r_2^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} \\
\le&\|u\|_\infty\sum_{i=1}^J\int_{\mathbb{R}^d}\int_{B_{r_i+\delta}(t_i,x_i)}\Big(\|\nabla_2\varphi\|_\infty|M(x)-M_i|+\langle M_i z,\nabla_2\varphi(x,z)\rangle\Big)\,d|D\boldsymbol{b}|(t,x)\,dz \\
&+\|u\|_\infty\|\nabla_2\varphi\|_\infty|D\boldsymbol{b}|\left(\bigcup_{i\ge J}B_i+B(0,\delta)\right) \\
\le&\|u\|_\infty\|\nabla_2\varphi\|_\infty\sum_{i=1}^J\int_{\mathbb{R}^d}\int_{B_i}|M(x)-M_i|\,d|D\boldsymbol{b}|(t,x)\,dz \\
&+\|u\|_\infty\sum_{i=1}^J\int_{\mathbb{R}^d}\int_{B_{(1-\delta)r_i}(t_i,x_i)}|\langle M_i z,\nabla_2\varphi(x,z)\rangle|\,d|D\boldsymbol{b}|(t,x)\,dz \\
&+\|u\|_\infty\sum_{i=1}^J\int_{\mathbb{R}^d}\int_{B_{r_i+\delta}(t_i,x_i)\setminus B_{(1-\delta)r_i}(t_i,x_i)}(\|\nabla_2\varphi\|_\infty|M(x)-M_i|+\langle M_i z,\nabla_2\varphi(x,z)\rangle)\,d|D\boldsymbol{b}|(t,x)\,dz \\
&+\|u\|_\infty\|\nabla_2\varphi\|_\infty|D\boldsymbol{b}|\left(\bigcup_{i\ge J}B_i+B(0,\delta)\right).
\end{aligned}
$$

and, taking into account that $\|M\|_\infty \le 1$ and that $\varphi_2$ is supported on $B_1$, we finally deduce

$$\begin{aligned}\|r_2^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} \le \|u\|_\infty\|\nabla_2\varphi\|_\infty \sum_{i=1}^J \int_{\mathbb{R}^d}\int_{B_i} |M(x)-M_i|\, d|D\boldsymbol{b}|(t,x)\, dz \\ + \|u\|_\infty \sum_{i=1}^J \int_{\mathbb{R}^d}\int_{B_{(1-\delta)r_i}(t_i,x_i)} |\langle M_i z, \nabla_2\varphi(x,z)\rangle|\, d|D\boldsymbol{b}|(t,x)\, dz \\ + 3\|u\|_\infty\|\nabla_2\varphi\|_\infty \sum_{i=1}^J |D\boldsymbol{b}|\Big(B_{r_i+\delta}(t_i,x_i)\setminus B_{(1-\delta)r_i}(t_i,x_i)\Big) \\ + \|u\|_\infty\|\nabla_2\varphi\|_\infty |D\boldsymbol{b}|\Big(\bigcup_{i\ge J} B_i + B(0,\delta)\Big).\end{aligned} \tag{2.23}$$

**2.3.6. Conclusion of the argument: bounds in $L^1$.** We are eventually ready to prove Theorem 2.4.

Proof. By combining estimates (2.19) and (2.23), we obtain that for every smooth kernel $\varphi$ satisfying (2.7) it holds

$$\begin{aligned}\|r^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} &\le \|r_1^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} + \|r_2^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} \\ &\le \|u\|_\infty\|\nabla_1\varphi\|_\infty\omega(\delta) \\ &\quad + \|u\|_\infty\|\nabla_2\varphi\|_\infty \sum_{i=1}^J \int_{\mathbb{R}^d}\int_{B_i} |M(x)-M_i|\, d|D\boldsymbol{b}|(t,x)\, dz \\ &\quad + \|u\|_\infty \sum_{i=1}^J \int_{\mathbb{R}^d}\int_{B_{(1-\delta)r_i}(t_i,x_i)} |\langle M_i z, \nabla_2\varphi(x,z)\rangle|\; d|D\boldsymbol{b}|(t,x)\, dz \\ &\quad + 3\|u\|_\infty\|\nabla_2\varphi\|_\infty \sum_{i=1}^J |D\boldsymbol{b}|\Big(B_{r_i+\delta}(t_i,x_i)\setminus B_{(1-\delta)r_i}(t_i,x_i)\Big) \\ &\quad + \|u\|_\infty\|\nabla_2\varphi\|_\infty |D\boldsymbol{b}|\Big(\bigcup_{i\ge J} B_i + B(0,\delta)\Big).\end{aligned} \tag{2.24}$$

We are now ready to choose parameters to have the desired bound in $L^1$. Let $\varepsilon > 0$ be fixed.

*Step 1.* Thanks to (2.12), we can find $R_\varepsilon > 0$ such that

$$\max_{\|M\|_\infty\le 1}\inf_{\varphi\in\mathcal{K}_{R_\varepsilon}} \Lambda(M,\varphi) \le \frac{\varepsilon}{5\|u\|_\infty|D\boldsymbol{b}|(I\times\mathbb{R}^d)}.$$

*Step 2.* Now we apply (2.20), choosing a covering made up of balls $B_i := B_{r_i}(t_i,x_i)$ such that $r_i \notin N$ and

$$\sum_i \int_{B_i} |M_t - M_i||D\boldsymbol{b}|(t,x) \le \frac{\varepsilon}{5R_\varepsilon\|u\|_\infty} \tag{2.25}$$

being $N$ the critical set of radii (the ones such that the spherical surface has positive $|D\boldsymbol{b}|$-measure). Given the covering above, being $\boldsymbol{b}\in L^1(\mathrm{BV})$, we can find $J_\varepsilon \gg 1$ such that

$$|D\boldsymbol{b}|\Big(\bigcup_{i\ge J_\varepsilon} B_i\Big) \le \frac{\varepsilon}{10\|u\|_\infty R_\varepsilon}.$$

In this way, being the family of radii $r_i$, with $i \le J_\varepsilon$, finite, we can assume that

$$\sum_{i=1}^{J_\varepsilon} |D\boldsymbol{b}|\Big(B_{r_i+\delta}(t_i,x_i) \setminus B_{(1-\delta)r_i}(t_i,x_i)\Big) \le \frac{\varepsilon}{15R_\varepsilon \|u\|_\infty}. \tag{2.26}$$

*Step 3.* By inner regularity, we can now pick a $\delta \ll 1$ such that

$$|D\boldsymbol{b}|\Big(\bigcup_{i\ge J_\varepsilon} B_i + B(0,\delta)\Big) \le \frac{\varepsilon}{5\|u\|_\infty R_\varepsilon}. \tag{2.27}$$

*Step 4.* Now let $\widehat{\varphi}_i$ be the minimizer of the functional $\Lambda(M_i,\cdot)$ over the compact $\mathcal{K}_{R_\varepsilon}$, whose existence is ensured by the discussion in paragraph 2.3.2. Define now $\mathcal{J}_\varepsilon := \{1,\dots,J_\varepsilon\}$ and the function

$$\varphi(t,x,z) := \begin{cases} \widehat{\varphi}_i(z) & \text{if } \exists i \in \mathcal{J}_\varepsilon : |(t,x)-(t_i,x_i)| \le (1-\delta)r_i, \\ \frac{|x|-(1-\delta)r_i}{\delta r_i}\widehat{\varphi}_i(z) + \frac{r_i-|x|}{\delta r_i}\varphi_{ext}(z) & \text{if } \exists i \in \mathcal{J}_\varepsilon : (1-\delta)r_i < |(t,x)-(t_i,x_i)| < r_i, \\ \psi_{ext}(z) & \text{otherwise} \end{cases} \tag{2.28}$$

where $\psi_{ext} \in \mathcal{K}$ is any smooth kernel. A direct computation shows that

$$\begin{aligned} \|\nabla_1 \varphi\|_\infty &\le \frac{1}{\delta \min_{i\in\mathcal{J}_\varepsilon} r_i}\big(\|\widehat{\varphi}_i\|_\infty + \|\varphi_2\|_\infty\big) \\ &\le \frac{C}{\delta \min_{i\in\mathcal{J}_\varepsilon} r_i}\big(\|\nabla_2\widehat{\varphi}_i\|_\infty + \|\nabla_2\varphi_2\|_\infty\big) \\ &\le \frac{2R_\varepsilon}{\delta \min_{i\in\mathcal{J}_\varepsilon} r_i}. \end{aligned}$$

Furthermore, being $\widehat{\varphi}_i$ the minimizer, it holds

$$\begin{aligned} &\sum_{i=1}^{J_\varepsilon} \int_{B_{(1-\delta)r_i}(t_i,x_i)} \Lambda(M_i,\varphi(x,z))\, d|D\boldsymbol{b}|(t,x) \\ =&\sum_{i=1}^{J_\varepsilon} \int_{B_{(1-\delta)r_i}(t_i,x_i)} \Lambda(M_i,\widehat{\varphi}_i(z))\, d|D\boldsymbol{b}|(t,x) \\ \le&\frac{\varepsilon}{5\|u\|_\infty |D\boldsymbol{b}|(I\times\mathbb{R}^d)} \sum_{i=1}^{J_\varepsilon} \int_{B_{(1-\delta)r_i}(t_i,x_i)} d|D\boldsymbol{b}|(t,x) \end{aligned}$$

hence

$$\sum_{i=1}^{J_\varepsilon} \int_{B_{(1-\delta)r_i}(t_i,x_i)} \Lambda(M_i,\varphi(x,z))\, d|D\boldsymbol{b}|(t,x) \le \frac{\varepsilon}{5\|u\|_\infty} \tag{2.29}$$

*Step 5.* Exploting the strong convergence of translations in $L^1$, there exists $\delta' < \delta$ sufficiently small such that

$$\omega(\delta') \le \frac{\delta \min_{i\in\mathcal{J}_\varepsilon} r_i}{10R_\varepsilon\|u\|_\infty}\varepsilon, \tag{2.30}$$

where $\omega$ is the function defined in (2.18). Recalling (2.24), taking $\varphi$ defined as in (2.28) and by redefining if necessary $\delta := \min\{\delta, \delta'\}$, we can finally write

$$
\begin{aligned}
\|r^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} &\le \|r_1^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} + \|r_2^{\varphi,\delta}\|_{L^1(I\times\mathbb{R}^d)} \\
&\le \|u\|_\infty \|\nabla_1\varphi\|_\infty \omega(\delta) \\
&\quad + \|u\|_\infty \|\nabla_2\varphi\|_\infty \sum_{i=1}^{J_\varepsilon} \int_{B_i} |M(x) - M_i| \, d|D\boldsymbol{b}|(t,x) \\
&\quad + \|u\|_\infty \sum_{i=1}^{J_\varepsilon} \int_{\mathbb{R}^d} \int_{B_{(1-\delta)r_i}(t_i,x_i)} |\langle M_i z, \nabla_2\varphi(x,z)\rangle| \, d|D\boldsymbol{b}|(t,x)\, dz \\
&\quad + 3\|u\|_\infty \|\nabla_2\varphi\|_\infty \sum_{i=1}^{J_\varepsilon} |D\boldsymbol{b}|\Big(B_{r_i+\delta}(t_i,x_i) \setminus B_{r_i-\delta}(t_i,x_i)\Big) \\
&\quad + \|u\|_\infty \|\nabla_2\varphi\|_\infty |D\boldsymbol{b}|\Big(\bigcup_{i\ge I} B_i + B(0,\delta)\Big) \\
\text{(2.30)}\ &\le \|u\|_\infty \frac{\delta \min_{i\in\mathcal{J}_\varepsilon} r_i}{10R_\varepsilon \|u\|_\infty} \varepsilon \frac{2R_\varepsilon}{\delta \min_{i\in\mathcal{J}_\varepsilon} r_i} \\
\text{(2.25)}\ &\quad + \|u\|_\infty R_\varepsilon \frac{\varepsilon}{5R_\varepsilon\|u\|_\infty} \\
\text{(2.29)}\ &\quad + \|u\|_\infty \frac{\varepsilon}{5\|u\|_\infty} \\
\text{(2.26)}\ &\quad + 3\|u\|_\infty R_\varepsilon \frac{\varepsilon}{15R_\varepsilon\|u\|_\infty} \\
\text{(2.27)}\ &\quad + \|u\|_\infty R_\varepsilon \frac{\varepsilon}{5\|u\|_\infty R_\varepsilon} \\
&\le \varepsilon.
\end{aligned}
$$

Hence we have shown that for every $\varepsilon > 0$ we can find $\delta > 0$ and construct a convolution kernel $\varphi$ (which depends on $\varepsilon$) such that for every $\delta' < \delta$ it holds $\|r^{\varphi,\delta'}\|_{L^1(I\times\mathbb{R}^d)} \le \varepsilon$, which is what we wanted. □

CHAPTER 3

# Lagrangian representations in linear transportation

ABSTRACT. In this chapter, we illustrate the concept of *Lagrangian representation* within linear transportation theory: this tool will play a significant role in the following chapters. We begin, in Section 3.1, by presenting a well-known theorem, the Ambrosio's Superposition Principle, which allows to represent non-negative (measure-valued) solutions to the continuity equation driven by some vector field $\boldsymbol{b}$ as a *superposition* of trajectories. Since it will be useful later on, we will give a proof of this theorem, using an analog decomposition result, which goes back to Smirnov [Smi94], formulated in terms of 1-dimensional normal currents. We will then show (Section 3.2) how the Superposition Principle allows to transfer well-posedness results from the PDE side to the ODE one, establishing a theory (existence, uniqueness, stability) of Regular Lagrangian Flows, as foreseen in Chapter 1. In the final part of the chapter, Section 3.3, we move to consider a more general class of vector fields, i.e. the *nearly incompressible* ones: we precise the definitions of weak solution and of renormalized solutions for the transport equation driven by such a field and we discuss their relevance in connection to Bressan's Compactness Conjecture, which is stated at the end of the chapter.

Before giving the precise definition of Lagrangian representation, which is the central topic of this chapter, we begin by presenting a theorem, due to Ambrosio [Amb04], which is known as *Superposition Principle*.

## 3.1. Ambrosio's Superposition Principle

Let us consider the continuity equation in the form

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0, \\ \mu_0 = \bar{\mu}, \end{cases} \tag{3.1}$$

where $[0,T] \ni t \mapsto \mu_t \in \mathscr{M}(\mathbb{R}^d)$ is a measurable, measure-valued function and $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded, Borel vector field. As usual, a solution to (3.1) is understood in distributional sense. We will consider the space of continuous curves $C([0,T];\mathbb{R}^d)$ equipped with the uniform norm; we will denote by $e_t\colon C([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ the evaluation map at time $t \in [0,T]$, i.e. $\gamma \mapsto \gamma(t)$. We have the following

THEOREM 3.1 (Ambrosio's Superposition Principle). *Let $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded, Borel vector field and let $[0,T] \ni t \mapsto \mu_t \in \mathscr{M}^+(\mathbb{R}^d)$ be a non-negative, locally finite, measure-valued solution of the continuity equation (3.1). Then there exists a family of probability measures $\{\eta_x\}_{x\in\mathbb{R}^d}$ on $C([0,T];\mathbb{R}^d)$ such that*

$$\mu_t = \int_{\mathbb{R}^d} (e_{t\sharp}\eta_x)\, \bar{\mu}(dx),$$

*for any $t \in (0,T)$ and $(e_0)_\sharp \eta_x = \delta_x$. Moreover, $\eta_x$ is concentrated on absolutely continuous integral solutions of the ODE starting from $x$, for $\bar{\mu}$-a.e. $x \in \mathbb{R}^d$, i.e.*

$$\int_\Gamma \left| \gamma(t) - x - \int_0^t \boldsymbol{b}(\tau,\gamma(\tau))\, d\tau \right| d\eta_x(\gamma) = 0, \qquad \text{for all } t \in [0,T], \qquad \bar{\mu}\text{-a.e. } x \in \mathbb{R}^d.$$

In other words, according to 3.1, any non-negative measure-valued solution $\mu_t$ of the continuity equation (3.1) can be represented as

$$\mu_t = e_{t\sharp}\eta,$$

where $\eta$ is some nonnegative measure on the space of continuous curves $C([0,T];\mathbb{R}^d)$, which is concentrated on the integral curves of the vector field $\boldsymbol{b}$. In terms of Theorem 3.1 this measure $\eta$ can be defined by

$$\eta := \int_{\mathbb{R}^d} \eta_x \, d\bar{\mu}(x),$$

which means that the family $\{\eta_x\}_{x\in\mathbb{R}^d}$ is the disintegration of $\eta$ under the map $e_0$.

Notice that Theorem 3.1 is indeed a bridge between the Eulerian and the Lagrangian formulations, out of the smooth setting, as it allows to represent (non-negative) distributional solutions to (3.1) as solutions transported by a family of trajectories. Before investigating how Theorem 3.1 transfers well-posedness from the PDE side to the Lagrangian one, we discuss a proof, which aims to enlighten the connection of Theorem 3.1 with a decomposition result for 1-dimensional normal currents.

**3.1.1. Normal currents and Smirnov's Theorem.** We briefly recall some results and terminology of the theory of currents, mainly following [**KP08**] (to whom we refer the reader for more details). We will write $\mathscr{D}^k(\mathbb{R}^n)$ to denote the space of smooth $k$-differential forms on $\mathbb{R}^d$ with compact support. The space of *$k$-dimensional currents* $\mathscr{D}_k(\mathbb{R}^d)$ is defined as the dual of $\mathscr{D}^k(\mathbb{R}^d)$.

On the space of currents it is defined a *boundary* operator, as the adjoint of De Rham's-Cartan differential: if $T \in \mathscr{D}_k(\mathbb{R}^n)$ is a $k$-current, then $\partial T \in \mathscr{D}_{k-1}(\mathbb{R}^d)$ is the $k-1$ current given by

$$\langle \partial C, \omega\rangle = \langle C, d\omega\rangle,$$

for any smooth compactly supported $(k-1)$-form $\omega$. We also recall that the *mass* of a current $T \in \mathscr{D}_k(\mathbb{R}^n)$ is defined as

$$\mathbb{M}(T) := \sup\left\{\langle T, \omega\rangle : \omega \in \mathscr{D}^k(\mathbb{R}^d), |\omega(x)| \leq 1, \ \forall x \in \mathbb{R}^d\right\}, \tag{3.2}$$

the duality pairing being denoted by $\langle\cdot,\cdot\rangle$. We will be dealing with a particular class of currents, which are defined as follows:

DEFINITION 3.2 (Normal currents). We say that a $k$-current $T \in \mathscr{D}_k(\mathbb{R}^d)$ is *normal* if both $T$ and $\partial T$ have finite mass, i.e. $\mathbb{M}(T) + \mathbb{M}(\partial T) < +\infty$.

We will consider the current associated to an oriented submanifold of $\Sigma \subset \mathbb{R}^d$: as customary (see [**KP08**]), the symbol $[\![S,\tau,\theta]\!]$ denotes the rectifiable current associated to the $k$-rectifiable surface $S \subset \mathbb{R}^d$, oriented by the unit simple $k$-vector $\tau$ with multiplicity $\theta \in L^1(S; \mathscr{H}^k \llcorner_S)$, which acts as

$$\langle [\![S,\tau,\theta]\!], \omega\rangle := \int_S \langle \omega(x), \tau(x)\rangle \theta(x)\, d\mathscr{H}^k(x), \qquad \forall \omega \in \mathscr{D}^k(S).$$

If $\gamma\colon [0,T] \to \mathbb{R}^d$ is a Lipschitz map, we associate to $\gamma$ the 1-current $T_\gamma := [\![\gamma([0,T]), \dot\gamma, 1]\!]$, which more explicitly (by Area Formula II.3) reads as

$$\langle T_\gamma, \omega\rangle := \int_0^T \omega(\gamma(t)) \cdot \dot\gamma(t)\, dt \tag{3.3}$$

which in turn yields

$$\mathbb{M}(T_\gamma) \leq \int_0^T |\dot\gamma(t)|\, dt = \mathscr{H}^1(\gamma([0,T])) \quad \text{and} \quad \partial T_\gamma = \delta_{\gamma(T)} - \delta_{\gamma(0)} \quad \text{as measures on } \mathbb{R}^d.$$

With a slight abuse of notation we will denote $T_\gamma$ simply by $[\![\gamma]\!]$. Finally, we recall the following

DEFINITION 3.3 (Subcurrents, cycles, a-cyclic currents). Let $T \in \mathscr{D}_k(\mathbb{R}^d)$ be a $k$-current in $\mathbb{R}^d$. We say that:

(1) $S \in \mathscr{D}_k(\mathbb{R}^d)$ is a *subcurrent* of $T$ and we write $S \leq T$ if
$$\mathbb{M}(T) - \mathbb{M}(S) = \mathbb{M}(T - S);$$
(2) $C \in \mathscr{D}_k(\mathbb{R}^d)$ is a *cycle* of $T$ if $C \leq T$ and $\partial C = 0$;
(3) $T$ is *a-cyclic* if $C = 0$ is the only cycle of $T$.

A well-known theorem, due to Smirnov [**Smi94**], asserts that all normal, a-cyclic, 1-currents in $\mathbb{R}^d$ can be decomposed into a superposition of curves, according to the following statement:

THEOREM 3.4 (Smirnov). *Let $T$ be a* normal, a-cyclic 1-*current in* $\mathbb{R}^d$. *Then there exist a non-negative measure $\eta \in \mathscr{M}^+(\mathrm{Lip}([0,1];\mathbb{R}^d))$ such that*
$$\langle T, \omega\rangle = \int_{\mathrm{Lip}([0,1];\mathbb{R}^d)} \langle [\![\gamma]\!], \omega\rangle \, d\eta(\gamma), \qquad \textit{for every } \omega \in \mathscr{D}^1(\mathbb{R}^d),$$
*where $[\![\gamma]\!]$ is the current associated to $\gamma \in \mathrm{Lip}([0,1];\mathbb{R}^d)$ according to (3.3). Furthermore, we have also a decomposition of the mass of $T$*
$$\mathbb{M}(T) = \int_{\mathrm{Lip}([0,1];\mathbb{R}^d)} \mathbb{M}([\![\gamma]\!]) \, d\eta(\gamma)$$
*and a decomposition of the boundary of $T$, i.e. splitting the measure $\partial T$ into its positive/negative part $(\partial T)^{\pm}$ it holds*
$$(\partial T)^+ = (e_1)_\sharp \eta, \qquad (\partial T)^- = (e_0)_\sharp \eta.$$

Theorem 3.4 has been proved first in the classical euclidean setting by Smirnov in [**Smi94**] but we would like to point out also the useful references [**PS12, PS13**], where an extension to Ambrosio-Kirchheim currents in a metric setting has been proved.

**3.1.2. A Geometric Measure Theoretic proof of Ambrosio's Superposition.** Using Smirnov's Theorem 3.4 we now want to prove the following version of the Superposition Principle.

THEOREM 3.5 (Superposition Principle). *Let $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field and let $[0,T] \ni t \mapsto \mu_t \in \mathscr{M}^+(\mathbb{R}^d)$ be a non-negative, measurable, measure-valued map such that*
$$\int_0^T \int_{\mathbb{R}^d} (1 + |\boldsymbol{b}(t,x)|) \, d\mu_t(x) \, dt < +\infty. \tag{3.4}$$
*Assume furthermore that $\{\mu_t\}_{t\in[0,T]}$ is a measure-valued solution of*
$$\partial_t \mu_t + \mathrm{div}(\boldsymbol{b}\mu_t) = 0 \qquad \textit{in } \mathscr{D}'((0,T)\times\mathbb{R}^d). \tag{3.5}$$
*Then there exists a finite, non-negative measure $\eta$ on $\Gamma := \mathrm{AC}((0,T);\mathbb{R}^d)$ such that $\mu_t = e_{t\sharp}\eta$ for $\mathscr{L}^1$-a.e. $t \in [0,T]$ and $\eta$ is concentrated on the integral curves of $\boldsymbol{b}$.*

Before entering in the technical proof, we set some notational conventions (valid within this paragraph) that we believe will make the argument easier to read and to understand. More precisely:

- we will denote by bold, capital letter ($\boldsymbol{B}$, $\boldsymbol{\Theta}$) functions which take values in $\mathbb{R}^{d+1}$;
- we will denote by bold, non-capital letter ($\boldsymbol{b}$, $\boldsymbol{\gamma}$) functions which take values in $\mathbb{R}^d$;
- we will denote by sans-serif, capital letters ($\mathsf{S}$) 1-dimensional currents in Euclidean space $(0,T)\times\mathbb{R}^d$ and we will use $[\![\boldsymbol{\Theta}]\!]$ to denote the current induced by the curve $\boldsymbol{\Theta}$;
- we will denote by $\cdot$ the scalar product over Euclidean space and by $\langle\cdot,\cdot\rangle$ any duality pairing (measures vs bounded, continuous functions, currents vs forms, distributions etc) ;

- we will not give any special name to the spaces of funtions and we will write them completely; we reserve the use of $\Gamma$ at the very last step of the proof.

Recall that, in view of Proposition 1.7, the function $t \mapsto \mu_t$ can be taken weakly-star in the sense of measures from $[0,T]$ to $\mathscr{M}^+((0,T)\times\mathbb{R}^d)$.

PROOF. We split the proof in differents steps.
*Step 1. Reduction to a 1-normal, a-cyclic current*. Introduce the vector field

$$\boldsymbol{B} := (1, \boldsymbol{b})$$

and the measure

$$\mu := \mu_t \otimes \mathscr{L}^1 \in \mathscr{M}^+([0,T]\times\mathbb{R}^d).$$

so that, by assumption, they satisfy

$$\operatorname{div}(\boldsymbol{B}\mu) = 0 \quad \text{in} \quad \mathscr{D}'((0,T)\times\mathbb{R}^d).$$

Introduce the 1-current $\mathsf{S} := \boldsymbol{B}\mu \in \mathscr{D}_1((0,T)\times\mathbb{R}^d)$. Assumption (3.4), together with the definition of mass of a current (3.2), gives that

$$\begin{aligned}
\mathbb{M}(\mathsf{S}) &= \sup\left\{\langle \mathsf{S}, \boldsymbol{\Phi}\rangle : \boldsymbol{\Phi} \in C_c^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^{d+1}), \|\boldsymbol{\Phi}\|_\infty \le 1\right\} \\
&= \sup\left\{\int_0^T\int_{\mathbb{R}^d} \boldsymbol{B}\cdot\boldsymbol{\Phi}\,d\mu_t\,dt : \boldsymbol{\Phi} \in C_c^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^{d+1}), \|\boldsymbol{\Phi}\|_\infty \le 1\right\} \\
&\le \int_0^T\int_{\mathbb{R}^d} (1+|\boldsymbol{b}|)\,d\mu_t\,dt < +\infty,
\end{aligned}$$

so that the current $\mathsf{S}$ has finite mass. Let us now compute the boundary of $\mathsf{S}$: the current $\partial\mathsf{S}$ will be a 0-current in $(0,T)\times\mathbb{R}^d$, hence for any test function $\phi \in C_c^\infty((0,T)\times\mathbb{R}^d;\mathbb{R})$ we have,

$$\begin{aligned}
\langle \partial\mathsf{S}, \phi\rangle = \langle \mathsf{S}, d\phi\rangle &= \int_0^T\int_{\mathbb{R}^d} \boldsymbol{B}\cdot\nabla_{t,x}\phi\,d\mu_t\,dt \\
&= \int_0^T\int_{\mathbb{R}^d} (1,\boldsymbol{b})\cdot\nabla_{t,x}\phi\,d\mu_t\,dt \\
&= \int_0^T\int_{\mathbb{R}^d} \partial_t\phi + \boldsymbol{b}\cdot\nabla_x\phi\,d\mu_t\,dt \\
&= \langle \delta_0\otimes\mu_t - \delta_T\otimes\mu_t, \phi\rangle,
\end{aligned} \tag{3.6}$$

in view of (3.5) (recall that $t\mapsto\mu_t$ is defined everywhere). In particular, the boundary has also finite mass, hence we deduce $\mathsf{S}$ is a 1-normal current in $(0,T)\times\mathbb{R}^d$. The fact that $\mathsf{S}$ is a-cyclic can be proved using the fact that the first component of $\mathsf{S}$, i.e. $\mu$, is non-negative: more precisely, it is easy to show that a subcurrent of $\mathsf{S}$ must necessarily have the form $\mathbf{C}\mu$, for some vector field $\boldsymbol{C} \in L^1((0,T)\times\mathbb{R}^d;\mu;\mathbb{R}^{d+1})$. This, combined with $\mu \ge 0$, leads to the desired a-cyclicity of $\mathsf{S}$.

*Step 2. Smirnov Theorem.* We are thus in position to apply Theorem 3.4, so that we can decompose $\mathsf{S}$ into curves: there exists a measure $\boldsymbol{\xi}$ on $\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)$ such that

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \langle [\![\boldsymbol{\Theta}]\!], \boldsymbol{\Phi}\rangle\,d\boldsymbol{\xi}(\boldsymbol{\Theta}) = \int_0^T\int_{\mathbb{R}^d} \boldsymbol{B}\cdot\boldsymbol{\Phi}\,d\mu_t\,dt$$

for every test vector field $\boldsymbol{\Phi} \in C_c^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^{d+1})$. This explicitly means that

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \boldsymbol{\Phi}(\boldsymbol{\Theta}(\tau))\cdot\dot{\boldsymbol{\Theta}}(\tau)d\tau\,d\boldsymbol{\xi}(\boldsymbol{\Theta}) = \int_0^T\int_{\mathbb{R}^d} \mathbf{B}\cdot\boldsymbol{\Phi}\,d\mu_t\,dt \tag{3.7}$$

for every test vector field $\boldsymbol{\Phi} \in C_c^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^{d+1})$. Furthermore, we also have that $\xi$-a.e. $\boldsymbol{\Theta}$ is injective and has finite lenght, since it holds the decomposition of the mass $\mathbb{M}(\mathsf{S})$:

$$\mathbb{M}(\mathsf{S}) = \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \mathbb{M}\big([\![\boldsymbol{\Theta}]\!]\big)\, d\boldsymbol{\xi}(\boldsymbol{\Theta}) = \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \left\| \frac{d}{d\tau}\boldsymbol{\Theta}(\tau) \right\| d\tau\, d\boldsymbol{\xi}(\boldsymbol{\Theta}). \tag{3.8}$$

We also have the decomposition of the boundary of $\mathsf{S}$ as

$$\partial\mathsf{S} = \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \left[\delta_{\boldsymbol{\Theta}(0)} - \delta_{\boldsymbol{\Theta}(1)}\right] d\boldsymbol{\xi}(\boldsymbol{\Theta}), \qquad \text{as measures on } (0,T)\times\mathbb{R}^d \tag{3.9}$$

and also

$$|\partial\mathsf{S}| = \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \left[\delta_{\boldsymbol{\Theta}(0)} + \delta_{\boldsymbol{\Theta}(1)}\right] d\boldsymbol{\xi}(\boldsymbol{\Theta}), \qquad \text{as measures on } (0,T)\times\mathbb{R}^d. \tag{3.10}$$

*Step 3. Direction of the curves* [**AB08**]. By polar decomposition of the current $\mathsf{S}$ (see, for instance, [**KP08**, §7.2]) we have

$$\mathsf{S} = \|\mathsf{S}\|\vec{\mathsf{S}},$$

where $\|\mathsf{S}\| \in \mathscr{M}^+((0,T)\times\mathbb{R}^d)$ is a non-negative measure and $\vec{\mathsf{S}}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^{d+1}$ is a Borel, unit vector field. Clearly, in our case we have

$$\vec{\mathsf{S}} = \frac{\boldsymbol{B}}{\|\boldsymbol{B}\|} = \frac{(1,\boldsymbol{b})}{\|(1,\boldsymbol{b})\|}, \qquad \|\mathsf{S}\| = \|\boldsymbol{B}\mu\| = \mu\|(1,\boldsymbol{b})\|.$$

Since $\mathsf{S}$ has finite mass, we can take as test $\vec{\mathsf{S}}$ into (3.7) (see again [**KP08**]) and we obtain

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \vec{\mathsf{S}}(\boldsymbol{\Theta}(\tau)) \cdot \dot{\boldsymbol{\Theta}}(\tau) d\tau\, d\boldsymbol{\xi}(\boldsymbol{\Theta}) = \int_0^T \int_{\mathbb{R}^d} \boldsymbol{B}\cdot\vec{\mathsf{S}}\, d\mu_t\, dt$$

which is

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \vec{\mathsf{S}}(\boldsymbol{\Theta}(\tau)) \cdot \dot{\boldsymbol{\Theta}}(\tau) d\tau\, d\boldsymbol{\xi}(\boldsymbol{\Theta}) = \int_0^T \int_{\mathbb{R}^d} \|\boldsymbol{B}\|\, d\mu_t\, dt.$$

So we have

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \vec{\mathsf{S}}(\boldsymbol{\Theta}(\tau)) \cdot \dot{\boldsymbol{\Theta}}(\tau) d\tau\, d\boldsymbol{\xi}(\boldsymbol{\Theta}) = \mathbb{M}(\mathsf{S})$$

which, in view of (3.4), can be written as

$$\begin{aligned}\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \vec{\mathsf{S}}(\boldsymbol{\Theta}(\tau)) \cdot \dot{\boldsymbol{\Theta}}(\tau) d\tau\, d\boldsymbol{\xi}(\boldsymbol{\Theta}) &= \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \mathbb{M}\big([\![\boldsymbol{\Theta}]\!]\big)\, d\boldsymbol{\xi}(\boldsymbol{\Theta}) \\ &= \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \left\| \frac{d}{d\tau}\boldsymbol{\Theta}(\tau) \right\| d\tau\, d\boldsymbol{\xi}(\boldsymbol{\Theta}).\end{aligned}$$

i.e.

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \left\{ \int_0^1 \left[ \vec{\mathsf{S}}(\boldsymbol{\Theta}(\tau)) \cdot \dot{\boldsymbol{\Theta}}(\tau) - \left\| \frac{d}{d\tau}\boldsymbol{\Theta}(\tau) \right\| \right] d\tau \right\} d\boldsymbol{\xi}(\boldsymbol{\Theta}) = 0.$$

Since $\|\vec{\mathsf{S}}\| = 1$ a.e., the integrand is non-negative and thus we obtain that for $\boldsymbol{\xi}$-a.e. $\boldsymbol{\Theta} \in \mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)$ it has to hold

$$\frac{d}{d\tau}\boldsymbol{\Theta}(\tau) = \left\| \frac{d}{d\tau}\boldsymbol{\Theta}(\tau) \right\| \vec{\mathsf{S}}(\tau, \boldsymbol{\Theta}(\tau)), \qquad \text{for } \mathscr{L}^1\text{ -a.e. } \tau \in [0,1]. \tag{3.11}$$

*Step 4. Reparametrizazion argument.* Writing equation (3.11) componentwise, i.e. writing the equations for $\boldsymbol{\Theta}(\tau) := (t(\tau), \boldsymbol{\vartheta}(\tau))$, we obtain:

$$\begin{cases} \dfrac{d}{d\tau}t(\tau) = \left\|\dfrac{d}{d\tau}\boldsymbol{\Theta}(\tau)\right\| \dfrac{1}{\|(1,\boldsymbol{b})(\boldsymbol{\Theta}(\tau))\|} \\ \dfrac{d}{d\tau}\boldsymbol{\theta}(\tau) = \left\|\dfrac{d}{d\tau}\boldsymbol{\Theta}(\tau)\right\| \dfrac{1}{\|(1,\boldsymbol{b})(\boldsymbol{\Theta}(\tau))\|}\boldsymbol{b}(\boldsymbol{\Theta}(\tau)), \end{cases} \quad \text{for } \mathscr{L}^1\text{-a.e. } \tau \in [0,1]. \tag{3.12}$$

*Step 4.1. Inverse map of $t$.* Observe that, from (3.11), we get in particular

$$\frac{d}{d\tau}t(\tau) = \left\|\frac{d}{d\tau}\boldsymbol{\Theta}(\tau)\right\| \frac{1}{\|(1,\boldsymbol{b})(\boldsymbol{\Theta}(\tau))\|} \geq 0,$$

which means that the Lipschitz function $[0,1] \ni \tau \mapsto t(\tau) \in [t(0), t(1)]$ is increasing, possibly not strictly. Let us denote by $s\colon [t(0), t(1)] \to [0,1]$ the generalized inverse function of $t$, i.e. $s(w) := \inf\{\tau : t(\tau) > w\}$ for any $w \in [t(0), t(1)]$: notice that $s$ may have jumps, i.e. $s$ is strictly increasing and, hence, of bounded variation. The points where $s$ has a jump correspond to the values of the flat parts of the function $t$: this means that if $t$ is such that $s^-(t) \neq s^+(t)$ (actually it has to be $s^-(t) < s^+(t)$) then the curve $\boldsymbol{\theta}$ is constant on the interval $\tau \in (s^-(t), s^+(t))$. More precisely, we make the following

**Claim.** Let $w \in \mathbb{R}$ such that $s^-(w) < s^+(w)$ and let $\tau_1, \tau_2 \in (s^-(w), s^+(w))$ with $\tau_1 \neq \tau_2$. Then $\boldsymbol{\theta}(\tau_1) = \boldsymbol{\theta}(\tau_2)$.

Let us prove the Claim: clearly, $t(\tau_1) = t(\tau_2) = w$ by definition, and the same is true for any point inside the interval $(\tau_1, \tau_2)$ (wlog we have assumed $\tau_1 < \tau_2$). Recall that $t$ is Lipschitz and we have just noticed that it is constant on the interval $(\tau_1, \tau_2)$: hence $\frac{d}{d\tau}t(\tau) = 0$ on this interval and using the first ODE in 3.12 we deduce that on $(\tau_1, \tau_2)$ it holds

$$\frac{d}{d\tau}\boldsymbol{\Theta}(\tau) = 0.$$

Plugging this into the second ODE of (3.12), we obtain the claim.

*Step 4.2 Area formula.* Now we are ready to apply Area Formula with the Lipschitz map $t = t(\tau)$: indeed we have for any $a, b \in [t(0), t(1)]$

$$\begin{aligned} \int_a^b \|\boldsymbol{B}(t(s(w)), \boldsymbol{\theta}(s(w)))\| dw &\overset{w=t(\tau)}{=} \int_{s(a)}^{s(b)} \|\boldsymbol{B}(t(\tau), \boldsymbol{\theta}(\tau))\| \frac{dt}{d\tau} d\tau \\ &= \int_{s(a)}^{s(b)} \left\|\frac{d}{d\tau}\boldsymbol{\Theta}(\tau)\right\| d\tau < +\infty \end{aligned}$$

because, in view of (3.8), $\boldsymbol{\xi}$-a.e. curve $\boldsymbol{\Theta}$ has finite length. Thus we have shown that for $\boldsymbol{\xi}$-a.e. $\boldsymbol{\Theta}$ the function $w \mapsto \boldsymbol{B}(\boldsymbol{\Theta}(s(w)))$ is $L^1([t(0), t(1)])$. Thanks to this fact we can apply Area Formula again, without modulus, and we finally deduce, by arguing componentwise, that $\boldsymbol{\xi}$-a.e. $\boldsymbol{\Theta}$ it holds

$$\frac{d}{dw}\boldsymbol{\theta}(s(w)) = \boldsymbol{b}(\boldsymbol{\Theta}(s(w))), \qquad \text{for } \mathscr{L}^1 \text{ -a.e. } w \in (a,b) \subset [t(0), t(1)].$$

*Step 4.3 Interval of definition $[t(0), t(1)]$.* Taking into account (3.9) and (3.10) we can show that for $\boldsymbol{\xi}$-a.e. $\Theta = (t, \boldsymbol{\theta})$ it has to hold $t(0) = 0$ and $t(1) = T$: indeed, recalling also (3.6), we have

$$\delta_T \otimes \mu_t = \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \delta_{\boldsymbol{\Theta}(1)}\, d\boldsymbol{\xi}(\boldsymbol{\Theta}), \qquad \text{as measures on } [0,T] \times \mathbb{R}^d$$

and

$$\delta_0 \otimes \mu_t = \int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \delta_{\boldsymbol{\Theta}(0)}\, d\boldsymbol{\xi}(\boldsymbol{\Theta}), \qquad \text{as measures on } [0,T] \times \mathbb{R}^d$$

and from this we easily deduce that for $\boldsymbol{\xi}$-a.e. $\boldsymbol{\Theta} = (t, \boldsymbol{\theta})$ is has to hold

$$t(0) = 0, \qquad t(1) = T.$$

*Step 4.4 Definition of curves $\boldsymbol{\gamma}$.* By Step 4.1 and in particular the Claim therein, and by Step 4.3, we are allowed to define for $\xi$-a.e. $(t, \boldsymbol{\theta})$ the following curve:

$$\begin{aligned} \boldsymbol{\gamma} \colon [t(0), t(1)] = [0, T] &\to \mathbb{R}^d \\ w &\mapsto \boldsymbol{\gamma}(w) = \boldsymbol{\theta}(s(w)). \end{aligned}$$

Indeed, the above definition makes sense and define an absolutely continuous curve: indeed, in view of Step 4.1 and Step 4.2 for $\boldsymbol{\xi}$-a.e. $\boldsymbol{\Theta}$ the associated $\boldsymbol{\gamma}$ solves

$$\frac{d}{dt}\boldsymbol{\gamma}(t) = \boldsymbol{b}(t, \boldsymbol{\gamma}(t)), \qquad \text{for } \mathscr{L}^1 \text{ a.e. } t \in [0, T],$$

and in particular $\boldsymbol{\gamma}$ is of class $W^{1,1}(0, T)$.

*Step 5. Conclusion of the proof.* Let now

$$\begin{aligned} \mathcal{R} \colon \mathrm{Lip}([0,1]; [0,T] \times \mathbb{R}^d) &\to \mathrm{AC}([0,T]; [0,T] \times \mathbb{R}^d) \\ \boldsymbol{\Theta} &\longmapsto \boldsymbol{\gamma} \end{aligned}$$

be the reparametrization map that to $\boldsymbol{\xi}$-a.e. curve $\boldsymbol{\Theta}$ associates its reparametrized form $\boldsymbol{\gamma}$. Let $\widehat{\boldsymbol{\xi}} := \mathcal{R}_\sharp \boldsymbol{\xi}$ and

$$\pi^2 \colon \mathrm{AC}([0,T]; [0,T] \times \mathbb{R}^d) = \mathrm{AC}([0,T]; [0,T]) \times \mathrm{AC}([0,T]; \mathbb{R}^d) \to \mathrm{AC}([0,T]; \mathbb{R}^d) =: \Gamma$$

be the canonical projection. Finally define

$$\eta := (\pi^2)_\sharp \widehat{\boldsymbol{\xi}} \in \mathscr{M}^+(\Gamma).$$

The measure $\eta$ is concentrated on integral curves of $\mathbf{b}$ by construction. On the other hand, for fixed $\bar{t} \in [0, T]$, let $\varphi_n \colon [0, T] \to \mathbb{R}$ be a sequence of functions in $C_c^\infty((0, T))$ which converges to a Dirac delta in $\bar{t}$ and let $\psi \in C_c^\infty(\mathbb{R}^d)$ be arbitrary. Putting $\boldsymbol{\Phi}_n(t, x) := (\psi(x)\varphi_n(t), \mathbf{0}) \in C_c^\infty((0, T) \times \mathbb{R}^d; \mathbb{R}^{d+1})$ as test function in (3.7), we get

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \boldsymbol{\Phi}_n(\boldsymbol{\Theta}(s)) \cdot \dot{\boldsymbol{\Theta}}(s) ds \, d\boldsymbol{\xi}(\boldsymbol{\Theta}) = \int_0^T \int_{\mathbb{R}^d} \mathbf{B} \cdot \boldsymbol{\Phi}_n \, d\mu_t \, dt$$

which is

$$\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \psi(\boldsymbol{\theta}(\tau))\varphi_n(t(\tau)) \frac{d}{d\tau} t(\tau) d\tau \, d\boldsymbol{\xi}(t, \boldsymbol{\theta}) = \int_0^T \int_{\mathbb{R}^d} \psi(x)\varphi_n(t) \, d\mu_t \, dt.$$

The RHS converges to

$$\int_0^T \int_{\mathbb{R}^d} \psi(x)\varphi_n(t) \, d\mu_t \, dt \to \int_{\mathbb{R}^d} \psi(x) d\mu_{\bar{t}} = \langle \mu_{\bar{t}}, \psi \rangle.$$

For the LHS, we have

$$\begin{aligned} &\int_{\mathrm{Lip}([0,1];[0,T]\times\mathbb{R}^d)} \int_0^1 \psi(\boldsymbol{\theta}(\tau))\varphi_n(t(\tau)) \frac{d}{d\tau} t(\tau) d\tau \, d\boldsymbol{\xi}(t, \boldsymbol{\theta}) \\ &= \int_{\mathrm{AC}([0,T];[0,T]\times\mathbb{R}^d)} \int_0^T \psi(\boldsymbol{\gamma}(t))\varphi_n(t) dt \, d\widehat{\boldsymbol{\xi}}(\boldsymbol{\gamma}) \end{aligned}$$

and, as $n \to +\infty$, the latter converges, being $t \mapsto \psi(\boldsymbol{\gamma}(t))$ continuous, to

$$\int_{\mathrm{AC}([0,T];[0,T]\times\mathbb{R}^d)} \psi(\boldsymbol{\gamma}(\bar{t})) \, d\widehat{\boldsymbol{\xi}}(\boldsymbol{\gamma}) = \int_\Gamma \psi(\boldsymbol{\gamma}(\bar{t})) \, d\eta(\boldsymbol{\gamma}) = \langle e_{\bar{t}\sharp}\eta, \psi \rangle.$$

Thus being $\psi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ and $\bar{t} \in [0, T]$ arbitrary, we have shown

$$\mu_t = e_{t\sharp}\eta$$

for every $t \in [0, T]$ and this concludes the proof. ☐

### 3.2. Lagrangian representations and Regular Lagrangian Flows

Since it will be useful later on, we now turn to consider the following more general setting: consider a vector field $\boldsymbol{b} \in L^1(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d)$, with compact support and assume there exists a non-negative function $\rho\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^+$ such that

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^d} \rho(1 + |\boldsymbol{b}|)\, dtdx < \infty.$$

Assume moreover that it holds in the sense of distributions

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{b}) = \mu \in \mathscr{M}(\mathbb{R}^{d+1}), \tag{3.13}$$

and, in order to avoid dealing with sets of $\mathscr{L}^{d+1}$-negligible measure, we assume that $\rho, \boldsymbol{b}$ are defined pointwise as Borel functions. For simplicity, we will often write in the following that the vector field $\rho(1, \boldsymbol{b})\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d+1}$ is a *measure divergence vector field* and (3.13) will be written shortly as

$$\operatorname{div}(\rho(1, \boldsymbol{b})) = \mu \tag{3.14}$$

where, with a slight abuse of notation, we think the divergence operator $\operatorname{div} = \operatorname{div}_{t,x} = \partial_t + \operatorname{div}_x$ acting also on the time variable. An absolutely continuous curve $\gamma\colon I_\gamma \to \mathbb{R}^d$, where $I_\gamma = (t_\gamma^-, t_\gamma^+)$ is a time interval, will be called *characteristic* if it solves the ODE

$$\frac{d}{dt}\gamma(t) = \boldsymbol{b}(t, \gamma(t)), \qquad \text{for } \mathscr{L}^1 \text{ -a.e. } t \in I_\gamma.$$

Accordingly, the space of curves we will work with is

$$\varUpsilon = \{(t_1, t_2, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R}^d),\, t_1 < t_2\}$$

and in particular its subset made up of characteristics

$$\varGamma = \Big\{(t_1, t_2, \gamma) \in \varUpsilon : \gamma \text{ characteristic in } (t_1, t_2)\Big\}.$$

We will tacitly identify the triplet $(t_\gamma^-, t_\gamma^+, \gamma) \in \varGamma$ with $\gamma$, so that we will usually write $\gamma \in \varGamma$; sometimes, to lighten the notation in the case $\mu = 0$, we will use $\Gamma$ (and not $\varGamma$) to denote the space of curves which are characteristics in $[0, T]$. It is possible to show that $\varGamma$ is a Borel subset of $\varUpsilon$. We now give the following

Definition 3.6. We say that a finite, non-negative measure $\eta \in \mathscr{M}_b^+(\varUpsilon)$ is a *Lagrangian representation of the measure-divergence vector field* $\rho(1, \boldsymbol{b})$ if the following conditions hold:

(1) $\eta$ is concentrated on the set $\varGamma$ of absolutely continuous solutions to the ODE

$$\dot\gamma(t) = \boldsymbol{b}(t, \gamma(t)),$$

which explicitly means for every $(s, t) \subseteq I_\gamma$

$$\int_\varGamma \left| \gamma(t) - \gamma(s) - \int_s^t \boldsymbol{b}(\tau, \gamma(\tau))\, d\tau \right| d\eta(\gamma) = 0;$$

(2) if $(\mathbb{I}, \gamma)\colon I_\gamma \to I_\gamma \times \mathbb{R}^d$ denotes the map defined by $t \mapsto (t, \gamma(t))$, then

$$\rho \mathscr{L}^{d+1} = \int_\varGamma \Big( (\mathbb{I}, \gamma)_\sharp \mathscr{L}^1 \Big)\, \eta(d\gamma); \tag{3.15}$$

(3) we can decompose the divergence $\mu$ defined in (3.13) as a local superposition of Dirac masses without cancellation, i.e.

$$\mu = \int_\varGamma \Big[ \delta_{t_\gamma^-, \gamma(t_\gamma^-)} - \delta_{t_\gamma^+, \gamma(t_\gamma^+)} \Big]\, \eta(d\gamma),$$

$$|\mu| = \int \Big[ \delta_{t_\gamma^-, \gamma(t_\gamma^-)} + \delta_{t_\gamma^+, \gamma(t_\gamma^+)} \Big]\, \eta(d\gamma),$$

where we recall that, for every $\gamma$, the interval in which it is a characteristic is denoted by $(t_\gamma^-, t_\gamma^+) = I_\gamma$.

The existence of such a measure $\eta$ follows from Theorem 3.4, repeating verbatim the proof presented above of Theorem 3.5. The only point which fails is Step 4.3, and this is the reason why we have introduced the interval $I_\gamma$ in which every $\gamma$ is a characteristic.

Notice furthermore that, in the case $\mu = 0$, Condition (3.15) is just another (more convenient for us) way of writing the usual $(e_t)_\sharp \eta = \rho(t)\mathscr{L}^d$ for a.e. $t \in (0,T)$ (actually, we can think the former as the "integrated" version of the latter). Explicitly, (3.15) means that for any bounded, continuous function $\varphi$ it holds

$$\iint_{\mathbb{R}\times\mathbb{R}^d} \rho(t,x)\varphi(t,x)\,\mathscr{L}^{d+1}(dtdx) = \int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \varphi(t,\gamma(t))\,\mathscr{L}^1(dt)\,\eta(d\gamma).$$

We conclude this section with two remarks.

Remark 3.7 (Disintegration). Observe that for all $\gamma$ the interval of definition is a bounded time interval (recall that we assume $\rho(1,\boldsymbol{b})$ with compact support), so that, denoting by $\mu^\pm$ the positive/negative part of the divergence, we can disintegrate $\eta$ in the following way:

$$\eta = \int_{\mathbb{R}^{d+1}} \eta_z\, d\mu^-(z), \qquad \mu^\pm = \int_\Gamma \delta_{t_\gamma^\mp,\gamma(t_\gamma^\mp)}\, d\eta(\gamma).$$

We will consider this disintegration several times in the following chapters. ♠

Remark 3.8 (Initial and final points). Notice that the curves $\gamma$ can be defined in the closed or open interval: indeed, by the first and second points of Definition 3.6, it follows that

$$\int_\Gamma\int_{I_\gamma} |\dot\gamma(t)|\,\mathscr{L}^1(dt)\,\eta(d\gamma) = \int_\Gamma\int_{I_\gamma} |\boldsymbol{b}(t,\gamma(t))|\,dt\,\eta(d\gamma) = \int \rho(t,x)|\boldsymbol{b}(t,x)|\mathscr{L}^{d+1}(dt\,dx),$$

so that the total variation of $\eta$-a.e. $\gamma$ is finite, and thus $\gamma(t_\gamma^\pm) \in \mathbb{R}^d$ exists. Adding or subtracting the end points does not change the representation. In the following chapters, we will usually consider the graph of $\gamma$ in the closed interval, writing

$$\mathrm{Graph}\,\gamma = \mathrm{clos}\,\mathrm{Graph}\,\gamma, \tag{3.16}$$

with a slight abuse of notation. ♠

**3.2.1. Lagrangian Flows.** As we have seen in Chapter 1, existence, uniqueness and stability properties of Regular Lagrangian Flows associated to a vector field $\boldsymbol{b}$ can be derived from corresponding results of the PDEs driven by $\boldsymbol{b}$. We now want to make this point rigorous, taking advantage of the concept of Lagrangian representation: the results we present here are due to Ambrosio (we refer to [Amb04] for the original approach in BV and to the lecture notes [AC08] or to the thesis [Cri09] for an account of the results). Let us start again from the continuity equation

$$\partial_t \mu_t + \mathrm{div}(\boldsymbol{b}\mu_t) = 0, \tag{3.17}$$

and assume $\mu_t \geq 0$ so that Theorem 3.1 applies: we deduce there exists a measure $\eta \in \mathscr{M}^+(\Gamma)$ such that

$$\mu_t = e_{t\sharp}\eta,$$

where $e_t$ is, as usual, the evaluation map and, as we saw, disintegrating the measure $\eta$ w.r.t. $\bar\mu = (e_0)_\sharp \eta$ we can write

$$\eta = \int_{\mathbb{R}^d} \eta_x d\bar\mu(x) \tag{3.18}$$

where $\eta_x$ is a probability measure concentrated on the set of integral curves of $\boldsymbol{b}$ which are at $x$ at time $t = 0$ for $\bar{\mu}$-a.e. $x$. Roughly speaking, the measures $\{\eta_x\}$ make up a *generalized, probabilistic* flow of $\boldsymbol{b}$. Thus we expect that, if $\eta_x$ is a Dirac mass for $\bar{\mu}$-a.e. $x \in \mathbb{R}^d$ then we have a unique, deterministic flow $\boldsymbol{X}$. This is indeed the case, as the following propositions show:

PROPOSITION 3.9 ([Cri09, Prop. 6.4.3]). *Let $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded vector field. Assume that, for every initial datum $\bar{\mu} = \bar{\rho}\mathscr{L}^d$, with $\bar{\rho} \in L^\infty(\mathbb{R}^d)$, the continuity equation (3.17) has a unique bounded solution starting from $\bar{\mu}$. Let $\{\eta_x\}_{x\in\mathbb{R}^d} \subset \mathscr{P}(\Gamma)$ be a family of probability measures concentrated on absolutely continuous integral solutions of the ordinary differential equation starting from $x$, for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$. Let $\eta$ be defined as in (3.18) and assume that $(e_t)_\sharp\eta = \rho(t,\cdot)\mathscr{L}^d$ for some $\rho \in L^\infty([0,T]\times\mathbb{R}^d)$. Then $\eta_x$ is a Dirac mass for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.*

As a corollary, we deduce

THEOREM 3.10 (Existence and uniqueness of the regular Lagrangian flow, [Cri09, Prop. 6.4.1]). *Let $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded vector field. Assume that, for every initial datum $\bar{\mu} = \bar{\rho}\mathscr{L}^d$, with $\bar{\rho} \in L^\infty(\mathbb{R}^d)$, the continuity equation (3.17) has a unique solution starting form $\bar{\mu}$. Then the regular Lagrangian flow associated to $\boldsymbol{b}$, if it exists, is unique. Assume in addition that the continuity equation (3.17) with initial data $\bar{\mu} = \mathscr{L}^d$ has a non-negative solution in $L^\infty([0,T] \times \mathbb{R}^d)$. Then we have existence of a regular Lagrangian flow relative to $\boldsymbol{b}$.*

Also stability results can be proved in this way, like for instance the following

THEOREM 3.11 (Stability of the regular Lagrangian flow [Cri09]). *Let $\{\boldsymbol{b}_k\}_{k\in\mathbb{N}}$ be a sequence of vector fields such that*

$$\|\boldsymbol{b}_k\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \|\operatorname{div}\boldsymbol{b}_k\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq C < +\infty$$

*and assume that for each $\boldsymbol{b}_k$ the continuity equation has a unique bounded solution for every bounded initial datum. Assume that the sequence $\{\boldsymbol{b}_k\}_{k\in\mathbb{N}}$ converges in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ to a vector field $\boldsymbol{b} \in L^\infty([0,T] \times \mathbb{R}^d)$ with $\operatorname{div}\boldsymbol{b} \in L^\infty([0,T] \times \mathbb{R}^d)$. Assume that the continuity equation with vector field $\boldsymbol{b}$ has a unique bounded solution for every bounded initial datum. Then the regular Lagrangian flows $\boldsymbol{X}_k$ associated to $\boldsymbol{b}_k$ converge strongly in $L^\infty([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to the regular Lagrangian flow $\boldsymbol{X}$ associated to $\boldsymbol{b}$.*

**3.2.2. Vector fields with special structure.** Before moving on, to illustrate how Lagrangian representations can be used, let us briefly consider, for the sake of completeness, a particular class of vector fields, which has been studied in [LBL04]. In the following we will think to the space $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d_1 + d_2 = d$ and accodingly we will write $\mathbb{R}^d \ni x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and a similar splitting holds for the differential operators $\operatorname{div}_x = \operatorname{div}_{x_1} + \operatorname{div}_{x_2}$ and $\nabla_x = (\nabla_{x_1}, \nabla_{x_2})$. With the theory of Regular Lagrangian Flow at our disposal we can prove the following

PROPOSITION 3.12 (Vector fields with special structure). *Let $\boldsymbol{b}\colon (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field and assume $\boldsymbol{b}(t, x_1, x_2) = \big(\boldsymbol{b}_1(t, x_1), \boldsymbol{b}_2(t, x_1, x_2)\big)$ where*

*(1) $\boldsymbol{b}$ is divergence-free;*
*(2) $\boldsymbol{b}_1$ has a unique Regular Lagrangian Flow $\boldsymbol{X}_1 = \boldsymbol{X}_1(t, x_1)$;*
*(3) for a.e. $x_1 \in \mathbb{R}^{d_1}$ the vector field $\boldsymbol{b}_2(\cdot, x_1, \cdot) \in L^1((0,T); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{d_2}))$.*

*Then $\boldsymbol{b}$ has a unique Regular Lagrangian Flow.*

PROOF. Let us fix a bounded initial datum $\bar{\rho} = \bar{\rho}(x_1, x_2)$ and let $\rho \in L^\infty([0,T] \times \mathbb{R}^d)$ be a bounded, non-negative solution starting from $\bar{\rho}$. By Disintegration Theorem (see

Thm. II)

$$\rho(t,\cdot)\mathscr{L}^d = \int_{\mathbb{R}^{d_1}} \rho^{x_1}(t,dx_2)\,\rho_1(t,dx_1), \qquad \text{where } \rho_1(t,dx_1) := (\mathrm{p}_{\mathbb{R}^{d_1}})_\sharp(\rho(t,\cdot)\mathscr{L}^d)$$

and similarly for the initial datum $\bar\rho\mathscr{L}^d = \int_{\mathbb{R}^{d_1}} \bar\rho^{x_1}(dx_2)\,\bar\rho_1(dx_1)$. Let $\eta_1$ be a Lagrangian representation of $\rho_1(1,\boldsymbol{b}_1)$: by assumption, $\eta_1$ has to be a superposition of Dirac masses, i.e. it holds

$$\rho_1(t,\cdot)\mathscr{L}^{d_1} = (e_t)_\sharp\eta_1, \qquad \eta_1 = \int_{\mathbb{R}^{d_1}} \delta_{\boldsymbol{X}_1(\cdot,x_1)}\,\big(\bar\rho_1\mathscr{L}^{d_1}\big)(dx_1).$$

For simplicity, let us denote by $\gamma_1$ the curves of the flow $\eta_1$; define now the vector field

$$\widetilde{\boldsymbol{b}}_2^{\gamma_1}(t,x_2) := \boldsymbol{b}_2(t,\gamma_1(t),x_2).$$

It is immediate to check that the vector field $\widetilde{\boldsymbol{b}}_2^{\gamma_1} \in L^1((0,T);W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{d_2}))$ for $\eta_1$-a.e. $\gamma_1$: indeed,

$$\begin{aligned}
&\int_{\Gamma_1}\left[\iint_{(0,T)\times\mathbb{R}^{d_1}} \big(|\widetilde{\boldsymbol{b}}_2^{\gamma_1}| + |\nabla_{x_2}\widetilde{\boldsymbol{b}}_2^{\gamma_1}|\big)\,\mathscr{L}^{d_2+1}(dtdx_2)\right]\eta_1(d\gamma_1)\\
\le&\|\rho\|_\infty \iint_{(0,T)\times\mathbb{R}^d} \big(|\boldsymbol{b}| + |\nabla_{x_2}\boldsymbol{b}_2|\big)\,\mathscr{L}^{d+1}(dtdx)
\end{aligned}$$

by projection, and the RHS is finite by assumption (3). In particular, we deduce that $\widetilde{\boldsymbol{b}}_2^{\gamma_1}$ has a unique Regular Lagrangian Flow $\boldsymbol{X}_2^{\gamma_1}$ for $\eta_1$-a.e. $\gamma_1$ and thus the associated Lagrangian representation is given by

$$\rho^{\gamma_1(t)}(t,dx_2)\mathscr{L}^{d_2} = (e_t)_\sharp\eta_2^{\gamma_1(t)}, \qquad \eta_2^{\gamma_1(t)} = \int_{\mathbb{R}^{d_2}} \delta_{\boldsymbol{X}_2^{\gamma_1(t)}(\cdot,x_2)}\,\big(\bar\rho^{\gamma_1(t)}\mathscr{L}^{d_2}\big)(dx_2).$$

By a disintegration argument, we can now easily show that every solution to the continuity equation driven by $\boldsymbol{b}$ is Lagrangian, in the sense that it is constant along the flow, and hence uniqueness of the Regular Lagrangian Flow follows. We have for any $\varphi \in C^\infty_c((0,T)\times\mathbb{R}^d)$

$$\begin{aligned}
0 &= \iint_{(0,T)\times\mathbb{R}^d} \rho(t,x)\big(\varphi_t(t,x) + \boldsymbol{b}(t,x)\cdot\nabla\varphi(t,x)\big)\,dtdx\\
&= \int_0^T\int_{\mathbb{R}^{d_1}}\left[\int_{\mathbb{R}^{d_2}}\Big(\varphi_t(t,x_1,x_2) + \boldsymbol{b}(t,x_1,x_2)\cdot\nabla\varphi(t,x_1,x_2)\Big)\,\rho^{x_1}(t,dx_2)\right]\rho_1(t,dx_1)\,dt\\
&= \int_0^T\int_{\Gamma_1}\left[\int_{\mathbb{R}^{d_2}}\Big(\varphi_t(t,\gamma_1(t),x_2) + \boldsymbol{b}(t,\gamma_1(t),x_2)\cdot\nabla\varphi(t,\gamma_1(t),x_2)\Big)\,\rho^{\gamma_1(t)}(t,dx_2)\right]\eta_1(d\gamma_1)\,dt\\
&= \int_0^T\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}^{d_2}} \varphi_t(t,\boldsymbol{X}_1(t,x_1),x_2)\,\rho^{\boldsymbol{X}_1(t,x_1)}(t,dx_2)\,\big(\bar\rho_1\mathscr{L}^{d_1}\big)(dx_1)\,dt\\
&\quad+\int_0^T\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}^{d_2}} \boldsymbol{b}(t,\boldsymbol{X}_1(t,x_1),x_2)\cdot\nabla\varphi(t,\boldsymbol{X}_1(t,x_1),x_2)\,\rho^{\boldsymbol{X}_1(t,x_1)}(t,dx_2)\,\big(\bar\rho_1\mathscr{L}^{d_1}\big)(dx_1)\,dt
\end{aligned}$$

where we have used that $\eta_1$ is a superposition of Dirac masses. Now we basically repeat the same procedure, taking advantage of the flow of $\boldsymbol{b}_2^{\gamma_1(t)}$: continuing from above we have

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \varphi_t(t, \boldsymbol{X}_1(t,x_1), x_2)\, \rho^{\boldsymbol{X}_1(t,x_1)}(t, dx_2)\, \big(\bar\rho_1 \mathscr{L}^{d_1}\big)(dx_1)\, dt \\
&\quad + \int_0^T \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \underbrace{\boldsymbol{b}(t, \boldsymbol{X}_1(t,x_1), x_2) \cdot \nabla\varphi(t, \boldsymbol{X}_1(t,x_1), x_2)}_{\boldsymbol{b}_1\cdot\nabla_{x_1}\varphi + \boldsymbol{b}_2\cdot\nabla_{x_2}\varphi}\, \rho^{\boldsymbol{X}_1(t,x_1)}(t, dx_2)\, \big(\bar\rho_1 \mathscr{L}^{d_1}\big)(dx_1)\, dt \\
&= \int_0^T \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \frac{d}{dt}\varphi(t, \boldsymbol{X}_1(t,x_1), x_2)\, \rho^{\boldsymbol{X}_1(t,x_1)}(t, dx_2)\, \big(\bar\rho_1 \mathscr{L}^{d_1}\big)(dx_1)\, dt \\
&\quad + \int_0^T \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \boldsymbol{b}_2(t, \boldsymbol{X}_1(t,x_1), x_2) \cdot \nabla_2\varphi(t, \boldsymbol{X}_1(t,x_1), x_2)\, \rho^{\boldsymbol{X}_1(t,x_1)}(t, dx_2)\, \big(\bar\rho_1 \mathscr{L}^{d_1}\big)(dx_1)\, dt \\
&= \iint_{(0,T)\times\mathbb{R}^d} \frac{d}{dt}\varphi(t, \boldsymbol{X}_1(t,x_1), \boldsymbol{X}_2^{X_1(t,x_1)}(t,x_2))\bar\rho(x_1,x_2)\, \mathscr{L}^{d+1}(dx, dt)
\end{aligned}
$$

hence the solution is the initial datum $\bar\rho$ transported along the flow. This concludes. □

REMARK 3.13. In view of the results we will present in the next chapters (in particular, after the proof of Bressan's Compactness Conjecture, stated in the next Section 3.3) one could relax the Assumption (3) of Proposition 3.12 to the following:

(3') for a.e. $x_1 \in \mathbb{R}^{d_1}$ the vector field $\boldsymbol{b}_2(\cdot, x_1, \cdot) \in L^1((0,T); \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{d_2}))$.

In particular, this is relevant in connection to [Ler04]. ♠

## 3.3. Nearly incompressible vector fields and Bressan's Compactness Conjecture

We now turn our attention to study a broad class of vector fields in which the usual assumption of boundedness of the divergence is replaced by the existence of a solution to the continuity equation which is bounded away from zero and infinity. As pointed out in the Introduction, this is particularly important in view of the applications, for instance to the Keyfitz and Kranzer system [KK80].

DEFINITION 3.14. Let $I \subset \mathbb{R}$ be an open interval and $\Omega \subset \mathbb{R}^d$ be an open set. A bounded, locally integrable vector field $\boldsymbol{b}\colon I \times \Omega \to \mathbb{R}^d$ is called *nearly incompressible* if there exists a function $\rho\colon I \times \Omega \to \mathbb{R}$ (called *density*) such that $\log\rho \in L^\infty(I \times \Omega)$ and

$$\partial_t\rho + \operatorname{div}(\rho\boldsymbol{b}) = 0 \qquad \text{in } \mathscr{D}'(I \times \Omega). \tag{3.19}$$

Thanks to Proposition 1.8, there exists a weakly-star continuous function $\widetilde\rho \in L^\infty(I \times \mathbb{R}^d)$ such that $\widetilde\rho(t,\cdot) = \rho(t,\cdot)$ for a.e. $t \in I$. Sometimes we will call $\widetilde\rho(t,\cdot)$ the *trace* of $\rho$ at time $t$.

REMARK 3.15. Using mollifications, one can easily prove that every vector field $\boldsymbol{b}$ with bounded divergence is nearly incompressible. In general, however, a nearly incompressible vector field does not need to have absolutely continuous divergence. This can be easily seen considering the autonomous one-dimensional vector field $\boldsymbol{b}(x) := 1 + f_c(x)$, for $x \in (0,1)$ where $f_c\colon (0,1) \to \mathbb{R}$ is the Cantor-Vitali staircase function. It is readily seen that $r(x) = \frac{1}{\boldsymbol{b}(x)}$ satisfies $\partial_x(r\boldsymbol{b}) = 0$ in the sense of distributions and $r(x) \in (\frac{1}{2}, 1)$ for every $x \in (0,1)$. On the other hand, clearly $\partial_x\boldsymbol{b}$ is a non-trivial singular measure on $(0,1)$. Near incompressibility can thus be considered as a relaxed version of the assumption $\operatorname{div}\boldsymbol{b} \in L^\infty(I \times \mathbb{R}^d)$. ♠

We want now to introduce the definition of renormalization property in this new context. As $\operatorname{div}\boldsymbol{b}$ may contain non trivial singular part, we have to specify what we mean by solution, as Definition 1.12 does not apply any more. We give the following

DEFINITION 3.16. Let $\boldsymbol{b}$ be a bounded nearly incompressible vector field with density $\rho$. We say that a function $u \in L^\infty(I \times \mathbb{R}^d)$ is a $\rho$-weak solution to the transport equation $\partial_t u + \boldsymbol{b} \cdot \nabla u = 0$ if we have

$$\partial_t(\rho u) + \operatorname{div}(\rho u \boldsymbol{b}) = 0 \quad \text{in } \mathscr{D}'(I \times \mathbb{R}^d).$$

As a consequence, we have

DEFINITION 3.17 (Renormalization property for nearly incompressible vector fields). We say that a bounded nearly incompressible vector field $\boldsymbol{b}\colon I \times \mathbb{R}^d \to \mathbb{R}^d$ has the *renormalization property* if, for some function $\rho$ as in Definition 3.14, every solution $u \in L^\infty(I \times \mathbb{R}^d)$ of

$$\partial_t(\rho u) + \operatorname{div}(\rho u \boldsymbol{b}) = 0 \quad \text{in } \mathscr{D}'(I \times \mathbb{R}^d)$$

satisfies

$$\partial_t\left(\rho\beta(u)\right) + \operatorname{div}\left(\rho\beta(u)\boldsymbol{b}\right) = 0 \quad \text{in } \mathscr{D}'(I \times \mathbb{R}^d)$$

for every function $\beta \in C^1(\mathbb{R})$.

It can be checked that the renormalization property is independent of the choice of the density $\rho$ used in Definition 3.17. Furthermore, for bounded nearly incompressible vector fields with the renormalization property it is possible to develop a well-posedness theory, in analogy to what done in Proposition 1.17.

REMARK 3.18. *Existence* of weak solutions of initial value problem for transport equation driven by a nearly incompressible vector field can be proved by a regularization argument similar to the one used in the proof of Proposition 1.13. For further details, see for instance [**DL07**]. ♠

**3.3.1. Bressan's compactness conjecture.** Nearly incompressible vector fields are related to a conjecture, raised by A. Bressan in [**Bre03**]. The original statement of the conjecture deals with ODEs:

CONJECTURE 3.19 (Bressan's Compactness Conjecture). *Let $\boldsymbol{b}_k\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $k \in \mathbb{N}$, be a sequence of smooth vector fields and denote by $\boldsymbol{X}_k$ the associated flows, i.e. the solutions of*

$$\begin{cases} \partial_t \boldsymbol{X}_k(t,x) = \boldsymbol{b}_k(t, \boldsymbol{X}_k(t,x)) \\ \boldsymbol{X}_k(0,x) = x. \end{cases}$$

*Assume that the quantity $\|\boldsymbol{b}_k\|_\infty + \|\nabla \boldsymbol{b}_k\|_{L^1}$ is uniformly bounded and that the flows $\boldsymbol{X}$ are* uniformly nearly incompressible, *in the sense that there exists $C > 0$ such that*

$$\frac{1}{C} \leq \det\left(\nabla_x \boldsymbol{X}_k(t,x)\right) \leq C.$$

*Then the sequence $\{\boldsymbol{X}_k\}_{k\in\mathbb{N}}$ is strongly precompact in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$.*

In particular, from standard compactness estimates, it has been proved in [**ABDL04**] that Conjecture 3.19 would follow from the following one, which amounts to establish a well-posedness theory for the PDE associated to a BV, nearly incompressible vector field:

CONJECTURE 3.20. *Any nearly incompressible vector field $\boldsymbol{b} \in L^1(\mathbb{R}; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$ has the renormalization property in the sense of Definition 3.17.*

In the next chapters we will address Conjecture 3.20, considering first the (autonomous) two-dimensional setting (Chapter 4) and then the general $d$-dimensional case (in Part 2, Chapters 6, 7 and 8).

CHAPTER 4

# The two-dimensional case

Abstract. In this chapter, we will study the problem of uniqueness of weak solutions to the transport equation in the two dimensional case. In this particular framework, one can take advantage of the Hamiltonian structure in $\mathbb{R}^2$ (in a sense that will be explained) to prove well-posedness results under more general assumptions. More precisely, we will begin presenting a result due to Alberti, Bianchini and Crippa that settles completely the divergence-free, autonomous case: in [**ABC14**], building also on the structure results of [**ABC13**], the authors are able to *characterize* completely the autonomous, bounded vector fields $\boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\operatorname{div} \boldsymbol{b} = 0$ in $\mathbb{R}^2$ for which the transport equation has a unique bounded (or even merely integrable) solution. After presenting, in Section 4.1 a heuristic description of the method of [**ABC13**], we present a variation which allows to handle also the case of nearly incompressible vector fields (not necessarily divergence free). A crucial ingredient for our proof is the Superposition Principle studied in Chapter 3. The results presented in this chapter have been obtained in collaboration with S. Bianchini and N.A. Gusev and have been published in [**BBG16**] (see also [**BBM17, Bon16**] for other shorter accounts): the main lines of the argument were already present in the author's Master Thesis [**Bon14**] but, since then, some simplifications (especially in the second part of the proof) have occurred. We propose it here for the sake of completeness, taking this opportunity to state and prove some disintegration Lemmata which will be useful in the examples presented in Chapter 5.

In this chapter we describe some well-posedness results that are available in the two-dimensional case.

## 4.1. Introduction: splitting the equation on the level sets

The starting point of our analysis is the paper [**ABC14**], where the authors *characterize* the autonomous, divergence-free vector fields $\boldsymbol{b}$ on the plane such that the Cauchy problem for the continuity equation $\partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0$ admits a unique bounded weak solution for every bounded initial datum.

Let $\boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a bounded, autonomous, divergence-free vector field with compact support and let us consider the initial value problem for the continuity equation:

$$\begin{cases} \partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0 \\ u(0,\cdot) = 0 \end{cases} \tag{4.1}$$

Since $\operatorname{div} \boldsymbol{b} = 0$ and $\mathbb{R}^2$ is simply connected, there exists a compactly supported Lipschitz function $H\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\boldsymbol{b}(x) = \nabla^\perp H(x), \qquad \mathscr{L}^2\text{-a.e. } x \in \mathbb{R}^2,$$

where $\nabla^\perp := (-\partial_2, \partial_1)$. Such $H$ is unique (being the support compact) and it is called the *Hamiltonian* associated to $\boldsymbol{b}$. One of the main ideas involved in the two-dimensional results we will present is the heuristic remark that the value of the Hamiltonian $H$ is constant on the trajectories of the vector field $\boldsymbol{b}$. Indeed, in the smooth setting, if $\dot\gamma = \boldsymbol{b}(\gamma(t))$ then

$$\frac{d}{dt} H(\gamma(t)) = \nabla H(\gamma(t)) \cdot \dot\gamma(t) = \nabla H(\gamma(t)) \cdot \boldsymbol{b}(\gamma(t)) = 0$$

since $\nabla H \perp \boldsymbol{b}$. This means that the trajectories "follow" the level sets of the Hamiltonian (this is true even in the Lipschitz case: for a precise statement of this property, we refer the reader to next Lemma 4.26).

Therefore, using the fact that the level sets are invariant under the action of the flow, we can reduce the equations onto the level sets: taking advantage of the Structure Theorem III of level sets of Lipschitz functions and using Disintegration Theorem II, we can study the equation on each level set as a one-dimensional problem. Indeed, we have that for a.e. $h \in \mathbb{R}$, every connected component $C$ of $E_h = H^{-1}(h)$ is a simple Lipschitz curve which admits a Lipschitz parametrization $\gamma\colon I \to C$, where $I \subset \mathbb{R}$ is an interval (or possibly $\mathbb{R}/\ell\mathbb{Z}$ for some $\ell > 0$). Under the change of variable $x = \gamma(s)$, the equation on $C$ becomes

$$\partial_t(\widehat{u}(1+\lambda)) + \partial_s \widehat{u} = 0 \qquad \text{in } \mathscr{D}'((0,T)\times I) \tag{4.2}$$

where $\lambda$ is a suitable singular measure on $I$ and $\widehat{u} = u \circ \gamma$. Note that, due to the particular choice of $\gamma$, the vector field $\boldsymbol{b}$ no longer appears in the equation; furthermore, one can prove that the Cauchy problem for (4.2) admits a unique bounded solution for every bounded initial datum if and only if the measure $\lambda$ is trivial (we will discuss this issue more in detail in the next Chapter 5, Section 5.4). Thus we conclude that uniqueness for (4.1) holds if and only if $\lambda = 0$ for every nontrivial connected component $C$ of a.e. level set $E_h$. It can be shown that this is equivalent to the following condition on the Hamiltonian $H$:

$$H_\sharp\left(\mathscr{L}^2 \llcorner_{S \cap E^\star}\right) \perp \mathscr{L}^1 \tag{4.3}$$

where, we recall, the set $E^\star$ is the union of all connected components with positive length of all level sets of $H$ (see again Theorem III). Condition (4.3) is usually called *Weak Sard Property*, as it is reminiscent of the Sard's property satisfied by all class of functions $C^2$ (in view of Sard's Lemma).

These informal considerations constitute a very general scheme of the proof of the following important

Theorem 4.1 ([**ABC14**, Theorem 4.7]). *Let $H$ and $\boldsymbol{b}$ defined as above. Then the following statements are* equivalent*:*

*(1) if $u\colon [0,T)\times\mathbb{R}^2 \to \mathbb{R}$ is a bounded solution of* (4.1) *then $u = 0$ for a.e. $(t,x) \in [0,T)\times\mathbb{R}^2$;*

*(2) the potential $H$ satisfies the Weak Sard Property* (4.3)*.*

At this point we point out a couple of remarks.

Remark 4.2. It is important to stress that Theorem 4.1 gives a *necessary and sufficient* condition, while the results in the literature usually give only sufficient conditions for uniqueness. As a consequence, the (rotated) gradient of every Lipschitz function without the Weak Sard Property (see, for instance, Chapter 5) is an example of divergence-free autonomous vector field in the plane for which there is no uniqueness of bounded weak solutions to (4.1). ♠

Remark 4.3. In [**ABC14**], the use of the renormalization property is completely avoided but can be obtained as an easy corollary. Indeed, in case of divergence-free and autonomous vector fields $\boldsymbol{b}$, the renormalization property for a weak solution $u$ of the continuity equation simply means that $\beta(u)$ is a weak solution the same equation for every $C^1$ function $\beta\colon \mathbb{R} \to \mathbb{R}$. When the potential $H$ of $\boldsymbol{b}$ satisfies the weak Sard property, this property can be deduced from the renormalization property for the one-dimensional equation $\partial_t \widehat{v} + \partial_s \widehat{v} = 0$. ♠

Remark 4.4. Theorem 4.1 actually can be improved to show uniqueness in the class of weak solutions that are merely *integrable* in space and time (instead of bounded). The key point is that uniqueness holds for the corresponding one-dimensional equation among solutions which are integrable in space and time. ♠

In [**BG16**], this Hamiltonian approach has been adapted to a more general setting, namely the one of *steady nearly incompressible* autonomous vector fields on $\mathbb{R}^2$. In general, a vector field $\boldsymbol{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ is said to be *steady nearly incompressible* when there exists a steady density $r\colon \mathbb{R}^d \to \mathbb{R}$, uniformly bounded from below and above by strictly positive constants, such that $\operatorname{div}(r\boldsymbol{b}) = 0$ in the sense of distributions over $\mathbb{R}^d$. In particular, it is easily seen that this class contains all divergence-free vector fields ($r \equiv 1$); moreover, any steady nearly incompressible vector field is nearly incompressible in the sense of Definition 3.14, but the inverse implication does not hold in general. For instance, consider the one-dimensional vector field $\boldsymbol{b}\colon (0,2) \to \mathbb{R}$ given by $\boldsymbol{b}(x) = |x-1| - 1$. If it were steady nearly incompressible, the function $r \cdot \boldsymbol{b}$ would be constant on $(0,2)$ and thus $r$ could not be uniformly bounded from above by a positive constant. On the other hand this vector field $\boldsymbol{b}$ is nearly incompressible: the solution to the continuity equation $\partial_t \rho + \partial_x(\rho \boldsymbol{b}) = 0$ with the initial condition $\rho|_{t=0} = 1$ satisfies $e^{-t} \le \rho(t,x) \le e^t$, as one can easily demostrate using the classical method of characteristics, since $\boldsymbol{b}$ is Lipschitz.

Adapting the splitting technique of [**ABC14**], in [**BG16**] it has been proved that any steady nearly incompressible vector field of class BV on $\mathbb{R}^2$ has the renormalization property (the assumption $\boldsymbol{b} \in \mathrm{BV}$ in [**BG16**] could be replaced, for instance, by the assumption $\boldsymbol{b} \neq 0$: see also Remark 4.25 in the following).

In the present chapter we want to extend the results of [**BG16**] to the non-steady case and more precisely we want to show the following theorem.

Theorem 4.5. *Every bounded, autonomous, compactly supported, nearly incompressible* BV *vector field on* $\mathbb{R}^2$ *has the renormalization property.*

As an immediate corollary, taking into account the link between renormalization property and uniqueness of weak solutions to transport equation we have

Corollary 4.6. *Let $I = (0,T)$ for some $T > 0$ and let $\boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a compactly supported, nearly incompressible* BV *vector field with density $\rho$. Then*

*(1) $\forall u_0 \in L^\infty(\mathbb{R}^2)$ there exists a unique ($\rho$-)weak solution $u \in L^\infty(I \times \mathbb{R}^2)$ to the transport equation*

$$\partial_t u + \boldsymbol{b} \cdot \nabla u = 0$$

*with the initial condition $u|_{t=0} = u_0$;*

*(2) $\forall u_0 \in L^\infty(\mathbb{R}^2)$ there exists a unique weak solution $u \in L^\infty(I \times \mathbb{R}^2)$ to the continuity equation*

$$\partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0$$

*with the initial condition $u|_{t=0} = u_0$.*

**4.1.1. Notation for this chapter.** Within this chapter we will adopt the following conventions:

- $\Gamma := C([0,T];\mathbb{R}^2)$ will denote the set of continuous curves in $\mathbb{R}^2$;
- $\dot\Gamma := \{\gamma \in \Gamma : \gamma(t) = \gamma(0),\ \forall t \in [0,T]\}$ denotes the set of constant curves (whose graphs are fixed points);
- $\widetilde\Gamma := \Gamma \setminus \dot\Gamma$ denotes the set of non-constant curves (whose graphs have positive length);
- $e_t\colon \Gamma \to \mathbb{R}^2$ is the *evaluation map* at time $t$, i.e. $e_t(\gamma) = \gamma(t)$.

Moreover, if $A \subset \mathbb{R}^2$ is a measurable set,

- $\Gamma_A := \left\{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\right\}$ denotes the set of curves which stay in $A$ for a positive amount of time;
- $\widetilde\Gamma_A := \widetilde\Gamma \cap \Gamma_A$ denotes the set of non-constant curves which stay in $A$ for a positive amount of time;
- $\dot\Gamma_A := \dot\Gamma \cap \Gamma_A$ denotes the set of constant curves which stay in $A$ for a positive amount of time.

- for every $s \in [0,T]$, we denote by

$$\Gamma_A^s := \{\gamma \in \Gamma : \gamma(s) \in A\},$$
$$\widetilde{\Gamma}_A^s := \left\{\gamma \in \widetilde{\Gamma} : \gamma(s) \in A\right\},$$
$$\dot{\Gamma}_A^s := \left\{\gamma \in \dot{\Gamma} : \ \gamma(s) \in A\right\}$$

  accordingly the sets of all curves, non-constant curves and constant curves, which at time $s$ belong to $A$;
- $\mathsf{T}_A := \{\gamma \in \Gamma_A : \ \gamma(0) \notin A, \gamma(T) \notin A\}$ denotes the set of curves which stay in $A$ for a positive amount of time and have the endpoints outside $A$.

Finally, if $A \subseteq \mathbb{R}^2$, we recall that we denote by

$$\mathrm{Conn}(A) := \left\{C \subset A : C \text{ is a connected component of } A\right\},$$
$$\mathrm{Conn}^\star(A) := \left\{C \in \mathrm{Conn}(A) : \mathscr{H}^1(C) > 0\right\},$$

and

$$A^\star := \bigcup_{C \in \mathrm{Conn}^\star(A)} C.$$

## 4.2. Partition of the plane and local disintegration

Let $\boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an *autonomous*, nearly incompressible vector field, with $\boldsymbol{b} \in \mathrm{BV}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$; we assume $\boldsymbol{b}$ is compactly supported (with support in the unit ball of $\mathbb{R}^2$, $\mathbb{B} := B(0,1)$), defined everywhere and Borel.

**4.2.1. Local reduction to the steady case.** Let us consider the countable covering $\mathscr{B}$ of $\mathbb{R}^2$ given by

$$\mathscr{B} := \left\{B(x,r) : \ x \in \mathbb{Q}^2, r \in \mathbb{Q}^+\right\}.$$

For each ball $B \in \mathscr{B}$, we are interested to the trajectories of $\boldsymbol{b}$ which cross $B$, staying inside $B$ for a positive amount of time. We therefore define, for every ball $B \in \mathscr{B}$ and for every rational numbers $s, t \in \mathbb{Q} \cap (0,T)$ with $s < t$, the sets

$$\mathsf{T}_{B,s,t} := \{\gamma \in \Gamma_B : \ \gamma(s) \notin B, \gamma(t) \notin B\},$$

where we recall

$$\Gamma_B = \left\{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in B\}) > 0\right\}.$$

In this first section we will work for simplicity with the sets $\mathsf{T}_B := \mathsf{T}_{B,0,T}$, where $B \in \mathscr{B}$ (and without any loss of generality we assume $T \in \mathbb{Q}$).

Remark 4.7. It is easy to see that

$$\bigcup_{B \in \mathscr{B}} \mathsf{T}_B = \widetilde{\Gamma}.$$

Indeed, for every curve which is moving there exists a point $\gamma(t) \neq \gamma(0), \gamma(T)$, so that one has only to choose a ball in $\mathscr{B}$ containing $\gamma(t)$ but not $\gamma(0)$, $\gamma(T)$. ♠

By Definition 3.14, there exists a function $\rho\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ which satisfies continuity equation (3.19) in $\mathscr{D}'((0,T) \times \mathbb{R}^2)$. Therefore, by Ambrosio's Superposition Principle (Thm. 3.1), there exists a measure $\eta$ on $\Gamma$, concentrated on the set of trajectories of $\boldsymbol{b}$, such that

$$\rho(t,\cdot)\mathscr{L}^2 = (e_t)_\sharp \eta, \tag{4.4}$$

where we recall that $e_t\colon \Gamma \to \mathbb{R}^2$ is the evaluation map $\gamma \mapsto \gamma(t)$. For a fixed ball $B \in \mathscr{B}$, we consider the measure $\eta_B := \eta \llcorner \mathsf{T}_B$ and we define $\rho_B$ by $\rho_B(t,\cdot)\mathscr{L}^2 = (e_t)_\sharp \eta_B$. See also Figure 1. Then we set

$$r_B(x) := \int_0^T \rho_B(t,x)dt, \qquad x \in B. \tag{4.5}$$

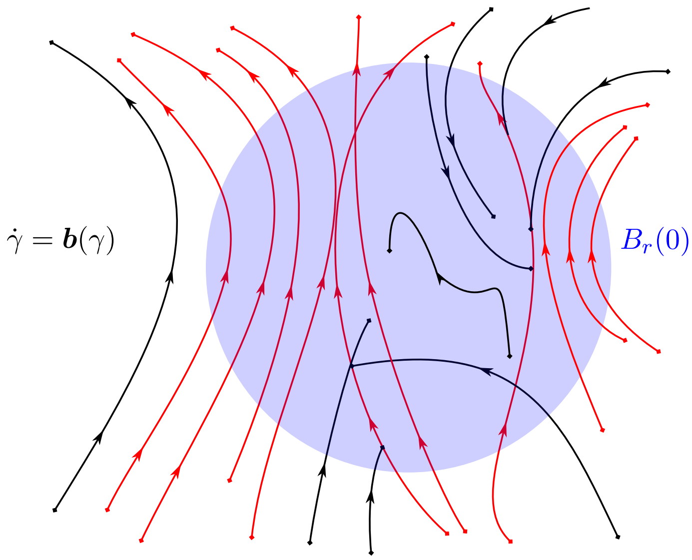



**Figure 1.** By means of Ambrosio's Superposition Principle we find a Lagrangian representation $\eta$ for the density $\rho$ and we further select a suitable family of characteristics crossing the ball $B$ (this produces the "steady" density $r$ defined in (4.5)).

LEMMA 4.8. *It holds* $\operatorname{div}(r_B \boldsymbol{b}) = 0$ *in* $\mathscr{D}'(B)$.

PROOF. For any $\phi \in C_c^\infty(B)$ we have

$$\begin{aligned}\int_B r_B(x)\boldsymbol{b}(x)\cdot\nabla\phi(x)dx &= \int_B\int_0^T \rho_B(t,x)\boldsymbol{b}(x)\cdot\nabla\phi(x)\,dt\,dx\\ &= \int_0^T\int_{\mathsf{T}_B} \boldsymbol{b}(\gamma(t))\cdot(\nabla\phi)(\gamma(t))\,d\eta_B\,dt\\ &= \int_0^T\int_{\mathsf{T}_B} \dot\gamma(t)\cdot(\nabla\phi)(\gamma(t))\,d\eta_B\,dt\\ &= \int_0^T\int_{\mathsf{T}_B} \frac{d}{dt}\phi(\gamma(t))\,d\eta_B\,dt\\ &= \int_{\mathsf{T}_B}\Big[\phi(\gamma(T))-\phi(\gamma(0))\Big]\,d\eta_B = 0.\end{aligned}$$

because for $\eta_B$-a.e. $\gamma \in \mathsf{T}_B$, $\gamma(0)\notin B$, $\gamma(T)\notin B$. □

**4.2.2. Disintegration with respect to Hamiltonians.** From Lemma 4.8 we have $\operatorname{div}(r\boldsymbol{b}) = 0$ in $B$; since $B$ is simply connected, there exists a Lipschitz potential $H_B\colon B \to \mathbb{R}$ such that

$$\nabla^\perp H_B(x) = r_B(x)\boldsymbol{b}(x), \qquad \text{for } \mathscr{L}^2\text{-a.e. } x \in B.$$

Using Theorem III on the Lipschitz function $H_B$, we can define the negligible set $N_1$ such that $E_h$ is regular in $B$ whenever $h \notin N_1$; moreover, let $N_2$ denote the negligible set on which the measure $((H_B)_\sharp\mathscr{L}^2)^{\mathrm{sing}}$ is concentrated, where $((H_B)_\sharp\mathscr{L}^2)^{\mathrm{sing}}$ is the singular part of $((H_B)_\sharp\mathscr{L}^2)$ with respect to $\mathscr{L}^1$. Then we set

$$N := N_1 \cup N_2 \qquad \text{and} \qquad E^* := \cup_{h\notin N} E_h^* \tag{4.6}$$

Therefore we can associate to $B$ a triple $(H_B, N, E)$. For any $x \in E$ let $C_x$ denote the connected component of $E$ such that $x \in C_x$. By definition of $E$ for any $x \in E$ the corresponding connected component $C_x$ has strictly positive length.

Let us fix an arbitrary ball $B \in \mathscr{B}$. For brevity let $H$ denote the corresponding Hamiltonian $H_B$.

LEMMA 4.9 ([ABC14, Lemma 2.8]). *There exist Borel families of measures $\sigma_h, \kappa_h$, $h \in \mathbb{R}$, such that*

$$\mathscr{L}^2 \llcorner_B = \int \left(c_h \mathscr{H}^1 \llcorner_{E_h} + \sigma_h\right) dh + \int \kappa_h \, d\zeta(h), \tag{4.7}$$

*where*

(1) *$c_h \in L^1(\mathscr{H}^1 \llcorner_{E_h^\star})$, $c_h > 0$ a.e.; moreover, by Coarea formula, we have $c_h = 1/|\nabla H|$ a.e. (w.r.t. $\mathscr{H}^1 \llcorner_{E_h^\star}$);*
(2) *$\kappa_h$ is concentrated on $E_h^\star \cap \{\nabla H = 0\}$;*
(3) *$\zeta := H_\sharp \mathscr{L}^2 \llcorner_{(B \setminus E^\star)}$ is concentrated on $N$ (hence $\zeta \perp \mathscr{L}^1$);*
(4) *$\sigma_h$ is concentrated on $E_h^\star \cap \{\nabla H = 0\}$;*
(5) *$\sigma_h \perp \mathscr{H}^1$ for $\mathscr{L}^1$-a.e. $h \notin N$;*
(6) *if $\boldsymbol{b} \in \mathrm{BV}$, then $\sigma_h$ is concentrated on $E_h \cap \{\boldsymbol{b} \neq 0, r_B = 0\}$.*

PROOF. Points (1)-(4) are exactly [ABC14, Lemma 2.8]. Concerning Claim (5), using Coarea formula (II.4)), we can show

$$\mathscr{H}^1(E_h \cap \{\nabla H = 0\}) = 0$$

for $\mathscr{L}^1$-a.e. $h \notin N$. Therefore $\sigma_h \perp \mathscr{H}^1$ for $\mathscr{L}^1$-a.e. $h \notin N$. Finally, Point (6) can be proved using minor modifications of the proof of [BG16, Theorem 8.2]: indeed, if $\boldsymbol{b}$ is of class BV and hence approximately differentiable a.e., then $H_\sharp \mathscr{L}^2 \llcorner_{\{\boldsymbol{b}=0\}} \perp \mathscr{L}^1$: by comparing two disintegrations of $\mathscr{L}^2 \llcorner_{\{\boldsymbol{b}=0\}}$ we conclude that $\sigma_h$ is concentrated on $\{\boldsymbol{b} \neq 0\}$ for a.e. $h$. □

REMARK 4.10. Thanks to (4.7) we always can add to $N$, if necessary, an $\mathscr{L}^1$-negligible set so that for any $h \notin N$ for $\mathscr{H}^1$-a.e. $x \in E_h^\star$ we have $r(x) > 0$, $\boldsymbol{b}(x) \neq 0$ and $r(x)\boldsymbol{b}(x) = \nabla^\perp H(x)$. ♠

**4.2.3. Reduction of the equation on the level sets.** Our goal is now to study the equation $\mathrm{div}(u\boldsymbol{b}) = \mu$, where $u$ is a bounded Borel function on $\mathbb{R}^2$ and $\mu$ is a Radon measure on $\mathbb{R}^2$, inside a ball from the collection $\mathscr{B}$.

LEMMA 4.11. *Suppose that $\mu$ is a Radon measure on $\mathbb{R}^2$ and $u \in L^\infty(\mathbb{R}^2)$. Then equation*

$$\mathrm{div}(u\boldsymbol{b}) = \mu \tag{4.8}$$

*holds in $\mathscr{D}'(B)$ if and only if:*

- *the disintegration of $\mu$ with respect to $H$ has the form*

$$\mu = \int \mu_h \, dh + \int \nu_h \, d\zeta(h), \tag{4.9}$$

  *where $\zeta$ is defined in Point (3) of Lemma 4.9;*
- *for $\mathscr{L}^1$-a.e. $h$*

$$\mathrm{div}\left(u c_h \boldsymbol{b} \mathscr{H}^1 \llcorner_{E_h}\right) + \mathrm{div}(u\boldsymbol{b}\sigma_h) = \mu_h; \tag{4.10}$$

- *for $\zeta$-a.e. $h$*

$$\mathrm{div}(u\boldsymbol{b}\kappa_h) = \nu_h. \tag{4.11}$$

PROOF. Let $\lambda^s$ be a measure on $\mathbb{R}$ such that $H_\sharp|\mu| \ll \mathscr{L}^1 + \zeta + \lambda^s$, where $\zeta$ is defined as in Lemma 4.9 and $\lambda^s \perp \mathscr{L}^1 + \zeta$. Applying the Disintegration Theorem, we have that

$$\mu = \int \mu_h dh + \int \nu_h d\zeta(h) + \int \lambda_h d\lambda^s(h), \tag{4.12}$$

with $\mu_h, \nu_h, \lambda_h$ concentrated on $\{H = h\}$. Writing equation (4.8) in distribution form we get

$$\int_{\mathbb{R}^2} u(\boldsymbol{b} \cdot \nabla\phi)\, dx + \int \phi d\mu = 0, \qquad \forall \phi \in C_c^\infty(B).$$

By an elementary approximation argument, it is clear that we can use as test functions $\phi$ Lipschitz with compact support.

Using the disintegration of Lebesgue measure (4.7) and the disintegration (4.12) we thus obtain

$$\begin{aligned}\int \left[\int_{\mathbb{R}^2} uc_h(\boldsymbol{b} \cdot \nabla\phi)\, d\mathscr{H}^1 \llcorner_{E_h} + \int_{\mathbb{R}^2} u(\boldsymbol{b} \cdot \nabla\phi)\, d\sigma_h\right] dh \\ + \int\int_{\mathbb{R}^2} u(\boldsymbol{b} \cdot \nabla\phi)\, d\kappa_h\, d\zeta(h) + \int\int_{\mathbb{R}^2} \phi\, d\mu_h\, dh \\ + \int\int_{\mathbb{R}^2} \phi\, d\nu_h\, d\zeta(h) + \int\int_{\mathbb{R}^2} \phi\, d\lambda_h\, d\lambda^s(h) = 0,\end{aligned} \tag{4.13}$$

for every $\phi \in \mathrm{Lip}_c(B)$. In particular we can take

$$\phi(x) = \psi(H(x))\varphi(x), \qquad \psi \in C^\infty(\mathbb{R}),\ \varphi \in C_c^\infty(B),$$

so that we can rewrite (4.13) as

$$\begin{aligned}\int \psi(h)\left[\int_{\mathbb{R}^2} uc_h(\boldsymbol{b} \cdot \nabla\varphi)\, d\mathscr{H}^1 \llcorner_{E_h} + \int_{\mathbb{R}^2} u(\boldsymbol{b} \cdot \nabla\varphi)\, d\sigma_h\right) dh \\ + \int \psi(h)\int_{\mathbb{R}^2} u(\boldsymbol{b} \cdot \nabla\varphi)\, d\kappa_h\, d\zeta(h) + \int \psi(h)\int_{\mathbb{R}^2} \varphi\, d\mu_h\, dh \\ + \int \psi(h)\int_{\mathbb{R}^2} \varphi\, d\nu_h\, d\zeta(h) + \int \psi(h)\int_{\mathbb{R}^2} \varphi\, d\lambda_h\, d\lambda^s(h) = 0,\end{aligned}$$

because

$$\boldsymbol{b}(x) \cdot \nabla\phi(x) = \psi(H(x))\boldsymbol{b}(x) \cdot \nabla\varphi(x)$$

for $\mathscr{L}^2$-a.e. $x \in \mathbb{R}^2$. Since the equalities above hold for all $\psi \in C^\infty(\mathbb{R})$ we have

$$\int \left[\int_{\mathbb{R}^2} uc_h(\boldsymbol{b} \cdot \nabla\varphi)\, d\mathscr{H}^1 \llcorner_{E_h} + \int_{\mathbb{R}^2} u(\boldsymbol{b} \cdot \nabla\varphi)\, d\sigma_h\right] dh + \int\int_{\mathbb{R}^2} \varphi\, d\mu_h\, dh = 0,$$

$$\int \left[\int_{\mathbb{R}^2} u(\boldsymbol{b} \cdot \nabla\varphi)\, d\kappa_h + \int_{\mathbb{R}^2} \varphi\, d\nu_h\right] d\zeta(h) = 0,$$

$$\int\int_{\mathbb{R}^2} \varphi\, d\lambda_h\, d\lambda^s(h) = 0,$$

which give, respectively, (4.10), (4.11) and (4.9).

□

**4.2.4. Reduction on connected components of level sets.** If $K \subset \mathbb{R}^d$ is a compact then, in general, not any connected component $C$ of $K$ can be separated from $K \setminus C$ by a smooth function. However, it can be separated by a sequence of such functions:

LEMMA 4.12 ([**ABC13**, Section 2.8], [**BG16**, Lemma 5.3]). *If $K \subset \mathbb{R}^d$ is compact then for any connected component $C$ of $K$ there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that*

*(1) $0 \le \phi_n \le 1$ on $\mathbb{R}^d$ and $\phi_n \in \{0,1\}$ on $K$ for all $n \in \mathbb{N}$;*
*(2) for any $x \in C$, we have $\phi_n(x) = 1$ for every $n \in \mathbb{N}$;*
*(3) for any $x \in K \setminus C$, we have $\phi_n(x) \to 0$ as $n \to +\infty$;*
*(4) for any $n \in \mathbb{N}$, we have* $\operatorname{supp} \nabla\phi_n \cap K = \emptyset$.

With the aid of this lemma we can now study the equation (4.10) on the nontrivial connected components of the level sets. In view of Lemma 4.11 in what follows we always assume that $h \notin N$ (see (4.6)).

LEMMA 4.13. *The equation* (4.10) *holds iff*

- *for any nontrivial connected component $C$ of $E_h$ it holds*
$$\operatorname{div}\left(uc_h \boldsymbol{b}\mathscr{H}^1 \llcorner_C\right) + \operatorname{div}(u\boldsymbol{b}\sigma_h \llcorner_C) = \mu_h \llcorner_C; \tag{4.14}$$
- *it holds*
$$\operatorname{div}(u\boldsymbol{b}\sigma_h \llcorner_{(E_h \setminus E_h^\star)}) = \mu_h \llcorner_{(E_h\setminus E_h^\star)}. \tag{4.15}$$

PROOF. For any Borel set $A \subset \mathbb{R}^2$ we introduce the following functional
$$\Lambda_A(\psi) := \int_A uc_h(\boldsymbol{b}\cdot\nabla\psi)\, d\mathscr{H}^1\llcorner_{E_h} + \int_A u(\boldsymbol{b}\cdot\nabla\psi)\, d\sigma_h + \int_A \psi\, d\mu_h,$$
for all $\psi \in C_c^\infty(B)$. Now fix a connected component $C$ of $E_h$ and take a sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ given by Lemma 4.12 (applied with $K := E_h$). By assumption, we have that $\Lambda(\psi\phi_n) = 0$ for every $\psi \in C_c^\infty(B)$ and for every $n$. Let us pass to the limit as $n \to \infty$. On one hand we have
$$\int \psi\phi_n\, d\mu_h = \int_C \psi\, d\mu + \int_{E_h\setminus C} \psi\phi_n\, d\mu \to \int_C \psi\, d\mu$$
because the second term converges to 0 since $\phi_n \to 0$ pointwise on $E_h \setminus C$. On the other hand $\nabla(\psi\phi_n) = \psi\nabla\phi_n + \phi_n\nabla\psi$. In the terms with $\phi_n\nabla\psi$ we pass to the limit as above. The terms with the product $\psi\nabla\phi_n$ identically vanish thanks to the condition (4) on $\phi_n$ in Lemma 4.12. Therefore, we have that for every $\psi \in C_c^\infty(B)$
$$\Lambda_{E_h}(\psi\phi_n) \to \int_C uc_h(\boldsymbol{b}\cdot\nabla\psi)\, d\mathscr{H}^1 + \int_C u(\boldsymbol{b}\cdot\nabla\psi)\, d\sigma_h + \int_C \psi\, d\mu_h = \Lambda_C(\psi),$$
as $n \to +\infty$. Since $\Lambda_{E_h}(\psi\phi_n) = 0$ for every $n$, we deduce that $\Lambda_C(\psi) = 0$ and this gives (4.14). In order to get (4.15), it is enough to observe that $E_h^\star$ is a countable union of connected components $C$, therefore (from the previous step) we deduce that
$$\int_{E_h^\star} uc_h(\boldsymbol{b}\cdot\nabla\psi)\, d\mathscr{H}^1 + \int_{E_h^\star} u(\boldsymbol{b}\cdot\nabla\psi)\, d\sigma_h + \int_{E_h^\star} \psi\, d\mu_h = 0, \qquad \forall\psi \in C_c^\infty(B).$$
Hence
$$\Lambda_{E_h\setminus E_h^\star} := \int_{E_h^\star\setminus E_h} uc_h(\boldsymbol{b}\cdot\nabla\psi)\, d\mathscr{H}^1 + \int_{E_h^\star\setminus E_h} u(\boldsymbol{b}\cdot\nabla\psi)\, d\sigma_h + \int_{E_h^\star\setminus E_h} \psi\, d\mu_h = 0,$$
for every $\psi \in C_c^\infty(B)$. Remembering that $\mathscr{H}^1(E_h^\star \setminus E_h) = 0$ by Theorem III we get (4.15) and this concludes the proof. The converse implication can be easily obtained by summing the equations (4.14) and (4.15). ☐

LEMMA 4.14. *Equation* (4.14) *holds iff*
$$\operatorname{div}\left(uc_h\boldsymbol{b}\mathscr{H}^1\llcorner_C\right) = \mu_h\llcorner_C, \tag{4.16a}$$
$$\operatorname{div}(u\boldsymbol{b}\sigma_h\llcorner_C) = 0. \tag{4.16b}$$

The proof of Lemma 4.14 would be fairly easy in the case when $C$ is a straight line. Roughly saying, in this case (4.14) would read as
$$\int u(x)c_h(x)\boldsymbol{b}(x)\psi'(x)\, dx + \int u(x)c_h(x)\boldsymbol{b}(x)\psi'(x)\, d\sigma_h(x) + \int \psi(x)\, d\mu(x) = 0,$$
$\psi \in C_0^\infty(\mathbb{R})$. Since $\sigma_h$ is concentrated on a $\mathscr{L}^1$-negligible set $S$, any $\phi \in C_0^1$ can be approximated in $C^0$-norm with a sequence of $C^1$-functions $\phi_n$ having 0-derivative on $S$. Consequently, $\phi_n'$ converge to $\phi'$ weak* in $L^\infty$ as $n \to \infty$. Then, substituting $\psi = \phi_n$ and passing to the limit as $n \to \infty$ we get
$$\int u(x)c_h(x)\boldsymbol{b}(x)\phi'(x)\, dx + \int \phi(x)\, d\mu(x) = 0.$$
Hence the only technicality here is to repeat this argument on a curve.

Before presenting the formal proof of Lemma 4.14 we would like to discuss the parametric version of the equation (4.16a). Let $\gamma\colon I \to \mathbb{R}^2$ be an injective Lipschitz parametrization of $C$, where $I = \mathbb{R}/\ell\mathbb{Z}$ or $I = (0,\ell)$ for some $\ell > 0$ is the domain of $\gamma$. In view of Remark 4.10) we can assume that the directions of $\boldsymbol{b}$ and $\nabla^\perp H$ agree $\mathscr{H}^1$-a.e. on $C$. So there exists a constant $\varpi \in \{+1, -1\}$ such that

$$\frac{\boldsymbol{b}(\gamma(s))}{|\boldsymbol{b}(\gamma(s))|} = \varpi \frac{\gamma'(s)}{|\gamma'(s)|} \tag{4.17}$$

for a.e. $s \in I$. We will say that $\gamma$ is an *admissible parametrization* of $C$ if $\varpi = +1$. In the rest of the text we will consider only admissible parametrizations of the connected components $C$.

Lemma 4.15. *Equation (4.16a) holds iff for any admissible parametrization $\gamma$ of $C$*

$$\partial_s(\widehat{u}\widehat{c}_h|\widehat{\boldsymbol{b}}|) = \widehat{\mu}_h$$

*where $\gamma_\sharp\widehat{\mu}_h = \mu_h \llcorner C$, $\widehat{u} = u\circ\gamma$, $\widehat{c}_h = c_h \circ \gamma$ and $\widehat{\boldsymbol{b}} = \boldsymbol{b}\circ\gamma$.*

In the proof of Lemma 4.15 we will use the following result:

Lemma 4.16 ([**ABC13**, Section 7]). *Let $a \in L^1(I)$ and $\mu$ a Radon measure on $I$, where $I = \mathbb{R}/\ell\mathbb{Z}$ or $I = (0,\ell)$ for some $\ell > 0$. Suppose that $\gamma\colon I \to \Omega$ is an injective Lipschitz function such that $\gamma' \neq 0$ a.e. on $I$ and $\gamma(0,\ell) \subset \Omega$. Consider the functional*

$$\Lambda(\phi) := \int_I \phi' a\, dt + \int_I \phi\, d\mu, \qquad \forall \phi \in \mathrm{Lip}_c(I).$$

*If $\Lambda(\varphi\circ\gamma) = 0$ for any $\varphi \in C_c^\infty(\Omega)$ then $\Lambda(\phi) = 0$ for any $\phi \in \mathrm{Lip}_c(I)$.*

Proof of Lemma 4.15. In view of Area Formula (II.3), if $\gamma\colon I \to \mathbb{R}^2$ is an injective Lipschitz parametrization of $C$ then

$$\mathscr{H}^1 \llcorner C = \gamma_\sharp \left(|\gamma'|\mathscr{L}^1\right).$$

Using this formula the distributional version of (4.16a),

$$\int_C u c_h \boldsymbol{b}\cdot\nabla\phi d\mathscr{H}^1 \llcorner C + \int_C \phi d\mu_h = 0, \quad \forall \phi \in C_c^\infty(B),$$

can be written as

$$\int_I u(\gamma(s))c_h(\gamma(s))\boldsymbol{b}(\gamma(s))\cdot(\nabla\phi)(\gamma(s))|\gamma'(s)|\,ds + \int_I \phi(\gamma(s))d\widehat{\mu}_h(s) = 0$$

where $\widehat{\mu}_h$ is defined by $\widehat{\mu}_h := \left(\gamma^{-1}\right)_\sharp \mu_h$.

Using (4.17) we can write the equation above as

$$\int_I u(\gamma(s))c_h(\gamma(s))\gamma'(s)(\nabla\phi)(\gamma(s))|\boldsymbol{b}(\gamma(s))|\,ds + \int_I \phi(\gamma(s))d\widehat{\mu}_h(s) = 0,$$

which reads as

$$\int_I \widehat{u}(s)\widehat{c}_h(s)\partial_s\phi(\gamma(s))|\widehat{\boldsymbol{b}}(s)|\,ds + \int_I \phi(\gamma(s))d\widehat{\mu}_h(s) = 0.$$

Since the equation above holds for any $\phi \in C_c^\infty(B)$ it remains to apply Lemma 4.16. ☐

Proof of Lemma 4.14. Let us write $\Lambda(\phi) = M(\phi) + N(\phi)$, where

$$M(\phi) := \int_C u c_h(\boldsymbol{b}\cdot\nabla\phi)\,d\mathscr{H}^1 + \int_C \phi\,d\mu_h$$

and

$$N(\phi) := \int_C u\boldsymbol{b}\cdot\nabla\phi d\sigma_h$$

for every $\phi \in C_c^\infty(B)$. Fix a test function $\phi$: we are going to “perturb” $\phi$ in such a way that $N(\phi)$ becomes arbitrarily small and $M(\phi)$ remains almost unchanged. Since $\Lambda(\phi) = 0$

we will obtain that $|M(\phi)| < \varepsilon$ and this will imply that $M(\phi) = N(\phi) = 0$. By Lemma 4.9, we have $\sigma_h \perp \mathscr{H}^1 \llcorner_C$ therefore there exists a $\mathscr{H}^1$-negligible set $S \subset C$ such that $\sigma_h$ is concentrated on $S$. Moreover, by inner regularity, for every $n \in \mathbb{N}$, we can find a compact $K \subset S$ such that

$$\sigma_h(S \setminus K) < \frac{1}{n}.$$

Using the fact that $\mathscr{H}^1(K) = 0$, for every $n \in \mathbb{N}$, we can find countably many open balls $\{B_{r_j}(z_j)\}_{j\in\mathbb{N}}$ which cover $K$ and whose radii $r_j$ satisfy

$$\sum_{j\in\mathbb{N}} r_j < \frac{1}{n}.$$

Furthermore, by compactness, we can extract from $\{B_{r_j}(z_j)\}_{l\in\mathbb{N}}$ a finite subcovering, $\{B_{r_j}(z_j)\}$ with $j = 1, \dots, \nu$ where $\nu = \nu(n) \in \mathbb{N}$ (we stress that $\nu$ depends on $n$). For every $j \in \{1, \dots, \nu\}$, let

$$P_i^{j,n} := (z_{j,i} - r_j, z_{j,i} + r_j)$$

denote the projection of $B_{r_j}(z_j)$ onto the $x_i$-axis, with $i = 1, 2$. Since $P_i^{j,n}$ is an open interval we can find a smooth function $\psi_i^{j,n} \colon \mathbb{R} \to \mathbb{R}$ such that

$$\psi_i^{j,n}(\xi) = \begin{cases} 0 & \xi \in P_i^{j,n}, \\ 1 & \operatorname{dist}(\xi, \partial P_i^{j,n}) > 2r_i, \end{cases}$$

and $0 \leq \psi_i^{j,n} \leq 1$ for every $\xi \in \mathbb{R}$. Now we consider the product $\psi_i^n := \psi_i^{1,n}\psi_i^{2,n} \cdots \psi_i^{\nu,n}$ and we define the functions $\chi_i^n \colon \mathbb{R} \to \mathbb{R}$ as

$$\chi_i^n(\xi) := \int_0^\xi \psi_i^n(w)\, dw$$

for $i = 1, 2$ and $n \in \mathbb{N}$. Now we set $\chi^n(x) := (\chi_1^n(x), \chi_2^n(x))$ and $\phi_n := \phi \circ \chi^n$. Since $\|\chi^n - \mathrm{id}\|_\infty \leq 4 \sum_i r_i \leq \frac{4}{n}$ we deduce that $\phi_n \to \phi$ uniformly in $C$ because

$$|\phi_n(x) - \phi(x)| \leq \|\nabla\phi\|_\infty \|\chi^n - \mathrm{id}\|_\infty \to 0$$

as $n \to +\infty$. Let us now take an admissible parametrization of $C$, $\gamma \colon I \to \mathbb{R}$, and let us introduce the functions $\widehat{\phi}_n := \phi_n \circ \gamma$. Using for instance the density of $C^1$ functions in $L^1(I)$, we can actually show that $\partial_s \widehat{\phi}_n \rightharpoonup^\star \partial_s \widehat{\phi}$ in weak$^\star$ topology of $L^\infty$. Passing to the parametrization as in the proof of Lemma 4.15 we get

$$\int_C u c_h (b \cdot \nabla\phi_n)\, d\mathscr{H}^1 = \int_I \widehat{u}\widehat{c}_h \widehat{b}\, \partial_s \widehat{\phi}_n\, ds,$$

where we denote by $\widehat{\cdot}$ the composition with $\gamma$. Using weak$^\star$ convergence, we obtain that

$$\int_C u c_h (b \cdot \nabla\phi_n)\, d\mathscr{H}^1 \to \int_C u c_h (b \cdot \nabla\phi)\, d\mathscr{H}^1.$$

On the other hand, by uniform convergence, we immediately get

$$\int \phi_n\, d\mu_h \to \int \phi\, d\mu_h,$$

as $n \to +\infty$. In particular, we have that $M(\phi_n) \to M(\phi)$. Now observe that $\nabla\phi_n = 0$ on $K$ by construction, hence we get

$$N(\phi_n) \leq \int_{S\setminus K} |ub||\nabla\phi_n| d\sigma_h \leq \|ub\|_\infty \|\nabla\phi\|_\infty \frac{1}{n} \to 0$$

and this implies that $N(\phi) = 0$. Therefore, $0 = \Lambda(\phi) = M(\phi)$, which concludes the proof. ☐

We note, in particular, that from (4.16b), being $\boldsymbol{b} \in \mathrm{BV}$ and taking $u \equiv 1$ in (4.8), we have that $\operatorname{div}(\boldsymbol{b}\sigma_{h \llcorner E_h}) = 0$ for a.e. $h$.

Let

$$F := \{\boldsymbol{b} \neq 0, r_B = 0\} \cap E. \tag{4.18}$$

By Point (5) of Lemma 4.9, $\sigma_h$ is concentrated on $F \cap E_h$ hence we have

$$\operatorname{div}(\mathbb{1}_F \boldsymbol{b}\sigma_h) = 0, \qquad \text{for } \mathscr{L}^1\text{-a.e. } h. \tag{4.19}$$

This important piece of information is very useful to prove the following

Lemma 4.17. *We have* $\operatorname{div}(\mathbb{1}_F \boldsymbol{b}) = 0$ *in* $\mathscr{D}'(B)$.

Proof. For every test function $\phi \in C_c^\infty(B)$, we have

$$\int_F \boldsymbol{b}(x) \cdot \nabla\phi(x)\, dx = \int \int_{F \cap E_h} \boldsymbol{b}(x) \cdot \nabla\phi(x)\, d\sigma_h(x) dh.$$

Using again Point (5) of Lemma 4.9 and (4.19), we get that

$$\int_{F \cap E_h} \boldsymbol{b}(x) \cdot \nabla\phi(x)\, d\sigma_h(x) = 0$$

and then we conclude. □

Finally, let us mention a covering property of the set $E^\star$:

Lemma 4.18. *Let* $E^\star$ *be the set defined in* (4.6). *Then*

$$E^\star \supset \{\nabla H \neq 0\} \mod \mathscr{L}^2.$$

Proof. Suppose that $P := \{\nabla H \neq 0\} \setminus E$ has positive measure. Then

$$0 < \int_P |\nabla H|\, dx = \int \int \mathbb{1}_P \, d\mathscr{H}^1 \llcorner_{E_h} \, dh = 0$$

where the first equality is due to Coarea Formula (II.4) and the second equality holds since $\mathbb{1}_P$ is zero on $E_h$ for a.e. $h$. □

Note that in general $E^\star$ can contain a subset of $\{\nabla H = 0\}$ with positive measure (see [**ABC13**]). However, in the next section we show that, if $H$ has the *weak Sard property*, then in fact $E^\star = \{\nabla H \neq 0\} \mod \mathscr{L}^2$.

## 4.3. Weak Sard Property of Hamiltonians

**4.3.1. Matching properties.** As we have seen at the beginning of Section 4.2.2, to every Hamiltonian $H$ we can associate a triple $(H, N, E)$ where $N$ is the set given by Theorem III and $E = \cup_{h \notin N} E_h^\star$.

Suppose now we have another triple $(\widetilde{H}, \widetilde{N}, \widetilde{E})$; we ask whether, given $x \in E \cap \widetilde{E}$ it is true that $C_x = \widetilde{C}_x$. This is essentially the definition of matching property; moreover, we will prove the "Matching Lemma", which states that gradients of $H$ and $\widetilde{H}$ being parallel (in a simply connected set) is a sufficient condition for matching.

**4.3.2. Matching of two Hamiltonians.** Let us consider two Lipschitz Hamiltonians $H_1$ and $H_2$, defined on the same open, simply connected set $A$; according to Theorem III, we have two negligible sets $N_1$ and $N_2$ such that the level sets $E_h^1$ and $E_{h'}^2$ of $H_1$ and $H_2$ are regular for $h \notin N_1$ and $h' \notin N_2$. We set $E_1 := \cup_{h \notin N_1} E_h^1$ and $E_2 := \cup_{h' \notin N_2} E_{h'}^2$.

Definition 4.19. The Hamiltonians $H_1$ and $H_2$ *match* in an open subset $A' \subset A$ if $C_x^1 = C_x^2$ for $\mathscr{L}^2$-a.e. $x \in A' \cap E_1 \cap E_2$, where $C_x^i$ denotes the connected component in $A'$ of the level sets $H_i^{-1}(H_i(x))$ which contains $x$.

As usual, given two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^2$ we write $\boldsymbol{v}_1 \parallel \boldsymbol{v}_2$ if $\boldsymbol{v}_1 = \alpha\boldsymbol{v}_2$ or $\boldsymbol{v}_2 = \alpha\boldsymbol{v}_1$ for some real number $\alpha$.

We now state and prove the following

Lemma 4.20 (Matching lemma). *Let $H_1, H_2$ be defined as above. If $\nabla H_1 \parallel \nabla H_2$ a.e. on $A' \subset A$ open, then the Hamiltonians $H_1$ and $H_2$ match in $A'$.*

Proof. Let $\boldsymbol{b}_1 := \nabla^\perp H_1$. Then $\operatorname{div} \boldsymbol{b}_1 = 0$. Let us prove that

$$\operatorname{div}(H_2 \boldsymbol{b}_1) = 0 \tag{4.20}$$

in the sense of distributions. Indeed, we have for every $\varphi \in \mathrm{Lip}_c(A')$

$$\int H_2(\boldsymbol{b}_1 \cdot \nabla\varphi)\, dx = \int \big[\boldsymbol{b}_1 \cdot \nabla(H_2\varphi) - \varphi(\boldsymbol{b}_1 \cdot \nabla H_2)\big]\, dx.$$

The first term is zero because $\operatorname{div} \boldsymbol{b}_1 = 0$ (and $\varphi H_2$ can be used as test function since it is Lipschitz); the second term is also zero because $\nabla H_2 \parallel \nabla H_1$ a.e. on $A'$, hence $\boldsymbol{b}_1 \perp \nabla H_2$ a.e. on $A'$.

From (4.20), using [**BG16**, Theorem 4.1 and 6.1], we obtain that there exists a $\mathscr{L}^1$ negligible set $N$ such that $H_2$ is constant on every non trivial connected components $C \cap A'$ of the level sets of $H_1$ which do not correspond to values in $N$. By disintegration, we have that the sets of points $x \in A' \cap E_1$ such that $H_1(x) \notin N$ are a negligible set and therefore we can infer that for a.e. $x \in A' \cap E_1$, $H_2$ is constant along the connected components in $A'$ of the level sets of $H_1$. By repeating the same argument for $H_2$ we get the claim. □

**4.3.3. The Weak Sard property.** Let us begin this section with the following remark concerning Weak Sard Property.

Remark 4.21. Informally, the Weak Sard Property means that the "good" level sets of $H$ do not intersect the critical set $S$, apart from a negligible set. In terms of the disintegration of the Lebesgue measure (4.7), we can say that $H$ has the weak Sard property if and only if $\sigma_h = 0$ for a.e. $h$. ♠

Now we give the following

Definition 4.22. We set

$$\widetilde{r}_B := r_B + \mathbb{1}_F,$$

where we recall that $r_B$ is the function defined in (4.5) and $F$ is the set defined in (4.18).

By linearity of divergence, by Lemma 4.8 and Lemma 4.17, we have

$$\operatorname{div}(\widetilde{r}_B \boldsymbol{b}) = 0$$

in $\mathscr{D}'(B)$. Therefore, we conclude that there exists a Lipschitz potential $\widetilde{H}$ such that $\nabla \widetilde{H}^\perp = \widetilde{r}_B b$.

Moreover, we observe that $\nabla H \parallel \nabla \widetilde{H}$ a.e. in $B$: therefore we can apply Matching Lemma 4.20 to get that the regular level sets of $H$ and of $\widetilde{H}$ agree. In particular, we obtain $E = \widetilde{E} \mod \mathscr{L}^2$, directly from the definition of $\widetilde{H}$. We note also that the function $\widetilde{H}$ has the Weak Sard property: indeed, directly from the construction, we have $\nabla \widetilde{H} \neq 0$ on $E$ hence, since $E = \widetilde{E} \mod \mathscr{L}^2$, it follows that $\mathscr{L}^2(\widetilde{E} \cap \widetilde{S}) = 0$.

Finally, disintegrating $\mathscr{L}^2 \llcorner_E$ with respect to $H$ we get

$$\mathscr{L}^2 \llcorner_E = \int_{\mathbb{R}} (c_h \mathscr{H}^1 \llcorner_{E_h} + \sigma_h)\, dh,$$

while using the Hamiltonian $\widetilde{H}$

$$\mathscr{L}^2 \llcorner_E = \int_{\mathbb{R}} \widetilde{c}_h \mathscr{H}^1 \llcorner_{\widetilde{E}_h}\, dh.$$

In particular, it follows that $\sigma_h = 0$ for a.e. $h$, which means that $H = \widetilde{H}$ (up to additive constants) and $H$ has the Weak Sard Property.

We collect this result in the following

Lemma 4.23. *The Hamiltonian $H_B$ has the weak Sard property.*

We conclude this section with the following corollary concerning the covering properties of the set $E^\star$ defined in (4.6):

COROLLARY 4.24. *Suppose that $H$ has the weak Sard property. Let $E^\star$ be the set defined in (4.6). Then*

$$E^\star = \{\nabla H \neq 0\} \mod \mathscr{L}^2.$$

PROOF. The argument is similar to Lemma 4.18. Let $Q = E^\star \setminus \{\nabla H \neq 0\}$. By (4.7)

$$\mathscr{L}^2(Q) = \int \left( \int_Q d\sigma_h \right) dh = 0$$

since by Remark 4.21 $\sigma_h = 0$ for a.e. $h$. □

REMARK 4.25 (On the BV assumption). If we do not assume BV regularity of $\boldsymbol{b}$, but $\boldsymbol{b}(x) \neq 0$ for $\mathscr{L}^2$-a.e. $x \in \mathbb{R}^2$ the conclusion of Lemma 4.23 still holds. This can be proved using minor modifications of the above argument. More precisely, since $\boldsymbol{b}$ is nearly incompressible the function $m(x) := \int_0^T \rho(\tau, x)\, d\tau$, where $\rho$ is the density of $\boldsymbol{b}$, solves

$$\operatorname{div}(m\boldsymbol{b}) = \rho(T, \cdot) - \rho(0, \cdot)$$

in $\mathscr{D}'(B)$, being $\rho(T, \cdot)$ and $\rho(0, \cdot)$ the weak-$\star$ limits in $L^\infty$ of $\rho(t, \cdot)$ as $t \to T$ and $t \to 0$ respectively. Applying Lemmata 4.11, 4.13, 4.14 with $u = m$, from (4.16b) we obtain

$$\operatorname{div}(m\boldsymbol{b}\sigma_{h \llcorner C}) = 0.$$

Hence Lemma 4.17 holds replacing $\mathbb{1}_F \boldsymbol{b}$ with $m\mathbb{1}_F \boldsymbol{b}$: in particular, setting

$$\widetilde{r}_B := r_B + m\mathbb{1}_F$$

we can repeat the argument of Section 4.3. ♠

## 4.4. Level sets and trajectories I

In this section, we assume that $H_B$ is defined on all $\mathbb{R}^2$ (using standard theorems for the extension of Lipschitz maps).

**4.4.1. Trajectories.** We now present some lemmata which relate the trajectories $\gamma \in \mathsf{T}_B$ to the level sets of the Hamiltonian. The first result we prove is that $\eta$-a.e. $\gamma$ is contained in a level set.

LEMMA 4.26. *Let $B \in \mathscr{B}$, $t_1, t_2 \in [0, T]$ and set $\mathsf{T} := \{\gamma : \gamma((t_1, t_2)) \subset B\}$. Then $\eta$-a.e. $\gamma \in \mathsf{T}$ we have $(t_1, t_2) \ni t \mapsto H(\gamma(t))$ is a constant function.*

PROOF. Let $(\varrho_\varepsilon)_\varepsilon$ be the standard family of convolution kernels in $\mathbb{R}^2$. We set $H_\varepsilon(x) := H * \varrho_\varepsilon(x)$ for any $x \in B$. For every $t \in [t_1, t_2]$ define

$$I(t) := \int_{\mathsf{T}} |H(\gamma(t)) - H(\gamma(0))| d\eta(\gamma)$$

and we will prove $I \equiv 0$. First note that $I$ is positive because the integrand is non-negative and $\eta$ is positive. On the other hand,

$$\begin{aligned} I(t) \leq & \underbrace{\int_{\mathsf{T}} |H(\gamma(t)) - H_\varepsilon(\gamma(t))| d\eta(\gamma)}_{I_1^\varepsilon} + \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H_\varepsilon(\gamma(0))| d\eta(\gamma)}_{I_2^\varepsilon} \\ & + \underbrace{\int_{\mathsf{T}} |H_\varepsilon(\gamma(0)) - H(\gamma(0))| d\eta(\gamma)}_{I_3^\varepsilon}. \end{aligned}$$

Now for a.e. $x \in \mathbb{R}^2$ we have $H_\varepsilon(x) \to H(x)$: hence

$$\int_{\mathsf{T}} |H_\varepsilon(\gamma(t)) - H(\gamma(t))|\, d\eta(\gamma) \leq \int_B |H_\varepsilon(x) - H(x)| \rho(t, x) dx \to 0$$

as $\varepsilon \to 0$. Therefore, we can infer that

$$I_1^\varepsilon \to 0, \qquad I_3^\varepsilon \to 0$$

as $\varepsilon \downarrow 0$. Let us study $I_2^\varepsilon$. We have

$$\begin{aligned}
I_2^\varepsilon(t) &\le \int_{\mathsf{T}} \int_{t_1}^t |\partial_s H_\varepsilon(\gamma(s))| \, ds \, d\eta(\gamma) \\
&= \int_{\mathsf{T}} \int_{t_1}^t |\nabla H_\varepsilon(\gamma(s)) \cdot \boldsymbol{b}(\gamma(s))| \, ds \, d\eta(\gamma) \\
&= \int_{t_1}^t \int |\nabla H_\varepsilon(x) \cdot \boldsymbol{b}(x)| \, d\left((e_t)_\sharp \eta \llcorner \mathsf{T}\right)(x) \, ds \\
&\le \int_0^T \int |\nabla H_\varepsilon(x) \cdot \boldsymbol{b}(x)| \rho_{\mathsf{T}}(t,x) \, dx \, ds \\
&= \int |\nabla H_\varepsilon(x) \cdot \boldsymbol{b}(x)| r_{\mathsf{T}}(x) \, dx \to \int |\nabla H(x) \cdot \boldsymbol{b}(x)| r_{\mathsf{T}}(x) \, dx = 0
\end{aligned}$$

where we have used $\nabla H_\varepsilon(x) \to \nabla H(x)$ for a.e. $x$. In the end, we have that $I_2^\varepsilon \to 0$ as $\varepsilon \downarrow 0$ and this concludes the proof. □

We now show that Lemma 4.26 can be improved, showing indeed that $\eta_B$-a.e. $\gamma$ is contained in a *regular* level set of $H$.

Lemma 4.27. *Up to a $\eta_B$ negligible set, the image of every $\gamma \in \mathsf{T}_B$ is contained in a connected component of a* regular *level set of $H_B$.*

Proof. Using Lemma 4.26, we remove $\eta_B$-negligible set of trajectories along which $H_B$ is not constant. Set $E^c := B \setminus E$ and consider the set

$$\mathscr{P} := \left\{ \gamma \in \mathsf{T}_B : \gamma\left((0,T)\right) \cap B \subset E^c \right\}.$$

It is enough to show that $\eta(\mathscr{P}) = 0$: this means that for $\eta$-a.e. $\gamma$ the image $\gamma(0,T)$ is not contained in the complement of $E$ and thus we must have (in the ball) $\gamma(0,T) \subset E$ for $\eta$-a.e. $\gamma \in \mathsf{T}_B$ (this follows remembering that a.e. $\gamma$ is contained in a level set). By Coarea formula (see Lemma 4.9), $|\nabla H| \mathscr{L}^2 \llcorner_{E^c} = 0$, i.e.

$$\int \mathbb{1}_{E^c}(x) |\nabla H(x)| \, dx = 0.$$

Since $\nabla H = r_B \boldsymbol{b}^\perp$ in $B$ and $r_B \ge 0$ (since $\rho_B > 0$), we have

$$\begin{aligned}
0 &= \int \mathbb{1}_{E^c}(x) |r_B(x) \boldsymbol{b}(x)| \, dx \\
&= \int \mathbb{1}_{E^c}(x) r_B(x) |\boldsymbol{b}(x)| \, dx \\
&= \int \int_0^T \mathbb{1}_{E^c}(x) \rho_B(t,x) |\boldsymbol{b}(x)| \, dx \, dt.
\end{aligned}$$

Using (4.4) we have

$$0 = \int_0^T \int \mathbb{1}_{E^c}(\gamma(t)) |\boldsymbol{b}(\gamma(t))| \, d\eta(\gamma) \, dt = \int_0^T \int_{\mathscr{P}} |\boldsymbol{b}(\gamma(t))| \, d\eta(\gamma) \, dt$$

which implies (by Fubini) that for $\eta$-a.e. $\gamma \in \mathscr{P}$ we have

$$\int_0^T |\boldsymbol{b}(\gamma(t))| \, dt = 0.$$

This gives $|\boldsymbol{b}(\gamma(t))| = 0$ for a.e. $t \in [0,T]$ and this contradicts the definition of $\mathsf{T}_B$. Hence $\eta(\mathscr{P}) = 0$. □

## 4.5. Locality of the divergence

In this section we prove that the if $\operatorname{div}(u\boldsymbol{b})$ is a measure, then it is 0 on the set

$$M := \left\{x \in \mathbb{R}^2 : \boldsymbol{b}(x) = 0,\, x \in \mathcal{D}_{\boldsymbol{b}} \text{ and } \nabla^{\text{appr}}\boldsymbol{b}(x) = 0\right\}, \tag{4.21}$$

where $\mathcal{D}_{\boldsymbol{b}}$ is the set of approximate differentiability points and $\nabla^{\text{appr}}\boldsymbol{b}$ is the approximate differential, according to Definition [**AFP00**, Def. 3.70]. For shortness, we will call this property *locality of the divergence*.

Let $U$ be an open set in $\mathbb{R}^d$, $d \in \mathbb{N}$. The main result of this section is the following

PROPOSITION 4.28. *Let $u \in L^\infty(U)$ and suppose that* $\operatorname{div}(u\boldsymbol{b}) = \lambda$ *in the sense of distributions, where $\lambda$ is a Radon measure on $U$. Then $|\lambda|\llcorner_M = 0$.*

Note that we do not assume any weak differentiability of $u$ or $u\boldsymbol{b}$, so the conclusion of Proposition 4.28 does not follow immediately from the standard locality properties of the approximate derivative (see e.g. [**AFP00**], Proposition 3.73). Moreover, we also mention a related counterexample (contained in [**ABC13**]), where the authors construct a bounded vector field $\boldsymbol{V}$ on the plane whose (distributional) divergence belongs to $L^\infty$, is non-trivial, and is supported in the set where $\boldsymbol{V}$ vanishes. Our proof is based on Besicovitch-Vitali covering Lemma ([**AFP00**, Thm. 2.19]) and uses some basic facts about the trace properties of $L^\infty$ vector fields whose divergence is a measure (recalled in the Preliminaries, see Section IV).

PROOF OF PROPOSITION 4.28. Fix an arbitrary $x \in M$. For brevity let $B_r := B_r(x)$. By (IV.7) with $F(y) := |x-y|^2$, there exists an $\mathscr{L}^1$-negligible set $N_x$ such that for any positive number $r \notin N_x$ we have

$$|\lambda(B_r)| = \left| \int_{\partial B_r} u\boldsymbol{b} \cdot \nu \, d\mathscr{H}^{d-1} \right| \le C \int_{\partial B_r} |\boldsymbol{b}| \, d\mathscr{H}^{d-1},$$

where $\nu$ denotes the exterior unit normal to $\partial B_r$. By a simple Fubini-type argument we have that

$$C \int_{\partial B_r} |\boldsymbol{b}| \, d\mathscr{H}^{d-1} \le \frac{C}{r} \int_{B_{2r}} |\boldsymbol{b}(x)| \, dx = o(r^d)$$

because, by definition of $M$, we have $⨍_{B_r} |\boldsymbol{b}| \, dx = o(r)$. Therefore

$$|\lambda(B_r)| = o(r^d). \tag{4.22}$$

Fix $\varepsilon > 0$. By (4.22) for any $x \in M$ there exists $\delta_x > 0$ such that for any positive number $r < \delta_x$ such that $r \notin N_x$ we have

$$|\lambda(B_r(x))| \le \varepsilon r^d. \tag{4.23}$$

Let $S \subset M$ be an arbitrary bounded subset. By regularity of $\lambda$, there exists a bounded open set $O \supset S$ such that $|\lambda|(O \setminus S) < \varepsilon$. Hence, for any $x \in S$ there exists $\rho_x > 0$ such that $B(x,r) \subset O$ for any positive number $r < \rho_x$. Consequently

$$\mathscr{F} := \left\{ B(x,r) : x \in S, r < \min(\rho_x, \delta_x), r \notin N_x \right\}$$

is a fine covering of $S$. Hence we can apply Besicovitch-Vitali covering Lemma ([**AFP00**, Thm. 2.19]): there exists a countable disjoint subfamily $\{B_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ such that

$$|\lambda| \left( S \setminus \bigcup_i B_i \right) = 0.$$

On the other hand, since $\bigcup_i B_i \subset O$ by construction, we have

$$|\lambda|\left(\bigcup_i B_i \setminus S\right) < \varepsilon.$$

Using (4.23), since the balls $B_i$ are disjoint, we have

$$\lambda\left(\bigcup_i B_i\right) = \sum_i \lambda(B_i) \le \varepsilon \mathscr{L}^2\left(\bigcup_i B_i\right).$$

Hence

$$\lambda(S) = \lambda\left(\bigcup_i B_i\right) - \lambda\left(\bigcup_i B_i \setminus S\right) \to 0$$

as $\varepsilon \downarrow 0$. Hence $\lambda \llcorner_S = 0$ and, by arbitrariness of $S \subset M$, $\lambda \llcorner_M = 0$. □

**4.5.1. Comparison between $\mathscr{L}^2$ and $\eta$.** We present here two general lemmata which relate the Lebesgue measure $\mathscr{L}^2$ and the measure $\eta$ and are based on nearly incompressibility of the vector field $\boldsymbol{b}$.

LEMMA 4.29. *Let $A \subset \mathbb{R}^2$ be a measurable set. Then $\mathscr{L}^2(A) = 0$ if and only if $\eta(\Gamma_A) = 0$ where*

$$\Gamma_A := \left\{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\right\}.$$

PROOF. Let us prove first that $\mathscr{L}^2(A) = 0$ implies $\eta(\Gamma_A) = 0$. We denote by $\rho_A$ the density such that $\rho_A(t,\cdot)\mathscr{L}^2 = e_{t\sharp}(\eta \llcorner \Gamma_A)$ and $r_A(x) := \int_0^T \rho_A(t,x)\,dt$. We have, using Fubini,

$$\begin{aligned}
0 = \mathscr{L}^2(A) = r_A \mathscr{L}^2(A) &= \int_0^T \int_\Gamma \mathbb{1}_A(x)\rho_A(t,x)\,dx\,dt \\
&= \int_0^T \int_\Gamma \mathbb{1}_A(\gamma(t))\,d\eta(\gamma)\,dt \\
&= \int_\Gamma \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\
&= \int_{\Gamma_A} \mathscr{L}^1\big(\{t \in [0,T] :\, \gamma(t) \in A\}\big)\,d\eta(\gamma),
\end{aligned}$$

hence, $\mathscr{L}^1(\{t \in [0,T] :\, \gamma(t) \in A\}) = 0$ for $\eta$-a.e. $\gamma \in \Gamma_A$.

For the opposite direction, using that $\rho$ is uniformly bounded from below by $1/C$, we get

$$\begin{aligned}\frac{T}{C}\mathscr{L}^2(A) &= \frac{T}{C}\int \mathbb{1}_A(x)\,dx = \frac{1}{C}\int_0^T\int \mathbb{1}_A(x)\,dx\,dt \\ &\le \int_0^T\int \mathbb{1}_A(x)\rho(t,x)\,dx\,dt \\ &= \int_0^T\int_\Gamma \mathbb{1}_A(\gamma(t))\,d\eta(\gamma)\,dt \\ &= \int_\Gamma\int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\ &= \int_{\Gamma_A}\int_0^T \mathbb{1}_A(\gamma(t))\,dt\,d\eta(\gamma) \\ &= \int_{\Gamma_A} \mathscr{L}^1(\{t\in[0,T]:\,\gamma(t)\in A\})\,d\eta(\gamma) = 0. \qquad \square\end{aligned}$$

LEMMA 4.30. *We have $\mathscr{L}^2(A) = 0$ if and only if $\eta(\Gamma_A^s) = 0$ for every $s \in [0,T]$.*

PROOF. For direct implication

$$\begin{aligned}0 = \mathscr{L}^2(A) &= \int \mathbb{1}_A(x)\rho(s,x)\,dx \\ &= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\ &= \int_{\Gamma_A^s} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma_A^s).\end{aligned}$$

For the opposite direction,

$$\begin{aligned}\frac{1}{C}\mathscr{L}^2(A) &= \frac{1}{C}\int \mathbb{1}_A(x)\,dx \\ &\le \int \mathbb{1}_A(x)\rho(s,x)\,dx \\ &= \int_\Gamma \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) \\ &= \int_{\Gamma_A^s} \mathbb{1}_A(\gamma(s))\,d\eta(\gamma) = \eta(\Gamma_A^s) = 0.\end{aligned}$$

$\square$

We now recall the set $M$, defined in (4.21) as

$$M := \Big\{x \in \mathbb{R}^2 :\, \boldsymbol{b}(x) = 0,\, x \in \mathcal{D}_{\boldsymbol{b}} \text{ and } \nabla^{\mathrm{appr}}\boldsymbol{b}(x) = 0\Big\},$$

and we consider the sets

$$\widetilde{\Gamma}_M := \widetilde{\Gamma} \cap \Gamma_M$$

and

$$\widetilde{\Gamma}_M^s := \Big\{\gamma \in \widetilde{\Gamma} : \gamma(s) \in M\Big\}.$$

Using Proposition 4.28, we can show the following

LEMMA 4.31. *Let $M$ be the set defined in (4.21) and for every fixed $s \in [0,T]$ let $\widetilde{\Gamma}_M^s := \{\gamma \in \widetilde{\Gamma} \colon \gamma(s) \in M\}$. Then:*

- $\eta(\widetilde{\Gamma}_M^s) = 0$ *for a.e. $s \in [0,T]$;*
- $\eta(\widetilde{\Gamma}_M) = 0$.

Proof. Let us denote by $\eta^s_M := \eta \llcorner_{\widetilde{\Gamma}^s_M}$ and consider the Borel function

$$\rho^s_M(t,\cdot)\mathscr{L}^2 = e_{t\sharp}\eta^s_M.$$

It is easy to see that $\rho^s_M$ solves continuity equation

$$\partial_t \rho^s_M + \operatorname{div}(\rho^s_M \boldsymbol{b}) = 0. \tag{4.24}$$

Integrating in time on $[0,t]$ we get

$$\operatorname{div}\left(\boldsymbol{b}\int_0^t \rho^s_M(\tau,\cdot)d\tau\right) = (\rho^s_M(t,\cdot) - \rho^s_M(0,\cdot))\mathscr{L}^2.$$

In particular, thanks to Proposition 4.28, we have that

$$\big(\rho^s_M(t,\cdot) - \rho^s_M(0,\cdot)\big)\mathscr{L}^2\llcorner_M = 0, \tag{4.25}$$

hence $\rho^s_M(t,\cdot) = \rho^s_M(0,\cdot)$, for a.e. $x$. Furthermore, integrating in space the continuity equation (4.24) we get the conservation of mass:

$$\frac{d}{dt}\int_{\mathbb{R}^2} \rho^s_M(t,x)\,dx = 0. \tag{4.26}$$

Therefore, using (4.25) and (4.26), we have

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus M} \rho^s_M(t,x)dx &= \int_{\mathbb{R}^2} \rho^s_M(t,x)dx - \int_M \rho^s_M(t,x)dx = \\
&= \int_{\mathbb{R}^2} \rho^s_M(s,x)dx - \int_M \rho^s_M(s,x)dx = \int_{\mathbb{R}^2\setminus M} \rho^s_M(s,x)dx = \\
&= \int \mathbb{1}_{\mathbb{R}^2\setminus M}(\gamma(s))d\eta_M(\gamma) = 0,
\end{aligned}$$

which gives us $\rho^s_M(t,\cdot) = 0$ a.e. on $\mathbb{R}^2\setminus M$. Hence

$$0 = \int_0^T \int_{\mathbb{R}^2\setminus M} \rho^s_M(t,x)\,dx = \int_0^T \int 1_{\mathbb{R}^2\setminus M}(\gamma(t))\,d\eta^s_M(\gamma)\,dt$$

and this implies that $\eta^s_M(\widetilde{\Gamma}^s_M) = 0$ for $s \in [0,T]$, since $\gamma \in \widetilde{\Gamma}$ are not constant functions (by definition) and $b = 0$ on $M$.

Now the second part easily follows from the first one by a Fubini-like argument: indeed, we set

$$I := \int_0^T \eta(\widetilde{\Gamma}^s_M)\,ds = 0.$$

Since $\eta(\widetilde{\Gamma}^s_M) = \int_{\widetilde{\Gamma}} 1_M(\gamma(s))\,d\eta(\gamma)$ and using Fubini's theorem we get

$$I = \int_{\widetilde{\Gamma}}\int_0^T 1_M(\gamma(s))\,ds\,d\eta(\gamma) = 0$$

i.e. $\mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in M\}) = 0$ for $\eta$-a.e. $\gamma \in \widetilde{\Gamma}_M$ and this concludes the proof. □

## 4.6. Level sets and trajectories II

The results obtained in the Section 4.5 provide us with a better description of the relantionship between the trajectories $\gamma \in \Gamma_B$ and the level sets of $H_B$, thus improving the results of Section 4.4.

**4.6.1. Trajectories and level sets coincide up to a translation in time.** Let $B \in \mathscr{B}$ a fixed ball of the collection and, as usual, let $H_B$ denote its Hamiltonian. Thanks to Lemma 4.27, there exists a $\eta$-negligible set $N$ such that for every $\gamma \in \Gamma_B \setminus N$ the image $\gamma(0,T)$ is contained in a connected component $\mathfrak{c}$ of a regular level set of $H_B$. Recalling Theorem III, there exists a parametrization $\gamma_\mathfrak{c}$ of $\mathfrak{c}$ with the following properties:

- $\gamma_\mathfrak{c} \colon I_\mathfrak{c} \to \mathbb{R}^2$ is a Lipschitz map, where $I_\mathfrak{c} = \mathbb{R}/\ell\mathbb{Z}$ or $I_\mathfrak{c} = [0,\ell]$ for some $\ell > 0$ is the domain of $\gamma$;
- $\gamma_\mathfrak{c}$ is injective;
- $\gamma_\mathfrak{c}'(s) = \boldsymbol{b}(\gamma_\mathfrak{c}(s))$ for $\mathscr{L}^1$-a.e. $s \in I_\mathfrak{c}$.

Thus it makes sense to wonder about the relationship between the trajectory $\gamma \in \Gamma_B \setminus N$ and the parametrization $\gamma_\mathfrak{c}$ of the corresponding connected component. The following proposition precises this relation, showing that $\gamma$ and $\gamma_\mathfrak{c}$ coincide up to a translation in time.

PROPOSITION 4.32. *Let $N$ be the set given by Lemma 4.27 and $\gamma \in \widetilde{\Gamma} \setminus N$. Then (a suitable restriction of) $\gamma$ coincides with $\gamma_\mathfrak{c}$ up to a translation in time.*

In order to prove Proposition 4.32, we need the following auxiliary

LEMMA 4.33. *Let $\gamma \colon I \to \mathbb{R}^2$ be a solution of the ordinary differential equation*

$$\gamma'(t) = \boldsymbol{b}(\gamma(t)), \qquad t \in I \subset \mathbb{R},$$

*where $I = [0,T]$ and $\frac{1}{|\boldsymbol{b}|} \in L^1_{\mathrm{loc}}(\mathscr{H}^1 \llcorner_{\gamma(I)})$. Assume that there exists a injective curve $\widehat{\gamma}$ defined on $I$ such that $\gamma(I) \subset \widehat{\gamma}(I)$ and that $\dot{\widehat{\gamma}} = b(\widehat{\gamma})$. Then*

$$\int_{\gamma([0,T])} \frac{d\mathscr{H}^1(w)}{|\boldsymbol{b}(w)|} = T - \mathscr{L}^1\big(\big\{t \in [0,T] : \gamma'(t) = 0\big\}\big).$$

PROOF. Observe that

$$\begin{aligned}
\int_{\gamma([0,T])} \frac{d\mathscr{H}^1(w)}{|\boldsymbol{b}(w)|} &\overset{(1)}{=} \int_{\gamma([0,T])} \frac{\mathbb{1}_{\{\boldsymbol{b}\neq 0\}}(w)\, d\mathscr{H}^1(w)}{|\boldsymbol{b}(w)|} \\
&\overset{(2)}{=} \int_{\{t\in[0,T]:\, \gamma'(t)\neq 0\}} \frac{|\gamma'(\tau)|}{|\boldsymbol{b}(\gamma(\tau))|} d\tau \\
&= T - \mathscr{L}^1\big(\big\{t \in [0,T] : \gamma'(t) = 0\big\}\big),
\end{aligned}$$

where

(1) follows by definition;
(2) is the Area formula, i.e. $\mathscr{H}^1 \llcorner_C = \gamma_\sharp(|\gamma'|\mathscr{L}^1)$, where $C = \gamma((0,T))$, which can be applied because there exists $\widehat{\gamma}$ by hypothesis.

This concludes the proof. □

Now we can prove Proposition 4.32.

PROOF. Let $\bar{t} \in [0,T]$ such that $\gamma_\mathfrak{c}(0) = \gamma(\bar{t})$. By Lemma 4.33, we have that for any $s$ in a suitable subinterval of $[0,T]$ it holds

$$\int_{\gamma([\bar{t},\bar{t}+s])} \frac{d\mathscr{H}^1(w)}{|\boldsymbol{b}(w)|} = (\bar{t}+s) - \bar{t} - \mathscr{L}^1([\bar{t},\bar{t}+s] \cap \gamma^{-1}(\{\boldsymbol{b}=0\})). \tag{4.27}$$

By Lemma 4.31 and the fact that $\mathscr{L}^2(\{\boldsymbol{b}=0\} \setminus M) = 0$, where $M$ is defined in (4.21), we know that for $\eta$-a.e. $\gamma \in \widetilde{\Gamma}$,

$$\mathscr{L}^1\big(\big\{t \in [0,T] : \gamma(t) \in \{\boldsymbol{b}=0\}\big\}\big) = 0,$$

hence (4.27) is actually

$$\int_{\gamma([\bar{t},\bar{t}+s])} \frac{d\mathscr{H}^1(w)}{|\boldsymbol{b}(w)|} = s. \tag{4.28}$$

On the other hand, applying again Lemma 4.33 to $\gamma_{\mathfrak{c}}$, which is injective, we get

$$\int_{\gamma_{\mathfrak{c}}(0,s)} \frac{d\mathscr{H}^1(w)}{|\boldsymbol{b}(w)|} = s. \tag{4.29}$$

Since, by definition, $\gamma_{\mathfrak{c}}(0) = \gamma(\bar{t})$, comparing (4.28) and (4.29) and using the fact that $|\boldsymbol{b}| > 0$ $\mathscr{H}^1$-a.e. on $\gamma$, we deduce that

$$\gamma(\bar{t} + s) = \gamma_{\mathfrak{c}}(s)$$

which means that $\gamma$ (restricted to a suitable time subinterval of $[0,T]$) and $\gamma_{\mathfrak{c}}$ coincide up to a translation in time. □

**4.6.2. Covering property of the regular level sets.** Let us recall that for each ball $B \in \mathscr{B}$ and for any rational numbers $s, t \in \mathbb{Q} \cap (0,T)$ with $s < t$ we have set

$$\mathsf{T}_{B,s,t} := \{\gamma \in \Gamma_B : \gamma(s) \notin B, \gamma(t) \notin B\}.$$

Remark 4.34. In the same way as in Remark 4.7, we can easily see that

$$\bigcup_{\substack{B\in\mathscr{B}\\ s,t\in\mathbb{Q}\cap[0,T]}} \mathsf{T}_{B,s,t} = \widetilde{\Gamma}.$$

♠

For each $B \in \mathscr{B}$, $s \in \mathbb{Q} \cap (0,T)$, $t \in \mathbb{Q} \cap (s,T)$ restricting $\eta$ to $\mathsf{T}_{B,s,t}$, we can construct the local Hamiltonian $H_{B,s,t}$ as in Sections 4.2.1-4.2.2.

We now set

$$\widehat{E} := \bigcup_{\substack{B\in\mathscr{B}\\ s,t\in\mathbb{Q}\cap[0,T]}} E^\star_{B,s,t}.$$

The following covering property is a global analog of Lemma 4.18:

Lemma 4.35. *It holds that* $\widehat{E} \supset \{\boldsymbol{b} \neq 0\} \mod \mathscr{L}^2$.

Proof. Let $P := \{\boldsymbol{b} \neq 0\} \setminus \widehat{E}$. Then for any $B \in \mathscr{B}$ it holds that $P \subset \{\nabla H_B = 0\}$ mod $\mathscr{L}^2$. Since $\boldsymbol{b} \neq 0$ on $P$ and $\nabla H^\perp = r_B \boldsymbol{b}$ it holds that $r_B = 0$ a.e. on $P$ for all $B \in \mathscr{B}$. Then for any $B \in \mathscr{B}$

$$\begin{aligned} 0 &= \int_{P\cap B} r_B \, dx \\ &= \int_0^T \int \mathbb{1}_{P\cap B}(x)\rho_B(t,x)\, dx\, dt \\ &= \int_{\widetilde{\Gamma}} \int_0^T \mathbb{1}_{P\cap B}(\gamma(t))\, d\eta(\gamma)\, dt, \end{aligned}$$

hence $\eta$-a.e. $\gamma \in \widetilde{\Gamma}$ spends zero amount of time in $P \cap B$. Since $B$ is arbitrary and $\mathscr{B}$ is countable, we can generalize this claim to the whole set $P$:

$$\int_{\widetilde{\Gamma}} \int_0^T \mathbb{1}_P(\gamma(t))\, dt\, d\eta(\gamma) = 0. \tag{4.30}$$

By nearly incompressibility

$$\begin{aligned}
\mathscr{L}^2(P) &\leq C\int_0^T \int \mathbb{1}_P(x)\rho(t,x)\,dx\,dt \\
&= C\int_0^T \int_{\dot\Gamma\cup\widetilde\Gamma} \mathbb{1}_P(\gamma(t))\,d\eta(\gamma)\,dt \\
&\overset{(*)}{=} C\int_0^T \int_{\dot\Gamma} \mathbb{1}_P(\gamma(t))\,d\eta(\gamma)\,dt \\
&\overset{(**)}{=} C\int_0^T \int_{\dot\Gamma} \mathbb{1}_P(\gamma(t))\mathbb{1}_{\{\boldsymbol{b}=0\}}(\gamma(t))\,d\eta(\gamma)\,dt \\
&\leq C\int_0^T \int \mathbb{1}_P(\gamma(t))\mathbb{1}_{\{\boldsymbol{b}=0\}}(\gamma(t))\,d\eta(\gamma)\,dt \\
&\leq C\int_0^T \int \mathbb{1}_P(x)\mathbb{1}_{\{\boldsymbol{b}=0\}}(x)\rho(t,x)\,dx\,dt \\
&\overset{(***)}{=} 0,
\end{aligned}$$

where

- (*) holds by (4.30);
- (**) holds because $\mathbb{1}_{\{\boldsymbol{b}=0\}}(\gamma(t)) = 1$ for any $t \in [0,T]$ and any $\gamma \in \dot\Gamma$: indeed, for any $\gamma \in \dot\Gamma$ which is an integral curve of $\boldsymbol{b}$ we have $0 = \gamma'(t) = \boldsymbol{b}(\gamma(t))$, hence $\gamma(t) \in \{b = 0\}$;
- (***) holds because $P$ and $\{\boldsymbol{b} = 0\}$ are disjoint. □

In view of Corollary 4.24 the proof above actually leads to a stronger statement:

Lemma 4.36. *The following holds true:* $\widehat{E} = \{\boldsymbol{b} \neq 0\} \mod \mathscr{L}^2$.

## 4.7. Solution of the transport equation on integral curves

We now pass to consider a general balance law associated to the Hamiltonian vector field $\boldsymbol{b}$, i.e. $\partial_t u + \mathrm{div}(u\boldsymbol{b}) = \nu$, being $\nu$ a Radon measure on $(0,T)\times\Omega$ and $u \in L^\infty((0,T)\times\Omega)$. A reduction on the connected components of the Hamiltonian $H$ can be performed, similarly to what we have done for equation $\mathrm{div}(u\boldsymbol{b}) = \mu$ to above. In some sense, we are presenting now the time-dependent version of Lemmata 4.11-4.13-4.14-4.15.

**4.7.1. Local disintegration of a balance law.** For the purposes of this Chapter, in the following Lemma, it would be enough to consider the simpler case where $\nu = 0$. Since we will need it later in this form we state and prove it as follows:

Lemma 4.37. *A function $u \in L^\infty([0,T]\times\Omega)$ is a solution to the problem*

$$\begin{cases} \partial_t u + \mathrm{div}(u\boldsymbol{b}) = \nu, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times\Omega) \tag{4.31}$$

*if and only if*

- $\widehat{u}_h(t,s) := u(t,\gamma_h(s))$ *solves*
$$\begin{cases} \partial_t \widehat{u}_h + \partial_s \widehat{u}_h = \widehat{\nu}_h \\ \widehat{u}_h(0,\cdot) = \widehat{u_0}_h(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times I)$$
- *it holds*
$$\mathrm{div}(u\boldsymbol{b}\sigma_h) = 0$$

*for $\mathscr{L}^1$-a.e. $h$, where $\gamma_h\colon I \to \mathbb{R}^2$ is an admissible parametrization of a connected component $C$ of the level set $E_h$ of the Hamiltonian $H$ and $\widehat{\nu}_h$ is a measure such that $\widehat{\nu}_h = (\gamma_h^{-1})_\sharp\nu$.*

PROOF. Multiplying equation in (4.31) by a function $\psi \in C_c^\infty([0,T))$ and formally integrating by parts we get

$$u_t\psi + \operatorname{div}(u\psi\boldsymbol{b}) = \psi\nu \Rightarrow \operatorname{div}\left(\int_0^T u\psi\,dt\,\boldsymbol{b}\right) = \int_0^T u\psi_t\,dt - \psi(0)u_0 + \left(\int_0^T \psi\,dt\right)\nu,$$

which can be written in the form

$$\operatorname{div}(w\boldsymbol{b}) = \mu, \tag{4.32}$$

where $w := \int_0^T u\psi\,dt$ and

$$\mu := \left(\int_0^T u\psi_t\,dt - \psi(0)v_0\right)\mathscr{L}^2 + \left(\int_0^T \psi\,dt\right)\nu.$$

Applying Lemma 4.11 and Lemma 4.14 to (4.32), we obtain that continuity equation is equivalent to

$$\operatorname{div}\left(wc_h\boldsymbol{b}\mathscr{H}^1\llcorner_{E_h}\right) = \mu_h \tag{4.33}$$

and

$$\operatorname{div}(u\boldsymbol{b}\sigma_h) = 0$$

for $\mathscr{L}^1$-a.e. $h$, where the measure $\mu_h$ can be computed explicitly, using Coarea Formula and Disintegration Theorem

$$\mu_h = \left(\int_0^T u\psi_t\,dt - \psi(0)v_0\right)\mathscr{H}^1\llcorner_{E_h} + \left(\int_0^T \psi\,dt\right)\nu_h.$$

Thanks to Lemma 4.15, equation (4.33) is *equivalent* to

$$\partial_s\widehat{u} = \widehat{\mu}_h,$$

in $\mathscr{D}'(I)$. Now being $\gamma_h$ Lipschitz and injective, we have

$$(\gamma_h^{-1})_\sharp(\mathscr{H}^1\llcorner_{E_h}) = |\gamma_h'|\mathscr{L}^1,$$

and this allows us to compute explicitly

$$\begin{aligned}\widehat{\mu}_h &= (\gamma_h^{-1})_\sharp\mu_h \\ &= (\gamma_h^{-1})_\sharp\left(\int_0^T u\psi_t\,dt\,c_h\mathscr{H}^1\llcorner_{E_h} - \int_{\mathbb{R}^2}\psi(0)v_0c_h d\mathscr{H}^1\llcorner_{E_h} + \int_0^T \psi\,dt\,\nu_h\right) \\ &= \int_0^T v(\tau,\gamma(s))\psi_\tau(\tau)\,d\tau - \psi(0)u_0(\gamma_h(s))c_h(\gamma(s)) + \left(\int_0^T \psi(\tau)d\tau\right)\widehat{\nu}_h,\end{aligned} \tag{4.34}$$

where

$$\widehat{\nu}_h = (\gamma_h^{-1})_\sharp\nu.$$

Formally, (4.34) means

$$\widehat{\mu}_h = -\int_0^T \partial_t\widehat{u} + \widehat{\nu}_h.$$

To sum up, we have obtained that Problem (4.31) is equivalent to

$$\begin{cases}\partial_t\widehat{u}_h + \partial_s\widehat{u}_h = \widehat{\nu}_h, \\ \widehat{u}_h(0,\cdot) = \widehat{u_0}_h(\cdot),\end{cases}$$

and

$$\operatorname{div}(u\boldsymbol{b}\sigma_h) = 0$$

in $\mathscr{D}'((0,T)\times I)$ for $\mathscr{L}^1$-a.e. $h \in \mathbb{R}$. □

Now, we show how Lemma 4.37 can be used in our setting to select a suitable family of trajectories on which the reduction can be performed.

LEMMA 4.38. *Fix $\sigma \in \mathbb{Q} \cap (0,T)$, $\theta \in \mathbb{Q} \cap (\sigma, T)$ and $B \in \mathscr{B}$. Let $H := H_{B,\sigma,\theta}$. Let $u \in L^\infty([0,T] \times \mathbb{R}^2)$ be a $\rho$-weak solution of the problem*

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2).$$

*Then there exists a negligible set $Z = Z_{B,\sigma,\theta} \subset \mathbb{R}$ such that*

- *for any $h \in Z$ the level set $E_h := H^{-1}(h)$ is regular;*
- *if $h \notin Z$ and $E_h$ is regular then for any nontrivial connected component $\mathfrak{c}$ of $E_h$ with admissible parametrization $\gamma_{\mathfrak{c}} \colon I \to \mathbb{R}^2$, any $t \in (0,T)$ and any $s \in I$ there exists a constant $w$ such that*

$$u(t+\xi, \gamma_{\mathfrak{c}}(s+\xi)) = w$$

*for a.e. $\xi \in \mathbb{R}$ such that $s + \xi \in I$ and $t + \xi \in (0,T)$.*
*In particular, for any $s \in I$ it holds that*

$$u(\xi, \gamma_{\mathfrak{c}}(s+\xi)) = u_0(s)$$

*for a.e. $\xi \in \mathbb{R}$ such that $s + \xi \in I$.*

PROOF. Setting $v := u\rho \in L^\infty([0,T] \times \mathbb{R}^2)$ and $v_0(\cdot) = u_0(\cdot)\rho(0,\cdot)$, by definition of $\rho$-weak solution we have

$$\begin{cases} \partial_t v + \operatorname{div}(v\boldsymbol{b}) = 0, \\ v(0,\cdot) = v_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2).$$

Hence we can apply Lemma 4.37 in $B$ to get

$$\begin{cases} \partial_t \big(\widehat{v}\widehat{c}_h |\widehat{\boldsymbol{b}}|\big) + \partial_s \big(\widehat{v}\widehat{c}_h |\widehat{\boldsymbol{b}}|\big) = 0, \\ \widehat{v}(0,\cdot) = \widehat{v}_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times I). \tag{4.35}$$

for all $h \in H(B) \setminus N_1$, where $\mathscr{L}^1(N_1) = 0$. From (4.35) it immediately follows that the function

$$\xi \mapsto \Big(\widehat{\rho}\widehat{u}\widehat{c}_h |\widehat{\boldsymbol{b}}|\Big)(t+\xi, s+\xi) \tag{4.36}$$

is equal a.e. to some constant $w_1$. Applying the same argument to the problem

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho\boldsymbol{b}) = 0, \\ \rho(0,\cdot) = \rho_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2),$$

(which holds thanks to nearly incompressibility assumption) we obtain a negligible set $N_2$ such that for all $h \in H(B) \setminus N_2$, for any connected component of $E_h$ the map

$$\xi \mapsto \Big(\widehat{\rho}\widehat{c}_h |\widehat{\boldsymbol{b}}|\Big)(t+\xi, s+\xi) \tag{4.37}$$

is equal a.e. to some constant $w_2$. Let $N := N_1 \cup N_2$ and fix $h \notin N$. Comparing (4.36) and (4.37), using that $\rho c_h |\boldsymbol{b}| > 0$ $\mathscr{H}^1$-a.e. on $E_h$ (for a.e. $h$), we obtain that

$$\xi \mapsto \widehat{u}(t+\xi, s+\xi)$$

is equal a.e. to the constant $w = w_1/w_2$ for a.e. $h \notin N$, which is what we wanted to prove. ☐

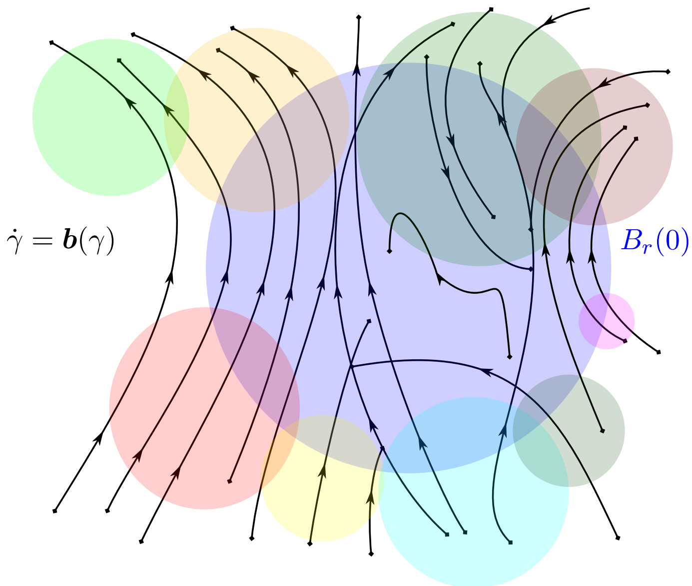



**Figure 2.** The passage from the local argument to the global one is made possible by the Matching Lemma 4.20 and the relation between trajectories and level sets.

#### 4.7.2. Selection of appropriate trajectories.

We state and prove the following

LEMMA 4.39. *There exists an $\eta$-negligible set $N \subset \Gamma$ such that any integral curve $\gamma \in \widetilde{\Gamma} \setminus N$ of the vector field $\boldsymbol{b}$ has the following properties:*

*(1) for any $B \in \mathscr{B}$, if $\gamma \in \mathsf{T}_{B,s,t}$ then each connected component of $\gamma([s,t]) \cap B$ is contained in a regular level set of $H_B$;*
*(2) for any $\tau \in (0,T)$ there exist a ball $B \in \mathscr{B}$, $s \in \mathbb{Q} \cap (0,T)$ and $t \in \mathbb{Q} \cap (\tau, T)$ such that $\gamma \in \mathsf{T}_{B,s,t}$.*

See also Figure 2.

PROOF. First of all, using Lemma 4.31 we can remove a negligible set of integral curves of $\boldsymbol{b}$ which stay in the set $\{\boldsymbol{b} = 0\}$ for a positive amount of time. Applying Lemmata 4.26 and 4.27 countably many times (for each ball $B \in \mathscr{B}$ and all rationals $s \in \mathbb{Q} \cap (0,T)$ and $t \in \mathbb{Q} \cap (s,T)$) we obtain the set $N \subset \Gamma$ such that the first property holds. Next, for any $\tau \in (0,T)$ there exists $s \in \mathbb{Q} \cap (0,\tau)$ such that $\gamma(s) \neq \gamma(\tau)$. (Otherwise, since $\gamma$ is an integral curve of $\boldsymbol{b}$, it would have to stay in $\{\boldsymbol{b} = 0\}$ for a positive amount of time). Similarly there exists $t \in (s,T)$ such that $\gamma(t) \neq \gamma(\tau)$. Then for any ball $B \in \mathscr{B}$ with sufficiently small radius, containing $\gamma(\tau)$ and not containing $\gamma(s)$ and $\gamma(t)$ it clearly holds that $\gamma \in \mathsf{T}_{B,s,t}$. □

LEMMA 4.40. *Let $Z_{B,s,t}$ denote negligible set given by Lemma 4.38. Then for $\eta$-a.e. $\gamma \in \widetilde{\Gamma}$ it holds that*

$$H_{B,s,t}(\gamma([0,T])) \cap Z_{B,s,t} = \emptyset.$$

PROOF. Set $A := H^{-1}_{B,s,t}(Z_{B,s,t})$: by Coarea Formula, $\mathscr{L}^2(A) = 0$. Applying Lemma 4.29 we deduce that

$$\eta\Big( \{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\} \Big) = 0.$$

On the other hand $\boldsymbol{b} \neq 0$ a.e. on $E_{B,s,t}$, hence

$$\begin{aligned}&\left\{\gamma \in \widetilde{\Gamma} \setminus N : \gamma([0,T]) \cap E_{B,s,t} \subset E_h,\ h \in Z\right\}\\&= \left\{\gamma \in \widetilde{\Gamma} \setminus N : \gamma([0,T]) \cap E_{B,s,t} \subset A\right\}\\&\subset \left\{\gamma \in \Gamma : \mathscr{L}^1(\{t \in [0,T] : \gamma(t) \in A\}) > 0\right\}.\end{aligned}$$ □

From the Lemma 4.40 it does not follow immediately that the endpoints $\gamma(0)$ and $\gamma(T)$ are contained in regular level sets of some Hamiltonians. But now we are going to establish this property. Being $Z_{B,s,t}$ given by Lemma 4.38, let $\widetilde{E}_{B,s,t} := E_{B,s,t} \setminus H^{-1}_{B,s,t}(Z_{B,s,t})$ and

$$\widetilde{E} := \bigcup_{\substack{B\in\mathscr{B},\\ s,t\in\mathbb{Q}\cap(0,T):\ s<t}} \widetilde{E}_{B,s,t}.$$

Note that since $\widetilde{E}_{B,s,t} = E_{B,s,t} \mod \mathscr{L}^2$ (by Coarea formula), it follows that $\widetilde{E} = \widehat{E} \mod \mathscr{L}^2$.

The following lemma shows that $\eta$-a.e. nontrivial trajectory of $\boldsymbol{b}$ starts from the set $\widetilde{E}$ (and also stops in $\widetilde{E}$):

LEMMA 4.41. *For $\eta$-a.e. $\gamma \in \widetilde{\Gamma}$ it holds that $\gamma(0) \in \widetilde{E}$ and $\gamma(T) \in \widetilde{E}$.*

PROOF. Consider the set $X$ of $\eta \in \widetilde{\Gamma}$ such that $\gamma(0) \notin \widetilde{E}$. By Lemma 4.36 it holds that $\boldsymbol{b} = 0$ a.e. on the complement of $\widetilde{E}$. Hence by Lemma 4.31 we have $\eta(X) = 0$. The argument for $\gamma(T)$ is similar. □

In the lemmata above we have been removing $\eta$-negligible sets of trajectories of $\boldsymbol{b}$. Let us summarize some properties of the remaining ones:

LEMMA 4.42. *There exists a $\eta$-negligible set $R \subset \widetilde{\Gamma}$ such that for any $\tau \in [0,T]$ and any $\gamma \in \widetilde{\Gamma} \setminus R$ there exist $s \in \mathbb{Q} \cap (0,T)$, $t \in \mathbb{Q} \cap (s,T)$ and $B \in \mathscr{B}$ such that $\gamma(\tau) \in \widetilde{E}_{B,s,t}$.*

PROOF. We define $R$ as the union of $\eta$-negligible sets given by Lemmata 4.39, 4.40 and 4.41. If $\tau \in (0,T)$ the claim follows from Lemma 4.39 since we can always find $s$ and $t$ such that $\tau \in (s,t)$ and the desired property holds. If $\tau = 0$ or $\tau = T$ then the result follows from Lemma 4.41. □

COROLLARY 4.43. *For any $\gamma \in \widetilde{\Gamma} \setminus R$ and any $\tau \in [0,T]$ there exists $\delta > 0$ and a constant $w$ such that the function $\xi \mapsto u(\xi, \gamma(\xi))$ is equal to $w$ for a.e. $\xi \in (\tau - \delta, \tau + \delta) \cap [0,T]$. Moreover, if $\tau = 0$ then the constant $w$ is equal to $u_0(\gamma(0))$.*

PROOF. The result follows directly from Lemma 4.42, Proposition 4.32 and Lemma 4.38. □

**4.7.3. Conclusion of the proof.** Now we are in a position to recover the method of characteristics in our weak setting:

LEMMA 4.44. *Suppose that $\boldsymbol{b}$ is a bounded, autonomous,* BV *compactly supported, nearly incompressible (with density $\rho$) vector field on $\mathbb{R}^2$ and let $u \in L^\infty([0,T] \times \mathbb{R}^2)$ be a $\rho$-weak solution of the problem*

$$\begin{cases}\partial_t u + \boldsymbol{b} \cdot \nabla u = 0,\\ u(0,\cdot) = u_0(\cdot),\end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2).$$

*Then for $\eta$-a.e. $\gamma \in \Gamma$ for a.e. $t \in [0,T]$ it holds that*

$$u(t, \gamma(t)) = u_0(\gamma(0)).$$

PROOF. It is clear that the thesis holds for any $\gamma \in \dot{\Gamma}$. Indeed, by Proposition 4.28

$$\partial_t(\rho u \mathbb{1}_M) = 0$$

in the sense of distributions, where the set $M$ is defined in (4.21). Hence it is sufficient to consider only the moving trajectories, i.e. $\gamma \in \widetilde{\Gamma}$. Let $R$ be the set given by Lemma 4.42. Let $\gamma \in \widetilde{\Gamma} \setminus R$. By Corollary 4.43 for any $\tau \in [0,T]$ there exists $\delta > 0$ such that the function $t \mapsto u(t,\gamma(t))$ is equal to some constant $w_\tau$ for a.e. $t \in (\tau - \delta, \tau + \delta) \cap [0,T]$. Moreover, if $\tau = 0$ then $w_\tau = u_0(\gamma(0))$. It remains to extract a finite covering of $[0,T]$. □

The following lemma is elementary, we prove it for sake of completeness.

LEMMA 4.45. *Let $u \in L^\infty([0,T] \times \mathbb{R}^2)$. If for $\eta$-a.e. $\gamma$ and a.e. $t \in [0,T]$ it holds that $u(t,\gamma(t)) = u_0(\gamma(t))$, then $u$ solves the transport equation with the initial condition $u_0$, i.e.*

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot). \end{cases}$$

PROOF. Let $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^2)$ be a smooth test function which vanishes at $T$. Then

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} (\rho u \varphi_t + \rho u \boldsymbol{b} \cdot \nabla \varphi) \, dx \, dt + \int_{\mathbb{R}^2} \rho(0,x) u_0(x) \varphi(0,x) \, dx \\
&= \int_0^T \int_\Gamma u(t,\gamma(t)) \partial_t \varphi(t,\gamma(t)) \, d\eta(\gamma) \, dt + \int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0)) \, d\eta(\gamma) \\
&= \int_0^T \int_\Gamma u_0(\gamma(0)) \partial_t \varphi(t,\gamma(t)) \, d\eta(\gamma) \, dt + \int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0)) \, d\eta(\gamma) \\
&= -\int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0)) \, d\eta(\gamma) + \int_\Gamma u_0(\gamma(0)) \varphi(0,\gamma(0)) \, d\eta(\gamma) = 0.
\end{aligned}$$

□

We are finally ready to prove the main theorem of this chapter, i.e. Theorem 4.5.

PROOF OF THEOREM 4.5. Let $u \in L^\infty([0,T] \times \mathbb{R}^2)$ be a solution of

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2).$$

By Lemma 4.44 the function $t \mapsto u(t,\gamma(t))$ is constant for $\eta$-a.e. $\gamma$. Then for any $\beta \in C^1(\mathbb{R},\mathbb{R})$ the function $t \mapsto \beta(u(t,\gamma(t)))$ is constant for $\eta$-a.e. $\gamma$. Hence by Lemma 4.45 the function $\beta(u)$ is a solution of

$$\begin{cases} \partial_t(\beta(u)) + \boldsymbol{b} \cdot \nabla \beta(u) = 0, \\ \beta(u)(0,\cdot) = \beta(u_0)(\cdot). \end{cases}$$

This concludes the proof. □

CHAPTER 5

# Further remarks on the two dimensional case

ABSTRACT. The aim of this chapter is to collect some examples and counterexamples related to the two dimensional case and to the Hamiltonian structure of the problem. More precisely, we will begin by discussing more in detail the Weak Sard Property (already introduced in Section 4.1): in particular, we present an example (taken from [**ABC13**]) of a Lipschitz function which does not have the Weak Sard Property. In the second part of the chapter, we will briefly discuss the Chain Rule problem (Section 5.2) and then present two counterexamples (still in $\mathbb{R}^2$) related to this problem: first, in Section 5.3, we will present a variant of a construction taken from [**BG16**], which allows to conclude that a priori estimates on the size of the *tangential set* of a BV vector fields in $\mathbb{R}^2$ are not available (thus answering in the negative to a question raised in [**ACM05**]). Then, in the final part of the chapter, in Section 5.4, we propose an example of a vector field in $\mathbb{R}^2$ which has a (non-steady) renormalization defect which is not a measure: this is related to the recent work [**CGSW17**], where the authors, by means of an abstract convex integration scheme, produce counterexamples of this kind in $\mathbb{R}^d$ for $d \geq 3$.

## 5.1. More on the Weak Sard Property

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function and let us consider the critical set $S$, defined as the set of all $x \in \mathbb{R}^2$ where $f$ is not differentiable or $\nabla f(x) = 0$. As already said in Section 4.1, we are interested in the following property: the push-forward according to $f$ of the restriction of $\mathscr{L}^2$ to $S$ is singular with respect to $\mathscr{L}^1$, that is

$$f_\sharp \left(\mathscr{L}^2 \llcorner_S\right) \perp \mathscr{L}^1. \tag{5.1}$$

This property clearly implies the following *Weak Sard Property*, which is the one used in [**ABC14**, Section 2.13]:

$$f_\sharp \left(\mathscr{L}^2 \llcorner_{S\cap E^\star}\right) \perp \mathscr{L}^1 \tag{5.2}$$

where the set $E^\star$ is the union of all connected components with positive length of all level sets of $f$ (recall Theorem III). We have already seen the connection of the Weak Sard Property in the framework of transport and continuity equation in Chapter 4. It is possible to prove that in some sense the Weak Sard Property is satisfied by a generic Lipschitz function (in Baire's category sense), as the class of all Lipschitz functions $H\colon \Omega \to \mathbb{R}$ satisfying the Weak Sard Property is residual in the Banach space of Lipschitz functions $\mathrm{Lip}(\Omega)$ (see [**ABC13**, Thm. 4]).

### 5.1.1. "Very" Weak Sard Property versus Weak Sard Property.

To begin, let us show a quick example of the fact that condition (5.2) is strictly *weaker* than (5.1).

We exhibit a Lipschitz function which has the Weak Sard Property but it does not satisfy (5.1).

PROPOSITION 5.1. *There exists a compactly supported Lipschitz function $H\colon \mathbb{R}^2 \to \mathbb{R}$ such that*

*(1) $S \cap E = \emptyset \mod \mathscr{L}^2$;*
*(2) $H_\sharp(\mathscr{L}^2 \llcorner_S) = \alpha \mathscr{L}^1 \llcorner_{[-1,1]}$,*

*where $S$ is the critical set of $H$, i.e. $S = \{x : \nabla H(x) = 0\}$, $E$ is the union of the regular level sets of $H$ and $\alpha$ is a strictly postitive real number.*

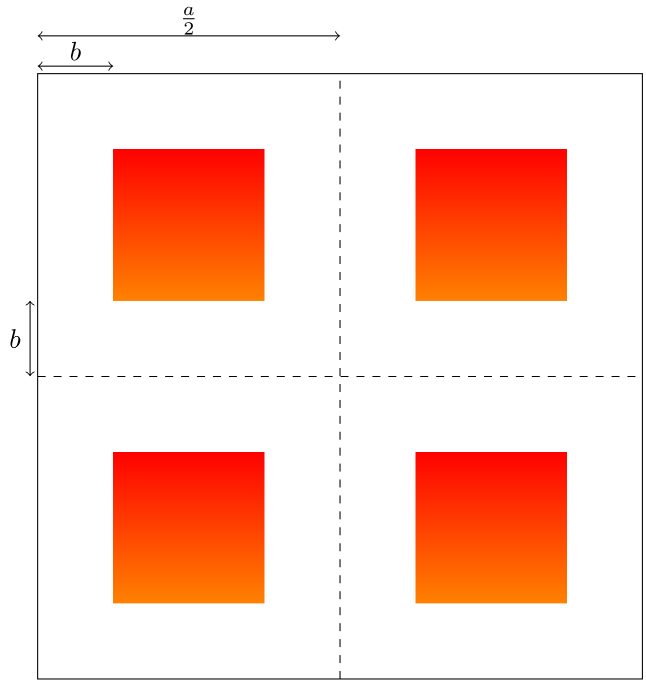



**Figure 1.** The sets $Q$ (black) and $Q_1, Q_2, Q_3, Q_4$ (red).

PROOF. For simplicity, we split the proof into several steps.

*Step 1. The construction of the set.* Consider a square $Q = [0,a] \times [0,a]$, $a > 0$. Suppose that $b \in (0, a/4)$. Let $Q_1$, $Q_2$, $Q_3$ and $Q_4$ denote the squares which are the connected components of $([b, a/2-b] \cup [a/2+b, a-b]) \times ([b, a/2-b] \cup [a/2+b, a-b])$ (see Figure 1). Define

$$F_b(Q) := \{Q_1, Q_2, Q_3, Q_4\}.$$

When $Q$ is a translation of $[0,a] \times [0,a]$ we define $F_b$ similarly. If $\mathcal{Q} = \{Q_1, \dots, Q_n\}$ where $Q_1, \dots, Q_n$ are disjoint squares with side $a$, let us define

$$F_b(\mathcal{Q}) := \bigcup_{i=1}^{n} F(Q_i).$$

*Step 2. The construction of the function.* Given two positive real numbers $0 < r < R$ we introduce the following cut-off auxiliary functions:

$$\zeta_{r,R}(t) := \begin{cases} 1, & t < r \\ \frac{R-t}{R-r}, & r \le t \le R \\ 0, & t > r \end{cases}, \qquad \chi_{r,R}(x) := \zeta_{r,R}(|x|_\infty),$$

where $|x|_\infty = \max(|x_1|, |x_2|)$, $x = (x_1, x_2) \in \mathbb{R}^2$. Now, given a square $Q = [-\ell, \ell] \times [-\ell, \ell]$ and $b \in (0, \ell)$ we define

$$h_{Q,b}(x) := \chi_{\ell-b,\ell}(x)$$

when $x \in Q$, and set $h_{Q,b}(x) := 0$ when $x \notin Q$. Finally, set

$$H_{Q,b}(x) := \frac{3}{4} h_{Q_1,b}(x) + \frac{1}{4} h_{Q_2,b}(x) - \frac{1}{4} h_{Q_3,b}(x) - \frac{3}{4} h_{Q_4,b}(x) \tag{5.3}$$

for $x \in Q$ and $H_{Q,b}(x) := 0$ for $x \notin Q$. We clearly have

$$\|H_{Q,b}\|_\infty = \frac{3}{4} \qquad \text{and} \qquad \|\nabla H_{Q,b}\|_\infty = \frac{3}{4}\frac{1}{b}.$$

*Step 3. Iteration.* Let now

$$b_n := \frac{1}{16}4^{-n}, a_0 := 1, \quad a_{n+1} := a_n/2 - 2b_n.$$

By induction it follows that

$$2^n a_n = a_0 - 4\sum_{k=0}^{n-1} 2^k b_k$$

Hence $2^n a_n \searrow a_0 - 4\frac{1}{1-\frac{1}{4}} \cdot \frac{1}{16} = \frac{2}{3}$, therefore

$$a_n \sim \frac{2}{3}2^{-n}, \quad \text{as} \quad n \to +\infty.$$

Define

$$\mathcal{Q}_0 := \{[0, a_0] \times [0, a_0]\}, \quad \mathcal{Q}_{n+1} := F_{b_n}(\mathcal{Q}_n)$$

Let

$$S_n := \{x \in Q \mid Q \in \mathcal{Q}_n\}, \quad S := \bigcap_{n=1}^{\infty} S_n.$$

*Step 4. Passage to the limit and conclusion.* Since $S_n$ consists of $4^n$ squares with side $a_n$,

$$\alpha := |S| = \lim_{n\to\infty} |S_n| = \lim_{n\to\infty} 4^n a_n^2 = \frac{4}{9}.$$

Finally, we define

$$\begin{aligned} h_n(x) &= 4^{-n} \sum_{Q\in\mathcal{Q}_n} H_{Q,b_n}(x), \\ H(x) &:= \sum_{n=0}^{\infty} h_n(x) \end{aligned} \tag{5.4}$$

where $H_{Q,b_n}$ is given by (5.3). The series in (5.4) converges uniformly because $\|h_n\|_\infty \le 4^{-n}$. Moreover,

$$\|\nabla h_n\|_\infty \le 4^{-n}\frac{1}{b_n} \le 16$$

and, since $\nabla h_i \mathscr{L}^2 \perp \nabla h_j \mathscr{L}^2$ for $i \neq j$, it follows that $\|\nabla H\|_\infty \le 16$, hence the function $H$ is Lipschitz. From the uniform convergence of the series (5.4), it follows that for any measure $\mu$

$$(H_n)_\sharp \mu \rightharpoonup H_\sharp \mu$$

as $n \to \infty$, where

$$H_n(x) := \sum_{k=0}^{n-1} h_k(x).$$

Finally, if we let $\mu := \mathscr{L}^2 \llcorner_S$, from the definition of $h_k$ it follows that

$$\nu_n := (H_n)_\sharp \mu = \alpha \cdot 4^{-n} \sum_{k=0}^{4^n-1} \delta_{x_k},$$

where $x_k = (2k+1)4^{-n} - 1$. For any $\varphi \in C_0(\mathbb{R})$, it is easy to recognize that the integral $\int \varphi \, d\nu_n$ is the Riemann sum (up to the constant factor $\alpha$), and thus one has that $\int \varphi \, d\nu_n \to \alpha \int_{-1}^1 \varphi(x)\, dx$. In conclusion, it holds

$$\nu_n \rightharpoonup \alpha \mathscr{L}^1 \llcorner_{[-1,1]},$$

which is what we wanted to prove. □

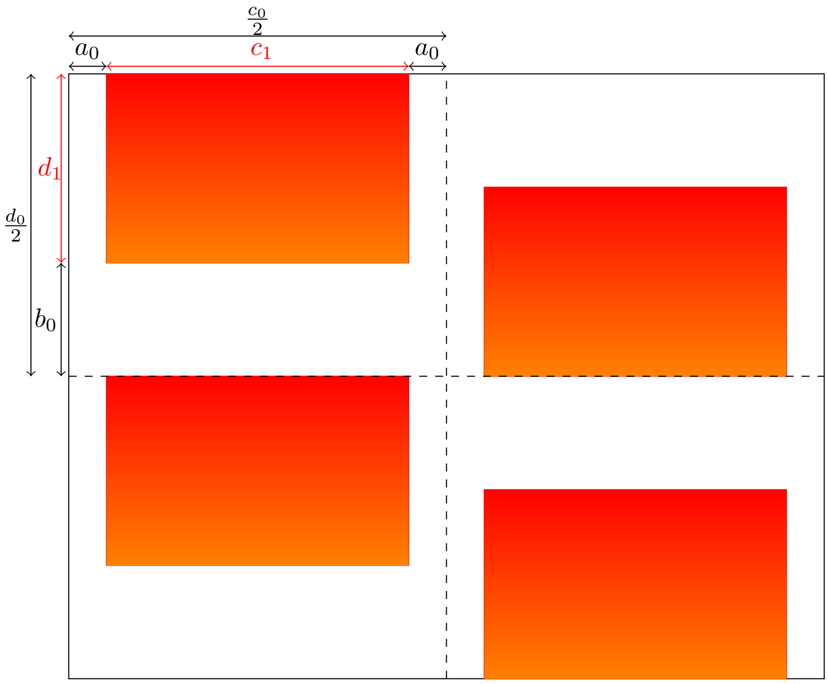



**Figure 2.** The sets $C_0$ (black) and $C_1$ (red).

**5.1.2. A function that does not have Weak Sard Property.** In this section, we present an example (which goes back to [**ABC13**]) of a Lipschitz function that does not have the Weak Sard Property (5.2). Although this example is well known, we want to recall it for the reader convenience, as it will be extensively used in Section 5.4 as the building block of a counterexample.

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be decreasing sequences of positive numbers with asymptotic behaviour given by

$$a_n \sim b_n \sim \frac{1}{n^2 2^n}.$$

Hence, the following quantities

$$\widehat{a} := \sum_{n=0}^{\infty} 2^{n+2} a_n, \qquad \widehat{b} := \sum_{n=0}^{\infty} 2^{n+1} b_n$$

are finite. Chosen a real number $\delta > 0$, we set

$$c_0 := \delta + \widehat{a}, \qquad d_0 := \delta + \widehat{b}.$$

*The construction of the set.* We consider the set $C_0$, which is the closed rectangle with width $c_0$ and height $d_0$. Then we define $C_1$ to be the union of 4 closed rectangles with sizes

$$c_1 := \frac{c_0}{2} - 2a_0, \qquad d_1 := \frac{d_0}{2} - b_0$$

like in Figure 2. If we iterate the above construction, we obtain a sequence of nested sets: more precisely, if $C_n$ is the union of $4^n$ pairwise disjoint, closed rectangles with width $c_n$ and height $d_n$, then $C_{n+1}$ is the union of $4^{n+1}$ pairwise disjoint closed rectangles with width

$$c_{n+1} := \frac{c_n}{2} - 2a_n, \qquad d_{n+1} := \frac{d_n}{2} - b_n.$$

It is easy to see that from this recursion we have

$$2^n c_n = c_0 - \sum_{m=0}^{n-1} 2^{m+2} a_m \searrow \delta \qquad \text{and} \qquad 2^n d_n = d_0 - \sum_{m=0}^{n-1} 2^{m+2} b_m \searrow \delta$$

which implies that $c_n, d_n$ are always strictly positive and satisfy

$$c_n \sim d_n \sim \frac{\delta}{2^n}.$$

If $C$ denotes the intersection of the closed sets $C_n$ we have

$$\mathscr{L}^2(C) = \lim_n \mathscr{L}^2(C_n) = \lim_n 4^n d_n c_n = \delta^2.$$

*Construction of the function.* We now turn to the construction of a suitable sequence of Lipschitz and piecewise smooth functions $f_n\colon \mathbb{R}^2 \to \mathbb{R}$. The function $f_0$ is defined by its level sets, drawn in Figure 3a. Let $s_n$ be the oscillation of the function $f_n$ on the component of $C_n$; it is clear from the picture that

$$s_{n+1} = \frac{s_n}{4},$$

hence $s_n = 4^{-n}s_0 = 4^{-n}d_0$.

*$L^\infty$ gradient estimates.* We can now estimate the gradient of the functions $f_n$. It is easy to see that the supremum of $|\nabla f_n|$ in the set $C_n$ is attained in the set $E$ defined in Figure 3a. Choosing the axes as in Figure 3b we can write an explicit formula for $f_n$; in particular, the line that passes through the points $(-a_n, b_n)$ and $(a_n, \frac{d_n}{2} - b_n)$ has equation

$$x_2 = b_n + \frac{1}{4a_n}(x_1 + a_n)(d_n - 4b_n).$$

Then if we pick a $\tau \in (0, b_n)$ we impose the similarity of the triangles, hence

$$\frac{\tau b_n}{b_n} = \frac{x_2}{b_n + \frac{1}{4a_n}(x_1 + a_n)(d_n - 4b_n)}$$

hence we get

$$\tau = \frac{4a_n x_2}{(d_n - 4b_n)x_1 + a_n d_n}.$$

Therefore, the function $f_n$ has the following explicit formula in $E$:

$$f_n(x_1, x_2) = (1-\tau)t + \tau\left(t + \frac{s_n}{4}\right) = t + \frac{s_n}{4}\tau = t + \frac{a_n s_n x_2}{(d_n - 4b_n)x_1 + a_n d_n}.$$

A direct computation shows that

$$\nabla f_n(x) = \frac{1}{a_n d_n + (d_n - 4b_n)x_1}\left(-(d_n - 4b_n)(f_n(x) - t), a_n s_n\right).$$

Taking into account that $x_1 \geq -a_n$ and that $d_n - 4b_n > 0$ (due to the asymptotic behaviour) we can estimate from below the denominator:

$$a_n d_n + (d_n - 4b_n)x_1 \geq 4a_n b_n.$$

On the other hand, we clearly have $|f_n - t| \leq s_n$ and thus we obtain the following estimate:

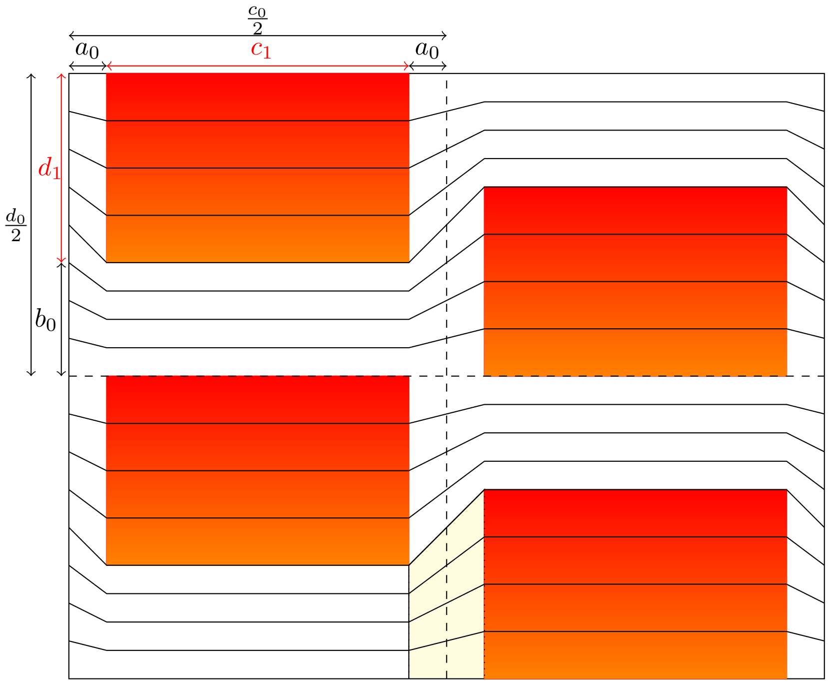
$$\|\nabla f_n\|_{L^\infty(C_n)} \leq \frac{(d_n - 4b_n)s_n + a_n s_n}{4a_n b_n} = \mathcal{O}(n^4 2^{-n}).$$

Now let us define the function $h_n := f_n - f_{n-1}$. Clearly, by definition of $f_n$, the support of $h_n$ lies in $C_n$; moreover,

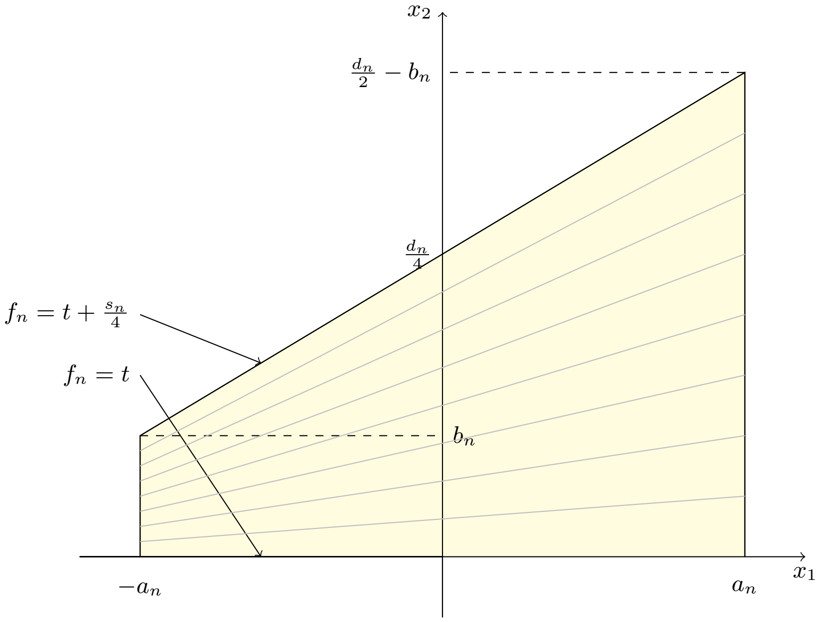
$$\|\nabla h_n\|_\infty \leq \|\nabla f_n\|_{L^\infty(C_n)} + \|\nabla f_{n-1}\|_{L^\infty(C_{n-1})} \sim n^4 2^{-n}.$$

Since the distance of a point in $C_n$ from $\mathbb{R}^2 \setminus C_n$ is of order $c_n \sim 2^{-n}$, by the Mean Value Theorem

$$\|h_n\|_{L^\infty} \sim n^4 4^{-n}.$$




**(a)** Level sets of the function $f_0$.




**(b)** Estimate of $|\nabla f_n|$: the level sets of $f_n$ in the set $E$.

**Figure 3.** Level sets of the function $f_0$ and estimates for $|\nabla f_n|$.

For every $x \in \mathbb{R}^2$ set

$$f(x) := \lim_{n \to +\infty} f_n(x) = f_0(x) + \sum_{n=1}^{\infty} h_n(x).$$

We sum up the properties of the function $f$ in the following

THEOREM 5.2 ([**ABC13**, Prop. 4.7]). *If $C$ is the set above and $f$ is the function built in the previous sections, then:*

*(i) $f$ is differentiable at every $x \in C$ with $\nabla f(x) = 0$;*
*(ii) $\mathscr{L}^1(f(C)) = d_0$;*
*(iii) $f_\sharp(\mathscr{L}^2 \llcorner_C) = m\mathscr{L}^1 \llcorner_{f(C)}$, where $m = \delta^2/d_0$; in particular, $f$ does not satisfy the Weak Sard Property.*

## 5.2. The Chain Rule problem

We now leave, for a moment, the two dimensional framework and we present in this section a problem which is closely related to the one of establishing the renormalization property for weakly differentiable vector fields. It is usually known as the *Chain rule problem* and it reads as follows:

PROBLEM 5.3 (Chain Rule). *Let $d \geq 2$ and assume that it is given a bounded, Borel vector field $\boldsymbol{b}\colon \mathbb{R}^d \to \mathbb{R}^d$, a bounded, scalar function $u\colon \mathbb{R}^d \to \mathbb{R}$ and Radon measures $\lambda, \mu \in \mathscr{M}(\mathbb{R}^d)$ such that*

$$\operatorname{div} \boldsymbol{b} = \lambda, \tag{5.5a}$$

$$\operatorname{div}(u\boldsymbol{b}) = \mu, \tag{5.5b}$$

*in the sense of distributions on $\mathbb{R}^d$. Characterize (compute) the distribution*

$$\nu := \operatorname{div}(\beta(u)\boldsymbol{b}),$$

*where $\beta\colon \mathbb{R} \to \mathbb{R}$ is a fixed $C^1$ function.*

In the smooth setting one can use the standard chain rule formula to get

$$\begin{aligned} \nu = \operatorname{div}(\beta(u)\boldsymbol{b}) &= \beta'(u)\operatorname{div}(u\boldsymbol{b}) + (\beta(u) - u\beta'(u))\operatorname{div} \boldsymbol{b} \\ &= \beta'(u)\mu + (\beta(u) - u\beta'(u))\lambda. \end{aligned} \tag{5.6}$$

The extension of (5.6) to a non-smooth setting is far from being trivial and this is exactly the aim of Problem 5.3.

Problem 5.3 arises naturally in the study of partial differential equations, like the transport equation, the continuity equation or, more generally, hyperbolic conservations laws: indeed, they all can be written in the form $\operatorname{div}(uB) = c$, where $B\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^d$ is vector field which has a space-time structure and $c \in \mathscr{D}'(\mathbb{R}^d)$ is some distribution. Specifically, let us assume $\boldsymbol{b}$ is a locally integrable, divergence-free vector field and $u \in L^\infty$ is a weak solution to the transport equation

$$\partial_t u + \boldsymbol{b} \cdot \nabla u = 0.$$

These two pieces of information can be written as

$$\operatorname{div}_{t,x} B = 0, \qquad \operatorname{div}_{t,x}(uB) = 0$$

where $B := (1, \boldsymbol{b}) \in \mathbb{R}^{d+1}$ and $\operatorname{div}_{t,x} := \partial_t + \operatorname{div}_x$. We thus see that this fits in the setting above and the renormalization property would follow by computing

$$\nu := \operatorname{div}_{t,x}(\beta(u)B)$$

and proving $\nu = 0$. More generally, considering Problem 5.3 for a particular choice of $B$ and $\beta$, one can establish uniqueness and comparison principles for weak solutions also to scalar conservation laws (in the spirit of Kružkov's theory, see [**Kru70**]).

As noted in [**ADLM07**], if one replaces "divergence" by "derivative", the problem boils down to the one of writing a chain rule for weakly differentiable functions (a theme that has been investigated in several papers, see e.g. [**Vol67, ADM90**] for the BV setting). However, the "divergence" problem seems to be harder than the "derivative" one, due to stronger cancellation effects.

**5.2.1. Positive results.** If we assume Sobolev regularity on the vector field, i.e. $\boldsymbol{b} \in W^{1,p}_{\rm loc}(\mathbb{R}^d)$ and $u \in L^q_{\rm loc}(\mathbb{R}^d)$ with $p, q$ dual exponents, the chain rule has been established in [DL89c]. In this case, it turns out that $\nu$ can be computed in terms of $\lambda$ and $\mu$ just as in the classical (smooth) setting: it holds

$$\nu = (\beta(u) - u\beta'(u))\lambda + \beta'(u)\mu,$$

provided $\mu = \operatorname{div}(u\boldsymbol{b})$ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^d$ on $\mathbb{R}^d$. As we have seen in the Introduction, this result has been extended in [ADLM07], using the commutator estimate due to Ambrosio [Amb04]: it is first proved that the distribution $\nu = \operatorname{div}(\beta(u)\boldsymbol{b})$ is a Radon measure which satisfies $\nu \ll |\lambda| + |\mu|$. Furthermore, by decomposing $\lambda, \mu, \nu$ into three parts (the *absolutely continuous* part $\lambda^a$, the *jump part* $\lambda^j$ and the *Cantor part* $\lambda^c$, as in the standard BV setting) they show that:

- the absolutely continuous part behaves as in the Sobolev case:
$$\nu^a = (\beta(u) - u\beta'(u))\lambda^a + \beta'(u)\mu^a, \qquad \text{as measures on } \mathbb{R}^d.$$
- For the jump part, they use the results obtained in [ACM05] to prove that $\nu^j$ can be computed in terms of the *traces* $u^+$ and $u^-$ of $u$ on the (countably) rectifiable set $\Sigma$ where $\lambda^j$ and $\mu^j$ are concentrated on.
- The Cantor part is harder and it is not characterized completely in [ADLM07], but only up to an error term. More precisely, it has been proved that
$$\nu^c = (\beta(\widetilde{u}) - \widetilde{u}\beta'(\widetilde{u}))\lambda^c \llcorner_{\Omega\setminus S_u} + \beta'(\widetilde{u})\mu^c \llcorner_{\Omega\setminus S_u} + \sigma$$
where $\widetilde{u}$ is the $L^1$ approximately continuous representative of $u$, $S_u$ is the set of points where the $L^1$ approximate limit does not exist and $\sigma$ is an error term (which is a measure concentrated on $S_u$, with $\sigma \ll \lambda^c + \mu^c$).

Thus it remains to characterize the error term $\sigma$ apperaring in the Cantor part. For, the approach (adopted in [ACM05] and in [ADLM07]) considers the tangential set of $\boldsymbol{b}$, according to the following definition.

Definition 5.4 (Tangential set). Suppose that $\boldsymbol{b} \in \mathrm{BV}(\Omega; \mathbb{R}^d)$ is a bounded vector field. Consider the Borel set $E$ of all points $x \in \Omega$ such that:

(1) there exists and it is finite the limit
$$M(x) := \lim_{t \to 0} \frac{D\boldsymbol{b}(B^d_r(x))}{|D\boldsymbol{b}|(B^d_r(x))}.$$
(2) the approximate $L^1$-limit $\widetilde{\boldsymbol{b}}(x)$ of $\boldsymbol{b}$ at $x$ exists.

Then we call *tangential set* of $\boldsymbol{b}$ (in $\Omega$) the set
$$\mathsf{T}_{\boldsymbol{b}} := \{x \in E : M(x) \cdot \widetilde{\boldsymbol{b}}(x) = 0\}.$$

Roughly speaking, the tangential set is made up of points at which the derivative is orthogonal to the vector field. Via a blow-up argument, in [ACM05] it has been shown the following

Proposition 5.5 ([ACM05, Thm. 6.5]). *Let $\boldsymbol{b} \in \mathrm{BV}_{\rm loc}(\Omega; \mathbb{R}^d)$ and let $u \in L^\infty_{\rm loc}(\Omega)$ such that $\operatorname{div}(u\boldsymbol{b})$ is a locally finite Radon measure on $\Omega$. Then the inclusion $S_u \subset \mathsf{T}_{\boldsymbol{b}}$ holds up to $|\operatorname{div}^c \boldsymbol{b}|$-negligible sets.*

A couple of questions which naturally arise at this point are thus the following:

(Q1) Let $\boldsymbol{b} \in \mathrm{BV}_{\rm loc} \cap L^\infty_{\rm loc}(\Omega; \mathbb{R}^d)$. Does the Cantor part of the divergence $|\operatorname{div}^c \boldsymbol{b}|$ vanish on the tangential set?

(Q2) Let $\boldsymbol{b} \in \mathrm{BV}_{\rm loc} \cap L^\infty_{\rm loc}(\Omega; \mathbb{R}^d)$ be a nearly incompressible vector field, i.e. assume there exists $\rho \in L^\infty(\Omega)$ so that $\ln \rho \in L^\infty(\Omega)$ and $\partial_t \rho + \operatorname{div}_x(\rho\boldsymbol{b}) = 0$ in the sense of distributions. Does the Cantor part of the divergence $|\operatorname{div}^c \boldsymbol{b}|$ vanish on the tangential set?

(Q3) Let $\boldsymbol{b} \in \mathrm{BV}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ be a nearly incompressible vector field, i.e. assume there exists $\rho \in L^\infty(\Omega)$ so that $\ln \rho \in L^\infty(\Omega)$ and $\partial_t \rho + \mathrm{div}_x(\rho \boldsymbol{b}) = 0$ in the sense of distributions. Is it true that the other inclusion

$$\mathsf{T}_{\boldsymbol{b}} \subset S_\rho$$

holds up to $|\mathrm{div}^c \boldsymbol{b}|$-negligible sets?

The answer to all the three questions is negative. More precisely, (Q1) has been addressed and solved in [ADLM07]. A counterexample concerning (Q2) has been proposed in [BG16], where the Chain Rule problem is completely solved in the case $d = 2$ for bounded variation vector fields.

In a sense, Question (Q3) aims to give a precise estimate of the size of the tangential set and it was raised in [ACM05]. We will address to it in the next section.

## 5.3. On the size of the tangential set

In this section, we want to exhibit an example of a bounded, autonomous vector field $\boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$, of class $\mathrm{BV}(\mathbb{R}^2)$ for which there exists a density $\rho\colon \mathbb{R}^2 \to \mathbb{R}$, with $\ln \rho \in L^\infty(\mathbb{R}^2)$ and $\mathrm{div}(\rho \boldsymbol{b}) = 0$ in the sense of distributions, for which the inclusion $\mathsf{T}_{\boldsymbol{b}} \subset S_\rho$ *does not* hold up to $|\mathrm{div}^c \boldsymbol{b}|$-negligible sets. This answers in the negative to (Q3); our construction is inspired by and based on [BG16].

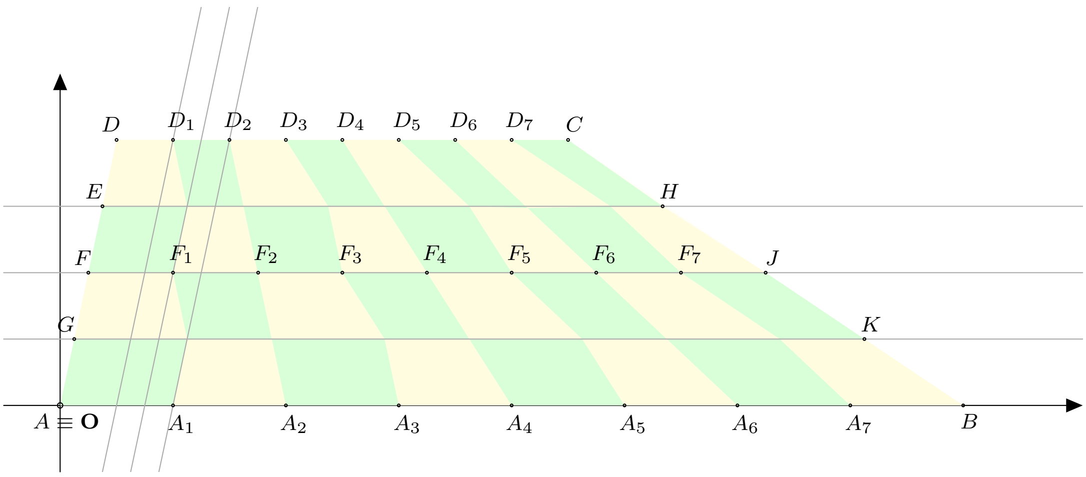



**Figure 4.** A cell $\mathcal{C} = ABCD$ and the set $\mathsf{Patch}(\mathcal{C})$.

### 5.3.1. Compressible and incompressible cells.

We begin by giving the following

Definition 5.6. A trapezium $ABCD$ in the plane $\mathbb{R}^2$ is called a *cell* if

- the bases $AB, CD$ are parallel to $\boldsymbol{e}_1$;
- $AD$ and $CB$ are the sides and $AD \cdot \boldsymbol{e}_2 > 0$;
- $|CD| \leq |AB|$.

In particular, if $|CD| = |AB|$ we say that the cell is *incompressible* while if $|CD| < |AB|$ we say the cell is *compressible*.

### 5.3.2. Patches and iterative construction.

Let $\mathcal{C} = ABCD$ be a *compressible* cell. We introduce the following auxiliary points (see Figure 4):

- $A_0, \dots, A_8 \in AB$ with $A_0 = A$, $A_8 = B$ and $|A_i A_{i+1}| = \frac{1}{8}|AB|$ for every $i = 0, \dots, 7$;
- $D_0, \dots, D_8 \in DC$ with $D_0 = D$, $D_8 = C$ and $|D_i D_{i+1}| = \frac{1}{8}|DC|$ for every $i = 0, \dots, 7$;

- $E, F, G \in AD$ and $H, J, K \in BC$ with
$$|AG| = |GF| = |FE| = |ED|, \qquad |BK| = |KJ| = |JH| = |HC|;$$
- $F_0, \dots, F_8 \in FJ$ with $F_0 = F$, $F_8 = J$ and
$$|F_i F_{i+1}| = \frac{1}{16} \cdot (|AB| + |CD|)$$
for every $i = 0, \dots, 7$.

Accordingly, we define the following "patch" operation:

Definition 5.7. Let $\mathcal{C} = ABCD$ be a cell. We define the map $\mathsf{Patch}$ as follows:
- if $\mathcal{C}$ is incompressible, then $\mathsf{Patch}(\mathcal{C}) := \mathcal{C}$;
- if $\mathcal{C}$ is compressible, then

$\mathsf{Patch}(\mathcal{C}) := \{$The 16 compressible and the 16 incompressible cells indicated in Figure 4$\}$;
- if $\{\mathcal{C}_i\}_{i=1}^N$ are $N$ disjoint cells, then
$$\mathsf{Patch}\left(\bigcup_{i=1}^N \mathcal{C}_i\right) := \bigcup_{i=1}^N \mathsf{Patch}(\mathcal{C}_i);$$

We denote by $\mathsf{Patch}^n$ the $n$-th iteration of the map $\mathsf{Patch}$. One can compute that, if $\mathcal{C}$ is a compressible cell, then $\mathsf{Patch}^n(\mathcal{C})$ is the union of $16^n$ compressible cells and $\frac{16}{15}(16^n - 1)$ incompressible cells.

**5.3.3. Construction of the associated vector fields.** We know want to use the geometric construction presented above to define a vector field associated to each cell.

Definition 5.8. Let $\mathcal{C} = ABCD$ be a cell. We define the vector field $\boldsymbol{v}_{\mathcal{C}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ associated with $\mathcal{C}$ as follows:

(1) if $\mathcal{C}$ is incompressible we set
$$\boldsymbol{v}_{\mathcal{C}}(x) := \mathbb{1}_{\operatorname{conv}\mathcal{C}}(x) \frac{AD}{AD \cdot \boldsymbol{e}_2};$$
(2) if $\mathcal{C}$ is compressible we set
$$\boldsymbol{v}_{\mathcal{C}}(x) := \mathbb{1}_{\operatorname{conv}\mathcal{C}}(x) \frac{x - M}{(x - M) \cdot \boldsymbol{e}_2},$$
where $M$ is the intersection of lines $AD$ and $BC$.

(3) if $\{\mathcal{C}_i\}_{i=1}^N$ are $N$ disjoint cells, then
$$\boldsymbol{v}_{\bigcup_i \mathcal{C}_i} := \sum_i \boldsymbol{v}_{\mathcal{C}_i}.$$

Notice that inside any compressible cell it holds
$$\langle \boldsymbol{v}_{\mathcal{C}}, \nabla \rangle \boldsymbol{v}_{\mathcal{C}} = 0, \tag{5.7}$$
and this means that $\mathcal{C} \subset \mathsf{T}_{\boldsymbol{v}_{\mathcal{C}}}$. On the other hand, a vector field associated with an incompressible cell has empty tangential set. Following [BG16], we then define
$$\mathsf{Range}(\mathcal{C}) := \operatorname{osc}_{\mathcal{C}} \boldsymbol{v}_{\mathcal{C}}$$
where we recall $\operatorname{osc}_E f = \sup_{x,y \in E} |f(x) - f(y)|$ is the oscillation of a function $f \colon E \to \mathbb{R}^n$. We have, by direct computation,
$$\mathsf{Range}(\mathcal{C}) = |\boldsymbol{v}_{\mathcal{C}}(D) - \boldsymbol{v}_{\mathcal{C}}(C)| = \frac{|AB| - |CD|}{\operatorname{dist}(AB, CD)}.$$
From Definition 5.7, one has that
$$\mathsf{Range}(\mathcal{C}_1) \;\leq\; 2^{-1} \cdot \mathsf{Range}(\mathcal{C}) \quad \forall \mathcal{C}_1 \in \mathsf{Patch}(\mathcal{C}). \tag{5.8}$$

On the other hand, setting for simplicity, $\boldsymbol{v} := \boldsymbol{v}_{\mathcal{C}}$ and $\boldsymbol{w} := \boldsymbol{v}_{\mathsf{Patch}(\mathcal{C})}$, we have that, if $\mathcal{C}$ is incompressible then $\mathcal{C} = \mathsf{Patch}(\mathcal{C})$ and hence

$$|\boldsymbol{v}(x) - \boldsymbol{w}(x)| \ = \ 0, \qquad \forall x \in \mathcal{C}\,.$$

On the other hand, if $\mathcal{C}$ is compressible, for any $x \in \mathcal{C}$, recalling Definition 5.7, we have that

$$-\cot(\angle CBA) \ \leq \ \boldsymbol{v}_1(x) \ \leq \ \cot(\angle DAB)\,,$$

where $\cot(\cdot)$ is the cotangent function. By observing that $x_1 \mapsto \boldsymbol{w}_1(x_1, x_2)$ are (non-strictly) decreasing function, we also have

$$-\cot(\angle CBA) \ \leq \ \boldsymbol{w}_1(x) \ \leq \ \cot(\angle DAB)\,.$$

This implies that

$$|\boldsymbol{v}(x) - \boldsymbol{w}(x)| \ = \ |\boldsymbol{v}_1(x) - \boldsymbol{w}_1(x)| \ = \ \cot(\angle DAB) + \cot(\angle CBA) \ = \ \mathsf{Range}(\mathcal{C})\,.$$

Therefore, in any case it holds

$$\sup_{x \in \mathcal{C}} |\boldsymbol{v}_{\mathcal{C}}(x) - \boldsymbol{v}_{\mathsf{Patch}(\mathcal{C})}(x)| \ \leq \ \mathsf{Range}(\mathcal{C})\,. \tag{5.9}$$

**5.3.4. Passage to the limit I.** Let un now fix a compressible cell $\mathcal{C} := ABCD$ and define the $n$-th approximation vector field as

$$\boldsymbol{b}_n \ := \ \boldsymbol{v}_{\mathsf{Patch}^n(\mathcal{C})}\,. \tag{5.10}$$

We have the following

LEMMA 5.9. *There exists a constant $C > 0$ and a bounded vector field $\boldsymbol{b} \colon \Omega \to \mathbb{R}^2$ such that*

*(1)* $\sup_\Omega |\boldsymbol{b}_n| \leq C$ *for all $n \in \mathbb{N}$;*
*(2)* $\boldsymbol{b}_n \to \boldsymbol{b}$ *uniformly in $\Omega$;*
*(3)* $\|\boldsymbol{b}_n\|_{\mathrm{BV}(\Omega)} \leq C$ *for all $n \in \mathbb{N}$.*

PROOF. For any $x \in \Omega$, let $\mathcal{C}_x$ be in $\mathsf{Patch}^n(\mathcal{C})$ such that $x \in \mathcal{C}_x$. Recalling (5.9), we have

$$|\boldsymbol{b}_{n+1}(x) - \boldsymbol{b}_n(x)| \ = \ |\boldsymbol{v}_{\mathsf{Patch}(\mathcal{C}_x)} - \boldsymbol{v}_{\mathcal{C}_x}(x)| \ \leq \ \mathsf{Range}(\mathcal{C}_x)\,.$$

On the other hand, by induction from (5.8) it follows that

$$\mathsf{Range}(C_x) \ \leq \ 2^{-n} \cdot \mathsf{Range}(\mathcal{C})\,.$$

This implies

$$\sup_{x \in \Omega} |\boldsymbol{b}_{n+1}(x) - \boldsymbol{b}_n(x)| \ \leq \ 2^{-n} \cdot \mathsf{Range}(\mathcal{C})\,.$$

Hence the series

$$\sum_{n=1}^{\infty} \sup_\Omega |\boldsymbol{b}_{n+1} - \boldsymbol{b}_n|$$

converges and thus the sequence $(\boldsymbol{b}_n)_{n \in \mathbb{N}}$ is uniformly bounded. This yields Points (1) and (2).

We now prove Point (3). For any compressible cell $\mathcal{C}' = A'B'C'D'$ in $\mathsf{Patch}^n(\mathcal{C})$, set $\boldsymbol{v} := \boldsymbol{v}_{\mathcal{C}'}$. Recalling (5.7), we have

$$\partial_2 \boldsymbol{v}_1(x) \ = \ -\boldsymbol{v}_1(x) \cdot \partial_2 \boldsymbol{v}_1(x), \qquad \forall x \in \mathcal{C}'\,.$$

This implies that

$$\|\nabla \boldsymbol{v}(x)\| \ \leq \ (1 + |\boldsymbol{v}_1(x)|) \cdot |\partial_1 \boldsymbol{v}_1(x)| \ \leq \ (1 + c) \cdot |\partial_1 \boldsymbol{v}_1(x)|, \qquad \forall x \in \mathcal{C}'$$

where $c \in \max\{|\cot(\angle DAB)|, |\cot(\angle ABC)|\}$. Thus,

$$\|\nabla \boldsymbol{v}\|_{L^1(\mathcal{C}')} \ \leq \ (1 + c) \cdot \int_{\mathcal{C}'} |\partial_1 \boldsymbol{v}_1(x)| dx \ = \ (1 + c) \cdot (|A'B'| - |C'D'|)\,.$$

Therefore,

$$|D^a \boldsymbol{b}_n|(\Omega) \;=\; \sum_{\mathcal{C}'\in \mathsf{Patch}^n(\mathcal{C})} \|\boldsymbol{v}_{\mathcal{C}'}\|_{L^1(\mathcal{C}')} \;\le\; (1+c)\cdot(|AB|-|CD|)\,. \tag{5.11}$$

Let us now turn to estimate the jump part of $\boldsymbol{b}_n$. Observe that the jumps of $\boldsymbol{b}_{n+1}-\boldsymbol{b}_n$ are concentrated on the union of upper and lower bases of compressible cells of $\mathsf{Patch}^n(\mathcal{C})$. For any compressible cell $\mathcal{C}' = A'B'C'D' \in \mathsf{Patch}^n(\mathcal{C})$, by recalling (5.9), we can estimate

$$\begin{aligned}|D^j\boldsymbol{b}_{n+1} - D^j\boldsymbol{b}_n|(\mathcal{C}') \;\le\; 3|A'B'|\cdot \mathsf{Range}(\mathcal{C}') &= \frac{3|A'B'|}{\mathrm{dist}(A'B',C'D')}(|A'B'|-|C'D')\\ &\le 2^{-n}\frac{3|AB|}{\mathrm{dist}(AB,CD)}(|A'B'|-|C'D'|)\,.\end{aligned}$$

This implies that

$$|D^j\boldsymbol{b}_{n+1} - D^j\boldsymbol{b}_n|(\mathcal{C}) \;\le\; 2^{-n}\frac{3|AB|\cdot(|AB|-|CD|)}{\mathrm{dist}(AB,CD)} \;=\; 2^{-n}3|AB|\cdot\mathsf{Range}(\mathcal{C})\,.$$

Since $|D^j\boldsymbol{b}_0|(\mathcal{C})=0$, we then obtain

$$|D\boldsymbol{b}_n|(\mathcal{C}) \;\le\; \left(\sum_{k=1}^{n} 2^{-k}\right)\cdot 3|AB|\cdot\mathsf{Range}(\mathcal{C}') \;\le\; 6|AB|\cdot\mathsf{Range}(\mathcal{C})\,. \tag{5.12}$$

Combining (5.11) and (5.12), we finally obtain

$$\|\boldsymbol{b}_n\|_{\mathrm{BV}(\Omega)} \;\le\; (1+c)\cdot(|AB|-|CD|) + 6|AB|\cdot\mathsf{Range}(\mathcal{C})\,.$$

and this completes the proof. □

**5.3.5. Passage to the limit II.** We now study some fine properties of the limit vector field $\boldsymbol{b}$. Assume that the origin coincides with point $A$ of the initial cell and let

$$L := \left\{\left(x_1, \frac{m}{4^n}\mathrm{dist}(AB,CD)\right) \mid x_1\in\mathbb{R},\, n\in\mathbb{N},\, m\in\mathbb{Z}\cap[0,4^n]\right\}$$

denote the union of the horizontal lines containing bases and midlines of all the cells obtained by iterations of the map $\mathsf{Patch}$. Let $\mathsf{Patch}^n_{\mathrm{comp}}(\mathcal{C})$ denote the union of all compressible cells in $\mathsf{Patch}^n(\mathcal{C})$. Let

$$S_n := \bigcup_{\mathcal{C}'\in\mathsf{Patch}^n_{\mathrm{comp}}(\mathcal{C})} \mathrm{conv}(\mathcal{C}'), \qquad S := \bigcap_{n\in\mathbb{N}} S_n. \tag{5.13}$$

Lemma 5.10. *The Hausdorff dimension of the set $S$ is equal to $\frac{5}{3}$. In particular, the Lebesgue measure of $S$ is 0.*

Proof. Recalling that $S_n$ is a union of $16^n$ disjoint compressible cells in $\mathsf{Patch}^n_{\mathrm{comp}}(\mathcal{C})$ with height $4^{-n}\cdot\mathrm{dist}(AB,CD)$. For each $\mathcal{C}'\in\mathsf{Patch}^n_{\mathrm{comp}}(\mathcal{C})$, $\mathcal{C}'$ can be covered by $2^n$ balls of radius $r_n = 8^{-n}d$ with $d:=\max\{|AB|,|BC|,|AD|\}$. Thus, for any $\alpha>0$,

$$\mathscr{H}^\alpha_{r_n}(S) \;\le\; \mathscr{H}^\alpha_{r_n}(S_n) \;\le\; 16^n\cdot 2^n 8^{-n\alpha}d^\alpha \;=\; 2^{5n-3\alpha n}d^\alpha.$$

This implies that

$$\mathscr{H}^{\frac{5}{3}}(S) \;=\; \lim_{n\to\infty}\mathscr{H}^{\frac{5}{3}}_{r_n}(S) \;\le\; d^{\frac{5}{3}}. \tag{5.14}$$

On the other hand, let $\{U_i\}$ be any finite cover of $S$. For each $U_i$, there exists $k\in\mathbb{N}$ such that

$$64^{-(k+1)}\cdot d_1 \;\le\; \mathrm{diam}(U_i) \;\le\; 64^{-k}\cdot d_1\,.$$

where $d_1 = \frac{1}{2}\cdot\min\{|AB|,|CD|,\mathrm{dist}(AB,CD)|\}$. For any $j>3k$, the number of trapeziums in $\mathsf{Patch}^j(\mathcal{C})$ that intersects $U_i$ is at most

$$16^{j-2k}\cdot 4^{-k} \;=\; 16^j\cdot 64^{-\frac{5}{3}k} \;\le\; 16^j\cdot\left(\frac{64}{d_1}\right)^{\frac{5}{3}}\cdot\mathrm{diam}(U_i)^{\frac{5}{3}}\,.$$

Since $\{U_i\}$ is a finite cover of $S$, there exists $j_0 \in \mathbb{N}$ sufficiently large such that $U_i$ intersects at most $16^{j_0} \cdot (\frac{64}{d_1})^{\frac{5}{3}} \cdot \operatorname{diam}(U_i)^{\frac{5}{3}}$ number of trapeziums in $\mathsf{Patch}^{j_0}(\mathcal{C})$. On the other hand, $\{U_i\}$ intersects all $16^{j_0}$ trapeziums of $\mathsf{Patch}^{j_0}(\mathcal{C})$. Thus,

$$16^{j_0} \ \leq \ 16^{j_0} \cdot \left(\frac{64}{d_1}\right)^{\frac{5}{3}} \cdot \sum_i \operatorname{diam}(U_i)^{\frac{5}{3}}.$$

This implies that

$$\sum_i \operatorname{diam}(U_i)^{\frac{5}{3}} \ \geq \ \left(\frac{d_1}{64}\right)^{\frac{5}{3}}.$$

Hence,

$$\mathscr{H}^{\frac{5}{3}}(S) \ \geq \ \left(\frac{d_1}{64}\right)^{\frac{5}{3}}. \tag{5.15}$$

The proof is complete by (5.14) and (5.15). □

We now can show the following fine properties of the vector field $\boldsymbol{b}$.

Lemma 5.11. *Let $\boldsymbol{b}$ denote the limit of $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ given by Lemma 5.9. Then $\boldsymbol{b} \in \mathrm{BV}(\Omega)$ and*

*(1) the absolutely continuous part $D^a\boldsymbol{b}$ is zero;*
*(2) the jump part $D^j\boldsymbol{b}$ is concentrated on $L$ and $D^j\boldsymbol{b}_n \overset{*}{\rightharpoonup} D^j\boldsymbol{b}$ locally in $\Omega$ as $n \to +\infty$;*
*(3) $\operatorname{div}^a \boldsymbol{b} = \operatorname{div}^j \boldsymbol{b} = 0$;*
*(4) the Cantor part $D^c\boldsymbol{b}$ is concentrated on the set $S$ (defined in (5.13)) and $D^a\boldsymbol{b}_n \overset{*}{\rightharpoonup} D^c\boldsymbol{b}$;*
*(5) the measure $\operatorname{div}^c \boldsymbol{b}$ is concentrated on the set $S$;*
*(6) it holds $|\operatorname{div}^c \boldsymbol{b}| \ll \mathscr{H}^{\frac{5}{3}} \llcorner S$.*

Proof. The vector field $\boldsymbol{b}$ is of class BV by Lemma 5.9 (using lower semicontinuity of total variation). We now address separately each point.

**1.** By Lemma 5.10, we have that $\mathscr{L}^2$-a.e. point $x \in \Omega$ belongs to the interior of some incompressible cell $\mathcal{I} \in \mathsf{Patch}^n(\mathcal{C})$ for some $n \in \mathbb{N}$. Inside this incompressible cell $\boldsymbol{b}$ coincides with the vector field associated with $\mathcal{I}$, which is constant (by definition). Hence the approximate differential of $\boldsymbol{b}$ is zero a.e. in $\Omega$ and, by Calderon-Zygmund Theorem, this gives $D^a\boldsymbol{b} = 0$ (see, for instance, [**AFP00**, Thm. 3.83]).

**2.** Observe that any $x \in \Omega \setminus L$ is a Lebesgue point of $\boldsymbol{b}$. Indeed, any such $x$ is a Lebesgue point of $\boldsymbol{b}_n$ for all $n \in \mathbb{N}$: we have

$$\begin{aligned}\int_{B_r(x)} |\boldsymbol{b}(y) - \boldsymbol{b}(x)|\, dy \ \leq \ & \int_{B_r(x)} |\boldsymbol{b}_n(y) - \boldsymbol{b}_n(x)|\, dy + \int_{B_r(x)} |\boldsymbol{b}(y) - \boldsymbol{b}_n(y)|\, dy \\ & + \int_{B_r(x)} |\boldsymbol{b}_n(x) - \boldsymbol{b}(x)|\, dy.\end{aligned}$$

Using uniform convergence one can show that $\boldsymbol{b}(x)$ is the Lebesgue value of $\boldsymbol{b}$ at $x$. Since all the points in $\Omega \setminus L$ are Lebesgue points of $\boldsymbol{b}$, the jump part $D^j\boldsymbol{b}$ is concentrated on $L$.

**3.** By (5.12), the jump part $D^j\boldsymbol{b}_n$ converges to some measure $\mu$ concentrated on $L$

$$D^j\boldsymbol{b}_n \overset{*}{\rightharpoonup} \mu$$

locally in $\Omega$ as $n \to +\infty$. Since $\boldsymbol{b}_n \to \boldsymbol{b}$ uniformly in $\Omega$, we have $D\boldsymbol{b}_n \overset{*}{\rightharpoonup} D\boldsymbol{b}$ locally in $\Omega$ as $n \to +\infty$. Thus, $D^a\boldsymbol{b}_n \overset{*}{\rightharpoonup} \nu := D\boldsymbol{b} - \mu$, locally in $\Omega$ as $n \to \infty$. On the other hand, recalling that $D\boldsymbol{b}^a = 0$, we have

$$\mu + \nu \ = \ D^c\boldsymbol{b} + D^j\boldsymbol{b}.$$

Since both $D^j\boldsymbol{b}$ and $\mu$ are concentrated on $L$, we obtain that $D^j\boldsymbol{b} = \mu + \nu \llcorner_L$. Therefore to prove that $D^j\boldsymbol{b} = \mu$ it is sufficient to show that $\nu\llcorner_L = 0$. For any $a, b \in I$ and $n \in \mathbb{N}$, by using the same computations as in derivation of (5.11) one can derive the estimate

$$|D^a\boldsymbol{b}_n|(\Omega \cap \mathbb{R} \times (a,b)) \;\le\; (1+c)\cdot(|A_1B_1| - |C_1D_1|)$$

where $A_1B_1C_1D_1 = \Omega \cap \mathbb{R} \times (a,b)$. Since

$$\begin{aligned} |A_1B_1| - |C_1D_1| \;&\le\; |b-a|\cdot(|\cot(\angle D_1A_1B_1)| + |\cot(\angle A_1B_1C_1)|) \\ &=\; |b-a|\cdot(|\cot(\angle DAB)| + |\cot(\angle ABC)|) \end{aligned}$$

we then obtain that

$$|D^a\boldsymbol{b}_n|(\Omega \cap \mathbb{R}\times(a,b)) \;\le\; (1+c)\cdot(|\cot(\angle DAB)| + |\cot(\angle ABC)|)\cdot(b-a)\,. \tag{5.16}$$

Covering $L$ by horizontal stripes with arbitrary small total projection on $x_2$ axis, it is possible to show that $|\nu|(L) < \varepsilon$ for any $\varepsilon > 0$. Hence indeed $\nu\llcorner_L = 0$.

**4.** We have proved $\mu = D^j\boldsymbol{b}$ and $\nu = D^c\boldsymbol{b}$. By the construction of $\boldsymbol{b}_n$ for any $n \in \mathbb{N}$, $D^j\boldsymbol{b}_n$ is concentrated on the union of upper and lower bases of compressible cells of $\mathsf{Patch}^n(\mathcal{C})$. Hence,

$$D^j\boldsymbol{b}_n \;=\; \Big[(\boldsymbol{b}_n^+ - \boldsymbol{b}_n^-) \;\otimes\; e_2\Big]\mathscr{H}^1\llcorner_L\,.$$

Since $\boldsymbol{b}_n^{2,+} - \boldsymbol{b}_n^{2,-} = 0$ and $\boldsymbol{e}_2 = (0,1)$, we then have $\operatorname{div}^j \boldsymbol{b}_n = 0$. But $\operatorname{div}^j \boldsymbol{b}_n \overset{*}{\rightharpoonup} \operatorname{div}^j \boldsymbol{b}$ as $n \to +\infty$, since $D^j\boldsymbol{b}_n \overset{*}{\rightharpoonup} D^j\boldsymbol{b}$ as $n \to +\infty$, therefore $\operatorname{div}^j \boldsymbol{b} = 0$.

**5.** From the steps above we already know that $D^c\boldsymbol{b}$ is concentrated on the set $S$ and that the measure $\operatorname{div}^c \boldsymbol{b}$ is the weak$^\star$ limit of the measures $\operatorname{div}^a \boldsymbol{b}_n$. For any $m \in \mathbb{N}$, for any $\mathcal{C}_m = A_mB_mC_mD_m \in \mathsf{Patch}^m(\mathcal{C})$, we have that

$$\operatorname{div}^a \boldsymbol{b}_m(x) \;=\; \operatorname{div}\, \boldsymbol{v}_{\mathcal{C}_m}(x) \;=\; \frac{\partial \boldsymbol{v}^1_{\mathcal{C}_m}}{\partial x_1}(x), \qquad \forall x \in \operatorname{int}(\mathcal{C}_m)\,.$$

This implies that

$$\operatorname{div}^a \boldsymbol{b}_m(\operatorname{int}\, \mathcal{C}_m) \;=\; \int_{\mathcal{C}_m} \frac{\partial \boldsymbol{v}^1_{\mathcal{C}_m}}{\partial x_1}(x)\, dx \;=\; |C_mD_m| - |A_mB_m|\,.$$

Thus, for any fixed $n \in \mathbb{N}$ and for any $\mathcal{C}_n = A_nB_mC_nD_n \in \mathcal{P}_n(\mathcal{C})$, it holds

$$\operatorname{div}^a \boldsymbol{b}_m(\operatorname{int}\, \mathcal{C}_n) \;=\; |C_nD_n| - |A_nB_n| \quad \text{for all } m \ge n\,.$$

and

$$|\operatorname{div}^a \boldsymbol{b}_m|(\operatorname{int}\, \mathcal{C}_n) \;\le\; |A_nB_n| - |C_nD_n|, \quad \text{for all } m \ge n\,.$$

By letting $m \to \infty$, we obtain

$$|\operatorname{div}^c \boldsymbol{b}|(\operatorname{int}\, \mathcal{C}_n) \;\le\; |A_nB_n| - |C_nD_n| \;\le\; 8^{-n}\cdot(|AB| - |CD|)\,.$$

Therefore, $\operatorname{div}^c \boldsymbol{b}$ is concentrated out of compressible cells (i.e. on the set $S$).

**6.** For any $S_1 \subset S$, let $U_i$ be any finite cover of $S$. For each $U_i$, there exists $k_i \in \mathbb{N}$ such that

$$8^{-k_i-1}d_1 \;\le\; \operatorname{diam}(U_i) \;\le\; 8^{-k_i}d_1\,.$$

Observe that $U_i$ has nonempty intersection with at most 2 compressible cells in $\mathsf{Patch}^{k_i}_{\text{comp}}(\mathcal{C})$. On the other hand, for a compressible cell $\mathcal{C}_{k_i} = A_{k_i}B_{k_i}C_{k_i}D_{k_i} \in \mathsf{Patch}^{k_i}_{\text{comp}}(\mathcal{C})$ with $U_i \cap \mathcal{C}_{k_i} \neq \varnothing$, there exists a trapezium $KLMN \subset A_{k_i}B_{k_i}C_{k_i}D_{k_i}$ with $K, N \in A_{k_i}D_{k_i}$, $L, N \in B_{k_i}C_{k_i}$, $KL \parallel MN$, and

$$\operatorname{dist}(MN, KL) \;=\; 8^{-k_i}d_1 \;\le\; 2^{-2k_i}\cdot \operatorname{dist}(A_{k_i}B_{k_i}, C_{k_i}D_{k_i})$$

such that $U_i \cap \mathcal{C}_{k_i} \subset KLMN$. With the same argument of Step 5, one can show that

$$\begin{aligned}|\operatorname{div}^c \boldsymbol{b}|(U_i \cap \mathcal{C}_{k_i}) &\leq |\operatorname{div}^c \boldsymbol{b}|(KLMN)\\ &\leq\ |KL| - |MN|\\ &= \frac{\operatorname{dist}(MN, KL)}{\operatorname{dist}(A_{k_i}B_{k_i}, C_{k_i}D_{k_i})} \cdot (|A_{k_i}B_{k_i}| - |C_{k_i}D_{k_i}|)\\ &\leq\ 2^{-2k_i} \cdot (|A_{k_i}B_{k_i}| - |C_{k_i}D_{k_i}|)\ =\ 2^{-5k_i} \cdot (|AB| - |CD|)\,.\end{aligned}$$

This implies that

$$|\operatorname{div}^c \boldsymbol{b}|(U_i)\ \leq\ 2 \cdot 2^{-5k_i} \cdot (|AB| - |CD|)\,.$$

Recalling that

$$\operatorname{diam}^{\frac{5}{3}}(U_i)\ \geq\ \frac{1}{32} d_1^{\frac{5}{3}} \cdot 2^{-5k_i}\,,$$

we get

$$|\operatorname{div}^c \boldsymbol{b}|(U_i)\ \leq\ \frac{64}{d_1^{\frac{5}{3}}} \cdot (|AB| - |DC|) \cdot \operatorname{diam}^{\frac{5}{3}}(U_i)\,.$$

Hence,

$$|\operatorname{div}^c \boldsymbol{b}|(S_1)\ \leq\ \frac{64}{d_1^{\frac{5}{3}}} \cdot (|AB| - |DC|) \cdot \sum_i \operatorname{diam}^{\frac{5}{3}}(U_i)\,.$$

Therefore

$$|\operatorname{div}^c \boldsymbol{b}|(S_1)\ \leq\ \frac{32}{d_1^{\frac{5}{3}}} \cdot (|AB| - |DC|) \cdot \mathscr{H}^{\frac{5}{3}}(S_1)\,,$$

and the proof is completed. □

**5.3.6. Nearly incompressibility.** We now want to prove that the vector fields $\boldsymbol{b}_n$ are uniformly (steady) nearly incompressible in the following sense: for every $n \in \mathbb{N}$, there exists a bounded function $\rho_n$ such that it holds $\operatorname{div}(\rho_n B_n) = 0$ in the sense of distributions on $\Omega$, for every $n$. Moreover, we need also some uniformity in the near incompressibility (to pass to the limit), i.e. we want that there exists a constant $C > 0$ (which does *not* depend on $n$) s.t.

$$0 < C^{-1} \leq \rho_n \leq C < +\infty$$

for every $n \in \mathbb{N}$.

*The base step.* Let us fix any compressible cell $\mathcal{C} = ABCD$ and divide the cells in $\mathsf{Patch}(\mathcal{C})$ into two blocks.

The cells in *blocks of the first kind* are characterized by the fact that their lowest cell is incompressible; analogously, the cells in *blocks of the second kind* are characterized by the property of having a compressible lowest cell: see Figure 5a. In what follows, we will work in the blocks of the first kind, the construction for the other cells being completely analogous.

Given a block of the first kind like the one in Figure 5a, we define the auxiliary function $f_1$ in the following way:

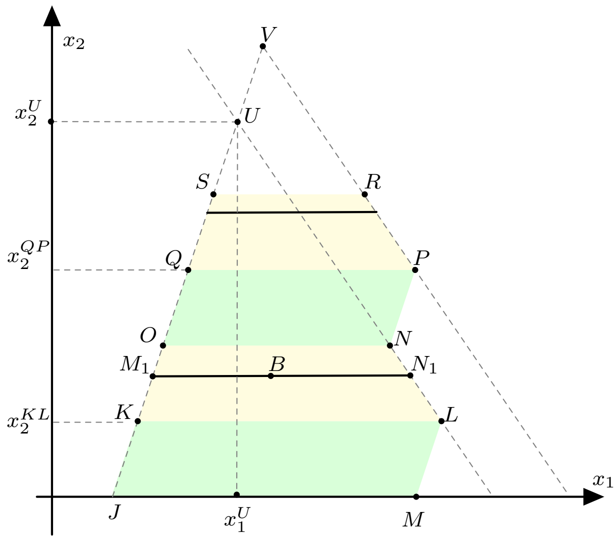
$$f_1(x_1, x_2) := \frac{|JM|}{|L_1(x_1, x_2)|} \qquad \text{where} \qquad L_1(x_1, x_2) := \{(y, x_2) \mid y \in \mathbb{R}\} \cap JMRS\,.$$

One can easily observe that the function $f_1$ is constant in the incompressible cells: more precisely, we have

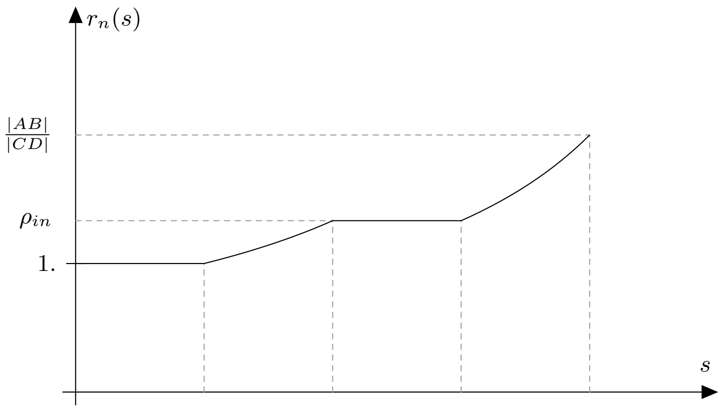
$$f_1 \equiv 1 \quad \text{in the cell } JMLK, \qquad f_1 \equiv \rho_{in} := \frac{2 \cdot |AB|}{|AB| + |CD|} \quad \text{in the cell } ONPQ.$$

On the other hand, in the compressible cells, we have

$$f_1(x_1, x_2) = \frac{x_2^U - x_2^{KL}}{x_2^U - x_2}, \quad \text{in the cell } KLNO$$




**(a)** A block of the first kind.




**(b)** The restriction of the function $\rho^n$ to a line parallel and sufficiently close to $JS$ behaves like the function $r_n$.

**Figure 5.** The density $\rho_n$ along the flow lines in blocks of the first kind.

and

$$f_1(x_1, x_2) = \rho_{in} \cdot \frac{x_2^V - x_2^{QP}}{x_2^V - x_2} \quad \text{in the cell } QPRS,$$

being $U = (x_1^U, x_2^U)$ the point where $KO$ and $LN$ intersect and $V = (x_1^V, x_2^V)$ the point where $QS$ and $PR$ intersect.

We remark that the restriction of $f_1$ along any line parallel (and sufficiently close) to $JS$ is an increasing function which takes the values from 1 to $\frac{|AB|}{|CD|}$ and looks like the function $r_n$ depicted in Figure 5b.

Finally, applying the map Patch, we end up with a scalar function $\rho_1^{\mathcal{C}}$ (defined in original cell $\mathcal{C}$) which coincides with $f_1$ in the blocks of the first kind and with the analogous function $f_2$ in the blocks of the second kind.

We sum up some useful properties of the function $\rho_1^{\mathcal{C}}$ in the following

LEMMA 5.12. *For a fixed compressible cell $\mathcal{C} = ABCD$, the function $\rho_1^{\mathcal{C}}$ satisfies the following properties:*

*(1) $\rho_1^{\mathcal{C}}$ takes the values from 1 to $\frac{|AB|}{|CD|}$ and is continuous in every $\mathcal{C}_1 \in \mathsf{Patch}(\mathcal{C})$. Moreover, for any $x \in \mathcal{C}$, it holds*

$$\rho_1^{\mathcal{C},-} \;\leq\; \rho_1(x) \;\leq\; \rho_1^{\mathcal{C},+} \quad \text{and} \quad \frac{\rho_1^{\mathcal{C},+}}{\rho_1^{\mathcal{C},-}} \;=\; \frac{|AB|}{|CD|}\,, \tag{5.17}$$

*where $\rho_1^{\mathcal{C},+}$ and $\rho_1^{\mathcal{C},-}$ are the values of $\rho_1^{\mathcal{C}}$ in the upper and lower bases of $\mathcal{C}$ respectively.*

*(2) The function $\rho_1^{\mathcal{C}}$ solves*

$$\operatorname{div}\left(\rho_1^{\mathcal{C}}\,\boldsymbol{v}_{\mathsf{Patch}(\mathcal{C})}\right) = 0 \qquad \text{in } \mathscr{D}'(\mathcal{C})\,.$$

PROOF. Point (1) follows immediately from the definition of the function $\rho_1^{\mathcal{C}}$ (and of the function $f_1$ and $f_2$). Let us prove Point (2): for any $\mathcal{C}_1 \in \mathcal{P}_1(\mathcal{C})$, we will show that

$$\operatorname{div}(\rho_1^{\mathcal{C}}\,\boldsymbol{v}_{\mathsf{Patch}(\mathcal{C})}) = 0 \qquad \text{in } \mathcal{C}_1\,. \tag{5.18}$$

If the cell $\mathcal{C}_1$ is incompressible, then (5.18) holds since both $\rho_1^{\mathcal{C}}$ and $V_{\mathsf{Patch}(\mathcal{C})}$ constant in $\mathcal{C}_1$. If $\mathcal{C}_1$ is compressible (say, $\mathcal{C}_1 = KLNO$), for any $(x_1, x_2) \in KLNO$, we have by definition

$$\boldsymbol{v}_{\mathsf{Patch}(\mathcal{C})}(x_1,x_2) = \left(\frac{x_1 - x_1^U}{x_2 - x_2^U}, 1\right)$$

and

$$\rho_1^{\mathcal{C}}(x_1,x_2) = \frac{x_2^U - x_2^{KL}}{x_2^U - x_2} = \frac{c}{x_2^U - x_2}\,,$$

being $c := x_2^U - x_2^{KL}$ a constant. We have thus

$$(\rho_1^{\mathcal{C}}\,\boldsymbol{v}_{\mathsf{Patch}(\mathcal{C})})(x_1,x_2) = c\cdot\left(-\frac{x_1 - x_1^U}{(x_2 - x_2^U)^2}, \frac{1}{x_2^U - x_2}\right)\,,$$

and a direct computation yields (5.18). □

*Inductive step.* We now define by induction a sequence of functions $(\rho_n)_{n\in\mathbb{N}}$ which will be the required steady densities for the vector fields $B_n$. For a fixed initial compressible $\mathcal{C} = ABCD$, we define

$$\rho_1(x) \;:=\; \rho_1^{\mathcal{C}}(x), \qquad \forall x \in \mathcal{C}\,.$$

Assume now that $\rho_n$ is constructed. For any $\mathcal{C}_n = A_nB_nC_nD_n \in \mathcal{P}_n(\mathcal{C})$, we define $\rho_{n+1}$ in the following way:

- If $\mathcal{C}_n$ is an incompressible cell then
$$\rho_{n+1}(x) := \rho_n(x) \qquad \forall x \in \mathcal{C}_n\,.$$
- If $\mathcal{C}_n$ is a compressible cell, denote by $\rho_n^{\mathcal{C}_n,-}$ the value of $\rho_n$ on the lower base of $\mathcal{C}_n$. Let $\rho_1^{\mathcal{C}_n}$ be the function which is constructed as in the previous step with the compressible cell $\mathcal{C}_n$. For any $x \in \mathcal{C}_n$,
$$\rho_{n+1}(x) := \rho_n^{\mathcal{C}_n,-}\cdot\rho_1^{\mathcal{C}_n}(x)\,. \tag{5.19}$$

From Lemma 5.12, for any $\mathcal{C}_n = A_nB_nC_nD_n \in \mathcal{P}_n(\mathcal{C})$, it holds

$$\rho_n(x) \;=\; \rho_n^{C_n,-}\cdot\frac{|A_nB_n|}{|L_{\mathcal{C}_n}(x)|} \qquad \forall x \in C_n\,, \tag{5.20}$$

where $\rho_n^{C_n,-}$ is the values of $\rho_n$ lower base of $\mathcal{C}_n$ and

$$L_{\mathcal{C}_n}(x_1,x_2) = \{(y,x_2) \mid y \in \mathbb{R}\} \cap \mathcal{C}_n\,.$$

This implies that

$$\frac{\rho_n^{C_n,+}}{\rho_n^{C_n,-}} \;=\; \frac{|A_nB_n|}{|C_nD_n|}$$

Hence, from (5.19) and (5.17), we have that

$$\rho_{n+1}(x) \;=\; \rho_n(x), \qquad \forall x \in A_nB_n \cup C_nD_n\,. \tag{5.21}$$

We have now the following

LEMMA 5.13. *For a fixed compressible cell $\mathcal{C} = ABCD$, the followings hold:*

*(1) the function $\rho_n$ takes the values from 1 to $\frac{|AB|}{|CD|}$ and is continuous in $\mathcal{C}_n \in \mathsf{Patch}_n(\mathcal{C})$.*

*(2) The function $\rho_n$ solves*

$$\operatorname{div}(\rho_n \boldsymbol{b}_n) \;=\; 0 \qquad \text{in } \mathscr{D}'(\mathcal{C})\,.$$

PROOF. Recalling lemma 5.12, (5.20), (5.21) and (5.19), one can obtain (1) by using the method of induction. To prove (2), we only need to show that for any $\mathcal{C}_n \in \mathcal{P}_n(\mathcal{C})$

$$\operatorname{div}(\rho_n \boldsymbol{b}_n) \;=\; 0 \qquad \text{in } \mathcal{C}_n\,. \tag{5.22}$$

If $\mathcal{C}_n$ is an incompressible cell, then (5.22) holds since $\rho_n$ and $\boldsymbol{b}_n$ are constants in $\mathcal{C}_n$. Assume that $\mathcal{C}_n$ is a compressible cell. Let $\mathcal{C}_{n-1} \in \mathcal{P}_{n-1}(\mathcal{C})$ be such that $\mathcal{C}_n \subset \mathcal{C}_{n-1}$. We have

$$(\rho_n \boldsymbol{b}_n)(x) \;=\; \rho_{n-1}^{\mathcal{C}_{n-1},-}(x) \cdot \rho_1^{\mathcal{C}_{n-1}}(x) \cdot V_{\mathsf{Patch}(\mathcal{C}_{n-1})}(x)\,.$$

Recalling (2) of Lemma 5.12, we obtain (5.22) and this concludes the proof.

□

**5.3.7. The counterexample.** We are eventually ready to prove the following theorem, which is analogue of Theorem 3.9 of [**BG16**] and ensures the existence of a bounded, nearly incompressible, BV vector field for which the inclusion

$$\mathsf{T}_{\boldsymbol{b}} \subset S_\rho$$

does *not* hold up to $|\operatorname{div}^c \boldsymbol{b}|$-negligible sets.

THEOREM 5.14. *There exists a bounded vector field $\boldsymbol{b} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ and a function $\rho\colon \Omega \to \mathbb{R}$ such that:*

*(1) $\operatorname{div}^c \boldsymbol{b}$ is a non-positive measure concentrated on the set $S$;*

*(2) $|\operatorname{div}^c \boldsymbol{b}|$-a.e. $x \in S$ belongs to the tangential set $\mathsf{T}_{\boldsymbol{b}}$ of $\boldsymbol{b}$;*

*(3) $\rho(\Omega \setminus S)$ is equal to a subset of null $\mathscr{L}^1$-measure in $\left[1, \frac{|AB|}{|CD|}\right]$.*

*(4) $\operatorname{div}(\rho \boldsymbol{b}) = 0$ in $\mathscr{D}'(\Omega)$ and $\log \rho \in L^\infty(\Omega)$;*

*(5) the function $\rho$ is ($L^1$-approximately) continuous $|\operatorname{div}^c \boldsymbol{b}|$-a.e $x \in S$.*

PROOF. Exactly as in [**BG16**], we consider the sequence of vector fields defined in (5.10). By Lemma 5.11, there exists a bounded BV vector field $\boldsymbol{b}\colon \Omega \to \mathbb{R}^2$ such that $\boldsymbol{b}_n \to \boldsymbol{b}$ uniformly in $\Omega$ as $n \to \infty$. We prove again separately each point of the statement.

**1.** This claim follows immediately from Lemma 5.11, Point 5.

**2.** If $V$ is a vector and $A$ is a matrix, we denote by $\langle V, A\rangle := V \otimes A = (v_i a^{ji})_j$ (sum over repeated indices). Let $\phi\colon \Omega \to \mathbb{R}$ be a bounded Borel function with compact support in $\Omega$. Due to Lemma 5.11, Point 4, we have that $D^a \boldsymbol{b}_n \rightharpoonup^\star D^c \boldsymbol{b}$ locally in $\Omega$: this gives

$$\lim_{n\to+\infty} \int_\Omega \phi \langle \boldsymbol{b}, D^a \boldsymbol{b}_n\rangle = \int_\Omega \phi \langle \boldsymbol{b}, D^c \boldsymbol{b}\rangle. \tag{5.23}$$

On the other hand

$$\int_\Omega \phi\langle \boldsymbol{b}, D^a\boldsymbol{b}_n\rangle = \int_\Omega \phi\langle \boldsymbol{b}-\boldsymbol{b}_n, D^a\boldsymbol{b}_n\rangle + \int_\Omega \phi\langle \boldsymbol{b}_n, D^a\boldsymbol{b}_n\rangle. \tag{5.24}$$

Now, by construction of $\boldsymbol{b}_n$ we have $\langle \boldsymbol{b}_n, D^a\boldsymbol{b}_n\rangle = 0$ (see (5.7)); moreover, $\boldsymbol{b}_n \to B$ uniformly in $\Omega$ and, by (5.16), the total variation of $D^a\boldsymbol{b}_n$ is uniformly bounded. Hence

$$\lim_{n\to+\infty} \int_\Omega \phi\langle \boldsymbol{b}_n - \boldsymbol{b}, D^a\boldsymbol{b}_n\rangle = 0. \tag{5.25}$$

Therefore, from (5.23), (5.24), (5.25) we conclude that

$$\int_\Omega \phi\langle \boldsymbol{b}, D^c\boldsymbol{b}\rangle = 0, \qquad \forall\phi \text{ bounded and Borel.} \tag{5.26}$$

Hence, writing the polar decomposition $D^c\boldsymbol{b} = M|D^c\boldsymbol{b}|$, from (5.26) we deduce

$$\int_\Omega \phi(x)\langle \boldsymbol{b}(x), M(x)\rangle\, d|D^c\boldsymbol{b}|(x) = 0, \qquad \forall\phi \text{ bounded and Borel}$$

and since (5.26) holds for arbitrary $\phi$, we deduce that

- $\langle \boldsymbol{b}(x), M(x)\rangle = 0$, for $|D^c\boldsymbol{b}|$-a.e. $x \in \Omega$;
- $|D^c\boldsymbol{b}|$ is concentrated on the set $\{x \in \Omega : \langle \boldsymbol{b}(x), M(x)\rangle = 0\}$.

The claim now easily follows remembering that, by definition, $|D^c\boldsymbol{b}|$ is concentrated on the set of Lebesgue points of $B$ (see [**AFP00**, pag. 184]) and moreover $|\operatorname{div}^c \boldsymbol{b}| \ll |D^c\boldsymbol{b}|$.

**3.** Let $(\rho_n)_{n\in\mathbb{N}}$ be the sequence constructed in Lemma 5.13. It is easy to see that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is Cauchy in $L^\infty(\Omega)$ (like the standard approximating sequence of the Cantor-Vitali funtion). In particular, one can verify that it holds

$$|\rho_{n+1}(x) - \rho_n(x)| \lesssim \omega(n) \tag{5.27}$$

where $\lesssim$ means that the inequality holds up to a positive constant (which depends only on the initial cell) and the sequence of real numbers $(\omega(n))_{n\in\mathbb{N}}$ is convergent (actually it is summable). Hence there exists a limit function $\rho \in L^\infty(\Omega)$ with $\ln\rho \in L^\infty(\Omega)$. Moreover, denoting by $A_n := \rho_n(\Omega \setminus S_n)$, for $n \in \mathbb{N}$, it is easy to see that each $A_n$ is finite and $A_n \subset \left[1, \frac{|AB|}{|CD|}\right]$. Thus the set $A := \rho(\Omega\setminus S) \subset \left[1, \frac{|AB|}{|CD|}\right]$ is at most countable and the claim now follows.

**4.** Since $\boldsymbol{b}_n \to \boldsymbol{b}$ in $L^1$ and $\rho_n \to \rho$ for *a.e.* $x \in \Omega$, we can pass to the limit in $\operatorname{div}(\rho_n\boldsymbol{b}_n) = 0$ and deduce that $\operatorname{div}(\rho\boldsymbol{b}) = 0$, hence $\boldsymbol{b}$ is nearly incompressible.

**5.** Denote by $L_f$ the set of ($L^1$-approximately) continuity points of a function $f$ we show

$$\bigcap_{n\in\mathbb{N}} L_{\rho_n} \subseteq L_\rho\,. \tag{5.28}$$

Fixed any $x \in \bigcap_{n\in\mathbb{N}} L_{\rho_n}$, we have

$$\fint_{B_s(x)} |\rho_n(y) - \rho_n(x)|\, dy \to 0 \qquad \text{as } s \to 0^+ \qquad \forall n \in \mathbb{N}\,.$$

Recalling (5.27), we have

$$\begin{aligned}\fint_{B_s(x)} |\rho(y) - \rho(x)|\, dy &\le \fint_{B_s(x)} |\rho(y) - \rho_n(y)|\, dy + \fint_{B_s(x)} |\rho_n(y) - \rho_n(x)|\, dy \\ &\quad + \fint_{B_s(x)} |\rho_n(x) - \rho(x)|\, dy \\ &\lesssim \omega(n) + \fint_{B_s(x)} |\rho_n(y) - \rho_n(x)|\, dy\,.\end{aligned}$$

This implies that

$$\lim_{s\to 0+} \fint_{B_s(x)} |\rho(y)-\rho(x)|\,dy \;=\; 0$$

i.e. $x \in L_\rho$. We now observe that, by the construction, each $\Omega\backslash L_{\rho_n}$ has finite length, hence it has Hausdorff dimension 1. In view of (5.28), it holds that

$$\Omega\backslash L_\rho \;\subseteq\; \Omega\backslash \bigcap_{n\in\mathbb{N}} L_{\rho_n}$$

so also the set $\Omega\backslash L_\rho$ has Hausdorff dimension at most 1. Since $|\operatorname{div}^c \boldsymbol{b}| \ll \mathscr{H}^{5/3}$, as shown in Lemma 5.11, this is enough to conclude the proof. □

## 5.4. Non-steady renormalization defects

We now come back to the Chain Rule problem discussed in Section 5.2. In particular, we want to briefly discuss the recent work [CGSW17] where, using the abstract machinery of convex integration, the authors construct examples of vector fields $\boldsymbol{b}\colon \mathbb{R}^d \to \mathbb{R}^d$ and densities $u\colon \mathbb{R}^d \to \mathbb{R}$ such that (in the language introducted in Problem 5.3) $\lambda = 0$, $\mu = 0$ but $\operatorname{div}(u^2\boldsymbol{b}) \neq 0$ in the sense of distributions in $\mathbb{R}^d$ for $d \geq 3$. More precisely, their main result reads as follows:

Theorem 5.15 ([CGSW17]). *Let $d \geq 3$ and $\Omega \subset \mathbb{R}^d$ a smooth domain. Let $f$ be a distribution such that the equation $\operatorname{div} \boldsymbol{w} = f$ admits a bounded, continuous solution $\boldsymbol{w}\colon \Omega \to \mathbb{R}^d$ on $\Omega$. Then there exist a bounded vector field $\boldsymbol{b} \in L^\infty(\Omega;\mathbb{R}^d)$ and a density $u\colon \mathbb{R}^d \to \mathbb{R}$, with $0 < C^{-1} \leq u \leq C$ a.e. for some constant $C > 0$, such that*

$$\operatorname{div} \boldsymbol{b} = 0$$
$$\operatorname{div}(u\boldsymbol{b}) = 0$$
$$\operatorname{div}\left(u^2\boldsymbol{b}\right) = f$$

*in the sense of distributions in $\Omega$.*

**5.4.1. The two-dimensional case.** The assumption $d \geq 3$ is essential in [CGSW17], in view of the result of [BG16]. More precisely, in [BG16], the authors proved that if $d = 2$, $\boldsymbol{b}$ is bounded and of class BV and $u\colon \mathbb{R}^d \to \mathbb{R}$, with $0 < C^{-1} \leq u \leq C$ a.e. for some constant $C > 0$, are such that

$$\operatorname{div} \boldsymbol{b} = 0$$
$$\operatorname{div}(u\boldsymbol{b}) = 0$$

then the Chain rule property holds, i.e. we have necessarily $\operatorname{div}\left(u^2\boldsymbol{b}\right) = 0$. Actually, the same conclusion is true if the assumption $\boldsymbol{b} \in \mathrm{BV}$ is replaced by $\boldsymbol{b} \neq 0$ a.e. in $\Omega$.

However, still remaining in the planar setting, in view of the results obtained in [BBG16], it seems reasonable to consider the Chain Rule problem also in the non steady setting, i.e. assuming that the vector field has a (special) space-time structure (and letting the divergence operator acting also on the time variable). More precisely, we are led to consider the following variant of Problem 5.3:

Problem 5.16 (Non-steady Chain Rule). *Let $T > 0$ be fixed and assume that it is given a bounded, Borel vector field $\boldsymbol{b}\colon (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$, a bounded, scalar function $u\colon (0,T) \times \mathbb{R}^d \to \mathbb{R}$ and Radon measures $\lambda$ and $\mu$ such that*

$$\operatorname{div} \boldsymbol{b} = \lambda,$$
$$\partial_t u + \operatorname{div}(u\boldsymbol{b}) = \mu,$$

*in the sense of distributions on $(0,T) \times \mathbb{R}^d$. Characterize (compute) the distribution*

$$\nu := \partial_t \beta(u) + \operatorname{div}(\beta(u)\boldsymbol{b}),$$

*where $\beta\colon \mathbb{R} \to \mathbb{R}$ is a fixed $C^1$ function.*

In this section we want to show the following

THEOREM 5.17 ([BB17a]). *There exists an autonomous, compactly supported vector field* $\boldsymbol{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\boldsymbol{b} \in L^\infty(\mathbb{R}^2)$, *and a bounded, scalar function* $u\colon (0,T) \times \mathbb{R}^2 \to \mathbb{R}$, *such that*

$$\operatorname{div} \boldsymbol{b} = 0,$$
$$\partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0,$$

*in* $\mathscr{D}'((0,T) \times \mathbb{R}^2)$ *but the distribution*

$$\partial_t \left(u^2\right) + \operatorname{div}\left(u^2 \boldsymbol{b}\right) \notin \mathscr{M}((0,T) \times \mathbb{R}^2)$$

*i.e. it is* not *(representable by) a Radon measure.*

The rest of this chapter is devoted to the proof of Theorem 5.17. We first collect some preliminary results.

**5.4.2. A particular change of variables.** As in [ABC14, §2.16], we will denote by $I$ the interval $[0, L]$, by $\mathscr{L}^1$ the Lebesgue measure on $I$ and, in general, $\lambda$ will be an arbitrary measure on $I$, which is singular with respect to $\mathscr{L}^1$ and has $\mathcal{A}$ as the set of its atoms (points with positive measure). We set $\widehat{L} := (\mathscr{L}^1 + \lambda)(I)$ and $\widehat{I} := [0, \widehat{L}]$. We denote by $\widehat{\mathscr{L}^1}$ the Lebesgue measure restricted to $\widehat{I}$. We denote by $\widehat{\sigma}$ the multifunction from $I$ to $\widehat{I}$ that to every $s \in I$ associates the interval

$$\widehat{\sigma}(s) := [\widehat{\sigma}_-(s), \widehat{\sigma}_+(s)]$$

where

$$\widehat{\sigma}_-(s) := (\mathscr{L}^1 + \lambda)\big([0,s)\big), \qquad \widehat{\sigma}_+(s) := (\mathscr{L}^1 + \lambda)\big([0,s]\big).$$

It is immediate to see that $\widehat{\sigma}$ is surjective on $I$, strictly increasing, and uni-valued for every $s \notin \mathcal{A}$, because $\sigma_-$ and $\sigma_+$ are strictly increasing, and $\sigma_-(s) = \sigma_+(s)$ whenever $s \notin \mathcal{A}$. Moreover it is obvious that the map is expanding, i.e.

$$s_2 - s_1 \leq \widehat{s}_2 - \widehat{s}_1 \tag{5.31}$$

for every $s_1, s_2 \in I$ with $s_1 < s_2$, and every $\widehat{s}_1 \in \widehat{\sigma}(s_1), \widehat{s}_2 \in \widehat{\sigma}(s_2)$. Accordingly $\sigma$ is surjective from $\widehat{I}$ onto $I$, uni-valued and 1-Lipschitz (because of (5.31)); furthermore, it is constant on the interval $\sigma(s)$ for every $s \in \mathcal{A}$ and strictly increasing at every point outside $\sigma(\mathcal{A})$.

We recall the following

LEMMA 5.18 ([ABC14, Lemma 2.17]). *Let* $F$ *a* $\mathscr{L}^1$*-null set* $I$ *on which the measure* $\lambda$ *is concentrated and let* $\widehat{F} := \widehat{\sigma}(F)$*. Then*

(1) *it holds* $\sigma_\sharp \widehat{\mathscr{L}^1} = \mathscr{L}^1 + \lambda$*;*
(2) *the derivative of* $\sigma$ *agrees with* $\mathbb{1}_{\widehat{I} \setminus \widehat{F}}$ *a.e. in* $\widehat{I}$*.*

**5.4.3. Solutions to singular, one-dimensional transport equations.** In the following we will be dealing with one-dimensional transport equations involving singular terms, i.e. equations of the form

$$\partial_t\big(v(1 + \mathscr{L}^1 \times \lambda)\big) + \partial_s v = 0, \tag{5.32}$$

where $v\colon [0,T] \times I \to \mathbb{R}$ is a function of $t, s$ and $\lambda$ is a singular measure on $I$. We explicity remark that we are now considering functions as equivalence classes modulo the measure $\mathscr{L}^1 \times \mathscr{L}^1 + \mathscr{L}^1 \times \lambda$. Clearly, equation (5.32) has to be understood in the sense of distributions on $(0,T)\times I$: we say that $v$ is a solution to (5.32) if for every $\phi\in C_c^\infty\big((0,T)\times I\big)$ it holds

$$\int_0^T\int_I v(t,s)\big(\phi_t(t,s)+\phi_s(t,s)\big)\,ds\,dt=-\int_0^T\int_I \phi_t(t,s)v(t,s)\,d\lambda(s)\,dt.$$

It is very well known that such equations present a severe phenomenon of non-uniqueness (for the associated initial value problem). In order to clarify what we mean, we begin by discussing an example.

Assume for simplicity that $I=\mathbb{R}$ and $\lambda$ is the Dirac mass at 0, so that we are considering the equation

$$\partial_t\big(v(\mathscr{L}^1\times\delta_0)\big)+\partial_s v=0 \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}). \tag{5.33}$$

If $v$ represents the density of a distribution of particles, then equation (5.33) is saying that each particle moves at constant speed 1 from left to right, except when it reaches the point 0, where it may stop for any given amount of time. Therefore, if $v_0$ is an arbitary, bounded initial datum (for simplicity, suppose its support is contained in $(-\infty,0)$), then a solution of (5.33) with initial condition $v(0,s)=v_0(s)$ is the function $v\colon[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ defined by

$$v(t,s)=\begin{cases} v_0(s-t) & s\neq 0,\\ 0 & s=0,\end{cases}$$

which physically means that no particle stops at 0. Another solution can be constructed by stopping all particles at 0, i.e.

$$\widetilde{v}(t,s)=\begin{cases} v_0(s-t) & s<0,\\ 0 & s=0,\\ \displaystyle\int_{-t}^0 u_0(\tau)\,d\tau & s>0.\end{cases}$$

More in general, for every $\alpha>0$ one can construct a solution for which the particles arrive at 0, stay there exactly for time $\alpha$ and then leave (see Figure 6):

$$u^\alpha(t,s):=\begin{cases} u_0(s-t) & s<0\\ \displaystyle\int_{-t}^{-t+\alpha} u_0(\tau)\,d\tau & s=0\\ u_0(s-t+\alpha) & s>0.\end{cases}$$

More precisely, we recall the following result, which is used in the proof of [**ABC14**, Lemma 4.5].

Lemma 5.19. *Let $\lambda$ be a non trivial measure on $[0,L]$, singular w.r.t. to $\mathscr{L}^1\llcorner_{[0,L]}$. Let furthermore $K\subset(0,L)$ be a closed, $\mathscr{L}^1$-negligible set, with $\lambda(K)>0$. Then the problem*

$$\begin{cases}\partial_t\big(v(1+\mathscr{L}^1\times\lambda)\big)+\partial_s v=0\\ v(0,\cdot)=\mathbb{1}_K(\cdot)\end{cases} \tag{5.34}$$

*admits a non trivial bounded solution.*

We recall here the main steps of the proof, as it will be useful in the following.

Proof. Clearly, the function $v(t,s):=\mathbb{1}_K(s)$ is a stationary solution of (5.34). Following [**ABC14**], we construct a second solution by exploiting the change of variable $s=\sigma(\widehat{s})$ defined in 5.4.2. We thus define

$$v(t,s):=\begin{cases} w(t,\widehat{\sigma}(s)) & \text{for } s\notin\mathcal{A},\\ \displaystyle⨍_{\widehat{\sigma}(s)} w(t,\widehat{s})\,d\widehat{s} & \text{for } s\in\mathcal{A},\end{cases} \tag{5.35}$$

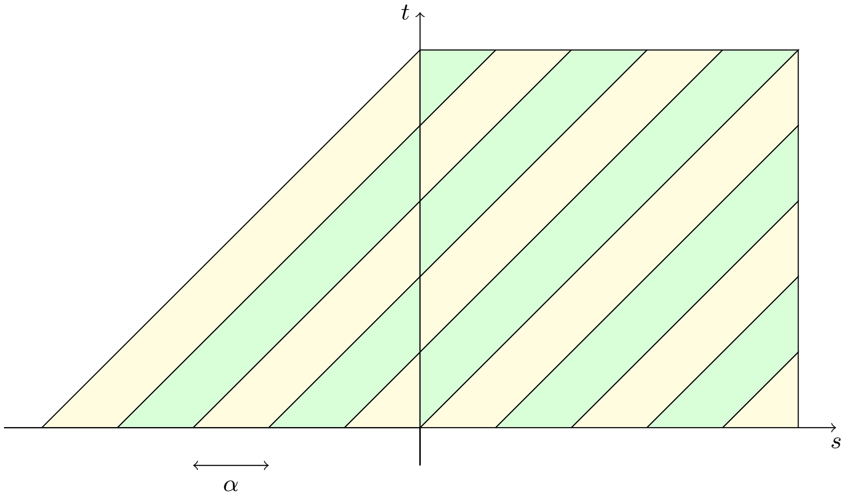



**Figure 6.** A particular solution to equation (5.33): the particles at the initial time are of two different colors (yellow and green): they start moving following characteristic lines, arrive at 0 and stay there for a prescribed time $\alpha$ before leaving.

where we recall $\mathcal{A}$ is the set of atoms of $\lambda$ and $w : [0, T) \times [0, L] \to \mathbb{R}$ is the (unique) bounded, distributional solution of

$$\begin{cases} \partial_t w + \partial_s w = 0 \\ w(0, \cdot) = \mathbb{1}_{\widehat{\sigma}(K)}(\cdot). \end{cases}$$

To see that (5.35) actually solves Problem 5.34 we proceed as follows: first observe that (5.34) can be explicitly written as

$$\int_0^T \int_0^L (\partial_t \phi + \mathbb{1}_{I \setminus F} \partial_s \phi) v \, d(\mathscr{L}^1 + \lambda) \, dt = \int_0^L \phi(0, \cdot) \mathbb{1}_K d(\mathscr{L}^1 + \lambda). \tag{5.36}$$

By changing variable $s = \sigma(\widehat{s})$, i.e. setting $\widehat{v}(t, \widehat{s}) := v(t, \sigma(\widehat{s}))$ and $\widehat{\phi}(t, \widehat{s}) := \phi(t, \sigma(\widehat{s}))$ and using Lemma 5.18, we can rewrite (5.36) as

$$\int_0^T \int_0^{\widehat{L}} (\partial_t \widehat{\phi} + \partial_{\widehat{s}} \widehat{\phi}) \widehat{v} \, d\widehat{s} \, dt = \int_0^{\widehat{L}} \widehat{\phi}(0, \cdot) \mathbb{1}_{\widehat{\sigma}(K)} d\widehat{s}.$$

Since on the complement of $\widehat{\sigma}(\mathcal{A})$ it holds $\widehat{v} = w$, to conclude we only need to show that

$$\int_{\widehat{\sigma}(\mathcal{A})} (\partial_t \widehat{\phi} + \partial_{\widehat{s}} \widehat{\phi}) \widehat{v} d\widehat{s} = \int_{\widehat{\sigma}(J)} (\partial_t \widehat{\phi} + \partial_{\widehat{s}} \widehat{\phi}) w d\widehat{s}.$$

Indeed,

$$\begin{aligned} \int_{\widehat{\sigma}(\mathcal{A})} (\partial_t \widehat{\phi} + \partial_{\widehat{s}} \widehat{\phi}) \widehat{v} d\widehat{s} &= \sum_{a \in \mathcal{A}} \int_{\widehat{\sigma}(a)} (\partial_t \widehat{\phi} + \partial_{\widehat{s}} \widehat{\phi}) \widehat{v} d\widehat{s} \\ &= \sum_{a \in \mathcal{A}} \partial_t \phi(t, s) \int_{\widehat{\sigma}(a)} \widehat{v} d\widehat{s} \\ &= \sum_{a \in \mathcal{A}} \partial_t \phi(t, s) \int_{\widehat{\sigma}(a)} w = \int_{\widehat{\sigma}(\mathcal{A})} (\partial_t \widehat{\phi} + \partial_{\widehat{s}} \widehat{\phi}) w d\widehat{s}, \end{aligned}$$

since $\partial_{\widehat{s}} \widehat{\phi}(t, \widehat{s}) = 0$ and $\partial_t \widehat{\phi}(t, \widehat{s}) = \partial_t \phi(t, s)$ for all $\widehat{s} \in \widehat{\sigma}(s)$ and by direct definition of $\widehat{v}$. To conclude the proof it is enough to show that the solution $\widehat{v}$ does not coincide with the stationary one, and for this a possible strategy is to show that the maximum $M(t)$ of the support of $v(t, \cdot)$ is strictly increasing at $t = 0$ (see [**ABC14**, Lemma 4.5])). □

**5.4.4. Further remarks on the Hamiltonian without Weak Sard Property.** In order to show Theorem 5.17, we need to perform some constructions involving an Hamiltonian which does not have Weak Sard Property, for instance the one whose construction has been recalled in Section 5.1.2. We will denote that function by $f_{c_0,d_0,\delta}$, since $c_0, d_0, \delta$ are free parameters in the construction. Recall also that $\operatorname{osc} f_{c_0,d_0,\delta} = d_0$ so that, up to a translation, we can suppose directly that

$$f_{c_0,d_0,\delta}(\mathbb{R}^2) = (0, d_0).$$

The critical set $S$ of $f_{c_0,d_0,\delta}$ has area $\mathscr{L}^2(S) = \delta^2$ and, as shown in Theorem 5.2

$$(f_{c_0,d_0,\delta})_\sharp(\mathscr{L}^2 \llcorner_C) = \frac{\delta^2}{d_0} \mathscr{L}^1 \llcorner_{f(C)}.$$

Therefore, we can apply Disintegration Theorem to the probability measure $\frac{1}{\delta^2}\mathscr{L}^2 \llcorner_C$ w.r.t. the map $f_{c_0,d_0,\delta}$. We thus write

$$\frac{1}{\delta^2}\mathscr{L}^2 \llcorner_C = \frac{1}{d_0}\int_{\mathbb{R}} \nu_h \, dh$$

where $h \mapsto \nu_h$ is a measurable measure-valued map, $\nu_h$ being a probability measure concentrated on $f^{-1}_{c_0,d_0,\delta}(h) \cap C$ for $\mathscr{L}^1$-a.e. $h \in \mathbb{R}$. We can actually say more, characterizing completely the measure $\nu_h$. In particular, we want to show that for a.e. $h$ the intersection

$$f^{-1}_{c_0,d_0,\delta}(h) \cap C$$

is a single point. We have indeed

$$f^{-1}_{c_0,d_0,\delta}(h) \cap C = \bigcap_n \left(f^{-1}_{c_0,d_0,\delta}(h) \cap C_n\right)$$

and for every $h$ it is possible to prove that $f^{-1}_{c_0,d_0,\delta}(h) \cap C_n$ is a sequence of nested intervals whose measure goes to 0 as $n \to +\infty$. For instance, if $h \in (d_0/2^n, d_0)$, we have that

$$f^{-1}(h) \cap C_n = \left(a_{n-1}, \frac{c_{n-1}}{2} - a_{n-1}\right) \times \{d_0\}.$$

The length of the interval is clearly $\frac{c_{n-1}}{2} - 2a_{n-1} = c_n \simeq \delta \cdot 2^{-n} \to 0$ as $n \to +\infty$. This shows that $f^{-1}(h) \cap C = \{x_h\}$ for every $h \in (0, d_0) = f([0, c_0] \times [0, d_0])$. So $\nu_h$ has to be $\delta_{x_h}$. Finally notice that we can write

$$\mathscr{L}^2 \llcorner_C = \int_{\mathbb{R}} \delta_{x_h} m \, dh.$$

*Scaling the Hamiltonian* $f_{1,1,\delta}$. Set now $H_1 := f_{1,1,\delta}$ whose range is $(0, 1)$. The disintegration now looks like

$$\mathscr{L}^2 \llcorner_C = \delta^2 \int_{\mathbb{R}} \delta_{x_h} \, dh = |S| \int_{\mathbb{R}} \delta_{x_h} \, dh$$

which will be written from now onwards as

$$\mathscr{L}^2 \llcorner_C = \int_{\mathbb{R}} c_h \delta_{x_h} \, dh,$$

where we have set for $\mathscr{L}^1$-a.e. $h$ the coefficient $c_h := |S|$. The map $h \mapsto c_h$ is thus constant and it simply represents the density of $f_\sharp(\mathscr{L}^2 \llcorner_C)$ along the level sets. We will see that this map plays a significant role in the construction: we will suitably modify it, in order to obtain a piecewise constant map which is integrable but not square-integrable. To do this, we perform some scaling transformations: for fixed $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ we first scale the *domain* of $H_1$ with the following linear map:

$$\mathcal{Q}_n \colon (x, y) \mapsto \left(x, \frac{y}{2^n}\right)$$

The area of the critical set was $|S| = \delta^2 = \int_0^1 c_h\,dh$, while after the operation the area becomes

$$\det \mathcal{Q}_n \cdot |S| = \frac{|S|}{2^n}$$

hence we set

$$c_h' := \frac{|S|}{2^n}.$$

Now we rescale the range $(0,1) \mapsto (0, 2^{-n\alpha})$ via a map $\mathcal{R}_{n,\alpha} \colon \mathbb{R} \to \mathbb{R}$ so that if we impose

$$\frac{|S|}{2^n} = \int_0^{2^{-n\alpha}} c_h''\,dh$$

we have to set accordingly

$$c_h'' := \frac{|S|}{2^n} 2^{n\alpha} = \frac{|S|}{2^{n(1-\alpha)}}$$

Finally, we define the translation operator $\mathcal{T}_{n,\alpha}$ which acts both in the domain and in the target in the following way: if a function is defined in the square $[0,1] \times \left[0, \frac{1}{2^n}\right] \subset \mathbb{R}^2$ with range $\left[0, \frac{1}{2^{n\alpha}}\right]$ then under the action of $\mathcal{T}_{n,\alpha}$ the domain becomes the rectangle $[0,1] \times \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ while the range turns to the interval $\left[\frac{1}{2^{n\alpha}}, \frac{1}{2^{n\alpha-1}}\right]$. We call the resulting function $\mathcal{T}_{n,\alpha} \circ \mathcal{R}_{n,\alpha} \circ \mathcal{Q}_{n,\alpha} \circ H_1 := H_{n,\alpha}$ and we define now

$$H_\alpha(x,y) := \sum_{n \in \mathbb{N}} H_{n,\alpha}(x,y), \qquad (x,y) \in D := \bigcup_{n \in \mathbb{N}} \left( [0,1] \times \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \right) = [0,1] \times [0,1].$$

In other words, we have "patched together" the rescaled Hamiltonians, one above the other, with ranges that are adiacent intervals. Notice that the function is well defined, as the domains of the different $H_{n,\alpha}$ are disjoint, so that for any $(x,y) \in D$ the sum is locally finite (actually it reduces to a single term).

*Properties of $H_\alpha$.* Some remarks about the properties of $H_\alpha$ are now in order.

- For $\alpha > 0$, the function $H_\alpha$ is bounded. Indeed, its range is

$$H_\alpha(D) = \bigcup_{n \in \mathbb{N}} \left[2^{-n\alpha}, 2^{-n\alpha+1}\right]$$

  whose measure is

$$\mathscr{L}^2(H_\alpha(D)) = \sum_n \frac{1}{2^{n\alpha}} =: \ell_\alpha < +\infty,$$

  for $\alpha > 0$.
- For any $\alpha \in \mathbb{R}$, the area of the critical set of $H_\alpha$ is always finite:

$$\int_0^{\ell_\alpha} c_h''\,dh = \sum_n \frac{|S|}{2^{n(1-\alpha)}} \times \frac{1}{2^{n\alpha}} = \sum_n \frac{|S|}{2^n} = |S| < +\infty.$$

- On the contrary, we have that

$$\int_0^{\ell_\alpha} (c_h'')^2\,dh = \sum_n \frac{|S|^2}{2^{2n(1-\alpha)}} \times \frac{1}{2^{n\alpha}} = \sum_n \frac{|S|^2}{2^{n(2-\alpha)}}.$$

  In particular, if we take $\alpha \geq 2$ we have that

$$\int_0^{\ell_\alpha} (c_h'')^2\,dh = +\infty.$$

  In other words, for $\alpha \geq 2$, the function $h \mapsto c_h''$ belongs to $L^1([0,\ell_\alpha]) \setminus L^2([0,\ell_\alpha])$ (it behaves essentially like $n\mathbb{1}_{[0,n^{-2}]}$ in $[0,1]$).

**5.4.5. Solution of the equation and structure of the defect.** We now fix $\alpha > 2$ and we consider the corresponding Hamiltonian $H_\alpha$ constructed in paragraph above and we set $\boldsymbol{b} := \nabla H_\alpha$. By construction, setting $\sigma_h := c''_h \delta_{x_h}$, we have that $H_\alpha$ satisfies the following

$$\mathscr{L}^2 = \int_{\mathbb{R}} \left( \frac{1}{|\nabla H|} \mathscr{H}^1 \llcorner E_h + \sigma_h \right) dh.$$

For typographical reasons, we will write from now onward simply $c_h$ instead of $c''_h$. By applying Lemma 4.37 to $H_\alpha$ we get at once the following

Proposition 5.20. *The problem*

$$\begin{cases} \partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0 \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

*is equivalent to*

$$\begin{cases} \partial_t \widehat{u}_h + \partial_s \widehat{u}_h + c_h \partial_t \widehat{u} \mathscr{L}^1 \otimes \delta_{s_h} = 0 \\ \widehat{u}_0(s) = u_{0h} \\ \partial_s \big( \widehat{u}_h c_h \mathscr{L}^1 \otimes \delta_{s_h} \big) = 0 \end{cases}$$

*for $\mathscr{L}^1$-a.e. $h$.*

Remark 5.21. Notice that, by splitting

$$u = m\mathbb{1}_S + u\mathbb{1}_{S^c}$$

the equation can be written as

$$\partial_t(u\mathbb{1}_{S^c}) + \partial_t(m\mathbb{1}_S) + \underbrace{\operatorname{div}(m\mathbb{1}_S\boldsymbol{b})}_{=0} + \operatorname{div}(u\mathbb{1}_{S^c}\boldsymbol{b}) = 0$$

because $\boldsymbol{b} = 0$ on $S$ by construction. Hence, taking into account that $\boldsymbol{b} = 0$ on the critical set, Proposition 5.20 is actually establishing that

$$\begin{cases} \partial_t(u\mathbb{1}_{S^c}) + \operatorname{div}(u\mathbb{1}_{S^c}\boldsymbol{b}) = -\partial_t(m\mathbb{1}_S) \\ u(0,\cdot) = u_0(\cdot) \end{cases}$$

is *equivalent* to

$$\begin{cases} \partial_t \widehat{u}_h + \partial_s \widehat{u}_h + c_h \partial_t \widehat{m}_h \mathscr{L}^1 \otimes \delta_{s_h} = 0 \\ \widehat{u}_0(\cdot) = u_{0h}(\cdot) \end{cases} \quad \text{for } \mathscr{L}^1\text{-a.e. } h.$$

♠

We are eventually ready to give the proof of the main result of this section, i.e. Theorem 5.17.

Proof (of Theorem 5.17). Take as $\boldsymbol{b} = \nabla H_\alpha$, with $\alpha > 2$ and consider the Cauchy problem for the transport equation associated to $\boldsymbol{b}$ with initial condition $u_0 := \mathbb{1}_S$:

$$\begin{cases} \partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0 \\ u(0,\cdot) = \mathbb{1}_S(\cdot) \end{cases}.$$

We disintegrate the equation on the level sets and we obtain, denoting for typographic simplicity by $v_h(t,s) := \widehat{u}_h(t,s)$, we have

$$\begin{cases} \partial_t v_h + \partial_s v_h = -c_h \partial_t (v_h \mathscr{L}^1 \times \delta_{s_h}) \\ v_h(0,\cdot) = c_h \mathbb{1}_{\{s_h\}}(\cdot) \end{cases} \quad \text{for } \mathscr{L}^1\text{-a.e. } h$$

i.e.

$$\begin{cases} \partial_t \big( v_h (1 + \mathscr{L}^1 \times c_h \delta_{s_h}) \big) + \partial_s v_h = 0 \\ v_h(0,\cdot) = c_h \mathbb{1}_{\{s_h\}}(\cdot) \end{cases} \quad \text{for } \mathscr{L}^1\text{-a.e. } h, \tag{5.37}$$

which is exactly of the form 5.34. Applying Lemma 5.19, we have that the function

$$v_h(t,s) := \begin{cases} c_h \mathbb{1}_{\widehat{\sigma}(s_h)}(\widehat{\sigma}(s) - t) & s \neq s_h \\ \fint_{\widehat{\sigma}(s_h)} c_h \mathbb{1}_{\widehat{\sigma}(s_h)}(\widehat{s} - t) d\widehat{s} & s = s_h \end{cases}$$

is a non-stationary solution to (5.37). Some easy computations show that

$$\begin{aligned} v_h(t, s_h) = \fint_{\widehat{\sigma}(s_h)} \mathbb{1}_{\widehat{\sigma}(s_h)}(\widehat{s} - t) d\widehat{s} &= \frac{1}{c_h} \int_{s_h}^{s_h + c_h} c_h \mathbb{1}_{[s_h, s_h + c_h]}(\widehat{s} - t) d\widehat{s} \\ &= \int_{s_h - t}^{s_h + c_h - t} \mathbb{1}_{[s_h, s_h + c_h]}(\tau) d\tau \\ &= \begin{cases} c_h - t & t < c_h \\ 0 & t > c_h \end{cases} \end{aligned}$$

In particular, we have that for a.e. $h \in \mathbb{R}$ and for every $t \in (0,T)$ it holds

$$\partial_t v_h(t, s_h) = -\mathbb{1}_{[0,c_h]}(t).$$

Hence, for this particular solution, the one dimensional equation on the level set $E_h$ is explicit:

$$\partial_t v_h + \partial_s v_h = c_h \mathbb{1}_{[0,c_h]},$$

which can be written also in the divergence form

$$\operatorname{div}_{t,s}(v_h(1,1)) = c_h \mathbb{1}_{[0,c_h]}. \tag{5.38}$$

From (5.38), we deduce immediately that, for a.e. $h \in \mathbb{R}$, the vector field $v_h(1,1)$ is a bounded, divergence-measure vector field in $(0,T) \times \mathbb{R}_s$. Applying Point 2 of Proposition XI we can write for a.e. $t \in (0,T)$

$$v_h^-(t) - v_h^+(t) = +c_h \mathbb{1}_{[0,c_h]}(t) \tag{5.39}$$

where $v_h^\pm$ are the ($L^\infty$ functions representing) Anzellotti traces on the surface $\Sigma_h := \{s = s_h\}$, defined as

$$v_h^\pm := \frac{\mathsf{Tr}^\pm\big(v_h(1,1), \Sigma_h\big)}{\mathsf{Tr}^\pm\big((1,1), \Sigma_h\big)} = \mathsf{Tr}^\pm\big(v_h(1,1), \Sigma_h\big).$$

We observe that by construction $v_h^- = 0$ a.e., hence (5.39) reduces to

$$-v_h^+ = c_h \mathbb{1}_{[0,c_h]}.$$

Taking now $\beta(\tau) = \tau^2$ and applying the Chain rule for Anzellotti traces (XII) (being the vector field $\boldsymbol{v} := (1,1)$ clearly of bounded variation) we obtain that for a.e. $h \in \mathbb{R}$ the vector field $w_h(1,1) := v_h^2(1,1)$ is still a divergence-measure vector field and it holds

$$w_h^- = 0, \qquad w_h^+ = +c_h^2 \mathbb{1}_{[0,c_h]},$$

i.e.

$$w_h^- - w_h^+ = -c_h^2 \mathbb{1}_{[0,c_h]}$$

so that, applying again Point 2 of Proposition XI, we can write

$$\operatorname{div}_{t,s}(w_h(1,1)) \llcorner_{\Sigma_h} = (\partial_t \widehat{w} + \partial_s \widehat{w}) \llcorner_{\Sigma_h} = -c_h^2 \mathbb{1}_{[0,c_h]}$$

which in turn can be written as (recall $\widehat{m}_h = c_h v_h \mathbb{1}_{s_h}$ )

$$\partial_t \widehat{w}_h + \partial_s \widehat{w}_h = -c_h \partial_t (\widehat{m}_h \mathscr{L}^1 \times \delta_{s_h}), \qquad \text{for a.e. } h \in \mathbb{R}.$$

Integrating and using Remark (5.21), we obtain the equation satisfied by $u^2$:

$$\partial_t(u^2) + \operatorname{div}(u^2 \boldsymbol{b}) = T,$$

being $T$ the distribution defined by

$$T := -\partial_t (u^2 \mathbb{1}_S). \tag{5.40}$$

To conclude, it is enough to prove the following

**Claim.** The distribution $T$ defined in (5.40) is not representable by a Radon measure.

By contradiction, assume that $T$ is induced by some measure $\xi$: being the divergence of the bounded, measure-divergence vector field $w(1, \boldsymbol{b})$, we would necessarily have $\xi \ll \mathscr{H}^d$. On the other hand, it is immediate to see, directly from the construction of the Hamiltonian, that for any $\phi \in C_c^\infty, \|\phi\|_\infty \leq 1$ we have

$$\int_{[0,T]\times\mathbb{R}^2} \phi(t,x)\, d\xi(t,x) = \int_0^T \int_S u^2(t,x)\phi_t(t,x)\, dtdx = \int_{\mathbb{R}} \int_0^T u(t,x_h)c_h^2\phi_t(t,x_h)\, dt\, dh$$

which diverges being $c_h \notin L^2(\mathbb{R})$. This shows that $T$ cannot be a distribution of order 0, hence it is not representable by a measure. □

# Part 2

# The general $d$-dimensional case

CHAPTER 6

# Localization method via proper sets and flow traces

ABSTRACT. In this chapter we begin to face the general, $d$-dimensional case of Bressan's Conjecture. As a starting point, we propose a method to *localize* the concept of Lagrangian representation explained in Chapter 3. Notice that a Lagrangian representation is by definition a *global* object, thus it is not immediate to relate it with suitable *local* estimates. In Section 6.1 we introduce a class of sets, which will be called *proper*, and we carefully study their main properties. In Section 6.2 we show how Lagrangian representations can be used to represent the *trace* of a measure-divergence vector field on an arbitrary closed set as a (possibly non-absolutely convergent) sum of measures. Finally in Section 6.3 we show how Lagrangian representations can be localized to proper sets.

Let us consider a vector field with compact support of the form

$$\rho(1, \boldsymbol{b}) \in L^1(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^{d+1}),$$

where

$$\rho\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^+, \qquad \boldsymbol{b}\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$$

and we assume that it holds in the sense of distributions

$$\operatorname{div}(\rho(1, \boldsymbol{b})) = \mu \in \mathscr{M}(\mathbb{R}^{d+1}).$$

From now onwards, if not otherwise stated, we will adopt the convention of writing div in view of $\operatorname{div}_{t,x}$, as in (3.14). As usual, to avoid dealing with sets of $\mathscr{L}^{d+1}$-negligible measure, we assume that $\rho, \boldsymbol{b}$ are defined pointwise as Borel functions. Let $\eta$ be a Lagrangian representation of $\rho(1, \boldsymbol{b})$ in the sense of Definition 3.6.

## 6.1. Proper sets and their perturbations

In the first paragraphs of this section we want to define a family of sets which have good trace properties for a given vector field $\rho(1, \boldsymbol{b}) \in L^1(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$ and we call these sets $\rho(1, \boldsymbol{b})$-*proper*. Their main properties are that their boundary $\partial\Omega$ is piecewise $C^1$, it is made of Lebesgue points of $\rho(1, \boldsymbol{b})$ and more importantly that the measure $\rho(1, \boldsymbol{b})\mathscr{H}^d \llcorner \partial\Omega$ is measuring the flux (in a sense that will be made precise) of $\rho(1, \boldsymbol{b})$ across $\partial\Omega$.

In the second part of this section we perturb these sets in order to take advantage of the fact that the vector field has the form $(1, \boldsymbol{b})$: the idea is to have the influx and outflux occurring on time-constant hyperplanes, i.e. regions of the boundary $\partial\Omega$ such that their outer normal is $\mathbf{n} = (\pm 1, 0)$. Also this step is done to avoid some technical computations later on.

### 6.1.1. Definition and basic properties of $\rho(1, \boldsymbol{b})$-proper sets.

We start by giving the following definition.

DEFINITION 6.1 (Proper sets). An open, bounded set $\Omega \subset \mathbb{R}^{d+1}$ is called $\rho(1, \boldsymbol{b})$-*proper* if:

(1) $\partial\Omega$ has finite $\mathscr{H}^d$-measure and it can be written as

$$\partial\Omega = \bigcup_{i \in \mathbb{N}} U_i \cup N,$$

where $N$ is a closed set with $\mathscr{H}^d(N) = 0$ and $\{U_i\}_{i\in\mathbb{N}}$ are countably many $C^1$-hypersurfaces such that the following holds: for every $(t,x) \in U_i$, there exists a ball $B_r^{d+1}(t,x)$ such that $\partial\Omega \cap B_r^{d+1}(t,x) = U_i$;

(2) the set of Lebesgue points of $\rho(1,\boldsymbol{b})$ has full measure w.r.t. $\mathscr{H}^d \llcorner_{\partial\Omega}$;

(3) if the functions $\phi^{\delta,\pm}$ are given by

$$\phi^{\delta,+}(t,x) := \max\left\{1 - \frac{\operatorname{dist}((t,x),\Omega)}{\delta}, 0\right\}, \quad \phi^{\delta,-}(t,x) := \min\left\{\frac{\operatorname{dist}((t,x),\mathbb{R}^{d+1}\setminus\Omega)}{\delta}, 1\right\}, \tag{6.1}$$

then

$$\lim_{\delta\searrow 0} |\rho(1,\boldsymbol{b})\cdot\nabla\phi^{\delta,\pm}|\,\mathscr{L}^{d+1} = |\rho(1,\boldsymbol{b})\cdot\mathbf{n}|\,\mathscr{H}^d\llcorner_{\partial\Omega}, \qquad w^\star\text{-}\mathscr{M}_b(\mathbb{R}^{d+1}).$$

In the following we will write *proper* instead of $\rho(1,\boldsymbol{b})$-proper when there is no ambiguity about the vector field.

Proposition 6.2. *Proper sets enjoy the following properties:*

*(1) the Lebesgue value* $\rho(1,\boldsymbol{b})\cdot\mathbf{n}\llcorner_{\partial\Omega}$ *belongs to* $L^1(\mathscr{H}^d\llcorner_{\partial\Omega})$*;*

*(2) it holds*

$$\lim_{\delta\searrow 0} \rho(1,\boldsymbol{b})\cdot\nabla\phi^{\delta,\pm}\mathscr{L}^{d+1} = \rho(1,\boldsymbol{b})\cdot\mathbf{n}\,\mathscr{H}^d\llcorner_{\partial\Omega}, \qquad w^\star\text{-}\mathscr{M}_b(\mathbb{R}^{d+1});$$

*(3)* $|\mu|(\partial\Omega) = 0$*, where* $\mu = \operatorname{div}_{t,x}(\rho(1,\boldsymbol{b}))$*.*

Proof. Point (1) follows from the well known fact that weakly convergent sequences are uniformly bounded.

To prove Point (2), let $\xi^+$ be a weak limit (up to subsequences) of the sequence $\rho(1,\boldsymbol{b})\cdot\nabla\phi^{\delta,+}\mathscr{L}^{d+1}$ and notice that, due to the weak l.s.c. of the norm, it holds

$$|\xi^+| \leq |\rho(1,\boldsymbol{b})\cdot\mathbf{n}|\mathscr{H}^d\llcorner_{\partial\Omega}.$$

For notational convenience we will write $\xi^+\mathscr{H}^d\llcorner_{\partial\Omega}$. It is thus enough to prove the statement locally inside each set $U_i$ for a fixed $i$: in particular, since the definition is invariant under $C^1$-diffeomorphisms as it can be easily checked, we can think $\Omega$ to be locally the set $\{s < 0\}$ in some coordinate system $(s,y) \in \mathbb{R}\times\mathbb{R}^d$.
For $\mathbf{a}\in\mathbb{R}$, $m\in\mathbb{N}$ set

$$E^m_{\mathbf{a}} = \Big\{y \in U_i : \big|\rho(1,\boldsymbol{b}) - \mathbf{a}\big| < 2^{-m}\Big\}.$$

Using the fact that $\mathscr{L}^d$-a.e. point $y \in E^m_{\mathbf{a}}$ is a Lebesgue for $\rho(1,\boldsymbol{b})$ w.r.t. the measure $\mathscr{L}^{d+1}$, for every $\varepsilon$ we can find $\bar r > 0$ and a compact subset $K^m_{\mathbf{a}} \subset E^m_{\mathbf{a}}$ such that $\mathscr{L}^d(E^m_{\mathbf{a}} \setminus K^m_{\mathbf{a}}) < \varepsilon$ and for every $y \in K^m_{\mathbf{a}}$, $0 < r < \bar r$ it holds

$$\frac{1}{r}\int_0^r \frac{1}{r^d}\int_{B^d_r(y)} \big|\rho(1,\boldsymbol{b})(y',s) - \mathbf{a}\big|\,dy'ds < (1+\varepsilon)2^{-m}.$$

Now, by Besicovitch' Theorem [**AFP00**, Theorem 2.17], we cover $K_{\mathbf{a}}$ with finitely many closed balls $B^d_r(y_j)$, $j = 1,\dots,N_r$, of radius $r < \bar r$ such that

$$N_r r^d \leq C_d\mathscr{L}^d(K^m_{\mathbf{a}}).$$

Then, since $\nabla\phi^{\delta,+} \simeq (1,0)$ by the $C^1$-regularity of the boundary, we have that

$$\begin{aligned}
&\int_{U_i\times\mathbb{R}} \big|\big(\rho(1,\boldsymbol{b})(s,y)-\mathbf{a}\big)\cdot\nabla\phi^{\delta,+}(s,y)\big|\,dyds \\
&\qquad\le \mathcal{O}(1)\sum_{i=1}^{N_r}\frac{1}{r}\int_0^r\int_{B^d_r(y_j)}\big|\rho(1,\boldsymbol{b})(s,y)-\mathbf{a}\big|\,dyds \\
&\qquad\quad+\frac{1}{r}\int_0^r\int_{U_i\setminus\bigcup_j B^d_r(y_j)}\big|\big(\rho(1,\boldsymbol{b})(s,y)-\mathbf{a}\big)\cdot\nabla\phi^{\delta,+}(s,y)\big|\,dyds \\
&\qquad\le \mathcal{O}(2^{-m})N_r r^d+\frac{\mathcal{O}(1)}{r}\int_0^r\int_{U_i\setminus K^m_{\mathbf{a}}}\Big[\big|\rho(1,\boldsymbol{b})(s,y)\cdot\nabla\phi^{\delta,+}(s,y)\big|+|\mathbf{a}|\Big]\,dyds \\
&\qquad\le \mathcal{O}(2^{-m})\mathscr{L}^d(K^m_{\mathbf{a}})+\frac{\mathcal{O}(1)}{r}\int_0^r\int_{U_i\setminus K^m_{\mathbf{a}}}\Big[\big|\rho(1,\boldsymbol{b})(s,y)\cdot\nabla\phi^{\delta,+}(s,y)\big|+|\mathbf{a}|\Big]\,dyds.
\end{aligned}$$

Passing to the limit in $r$ one concludes that for a test function $\psi$ whose support is in $U_i\times(-c,c)$ with $c<\bar r$ it holds

$$\begin{aligned}
&\left|\int_{s=0}\psi(0,y)\,\xi^+(y)\,\mathscr{L}^d(dy)-\int_{s=0}\psi(0,y)\,\mathbf{a}\cdot\mathbf{n}\,\mathscr{L}^d\right| \\
&\qquad\le \|\psi\|_\infty\liminf_{\delta\searrow 0}\int_{U_i\times\mathbb{R}}\Big|\big(\rho(1,\boldsymbol{b})(s,y)-\mathbf{a}\big)\cdot\nabla\phi^{\delta,+}(s,y)\Big|\,dyds \\
&\qquad\le \mathcal{O}(2^{-m}\|\psi\|_\infty)\mathscr{L}^d(K^m_{\mathbf{a}})+\|\psi\|_\infty\int_{U_i\setminus K^m_{\mathbf{a}}}\Big[\big|\rho(1,\boldsymbol{b})(0,y)\cdot\mathbf{n}(y)\big|+|\mathbf{a}|\Big]\,dy.
\end{aligned}$$

By considering a sequence of $\psi\le 1$ converging to $\mathbb{1}_{K^m_{\mathbf{a}}}$ and whose support is a subset of $V$ open, $V\supset K^m_{\mathbf{a}}$, the above inequality gives that for every open set

$$\left|\int_{K^m_{\mathbf{a}}}\big(\xi^+(y)-\mathbf{a}\cdot\mathbf{n}\big)\,\mathscr{L}^d(dy)\right|\le\mathcal{O}(2^{-m})\mathscr{L}^d(K^m_{\mathbf{a}})+\int_{V\setminus K^m_{\mathbf{a}}}\Big[\big|\rho(1,\boldsymbol{b})(0,y)\cdot\mathbf{n}(y)\big|+|\mathbf{a}|\Big]\,dy.$$

Letting now $V\searrow K^m_{\mathbf{a}}$ and then $\varepsilon\to 0$, we obtain that

$$\big|\xi^+(y)-\rho(1,\boldsymbol{b})(0,y)\cdot\mathbf{n}\big|\le C_d 2^{1-m}\qquad\mathscr{H}^d\text{-a.e. on } K^m_{\mathbf{a}}.$$

In particular the same holds in $E^m_{\mathbf{a}}$, by inner regularity of Radon measures. In particular by letting $m\to\infty$ we conclude that $\xi^+(y)=\rho(1,\boldsymbol{b})\cdot\mathbf{n}(y)$ for $\mathscr{H}^d$-a.e. $y\in V$.
The proof for the other case is completely similar.

The last point is a consequence of the second, as it holds

$$\xi^+=\xi^-=\rho(1,\boldsymbol{b})\cdot\mathbf{n}\mathscr{H}^d\llcorner\partial\Omega,$$

thus $|\mu|(\partial\Omega)=0$, where $\xi^-$ is the weak limit of the sequence $\rho(1,\boldsymbol{b})\cdot\nabla\phi^{\delta,-}$ as $\delta\searrow 0$. □

We now present a couple of remarks, discussing in particular some possible extensions of the concept of proper set.

Remark 6.3 (More general proper sets). It is possible to provide a more general class of proper sets as follows: let $f:\mathbb{R}^{d+1}\to\mathbb{R}$ be a Lipschitz function whose level sets $E_h=f^{-1}((h,+\infty))$ are compact: assume that there exists a closed set $N\subset\mathbb{R}^{d+1}$, with $\mathscr{H}^d(N)=0$, such that $f\in C^1(\mathbb{R}^{d+1}\setminus N)$ and $\nabla f\neq 0$ in $\mathbb{R}^{d+1}\setminus N$. By Coarea Formula (Theorem V) and the local invertibility of $C^1$-functions outside critical points, it follows that for $\mathscr{L}^1$-a.e. $h$ the set $E_h$ satisfies Point (1) and Point (2). Define now, for $h\in\mathbb{R}$, the functions

$$\phi^{\delta,+}_h=\left[1-\frac{1}{\delta}[h-f]^+\right]^+,\qquad\phi^{\delta,-}_h=\min\left\{1,\frac{1}{\delta}[f-h]^+\right\}.$$

Condition (3) of Definition 6.1 is then replaced by

$$\lim_{\delta\searrow 0} \left|\rho(1,\boldsymbol{b})\cdot\nabla\phi_h^{\delta,\pm}\right| \mathscr{L}^{d+1} = \left|\rho(1,\boldsymbol{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\right| \mathscr{H}^d \llcorner_{\partial E_h}. \tag{6.2}$$

in the weak-star convergence of measures. We now make the following

**Claim.** The equality (6.2) holds for $\mathscr{L}^1$-a.e. $h$.

This can be seen as an easy consequence of Lusin's Theorem, being

$$\mathbb{R}\ni h\mapsto \left|\rho(1,\boldsymbol{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\right| \mathscr{H}^d\llcorner_{\partial E_h}\in \mathscr{M}_b(\mathbb{R}^{d+1})$$

an integrable map. For completeness, let us give a complete proof of the claim (for furhter generalization we refer the reader to [Fre06, Theorem 4.18J]) . Consider the measure $m$ on $\mathbb{R}$ defined by

$$m(dh) = \left(\int_{\partial E_h}\left|\rho(1,\boldsymbol{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\right|\mathscr{H}^d\right)\mathscr{L}^1(dh).$$

If $\psi_n\in C^0(\mathbb{R}^{d+1},\mathbb{R})$, $n\in\mathbb{N}$, is a dense sequence of test functions, then by the standard Lusin's Theorem in $\mathbb{R}$ we obtain that up to an open set $N_n$ such that $m(N_n)<\varepsilon 2^{-n}$ the function

$$h\mapsto d_{\psi_n}(h) := \int_{\partial E_h}\left|\rho(1,\boldsymbol{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\right|\psi_n\,\mathscr{H}^d$$

is continuous. By closure of the set $\{\psi_n\}_n$, it follows that

$$h\mapsto d_{\psi}(h) := \int_{\partial E_h}\left|\rho(1,\boldsymbol{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\right|\psi\,\mathscr{H}^d$$

is continuous in $\mathbb{R}\setminus\bigcup_n N_n$, and being

$$d_\psi\,\mathscr{L}^1\le\|\psi\|_\infty\, m$$

it follows that every Lebesgue density point of $\mathbb{R}\setminus\bigcup_n N_n$ w.r.t. the measure $m$ is a Lebesgue point of $d_\psi$. Being $\mathscr{L}^1(\bigcup_n N_n)<\varepsilon$, the conclusion follows and the claim is proved. ♠

Remark 6.4. By means of the notion of trace introduced in following Section 6.2, it is also possible to refine the definition of proper sets as follows:

Definition 6.5 (Inner proper sets). An open, bounded set $\Omega\subset\mathbb{R}^{d+1}$ is called $\rho(1,\boldsymbol{b})$-*inner proper* if:

(1) $\partial\Omega$ has finite $\mathscr{H}^d$-measure and it is piecewise $C^1$, i.e.

$$\partial\Omega = \bigcup_{i\in\mathbb{N}} U_i\cup N,$$

where $N$ is a closed set with $\mathscr{H}^d(N)=0$ and $\{U_i\}_{i\in\mathbb{N}}$ are countably many $C^1$-hypersurfaces such that the following holds: for every $(t,x)\in U_i$, there exists a ball $B_r^{d+1}(t,x)$ such that $\partial\Omega\cap B_r^{d+1}(t,x)=U_i$;

(2) the distributional inner normal trace $\mathsf{Tr}^{\mathrm{in}}(\rho(1,\boldsymbol{b}),\Omega)\cdot\mathbf{n}$ of the vector field $\rho(1,\boldsymbol{b})$ is a measure and satisfies

$$\mathsf{Tr}^{\mathrm{in}}\big(\rho(1,\boldsymbol{b}),\Omega\big)\cdot\mathbf{n}\ll\mathscr{H}^d\llcorner_{\partial\Omega}.$$

As in the next section, in this case we will denote the trace as

$$\mathsf{Tr}^{\mathrm{in}}\big(\rho(1,\boldsymbol{b}),\Omega\big)\cdot\mathbf{n}\,\mathscr{H}^d\llcorner_{\partial\Omega},$$

i.e. as a function in $L^1(\mathscr{H}^d\llcorner_{\partial\Omega})$;

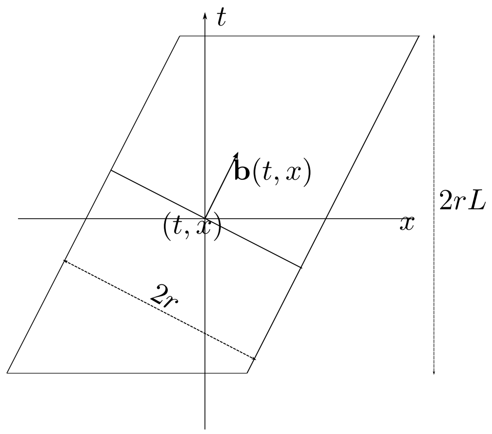



**Figure 1.** The cylinder $\mathrm{Cyl}^{r,L}_{t,x}$.

(3) if

$$\phi^{\delta,-}(x) := \min\left\{\frac{\mathrm{dist}(x, \mathbb{R}^{d+1} \setminus \Omega)}{\delta}, 1\right\},$$

then

$$\lim_{\delta \searrow 0} \left|\rho(1, \boldsymbol{b}) \cdot \nabla \phi^{\delta,-}\right| \mathscr{L}^{d+1} = \left| \mathsf{Tr}^{\mathrm{in}} \left(\rho(1, \boldsymbol{b}), \Omega\right) \cdot \mathbf{n}\right| \mathscr{H}^d \llcorner_{\partial\Omega}, \qquad w^\star\text{-}\mathscr{M}_b(\mathbb{R}^{d+1}).$$

A similar definition for $\rho(1, \boldsymbol{b})$-*outer proper*, i.e. $\mathbb{R}^{d+1} \setminus \operatorname{clos} \Omega$ is $\rho(1, \boldsymbol{b})$-inner proper. If the outer and inner normal traces coincide and the boundary $\partial\Omega$ is made of Lebesgue points, then $\Omega$ is $\rho(1, \boldsymbol{b})$-proper.

Notice, finally, that one can extend the definition of proper sets to sets with Lipschitz boundary (i.e. locally graph of Lipschitz functions), being the relevant quantities (i.e. Conditions (2), (3) of Definition 6.1) still meaningful. ♠

**6.1.2. Example of proper sets.** We now turn to show that there are sufficiently many proper sets. As usual we assume that $\boldsymbol{b}$ is a Borel function, hence defined everywhere.

Definition 6.6. For every fixed $(t, x) \in \mathbb{R}^{d+1}$ and $r, L > 0$, the *cylinder of center* $(t, x)$ *and sizes* $r, L$ (see Figure 1) is defined by

$$\mathrm{Cyl}^{r,L}_{t,x} = \Big\{(\tau, y) : |\tau - t| \leq Lr, \left|y - x - \boldsymbol{b}(t, x)(\tau - t)\right| < r\Big\}.$$

We now show that almost all balls and cylinders are proper sets: indeed, we have the following

Lemma 6.7. *For every $(t, x)$ consider the family of balls $\{B^{d+1}_r(t, x)\}_{r>0}$ and the family of cylinders $\{\mathrm{Cyl}^{r,L}_{t,x}\}_{r>0}$ with $L > 0$ fixed. Then for $\mathscr{L}^1$-a.e. $r > 0$ the ball $B^{d+1}_r(t, x)$ and the cylinder $\mathrm{Cyl}^{r,L}_{t,x}$ are proper sets.*

Proof. The statement is a consequence of Remark 6.3, respectively using the Lipschitz functions

$$(\tau, y) \mapsto \left|(\tau, y) - (t, x)\right|, \quad (\tau, y) \mapsto \max\Big\{\left|y - x - \boldsymbol{b}(t, x)(\tau - t)\right|, |\tau - t|/L\Big\}. \qquad \square$$

Proposition 6.8. *If $\Omega_1, \Omega_2$ are proper sets with*

$$\mathscr{H}^d\Big(\operatorname{Fr}\left(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega_1 \cup \partial\Omega_2\right)\Big) = 0, \tag{6.3}$$

*then $\Omega := \Omega_1 \cup \Omega_2$ is proper.*

Proof. Clearly, the set $\Omega$ is piecewise $C^1$ and the set of Lebesgue points of $\rho(1, \boldsymbol{b})$ has full measure. It remains to prove Condition (3) of Definition 6.1. We will study only $\phi^{\delta,+}$.

If $\phi_i^{\delta,+}$ is the function given by the first formula of (6.1) for $\Omega_i$, with $i = 1, 2$, observe that

$$\phi^{\delta,+} = \max\left\{\phi_1^{\delta,+}, \phi_2^{\delta,+}\right\}$$

and we write for any continuous function $\psi$

$$\begin{aligned}
&\int \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi^{\delta,+}\right| \psi\, \mathscr{L}^{d+1} \\
&\quad = \left[\int_{A_1} + \int_{A_2} + \int_{A_3}\right] \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi^{\delta,+}\right| \psi\, \mathscr{L}^{d+1} \\
&\quad = \int_{A_1} |\rho(1,\boldsymbol{b}) \cdot \nabla\phi_1^{\delta,+}| \psi\, \mathscr{L}^{d+1} + \int_{A_2} |\rho(1,\boldsymbol{b}) \cdot \nabla\phi_2^{\delta,+}| \psi\, \mathscr{L}^{d+1} + \int_{A_3} |\rho(1,\boldsymbol{b}) \cdot \nabla\phi^{\delta,+}| \psi\, \mathscr{L}^{d+1}
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \Big\{(t,x) : \operatorname{dist}\big((t,x),\Omega_1\big) < \operatorname{dist}\big((t,x),\Omega_2\big)\Big\}, \\
A_2 &= \Big\{(t,x) : \operatorname{dist}\big((t,x),\Omega_2\big) < \operatorname{dist}\big((t,x),\Omega_1\big)\Big\}, \\
A_3 &= \Big\{(t,x) : \operatorname{dist}\big((t,x),\Omega_1\big) = \operatorname{dist}\big((t,x),\Omega_2\big)\Big\}.
\end{aligned}$$

We prove that

$$\int_{A_1} \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi_1^{\delta,+}\right| \psi\, \mathscr{L}^{d+1} \to \int_{\partial\Omega_1 \setminus \operatorname{clos}\Omega_2} \left|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\right| \psi\, \mathscr{H}^d. \tag{6.4}$$

Consider the set $\operatorname{int}(A_1, \partial\Omega)$ which is relatively open by definition, so that by l.s.c. of the weak convergence on open sets we deduce

$$\left|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\right| \mathscr{H}^d \llcorner_{\operatorname{int}(A_1;\partial\Omega)} \leq \liminf_{\delta\to 0} \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi_1^{\delta,+}\right| \mathscr{L}^{d+1} \llcorner_{A_1}.$$

On the other hand,

$$\operatorname{clos}(A_1, \partial\Omega) \subset \operatorname{int}(A_1, \partial\Omega) \cup \operatorname{Fr}(A_3, \partial\Omega)$$

and

$$\operatorname{Fr}(A_3, \partial\Omega) = \operatorname{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega) = \operatorname{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega_1 \cup \partial\Omega_2).$$

Being the latter sets $\mathscr{H}^d$-negligible (by assumption) and using the u.s.c. of the weak convergence on closed set ($\operatorname{clos}(A_1, \partial\Omega)$ in this case), we get

$$\begin{aligned}
\limsup_{\delta\to 0} \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi_1^{\delta,+}\right| \mathscr{L}^{d+1} \llcorner_{A_1} &\leq \left|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\right| \mathscr{H}^d \llcorner_{\operatorname{clos}(A_1,\partial\Omega)} \\
&\leq \left|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\right| \mathscr{H}^d \llcorner_{\operatorname{int}(A_1,\partial\Omega) \cup \operatorname{Fr}(A_3,\partial\Omega)} \\
&= \left|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\right| \mathscr{H}^d \llcorner_{\operatorname{int}(A_1,\partial\Omega)}.
\end{aligned}$$

This gives (6.4).
The proof for $A_2$ is analogous, i.e.

$$\int_{A_2} \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi_2^{\delta,+}\right| \psi\, \mathscr{L}^{d+1} \to \int_{\partial\Omega_2 \setminus \operatorname{clos}\Omega_1} \left|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\right| \psi\, \mathscr{H}^d. \tag{6.5}$$

Finally it holds

$$\phi^{\delta,+} \llcorner_{\operatorname{int} A_3} = \phi_1^{\delta,+} \llcorner_{\operatorname{int} A_3} = \phi_2^{\delta,+} \llcorner_{\operatorname{int} A_3},$$

and then in a completely similar way for $A_3$

$$\int_{\operatorname{int} A_3} \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi_1^{\delta,+}\right| \psi\, \mathscr{L}^{d+1} \to \int_{\partial\Omega_3 \setminus \operatorname{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega)} \left|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\right| \psi\, \mathscr{H}^d. \tag{6.6}$$

Concerning the set of point on $\partial A_3$, it follows that for $\delta \ll 1$

$$\int_{\partial A_3} \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi^{\delta,+}\right| \psi\, \mathscr{L}^{d+1} \leq \int_O \left|\rho(1,\boldsymbol{b}) \cdot \nabla\phi_1^{\delta,+}\right| \psi \mathscr{L}^{d+1},$$

where $O$ is an open neighborhood in $\mathbb{R}^{d+1}$ of $\operatorname{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega)$ containing the support of $\psi$. Hence

$$\begin{aligned}\limsup_{\delta\to 0} \int_{\partial A_3} \left|\rho(1,\boldsymbol{b})\cdot\nabla\phi^{\delta,+}\right|\psi\,\mathscr{L}^{d+1} &\le \int_{O\cap\partial\Omega} \left|\rho(1,\boldsymbol{b})\cdot\mathbf{n}\right||\psi|\,\mathscr{H}^d \\ &\le \|\psi\|_\infty \int_{O\cap\partial\Omega} \left|\rho(1,\boldsymbol{b})\cdot\mathbf{n}\right|\mathscr{H}^d,\end{aligned}$$

and by the assumption on the $\mathscr{H}^d$-negligibility of $\operatorname{Fr}(\partial\Omega_1\cap\partial\Omega_2,\partial\Omega)$ one obtains that this integral is arbitrarily small. Adding (6.4), (6.5) and (6.6) the conclusion follows. □

The above proposition allows to construct sufficiently many proper sets for our purposes, starting from Lemma 6.7.

Corollary 6.9. *The finite union of proper balls and proper cylinders is proper.*

Proof. Indeed their intersection has the property (6.3) by elementary geometry. □

**6.1.3. Perturbation of proper sets.** We now would like to explain how proper sets can be *perturbed* in order to take advantage of the fact that the vector field under study has the form $(1,\boldsymbol{b})$: in particular, we would like to have almost all the influx and outflux occurring on time-constant hyperplanes, i.e. regions of the boundary $\partial\Omega$ such that their outer normal is $\mathbf{n} = (\pm 1, 0)$. This step is done to avoid some technical computations later on.

Let us take $\Omega \subset \mathbb{R}^{d+1}$ to be a $\rho(1,\boldsymbol{b})$-proper set. We begin by proving the following

Lemma 6.10. *For every $\varepsilon > 0$ there exist a compact set $K^\varepsilon \subset \partial\Omega \setminus N$ and $\alpha > 0$ with the following properties:*

*(1) $\alpha^{-1} < \rho, |(1,\boldsymbol{b})\cdot\mathbf{n}|$ and $\rho, |\boldsymbol{b}| < \alpha$ for $\mathscr{H}^d$-a.e. $(t,x)\in K^\varepsilon$;*
*(2) the remaining set has small normal trace, i.e.*

$$\int_{\partial\Omega\setminus K^\varepsilon} \rho|(1,\boldsymbol{b})\cdot\mathbf{n}|\,\mathscr{H}^d < \varepsilon.$$

Proof. It is enough to observe that

$$\lim_{\alpha\to+\infty} \int_{\partial\Omega\cap\{\alpha^{-1}<\rho,|(1,\boldsymbol{b})\cdot\mathbf{n}|\}\cap\{\rho,|\boldsymbol{b}|<\alpha\}} \rho(t,x)|(1,\boldsymbol{b}(t,x))\cdot\mathbf{n}|\,\mathscr{H}^d(dtdx) = \int_{\partial\Omega}\rho|(1,\boldsymbol{b})\cdot\mathbf{n}|\,\mathscr{H}^d,$$

since $\rho|(1,\boldsymbol{b})\cdot\mathbf{n}|$ is an $L^1$-function w.r.t. $\mathscr{H}^d\llcorner_{\partial\Omega}$ and

$$\partial\Omega \subset \big\{\rho, |(1,\boldsymbol{b})\cdot\mathbf{n}|, |\boldsymbol{b}| = 0\big\} \cup \bigcup_\alpha \big\{\alpha^{-1} < \rho, |(1,\boldsymbol{b})\cdot\mathbf{n}|\big\} \cap \big\{\rho, |\boldsymbol{b}| < \alpha\big\}$$

being Borel functions. □

By simple geometric manipulation, it follows that for $r$ sufficiently small and $L > 2\alpha^2$ ($L > \alpha^2$ would be enough for most of the theorems, but later we need some extra room) the cylinder

$$\mathrm{Cyl}^{r,L}_{t,x} = \Big\{(\tau,y) : |\tau - t| < Lr, \big|y - x - \boldsymbol{b}(t,x)(\tau - t)\big| < r\Big\}$$

has top and bottom faces contained one inside $\Omega$ and the other outside, for every point in $(t,x)\in K^\varepsilon$: more precisely, if $(1,\boldsymbol{b})\cdot\mathbf{n} > 0$ then

$$\Big\{(t+Lr,y) : \big|y - x - L\boldsymbol{b}(t,x)r\big| < r\Big\} \subset \mathbb{R}^{d+1}\setminus \operatorname{clos}\Omega,$$

$$\Big\{(t-Lr,y) : \big|y - x + L\boldsymbol{b}(t,x)r\big| < r\Big\} \subset \Omega.$$

The opposite relations hold for $(1,\boldsymbol{b})\cdot\mathbf{n} < 0$. Moreover, being $\partial\Omega$ of class $C^1$ in a neighborhood of $\cap K^\varepsilon$, we have

$$B^{d+1}_{r/L}(t,x)\cap\partial\Omega \subset \mathrm{Cyl}^{r,L}_{t,x}\cap\partial\Omega \subset B^{d+1}_{Lr}(t,x)\cap\partial\Omega, \tag{6.8a}$$

$$\mathscr{H}^d(\partial \mathrm{Cyl}^{r,L}_{t,x} \cap \partial\Omega) = 0, \tag{6.8b}$$

again by simple geometrical arguments.

We now recall the following elementary

Lemma 6.11. *If $(t,x)$ is a Lebesgue point for $\rho(1,\boldsymbol{b})$, then for every $L>0$ fixed it holds*

$$\lim_{r\to 0} \frac{1}{r}\int_0^r \left[\frac{1}{r^d}\int_{\partial \mathrm{Cyl}^{r,L}_{t,x}} \big|\rho(1,\boldsymbol{b})(\tau,y) - \rho(1,\boldsymbol{b})(t,x)\big| \mathscr{H}^{d-1}(dy)d\tau\right] ds = 0.$$

Proof. We have, using Fubini's Theorem,

$$\begin{aligned}
&\frac{1}{r}\int_0^r \left[\frac{1}{r^d}\int_{\partial \mathrm{Cyl}^{r,L}_{t,x}} \big|\rho(1,\boldsymbol{b})(\tau,y) - \rho(1,\boldsymbol{b})(t,x)\big| \mathscr{H}^{d-1}(dy)d\tau\right] ds \\
&\quad = \frac{1}{r}\int_0^r \left[\frac{1}{r^d}\int_{t-Lr}^{t+Lr}\int_{\partial B^d_r(x-\boldsymbol{b}(t,x)(\tau-t))} \big|\rho(1,\boldsymbol{b})(\tau,y) - \rho(1,\boldsymbol{b})(t,x)\big| \mathscr{H}^{d-1}(dy)d\tau\right] ds \\
&\quad = \frac{1}{r^{d+1}}\int_{t-Lr}^{t+Lr}\int_{B^d_r(x-\boldsymbol{b}(t,x)(\tau-t))} \big|\rho(1,\boldsymbol{b})(\tau,y) - \rho(1,\boldsymbol{b})(t,x)\big| dy d\tau \\
&\quad \leq \frac{1}{r^{d+1}}\int_{B^d_{(1+L|\boldsymbol{b}|(t,x))r}(t,x)} \big|\rho(1,\boldsymbol{b})(\tau,y) - \rho(1,\boldsymbol{b})(t,x)\big| dy d\tau \\
&\quad = \omega_{d+1}(1+L|\boldsymbol{b}|(t,x))^{d+1} \fint_{B^d_{(1+L|\boldsymbol{b}|(t,x))r}(t,x)} \big|\rho(1,\boldsymbol{b})(\tau,y) - \rho(1,\boldsymbol{b})(t,x)\big| dy d\tau \to 0,
\end{aligned}$$

since $(t,x)$ is a Lebesgue point for $\rho(1,\boldsymbol{b})$. This implies the statement. □

Using Lemma 6.7 and 6.11, we have that for every fixed $\varepsilon'>0$, for any $(t,x)\in K^\varepsilon$ Lebesgue point for $\rho(1,\boldsymbol{b})$, we can choose the $r<\varepsilon'$ such that:

- $\mathrm{Cyl}^{r,L}_{t,x}$ is proper;
- it holds
$$\frac{1}{r^d}\int_{t-Lr}^{t+Lr}\int_{\partial B^d_r(x-\boldsymbol{b}(t,x)(\tau-t))} \big|\rho(1,\boldsymbol{b})(\tau,y) - \rho(1,\boldsymbol{b})(t,x)\big| \mathscr{H}^{d-1}(dy)d\tau < \varepsilon'; \tag{6.9}$$
- conditions (6.7) hold;
- $\mathrm{Cyl}^{r,L}_{t,x}\cap\partial\Omega$ is equivalent to a ball and its boundary is $\mathscr{H}^d$ negligible, i.e. (6.8) hold.

In the following we will call a cylinder satisfying the above condition $\rho(1,\boldsymbol{b})$-*proper* $(\varepsilon',\Omega)$-*regular cylinder*, or for brevity *proper regular* whenever the vector field $\rho(1,\boldsymbol{b})$ and dependence on the $\varepsilon'$ or $\Omega$ is clear from the context or not essential to the computation.

We can proceed further by observing that 0 is a Lebesgue density point for the set satisfying (6.9) for all $\varepsilon'>0$. On the other hand, it is easy to see that the other three properties are verified $\mathscr{L}^1$-a.e. $r>0$. We state it in the following lemma.

Lemma 6.12. *If $(t,x)\in K^\varepsilon$ is a Lebesgue point for $\rho(1,\boldsymbol{b})$, the set of $r$ such that $\mathrm{Cyl}^{r,L}_{t,x}$ satisfies the above condition has 0 as a Lebesgue point w.r.t. the measure $\mathscr{L}^1$:*

$$\lim_{r\searrow 0}\frac{1}{r}\mathscr{L}^1\Big(\Big\{r'\in(0,r) : \mathrm{Cyl}^{r',L}_{t,x} \text{ is proper } (\varepsilon',\Omega)\text{-regular}\Big\}\Big) = 1.$$

Thus we obtain the following extension of Lemma 6.10:

Lemma 6.13. *For every $\varepsilon'>0$, there exists $\bar r>0$ and a compact set $K^{\varepsilon,\varepsilon'}_{\bar r}\subset K^\varepsilon$ made of Lebesgue points of $\rho(1,\boldsymbol{b})$ such that*

*(1) $\alpha^{-1}<\rho, |(1,\boldsymbol{b})\cdot\mathbf{n}|$ and $\rho,|\boldsymbol{b}|<\alpha$ for $\mathscr{H}^d$-a.e. $(t,x)\in K^{\varepsilon,\varepsilon'}_{\bar r}$;*

*(2) the remaining set has small normal trace,*

$$\int_{\partial\Omega\setminus K^{\varepsilon,\varepsilon'}_{\bar r}} \rho|(1,\boldsymbol{b})\cdot \mathbf{n}|\,\mathscr{H}^d < 2\varepsilon,$$

*and for every* $(t,x)\in K^{\varepsilon,\varepsilon'}_{\bar r}$, $r'\le \bar r$ *there exists a proper* $(\varepsilon',\Omega)$*-regular cylinder* $\mathrm{Cyl}^{r,L}_{t,x}$ *with* $r'/2<r<r'$.

By (6.8) we get the next proposition.

PROPOSITION 6.14. *For every* $r'\le\bar r$, *there exists a finite covering of* $K^{\varepsilon,\varepsilon'}_{\bar r}$ *with cylinders* $\{\mathrm{Cyl}^{r_i,L}_{t_i,x_i}\}_{i=1}^{N_{r'}}$ *with* $L>2\alpha^2$ *and* $r'/2<r_i<r'$, *such that*

- *they are all proper* $(\varepsilon',\Omega)$*-regular,*
- *it holds*

$$N_{r'}(r')^d \le C_d L^d \mathscr{H}^d(K^{\varepsilon,\varepsilon'}_{\bar r})$$

*and*

$$\sum_{i=1}^{N_{r'}} \int_{t_i-Lr_i}^{t_i+Lr_i} \int_{\partial B^d_{r_i}(x_i-\boldsymbol{b}(t_i,x_i)(t-t_i))} \big|\rho(1,\boldsymbol{b})\cdot\mathbf{n}\big|\,\mathscr{H}^d \le (1+\alpha)C_d\varepsilon' L^d\mathscr{H}^d(K^{\varepsilon,\varepsilon'}_{\bar r}). \tag{6.10}$$

PROOF. By Lemma 6.12 for every point of $K^{\varepsilon,\varepsilon'}_{\bar r}$, $r'\le\bar r$ we can find cylinders $\mathrm{Cyl}^{r,L}_{t,x}$ which are proper sets with $r'/2<r<r'$ and by (6.8a) their intersection with $\partial\Omega$ is equivalent to balls (by the assumption $L>2\alpha^2$), so that by Besicovitch Theorem [**AFP00**, Theorem 2.17] we can take a covering $\{\mathrm{Cyl}^{r_i,L}_{t_i,x_i}\}_{i=1}^{N_{r'}}$ satisfying

$$2^{-d}N_{r'}\left(\frac{r'}{L}\right)^d \le \sum_{i=1}^{N_{r'}} \mathscr{H}^d\big(\mathrm{Cyl}^{r_i,L}_{t_i,x_i}\cap\partial\Omega\big) \le C_d\mathscr{H}^d(K^{\varepsilon,\varepsilon'}_{\bar r}),$$

with $C_d$ constant depending only on the dimension. The constant $L^d$ is a consequence of (6.8a). The other claim follows from (6.9), because of the triangle inequality

$$\begin{aligned}
&\int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\boldsymbol{b}(t_i,x_i)(t-t_i))} \rho\big|(1,\boldsymbol{b})\cdot\mathbf{n}\big|\,\mathscr{H}^d \\
&\qquad\le \int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\boldsymbol{b}(t_i,x_i)(t-t_i))} \rho(\tau,x)\big|\boldsymbol{b}(\tau,x)-\boldsymbol{b}(t_i,x_i)\big|\,\mathscr{H}^{d-1}(dx)d\tau \\
&\qquad\le \int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\boldsymbol{b}(t_i,x_i)(t-t_i))} \big|\rho(\tau,x)\boldsymbol{b}(\tau,x)-\rho(t_i,x_i)\boldsymbol{b}(t,x)\big|\,\mathscr{H}^{d-1}(dx)d\tau \\
&\qquad\quad+|\boldsymbol{b}(t_i,x_i)|\int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\boldsymbol{b}(t_i,x_i)(t-t_i))} \big|\rho(t,x)-\rho(t_i,x_i)\big|\,\mathscr{H}^{d-1}(dx)d\tau \\
&\qquad\le (1+\alpha)\varepsilon' r_i^d.
\end{aligned}$$

□

We thus obtain the main result of this section.

THEOREM 6.15 (Perturbation of proper sets). *For every* $\varepsilon>0$ *there exists a proper set* $\Omega^\varepsilon$ *such that*

*(1)* $\Omega\subset\Omega^\varepsilon\subset\Omega+B^{d+1}_\varepsilon(0)$;
*(2) if*

$$\partial\Omega^\varepsilon_1=\Big\{(t,x)\in\partial\Omega^\varepsilon : \mathbf{n}=(1,0)\ \textit{in a neigborhood of}\ (t,x)\Big\},$$

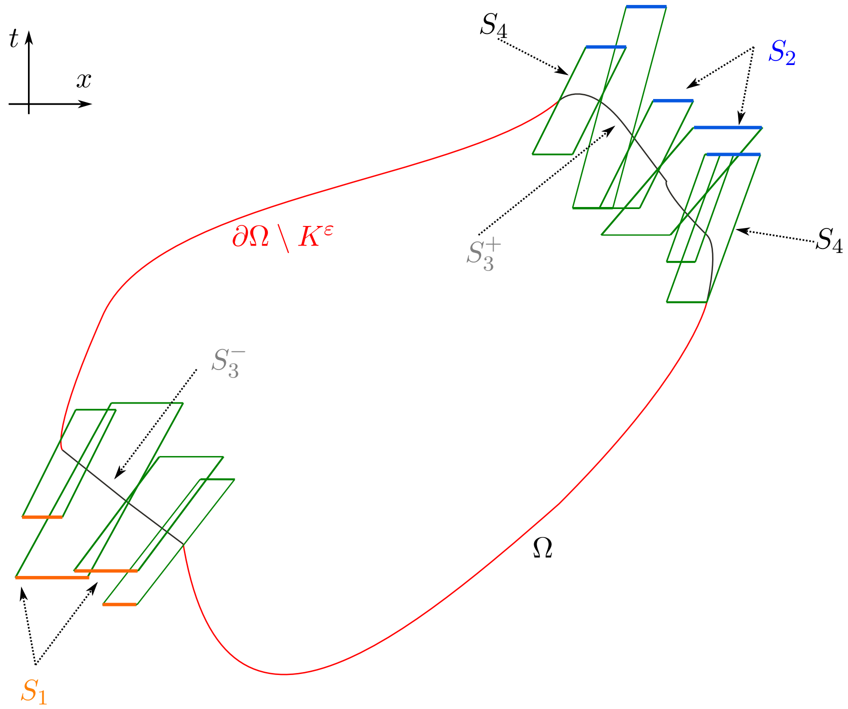



**Figure 2.** Perturbation of the proper set $\Omega$ constructed in Theorem 6.15.

*then* $\partial\Omega_1^\varepsilon$ *is made of Lebesgue points of* $\rho(1, \boldsymbol{b})$ *and*

$$\left| \int_{\partial\Omega_1^\varepsilon} \rho \mathscr{H}^d - \int_{\partial\Omega} \rho[(1, \boldsymbol{b}) \cdot \mathbf{n}]^+ \mathscr{H}^d \right| < \varepsilon;$$

*(3) if*

$$\partial\Omega_2^\varepsilon = \Big\{(t, x) \in \partial\Omega^\varepsilon : \mathbf{n} = (-1, 0) \text{ in a neigborhood of } (t, x)\Big\},$$

*then* $\partial\Omega_2^\varepsilon$ *is made of Lebesgue points of* $\rho(1, \boldsymbol{b})$ *and*

$$\left| \int_{\partial\Omega_2^\varepsilon} \rho \mathscr{H}^d - \int_{\partial\Omega} \rho[(1, \boldsymbol{b}) \cdot \mathbf{n}]^- \mathscr{H}^d \right| < \varepsilon.$$

Additionally to the fact that proper sets can be perturbed, the advantage of the perturbations considered here is that essentially all inflow and outflow of $\rho(1, \boldsymbol{b})$ are occurring on open sets which are contained in countably many time-flat hyperplanes (see Figure 2). Due to the special form of the vector field, many computations occurring in the next sections are greatly simplified.

Proof. First we find a compact set $K^{\varepsilon/7}$ such that Properties (1), (2) of the statement of Lemma 6.10 hold for $\varepsilon/7$. By inner regularity of the measure $\mathscr{H}^d$, we can further find two disjoint compact sets $K^{\varepsilon/6,\pm}$ such that $K^{\varepsilon/6} := K^{\varepsilon/6,+} \cup K^{\varepsilon/6,-}$ satisfies again Lemma 6.10 but

$$(1, \boldsymbol{b}) \cdot \mathbf{n} \llcorner_{K^{\varepsilon/6,\pm}} \gtrless 0.$$

Choose $\varepsilon'$ such that

$$(1 + \alpha) C_d \varepsilon' (2\alpha)^d \mathscr{H}^d(\partial\Omega) < \frac{\varepsilon}{3}.$$

We apply Lemma 6.13 in order to obtain a a compact set $K^{\varepsilon/6,\varepsilon'}_{\bar r} \subset K^{\varepsilon/6}$ such that

$$\int_{\partial\Omega\setminus K^{\varepsilon/6,\varepsilon'}_{\bar r}} \rho|(1,\boldsymbol{b})\cdot\mathbf{n}|\,\mathscr{H}^d < \frac{\varepsilon}{3}.$$

Next, by Proposition 6.14 with

$$r' < \frac{\operatorname{dist}(K^{\varepsilon/6,+},K^{\varepsilon/6,-})}{2(1+2\alpha^2)} \quad \text{such that} \quad |\mu|\big((\Omega + B^{d+1}_{r'}(\mathbf{0}))\setminus\Omega\big) < \frac{\varepsilon}{3}, \tag{6.11}$$

we conclude that there exists a covering of $K^{\varepsilon/6,\varepsilon'}_{\bar r}$ with finitely many $\varepsilon'$-proper regular cylinders $\{\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\}_{i=1}^{N'_r}$, with $r'/2 < r_i < r'$ such that (6.10) holds. By the choice (6.11) it follows that the coverings of $K^{\varepsilon/6,+}\cap K^{\varepsilon/6,\varepsilon'}_{\bar r}$ and of $K^{\varepsilon/6,-}\cap K^{\varepsilon/6,\varepsilon'}_{\bar r}$ are disjoint.

Define

$$\Omega^\varepsilon := \Omega \cup \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}.$$

By Proposition 6.8 and Corollary 6.9 the set $\Omega^\varepsilon$ is proper and Point (1) is clearly satisfied.

To prove Point (2), partition the boundary of $\Omega^\varepsilon\setminus\Omega$ as

$$\begin{aligned}
\partial(\Omega^\varepsilon\setminus\Omega) = &\Bigg[\partial\Omega^\varepsilon \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,+}} \Big\{(t_i+2\alpha^2 L r_i, y) : \big|y - x_i - 2\alpha^2\boldsymbol{b}(t_i,x_i)\big| < r_i\Big\}\Bigg] \\
&\cup \Bigg[\partial\Omega^\varepsilon \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,-}} \Big\{(t_i-2\alpha^2 L r_i, y) : \big|y - x_i + 2\alpha^2\boldsymbol{b}(t_i,x_i)\big| < r_i\Big\}\Bigg] \\
&\cup \Bigg[\partial\Omega \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,+}} \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Bigg] \cup \Bigg[\partial\Omega \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,-}} \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Bigg] \cup S_4 \\
= &\; S_1 \cup S_2 \cup S_3^+ \cup S_3^- \cup S_4.
\end{aligned} \tag{6.12}$$

The set $S_4$ satisfies

$$S_4 \subset \bigcup_i \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} := \bigcup_i \Big\{(\tau,y) : |\tau - t_i| \leq 2\alpha^2 r_i, \big|y - x_i - \boldsymbol{b}(t_i,x_i)(\tau - t)\big| = r_i\Big\},$$

so that from (6.10)

$$\int_{S_4} \rho\big|(1,\boldsymbol{b})\cdot\mathbf{n}\big|\,\mathscr{H}^d \leq (1+\alpha)C_d\varepsilon' L^d \mathscr{H}^d(\partial\Omega) < \frac{\varepsilon}{3}, \tag{6.13}$$

by the choice of $\varepsilon'$.

The balance of the equation $\operatorname{div}_{t,x}(\rho(1,\boldsymbol{b})) = \mu$ for the covering of $K^{\varepsilon/6,\varepsilon'}_{\bar r}\cap K^{\varepsilon/6,+}$ and the continuity property (6.11) give

$$\left|\int_{S_1} \rho\,\mathscr{H}^d - \int_{S_3^+} \rho[(1,\boldsymbol{b})\cdot\mathbf{n}]\,\mathscr{H}^d\right| \leq \int_{S_4} \rho|(1,\boldsymbol{b})\cdot\mathbf{n}|\,\mathscr{H}^d + |\mu|(\Omega^\varepsilon\setminus\Omega) < \frac{2\varepsilon}{3}$$

and, from the properties of $K^{\varepsilon/6,\varepsilon'}_{\bar r}$, we eventually get

$$\left|\int_{S_3^+} \rho(1,\boldsymbol{b})\cdot\mathbf{n}\,\mathscr{H}^d - \int \rho[(1,\boldsymbol{b})\cdot\mathbf{n}]^+\,\mathscr{H}^d\right| \leq \int_{\partial\Omega\setminus K^{\varepsilon/6,\varepsilon'}_{\bar r}} \rho|(1,\boldsymbol{b})\cdot\mathbf{n}|\,\mathscr{H}^d < \frac{\varepsilon}{3}.$$

This concludes the proof of Point (2) because $S_1 \subset \partial\Omega^\varepsilon_1$. The proof of Property (3) is similar and it is omitted. □

## 6.2. Flow traces

We now turn our attention to study how, using Lagrangian representations, it is possible to represent the normal trace over a generic closed set of a measure-divergence vector field $B = \rho(1, \boldsymbol{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$ as a (possibly non-absolutely convergent) sum of signed measures. In the case of a compact set $\Omega \subset \mathbb{R}^{d+1}$ with Lipschitz boundary and when $\boldsymbol{b}$ enjoys for instance BV bounds (or even BD, as shown in [ACM05]) and $\rho$ is bounded, the series turns out to be strongly convergent and thus gives back the usual definition of trace as a measure (absolutely continuous w.r.t $\mathscr{H}^d \llcorner \partial\Omega$) as recalled in the Preliminaries (see Section IV). For general measure-divergence vector fields, the same conclusion can be obtained when the set $\Omega$ is $\rho(1, \boldsymbol{b})$-proper, and it will be addressed in the next section.

We start by recalling some well known definitions.

**6.2.1. Definition of normal traces.** Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set and let $B \colon \Omega \to \mathbb{R}^{d+1}$ be a locally integrable vector field with measure divergence, i.e.

$$B \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}), \qquad \mathrm{div}_{t,x} B \in \mathscr{M}_b(\mathbb{R}^{d+1}).$$

Definition 6.16. The *inner normal trace* of $B$ over $\partial\Omega$ is the distribution denoted by $\mathsf{Tr}^{\mathrm{in}}(B, \Omega) \cdot \mathbf{n}$ and defined by

$$\left\langle \mathsf{Tr}^{\mathrm{in}}(B, \Omega) \cdot \mathbf{n}, \psi \right\rangle := \int_\Omega \psi(t,x)\,(\mathrm{div}\, B)(dt, dx) + \int_\Omega B \cdot \nabla_{t,x}\psi(t,x)\, \mathscr{L}^{d+1}(dt, dx)$$

for every compactly supported smooth test function $\psi \in C^\infty_c(\mathbb{R}^{d+1})$. Similarly, we define the *outer normal trace* by

$$\mathsf{Tr}^{\mathrm{out}}(B, \Omega) \cdot \mathbf{n} := -\,\mathsf{Tr}^{\mathrm{in}}(B, \mathbb{R}^{d+1} \setminus \mathrm{clos}\, \Omega) \cdot \mathbf{n}.$$

Notice that

$$\left\langle \mathsf{Tr}^{\mathrm{out}}(B, \Omega) \cdot \mathbf{n}, \psi \right\rangle - \left\langle \mathsf{Tr}^{\mathrm{in}}(B, \Omega) \cdot \mathbf{n}, \psi \right\rangle = \int_{\partial\Omega} \psi\,(\mathrm{div}\, B) + \int_{\partial\Omega} B \cdot \nabla\psi\, \mathscr{L}^{d+1}.$$

In particular they coincide if $\partial\Omega$ is negligible w.r.t. both $\mathscr{L}^{d+1}$ and $\mathrm{div}\, B$.

Remark 6.17. We explicitly want to remark that in general $\mathbf{n}$ is not well defined, without further assumptions on the set $\Omega$: we use it only to keep the notation similar to the smooth case, where the value of $B$ on $\partial\Omega$ is defined. Later on we will show that, in the case of a proper set, $\mathbf{n}$ coincides with the unit outer normal, and $\mathsf{Tr}^{\mathrm{in/out}}(\rho(1, \boldsymbol{b}), \Omega)$ will be the Lebesgue value of the vector field on $\partial\Omega$, both defined $\mathscr{H}^d$-a.e.. ♠

**6.2.2. The non smooth setting.** In the case where the domain $\Omega$ has Lipschitz boundary, $\boldsymbol{b} \in L^1(\mathbb{R}^+; \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^d))$ and $\rho \in L^\infty$, we have seen in Section IV that there are well known results that allows to characterize the trace.

We now drop the assumption that $\Omega$ has a regular boundary and we assume only that $\mathrm{div}\, B$ is a measure. We are going to prove (using Lagrangian representations) that the traces $\mathsf{Tr}^-(B, \Omega) \cdot \mathbf{n}$ can be represented by a countable sum of Radon measures.

**The case of one hitting time.** To begin with, let us consider a simplified setting, i.e. assume that $|\mu|(\partial\Omega) = 0$ and that there exists a well defined map

$$\begin{array}{cccc} \mathtt{T} & : & \Gamma \supset \mathcal{D}(\mathtt{T}) & \to & I \times \partial\Omega \\ & & \gamma & \mapsto & \mathtt{T}(\gamma) := (t_\gamma, \gamma(t_\gamma)) \end{array} \tag{6.14}$$

such that $\gamma(t_\gamma)$ the unique point along the trajectory belonging to $\partial\Omega$ with (for the orientation)

$$(\mathbb{I}, \gamma)([t^-_\gamma, t_\gamma)) \in \Omega, \qquad (\mathbb{I}, \gamma)((t_\gamma, t^+_\gamma]) \in [0, T] \times \mathbb{R}^{d+1} \setminus \mathrm{clos}\, \Omega.$$

We assume moreover that a Lagrangian representation $\eta$ is concentrated on $\mathcal{D}(\mathtt{T})$. In this case, we can prove the following

PROPOSITION 6.18. *The distributions* $\mathsf{Tr}^{\text{in}}(B,\Omega)\cdot\mathbf{n}$ *and* $\mathsf{Tr}^{\text{out}}(B,\Omega)\cdot\mathbf{n}$ *are induced by a measure, i.e.*

$$\mathsf{Tr}^{\text{in}}(B,\Omega)\cdot\mathbf{n} = \mathsf{Tr}^{\text{out}}(B,\Omega)\cdot\mathbf{n} = \mathtt{T}_\sharp\eta,$$

*where* $\mathtt{T}$ *is the map defined in* (6.14).

PROOF. By a direct computation, for any test function $\psi \in C_c^\infty((0,T)\times\mathbb{R}^d)$ it holds

$$\begin{aligned}\big\langle \mathsf{Tr}^{\text{in}}(B,K)\cdot\mathbf{n},\psi\big\rangle &= \int_\Omega \psi\, \operatorname{div} B + \int_\Omega B\cdot\nabla\psi\,\mathscr{L}^{d+1} \\ &= \int_\Omega \psi\, \operatorname{div}(\rho(1,\boldsymbol{b})) + \int_\Omega \rho(1,\boldsymbol{b})\cdot\nabla\psi\,\mathscr{L}^{d+1} \\ &= \int \psi(t_\gamma^-,\gamma(t_\gamma^-))\,\eta(d\gamma) + \int\bigg[\int_{t_\gamma^-}^{t_\gamma}(1,\boldsymbol{b}(t,\gamma(t)))\cdot\nabla_{t,x}\psi(t,\gamma(t))\,dt\bigg]\,\eta(d\gamma) \\ &= \int \psi(t_\gamma,\gamma(t_\gamma))\,\eta(d\gamma),\end{aligned}$$

where we have used that $\eta$ is a Lagrangian representation of $\rho(1,\boldsymbol{b})$. □

<u>**The general case: multiple hitting times.**</u> In the general case consider the open set

$$O := \big\{(t,\gamma) : \gamma\in\mathit{\Gamma}, (t,\gamma(t))\in\Omega\big\} \subset \mathbb{R}\times\mathit{\Gamma}.$$

and decompose it as

$$O = \bigcup_{i,j\in\mathbb{N}} \big\{|t-t_i|<r_i\big\}\times\big\{\|\gamma-\gamma_j\|_{C^0}<r_j\big\} = \bigcup_{i,j\in\mathbb{N}} B^1_{r_i}(t_i)\times B_{r_j}(\gamma_j).$$

For $\gamma\in B_{r_j}(\gamma_j)$ let $(t_\gamma^{i,-},t_\gamma^{i,+})$ be the connected component of $(\mathbb{I},\gamma)^{-1}(\Omega)$ such that

$$t_i \in \big(t_\gamma^{i,-},t_\gamma^{i,+}\big).$$

It is elementary to show that $t_\gamma^{i,+}$ is l.s.c. and $t_\gamma^{i,-}$ is u.s.c. on $B_{r_j}(\gamma_j)$. We thus conclude that

LEMMA 6.19. *There exists countably many Borel functions*

$$D_i \ni \gamma \mapsto t_\gamma^{i,-}, t_\gamma^{i,+}$$

*such that*

$$(\mathbb{I},\gamma)^{-1}(\Omega) = (t_\gamma^-,t_\gamma^{+,0})\cup(t_\gamma^{-,0},t_\gamma^+)\cup\bigcup_i(t_\gamma^{i,-},t_\gamma^{i,+}),$$

*where the first two intervals may be empty.*

PROOF. The only additional step is to relabel the intervals of $(\mathbb{I},\gamma)^{-1}(\Omega)$ which contains the initial time $t_\gamma^-$ and the final time $t_\gamma^+$ as $t_\gamma^{+,0}$, $t_\gamma^{-,0}$, respectively. By the topology of $\mathit{\Upsilon}$ this relabeling is still Borel. □

Trivially it holds for any test function $\psi\in C^\infty$

$$\begin{aligned}\int_{(\mathbb{I},\gamma)^{-1}(\Omega)} \frac{d}{dt}\psi(t,\gamma(t))\,dt = &\Big[\psi(t_\gamma^{+,0},\gamma(t_\gamma^{+,0})) - \psi(t_\gamma^-,\gamma(t_\gamma^-))\Big] \\ &+ \Big[\psi(t_\gamma^+,\gamma(t_\gamma^+)) - \psi(t_\gamma^{-,0},\gamma(t_\gamma^{-,0}))\Big] \\ &+ \sum_i \Big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+})) - \psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\Big],\end{aligned}$$

where the sum converges (as it is written) due to the estimate

$$\begin{aligned}\Big|\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+}))-\psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\Big| &\le \|\nabla_{t,x}\psi\|_\infty\Big(\big(t_\gamma^{i,+}-t_\gamma^{i,-}\big)+\big|\gamma(t_\gamma^{i,+})-\gamma(t_\gamma^{i,-})\big|\Big)\\ &\le \|\nabla_{t,x}\psi\|_\infty\int_{t_\gamma^{i,-}}^{t_\gamma^{i,+}}\big(1+|\dot\gamma(s)|\big)\,ds.\end{aligned} \tag{6.15}$$

It thus follows that

$$\begin{aligned}&\int_\Omega B\cdot\nabla_{t,x}\psi\,\mathscr{L}^{d+1}+\int_\Omega\psi\ \mathrm{div}\,B\\ &\quad=\int\bigg[\int_{(\mathbb{I},\gamma)^{-1}(\Omega)}\frac{d}{dt}\psi(t,\gamma(t))\,dt\bigg]\eta(d\gamma)\\ &\qquad+\int\Big[\psi(t_\gamma^-,\gamma(t_\gamma^-))\mathbb{1}_\Omega(t_\gamma^-,\gamma(t_\gamma^-))-\psi(t_\gamma^+,\gamma(t_\gamma^+))\mathbb{1}_\Omega(t_\gamma^+,\gamma(t_\gamma^+))\Big]\eta(d\gamma)\\ &\quad=\int\bigg[\big(\psi(t_\gamma^{+,0},\gamma(t_\gamma^{+,0}))-\psi(t_\gamma^{-,0},\gamma(t_\gamma^{-,0})\big)+\sum_i\Big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+}))-\psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\Big]\bigg]\eta(d\gamma).\end{aligned}$$

Thanks to (6.15), we can partition the last sum as

$$\begin{aligned}&\int\sum_i\big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+}))-\psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\big]\eta(d\gamma)\\ &=\sum_i\int\big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+}))-\psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\big]\eta(d\gamma)\\ &=\sum_i\big\langle(\mathtt{T}_\Omega^{i,+})_\sharp\eta-(\mathtt{T}_\Omega^{i,-})_\sharp\eta,\psi\big\rangle,\end{aligned}$$

where

$$\mathtt{T}_\Omega^{i,\pm}\colon\gamma\mapsto(t_\gamma^{i,\pm},\gamma(t_\gamma^{i,\pm}))\in\partial\Omega. \tag{6.16}$$

We thus have obtained the following lemma.

LEMMA 6.20. *The distributional trace of $B=\rho(1,\boldsymbol{b})$ on $\partial\Omega$ can be represented as the countable sum of measures supported on $\partial\Omega$, namely*

$$\mathsf{Tr}^{\mathrm{in}}(\rho(1,\boldsymbol{b}),\Omega)\cdot\mathbf{n}=\sum_{i=0}^\infty(\mathtt{T}_\Omega^{i,+})_\sharp\eta-(\mathtt{T}_\Omega^{i,-})_\sharp\eta \tag{6.17}$$

*where the series converges in the sense of distributions.*

Define now the *restriction operators* $\mathtt{R}_\Omega^i$, $\mathtt{R}_\Omega$ as

$$\mathtt{R}_\Omega^i\gamma:=\gamma\llcorner_{(t_\gamma^{i,-},t_\gamma^{i,+})},\qquad\mathtt{R}_\Omega\gamma=\big\{\mathtt{R}_\Omega^i\gamma\big\}_i, \tag{6.18}$$

and the measures $\eta_\Omega^i$ as

$$\eta_\Omega^i:=(\mathtt{R}_\Omega^i)_\sharp\eta. \tag{6.19}$$

See Figure 3. It is clear that if

$$\rho_\Omega^i(1,\boldsymbol{b})\,\mathscr{L}^{d+1}:=\int(\mathbb{I},\gamma)_\sharp\big((1,\dot\gamma)\,\mathscr{L}^1\big)\,\eta_\Omega^i(d\gamma), \tag{6.20}$$

then in $\Omega$

$$\rho(1,\boldsymbol{b})=\sum_i\rho_\Omega^i(1,\boldsymbol{b})$$

and

$$\mathsf{Tr}^{\mathrm{in}}(\rho_\Omega^i(1,\boldsymbol{b}),\Omega)=(\mathtt{T}_\Omega^{i,+})_\sharp\eta_\Omega^i-(\mathtt{T}_\Omega^{i,-})_\sharp\eta_\Omega^i=\mathrm{div}_{t,x}\big(\rho_\Omega^i(1,\boldsymbol{b})\big).$$

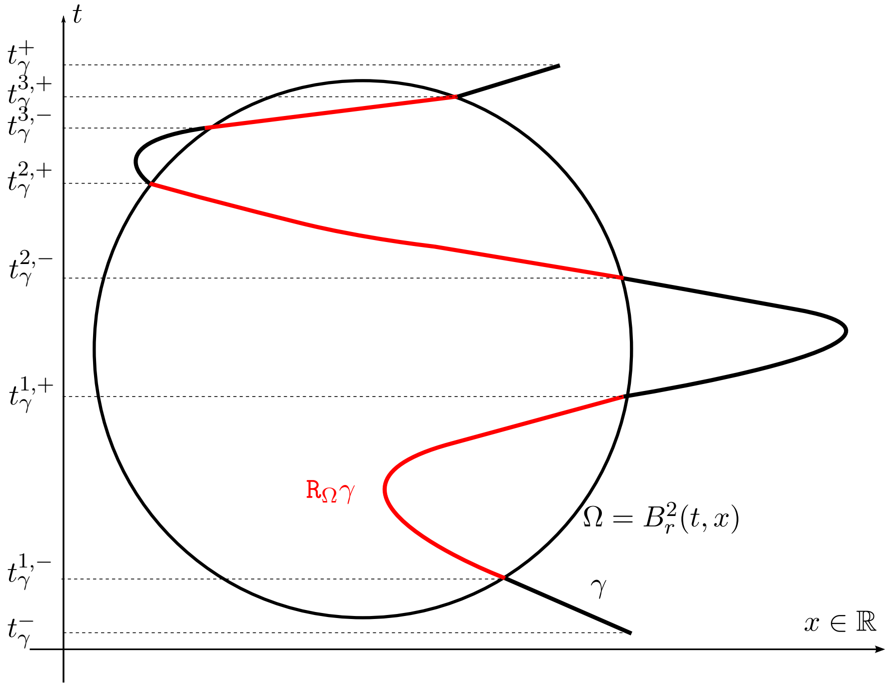



**Figure 3.** Restriction operator $\mathtt{R}_\Omega$ in the case $\Omega$ is a ball $B_r^2(t,x)$. The curve $\gamma$ (depicted in black) is cut into the three red pieces which make up $\mathtt{R}_\Omega\gamma$.

We remark that even if for $\eta_\Omega^i$ the series in (6.17) reduces to a finite sum of measures, the measure $\eta_\Omega^i$ is not in general a Lagrangian representation of $\rho_\Omega^i(1,\boldsymbol{b})$, unless

$$(\mathtt{T}_\Omega^{i,+})_\sharp \eta_\Omega^i \perp (\mathtt{T}_\Omega^{i,-})_\sharp \eta_\Omega^i.$$

Roughly speaking, cancellation effects may occur between the positive and negative parts of the traces: thus, even if $\eta_\Omega^i$ are Lagrangian representations of $\rho_\Omega^i(1,\boldsymbol{b})\llcorner_\Omega$, their sum may not be a Lagrangian representation of $\rho(1,\boldsymbol{b})$, as Point (3) of Definition 3.6 fails. A quick sketch of an example of this behaviour is proposed in the following remark.

Remark 6.21. One can construct a vector field $\boldsymbol{b} \in L^\infty(\mathbb{R}^3)$ supported in $[-1,0] \times [0,1]^2$ with the following properties:

(1) it is divergence-free, smooth outside $\{x_1 = 1\}$ and of the form $(1,\tilde{b}(x_1,x^\perp))$, $(x,x^\perp) \in \mathbb{R}\times\mathbb{R}^2$;
(2) the flow $\tilde{X}$ generated by the ODE

$$\frac{d\tilde{X}}{dx_1} = \boldsymbol{b}(x_1,\tilde{X}), \qquad \tilde{X}(-1,x^\perp) = x^\perp,$$

has the property that it can be extended by continuity to $x_1 = 0$ and it holds

$$(\tilde{X}(0))_\sharp\big(\mathscr{L}^2\llcorner_{(0,1/2)\times(0,1)}\big) = (\tilde{X}(0))_\sharp\big(\mathscr{L}^2\llcorner_{(1/2,1)\times(0,1)}\big) = \frac{1}{2}\mathscr{L}^2\llcorner_{(0,1)^2}.$$

The above assumptions yields that there exists a solution to

$$\mathrm{div}_x\,\big(\tilde{\rho}(1,\tilde{\boldsymbol{b}})\big) = 0$$

which is $w^*$-continuous in $L^\infty$ w.r.t. $x_1$ and such that

$$\tilde{\rho}(-1,x^\perp) = \mathbb{1}_{(1/2,1)\times(0,1)}(x^\perp) - \mathbb{1}_{(0,1/2)\times(0,1)}(x^\perp), \quad \tilde{\rho}(x_1<0)\in\{-1,1\}, \quad \tilde{\rho}(x_1>0)=0.$$

An example of a construction can be found in [**ACM05**, Example 3.8], see also [**Dep03b**]. Define now the vector field

$$\boldsymbol{b}(x_1,x^\perp) = (\tilde{\rho}\tilde{\boldsymbol{b}})(x_1,x^\perp),$$

so that it is divergence free, and its trace on $\{x_1 = 0\}$ is 0. In particular $\rho^- := \mathbb{1}_{\{x_1<0\}}$ is a solution to $\mathrm{div}_{t,x}(\rho(1,\boldsymbol{b})) = 0$.

Let $\eta^-$ be a Lagrangian representation for $\rho^-(1,\boldsymbol{b})$: due to the uniqueness of $\bar{X}$, the set of curves on which $\eta$ is concentrated is the set of curves such that, if $t_\gamma$ is the time where $\gamma(t_\gamma) \in \{x_1 = 0\}$, then

$$\gamma(t) = \begin{cases} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma, \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma, \end{cases}$$

with $x_\gamma^{\perp,-} \in (1/2, 1) \times (0, 1)$, $x_\gamma^{\perp,+} \in (0, 1/2) \times (0, 1)$. If now we extend the vector field $\boldsymbol{b}$ to the region $x_1 > 0$ by symmetry

$$\boldsymbol{b}(x_1, x^\perp) = -\boldsymbol{b}(-x_1, x^\perp),$$

then a Lagrangian representation $\eta$ is obtained by gluing $\eta^-$ with

$$\eta^+ := \mathtt{S}_\sharp \eta^-,$$

where $\mathtt{S}(\gamma)$ is the symmetric curve w.r.t. $\{x_1 = 0\}$,

$$\mathtt{S}(\gamma)(t) = (-\gamma_1, \gamma^\perp)(t).$$

Now we can construct a new Lagrangian representation $\eta'$ for the extended $(1,\boldsymbol{b})$ by piecing together the curves $\gamma$ and $\mathtt{S}(\gamma)$ in order to let both cross the surface: more precisely, defining the maps

$$\left.\begin{array}{ll} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{array}\right\} = \gamma \mapsto G_1(\gamma) = \begin{cases} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ (-\bar{X}, \bar{X}^\perp)\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{cases}$$

$$\left.\begin{array}{ll} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{array}\right\} = \gamma \mapsto G_2(\gamma) = \begin{cases} (-\bar{X}, \bar{X}^\perp)\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{cases}$$

the Lagrangian representation is now given by

$$\eta' := (G_1)_\sharp \eta^- + (G_2)_\sharp \eta^-.$$

A simple computation yields for $\eta'$ it holds

$$(\mathtt{T}^{0,+}_{\{x_1<0\}})_\sharp \eta' = (\mathtt{T}^{0,-}_{\{x_1<0\}})_\sharp \eta' = \|\eta'\|,$$

while being $\mathsf{Tr}^{\mathrm{in}}(\boldsymbol{b}, \{x_1 < 0\}) \cdot \mathbf{n} = 0$ both terms should be 0. A small variation of the above example (i.e. letting the curves cross the surface several times) shows also that the sum (6.17) is diverging in the general case. ♠

**6.2.3. Bounded variation vector fields.** Let us conclude this section considering a relevant case, namely when we improve the regularity of $\boldsymbol{b}$ w.r.t. the space variable (we will assume it enjoys BV-BD bounds): within this setting, the restriction operator $\mathtt{R}_\Omega$ preserves the property of being a Lagrangian representation, for every Lipschitz set $\Omega \subset \mathbb{R}^{d+1}$. The general case of a vector field $\rho(1,\boldsymbol{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$ and a $\rho(1,\boldsymbol{b})$-proper set will then be addressed in Section 6.3.

Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set with a Lipschitz boundary $\partial\Omega$ and let $\mathbf{n}$ be the outer normal defined $\mathscr{H}^d \llcorner \Omega$-a.e.. Consider a vector field $\boldsymbol{b} \in L^1(\mathbb{R}; \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^d))$ and $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ be a non-negative solution to $\mathrm{div}_{t,x}(\rho(1,\boldsymbol{b})) = \mu$ and let $\eta$ be an associated Lagrangian representation. Building on the Chain Rule Formula for traces (see Theorem XII and more precisely Proposition XIII), we show that the restriction of a Lagrangian representation in the sense of (6.18) is a Lagrangian representation of the vector field $\rho(1,\boldsymbol{b})\mathscr{L}^d \llcorner \Omega$.

PROPOSITION 6.22. *Let* $\boldsymbol{b} \in L^1(\mathbb{R}; \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^d))$, $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ *so that* $\mathrm{div}(\rho(1, \boldsymbol{b})) = \mu \in \mathscr{M}(\mathbb{R}^+ \times \mathbb{R}^d)$. *Then for any* $\Omega \subset \mathbb{R}^{d+1}$ *Lipschitz, the measure*

$$(\mathtt{R}_\Omega)_\sharp \eta := \sum_i \eta^i_\Omega = \sum_i (\mathtt{R}^i_\Omega)_\sharp \eta$$

*is a Lagrangian representation of* $\rho(1, \boldsymbol{b}) \mathscr{L}^{d+1} \llcorner \Omega$.

PROOF. Let $\rho^i_\Omega$ be defined as in (6.20); in particular, the distribution $\mathsf{Tr}^{\mathrm{in}}(\rho^i_\Omega(1, \boldsymbol{b}), \Omega) \cdot \mathbf{n}$ is now representable as sum of two Radon measures $(\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta^i_\Omega$, being $\mathtt{T}^{i,\pm}_\Omega$ defined as in (6.16). By applying now Proposition XIII with $\boldsymbol{v} := (1, \boldsymbol{b})$, $\boldsymbol{V} := \rho(1, \boldsymbol{b})$ and $\beta(\cdot) := |\cdot|$, we deduce that

$$\begin{aligned} \mathsf{Tr}^{\mathrm{in}} \left(\rho^i_\Omega(1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} &= \mathsf{Tr}^{\mathrm{in}} \left(|\rho^i_\Omega|(1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} \\ &= \left| \frac{\mathsf{Tr}^{\mathrm{in}} \left(\rho^i_\Omega(1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n}}{\mathsf{Tr}^{\mathrm{in}} \left((1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n}} \right| \mathsf{Tr}^{\mathrm{in}} \left((1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} \end{aligned}$$

because $\rho^i_\Omega \geq 0$. It thus follows that $\mathsf{Tr}^{\mathrm{in}}(\rho^i_\Omega(1, \boldsymbol{b}), \partial\Omega) \cdot \mathbf{n}$ has the same sign of $\mathsf{Tr}^{\mathrm{in}}((1, \boldsymbol{b}), \partial\Omega) \cdot \mathbf{n}$, which means that $(\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta^i_\Omega$ are orthogonal.
Hence there exists two disjoint Borel sets $A^\pm$ such that for all $i \in \mathbb{N}$

$$\mathsf{Tr}^{\mathrm{in},\pm} \left(\rho^i_\Omega(1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} = \left((\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta^i_\Omega\right) \llcorner_{A^\pm},$$

where $A^\pm$ are determined by

$$\mathsf{Tr}^{\mathrm{in},\pm} \left((1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} = \mathsf{Tr}^{\mathrm{in}} \left((1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} \llcorner_{A^\pm},$$

up to $\mathsf{Tr}^{\mathrm{in}}((1, \boldsymbol{b}), \partial\Omega) \cdot \mathbf{n}$-negligible sets. Here the apex $\pm$ means the positive/negative part of the trace.
Furthermore, repeating the argument for a finite sum of $\eta^i_\Omega$ it follows

$$\sum_i^N \rho^i_\Omega \leq \rho,$$

and

$$\begin{aligned} \sum_i^N \mathsf{Tr}^{\mathrm{in},\pm} \left(\rho^i_\Omega(1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} &= \sum_i^N \mathsf{Tr}^{\mathrm{in}} \left(\rho^i_\Omega(1, \boldsymbol{b}), \partial\Omega\right) \cdot \mathbf{n} \llcorner_{A^\pm} \\ &= \left( \sum_i^N \mathsf{Tr}^{\mathrm{in}} \left(\rho^i_\Omega(1, \boldsymbol{b}), \Omega\right) \cdot \mathbf{n} \right) \llcorner_{A^\pm} \\ &= \left( \mathsf{Tr}^{\mathrm{in}} \left( \sum_i^N \rho^i_\Omega(1, \boldsymbol{b}), \Omega \right) \cdot \mathbf{n} \right) \llcorner_{A^\pm} \\ &= \mathsf{Tr}^{\mathrm{in},\pm} \left( \sum_i^N \rho^i_\Omega(1, \boldsymbol{b}), \Omega \right) \cdot \mathbf{n} \\ &\leq \mathsf{Tr}^{\mathrm{in},\pm} \left(\rho(1, \boldsymbol{b}), \Omega\right) \cdot \mathbf{n}, \end{aligned}$$

where we have used the monotonicity of the trace (consequence of Theorem XII). It follows that

$$\sum_i \mathsf{Tr}^{\mathrm{in},\pm} \left(\rho^i_\Omega(1, \boldsymbol{b}), \Omega\right) \cdot \mathbf{n} = \mathsf{Tr}^{\mathrm{in},\pm} \left(\rho(1, \boldsymbol{b}), \Omega\right) \cdot \mathbf{n} < +\infty,$$

where the equality follows from the weak convergence of the sum to the trace. ☐

### 6.3. Restriction operator R and proper sets

We now show that for generic vector fields $\rho(1,\boldsymbol{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$, if $\Omega$ is a $\rho(1,\boldsymbol{b})$-proper set, then the reduction operator $\mathtt{R}_\Omega$ introduced in Proposition 6.22, namely

$$(\mathtt{R}_\Omega)_\sharp \eta := \sum_i (\mathtt{R}^i_\Omega)_\sharp \eta,$$

generates a Lagrangian representation of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1} \llcorner \Omega$. The idea of the proof is to show that there are two disjoint sets where $\eta$-a.e. curve $\gamma$ is only entering or exiting, as we have done in the BD case in the proof of Proposition 6.22. We conclude this section with some useful properties of the operator $\mathtt{R}_\Omega$ for proper sets.

We begin with the following elementary lemma.

Lemma 6.23. *For every Lipschitz function $0 \leq \psi \leq 1$ it holds*

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \{\psi = 1\} \neq \emptyset,\ \mathrm{Graph}\,\gamma \cap \{\psi = 0\} \neq \emptyset\Big\}\Big) \leq \int \rho\big|(1,\boldsymbol{b}) \cdot \nabla\psi\big| \mathscr{L}^{d+1}.$$

Proof. Setting

$$A := \Big\{\gamma : \mathrm{Graph}\,\gamma \cap \{\psi = 1\} \neq \emptyset,\ \mathrm{Graph}\,\gamma \cap \{\psi = 0\} \neq \emptyset\Big\},$$

one has for $\gamma \in A$

$$\int_{t^-_\gamma}^{t^+_\gamma} \big|(1,\boldsymbol{b}) \cdot \nabla\psi\big|\, dt = \mathrm{Tot.Var.}\psi(\gamma) \geq 1,$$

so that

$$\eta(A) \leq \int_A \mathrm{Tot.Var.}\big(\psi \circ \gamma\big)\, \eta(d\gamma) \leq \int \rho\big|(1,\boldsymbol{b}) \cdot \nabla\psi\big|\, \mathscr{L}^{d+1}$$

which concludes the proof. □

Applying Lemma 6.23 to a proper set $\Omega$ with the functions $\phi^{\delta,\pm}$ and passing to the limit as $\delta \to 0$ we obtain the following

Proposition 6.24. *It holds*

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \mathrm{clos}\,\Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1} \setminus \mathrm{clos}\,\Omega \neq \emptyset\Big\}\Big) \leq \int_{\partial\Omega} \rho\big|(1,\boldsymbol{b}) \cdot \mathbf{n}\big|\, \mathscr{H}^d,$$

*and*

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1} \setminus \Omega \neq \emptyset\Big\}\Big) \leq \int_{\partial\Omega} \rho\big|(1,\boldsymbol{b}) \cdot \mathbf{n}\big|\, \mathscr{H}^d. \tag{6.21}$$

In particular, for every proper set $\Omega$ we deduce that

$$\begin{aligned}
&\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial\Omega \neq \emptyset, \mathrm{Graph}\,\gamma \not\subseteq \partial\Omega\Big\}\Big) \\
&\quad\leq \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1} \setminus \Omega \neq \emptyset\Big\}\Big) \\
&\qquad + \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \mathrm{clos}\,\Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1} \setminus \mathrm{clos}\,\Omega \neq \emptyset\Big\}\Big) \\
&\quad\leq 2 \int_{\partial\Omega} \rho\big|(1,\boldsymbol{b}) \cdot \mathbf{n}\big|\mathscr{H}^d.
\end{aligned} \tag{6.22}$$

At the end of this section Corollary 6.31 gives that the constant 2 can be replaced with 1.

Let $\Omega$ be a proper set and let $\Omega^\varepsilon$ its perturbation constructed in Theorem 6.15: moreover, if $K^{\varepsilon,\varepsilon'}_{\bar r} \subset \partial\Omega$ is the compact set constructed in Lemma 6.10, w.l.o.g. we can assume that $\rho(1,\boldsymbol{b})\llcorner_{K^{\varepsilon,\varepsilon'}_{\bar r}}$ is continuous. Recall the decomposition

$$\partial\big(\Omega^\varepsilon \setminus \Omega\big) = S_1 \cup S_2 \cup S^+_3 \cup S^-_3 \cup S_4$$

given in (6.12), where $S_1$, $S_2$ are subset of finitely many hyperplanes $\{t = \mathrm{const}\}$, and $S_4$ is a subset of the lateral faces of the cylinders given by Proposition 6.14.

Applying (6.22) to the lateral boundary of a cylinder

$$\partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} = \Big\{(s,y) : |s-t_i| \leq 2\alpha^2 r_i, |y - x_i - \boldsymbol{b}(t_i,x_i)(s-t_i)| = r_i\Big\},$$

and considering the trajectories restricted to

$$J^i_\gamma := [t^-_\gamma, t^+_\gamma] \cap \big[t_i - 2\alpha^2 r_i, t_i + 2\alpha^2 r_i\big],$$

we obtain

$$\begin{aligned}
&\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma \cap (J^i_\gamma \times \mathbb{R}^d) \nsubseteq \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\}\Big) \\
&\leq \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \big(\mathbb{R}^{d+1} \setminus \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\big) \neq \emptyset\Big\}\Big) \\
&\quad + \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \mathrm{clos}\,\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \big(\mathbb{R}^{d+1} \setminus \mathrm{clos}\,\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\big) \neq \emptyset\ \Big\}\Big) \\
&\leq 2 \int_{\partial \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \rho \big|(1,\boldsymbol{b}) \cdot \mathbf{n}\big| \mathscr{H}^d.
\end{aligned} \tag{6.23}$$

Then we can prove the following.

Lemma 6.25. *It holds*

$$\eta\big(\{\gamma : \mathrm{Graph}\,\gamma \cap S_4 \neq \emptyset\}\big) \leq 2(1+2\alpha) C_d \varepsilon' L^d \mathscr{H}^d(\partial\Omega).$$

Proof. We observe that

$$\begin{aligned}
\big\{\gamma : \mathrm{Graph}\gamma \cap S_4 \neq \emptyset\big\} \subset \bigcup_i &\Big\{\gamma :\ \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \subset \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\} \\
&\cup \Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \big(\mathbb{R}^{d+1} \setminus \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\big) \neq \emptyset\Big\}
\end{aligned}$$

The curves in the first set are curves are the ones which lie on the lateral boundaries of a cylinder for a positive set of times: thus they have $\eta$ measure 0 because $\mathscr{L}^{d+1}(\partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}) = 0$ for every $i \in \mathbb{N}$. For the other set, the computation leading to (6.13) yields

$$\begin{aligned}
&\eta\Big(\bigcup_i \{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \mathbb{R}^{d+1} \setminus \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset\}\Big) \\
&\leq 2(1+2\alpha) C_d \varepsilon' L^d \mathscr{H}^d(\partial\Omega),
\end{aligned}$$

where we have used (6.23). □

We now estimate the flux across the region $\partial\Omega \setminus K^{\varepsilon,\varepsilon'}_{\bar r}$.

Lemma 6.26. *It holds for* $\varepsilon' \ll 1$

$$\eta\big(\{\gamma : \mathrm{Graph}\,\gamma \cap (\partial\Omega \setminus K^{\varepsilon,\varepsilon'}_{\bar r})\}\big) < 5\varepsilon.$$

Proof. As before we observe that

$$\begin{aligned}\Big\{\gamma : \operatorname{Graph}\gamma \cap \Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big) \neq \emptyset\Big\} \\ \subset \Big\{\gamma : \operatorname{Graph}\gamma \subset \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\} \\ \cup \big\{\gamma : \operatorname{Graph}\gamma \cap S_4 \neq \emptyset\big\} \\ \cup \bigcup_{n\in\mathbb{N}} \Big\{\gamma : \operatorname{Graph}\gamma \cap \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} : \mathrm{Tot.Var.}\big(\phi^{2^{-n},+} \circ \gamma\big) \geq 1\Big\} \\ \cup \bigcup_{n\in\mathbb{N}} \Big\{\gamma : \operatorname{Graph}\gamma \cap \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} : \mathrm{Tot.Var.}\big(\phi^{2^{-n},-} \circ \gamma\big) \geq 1\Big\},\end{aligned}$$

where the functions $\phi^{2^{-n},\pm}$ have been introduced in (6.1).

For the first term, as in the proof of the previous lemma, we have that (having all curves in $\Gamma$ a positive length)

$$\eta\bigg(\Big\{\gamma : \gamma \subset \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\}\bigg) = \eta\bigg(\Big\{\gamma : \operatorname{int}\Big((\mathbb{I},\gamma)^{-1}\Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big)\Big) \neq \emptyset\Big\}\bigg) = 0.$$

For the second term, by Lemma 6.25, we infer

$$\big\{\gamma : \operatorname{Graph}\gamma \cap S_4 \neq \emptyset\big\} \leq 2(1+2\alpha) C_d \varepsilon' L^d \mathscr{H}^d(\partial\Omega).$$

Finally, to settle the last terms we argue as in Proposition 6.8: using condition (6.8b) and the fact that

$$\big|\rho(1,\boldsymbol{b}) \cdot \big(\nabla\phi^{2^{-n},\pm}\big)\big| \mathscr{L}^{d+1} \rightharpoonup \big|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\big| \mathscr{H}^d \llcorner_{\partial\Omega},$$

we deduce that

$$\big|\rho(1,\boldsymbol{b}) \cdot \big(\nabla\phi^{2^{-n},\pm}\big)\big| \mathscr{L}^{d+1}\Big(\mathbb{R}^{d+1} \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big) \to \int_{\mathbb{R}^{d+1}\setminus\bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \big|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\big|\, \mathscr{H}^d \llcorner_{\partial\Omega}.$$

Now we have

$$\int_{\mathbb{R}^{d+1}\setminus\bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \big|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\big|\, \mathscr{H}^d \llcorner_{\partial\Omega} \leq \int_{\partial\Omega\setminus K^{\varepsilon,\varepsilon'}_{\bar r}} \big|\rho(1,\boldsymbol{b}) \cdot \mathbf{n}\big| \mathscr{H}^d \llcorner_{\partial\Omega} < 2\varepsilon.$$

Summing up, and using for the last term Lemma 6.23, we get

$$\begin{aligned}&\eta\bigg(\Big\{\gamma : \operatorname{Graph}\gamma \cap \Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big)\Big\}\bigg) \\ &\leq \eta\big(\{\gamma : \operatorname{Graph}\gamma \cap S_4 \neq \emptyset\}\big) \\ &\quad + \sum_{n\in\mathbb{N}} \eta\bigg(\Big\{\gamma : \operatorname{Graph}\gamma \cap \Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big) : \mathrm{Tot.Var.}\big(\phi^{2^{-n},+} \circ \gamma\big) \geq 1\Big\}\bigg) \\ &\quad + \sum_{n\in\mathbb{N}} \eta\bigg(\Big\{\gamma : \operatorname{Graph}\gamma \cap \Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big) : \mathrm{Tot.Var.}\big(\phi^{2^{-n},-} \circ \gamma\big) \geq 1\Big\}\bigg) \\ &\leq 2(1+2\alpha) C_d \varepsilon' L^d \mathscr{H}^d(\partial\Omega) + 2\int_{\partial\Omega\setminus K^{\varepsilon,\varepsilon'}_{\bar r}} |\rho(1,\boldsymbol{b}) \cdot \mathbf{n}| \mathscr{H}^d \llcorner_{\partial\Omega} \\ &\leq 2(1+2\alpha) C_d \varepsilon' L^d \mathscr{H}^d(\partial\Omega) + 4\varepsilon.\end{aligned}$$

Choosing now $\varepsilon' \ll 1$ we obtain that

$$\eta\bigg(\Big\{\gamma : \operatorname{Graph}\gamma \cap \Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big) \neq \emptyset\Big\}\bigg) \leq 5\varepsilon.$$

Being a covering of $K_{\bar r}^{\varepsilon,\varepsilon'}$ we conclude that the statement holds. □

With the same tools we have also the following result.

LEMMA 6.27. *It holds*

$$\sum_i \eta\Big(\Big\{\gamma : \exists t, |s| \le \alpha^2 r_i : \Big(\gamma(t) \in \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \wedge \big|\gamma(t+s) - \gamma(t) - \boldsymbol{b}(t_i,x_i)s\big| > 4r_i\Big)\Big\}\Big)$$
$$\le (1+\alpha)C_d\varepsilon'(2\alpha)^{2d}\mathscr{H}^d(\partial\Omega).$$

PROOF. By (half of) (6.21) we have

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \not\subseteq \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\}\Big) \le \int_{\partial \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \rho\big|(1,\boldsymbol{b})\cdot \mathbf{n}\big|\mathscr{H}^d.$$

Now, observe that

$$\Big\{\gamma : \exists t, |s| \le \alpha^2 r_i\Big(\gamma(t) \in \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \ \wedge \ \big|\gamma(t+s) - \gamma(t) - \boldsymbol{b}(t_i,x_i)s\big| \ge 2r_i\Big)\Big\}$$
$$\subseteq \Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \not\subseteq \mathrm{Cyl}^{r_i,\alpha^2}_{t_i,x_i}\Big\}.$$

Summing over $i$ we get

$$\sum_i \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset : \exists |s| \le \alpha^2 r_i\big(\big|\gamma(t+s) - \gamma(t) - \boldsymbol{b}(t_i,x_i)s\big| > 2r_i\big)\Big\}\Big)$$
$$\le \sum_i \int_{\partial \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \rho\big|(1,\boldsymbol{b})\cdot\mathbf{n}\big|\mathscr{H}^d \le (1+\alpha)C_d\varepsilon'(2\alpha)^{2d}\mathscr{H}^d(\partial\Omega),$$

because of (6.10). □

From Lemma 6.27 we can prove the following weak differentiability of the curves:

COROLLARY 6.28. *For all $\alpha > 0$ it holds*

$$\lim_{s\to 0} \eta\left(\left\{\gamma :\ \gamma(t) \in K_{\bar r}^{\varepsilon,\varepsilon'},\ \left|\frac{\gamma(t+s)-\gamma(t)}{s} - \boldsymbol{b}(t,\gamma(t))\right| > \frac{8}{\alpha^2}\right\}\right) = 0.$$

PROOF. By Lemma 6.13, we can assume that $s < \bar r$, and that there are regular cylinders in all points of $K_{\bar r}^{\varepsilon,\varepsilon'}$ with radius $r$ such that $\frac{\alpha^2 r}{2} \le s \le \alpha^2 r$. Then, using these cylinders for the covering $\{\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\}_{i=1}^{N_s}$ of $K_{\bar r}^{\varepsilon,\varepsilon'}$, we deduce that

$$\left\{\gamma :\ \gamma(t) \in K_{\bar r}^{\varepsilon,\varepsilon'},\ \left|\frac{\gamma(t+s)-\gamma(t)}{s} - \boldsymbol{b}(t,\gamma(t))\right| > \frac{8}{\alpha^2}\right\}$$

is a subset of

$$\bigcup_{i=0}^{N_s}\left\{\gamma : \exists t, \frac{\alpha^2 r_i}{2} \le s \le \alpha^2 r_i : \begin{pmatrix} \gamma(t) \in \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}, \\ \big|\gamma(t+s)-\gamma(t)-\boldsymbol{b}(t_i,x_i)s\big| > 4r_i \end{pmatrix}\right\}.$$

Applying Lemma 6.27 and then letting $\varepsilon' \to 0$ the proof is concluded. □

We now present the following proposition which plays the role of the first part of the proof of Proposition 6.22. Recall the definition of the measures

$$\eta^i_\Omega = (\mathtt{R}^i_\Omega)_\sharp \eta, \qquad \rho_i(1,\boldsymbol{b})\,\mathscr{L}^{d+1} := \int (\mathbb{I},\gamma)_\sharp\big((1,\dot\gamma)\mathscr{L}^1\big)\,\eta^i_\Omega(d\gamma),$$

given in (6.19), (6.20).

PROPOSITION 6.29. *If $\mathtt{T}^{i,\pm}_\Omega$ are the operators defined in (6.16), then it holds*

$$(\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta^i_\Omega \le \rho[(1,\boldsymbol{b})\cdot\mathbf{n})]^\pm\,\mathscr{H}^d\llcorner_{\partial\Omega}.$$

PROOF. First of all observe that the results obtained in this section so far holds also for $\eta^i_\Omega$: indeed all proofs depend only on the quantity $\rho|(1,\boldsymbol{b})\cdot\mathbf{n}|$, which is monotone in $\rho$.

By Lemma 6.26 it is enough to prove the statement in $K^{\varepsilon,\varepsilon'}_{\bar r}$, and assume that the interval of definition of $\gamma$ has length at least $2\tau$. Hence for $r_i < \tau/\alpha^2$, up to a set of trajectories of $\eta^i_\Omega$-measure of the order of $\varepsilon'$ obtained by Lemma 6.25 when applied to $\mathbb{R}^{d+1}\setminus\operatorname{clos}\Omega$, all trajectories of $\eta^i_\Omega$ starting from $K^{\varepsilon,\varepsilon'}_{\bar r}\cap\mathrm{Cyl}^{r,2\alpha^2}_{t,x}$ exit the cylinder by crossing one of the flat bases. In particular we deduce that up to $\mathcal{O}(\varepsilon+\varepsilon')$ trajectories, $(\mathtt{T}^{i,\pm}_\Omega)_\sharp\eta^i_\Omega$ is concentrated on $K^{\varepsilon,\varepsilon'}_{\bar r}\cap\{(1,\boldsymbol{b})\cdot\mathbf{n}\gtrless 0\}$. Hence $(\mathtt{T}^{i,\pm}_\Omega)_\sharp\eta^i_\Omega$ are orthogonal. Since it holds

$$\begin{aligned}0 &\le \int \rho_i\Big(\big|(1,\boldsymbol{b})\cdot\nabla_{t,x}\phi^{\delta,-}\big| - (1,\boldsymbol{b})\cdot\nabla_{t,x}\phi^{\delta,-}\Big)\,\mathscr{L}^{d+1}\\ &\le \int \rho\Big(\big|(1,\boldsymbol{b})\cdot\nabla_{t,x}\phi^{\delta,-}\big| - (1,\boldsymbol{b})\cdot\nabla_{t,x}\phi^{\delta,-}\Big)\,\mathscr{L}^{d+1},\end{aligned}$$

using the weak convergence of $\rho(1,\boldsymbol{b})\cdot\nabla_{t,x}\phi^{\delta,-}$ and $\rho|(1,\boldsymbol{b})\cdot\nabla_{t,x}\phi^{\delta,-}|$ together with the fact $\rho_i\le\rho$ we obtain the statement. □

In particular the behavior (entering/exiting) of trajectories which cross $\Omega$ does not depend on the particular characteristic, but only on the sign of $(1,\boldsymbol{b})\cdot\mathbf{n}$. It follows from the trace analysis that the same property of BD vector fields (see proof of Proposition 6.22) holds also for proper sets.

THEOREM 6.30. *If $\Omega$ is a proper set, the restriction operator $\mathtt{R}_\Omega$ maps a Lagrangian representation of $\rho(1,\boldsymbol{b})$ to a Lagrangian representation of $\rho(1,\boldsymbol{b})\llcorner\Omega$.*

PROOF. Using Proposition 6.29 we can define the sets

$$A^\pm = \big\{(t,x)\in\partial\Omega : (1,\boldsymbol{b})\cdot\mathbf{n}(t,x)\gtrless 0\big\}.$$

Now it is sufficient to repeat the proof of Proposition 6.22. □

COROLLARY 6.31. *A Lagrangian representation $\eta$ of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}$ is concentrated on the set*

$$\bigcup_{N\in\mathbb{N}}\bigg\{\gamma : (\mathbb{I},\gamma)^{-1}(\Omega) = \bigcup_{i=1}^N\big(t^{i,-}_\gamma,t^{i,+}_\gamma\big),\ (\mathbb{I},\gamma)^{-1}(\operatorname{clos}\Omega) = \bigcup_{i=1}^N\big[t^{i,-}_\gamma,t^{i,+}_\gamma\big]\ \text{with}\ t^{i,+}_\gamma < t^{i+1,-}_\gamma\bigg\}.$$

*Moreover, if $\eta^\varepsilon$ is a Lagrangian representation of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}\llcorner\Omega^\varepsilon$, then*

$$\lim_{\varepsilon\to0}\eta^\varepsilon\big(\big\{\gamma : (\mathbb{I},\gamma)^{-1}(\Omega)\ \text{is not an interval}\big\}\big) = 0.$$

PROOF. For the first part of the statement, observe that by the absolute convergence of the series $\sum(\mathtt{T}^{i,\pm}_\Omega)_\sharp\eta$ it follows that $\eta$ is concentrated on the set

$$\bigcup_{N\in\mathbb{N}}\bigg\{\gamma : (\mathbb{I},\gamma)^{-1}(\Omega) = \bigcup_{i=1}^N\big(t^{i,-}_\gamma,t^{i,+}_\gamma\big)\bigg\}.$$

On the other hand, since the set of curves which lie on $\partial\Omega$ for a positive amount of time is negligible, it follows that

$$(\mathbb{I},\gamma)^{-1}(\operatorname{clos}\Omega) = \bigcup_{i=1}^N\big[t^{i,-}_\gamma,t^{i,+}_\gamma\big]$$

for $\eta$-a.e. curve such that $(\mathbb{I},\gamma)^{-1}(\Omega)$ is made of finitely many open intervals. Finally, by Corollary 6.28 the set of curves which have $t^{i,-}_\gamma = t^{i+1,-}_\gamma$ is negligible.

The second part of the statement follows by observing that if a curve $\gamma$ is such that $\operatorname{Graph}\gamma \in \Omega^\varepsilon$ and $(\mathbb{I},\gamma)^{-1}(\Omega)$ is not an interval, then up to a measure of order $\varepsilon'$ it must re-enter in $\Omega$ (re-exit from $\Omega$) in the same cylinder $\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}$ where it just exited (entered). By Proposition 6.29, this is controlled by the entering (exiting) flow in a neighborhood of $K^{\varepsilon,\varepsilon'}_{\tilde r} \cap \{(1,\boldsymbol{b})>0\}$, this can be made arbitrarily small as $\varepsilon \to 0$. □

To end this section we present the following

Proposition 6.32. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and $N \subset \Gamma$ a Borel set. It holds*

$$\eta\Big(\Big\{\gamma : \exists i \text{ s.t. } \mathtt{R}^i_\Omega\gamma = \gamma \llcorner_{(t^{i-}_\gamma, t^{i+}_\gamma)} \in N\Big\}\Big) \leq (\mathtt{R}_\Omega)_\sharp\eta(N).$$

Proof. Let $\widetilde{N}$ be the set given by

$$\widetilde{N} := \Big\{(\gamma,i) \in \Gamma \times \mathbb{N} : \mathtt{R}^i_\Omega\gamma = \gamma \llcorner_{(t^{i-}_\gamma, t^{i+}_\gamma)} \in N\Big\},$$

which is a Borel set because the map $\mathtt{R}^i_\Omega$ is Borel (see Lemma 6.19). Let

$$\pi_1(\widetilde{N}) \ni \gamma \mapsto i(\gamma)$$

be a Borel selection which exists because $\widetilde{N}$ is countable union of Borel graphs. We estimate by using the definition of $\mathtt{R}_\Omega$

$$\begin{aligned}
(\mathtt{R}_\Omega)_\sharp\eta(N) &= \sum_j (\mathtt{R}^j_\Omega)_\sharp\eta(N) \\
&\geq \sum_j (\mathtt{R}^j_\Omega)_\sharp\eta(\{\gamma : i(\gamma) = j\}) \\
&= \sum_j \eta(\{\gamma : i(\gamma) = j\}) \\
&= \eta(\pi_1(\widetilde{N})).
\end{aligned}$$
□

Together with Corollary 6.31 we deduce

Corollary 6.33. *For all $N \subset \Gamma$ is holds*

$$\lim_{\varepsilon \searrow 0} (\mathtt{R}_{\Omega^\varepsilon})_\sharp\eta\big(\big\{\gamma : \exists i \text{ s.t. } \mathtt{R}^i_{\Omega^\varepsilon}\gamma \in N\big\}\big) = (\mathtt{R}_\Omega)_\sharp\eta(N). \tag{6.24}$$

Proof. Just observe that the equality in (6.24) above holds when $(\mathbb{I},\gamma)^{-1}(\Omega)$ is a single interval, and apply Corollary 6.31. □

CHAPTER 7

# Cylinders of approximate flow and untangling of trajectories

Abstract. This chapter contains the core of our strategy to prove Bressan's Conjecture.

Building on the localization method developed in Chapter 6, we give a local condition on the vector field $\rho(1,\boldsymbol{b})$ in order to have that the representation $\eta$ is *untangled*: this means that there exists a partition of the space-time $\mathbb{R}^+ \times \mathbb{R}^d$ made up of disjoint trajectories such that $\eta$-a.e. $\gamma$ is a subset of these curves.

The condition we give is quite general and is presented in Section 7.1: it can be resumed by saying that we control the measure of trajectories entering and exiting from arbitrarily small cylinders (that we call *cylinders of approximate flow*) around $\eta$-a.e. trajectory $\gamma$ in terms of the $\mathscr{L}^d$-measure of their base. This yields a control of the amount of trajectories which bifurcate in the future or in the past from a given trajectory, and it can be nicely expressed in terms of transference plans.

By means of a duality result (borrowed from Optimal Transportation Theory), we show that a control on the flow across the boundary of these cylinders yields an estimate of the amount of trajectories which have a common point but are not subsets of a unique trajectory. This leads, in Section 7.2, to the introduction of the *untangling functional*, which measures the minimal amount of trajectories one has to remove in order to obtain a disjoint set of trajectories such that $\eta$-a.e. $\gamma$ is a subset of these. This functional turns out to be subadditive, allowing a natural condition in order to extend a local estimate to a global one.

The last part of the chapter, namely Section 7.3, shows that in the case of untangling, the structure of the representation allows the complete description of the disintegration of the PDE, in particular the computation of the chain rule.

Consider a proper set $\Omega \subset \mathbb{R}^{d+1}$, and let $\Omega^\varepsilon$ be the pertubed set constructed in Theorem 6.15. For convenience, in the first part of this chapter we will drop the index $\varepsilon$ and refer to $\Omega^\varepsilon$ directly as $\Omega$. Furthermore, $\eta$ will denote a Lagrangian representation of $\mathrm{div}(\rho(1,\boldsymbol{b})) = \mu$ in $\Omega$ (which can be taken as the restriction of a Lagrangian representation in $\mathbb{R}^{d+1}$, in view of Theorem 6.30).

Recall that the set $S_1$ is defined in (6.12), so that essentially all inflow and outflow of $\rho(1,\boldsymbol{b})$ are occurring on open sets which are contained in finitely many time-flat hyperplanes $\{t = t_i\}$. We can assume without loss of generality that $\mathtt{p}_t(S_1) \subset \{\{t = t_i\}$ is locally proper$\}$. Define now

$$\eta^{\mathrm{in}} := \int_{S_1} \eta^{\mathrm{in}}_z \rho(z)\, \mathscr{H}^d(dz) = \eta \llcorner_{\{\mathrm{Graph}\,\gamma \cap S_1 \neq \emptyset\}}.$$

according to Remark 3.7.

## 7.1. Cylinders of approximate flow and transference plans

We consider the following assumption.

Assumption 7.1. There are constants $\mathtt{M}, \varpi > 0$ and a family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ such that:

(1) for every $\gamma \in \Gamma, \ell \in \mathbb{R}^+$, the function $\phi^\ell_\gamma \colon [t^-_\gamma, t^+_\gamma] \times \mathbb{R}^d \to [0,1]$ is Lipschitz;

(2) for $t \in [t^-_\gamma, t^+_\gamma]$, $x \in \mathbb{R}^d$

$$\mathbb{1}_{\gamma(t)+B^d_{\ell/\mathtt{M}}(0)}(x) \leq \phi^\ell_\gamma(t,x) \leq \mathbb{1}_{\gamma(t)+B^d_{\mathtt{M}\ell}(0)}(x);$$

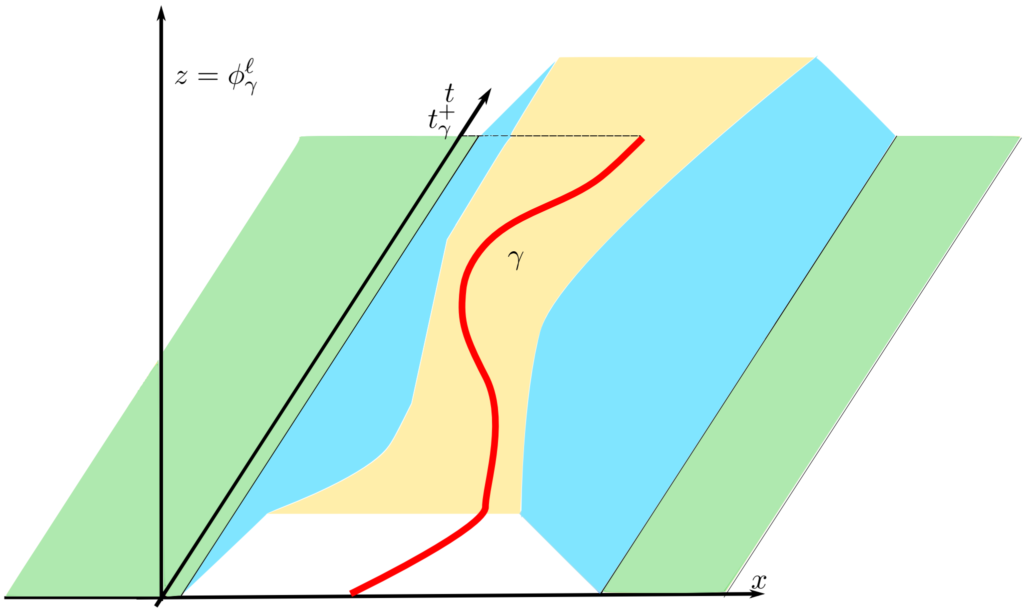



**Figure 1.** A cylinder of approximate flow $\phi^\ell_\gamma$.

(3) it holds

$$\int \left\{ \frac{1}{\sigma(\phi^\ell_\gamma(t^-_\gamma))} \int_{t^-_\gamma}^{t^+_\gamma} \left[ \int \rho(t) \big|(1, \boldsymbol{b}) \cdot \nabla \phi^\ell_\gamma(t)\big| \mathscr{L}^d \right] dt \right\} \eta^{\mathrm{in}}(d\gamma) \leq \varpi, \tag{7.1}$$

where

$$\sigma(f(t)) = \int f(t,x) \rho(t,x) \, \mathscr{L}^d(dx), \tag{7.2}$$

for every $t \in \mathtt{p}_1(S_1)$.

From now onwards we will often refer to the family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ as *cylinders of approximate flow*: indeed, if $\gamma$ is a characteristic of the vector field $\boldsymbol{b}$, the function $\phi^\ell_\gamma$ can be thought as generalized, smoothed cylinder centered at $\gamma$ (see Fig. 1). In particular, Point (3) is saying that the flow through the "lateral boundary of the cylinder" is controlled by the quantity $\varpi$.

Introduce the set

$$W := W_1 \cup W_2 \subset \Gamma \times \Gamma$$

where $W_1$ is the open set

$$W_1 := \big\{ (\gamma,\gamma') : \mathrm{Graph}\, \gamma \cap \mathrm{Graph}\, \gamma' = \emptyset \big\},$$

while $W_2$ is the closed set

$$W_2 := \Big\{ (\gamma,\gamma') : \mathrm{Graph}\, \gamma \cap \mathrm{Graph}\, \gamma' = \mathrm{Graph}\, \big( \gamma \llcorner_{[\max\{t^-_\gamma, t^-_{\gamma'}\}, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \big) \Big\}.$$

Thus the set $W$ is a Borel set (we recall that $\mathrm{Graph}\, \gamma$ is the set of points $(t,\gamma(t))$ for $t$ in the closed interval $[t^-_\gamma, t^+_\gamma]$, see (3.16)).

PROPOSITION 7.2. *Under Assumption 7.1, it holds*

$$\int_{S_1} \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z (\Gamma^2 \setminus W_2) \rho(z) \, \mathscr{H}^d(dz) \leq \varpi.$$

PROOF. We split the proof in several steps.

*Step 1.* For fixed $\ell > 0$ and $\gamma \in \Gamma$ we introduce the following set

$$E^\ell_\gamma := \big\{\gamma' : \mathrm{Graph}\,\gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{supp}\,\phi^\ell_\gamma\big\} \subset \Gamma$$

and consider the functional

$$\Phi^\ell_{\mathrm{exit}}(\gamma) := \int_{S_1 \cap \{t = t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}(E^\ell_\gamma)\phi^\ell_\gamma(t^-_\gamma, z')\rho(z')\,\mathscr{H}^d(dz').$$

This functional computes the weighted amount of curves $\gamma'$ starting inside $\mathrm{supp}\,\phi^\ell_\gamma \cap S_1$ and exiting from the cylinder.

Noticing that

$$\mathrm{Tot.Var.}\big(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}\big) \geq \phi^\ell_\gamma(z') \qquad \text{when } \gamma'(t^-_\gamma = t^-_{\gamma'}) = z',\ \mathrm{Graph}\,\gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{supp}\,\phi^\ell_\gamma,$$

we have

$$\begin{aligned}
\Phi^\ell_{\mathrm{exit}}(\gamma) &= \int_{S_1 \cap \{t = t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}(E^\ell_\gamma)\phi^\ell_\gamma(z')\rho(z')\,\mathscr{H}^d(dz') \\
&= \int_{\{\gamma' : t^-_\gamma = t^-_{\gamma'},\, \mathrm{Graph}\,\gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{supp}\,\phi^\ell_\gamma,\, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'})) > 0\}} \phi^\ell_\gamma\big(t^-_{\gamma'}, \gamma'(t^-_{\gamma'})\big)\,\eta^{\mathrm{in}}(d\gamma') \\
&\leq \int_{\{\gamma' : t^-_\gamma = t^-_{\gamma'},\, \mathrm{Graph}\,\gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{supp}\,\phi^\ell_\gamma,\, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'})) > 0\}} \mathrm{Tot.Var.}\big(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}\big)\,\eta^{\mathrm{in}}(d\gamma') \\
&\leq \int \mathrm{Tot.Var.}\big(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}\big)\,\eta(d\gamma') \\
&\leq \int_{t^-_\gamma}^{t^+_\gamma} \bigg[\int \rho(t,x)\big|(1, \boldsymbol{b})(t,x) \cdot \nabla_{t,x}\phi^\ell_\gamma(t,x)\big|\,\mathscr{L}^d(dx)\bigg]\,dt,
\end{aligned}$$

so that using Point (3), we deduce

$$\int_\Gamma \frac{1}{\sigma(\phi^\ell_\gamma(t^-_\gamma))}\Phi^\ell_{\mathrm{exit}}(\gamma)\,\eta^{\mathrm{in}}(d\gamma) \leq \varpi.$$

*Step 2.* Consider now a sequence $\ell_i \to 0$ such that

$$\phi^{\ell_i}_\gamma \geq \phi^{\ell_{i+1}}_\gamma. \tag{7.3}$$

Due to Point (2), Assumption 7.1 this can be achieved if

$$\ell_{i+1} \leq \frac{\ell_i}{\mathtt{M}^2},$$

because with this choice

$$\mathrm{supp}\,\phi^{\ell_{i+1}}_\gamma(t) \subset \gamma(t) + B^d_{\mathtt{M}\ell_{i+1}} \subset \gamma(t) + B^d_{\ell_i/\mathtt{M}} \subset \big\{\phi^{\ell_i}_\gamma(t) = 1\big\}. \tag{7.4}$$

*Step 3.* Thanks to the choice of the sequence $\ell_i$ in Step 2, we can estimate for $j < i$

$$\begin{aligned}
\varpi &\geq \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\Phi^{\ell_j}_{\mathrm{exit}}(\gamma)\,\eta^{\mathrm{in}}(d\gamma) \\
&= \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\bigg\{\int_{S_1 \cap \{t = t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}(E^{\ell_j}_\gamma)\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathscr{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma) \\
(E^{\ell_i}_\gamma \subset E^{\ell_j}_\gamma) &\geq \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\bigg\{\int_{S_1 \cap \{t = t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}(E^{\ell_i}_\gamma)\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathscr{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma).
\end{aligned}$$

Now, for fixed $i$, we pass to the limit as $j \to +\infty$ and we observe that

$$\frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\int_{S_1 \cap \{t = t^-_\gamma\}} \eta^{\mathrm{in}}_z \phi^{\ell_j}_\gamma(z)\rho(z)\,\mathscr{H}^d(dz) \rightharpoonup \eta^{\mathrm{in}}_{\gamma(t^-_\gamma)} \qquad \text{weakly}^*$$

in duality w.r.t. continuous, bounded functions for $\eta^{\mathrm{in}}$-a.e. $\gamma$. This follows from the fact that $\rho\mathscr{H}^d$-a.e. $z' \in S_1$ is a Lebesgue point for the map $z' \mapsto \eta^{\mathrm{in}}_{z'}$ and the set of $\gamma$ starting in a negligible set in $S_1$ is $\eta^{\mathrm{in}}$ negligible. Notice that for every $i \in \mathbb{N}$ the set $E^{\ell_i}_\gamma$ is open, so that thanks to the l.s.c. of the weak convergence on open sets, we have

$$\eta^{\mathrm{in}}_{\gamma(t^-_\gamma)}\big(E^{\ell_i}_\gamma\big) \leq \liminf_{j\to\infty} \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \int_{S_1\cap\{t=t^-_\gamma\}} \phi^{\ell_j}_\gamma(z')\eta_{z'}\big(E^{\ell_i}_\gamma\big)\rho(z')\,\mathscr{H}^d(dz').$$

*Step 4.* Using Fatou's Lemma, we conclude that

$$\begin{aligned}
\varpi &\geq \liminf_{j\to\infty} \int_{S_1}\bigg\{\int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\bigg[\int_{\operatorname{supp}\phi^{\ell_j}_\gamma(t^-_\gamma)} \phi^{\ell_j}_\gamma(z)\eta_z(E^{\ell_i}_\gamma)\rho(z)\,\mathscr{H}^d(dz)\bigg]\,\eta_{z'}(d\gamma)\bigg\}\rho(z')\,\mathscr{H}^d(dz') \\
&\geq \int_{S_1}\bigg\{\int \liminf_j \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\bigg[\int_{\operatorname{supp}\phi^{\ell_j}_\gamma(t^-_\gamma)} \phi^{\ell_j}_\gamma(z)\eta_z(E^{\ell_i}_\gamma)\rho(z)\,\mathscr{H}^d(dz)\bigg]\,\eta_{z'}(d\gamma)\bigg\}\rho(z')\,\mathscr{H}^d(dz') \\
&\geq \int_{S_1}\bigg\{\int \eta_{z'}(E^{\ell_i}_\gamma)\,\eta_{z'}(d\gamma)\bigg\}\rho(z')\,\mathscr{H}^d(dz') \\
&= \int_{S_1} \eta_{z'}\otimes\eta_{z'}\big(\big\{(\gamma,\gamma') : \gamma' \in E^{\ell_i}_\gamma\big\}\big)\rho(z')\,\mathscr{H}^d(dz').
\end{aligned} \tag{7.5}$$

Observe now that when $i \to \infty$

$$\big\{(\gamma,\gamma') : \gamma' \in E^{\ell_i}_\gamma\big\} \nearrow \Gamma^2 \setminus W_2.$$

By the Monotone Convergence Theorem, we then conclude

$$\begin{aligned}
&\int_{S_1} \eta_{z'}\otimes\eta_{z'}(\Gamma^2\setminus W_2)\rho(z')\,\mathscr{H}^d(dz') \\
&= \lim_i \int_{S_1} \eta_{z'}\otimes\eta_{z'}\big(\big\{(\gamma,\gamma') : \gamma' \in E^{\ell_i}_\gamma\big\}\big)\rho(z')\,\mathscr{H}^d(dz') \leq \varpi,
\end{aligned}$$

which concludes the proof. □

To analyze the trajectories which are entering into the cylinder $\phi^\gamma_\ell$, we have to introduce a new object. Let $\pi \in \mathsf{Adm}(\eta^{\mathrm{in}},\eta)$ be an admissible plan between the measures $\eta^{\mathrm{in}}$ and $\eta$: this means that

$$(\mathtt{p}_1)_\sharp\pi = g_1\eta^{\mathrm{in}}, \qquad (\mathtt{p}_2)_\sharp\pi = g_2\eta,$$

with $0 \leq g_1, g_2 \leq 1$ are Borel functions. Observe that by disintegration we have

$$\pi = \int \pi_\gamma\,\eta^{\mathrm{in}}(d\gamma) = \int_{S_1}\bigg[\int \pi_\gamma\,\eta^{\mathrm{in}}_z(d\gamma)\bigg]\rho(z)\,\mathscr{H}^d(dz),$$

with $\|\pi_\gamma\| = g_1(\gamma)$, and similarly for the disintegration w.r.t. the second marginal $\eta$.

The following proposition is the analogue of Proposition 7.2 for the plan $\pi$.

Proposition 7.3. *Under Assumption 7.1, it holds*

$$\int\bigg\{\int \pi_{\gamma'}\left(\left\{\begin{array}{l}(\gamma',\gamma'') : \gamma''(t^-_{\gamma''}) \notin \operatorname{Graph}\gamma, \\ \big(\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}, \gamma''\llcorner_{[t^-_\gamma,t^+_\gamma]}\big) \in \Gamma^2\setminus W_1\end{array}\right\}\right)\eta^{\mathrm{in}}_z\otimes\eta^{\mathrm{in}}_z(d\gamma d\gamma')\bigg\}\rho(z)\,\mathscr{H}^d(dz) \leq \varpi.$$

Proof. We split the proof in several steps.

*Step 1.* For fixed $\ell > 0$ and $\gamma \in \Gamma$ we introduce the following set

$$A^\ell_\gamma := \Big\{(\gamma',\gamma'') : \phi^\ell_\gamma(\gamma''(\max\{t^-_{\gamma''},t^-_\gamma\})) = 0,\ \big(\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}, \gamma''\llcorner_{[t^-_\gamma,t^+_\gamma]}\big) \in \Gamma^2\setminus W_1\Big\}, \tag{7.6}$$

and consider the functional

$$\Phi^\ell_{\mathrm{enter}}(\gamma) := \int_{S_1\cap\{t=t^-_\gamma\}}\bigg[\int \pi_{\gamma'}(A^\ell_\gamma)\,\eta^{\mathrm{in}}_{z'}(d\gamma')\bigg]\phi^\ell_\gamma(z')\rho(z')\,\mathscr{H}^d(dz').$$

This integral computes the weighted amount of curves $\gamma''$ starting outside the cylinder $\phi^\ell_\gamma$ and touching a curve $\gamma'$ which starts inside the cylinder in the time interval $[t^-_\gamma, t^+_\gamma]$. We observe that for every $(\gamma', \gamma'') \in A^\ell_\gamma$ it holds

$$\mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}) + \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \geq \phi^\ell_\gamma(z'), \tag{7.7}$$

when $\gamma'(t^-_{\gamma'} = t^-_\gamma) = z'$. Then we have, by integration,

$$\begin{aligned}
\Phi^\ell_{\mathrm{enter}}(\gamma) &= \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[ \int \pi_{\gamma'}(A^\ell_\gamma)\, \eta^{\mathrm{in}}_{z'}(d\gamma') \bigg] \phi^\ell_\gamma(z') \rho(z')\, \mathscr{H}^d(dz') \\
&= \int_{A^\ell_\gamma \cap \{(\gamma',\gamma''):\, t^-_\gamma = t^-_{\gamma'}\}} \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))\, \pi(d\gamma' d\gamma'') \\
&= \int_{A^\ell_\gamma \cap \{(\gamma',\gamma''):\, t^-_\gamma = t^-_{\gamma'},\, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))\, \pi(d\gamma' d\gamma'') \boxed{\leq}
\end{aligned}$$

so that, taking into account (7.7), we get

$$\begin{aligned}
\boxed{\leq} \int_{A^\ell_\gamma \cap \{(\gamma',\gamma''):\, t^-_\gamma = t^-_{\gamma'},\, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} \Big[ & \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \\
& + \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \Big] \pi(d\gamma' d\gamma'') \\
= \int_{\{\gamma':t^-_\gamma = t^-_{\gamma'}, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} & \pi_{\gamma'}\big(A^\ell_\gamma\big) \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]})\, \eta^{\mathrm{in}}(d\gamma') \\
+ \int_{\{\gamma'':\, \phi^\ell_\gamma(\gamma''(\max\{t^-_\gamma, t^-_{\gamma''}\}))=0\}} & \pi_{\gamma''}\left( \left\{ \begin{array}{l} \gamma' \in A^\ell_\gamma,\ t^-_\gamma = t^-_{\gamma'}, \\ \phi^\ell_\gamma(\gamma'(t^-_{\gamma'})) > 0 \end{array} \right\} \right) \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]})\, \eta(d\gamma'') \\
\leq \int_{\{\gamma':t^-_\gamma = t^-_{\gamma'}, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} & \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]})\, \eta^{\mathrm{in}}(d\gamma') \\
+ \int_{\{\gamma'':\, \phi^\ell_\gamma(\gamma''(\max\{t^-_\gamma, t^-_{\gamma''}\}))=0\}} & \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]})\, \eta(d\gamma'') \\
\leq \int \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]})\, \eta(d\gamma') & \\
\leq \int_{t^-_\gamma}^{t^+_\gamma} \bigg[ \int \rho(t,x) \big| (1, \boldsymbol{b}(t,x)) \cdot \nabla_{t,x} \phi^\ell_\gamma(t,x) \big| \, \mathscr{L}^d(dx) \bigg] dt. &
\end{aligned}$$

Integrating in $\gamma$ and using Point (3), we deduce

$$\int \frac{1}{\sigma(\phi^\ell_\gamma(t^-_\gamma))} \Phi^\ell_{\mathrm{enter}}(\gamma)\, \eta^{\mathrm{in}}(d\gamma) \leq \varpi. \tag{7.8}$$

*Step 2.* Consider now a sequence $\ell_i \to 0$ such that

$$\{\phi^{\ell_i}_\gamma < a\} \subset \{\phi^{\ell_j}_\gamma = 0\} \tag{7.9}$$

for every $i < j$. For instance, the same choice as in Step 2 of Proposition 7.2 is sufficient for $a = 1$, thanks to (7.4).

*Step 3.* We now pass to the limit. By (7.8), we have

$$\begin{aligned}
\varpi &\geq \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \Phi^{\ell_j}_{\mathrm{enter}}(\gamma)\, \eta^{\mathrm{in}}(d\gamma) \\
&= \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \bigg\{ \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[ \int \pi_{\gamma'}(A^{\ell_j}_\gamma)\, \eta^{\mathrm{in}}_{z'}(d\gamma') \bigg] \phi^{\ell_j}_\gamma(z') \rho(z')\, \mathscr{H}^d(dz') \bigg\} \eta^{\mathrm{in}}(d\gamma).
\end{aligned}$$

where we recall the set $A^\ell_\gamma$ is defined in (7.6) as

$$A^\ell_\gamma = \Big\{(\gamma',\gamma'') : \phi^\ell_\gamma(\gamma''(\max\{t^-_{\gamma''},t^-_\gamma\})) = 0,\ \big(\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]},\gamma''\llcorner_{[t^-_\gamma,t^+_\gamma]}\big) \in \Gamma^2\setminus W_1\Big\}.$$

To overcome the difficulty given by the fact that $A^\ell_\gamma$ is not open, we take into account Step 2 and define the open set

$$A^{\ell,a}_\gamma := \Big\{(\gamma',\gamma'') : \phi^\ell_\gamma(\gamma''(\max\{t^-_{\gamma''},t^-_\gamma\})) < a,\ \big(\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]},\gamma''\llcorner_{[t^-_\gamma,t^+_\gamma]}\big) \in \Gamma^2\setminus W_1\Big\}.$$

Notice so that, thanks to the (7.4), $A^{\ell_i,a}_\gamma \subset A^{\ell_j}_\gamma$ for $i<j$ and hence

$$\begin{aligned}
&\frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\int_{S_1\cap\{t=t^-_\gamma\}}\bigg[\int \pi_{\gamma'}(A^{\ell_j}_\gamma)\,\eta^{\rm in}_{z'}(d\gamma')\bigg]\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathscr{H}^d(dz')\\
&\geq \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\int_{S_1\cap\{t=t^-_\gamma\}}\bigg[\int \pi_{\gamma'}(A^{\ell_i,a}_\gamma)\,\eta^{\rm in}_{z'}(d\gamma')\bigg]\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathscr{H}^d(dz')\\
&= \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\int_{S_1\cap\{t=t^-_\gamma\}}\bigg\{\int \Big[\mathbf{I}(\gamma,\gamma',\ell_i)\Big]\,\eta^{\rm in}_{z'}(d\gamma')\bigg\}\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathscr{H}^d(dz'),
\end{aligned}$$

where

$$\Gamma^2\setminus W_1(\gamma) = \Big\{(\gamma',\gamma'') : \big(\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]},\gamma''\llcorner_{[t^-_\gamma,t^+_\gamma]}\big)\in\Gamma^2\setminus W_1\Big\}$$

and

$$\mathbf{I}(\gamma,\gamma',\ell_i) := \pi\llcorner_{\Gamma^2\setminus W_1(\gamma)})_{\gamma'}\big(\{\gamma'' :\ \phi^{\ell_i}_\gamma(\gamma''(\max\{t^-_{\gamma''},t^-_\gamma\})) < a\}\big)$$

*Step 4.* Define for $t_1<t_2$ the set

$$\begin{aligned}
\Gamma^2\setminus W_1(t_1,t_2) &= \Big\{(\gamma',\gamma'') : \big(\gamma'\llcorner_{[t_1,t_2]},\gamma''\llcorner_{[t_1,t_2]}\big)\in\Gamma^2\setminus W_1\Big\}\\
&= \Big\{(\gamma',\gamma'') : \mathrm{Graph}\,\gamma'\llcorner_{[t_1,t_2]}\cap\mathrm{Graph}\,\gamma''\llcorner_{[t_1,t_2]}\neq\emptyset\Big\},
\end{aligned}$$

and accordingly let

$$\mathbf{I}^{t_2}_{t_1}(\gamma',\ell_i) := (\pi\llcorner_{\Gamma^2\setminus W_1(t_1,t_2)})_{\gamma'}\big(\{\gamma'' :\ \phi^{\ell_i}_\gamma(\gamma''(\max\{t^-_{\gamma''},t^-_\gamma\})) < a\}\big).$$

Now $\rho\mathscr{H}^d$-a.e. $z'\in S_1$ is a Lebesgue point for the map

$$z'\mapsto\int\big[(\pi\llcorner_{\Gamma^2\setminus W_1(t_1,t_2)})_{\gamma'}\big]\,\eta^{\rm in}_{z'}(d\gamma'),$$

w.r.t. the weak* topology, and hence, arguing as in Proposition 7.2, passing to the limit in $j$ and using the l.s.c. on open sets (i.e. $\{\gamma'' :\ \phi^{\ell_i}_\gamma(\gamma''(\max\{t^-_{\gamma''},t^-_\gamma\})) < a\}$) we deduce

$$\begin{aligned}
&\int I^{t_2}_{t_1}(\gamma',\ell_i)\,\eta^{\rm in}_{\gamma(t^-_\gamma)}(d\gamma')\\
&\leq \liminf_{j\to+\infty}\frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\int_{S_1\cap\{t=t^-_\gamma\}}\bigg[\int \mathbf{I}^{t_2}_{t_1}(\gamma',\ell_i)\,\eta^{\rm in}_{z'}(d\gamma')\bigg]\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathscr{H}^d(dz')
\end{aligned}$$

for $\eta^{\rm in}$-a.e. $\gamma$.

*Step 5.* Take a partition of a set where $\eta^{\rm in}$ is concentrated into finitely many disjoint sets $\{A^{\rm in}_{k,n}\}_{n=1}^{N_k}$ so that

$$A^{\rm in}_{k,n}\subset\Big\{\gamma\in\Gamma : t^-_n - 2^{-k} < t^-_\gamma < t^-_n,\ t^+_n \leq t^+_\gamma \leq t^+_n + 2^{-k}\Big\}$$

and a set $A^{\rm in}_{k,0}$ whose measure is arbitrarily small for $k \to \infty$. Step 3 above gives

$$\begin{aligned}
\varpi \geq{}& \int_\Gamma \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg\{ \int \mathbf{I}(\gamma,\gamma',\ell_i)\, \eta^{\rm in}_{z'}(d\gamma') \bigg\} \phi^{\ell_j}_\gamma(z') \rho(z')\, \mathscr{H}^d(dz') \bigg\}\, \eta^{\rm in}(d\gamma) \\
\geq{}& \sum_{n=1}^{N_k} \int_{A^{\rm in}_{k,n}} \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \bigg\{ \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[ \int \mathbf{I}^{t^+_n}_{t^-_n}(\gamma',\ell_i)\, \eta^{\rm in}_{z'}(d\gamma') \bigg] \phi^{\ell_j}_\gamma(z') \rho(z')\, \mathscr{H}^d(dz') \bigg\}\, \eta^{\rm in}(d\gamma),
\end{aligned}$$

because $\mathbf{I}(\gamma,\gamma',\ell_i) \supset \mathbf{I}^{t^+_n}_{t^-_n}(\gamma',\ell_i)$ when $\gamma \in A^{\rm in}_{k,n}$. Using Fatou's Lemma, we conclude that

$$\begin{aligned}
&\liminf_{j \to +\infty} \sum_{k=1}^{N_k} \int_{A^{\rm in}_{k,n}} \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \bigg\{ \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[ \int \mathbf{I}^{t^+_n}_{t^-_n}(\gamma',\ell_i)\, \eta^{\rm in}_{z'}(d\gamma') \bigg] \phi^{\ell_j}_\gamma(z') \rho(z')\, \mathscr{H}^d(dz') \bigg\}\, \eta^{\rm in}(d\gamma) \\
&\geq \sum_k \int_{A^{\rm in}_{k,n}} \liminf_{j \to +\infty} \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \bigg\{ \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[ \int \mathbf{I}^{t^+_n}_{t^-_n}(\gamma',\ell_i)\, \eta^{\rm in}_{z'}(d\gamma') \bigg] \phi^{\ell_j}_\gamma(z') \rho(z')\, \mathscr{H}^d(dz') \bigg\}\, \eta^{\rm in}(d\gamma) \\
&\geq \sum_k \int_{A^{\rm in}_{k,n}} \bigg[ \int \mathbf{I}^{t^+_n}_{t^-_n}(\gamma',\ell_i)\, \eta^{\rm in}_{\gamma(t^-_\gamma)}(d\gamma') \bigg]\, \eta^{\rm in}(d\gamma),
\end{aligned}$$

hence

$$\varpi \geq \sum_k \int_{A^{\rm in}_{k,n}} \bigg[ \int \mathbf{I}^{t^+_n}_{t^-_n}(\gamma',\ell_i)\, \eta^{\rm in}_{\gamma(t^-_\gamma)}(d\gamma') \bigg]\, \eta^{\rm in}(d\gamma). \tag{7.10}$$

By taking $t^+_n$ increasing and $t^+_n$ decreasing for $\eta^{\rm in}$-a.e. $\gamma$, when $k \to \infty$ we have for every $\gamma'$

$$\sum_n \mathbf{I}^{t^+_n}_{t^-_n}(\gamma',\ell_i) \mathbb{1}_{A^{\rm in}_{k,n}} \nearrow \mathbf{I}(\gamma,\gamma',\ell_i),$$

on a $\eta$-conegligible set, so that by passing to the limit in $n$ we conclude by monotonicity that

$$\varpi \geq \int_\Gamma \bigg[ \int \mathbf{I}(\gamma,\gamma',\ell_i)\, \eta^{\rm in}_{\gamma(t^-_\gamma)}(d\gamma') \bigg]\, \eta^{\rm in}(d\gamma).$$

Observe now that when $i \to +\infty$

$$\begin{aligned}
\big\{(\gamma',\gamma'') : \phi^{\ell_i}_\gamma(\gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) < a\big\} &\nearrow \{(\gamma',\gamma'') : t^-_{\gamma''} \leq t^+_\gamma, \gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) \notin \mathrm{Graph}\,\gamma\}) \\
&= \{(\gamma',\gamma'') : t^-_{\gamma''} \leq t^+_\gamma, \gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma\})
\end{aligned}$$

because for $\eta^{\rm in}$-a.e. $\gamma$ we have $\gamma(t^-_\gamma) \in S_1$. By Monotone Convergence Theorem, we then have

$$\begin{aligned}
\varpi \geq{}& \int \bigg\{ \int \Big[ \pi_{\gamma'}\big(\{(\gamma',\gamma'') : t^-_{\gamma''} \leq t^+_\gamma, \gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma\} \cap \Gamma^2 \setminus W_1(\gamma)\big) \Big]\, \eta^{\rm in}_{\gamma(t^-_\gamma)}(d\gamma') \bigg\}\, \eta^{\rm in}(d\gamma) \\
\geq{}& \int \bigg\{ \int \Big[ \pi_{\gamma'}\big(\{(\gamma',\gamma'') : \gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma\} \cap \Gamma^2 \setminus W_1(\gamma)\big) \Big]\, \eta^{\rm in}_z \otimes \eta^{\rm in}_z(d\gamma d\gamma') \bigg\} \rho(z)\, \mathscr{H}^d(dz),
\end{aligned} \tag{7.11}$$

which is what we wanted to prove taking into account the definition of $\Gamma^2 \setminus W_1(\gamma)$. □

Remark 7.4. In general Proposition 7.2 is sharp and it holds

$$\eta^{\rm in} \otimes \eta^{\rm in}(\Gamma^2 \setminus W) < \pi(\Gamma^2 \setminus W),$$

so that we cannot expect a control on the quantity $\pi(\Gamma^2 \setminus W)$. For example, consider three curves $\gamma_a$, $\gamma_b$ and $\gamma_c$ starting at the same time ($t = 0$) such that

$$\gamma_a = \gamma_b \cap \gamma_c, \qquad \gamma_b \neq \gamma_c,$$

with weight $a, b, c > 0$ (see Figure 2). Then one has $\eta^{\rm in} \otimes \eta^{\rm in}(\Gamma^2 \setminus W) = 2bc$, while by duality $\max \pi(\Gamma^2 \setminus W) = 2 \min\{b, c\}$. ♠

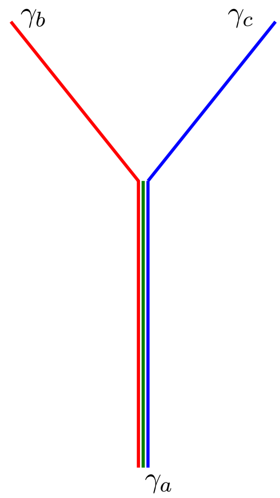



**Figure 2.** The discrete case described in Example 7.4: $\eta = \eta^{\rm in} = a\delta_{\gamma_a} + b\delta_{\gamma_b} + c\delta_{\gamma_c}$, where $a, b, c > 0$ are positive real numbers. The red and blue curves (resp. $\gamma_b, \gamma_c$ are distinct but they have non trivial intersection, which coincides with $\gamma_a$, the green curve. It is clear that $\eta^{\rm in} \otimes \eta^{\rm in}(\Gamma^2 \setminus W) = bc + cb = 2bc$. On the other hand, if e.g. $b < c$, we can construct a plan which moves $b\delta_{\gamma_b}$ to $b\delta_{\gamma_c}$ and $b\delta_{\gamma_c}$ to $b\delta_{\gamma_b}$ (leaving the remaining $(c-b)\delta_{\gamma_c}$ fixed) and thus it holds $\pi(\Gamma^2 \setminus W) = b + b = 2b = 2\min\{b, c\}$.

Remark 7.5. By inspection, one can observe that to deduce Propositions 7.2 and 7.3 one can relax Point 2 to the following:

(2') for $\eta^{\rm in}$-a.e. $\gamma$ there are two sequences of Lipschitz functions $\phi^{\ell_i}_\gamma$, $\phi^{\ell_{i'}}_\gamma$ such that
  (a) (7.3) is satisfied by $\phi^{\ell_i}_\gamma$ and
$$\phi^{\ell_i}_\gamma(\mathrm{Graph}\,\gamma) = 1, \qquad \lim_{i\to\infty} \mathrm{supp}\,\phi^{\ell_i}_\gamma = \mathrm{Graph}\,\gamma, \tag{7.12}$$
  (b) (7.9) is satisfied by $\phi^{\ell_{i'}}_\gamma$ and
$$\phi^{\ell_{i'}}_\gamma(\mathrm{Graph}\,\gamma) = 1, \qquad \lim_{i'\to\infty} \mathrm{supp}\,\phi^{\ell_{i'}}_\gamma = \mathrm{Graph}\,\gamma, \tag{7.13}$$
  (c) it holds
$$\lim_{\ell_i} \frac{\sigma((f\phi^{\ell_i}_\gamma)(t^-_\gamma))}{\sigma(\phi^{\ell_i}_\gamma(t^-_\gamma))} = f(\gamma(t^-_\gamma)), \qquad \lim_{\ell_{i'}} \frac{\sigma((f\phi^{\ell_{i'}}_\gamma)(t^-_\gamma))}{\sigma(\phi^{\ell_{i'}}_\gamma(t^-_\gamma))} = f(\gamma(t^-_\gamma)),$$
  for all integrable functions $f$ and $\eta^{\rm in}$-a.e. $\gamma$, where $\sigma(\cdot)$ is defined in (7.2).

On can further require that (7.3), (7.9) hold up to a set of trajectories which vanishes when computing the limits (7.5), (7.10), and the same requirement for (7.12), (7.13).

Finally, in some cases it is easier to have replace $\phi^\ell_\gamma$ with the characteristic function of an inner/outer proper set, replacing the integral of $\rho|(1, \boldsymbol{b}) \cdot \mathbf{n}|$ with the inner/outer trace as follows.

Assumption 7.6 (Inner proper cylinders). There are constants $\mathtt{M}, \varpi > 0$ and a family of sets $\{Q^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ such that:

(1) for every $\gamma \in \Gamma, \ell \in \mathbb{R}^+$, the set $Q^\ell_\gamma \subset \mathbb{R}^{d+1}$ is $\rho(1, \boldsymbol{b})$-inner proper;
(2) for $t \in (t^-_\gamma, t^+_\gamma)$
$$\gamma(t) + B^d_{\ell/\mathtt{M}}(0) \subseteq Q_t \subseteq \gamma(t) + B^d_{\mathtt{M}\ell}(0);$$

(3) it holds

$$\int \left[ \frac{1}{\sigma(\mathbb{1}_{Q^\ell_\gamma}(t^-_\gamma))} \int_{t\in(t^-_\gamma,t^+_\gamma)} \mathsf{Tr}\left(\rho(1,\boldsymbol{b}),Q\right)\mathscr{H}^d \llcorner_{\partial Q}\right] \eta^{\mathrm{in}}(d\gamma) \leq \varpi, \tag{7.14}$$

where $\sigma$ is given by (7.2).

The key observation is that being inner proper, up to an arbitrarily small quantity one can replace (7.14) with (7.1) because of Condition (3) of Definition 6.5. The two definitions are essentially equivalent because of Remark 6.3. The assumption in the case of outer proper cylinders is analogous, and one can image also combinations of the two cases. ♠

**7.1.1. Forward uniqueness.** We now turn our attention to the set of *crossing trajectories*, i.e. the trajectories which enter from $S_1$ and leave the domain $\Omega$: set

$$\Gamma^{\mathrm{cr}} := \left\{\gamma : \gamma(t^-_\gamma) \in S_1, \gamma(t^+_\gamma) \in \partial\Omega\right\}$$

and define accordingly the measures

$$\eta^{\mathrm{cr}} := \eta\llcorner_{\Gamma^{\mathrm{cr}}}, \qquad \eta^{\mathrm{cr}}_z := \eta_z\llcorner_{\Gamma^{\mathrm{cr}}}.$$

Remark 7.7. Notice that $\|\eta^{\mathrm{cr}}_z\|$ may be less than 1, hence it is not the standard normalized disintegration of $\eta^{\mathrm{cr}}$ w.r.t. $\rho\mathscr{H}^d\llcorner_{S_1}$. By projection, the corresponding density $\rho^{\mathrm{cr}} \geq 0$, defined by

$$\rho^{\mathrm{cr}}(t,\cdot)\,\mathscr{L}^d = (e_t)_\sharp \eta^{\mathrm{cr}}$$

satisfies

$$\mathrm{div}\left(\rho^{\mathrm{cr}}(1,\boldsymbol{b})\right) = \rho^{\mathrm{cr}}\mathscr{H}^d\llcorner_{S_1} - \rho^{\mathrm{cr}}[(1,\boldsymbol{b})\cdot\mathbf{n}]^+\mathscr{H}^d\llcorner_{\partial\Omega}.$$

Furthermore, for $\mathscr{H}^d$-a.e. $z \in \partial\Omega$ it holds

$$\rho^{\mathrm{cr}}(z) = \|\eta^{\mathrm{cr}}_z\|\rho(z).$$

♠

We start by observing that if $\gamma(t^+_\gamma) \in \partial\Omega$, then one can replace the requirement

$$\left(\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}, \gamma''\llcorner_{[t^-_\gamma,t^+_\gamma]}\right) \in \Gamma^2 \setminus W_1$$

with

$$(\gamma',\gamma'') \in \Gamma^2 \setminus W_1,$$

because in this case either $\gamma' \neq \gamma$ or $(\gamma,\gamma'') \in \Gamma^2 \setminus W_1$: in particular (7.7) holds for all $(\gamma',\gamma'') \in \Gamma^2 \setminus W_1$ for $\ell \ll 1$. By restricting the estimate in Proposition 7.3 to $\eta^{\mathrm{cr}}$, we then deduce the following.

Corollary 7.8. *For any transport plan $\pi \in \mathsf{Adm}(\eta^{\mathrm{cr}},\eta^{\mathrm{in}})$ it holds*

$$\pi\left(\left\{(\gamma',\gamma'') : \gamma''(t^-_{\gamma''}) \neq \gamma'(t^-_{\gamma'})\right\} \cap \Gamma^2 \setminus W_1\right) \leq \varpi. \tag{7.15}$$

Proof. Starting from (7.11), using the observation above and integrating, we obtain

$$\begin{aligned}
\varpi &\geq \int \left\{ \int \left[\pi_{\gamma'}\left(\left\{(\gamma',\gamma'') : \gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma\right\} \cap \Gamma^2 \setminus W_1\right)\right] \eta^{\mathrm{in}}_{\gamma(t^-_\gamma)}(d\gamma') \right\} \eta^{\mathrm{in}}(d\gamma) \\
&= \int \left\{ \int \left[\pi_{\gamma'}\left(\left\{(\gamma',\gamma'') : \gamma''(t^-_{\gamma''}) \neq \gamma(t^-_\gamma)\right\} \cap \Gamma^2 \setminus W_1\right)\right] \eta^{\mathrm{in}}_{\gamma(t^-_\gamma)}(d\gamma') \right\} \eta^{\mathrm{in}}(d\gamma) \\
&= \int \left\{ \int \left[\pi_{\gamma'}\left(\left\{(\gamma',\gamma'') : \gamma''(t^-_{\gamma''}) \neq \gamma'(t^-_{\gamma'})\right\} \cap \Gamma^2 \setminus W_1\right)\right] \eta^{\mathrm{in}}_z(d\gamma') \right\} \rho(z)\mathscr{H}^d(dz) \\
&= \pi\left(\left\{(\gamma',\gamma'') : \gamma''(t^-_{\gamma''}) \neq \gamma'(t^-_{\gamma'})\right\} \cap \Gamma^2 \setminus W_1\right),
\end{aligned}$$

where we have used the observation that if $\gamma, \gamma''$ start on $\partial\Omega$ then the condition $\gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma$ reduces to $\gamma''(t^-_{\gamma''}) \neq \gamma(t^-_\gamma) = \gamma'(t^-_{\gamma'}) = z$ by the domain of integration. □

Our goal now is to estimate in a quantitative way how much $\eta^{\rm cr}$ differs from a superposition of Dirac masses. This will be achieved using two main ingredients: on the one hand, we will use the estimates given by Proposition 7.2 and Proposition 7.3; on the other hand we will get rid of the divergence $\mu$ inside the domain $\Omega$ (which is the quantity which measures how many trajectories start or finish inside $\Omega$) playing with constants.

LEMMA 7.9. *It holds*

$$\int_{S_1} \big(\rho(z) - \rho^{\rm cr}(z)\big) \mathscr{H}^d(dz) \le \mu^-(\Omega).$$

PROOF. The balance of the divergence gives

$$\begin{aligned}\int_{S_1} \big(\rho(z) - \rho^{\rm cr}(z)\big) \mathscr{H}^d(dz) &= \int_{S_1} \big(1 - \|\eta^{\rm cr}_z\|\big)\rho(z) \mathscr{H}^d(dz) \\ &= \int_{S_1} \eta_z(\Gamma \setminus \Gamma^{\rm cr})\rho(z) \mathscr{H}^d(dz) \le \mu^-(\Omega),\end{aligned}$$

because the curves which enter in $S_1$ but do not exit from $\Omega$ necessarily have the final point $\gamma(t^+_\gamma)$ inside $\Omega$. □

Since clearly $\eta^{\rm cr} \le \eta^{\rm in}$, by Proposition 7.2 we deduce the estimate

$$\int_{S_1} \eta^{\rm cr}_z \otimes \eta^{\rm cr}_z(\mathit{\Gamma}^2 \setminus W_2)\rho(z)\mathscr{H}^d(dz) \le \varpi. \tag{7.16}$$

Observe now that, when we restrict to $\mathit{\Gamma}^{\rm cr}$, the following equality holds:

$$(\mathit{\Gamma}^{\rm cr})^2 \setminus W_2 = \big\{(\gamma,\gamma') \in (\mathit{\Gamma}^{\rm cr})^2 : \gamma \ne \gamma'\big\}.$$

Thus, we can rewrite (7.16) as

$$\int_{S_1} \eta^{\rm cr}_z \otimes \eta^{\rm cr}_z\big(\big\{(\gamma,\gamma') \in (\mathit{\Gamma}^{\rm cr})^2 : \gamma \ne \gamma'\big\}\big)\rho(z)\mathscr{H}^d(dz) \le \varpi. \tag{7.17}$$

To proceed further, we need the following elementary lemma.

LEMMA 7.10. *For any finite, non negative measure* $\mathsf{m}$ *on a Polish space* $Y$ *it holds*

$$\|\mathsf{m}\|\big(\|\mathsf{m}\| - \max_{y\in Y} \mathsf{m}(\{y\})\big) \le \mathsf{m} \otimes \mathsf{m}(\{(y,y') : y \ne y'\}).$$

*In particular, for probability measures*

$$1 - \max_{y\in Y} \mathsf{m}(\{y\}) \le \mathsf{m} \otimes \mathsf{m}(\{(y,y') : y \ne y'\}).$$

PROOF. Decompose

$$\mathsf{m} = \mathsf{m}^{\rm cont} + \sum_n c_n \delta_{y_n},$$

so that

$$\mathsf{m} \otimes \mathsf{m}(\{(y,y') : y \ne y'\}) = \|\mathsf{m}\|^2 - \sum_n c_n^2.$$

Assume that

$$n \mapsto c_n$$

is decreasing, and estimate

$$\sum_n c_n^2 \le c_1 \sum_n c_n \le c_1 \|\mathsf{m}\|.$$

Hence

$$\mathsf{m} \otimes \mathsf{m}(\{(y,y') : y \ne y'\}) \ge \|\mathsf{m}\|(\|\mathsf{m}\| - c_1),$$

with

$$c_1 = \max_n c_n$$

which is the claim. □

Combining Proposition 7.2 (which gives (7.17)) with Lemma 7.10, we deduce the following proposition.

PROPOSITION 7.11. *For any real constant $C > 1$, we have the estimate*

$$\int_{S_1} \big(\|\eta_z^{\rm cr}\| - \max_{\gamma\in\Gamma} \eta_z^{\rm cr}(\{\gamma\})\big)\rho(z)\,\mathscr{H}^d(dz) < C\varpi + \frac{\mu^-(\Omega)}{C-1}.$$

PROOF. Write for $C > 1$

$$\begin{aligned}
&\int_{S_1} \bigg\{ \frac{\eta_z^{\rm cr}}{\|\eta_z^{\rm cr}\|} \otimes \frac{\eta_z^{\rm cr}}{\|\eta_z^{\rm cr}\|} \big(\{(\gamma,\gamma') : \gamma \neq \gamma'\}\big) \bigg\} \rho^{\rm cr}(z)\,\mathscr{H}^d(dz) \\
&\quad = \bigg[ \int_{\rho^{\rm cr}\geq\rho/C} + \int_{\rho^{\rm cr}<\rho/C} \bigg] \bigg\{ \frac{\eta_z^{\rm cr}}{\|\eta_z^{\rm cr}\|} \otimes \eta_z^{\rm cr} \big(\{(\gamma,\gamma') : \gamma \neq \gamma'\}\big) \bigg\} \rho(z)\,\mathscr{H}^d \llcorner_{S_1}(dz) \\
&\quad \leq C \int_{\rho^{\rm cr}\geq\rho/C} \Big\{ \eta_z^{\rm cr} \otimes \eta_z^{\rm cr} \big((\Gamma^{\rm cr})^2 \setminus W_2\big) \Big\} \rho(z)\,\mathscr{H}^d \llcorner_{S_1}(dz) + \int_{\rho^{\rm cr}<\rho/C} \rho^{\rm cr}(z) \mathscr{H}^d \llcorner_{S_1}(dz) \\
&\quad < C\varpi + \frac{1}{C} \frac{\mu^-(\Omega)}{1-1/C},
\end{aligned}$$

where in the last passage we have used Lemma 7.9. Now the conclusion follows directly applying Lemma 7.10. □

From Proposition 7.11, we deduce that, up to a set of trajectories whose $\eta$-measure is controlled, the measure $\eta^{\rm cr}$ is essentially a superposition of Dirac deltas. More precisely, we can find a family of crossing trajectories $\Xi \subset \Gamma^{\rm cr}$ such that

$$\eta^{\rm cr}(\Gamma^{\rm cr} \setminus \Xi) < C\varpi + \frac{\mu^-(\Omega)}{C-1}$$

and

$$(\eta^{\Xi})_z = \eta_z^{\rm cr} \llcorner_{\Xi} := m_z \delta_{\gamma_z}, \qquad \gamma_z \in \Gamma^{\rm cr}. \tag{7.18}$$

This additional piece of information can be combined together with Proposition 7.2 in the following way.

Consider an admissible plan $\tilde\pi \in \mathsf{Adm}(\eta^{\Xi}, \eta^{\rm in})$. We have the following lemma.

LEMMA 7.12. *Let*

$$\mathcal{S} := \big\{(\gamma,\gamma') : \gamma(t_\gamma^-) = \gamma'(t_{\gamma'}^-)\big\} \subset \Gamma^2,$$

*i.e. the set of curves which start from the same point. Then*

$$\tilde\pi \llcorner_{\mathcal{S}} \big(\Gamma^2 \setminus W_2\big) \leq \varpi. \tag{7.19}$$

PROOF. By Disintegration Theorem (applied w.r.t. the map $\mathcal{S} \ni (\gamma,\gamma') \mapsto \gamma(t_\gamma^-)$), we have

$$\tilde\pi \llcorner_{\mathcal{S}} = \int_{S_1} \big(\tilde\pi \llcorner_{\mathcal{S}}\big)_z \rho(z)\,\mathscr{H}^d(dz),$$

where $(\tilde\pi \llcorner_{\mathcal{S}})_z \in \mathsf{Adm}(\eta_z^{\Xi}, \eta_z^{\rm in})$ for $\mathscr{H}^d$-a.e. $z \in S_1$. Being $\eta_z^{\Xi}$ the Dirac delta $m_z\delta_{\gamma_z}$ in view of (7.18), it follows that every transference plan in $\tilde\pi_z \in \mathsf{Adm}(\eta_z^{\Xi}, \eta_z^{\rm in})$ satisfies

$$\tilde\pi_z \leq \eta_z^{\Xi} \otimes \eta_z^{\rm in} \leq \eta_z^{\rm in} \otimes \eta_z^{\rm in},$$

so that Proposition 7.2 directly implies the statement. □

By summing up the results in Lemma 7.12 and Corollary 7.8 we deduce the following corollary.

COROLLARY 7.13. *For any admissible transport plan $\pi \in \mathsf{Adm}(\eta^{\rm cr}, \eta^{\rm in})$, it holds*

$$\pi(\Gamma^2 \setminus W) < 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1}.$$

PROOF. For any plan $\pi$ we have

$$\begin{aligned}\pi(\Gamma^2 \setminus W) &= \pi\big((\Xi \times \Gamma) \setminus W\big) + \pi\big(((\Gamma \setminus \Xi) \times \Gamma) \setminus W\big)\\ &\leq \pi((\Xi \times \Gamma) \setminus W) + \eta^{\mathrm{cr}}(\Gamma \setminus \Xi)\\ \text{by (7.18) and Proposition 7.11 } &\leq \pi((\Xi \times \Gamma) \setminus W) + C\varpi + \frac{\mu^-(\Omega)}{C-1}\\ &\leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1},\end{aligned}$$

where in the last line we have use the fact that $\pi \llcorner_{\Xi\times\Gamma} \in \mathsf{Adm}(\eta^{\Xi}, \eta^{\mathrm{in}})$ so that (7.15) and (7.19) give the estimate. □

Notice that we can rephrase Corollary 7.13 by saying that

$$\sup_{\pi \in \mathsf{Adm}(\eta^{\mathrm{cr}},\eta^{\mathrm{in}})} \pi(\Gamma^2 \setminus W) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1}. \tag{7.20}$$

for all $C > 1$.

Invoking the deep duality results of [Kel84] recalled in the preliminaries (Section V), we can prove the following

THEOREM 7.14. *There exist Borel sets $N_1 \subset \Gamma^{\mathrm{cr}}, N_2 \subset \Gamma^{\mathrm{in}}$ such that*

$$\eta^{\mathrm{cr}}(N_1) + \eta^{\mathrm{in}}(N_2) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1},$$

*and for every $(\gamma, \gamma') \in (\Gamma^{\mathrm{cr}} \setminus N_1) \times (\Gamma^{\mathrm{in}} \setminus N_2)$ either* $\mathrm{Graph}\,\gamma' \subset \mathrm{Graph}\,\gamma$ *or* $\mathrm{Graph}\,\gamma \cap \mathrm{Graph}\,\gamma' = \emptyset$.

Equivalently we can say that

$$(\Gamma^{\mathrm{cr}} \setminus N_1) \times (\Gamma^{\mathrm{in}} \setminus N_2) \subset W.$$

PROOF. Taking into account Theorem XV and Proposition XVI, we have that there exist Borel sets $N_1, N_2$ such that

$$\mathbb{1}_{N_1} + \mathbb{1}_{N_2} \geq \mathbb{1}_{(\Gamma^{\mathrm{cr}} \times \Gamma) \setminus W}$$

and

$$\eta^{\mathrm{cr}}(N_1) + \eta^{\mathrm{in}}(N_2) = \sup_{\pi \in \mathsf{Adm}(\eta^{\mathrm{cr}},\eta^{\mathrm{in}})} \pi(\Gamma^2 \setminus W) \overset{(7.20)}{\leq} 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1},$$

which is exactly the claim. □

Recall now that, so far, we have been working with $\Omega = \Omega^{\varepsilon}$, being $\Omega$ a proper set and $\Omega^{\varepsilon} \supset \Omega$ the perturbed set constructed in Proposition 6.14. In some sense, we now want to pass to the limit the above estimates as $\varepsilon \to 0$.

Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and $\eta$ be a Lagrangian representation of $\rho(1, \boldsymbol{b})\mathscr{L}^{d+1}$. Set

$$\Gamma^{\mathrm{cr}}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t^{\pm}_{\gamma}) \in \partial\Omega\big\}, \qquad \Gamma^{\mathrm{in}}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t^{-}_{\gamma}) \in \partial\Omega\big\}.$$

Assume that Theorem 7.14 holds for a family of perturbations $\Omega^{\epsilon_n}$ with constant $\varpi$.

THEOREM 7.15. *There exist $N_1 \subset \Gamma^{\mathrm{cr}}(\Omega), N_2 \subset \Gamma^{\mathrm{in}}(\Omega)$ such that*

$$(\mathtt{R}_{\Omega})_\sharp\eta^{\mathrm{cr}}(N_1) + (\mathtt{R}_{\Omega})_\sharp\eta^{\mathrm{in}}(N_2) \leq \inf_{C>1}\left\{2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1}\right\}$$

*and for every $(\gamma, \gamma') \in (\Gamma^{\mathrm{cr}} \setminus N_1) \times (\Gamma^{\mathrm{in}} \setminus N_2)$ either*

$$\mathrm{Graph}\,\gamma' \llcorner_{\mathrm{clos}\,\Omega} \subset \mathrm{Graph}\,\gamma \llcorner_{\mathrm{clos}\,\Omega} \quad \textit{or} \quad \mathrm{Graph}\,\gamma \llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\,\gamma' \llcorner_{\mathrm{clos}\,\Omega} = \emptyset.$$

PROOF. From Theorem 7.14 applied to every $\Omega^{\varepsilon_n}$, we obtain two sets $N_1^{\varepsilon_n}$ and $N_2^{\varepsilon_n}$ such that

$$(\mathtt{R}_{\Omega^{\epsilon_n}})_\sharp\eta^{\mathrm{cr}}(N_1^{\varepsilon_n}) + (\mathtt{R}_{\Omega^{\epsilon_n}})_\sharp\eta^{\mathrm{in}}(N_2^{\varepsilon_n}) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega^{\epsilon_n})}{C-1},$$

and for every $(\gamma,\gamma') \in (\Gamma^{\mathrm{cr}}(\Omega^{\varepsilon_n}) \setminus N_1^{\varepsilon_n}) \times (\Gamma^{\mathrm{in}}(\Omega^{\varepsilon_n}) \setminus N_2^{\varepsilon_n})$ either

$$\mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} \subset \mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} \quad \textit{or} \quad \mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} \cap \mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} = \emptyset.$$

Now $\mathtt{R}_\Omega(\Gamma^{\mathrm{cr}}(\Omega^{\varepsilon_n})) \subset \Gamma^{\mathrm{cr}}(\Omega)$ and

$$\big|(\mathtt{R}_\Omega)_\sharp\eta(\Gamma^{\mathrm{cr}}(\Omega)) - (\mathtt{R}_{\Omega^{\varepsilon_n}})_\sharp\eta(\Gamma^{\mathrm{cr}}(\Omega^{\varepsilon_n}))\big| < \mathcal{O}(\varepsilon_n)$$

from Theorem 6.15 and the estimates therein. In the same way, $\mathtt{R}_\Omega(\Gamma^{\mathrm{in}}(\Omega^{\varepsilon_n})) \subset \Gamma^{\mathrm{in}}(\Omega)$ and

$$\big|(\mathtt{R}_\Omega)_\sharp\eta(\Gamma^{\mathrm{in}}(\Omega)) - (\mathtt{R}_{\Omega^{\varepsilon_n}})_\sharp\eta(\Gamma^{\mathrm{in}}(\Omega^{\varepsilon_n}))\big| < \mathcal{O}(\varepsilon_n).$$

If we now consider the sets

$$\widetilde{N}_1^{\varepsilon_n} := \mathtt{R}_\Omega(N_1^{\varepsilon_n}) \cup \big(\Gamma^{\mathrm{cr}}(\Omega) \setminus \mathtt{R}_\Omega\big(\Gamma^{\mathrm{cr}}(\Omega^{\varepsilon_n})\big)\big) \quad \text{and} \quad \widetilde{N}_2^{\varepsilon_n} := \mathtt{R}_\Omega(N_2^{\varepsilon_n}) \cup \big(\Gamma^{\mathrm{in}}(\Omega) \setminus \mathtt{R}_\Omega\big(\Gamma^{\mathrm{in}}(\Omega^{\varepsilon_n})\big)\big),$$

we have by Corollary 6.33 as $\varepsilon_n \to 0$ that

$$(\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{cr}}(\widetilde{N}_1^{\varepsilon_n}) + (\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{in}}(\widetilde{N}_2^{\varepsilon_n}) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega^{\epsilon_n})}{C-1} + o(1) = 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1} + o(1),$$

and for every $(\gamma,\gamma') \in (\Gamma^{\mathrm{cr}}(\Omega) \setminus \widetilde{N}_1^{\varepsilon_n}) \times (\Gamma^{\mathrm{in}}(\Omega) \setminus \widetilde{N}_2^{\varepsilon_n})$ either

$$\mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega} \subset \mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega} \quad \text{or} \quad \mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega} = \emptyset.$$

In particular, it follows that

$$\inf\Big\{(\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{cr}}(N_1) + (\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{in}}(N_2) : (\Gamma^{\mathrm{cr}} \setminus N_1) \times (\Gamma^{\mathrm{in}} \setminus N_2) \subset W\Big\} \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1},$$

and we apply again Proposition XVI in order to find two actual minimizers. □

### 7.2. Untangling functional and untangled Lagrangian representations

This section is divided into two parts. In the first part, following the analysis of Theorem 7.15, we define two functionals on the family of proper sets which measure how much the trajectories used by a Lagrangian representation $\eta$ cross each other. The main result is that these functionals are subadditive, so that it seems natural to compare them with a measure $\varpi^\tau$. This is the main result of the second part, which shows that if one can bound the untangling functional in sufficiently many sets by a given measure, then we can have an estimate on how many trajectories one has to remove in order to obtain an untangled set of trajectories, i.e. trajectories which do not cross each other.

**7.2.1. Subadditivity of untangling functional.** For $\Omega \subset \mathbb{R}^{d+1}$ proper set we give the following definition.

DEFINITION 7.16. The *untangling functional for $\eta^{\mathrm{in}}$* is defined as

$$\mathfrak{f}^{\mathrm{in}}(\Omega) := \inf\Big\{(\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{cr}}(N_1) + (\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{in}}(N_2) :\ (\Gamma \setminus N_1) \times (\Gamma \setminus N_2) \subset W\Big\}. \tag{7.21}$$

Setting

$$(\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{out}} := \int_{\partial\Omega} \eta_z \rho(z)[(1,\boldsymbol{b}(z))\cdot\mathbf{n}(z)]^+ \mathscr{H}^d(dz),$$

we can define analogously the untangling functional for $\eta^{\mathrm{out}}$.

DEFINITION 7.17. The *untangling functional for $\eta^{\mathrm{out}}$* is defined as

$$\mathfrak{f}^{\mathrm{out}}(\Omega) := \inf\Big\{(\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{cr}}(N_1) + (\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{out}}(N_2) :\ (\Gamma \setminus N_1) \times (\Gamma \setminus N_2) \subset W\Big\}. \tag{7.22}$$

As noticed before, the condition $(\gamma,\gamma') \in (\Gamma^{\mathrm{cr}}(\Omega)\times\Gamma^{\mathrm{in}}(\Omega))\cap W$ is equivalent to say that

$$\text{either} \quad \operatorname{Graph}\gamma'\llcorner_{\operatorname{clos}\Omega}\subset \operatorname{Graph}\gamma\llcorner_{\operatorname{clos}\Omega} \quad \text{or} \quad \operatorname{Graph}\gamma\llcorner_{\operatorname{clos}\Omega}\cap \operatorname{Graph}\gamma'\llcorner_{\operatorname{clos}\Omega}=\emptyset,$$

and similarly for $(\gamma,\gamma')\in(\Gamma^{\mathrm{cr}}\times\Gamma^{\mathrm{out}})\cap W$. Recalling now Theorem 7.15 we can infer that the infima in (7.21) and (7.22) are actually minima.

We now show the following remarkable property of the untangling functionals:

Proposition 7.18. *The functionals $\mathfrak{f}^{\mathrm{in}}$ and $\mathfrak{f}^{\mathrm{out}}$ are subadditive on the class of proper sets. More precisely, if* $\mathrm{U},\mathrm{V}\subset\mathbb{R}^{d+1}$ *are proper sets whose union* $\Omega := \mathrm{U}\cup\mathrm{V}$ *is proper, then*

$$\mathfrak{f}^{\mathrm{in}}(\Omega)\le \mathfrak{f}^{\mathrm{in}}(\mathrm{U})+\mathfrak{f}^{\mathrm{in}}(\mathrm{V}),\qquad \mathfrak{f}^{\mathrm{out}}(\Omega)\le \mathfrak{f}^{\mathrm{out}}(\mathrm{U})+\mathfrak{f}^{\mathrm{out}}(\mathrm{V}).$$

Proof. We prove the assertion only for the functional $\mathfrak{f}^{\mathrm{in}}$, being the other case completely similar. By definition, there exist sets $N_1(\mathrm{U})\subset\Gamma^{\mathrm{cr}}(\mathrm{U})$ and $N_2(\mathrm{U})\subset\Gamma^{\mathrm{in}}(\mathrm{U})$ such that

$$\mathfrak{f}^{\mathrm{in}}(\mathrm{U}) = (\mathtt{R}_{\mathrm{U}})_\sharp\eta^{\mathrm{cr}}(N_1(\mathrm{U})) + (\mathtt{R}_{\mathrm{U}})_\sharp\eta^{\mathrm{in}}(N_2(\mathrm{U}))$$

and

$$\big(\Gamma^{\mathrm{cr}}(\mathrm{U})\setminus N_1(\mathrm{U})\big)\times\big(\Gamma^{\mathrm{in}}(\mathrm{U})\setminus N_2(\mathrm{U})\big)\subset W.$$

Let $N_1(\mathrm{V}), N_2(\mathrm{V})$ be a corresponding couple of sets for V. Set

$$N_1 := \big\{\gamma\in\Gamma^{\mathrm{cr}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{U}}\gamma\in N_1(\mathrm{U})\big)\big\}\cup\big\{\gamma\in\Gamma^{\mathrm{cr}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{V}}\gamma\in N_1(\mathrm{V})\big)\big\}$$

and

$$N_2 := \big\{\gamma\in\Gamma^{\mathrm{in}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{U}}\gamma\in N_2(\mathrm{U})\big)\big\}\cup\big\{\gamma\in\Gamma^{\mathrm{in}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{V}}\gamma\in N_2(\mathrm{V})\big)\big\}.$$

By Proposition 6.32

$$\begin{aligned}\eta(N_1)+\eta(N_2) \le{}& \eta\big(\big\{\gamma\in\Gamma^{\mathrm{cr}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{U}}(\gamma)\in N_1(\mathrm{U})\big)\big\}\big)\\ &+\eta\big(\big\{\gamma\in\Gamma^{\mathrm{cr}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{U}}(\gamma)\in N_2(\mathrm{U})\big)\big\}\big)\\ &+\eta\big(\big\{\gamma\in\Gamma^{\mathrm{cr}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{V}}(\gamma)\in N_1(\mathrm{V})\big)\big\}\big)\\ &+\eta\big(\big\{\gamma\in\Gamma^{\mathrm{cr}}(\Omega) : \exists i\,\big(\mathtt{R}^i_{\mathrm{V}}(\gamma)\in N_2(\mathrm{V})\big)\big\}\big)\\ \le{}&(\mathtt{R}_{\mathrm{U}})_\sharp\eta(N_1(\mathrm{U}))+(\mathtt{R}_{\mathrm{U}})_\sharp\eta(N_2(\mathrm{U}))+(\mathtt{R}_{\mathrm{V}})_\sharp\eta(N_1(\mathrm{V}))+(\mathtt{R}_{\mathrm{V}})_\sharp\eta(N_2(\mathrm{V}))\\ ={}&\mathfrak{f}^{\mathrm{in}}(\mathrm{U})+\mathfrak{f}^{\mathrm{in}}(\mathrm{V}).\end{aligned}$$

It remains to show $(\Gamma^{\mathrm{cr}}(\Omega)\setminus N_1)\times(\Gamma^{\mathrm{in}}(\Omega)\setminus N_2)\subset W$: this follows from the observation

$$\mathtt{R}_{\mathrm{U}}(\Gamma^{\mathrm{cr}}(\Omega))\subset\Gamma^{\mathrm{cr}}(\mathrm{U}),$$

and

$$\mathtt{R}_{\mathrm{U}}(\Gamma^{\mathrm{in}}(\Omega))\subset\Gamma^{\mathrm{in}}(\mathrm{U})$$

and the same for V. Hence, if $\operatorname{Graph}\gamma\llcorner_{\operatorname{clos}\Omega}\cap\operatorname{Graph}\gamma'\llcorner_{\operatorname{clos}\Omega}\neq\emptyset$ then they must coincide either in $\operatorname{clos}\mathrm{U}$ or $\operatorname{clos}\mathrm{V}$ and, by elementary arguments, in $\operatorname{clos}\mathrm{U}\cup\operatorname{clos}\mathrm{V}=\operatorname{clos}\Omega$. □

We conclude this paragraph with the following lemma, which shows that $\mathfrak{f}^{\mathrm{in}}$ and $\mathfrak{f}^{\mathrm{out}}$ are related.

Lemma 7.19. *It holds*

$$\mathfrak{f}^{\mathrm{in}}(\Omega)-\mu^-(\Omega)\le\mathfrak{f}^{\mathrm{out}}(\Omega)\le\mathfrak{f}^{\mathrm{in}}(\Omega)+\mu^+(\Omega)$$

*where we recall that $\mu^+,\mu^-$ are the positive/negative part of the measure $\mu=\operatorname{div}(\rho(1,\boldsymbol{b}))$.*

PROOF. We prove only $\mathfrak{f}^{\rm out}(\Omega) \leq \mathfrak{f}^{\rm in}(\Omega) + \mu^+(\Omega)$, the other case being analogous. Let $\eta$ be a Lagrangian representation of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1} \llcorner \Omega$, and $N_1$, $N_2$ a minimal couple for $\mathfrak{f}^{\rm in}$. Since

$$\eta^{\rm cr}(N_2) \leq \eta^{\rm in}(N_2),$$

then it follows that $(\Gamma^{\rm cr}(\Omega) \setminus (N_1 \cup N_2))^2 \subset W$. As already observed in Lemma 7.9,

$$\|\eta^{\rm out} - \eta^{\rm cr}\| \leq \mu^+(\Omega),$$

so that the conclusion follows by considering the couple $N_1' = N_1 \cup N_2$ and $N_2' = \{\gamma : \gamma(t_\gamma^-) \in \Omega\}$. □

**7.2.2. Untangled Lagrangian representations.** Assume the following:

ASSUMPTION 7.20. Let $\tau > 0$ and $C > 1$ be such that

(1) there exist $K^{\tau,\pm}$ compact sets satisfying

$$\mu^\pm(K^{\tau,\mp}) = 0, \qquad \mu^\pm\big(\mathbb{R}^{d+1} \setminus K^{\tau,\pm}\big) < \tau;$$

(2) there exists a positive measure $\varpi^\tau$ such that

(a) for all $(t,x) \in K^{\tau,-}$ there exists a family of proper balls $\{B^{d+1}_r(t,x)\}_r$ with 0 as Lebesgue density point and such that it holds

$$\mathfrak{f}^{\rm in}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{\mu^-(B^{d+1}_r(t,x))}{C-1},$$

(b) for all $(t,x) \in K^{\tau,+}$ there exists a family of proper balls $\{B^{d+1}_r(t,x)\}_r$ with 0 as Lebesgue density point and such that it holds

$$\mathfrak{f}^{\rm out}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{\mu^+(B^{d+1}_r(t,x))}{C-1},$$

(c) for all $(t,x) \in \mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ there exists a family of proper balls $\{B^{d+1}_r(t,x)\}_r$ with 0 as Lebesgue density point and such that it holds

$$\min\big\{\mathfrak{f}^{\rm in}(B^{d+1}_r(t,x)), \mathfrak{f}^{\rm out}(B^{d+1}_r(t,x))\big\} \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{|\mu|(B^{d+1}_r(t,x))}{C-1}.$$

By the choice of the sets $K^{\tau,\pm}$ we can have in a sufficiently small ball the following estimate.

PROPOSITION 7.21. *For every $(t,x) \in \mathbb{R}^{d+1}$ there exists $r_{t,x}$ such that for the families of balls $\{B^{d+1}_r(t,x)\}_r$ as above and for $r < r_{t,x}$ it holds*

$$\begin{aligned} \mathfrak{f}^{\rm in}(B^{d+1}_r(t,x)), \mathfrak{f}^{\rm out}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{|\mu|(B^{d+1}_r(t,x))}{C-1} \\ + \frac{C}{C-1}|\mu|(B^{d+1}_r(t,x) \setminus K^{\tau,+} \cup K^{\tau,-}). \end{aligned} \tag{7.23}$$

PROOF. It $(t,x) \in K^{\tau,-}$, then by Point (2a) of Assumption 7.20

$$\mathfrak{f}^{\rm in}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{\mu^-(B^{d+1}_r(t,x))}{C-1},$$

and since $(t,x) \in K^{\tau,-}$, by Point (1) we can take $r \ll 1$ such that

$$\mu^+(B^{d+1}_r(t,x)) \leq \frac{\mu^-(B^{d+1}_r(t,x))}{C-1}.$$

One thus applies the Lemma 7.19 above. A completely similar computation holds for $K^+$.

For points in the open set $\mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ just take a ball $B^{d+1}_r(t,x) \subset \mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ and combine Point (2c) and Lemma 7.19. □

For future reference let us define the measure

$$\zeta_C^\tau := (C+2)\varpi^\tau + \frac{|\mu|}{C-1} + \frac{C}{C-1}|\mu|\llcorner_{\mathbb{R}^{d+1}\setminus K^{\tau,+}\cup K^{\tau,-}}.$$

A covering argument yields the following global estimate.

Corollary 7.22. *If $\Omega \subset \mathbb{R}^{d+1}$ is a proper set with compact closure, then*

$$\mathfrak{f}^{\mathrm{in}}(\Omega), \mathfrak{f}^{\mathrm{out}}(\Omega) \leq C_d \zeta_C^\tau(\operatorname{clos}\Omega), \tag{7.24}$$

*where $C_d$ is a dimensional constant.*

Proof. Thanks to Proposition 7.21 and Vitali Theorem, for any $\varepsilon > 0$, we can cover the compact set $\operatorname{clos}\Omega$ with finitely many proper balls $B_i$ such that the estimates (7.23) hold and

$$\sum_i \zeta_C^\tau(B_i) \leq C_d \zeta_C^\tau(\operatorname{clos}\Omega) + \varepsilon.$$

Thanks to the subadditivity (and the monotonicity) of $\mathfrak{f}^{\mathrm{in}}$ we can thus write

$$\mathfrak{f}^{\mathrm{in}}(\Omega) \leq \mathfrak{f}^{\mathrm{in}}\bigg(\bigcup_i B_i\bigg) \leq \sum_i \mathfrak{f}^{\mathrm{in}}(B_i) \leq C_d\zeta_C^\tau(\operatorname{clos}\Omega) + \varepsilon.$$

Sending $\varepsilon \to 0$ we obtain (7.24). The same proof holds for the functional $\mathfrak{f}^{\mathrm{out}}$. □

Let now $N \subset \Gamma$ be a set such that

$$(\Gamma \setminus N)^2 \subset \mathring{W},$$

where

$$\mathring{W} = \Big\{(\gamma,\gamma') : \operatorname{Graph}\gamma\llcorner_{(t_\gamma^-,t_\gamma^+)} \cap \operatorname{Graph}\gamma'_{(t_{\gamma'}^-,t_{\gamma'}^+)} = \emptyset\Big\}$$
$$\cup \Big\{(\gamma,\gamma') : \operatorname{Graph}\gamma \cap \operatorname{Graph}\gamma' = \operatorname{Graph}\big(\gamma\llcorner_{[\max\{t_\gamma^-,t_{\gamma'}^-\},\min\{t_\gamma^+,t_{\gamma'}^+\}]}\big)\Big\}.$$

In the last part of this section we want estimate the measure $\eta(N)$ in terms of $\zeta_C^\tau(\mathbb{R}^{d+1}) = \|\zeta_C^\tau\|$. To this aim, define the compact sets (recall we consider solutions in a bounded domain)

$$\mathcal{K}^n := \big\{\gamma \in \Gamma : t_\gamma^+ - t_\gamma^- \geq 2^{1-n}\big\}$$

and observe that, given $\varepsilon > 0$ there exists $n \gg 1$ such that

$$\eta(\Gamma \setminus \mathcal{K}^n) \leq \epsilon.$$

If $(\gamma,\gamma') \in \Gamma^2 \setminus \mathring{W}$, then there exists $n \in \mathbb{N}$ such that $\gamma, \gamma' \in \mathcal{K}^n$ and

$$\operatorname{Graph}\gamma\llcorner_{[t_\gamma^- + 2^{-n}, t_\gamma^+ - 2^{-n}]} \cap \operatorname{Graph}\gamma' \neq \emptyset, \tag{7.25a}$$

$$\sup\Big\{|\gamma(t) - \gamma'(t)|, t \in \big[\max\{t_\gamma^- + 2^{-n}, t_{\gamma'}^-\}, \min\{t_\gamma^+ - 2^{-n}, t_{\gamma'}^+\}\big]\Big\} > 0, \tag{7.25b}$$

so that we can write

$$\Gamma^2 \setminus \mathring{W} = \bigcup_n Z^n$$

where

$$Z^n := \big\{(\gamma,\gamma') \in (\mathcal{K}^n)^2 : (7.25) \text{ holds}\big\}.$$

Now consider a covering of the compact set

$$K^n := \bigcup_{\gamma \in \mathcal{K}^n} \operatorname{Graph}\gamma\llcorner_{[t_\gamma^- + 2^{-n}, t_\gamma^+ - 2^{-n}]}$$

made up of finitely many proper balls $B_i := B^{d+1}_{r_i}(t_i, x_i)$ with radius less than $2^{-n}$, for which Proposition 7.21 holds together with $\zeta^\tau_C(\partial B_i) = 0$, and define

$$O^n := \bigcup_i B_i.$$

We now have the following lemma, whose proof is elementary.

LEMMA 7.23. *If* $(\gamma, \gamma') \in Z^n$ *then*

*a) if* $\mathrm{Graph}\,\gamma \cap B_i \neq \emptyset$ *then* $\mathtt{R}_{B_i}\gamma \in \Gamma^{\mathrm{cr}}(B_i)$*;*
*b) if* $\mathrm{Graph}\,\gamma' \cap B_i \neq \emptyset$ *then* $\mathtt{R}_{B_i}\gamma \in \Gamma^{\mathrm{in}}(B_i) \cup \Gamma^{\mathrm{out}}(B_i)$*;*
*c) there exists* $i$ *such that* $(\mathtt{R}_{B_i}\gamma, \mathtt{R}_{B_i}\gamma') \notin W$*.*

Applying Corollary 7.22, we obtain $N^n_1 \subset \mathcal{K}^n$ and $N^n_2 \subset \mathcal{K}^n$ such that

$$\eta(N^n_1) + \eta(N^n_2) \leq C_d \zeta^\tau_C(\mathrm{clos}\, O^n) = C_d \zeta^\tau_C(O^n)$$

and

$$\mathtt{R}_{\mathrm{clos}\, O^n}(\mathcal{K}^n \setminus N^n_1) \times \mathtt{R}_{\mathrm{clos}\, O^n}(\mathcal{K}^n \setminus N^n_2) \subset \Gamma^2 \setminus Z^n.$$

Now send $n \to +\infty$ with the same reasoning of Theorem 7.14 we finally obtain the following result.

THEOREM 7.24. *There exists a set* $N \subset \Gamma$ *such that*

$$\eta(N) \leq C_d \zeta^C_\tau(\mathbb{R}^{d+1})$$

*and*

$$(\Gamma \setminus N)^2 \subset \mathring{W}.$$

The following definition seems now natural:

DEFINITION 7.25. A Lagrangian representation $\eta$ is called *untangled* if there exists a set $\Delta \subset \Gamma$ such that

a) $\Delta \times \Delta \subset \mathring{W}$ and
b) $\eta$ is concentrated on $\Delta$.

By inner regularity we can assume $\Delta$ to be $\sigma$-compact. We conclude by pointing out the following important point.

COROLLARY 7.26. *Suppose there exist sequences* $\tau_i \searrow 0$ *and* $C_i \nearrow +\infty$ *such that Assumption 7.20 holds for* $\tau_i, C_i$ *and moreover*

$$C_i \|\varpi^{\tau_i}\| \to 0.$$

*Then* $\eta$ *is untangled.*

PROOF. It is enough to observe that $\zeta^{\tau_i}_{C_i} \to 0$. □

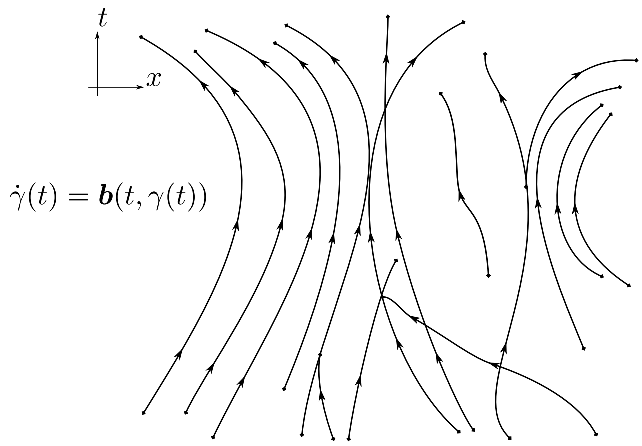
Notice that the assumptions of the above corollary are satisfied if one assumes that in each point of the compact sets $K^{\tau,\pm}$ (of Point (1) of Assumption 7.20) there exists a family of proper balls $B_r$ such that Assumption 7.1 or Assumption 7.6 holds in $B_r$ (with arbitrarily small $\tau$): basically, we are replacing the assumption of the control of the functionals with the existence of (local) cylinders of approximate flow. The precise assumptions reads as follows:

ASSUMPTION 7.27. For all $\tau > 0$

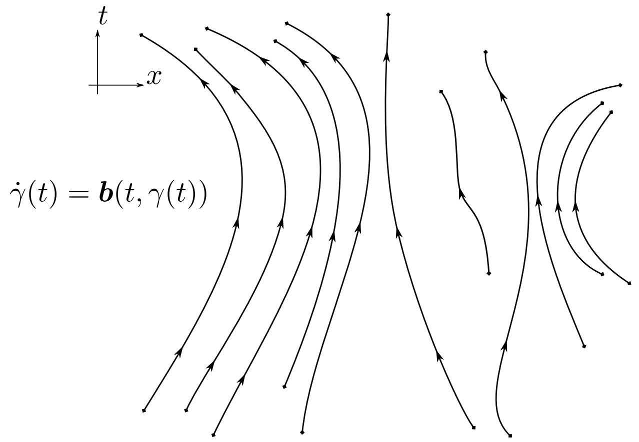
(1) there exist $K^{\tau,\pm}$ compact sets such that

$$\mu^\pm(\mathbb{R}^{d+1} \setminus K^{\tau,\pm}) < \tau;$$

(2) there exists a measure $\varpi^\tau$ such that




**(a)** Initial configuration: the curves may intersect several times, overlap and bifurcate.




**(b)** Final configuration: after the untangling, the curves are disjoint, thus forming a partition $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}}$ of $\mathbb{R}^{d+1}$ up to a set $\rho\mathscr{L}^{d+1}$-negligible.

**Figure 3.** Visual effect of the *untangling* of trajectories: we start by removing locally a set of curves, whose $\eta$ measure is controlled, in such a way that the curves are disjoint in a small ball. Iterating this step - thanks to subadditivity - we end up with a family of disjoint, untangled trajectories.

(a) for all $(t, x) \in K^{\tau,-}$ there exists a family of proper balls $\{B^{d+1}_r(t, x)\}_r$ with 0 as Lebesgue density point and such that Assumption 7.1 or Assumption 7.6 holds forward in $B^{d+1}_r(t, x)$,

(b) for all $(t, x) \in K^{\tau,+}$ there exists a family of proper balls $\{B^{d+1}_r(t, x)\}_r$ with 0 as Lebesgue density point and such that Assumption 7.1 or Assumption 7.6 holds backward in $B^{d+1}_r(t, x)$,

(c) for all $(t, x) \in \mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ there exists a family of proper balls $\{B^{d+1}_r(t, x)\}_r$ with 0 as Lebesgue density point and such that Assumption 7.1 or Assumption 7.6 holds either backward or forward in $B^{d+1}_r(t, x)$;

(3) it holds $\|\varpi^\tau\| \leq \tau$.

Indeed, for all $(t,x) \in K^-$, by Theorem 7.15 and monotonicity of $\mathfrak{f}^{\mathrm{in}}$, for $\mathscr{L}^1$-a.e. proper balls $B_r^{d+1}(t,x)$ of the family and for all $C > 1$ it holds

$$\mathfrak{f}^{\mathrm{in}}(B_r^{d+1}(t,x)) \leq (C+2)\varpi^\tau(B_r^{d+1}(t,x)) + \frac{\mu^-(B_r^{d+1}(t,x))}{C-1}.$$

The other cases are completely similar. The choice $C = \tau^{-1/2}$ thus suffices.

REMARK 7.28. We point out that one can consider also the equation $\mathrm{div}(\rho \boldsymbol{b}) = \mu \in \mathscr{M}(\mathbb{R}^d)$, denoting by div the divergence operator in the spatial variables only. By techniques similar to the ones used in Chapter 4 (essentially Lemma 4.8 in the opposite direction) one can obtain the untangling of the trajectories of the Lagrangian representation of $\rho\boldsymbol{b}$ from the corresponding statements of $(1, \rho\boldsymbol{b})$. ♠

## 7.3. Partition via characteristics and consequences

In this section we use the assumption that the representation $\eta$ is untangled to show that a partition of $\mathbb{R}^{d+1}$ made of characteristics $\wp_\alpha$ such that each $\gamma$ is a subset of these. By disintegrating w.r.t. this partition one can show that the PDE reduces to a one-dimensional ODE with measure r.h.s., and thus a complete description of the solution can be obtained. Moreover, if $\rho' \in L^\infty(\rho\mathscr{L}^{d+1})$ solves $\mathrm{div}(\rho'(1,\boldsymbol{b})) = \mu'$, then the trajectories of its Lagrangian representation $\eta'$ are subsets of the same partition $\wp_\alpha$. In particular the explicit form of distribution $\mathrm{div}(\beta(\rho)(1,\boldsymbol{b}))$ is obtained, settling the Chain Rule Problem.

**7.3.1. Construction of the partition and disintegration.** Let $\eta$ be an untangled Lagrangian representation and $\varDelta$ a $\sigma$-compact set as in Definition 7.25, and consider the following relation $\varDelta$:

$$\gamma \sim \gamma' \quad \Longleftrightarrow \quad \exists N \in \mathbb{N}, \{\gamma_i\}_{i=1}^N \subset \varDelta : \ \Big(\gamma = \gamma_1, \gamma_N = \gamma' \ \wedge \ \sharp\big(\mathrm{Graph}\,\gamma \cap \mathrm{Graph}\,\gamma'\big) > 1\Big).$$

It is standard to check that this is an equivalence relation: let $E_\mathfrak{a}$, $\mathfrak{a} \in \mathfrak{A}$, be the equivalence classes, being $\mathfrak{A}$ an appropriate set of indexes. Define now $\wp_\mathfrak{a}$ as the curve defined in an open interval of time whose graph is

$$\mathrm{Graph}\,\wp_\mathfrak{a} := \bigcup_{\gamma \in E_\mathfrak{a}} \mathrm{Graph}\,\gamma\llcorner_{(t_\gamma^-, t_\gamma^+)}.$$

One can check that $\wp_\mathfrak{a}$ is an absolutely continuous curve in $\varGamma$ for every $\mathfrak{a}$ and furthermore it holds

$$\mathrm{Graph}\,\wp_\mathfrak{a} \cap \mathrm{Graph}\,\wp_{\mathfrak{a}'} = \emptyset$$

for every $\mathfrak{a} \neq \mathfrak{a}'$ (see also Figure 3). We now show that the partition induced by the equivalence classes of this relation is a Borel partition, according to the following

PROPOSITION 7.29. *There exists a Borel map* $\mathtt{f} \colon \mathbb{R}^{d+1} \to \mathbb{R}$ *such that* $\mathtt{f}^{-1}(\mathfrak{a}) = \mathrm{Graph}\,\wp_\mathfrak{a}$.

PROOF. It is enough to construct the map restricted to the set of curves $\wp_\mathfrak{a}$ whose interval of existence contains a fixed time $\bar t$: by repeating the process for a countable set of times one constructs the map in the general case.

The equivalence classes intersecting $A \subset \{t = \bar t\}$ can be written as

$$\mathtt{S}(A) = \bigcup_n \mathtt{S}_n(A),$$

where $\mathtt{S}_0(A) = A$ and recursively

$$\mathtt{S}_n(A) = \big\{\gamma \in \varDelta : \mathrm{Graph}\,\gamma\llcorner_{(t_\gamma^-, t_\gamma^+)} \cap \mathtt{S}_{n-1}(A)\big\}.$$

Being the valuation map $\gamma \mapsto e_t(\gamma) = \gamma(t)$ continuous, it follows that each $\mathtt{S}_n(A)$ is Borel if $A$ is Borel, and then the conclusion follows. □

Using again that the evaluation map is Borel, we deduce also

COROLLARY 7.30. *There exists a Borel map* $\widehat{\mathtt{f}}\colon \Delta \to \mathbb{R}$ *such that* $\widehat{\mathtt{f}}^{-1}(\mathfrak{a}) = E_{\mathfrak{a}}$.

7.3.1.1. *Disintegration.* Having at our disposal a partition of the space-time into trajectories, one can try to disintegrate the equation $\operatorname{div}(\rho(1, \boldsymbol{b})) = \mu$ over this partition obtaining a family of one-dimensional equations (like in the Hamiltonian setting examined in Chapter 4): this is the aim of this paragraph.

First, using the fact that $\widehat{\mathtt{f}}$ is a Borel map, we can disintegrate $\eta$ w.r.t. the measure $m := \widehat{\mathtt{f}}_\sharp \eta$, so that we write:

$$\eta = \int_{\mathfrak{A}} \eta_{\mathfrak{a}}\, m(d\mathfrak{a})$$

with the property that, for $m$-a.e. $\mathfrak{a} \in \mathfrak{A}$ the measure $\eta_{\mathfrak{a}}$ is concentrated on $\operatorname{Graph} \wp_{\mathfrak{a}}$. Recall that, by definition of Lagrangian Representation 3.6, it holds

$$\rho \mathscr{L}^{d+1} = \int_{\Gamma} \Big( (\mathbb{I}, \gamma)_\sharp \mathscr{L}^1 \Big)\, \eta(d\gamma), \qquad \mu = \int_{\Gamma} \big( \delta_{(\mathbb{I},\gamma)(t_\gamma^-)} - \delta_{(\mathbb{I},\gamma)(t_\gamma^+)} \big)\, \eta(d\gamma).$$

Thus, we have

$$\begin{aligned} \rho \mathscr{L}^{d+1} &= \int_{\mathfrak{A}} \bigg[ \int_{\Gamma} \Big( (\mathbb{I}, \gamma)_\sharp \mathscr{L}^1 \Big) \eta_\alpha(d\gamma) \bigg] m(d\mathfrak{a}), \\ \mu &= \int_{\mathfrak{A}} \bigg[ \int_{\Gamma} \big( \delta_{(\mathbb{I},\gamma)(t_\gamma^-)} - \delta_{(\mathbb{I},\gamma)(t_\gamma^+)} \big) \eta_\alpha(d\gamma) \bigg]\, m(d\mathfrak{a}). \end{aligned}$$

Using the property that for $m$-a.e. $\mathfrak{a} \in \mathfrak{A}$ the measure $\eta_{\mathfrak{a}}$ is concentrated on $\operatorname{Graph} \wp_{\mathfrak{a}}$ we have, by Fubini Theorem, for any bounded continuous function $\varphi$

$$\begin{aligned} &\iint_{\mathbb{R}^+ \times \mathbb{R}^d} \varphi(t,x) \rho(t,x)\, \mathscr{L}^{d+1}(dt\, dx) \\ &= \int_{\mathfrak{A}} \bigg[ \int_{\Gamma} \int_{t_\gamma^-}^{t_\gamma^+} \varphi(t, \gamma(t))\, \mathscr{L}^1(dt)\, \eta_{\mathfrak{a}}(d\gamma) \bigg]\, m(d\mathfrak{a}) \\ &= \int_{\mathfrak{A}} \bigg[ \iint_{\mathbb{R}^+ \times \Gamma} \varphi(t, \gamma(t)) \mathbb{1}_{(t_\gamma^-, t_\gamma^+)}(t)\, \mathscr{L}^1 \times \eta_{\mathfrak{a}}(dt\, d\gamma) \bigg]\, m(d\mathfrak{a}) \\ &= \int_{\mathfrak{A}} \bigg[ \int_{\mathbb{R}^+} \varphi(t, \wp_{\mathfrak{a}}(t)) \bigg( \int_{\Gamma} \mathbb{1}_{(t_\gamma^-, t_\gamma^+)}(t) \eta_{\mathfrak{a}}(d\gamma) \bigg)\, \mathscr{L}^1(dt) \bigg]\, m(d\mathfrak{a}) \\ &= \int_{\mathfrak{A}} \bigg[ \int_{\mathbb{R}^+} \varphi(t, \wp_{\mathfrak{a}}(t)) w_{\mathfrak{a}}(t)\, \mathscr{L}^1(dt) \bigg]\, m(d\mathfrak{a}) \end{aligned}$$

where we have set

$$w_{\mathfrak{a}}(t) := \int_{\Gamma} \mathbb{1}_{(t_\gamma^-, t_\gamma^+)}(t) \eta_{\mathfrak{a}}(d\gamma) = \eta_{\mathfrak{a}}\big( \{ \gamma \in \Gamma :\, \gamma \text{ is defined in } t, \text{ i.e. } t \in (t_\gamma^-, t_\gamma^+) \} \big).$$

Thus, in view of the computation above we have obtained the following decomposition for $\rho \mathscr{L}^{d+1}$:

$$\rho \mathscr{L}^{d+1} = \int_{\mathfrak{A}} (\mathbb{I}, \wp_{\mathfrak{a}})_\sharp (w_{\mathfrak{a}} \mathscr{L}^1)\, m(d\mathfrak{a}). \tag{7.26}$$

In a similar fashion, we define for $\mu$

$$\mu_{\mathfrak{a}} := \int_{\Gamma} \Big[ \delta_{(\mathbb{I},\gamma)(t_\gamma^-)} - \delta_{(\mathbb{I},\gamma)(t_\gamma^+)} \Big] \eta_{\mathfrak{a}}(d\gamma),$$

so that

$$\mu = \int_{\mathfrak{A}} \mu_{\mathfrak{a}}\, m(d\mathfrak{a}) \tag{7.27}$$

Notice that the above formula is not a disintegration of $\mu$ because the sets of starting and ending points may be not disjoint in general. However, there is no cancellation of mass, since it holds

$$|\mu| = \int_{\mathfrak{A}} |\mu_{\mathfrak{a}}|\, m(d\mathfrak{a}),$$

consequence of the fact that $\mu^\pm$ are orthogonal and $\eta$ is a Lagrangian representation. By putting together the equation $\operatorname{div}(\rho(1,\boldsymbol{b})) = \mu$ with the decompositions (7.26) and (7.27), we thus have proved the following

PROPOSITION 7.31. *There exists a measure $m$ on the set $\mathfrak{A}$ such that the decompositions* (7.26) *and* (7.27) *hold and*

$$\frac{d}{dt} w_{\mathfrak{a}} = \mu_{\mathfrak{a}}, \qquad \textit{for } m\textit{-a.e. } \mathfrak{a} \in \mathfrak{A}, \tag{7.28}$$

*where we consider $w_{\mathfrak{a}}$ extended to 0 outside the domain of $\wp_{\mathfrak{a}}$.*

Since it will be useful later, we want to give a special name to the partitions of the space-time on which one can split the equation $\operatorname{div}(\rho(1,\boldsymbol{b})) = \mu$ as in Proposition 7.31.

DEFINITION 7.32. We will call a Borel map $\mathtt{g}\colon \mathbb{R}^{d+1} \to \mathfrak{A}$ a *partition via characteristics* of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}$ if:

- $\wp_{\mathfrak{a}} := \mathtt{g}^{-1}(\mathfrak{a})$ is a characteristic in some open domain $I_{\mathfrak{a}}$;
- if $\widehat{\mathtt{g}}$ denotes the corresponding map $\widehat{\mathtt{g}}\colon \varDelta \to \mathfrak{A}$, $\widehat{\mathtt{g}}(\gamma) := \mathtt{g}(\operatorname{Graph}\gamma)$, setting $m := \widehat{\mathtt{g}}_\sharp \eta$ and letting $w_{\mathfrak{a}}$ be the disintegration
$$\rho\,\mathscr{L}^{d+1} = \int_{\mathfrak{A}} (\mathbb{I}, \wp_{\mathfrak{a}})_\sharp (w_{\mathfrak{a}} \mathscr{L}^1)\, m(d\mathfrak{a})$$
then
$$\frac{d}{dt} w_{\mathfrak{a}} = \mu_{\mathfrak{a}} \in \mathscr{M}(\mathbb{R}), \qquad \text{for } m\text{-a.e. } \mathfrak{a} \in \mathfrak{A},$$
where $w_{\mathfrak{a}}$ is considered extended to 0 outside the domain of $\wp_{\mathfrak{a}}$;
- it holds
$$\mu = \int_{\mathfrak{A}} (\mathbb{I}, \wp_{\mathfrak{a}})_\sharp \mu_{\mathfrak{a}}\, m(d\mathfrak{a}) \qquad \text{and} \qquad |\mu| = \int_{\mathfrak{A}} (\mathbb{I}, \wp_{\mathfrak{a}})_\sharp |\mu_{\mathfrak{a}}|\, m(d\mathfrak{a}).$$

We will say the partition is *minimal* if moreover

$$\lim_{t\to \bar{t}\pm} w_{\mathfrak{a}}(t) > 0 \qquad \forall \bar{t} \in I_{\mathfrak{a}}.$$

Thus, one can rephrase Proposition 7.31 by saying that the map $\mathtt{f}$ is a partition via characteristics of $\rho(1,\boldsymbol{b})$. Moreover, taking into account the BV regularity of the functions $w_{\mathfrak{a}}$ (for $m$-a.e. $\mathfrak{a} \in \mathfrak{A}$, in view of (7.28)), we have that $\mathtt{f}$ is also a *minimal* partition via characteristics.

THEOREM 7.33. *There exists a minimal partition via characteristics of $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}$.*

PROOF. From Proposition 7.31, we get $w_{\mathfrak{a}} \in \mathrm{BV}(\mathbb{R})$ for $m$-a.e. $\mathfrak{a} \in \mathfrak{A}$: hence, we can decompose $\mathbb{R}$ into countably many open intervals $I^n_{\mathfrak{a}} := (t^{n,-}_{\mathfrak{a}}, t^{n,+}_{\mathfrak{a}})$, with $n \in \mathbb{N}$, such that $w_{\mathfrak{a}} > 0$ in each $I^n_{\mathfrak{a}}$ and

$$\lim_{t\to (t^{n,+}_{\mathfrak{a}})-} w_{\mathfrak{a}}(t) = 0 \qquad \text{or} \qquad \lim_{t\to (t^{n,-}_{\mathfrak{a}})+} w_{\mathfrak{a}}(t) = 0.$$

Accordingly, we can define a new partition by further decomposing $\wp_{\mathfrak{a}}$ into countably many curves $\wp^n_{\mathfrak{a}} := \wp_{\mathfrak{a}}|_{I^n_{\mathfrak{a}}}$. By construction, this new partition is again a partition via characteristics of $\rho(1,\boldsymbol{b})$ and it is indeed minimal. □

**7.3.2. Uniqueness of partition via characteristics and consequences.** Having proved *existence* of a minimal partition via characteristics of a vector field of the form $\rho(1,\boldsymbol{b})$, with $\operatorname{div}(\rho(1,\boldsymbol{b})) = \mu \in \mathscr{M}$, we now face the problem of *uniqueness* of such partition. In this Section, we will show that the partition constructed in Theorem 7.33 is *unique* in a suitable sense, provided every Lagrangian representation of $\rho(1,\boldsymbol{b})$ is untangled. More precisely, assume that $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}$ satisfies Assumption 7.27, and consider $\rho' \in L^\infty(\rho\mathscr{L}^{d+1})$ with

$$\operatorname{div}\big(\rho'(1,\boldsymbol{b})\big) = \mu'.$$

Without loss of generality, being $\rho' \in L^\infty(\rho \mathscr{L}^{d+1})$, we can assume that $|\rho'| \leq \frac{\rho}{2}$ so that

$$\frac{\rho}{2} \leq \rho + \rho' \leq \frac{3\rho}{2}. \tag{7.29}$$

Let $\eta'$ be a Lagrangian representation of $(\rho+\rho')(1,\boldsymbol{b})$, which exists because $\rho+\rho' \geq 0$. We now repeat the analysis above considering $(\rho+\rho')(1,\boldsymbol{b})\mathscr{L}^{d+1}$: notice that, in view of the bounds (7.29), the vector field $(\rho+\rho')(1,\boldsymbol{b})\mathscr{L}^{d+1}$ still satisfies Point 2 of Assumption 7.27 if $\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}$ does: indeed, the lateral flux of $\rho+\rho'$ (in Assumption 7.1) is controlled by $3/2$ of the lateral flux of $\rho$.

As before, we thus find a partition of $\mathbb{R}^{d+1}$ (up to a $\rho\mathscr{L}^{d+1}$-null set) into classes $(\widetilde{\wp}_{\mathfrak{b}})_{\mathfrak{b}\in\mathfrak{B}}$. If now we consider the function $u \in L^\infty$ such that $u\rho = \rho + \rho'$ we have

$$\operatorname{div}\big(u\rho(1,\boldsymbol{b})\big) = \mu + \mu' =: \nu.$$

By applying Proposition 7.31 with the classes $\widetilde{\wp}_{\mathfrak{b}}$ we deduce

$$u\rho\,\mathscr{L}^{d+1} = \int (\mathbb{I}, \widetilde{\wp}_{\mathfrak{b}})_\sharp \big(u \circ \widetilde{\wp}_{\mathfrak{b}} w_{\mathfrak{b}}\, \mathscr{L}^1\big) m(d\mathfrak{b}), \qquad \nu = \int (\mathbb{I}, \widetilde{\wp}_{\mathfrak{b}})_\sharp \nu_{\mathfrak{b}}\, m(d\mathfrak{b}),$$

and

$$\frac{d}{dt}(u_{\mathfrak{b}} w_{\mathfrak{b}}) = \nu_{\mathfrak{b}}, \qquad \text{where } u_{\mathfrak{b}} := u \circ \wp_{\mathfrak{b}}.$$

Notice that the density $w_{\mathfrak{b}}$ appearing in the disintegration is controlled (up to constants) from below and from above by $w_{\mathfrak{a}}$ in view of (7.29). This means that the graph of the classes $\wp_{\mathfrak{b}}$ graph contains the graph of the equivalence relation induced by $\wp_\alpha$, i.e. it has to hold

$$\widetilde{\wp}_{\mathfrak{b}} = N_{\mathfrak{b}} \cup \bigcup_n \wp_{\mathfrak{a}_n^{\mathfrak{b}}},$$

where $N_{\mathfrak{b}}$ is a possibly non-empty closed set. Furthermore, it holds

$$w_{\mathfrak{b}} = \sum_n w_{\mathfrak{a}_n^{\mathfrak{b}}} \qquad \text{and} \qquad \text{Tot.Var.}(w_{\mathfrak{b}}) = \sum \text{Tot.Var.}(w_{\mathfrak{a}_n^{\mathfrak{b}}})$$

because $\wp_{\mathfrak{a}}$ is a partition via characteristics. Then since $u_{\mathfrak{b}} \in L^\infty$ and $w_{\mathfrak{b}} > 0$ inside $I_{\mathfrak{a}_n^{\mathfrak{b}}}$, it follows that $u \circ \widetilde{\wp}_{\mathfrak{b}}$ is BV and at the endpoints

$$\liminf_{t\to\bar t} |u_{\mathfrak{b}} w_{\mathfrak{b}}| \leq \|u\|_\infty \liminf_{t\to\bar t} |w_{\mathfrak{b}}| = 0.$$

Then it is fairly easy to see that

$$\text{Tot.Var.}(u_{\mathfrak{b}} w_{\mathfrak{b}}) = \sum_n \text{Tot.Var.}(u_{\mathfrak{b}} w_{\mathfrak{a}_n^{\mathfrak{b}}})$$

and thus we conclude with the following universality result.

Theorem 7.34. *If $\rho' \in L^\infty(\rho\mathscr{L}^{d+1})$ then the map $\mathtt{f}$ is a partition via characteristics of $\rho'(1,\boldsymbol{b})\mathscr{L}^{d+1}$.*

In particular one can deduce that

Corollary 7.35. *The minimal partition of characteristic is unique up to a $\eta$-negligible set of trajectories.*

Proof. The set of equivalence classes must be the same up to $\eta$-negligible sets, because every representation is untangled. Being the $\mu_{\mathfrak{a}}$ determined up to $m$-negligible sets, it follows that $\wp_{\mathfrak{a}}$ are uniquely determined too, and the in particular the intervals where $w_{\mathfrak{a}} > 0$. ∎

7.3.2.1. *Chain rule.* Using Vol'pert's Chain Rule we obtain the following: for any $\beta \in C^1(\mathbb{R})$ with $\beta(0) = 0$ the distribution

$$\mu_{\mathfrak{a}}^{\beta} := \frac{d}{dt}\big(\beta(u)w_{\mathfrak{a}}\big)$$

is a measure given by

$$\begin{aligned}\mu_{\mathfrak{a}}^{\beta} &:= \sum_{t_i \text{ jump}} \Big[\beta(u_{\mathfrak{a}}(t_i^+))w_{\mathfrak{a}}(t_i^+) - \beta(u_{\mathfrak{a}}(t_i^-))w_{\mathfrak{a}}(t_i^-)\Big] + \beta'(u_{\mathfrak{a}})(D^{\text{cont}}u_{\mathfrak{a}})w_{\mathfrak{a}} + \beta(u_{\mathfrak{a}})D^{\text{cont}}w_{\mathfrak{a}} \\ &= \sum_{t_i \text{ jump}} \Big[\beta(u_{\mathfrak{a}}(t_i^+))w_{\mathfrak{a}}(t_i^+) - \beta(u_{\mathfrak{a}}(t_i^-))w_{\mathfrak{a}}(t_i^-)\Big] + \beta'(u_{\mathfrak{a}})(\nu_{\mathfrak{a}})^{\text{cont}} + \big(\beta(u_{\mathfrak{a}}) - u_{\alpha}\beta'(u_{\mathfrak{a}})\big)\mu_{\mathfrak{a}}^{\text{cont}}.\end{aligned} \tag{7.30}$$

A simple computations yields that

$$\|\mu_{\mathfrak{a}}^{\beta}\| \leq \|\beta'\|_{\infty}\|\nu_{\mathfrak{a}}\| + \|\beta'\|_{\infty}\|u\|_{\infty}\|\mu_{\mathfrak{a}}\|.$$

The above estimate allows to conclude with the following proposition.

PROPOSITION 7.36. *For any $\beta \in C^1$ the distribution*

$$\operatorname{div}\big(\beta(u)\rho(1,\boldsymbol{b})\mathscr{L}^{d+1}\big) = \mu^{\beta},$$

*where the measure $\mu^{\beta}$ is given by*

$$\mu^{\beta} := \int_{\mathfrak{A}} \mu_{\mathfrak{a}}^{\beta}\, m(d\mathfrak{a}),$$

*with $\mu_{\mathfrak{a}}^{\beta}$ defined in (7.30).*

In particular, Proposition 7.36 establishes completely the chain rule formula (and, as a consequence, renormalization property) for vector fields $\rho(1,\boldsymbol{b})$ satisfying Assumption 7.27.

CHAPTER 8

# The $L^1_{\text{loc}}(\mathbb{R}; \text{BV}_{\text{loc}}(\mathbb{R}^d))$ case and Bressan's Compactness Conjecture

ABSTRACT. This chapter concludes the proof of Bressan's Conjecture, showing that the vector field $(1, \boldsymbol{b})$ satisfies Assumption 7.27 if $\boldsymbol{b} \in L^1((0,T); \text{BV}(\mathbb{R}^d))$: in particular, it has a minimal partition via characteristic and the disintegration argument discussed in Chapter 7 can be performed. The construction of the approximate cylinders of flow in the BV setting depends on the local structure of the vector fields: in particular, in Section 8.1, using the Rank-One Theorem and Coarea formula, we construct an approximate vector field which will be then used in Section 8.2 to construct the cylinders. For the reader's convenience, the computations of the flux estimates have been collected in Section 8.3.

## 8.1. A covering of the singular part of the derivative

The aim of this section it to construct a decomposition of the set where the singular part of the derivative of $\boldsymbol{b}$ lives into a family of Lipschitz surfaces: we approximate the component of $\boldsymbol{b}$ in a particular direction with a function whose super-level sets are regular and share essentially a common direction. This will be useful in the following sections to construct the cylinders of approximate flow in the $L^1((0,T); \text{BV}(\mathbb{R}^d))$ setting.

The decomposition we present here relies essentially on Alberti's Rank-One Theorem (and ultimately on the properties of sets of finite perimeter, in particular the De Giorgi Rectifiability Theorem).

**8.1.1. BV functions and cones.** For $\boldsymbol{e} \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$ and $0 < a < 1$, let

$$C(\boldsymbol{e}, a; x) := \big\{ y \in \mathbb{R}^d : |(y - x) \cdot \boldsymbol{e}| \geq a|y - x| \big\}.$$

be the closed, convex cone around $\boldsymbol{e}$ of vertex $x$ and opening $a$. We will often think $x$ to be the origin, so we will often write $C(\boldsymbol{e}, a)$ to denote $C(\boldsymbol{e}, a; 0)$. The following proposition is well known:

PROPOSITION 8.1. [**DL08**, Prop. 5.1] *Let $C = C(\boldsymbol{e}, a)$ be a closed convex cone and $v \in \text{BV}(\mathbb{R}^d; \mathbb{R})$. Set*

$$G := \left\{ x : \frac{Dv}{|Dv|}(x) \in C \right\}.$$

*For any closed convex cone $C' := C(\boldsymbol{e}, a')$ with $a' < a$ there exists $w \in \text{BV}(\mathbb{R}^d; \mathbb{R})$ such that $|Dv|\llcorner_G \ll |Dw|$ and*

$$\frac{Dw}{|Dw|}(x) \in C' \qquad \text{for } |Dw|\text{-a.e. } x \in \mathbb{R}^d.$$

For our purposes, we need a slight modification of Proposition 8.1. More precisely, we show

PROPOSITION 8.2. *Let $C = C(\boldsymbol{e}, a)$ be a closed convex cone and $v \in \text{BV}(\mathbb{R}^d; \mathbb{R})$. Set*

$$G := \left\{ x : \frac{Dv}{|Dv|}(x) \in C \right\}.$$

*For any closed convex cone $C' := C(\boldsymbol{e}, a')$ with $a' < a$ and for any $\varepsilon > 0$ there exist $\bar{r} > 0$ and $w \in \text{BV}(\mathbb{R}^d; \mathbb{R})$ such that:*

- $|Dv|\llcorner_G \ll |Dw|$ *and*
$$\frac{Dw}{|Dw|}(x) \in C' \qquad \textit{for } |Dw|\textit{-a.e. } x;$$
- *there exists a family of $a'$-Lipschitz functions $(L_{i,j})_{i,j\in\mathbb{N}}$ such that, set $E^h_{i,j} := \{L_{i,j} > h\}$, then*
$$|Dw| = \int_{\mathbb{R}} \sum_{i,j} \mathscr{H}^{d-1}\llcorner_{\partial^\star E^h_{i,j}}\, dh.$$
*Furthermore, there exist a family of compact sets $(K_i')_{i\in\mathbb{N}} \subset \mathbb{R}^d$ such that for $r < \bar{r}$ it holds*
$$\left| Dv\llcorner_{G_i} - \int_{\mathbb{R}} \sum_{i,j} \nu^h_{i,j}\, \mathscr{H}^{d-1}\llcorner_{E^h_{i,j}}\, dh \right| (B^d_r(x)) < \varepsilon |Dv|(B^d_r(x))$$
*for every $x \in K_i'$, where $\nu^h_{i,j}(\cdot)$ denotes the outer measure theoretic normal to $E^h_{i,j}$ and $G_i \subset G$ are suitable subsets of $G$ introduced in the proof.*

Following [DL08], we decide to present first the proof of Proposition 8.2 in special case, i.e. when $v$ is the characteristic function of a set (which therefore is a set of finite perimeter). This case turns out to be the building block to prove the Proposition in its full generality, via Coarea formula.

### 8.1.2. Proof of Proposition 8.2 in the case of a set of finite perimeter.

PROPOSITION 8.3. *Let $C = C(\boldsymbol{e}, a)$ be a closed convex cone and $E \subset \mathbb{R}^d$ be a set of finite perimeter. Set $v = \mathbb{1}_E$ and*
$$G := \left\{ x : \frac{Dv}{|Dv|}(x) \in C \right\}.$$
*For any $a' < a$ and for any $\varepsilon > 0$ there exist $\bar{r} > 0$ and $w \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ such that*

- $|Dv|\llcorner_G \ll |Dw|$ *and*
$$\frac{Dw}{|Dw|}(x) \in C' \qquad \textit{for } |Dw|\textit{-a.e. } x;$$
- *there exist a family of open, $C^1$ domains $(\Omega_{i,j})_{i,j\in\mathbb{N}} \subset \mathbb{R}^d$ and real non-negative numbers $\lambda_{i,j} \geq 0$ such that*
$$|Dw| = \sum_{i,j} \lambda_{i,j} \mathscr{H}^{d-1}\llcorner_{\partial\Omega_{i,j}}.$$
*Furthermore, there exist compact sets $K_i \subset \bigcup_j \partial\Omega_{i,j}$ such that for $r < \bar{r}$ it holds*
$$\left| D\mathbb{1}_E\llcorner_{G_i} - \sum_j \nu_{i,j} \mathscr{H}^{d-1}\llcorner_{\partial^\star\Omega_{i,j}} \right| \big(B^d_r(x)\big) \leq C_{d-1}\varepsilon |D\mathbb{1}_E|(B^d_r(x))$$
*for any $x \in K_i$, where $\nu_{i,j}(\cdot)$ is the outer unit normal to $\Omega_{i,j}$ and $G_i \subset G$ are suitable subsets of $G$ introduced in the proof.*

PROOF. Let $v, E$ be as in the statement. We denote by $\partial^\star E$ the reduced boundary of $E$ (see Preliminaries, Section III) and let $\nu$ be the approximate exterior unit normal to $\partial^\star E$, so that we can write
$$Dv = \nu \mathscr{H}^{d-1}\llcorner_{\partial^\star E}$$
and accordingly the set $G$ is
$$G = \big\{ x \in \partial^\star E : \nu(x) \in C \big\}.$$

Being $\partial^\star E$ rectifiable, in view of Theorem VII, we have that $G$ can be decomposed as

$$G = G_0 \cup \bigcup_{i=1}^{\infty} G_i$$

where:

- $\mathscr{H}^{d-1}(G_0) = 0$ and for $i \geq 1$ each $G_i$ is a subset of a $(d-1)$-dimensional $C^1$ manifold $M_i$;
- $\nu|_{G_i}$ coincides with the normal vector $\mathbf{n}_i$ to the manifold $M_i$.

We now split the argument into steps:

*Step 1.* For each $i \geq 1$ we claim that there are $C^1$ open sets $\{\Omega_{i,j}\}_{j\in\mathbb{N}}$ such that, having set $S_{i,j} := \partial\Omega_{i,j}$ the following conditions hold: the exterior normal to $S_{i,j}$ belongs $\mathscr{H}^{d-1}$-a.e. to $C'$ and $\{S_{ij}\}_{j\in\mathbb{N}}$ is a covering of $G_i$.

Indeed, recall that $C' = C(\boldsymbol{e}, a')$ and, up to a change of coordinates, we may assume that $\boldsymbol{e} = \boldsymbol{e}_d = (0, 0, \ldots, 1)$. For any $x \in G_i$, the normal $\mathbf{n}_i(x)$ belongs to $C(\boldsymbol{e}, a)$, and thus it is transversal to $\boldsymbol{e}_d^\perp := \operatorname{span}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d-1})$. This implies that we can choose an open ball $B_r^d(x)$ centered at $x$ such that

$$M_i \cap B_r^d(x) = \{(x^\perp, x) : x = f_i(x^\perp))\}$$

i.e. $M_i \cap B_r^d(x)$ coincides with the graph of a $C^1$ function $f_i \colon O_i \subset \mathbb{R}^{n-1} \to \mathbb{R}$ where $O_i$ is some bounded open set in $\mathbb{R}^{d-1}$. Moreover, by continuity of the normal $\mathbf{n}_i$, we can choose $B_r^d(x)$ so that $\mathbf{n}_i(y) \in C'$ for every $y \in M_i \cap B_R^d(x)$. By defining

$$\Omega_x := \{(x^\perp, x) : x < f_i(x^\perp))\}$$

then $\Omega_x$ turns to be a $C^1$ open set, the normal to $S_x := \partial\Omega_x$ belongs to the cone $C'$ and $S_x$ covers $B_r^d(x) \cap M_i$. Since we can cover $M_i$ with a countable family of these balls $B_r^d(x)$, the corresponding $S_x$ form the desired countable covering $S_{i,j}$.

*Step 2.* We now consider the sets $S_{i,j}$. They have all finite $\mathscr{H}^{d-1}$ measure, which we denote by $\ell_{i,j}$ and they cover $\mathscr{H}^{d-1}$-a.e. $G$. Take any collection $\lambda_{i,j}$ of positive real numbers such that $\sum_{i,j} \lambda_{i,j} \leq 1$ and $\sum_{i,j} \lambda_{i,j}\ell_{i,j} \leq 1$ and finally set

$$w := \sum_{i,j} \lambda_{i,j} \mathbb{1}_{\Omega_{i,j}}.$$

It is immediate to see that $w$ is bounded and of bounded variation since

$$\|w\|_\infty \leq \sum_{i,j} \lambda_{i,j} \leq 1, \qquad |Dw| = \sum_{i,j} \lambda_{i,j} \mathscr{H}^{d-1} \llcorner_{S_{i,j}} \leq 1.$$

For more details, see [DL08].

*Step 3.* We now exploit some further properties of points in the reduced boundary. Recall that for sets of finite perimeter for every $x \in \partial^\star E$ it holds

$$\lim_{r\to 0} \frac{|D\mathbb{1}_E|(B_r^d(x))}{\omega_{d-1} r^{d-1}} = 1. \tag{8.1}$$

On the other hand, by Lebesgue's differentiation theorem and Area formula, for every $i, j \in \mathbb{N}$, $\mathscr{H}^{d-1}$-a.e. $x \in S_{i,j}$ is a $\mathscr{L}^{d-1}$-density point for the corresponding open set $O_{i,j}$, given by

$$O_{i,j} = (\mathbb{I}, f_{i,j})^{-1}(S_{i,j}) \subset \mathbb{R}^{d-1},$$

which explicitly means that

$$\lim_{r\to 0} \frac{\mathscr{L}^{d-1}(O_{i,j} \cap B_r^{d-1}(x^\perp))}{\omega_{d-1} r^{d-1}} = 1. \tag{8.2}$$

We now apply Egorov's Theorem to the two limits (8.1), (8.2) (for each $i, j$): for every $\varepsilon > 0$, there exists $\bar{r} > 0$ and a compact set $F_{i,j}(\varepsilon, \bar{r}) \subset O_{i,j}$, covering $O_{i,j}$ up to a set of $\mathscr{H}^{d-1}$ measure less than $\varepsilon$, such that for any $r < \bar{r}$ it holds

$$\left| \frac{|D\mathbb{1}_E|(B^d_r(f_{i,j}(x^\perp)))}{r^{d-1}} - \omega_{d-1} \right| < \omega_{d-1}\varepsilon, \tag{8.3a}$$

$$\left| \frac{1}{r^{d-1}} \mathscr{L}^{d-1}\big(O_{i,j} \cap B^{d-1}_r(x^\perp))\big) - \omega_{d-1} \right| < \omega_{d-1}\varepsilon \tag{8.3b}$$

for any $x^\perp \in F_{i,j}(\varepsilon, \bar{r})$. We now introduce the following compact set:

$$K_i(\varepsilon, \bar{r}) := \bigcup_{j \in \mathbb{N}} \mathrm{Graph}\left(f_{i,j} \llcorner_{F_{i,j}(\varepsilon,\bar{r}))}\right).$$

For any $x \in K_i(\varepsilon, \bar{r})$, thanks to (8.3a) it holds for $r < \bar{r}$

$$\omega_{d-1}(1-\varepsilon) r^{d-1} \le |D\mathbb{1}_E|(B^d_r(x)) \le \omega_{d-1}(1+\varepsilon) r^{d-1}$$

and, on the other hand, using (8.3b) and being the projection 1-Lipschitz

$$|D\mathbb{1}_E|\llcorner_{G_i}\big(S_{i,j} \cap B^d_r(x)\big) \ge \mathscr{L}^{d-1}\big(O_{i,j} \cap B^{d-1}_r(x^\perp)\big) \ge \omega_{d-1}(1-\varepsilon) r^{d-1}$$

for every $j$. Thus we get that, for any $x \in K_i(\varepsilon, \bar{r})$ and any $r < \bar{r}$ we have

$$\begin{aligned}
|D\mathbb{1}_E|\llcorner_{G_i}\big(B^d_r(x) \setminus S_{i,j}\big) &\le |D\mathbb{1}_E|\big(B^d_r(x)\big) - |D\mathbb{1}_E|\llcorner_{G_i}\big(S_{i,j} \cap B^d_r(x)\big) \\
&\le \omega_{d-1}(1+\varepsilon) r^{d-1} - \omega_{d-1}(1-\varepsilon) r^{d-1} \\
&= 2\varepsilon\omega_{d-1} r^{d-1} \\
&\le \frac{2\varepsilon}{1-\varepsilon} |D\mathbb{1}_E|(B^d_r(x)).
\end{aligned}$$

To sum up, the set $K_i(\varepsilon, \bar{r})$ is the set of points $x \in \partial^\star E$ for which it holds for $r < \bar{r}$

$$|D\mathbb{1}_E|\big(B^d_r(x) \setminus S_{i,j}\big) \le C_{d-1}\varepsilon |D\mathbb{1}_E|(B^d_r(x))$$

and

$$\left| \mathscr{H}^{d-1}\big(B^d_r(x) \cap S_{i,j}\big) - |D\mathbb{1}_E|\big(B^d_r(x)\big) \right| \le C_{d-1}\varepsilon |D\mathbb{1}_E|(B^d_r(x)), \tag{8.4}$$

which comes from (8.3b). Finally, by integration of the normal vector, from (8.4), we obtain that for every $x \in K_i(\varepsilon, \bar{r})$ and $r < \bar{r}$ the desired estimate

$$\left| D\mathbb{1}_E \llcorner_{G_i} - \sum_j \nu_{i,j} \mathscr{H}^{d-1} \llcorner_{S_{i,j}} \right| \big(B^d_r(x)\big) \le C_{d-1}\varepsilon |D\mathbb{1}_E|(B^d_r(x))$$

holds and this concludes the proof. □

**8.1.3. Proposition 8.2 in the general case.** To prove the general case we exploit Coarea formula, as done in [DL08].

Proof. For every $h \in \mathbb{R}$ we consider the function $v_h := \mathbb{1}_{\{v>h\}}$ and, for future reference, we define the measure $\mathcal{M} \in \mathscr{M}(\mathbb{R}^d \times \mathbb{R})$ as $\mathcal{M} := |Dv_h| \otimes \mathscr{L}^1(dh)$, which explicitly means that, for every continuous function $\phi \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, it holds

$$\begin{aligned}
\int_{\mathbb{R}\times\mathbb{R}^d} \phi(x,h)\, \mathcal{M}(dh dx) &= \int \left[ \int \phi(x,h) \big|D\mathbb{1}_{\{v>h\}}\big|(dx) \right] \mathscr{L}^1(dh) \\
&= \int \left[ \int_{\partial^*\{v>h\}} \phi(x,h)\, \mathscr{H}^d(dx) \right] \mathscr{L}^1(dh).
\end{aligned}$$

From Coarea formula V we have that:

- $v_h$ is a BV function for $\mathscr{L}^1$-a.e. $h$, i.e. $\{v > h\}$ is a set of finite perimeter. Let $\nu_h$ be its exterior unit normal;

- it holds

$$\nu_h(x) = \frac{Dv}{|Dv|}(x)$$

for $\mathscr{L}^1$-a.e. $h \in \mathbb{R}$ and $\mathscr{H}^{d-1}$-a.e. $x \in \partial^\star\{v > h\}$, i.e. for $\mathcal{M}$-a.e. $(x, h)$;
- it holds

$$|Dv| = \int_{\mathbb{R}} |Dv_h| \mathscr{L}^1(dh);$$

- it holds $\mathtt{p}_{\mathbb{R}^d}(\mathcal{M}) = |Dv|$, hence $\mathcal{M}$ can be disintegrated as

$$\mathcal{M} = \int \mathcal{M}_x |Dv|(dx), \tag{8.5}$$

Therefore, for $\mathscr{L}^1$-a.e. $h$ we can apply Proposition 8.3. We denote by $w_h$ the corresponding bounded, BV function given by Proposition 8.3 and we set

$$w(x) := \int_{\mathbb{R}} w_h(x)\, dh. \tag{8.6}$$

Notice that, in order to write (8.6), we have to be sure that the map $h \mapsto w_h$ enjoys some measurability properties. To show the existence of such a selection, one can use the Aumann Measurable Selection Theorem (for the precise argument we refer again the reader to [**DL08**]). Then it is immediate to see that $w$ satisfies $|Dv|\llcorner_G \ll |Dw|$ and

$$\frac{Dw}{|Dw|}(x) \in C' \qquad \text{for } |Dw|\text{-a.e. } x.$$

Furthermore, denoting by $S^h_{i,j}$ and $K^h_i(\varepsilon, \bar{r})$ the corresponding sets for $w_h$ (obtained via Proposition 8.3), we have that for any $x \in K^h_i(\varepsilon, \bar{r})$ for $r < \bar{r}$ it holds

$$|Dv_h|\big(B^d_r(x) \setminus S^h_{i,j}\big) \leq C_{d-1}\varepsilon |Dv_h|(B^d_r(x))$$

and

$$\Big|\mathscr{H}^{d-1}\big(B^d_r(x) \cap S^h_{i,j}\big) - |Dv_h|\big(B^d_r(x)\big)\Big| \leq C_{d-1}\varepsilon |Dv_h|(B^d_r(x)).$$

By means of measurable selection, we can define now the measurable sets

$$\widetilde{K}_i(\varepsilon, \bar{r}) := \{(x, h) : x \in K^h_i(\varepsilon, \bar{r})\} \subset \mathbb{R}^d \times \mathbb{R}, \qquad i \in \mathbb{N}$$

so that for every $h$ we have $\widetilde{K}_i(\varepsilon, \bar{r}; h) = K^h_i(\varepsilon, \bar{r})$; observe that, by construction, they cover $\mathbb{R}^d \times \mathbb{R}$ up to a set of $\mathcal{M}$ - measure less than $\varepsilon$. In view of the disintegration (8.5) we thus can write for all $R > 0$

$$\int_{B^d_R(0)} \mathcal{M}_x\Big((\mathbb{R}^d \times \mathbb{R}) \setminus \widetilde{K}_i(\varepsilon, \bar{r})\Big)|Dv|(dx) < \varepsilon.$$

Thus, by Chebyshev inequality, we deduce that $\widetilde{K}_i(\varepsilon, \bar{r})$ covers almost all the fiber of an arbitrary large fraction of points $x$ (in any ball $B^d_R(0)$)): in other words, for every fixed $\delta > 0$, there is a set $N^i_\delta \subset B^d_R(0)$ such that

$$|Dv|(N^i_\delta) < \frac{\varepsilon}{1 - \delta}$$

and

$$\mathcal{M}_x\Big(\widetilde{K}_i(\varepsilon, \bar{r})\Big) > 1 - \delta, \qquad \forall x \in B^d_R(0) \setminus N^i_\delta.$$

Taking a compact set $K'_i \subset B^d_R(0) \setminus N^i_\delta \subset \mathtt{p}_{\mathbb{R}^d}(\widetilde{K}_i(\varepsilon, \bar{r}))$ we obtain that for every $x \in K'_i$ and $r < \bar{r}$ it holds

$$\big|Dw\llcorner_{G_i} - Dv\big|\big(B^d_r(x)\big) \leq C_{d-1}\varepsilon |Dv|(B^d_r(x)),$$

which is the claim. □

It is now clear that we can repeat finitely many times the above constructions in order to cover all the reduced boundary. More precisely, given any $\delta_c > 0$, we pick a set of unit vectors $\{\mathbf{n}_s, s = 1, \dots, J_{\delta c}\} \subset \mathbb{R}^d$ in such a way that

$$B_1^d(0) \subset \bigcup_{s=1}^{J_{\delta_c}} C(\mathbf{n}_s, \delta_c).$$

By choosing $a' = \delta_c/2$ and applying Proposition 8.2, we obtain the following

Corollary 8.4. *Let $v \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ and for every $s = 1, \dots, J_{\delta_c}$ set*

$$G_s := \left\{x : \frac{Dv}{|Dv|}(x) \in C(\mathbf{n}_s, \delta_c)\right\}.$$

*For every $\varepsilon > 0$ there exist $\bar r > 0$ and $w \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ such that:*

- $|Dv|\llcorner_{G_s} \ll |Dw|$ *for every $s = 1, \dots, J_{\delta c}$ and*

$$\frac{Dv}{|Dv|}(x) \in C(\mathbf{n}_s, \delta_c) \Rightarrow \frac{Dw}{|Dw|}(x) \in C\left(\mathbf{n}_s, \frac{\delta_c}{2}\right) \qquad \text{for } |Dw|\text{-a.e. } x;$$

- *for every $s = 1, \dots, J_{\delta c}$ there exists a family of $C^1$ functions $(L_{i,j,s})$ for $i, j \in \mathbb{N}$, with Lipschitz constant $\delta_c$, such that, setting $E^h_{i,j,s} := \{L_{i,j,s} > h\}$, then*

$$|Dw|\llcorner_{G_s} = \int_{\mathbb{R}} \sum_{i,j} \mathscr{H}^{d-1}\llcorner_{\partial^\star E^h_{i,j,s}} \, dh.$$

  *Furthermore, there exists a family of compact sets $(K'_{s,i}) \subset \mathbb{R}^d$ with $i \in \mathbb{N}$ and $s \in \{1, \dots, J_{\delta c}\}$ such that for $r < \bar r$ it holds*

$$\left| Dv\llcorner_{G_{s_i}} - \int_{\mathbb{R}} \sum_{i,j} \nu_{j,s} \mathscr{H}^{d-1}\llcorner_{\partial^\star E^h_{i,j,s}} \, dh \right| (B_r^d(x)) < \varepsilon |Dv|(B_r^d(x))$$

  *for every $x \in K'_{s,i}$, where $\nu^h_{i,j,s}(\cdot)$ is the outer unit normal to $E^h_{i,j,s}$ and $G_{s_i} \subset G_s$ are suitable subsets of $G_s$.*

**8.1.4. Decomposition for vector fields $L^1_{\mathrm{loc}}(\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$.** We now consider the vector-valued case, i.e. we take $\boldsymbol{b} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$ and we are interested in covering the singular part of $D\boldsymbol{b}$: in order to achieve this, we have to exploit Alberti's Rank one Theorem VI.

More precisely, let us denote by $\mathsf{n}, \mathsf{m}$ the two unit vectors given by Rank one property, i.e. such that

$$D^{\mathrm{s}}\boldsymbol{b} = \mathsf{m} \otimes \mathsf{n} |D^{\mathrm{s}}\boldsymbol{b}|.$$

Consider the points $(\bar t, \bar x)$ with the following properties:

- $(\bar t, \bar x)$ is a point where the measure $D\boldsymbol{b}$ is essentially singular, i.e. it is a density point for $D^{\mathrm{s}}\boldsymbol{b}$. More precisely, $(\bar t, \bar x)$ is such that for every $\varepsilon > 0$ there exists $\bar r(\varepsilon, \bar t, \bar x) > 0$ such that for $0 < r < \bar r$ it holds

$$|D^{\mathrm{s}}\boldsymbol{b}|(B_r^{d+1}(\bar t, \bar x)) > (1 - \varepsilon)|D\boldsymbol{b}|(B_r^{d+1}(\bar t, \bar x)); \tag{8.7}$$

- $(\bar t, \bar x)$ is a Lebesgue point of the matrix valued map $(t, x) \mapsto \mathsf{m} \otimes \mathsf{n}(t, x)$, which is defined $|D^{\mathrm{s}}\boldsymbol{b}|$-a.e., that is to say for every $\varepsilon > 0$ there exists $\bar r'(\varepsilon, \bar t, \bar x) > 0$ such that for $0 < r < \bar r'$ it holds

$$\int_{B_r^{d+1}(\bar t, \bar x)} |\mathsf{m} \otimes \mathsf{n} - \overline{\mathsf{m}} \otimes \overline{\mathsf{n}}| |D^{\mathrm{s}}\boldsymbol{b}|(dtdx) < \varepsilon |D^{\mathrm{s}}\boldsymbol{b}|(B_r^{d+1}(\bar t, \bar x)), \tag{8.8}$$

  having denoted by $\overline{\mathsf{m}} \otimes \overline{\mathsf{n}}$ the Lebesgue value in $(\bar t, \bar x)$.

By a standard application of Egorov Theorem, for every fixed $\varepsilon > 0$ we can find a sequence $(\bar{r}_i)_{i\in\mathbb{N}}$ (where $\bar{r}_i$ depend only on $\varepsilon$) and a family of compact sets $\big(G(\varepsilon, r_i)\big)_{i\in\mathbb{N}} \subset \mathbb{R}^{d+1}$ covering almost all the set where $D^s\boldsymbol{b}$ is concentrated and such that the limits (8.7) and (8.8) are uniform on each $G(\varepsilon, r_i)$. Moreover, we can further split the compact sets $G(\varepsilon, r_i)$ according to the direction: indeed, we denote by $G(\varepsilon, r_i, s)$ the set of points $(t, x) \in G(\varepsilon, r_i)$ such that $\mathsf{m}(t, x) \in C(\mathbf{n}_s, \delta_c)$, for $s \in \{1, \dots, J_{\delta_c}\}$.

Now, we denote by $b_{\mathsf{n}} := \boldsymbol{b} \cdot \bar{\mathsf{n}}$ the component of $\boldsymbol{b}$ along $\bar{\mathsf{n}}$. By Rank one, the (scalar) function $b_{\bar{\mathsf{n}}}$ has polar vector $\bar{\mathsf{m}}$ in $(\bar{t}, \bar{x})$. Thus, by Chebyshev inequality, we can say that for an arbitrary large fraction (w.r.t $D^s\boldsymbol{b}$) of points $(t, x) \in G(\varepsilon, r_i, s)$ it holds

$$\frac{Db_{\mathsf{n}}}{|Db_{\mathsf{n}}|}(t, x) = \mathsf{m} \in C(\mathbf{n}_s, \delta_c)$$

since $\mathsf{m}$ is close to $\bar{\mathsf{m}}$ in view of (8.8). Therefore, we are in position to apply Corollary 8.4: there are a BV function $\mathcal{U}_{\bar{\mathsf{n}}}$ and $C^1$ functions (with Lipschitz constant less than $\delta_c$) $(L^{\bar{\mathsf{n}}}_{i,j,s})$ for $i, j \in \mathbb{N}$ and $s \in \{1, \dots, J_{\delta_c}\}$ such that, set $E^{\bar{\mathsf{n}},h}_{i,j,s} := \{L^{\bar{\mathsf{n}}}_{i,j,s} > h\}$, then the derivative of $\mathcal{U}_{\bar{\mathsf{n}}}$ can be written as

$$|D\mathcal{U}_{\bar{\mathsf{n}}}|\llcorner_{G_s} = \int_{\mathbb{R}} \sum_{i,j} \mathscr{H}^{d-1}\llcorner_{\partial^\star E^{s,\bar{\mathsf{n}},h}_{i,j}}\, dh.$$

Furthermore, there exist $\bar{r} > 0$ and a family of sets $(K^{\bar{\mathsf{n}}}_s)_s \subset \mathbb{R} \times \mathbb{R}^d$ such that for $r < \bar{r}$ it holds

$$\left| Db_{\mathsf{n}} - \int_{\mathbb{R}} \sum_{i,j} \nu^{\bar{\mathsf{n}},h}_{i,j,s} \mathscr{H}^{d-1}\llcorner_{\partial^\star E^{\bar{\mathsf{n}},h}_{i,j,s}}\, dh \right| (B^d_r(x)) < \varepsilon |Db_{\mathsf{n}}|(B^d_r(x))$$

for every $x \in K^{\bar{\mathsf{n}}}_s$ where $\nu^{\bar{\mathsf{n}},h}_{i,j,s}$ is the outer unit normal to $E^{\bar{\mathsf{n}},h}_{i,j,s}$.

Finally if we multiply back times $\bar{\mathsf{m}}$ we end up with a matrix valued measure which is the derivative of an approximated BV vector field: this yields a sort of vectorial analog of Corollary 8.4. By expliciting the normal to the set $E^{\bar{\mathsf{n}},h}_{i,j,s}$, observing that the map $(t, x, h) \mapsto \mathbb{1}_{\{\boldsymbol{b}_t\cdot\bar{\mathsf{n}} > h\}}(x)$ is measurable and using again a measurable selection argument, we can finally state the following

Corollary 8.5. *Let $\boldsymbol{b} \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$. Then for every $\varepsilon > 0$ and $\delta_c > 0$ there exists compact sets $K^{\varepsilon,j}_{\delta_c,r_i}$ the such that if $(\bar{t}, \bar{x}) \in K^{\varepsilon,j}_{\delta_c,r_i}$ then there exist a family of Lipschitz functions $\{y_{\mathbf{n}_j} = L_{t,h}(y^\perp_{\mathbf{n}_j})\}_{t,h}$ with Lipschitz constant less than $\delta_c$ such that*

$$\left| D\boldsymbol{b} - \int \left\{ \left[\mathsf{m}(\bar{t}, \bar{x}) \otimes (1, -\nabla_{y^\perp_{\bar{\mathsf{n}}}} L_{t,h})\right] \delta_t \otimes \big((\mathbb{I}, L_{t,h})_\sharp \mathscr{L}^{d-1}\big) \right\} dt dh \right| (B) < C\varepsilon |D^s\boldsymbol{b}|(B),$$

*where $B = B^{d+1}_r(\bar{t}, \bar{x})$.*

## 8.2. Construction of approximate cylinders of flow in the BV setting

The aim of this section is to construct locally some approximate flow cylinders, which maintain a quite regular shape and have a small boundary flow. We want to verify that Assumption 7.1 or Assumption 7.6 holds in a neighborhood of every point $(t, x)$, and that then Assumption 7.27 is valid.

As observed in Remark 6.4, one has to control the lateral flow either for a family of smooth Lipschitz functions $\phi^\ell_\gamma$ or Lipschitz sets $Q^\ell_\gamma$, the two conditions being equivalent.

**8.2.1. Estimates for the absolutely continuous part.** Fix a matrix $A$. For $\gamma \in \mathit{\Gamma}$ define the cylinder

$$\phi^{\ell,\delta_1}_\gamma\big(t, \gamma(t) + e^{At}y\big) = \left[1 - \frac{1}{\delta_1 \ell}\,\mathrm{dist}\,\big(y, B^d_\ell(0)\big)\right]^+,$$

and the normalization constant

$$\sigma^{\ell,\delta_1} = \int \bigg[1 - \frac{1}{\delta_1 \ell} \operatorname{dist}\big(x, B^d_\ell(0)\big)\bigg]^+ \mathscr{L}^d(dx).$$

A standard computation gives

$$\begin{aligned}
&\frac{1}{\sigma^{\ell,\delta_1}} \int_{t^-_\gamma}^{t^+_\gamma} \int \big|(1,\boldsymbol{b}) \cdot \nabla_{t,x} \phi^{\ell,\delta_1}_\gamma\big| \mathscr{L}^{d+1} \\
=&\frac{1}{\sigma^{\ell,\delta_1}} \int_{t^-_\gamma}^{t^+_\gamma} \Big| - \nabla_x \phi^{\ell,\delta_1}_\gamma \cdot \big(\boldsymbol{b}(t,\gamma(t)) + Ae^{At}y\big) + \boldsymbol{b}(t,x) \cdot \nabla_x \phi^{\ell,\delta_1}_\gamma(t,x)\Big| \, dy \\
=&\int_{t^-_\gamma}^{t^+_\gamma} \frac{1}{\delta_1 \ell} \int_{|y| \in \ell(1,1+\delta_1)} \bigg|\Big(\boldsymbol{b}(t,\gamma(t)+e^{At}y) - \boldsymbol{b}(t,\gamma(t)) - Ae^{At}y\Big) \cdot e^{-At} \frac{y}{\sigma^{\ell,\delta_1}|y|}\bigg| e^{\mathsf{tr}At} \, dy,
\end{aligned}$$

so that if $\eta$ is a Lagrangian representation for $(1,\boldsymbol{b})\mathscr{L}^{d+1} \llcorner \Omega$ with $\Omega$ Lipschitz (see Proposition 6.22)

$$\begin{aligned}
&\int \frac{1}{\sigma^{\ell,\delta_1}} \bigg[\int_{t^-_\gamma}^{t^+_\gamma} \int \big|(1,\boldsymbol{b}) \cdot \nabla_{t,x} \phi^{\ell,\delta_1}_\gamma\big| \mathscr{L}^{d+1}\bigg] \eta(d\gamma) \\
\leq& \int \bigg[\int_{t^-_\gamma}^{t^+_\gamma} \frac{1}{\sigma^{\ell,\delta_1}\delta_1 \ell} \int_{|y| \in \ell(1,1+\delta_1)} \Big|\boldsymbol{b}\big(t,\gamma(t)+e^{At}y\big) - \boldsymbol{b}\big(t,\gamma(t)\big) - Ae^{At}y\Big| \frac{|e^{-At}y|}{|y|} e^{\mathsf{tr}At} \, dt \, dy\bigg] \eta(d\gamma) \\
\leq& \frac{1}{\delta_1 \omega_d \ell^{d+1}} \int_{|e^{-At}z| \in \ell(1,1+\delta_1)} \frac{|e^{-2At}z|}{|e^{-At}z|} \int_\Omega \big|\boldsymbol{b}(t,x+z) - \boldsymbol{b}(t,x) - Az\big| \mathscr{L}^{2d+1}(dtdxdz) \\
\leq& \frac{1}{\delta_1 \omega_d \ell^{d+1}} \int_{|e^{-At}z| \in \ell(1,1+\delta_1)} \frac{|e^{-2At}z|}{|e^{-At}z|} |z| \big|D\boldsymbol{b}_t - A\mathscr{L}^d\big| \big(\Omega_t + B^d_{(1+\delta_1)e^{\|A\|t}\ell}(0)\big) \mathscr{L}^{d+1}(dtdx) \\
\leq& C_d \|e^{2\|A\|t}\|_{L^\infty(\mathtt{p}_t\Omega)} \big|D\boldsymbol{b} - A\mathscr{L}^{d+1}\big| \big(\Omega + \{t=0\} \times B^d_{(1+\delta_1)\|e^{\|A\|t}\|_{L^\infty(\mathtt{p}_t\Omega)}\ell}(0)\big).
\end{aligned}$$

Letting $\ell, \delta_1 \to 0$ and choosing the matrices $A$ in order to approximate the a.c. part of $D\boldsymbol{b}$, we conclude with the following proposition.

PROPOSITION 8.6. *For every point $(t,x)$ there exists $\bar{r}_{t,x}$ such that for $\mathscr{L}^1$-a.e. $0 < r < \bar{r}_{t,x}$ the ball $B^{d+1}_r(t,x)$ is $(1,\boldsymbol{b})$-proper and Assumption 7.1 holds with constant $\varpi_r(t,x)$ such that*

$$\varpi_r(t,x) \leq \begin{cases} \tau|D\boldsymbol{b}|(B^{d+1}_r(t,x)) & (t,x) \text{ Lebesgue point for } |D^{\mathrm{a.c.}}\boldsymbol{b}|, \\ C_d|D\boldsymbol{b}|(B^{d+1}_r(t,x)) & \text{otherwise.} \end{cases}$$

The proof is just an application of the Radon-Nikodym theorem, and it will be omitted.

**8.2.2. Estimates for the singular part.** Fix $0 < \tau \ll 1$, and set

$$\delta_c = \frac{\tau^2}{2}$$

By Corollary 8.5 of Section 8.1, there exists a compact set $K^\tau_{\delta_c,\bar{r}}$ such that

(1) its complement has small measure

$$|D^{\mathrm{s}}\boldsymbol{b}|\big(\mathbb{R}^{d+1} \setminus K^\tau_{\delta_c,\bar{r}}\big) < \tau;$$

(2) each $(\bar{t},\bar{x}) \in K^\tau_{\delta_c,\bar{r}}$ is a Lebesgue point for $\mathsf{m} \otimes \mathsf{n}$: denote by

$$\overline{\mathsf{m}} \otimes \overline{\mathsf{n}} = \mathsf{m} \otimes \mathsf{n}(\bar{t},\bar{x})$$

its value, and for every $r < \bar{r}$ it holds

$$|D^{\mathrm{a.c.}}\boldsymbol{b}|(B^{d+1}_r(\bar{t},\bar{x})), \int_{B^{d+1}_r(\bar{t},\bar{x})} |\mathsf{m} \otimes \mathsf{n} - \overline{\mathsf{m}} \otimes \overline{\mathsf{n}}| |D^{\mathrm{s}}\boldsymbol{b}|(dtdx) < \tau^2 |D^{\mathrm{s}}\boldsymbol{b}|(B^{d+1}_r(\bar{t},\bar{x})); \quad (8.9)$$

(3) for every $(\bar t, \bar x) \in K^{j,\tau}, r < \bar r$ there exists a compact family of $\delta_c$-Lipschitz functions $\{y_{\mathsf{n}} = L_{t,h}(y_{\mathsf{n}}^\perp)\}_{h \in H}$ such that defining the function $\mathcal{U}$ by

$$\mathcal{U}(t, \bar x) = 0, \quad D\mathcal{U}(t) = \int_H \Big\{ \big[\bar{\mathsf{m}} \otimes (1, -\nabla_{y_{\mathsf{n}}^\perp} L_{t,h})\big] \delta_t \otimes \big((\mathbb{I}, L_{t,h})_\sharp \mathscr{L}^{d-1}\big) \Big\} \, dt dh,$$

then it holds

$$|D\mathcal{U} - D\boldsymbol{b}|(B^{d+1}_r(\bar t, \bar x)) < \tau^2 |D\boldsymbol{b}|(B^{d+1}_r(\bar t, \bar x)). \tag{8.10}$$

This compact set is obtained by the union of the compact sets $K^{\tau,j}_{\delta_c,\bar r}$ of Corollary 8.5, with $\bar r \ll 1$.

8.2.2.1. *Construction of the approximate cylinders of flow.* We can assume that $\bar{\mathsf{n}} = \boldsymbol{e}_1$, and write $y = (y_1, y^\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$ for the corresponding coordinates. Set

$$\bar\ell_1 := \tau\ell, \qquad \delta_1 := \tau^2, \qquad \ell > 0,$$

and let $\eta$ be a Lagrangian representation of $\rho(1, \boldsymbol{b})\mathscr{L}^{d+1} \llcorner_{B^{d+1}_{\bar r}(\bar t, \bar x)}$.

We consider three cases.

**Case 1:** $\bar{\mathsf{m}}_1 = \bar{\mathsf{m}} \cdot \bar{\mathsf{n}} = \bar{\mathsf{m}} \cdot \boldsymbol{e}_1 < -\tau$**.** For every $\gamma \in \Gamma$, define the functions $\ell^\pm_{1,\gamma} : [t^-_\gamma, t^+_\gamma] \times B^{d-1}_\ell \to \mathbb{R}$ by solving the following ODEs:

$$\partial_t \ell^-_{1,\gamma}(t, y^\perp) = -\mathcal{U}_1\big(t, \gamma(t) + (-\ell^-_{1,\gamma}(t, y^\perp)+, y^\perp)\big) + \mathcal{U}_1\big(t, \gamma(t) + ((-\delta_1 - \delta_c)\ell+, 0)\big), \tag{8.11a}$$

$$\partial_t \ell^+_{1,\gamma}(t, y^\perp) = \mathcal{U}_1\big(t, \gamma(t) + (\ell^+_{1,\gamma}(t, y^\perp)-, y^\perp)\big) - \mathcal{U}_1\big(t, \gamma(t) + ((\delta_1 + \delta_c)\ell-, 0)\big), \tag{8.11b}$$

with initial data $\ell^\pm_{1,\gamma}(t^-_\gamma, y^\perp) = \bar\ell_1$. We recall that $\mathcal{U}_1 = \mathcal{U} \cdot \boldsymbol{e}_1 = \mathcal{U} \cdot \bar{\mathsf{n}}$, and we have denoted with $\pm$ the right/left limits of 1-d BV functions.

Lemma 8.7. *The solutions to* (8.11) *satisfy*

*(1)* $[t^-_\gamma, t^+_\gamma] \ni t \mapsto \ell^\pm_{1,\gamma}(t, y^\perp)$ *is decreasing;*
*(2)* $B^{d-1}_\ell(0) \ni y^\perp \mapsto \ell^\pm_{1,\gamma}(t, y^\perp)$ *is* $\delta_c$*-Lipschitz continuous;*
*(3)* $\delta_1 \ell \leq \ell^\pm_{1,\gamma}(t, y^\perp) \leq \bar\ell_1$ *for all* $(t, y^\perp) \in [t^-_\gamma, t^+_\gamma] \times B^{d-1}_\ell(\bar t, \bar x)$.

Proof. We prove the lemma for $\ell^+_{1,\gamma}$, being the analysis of $\ell^-_{1,\gamma}$ equivalent. The existence of a unique solution which is decreasing in time is standard, see for example [BG11]: indeed for fixed $(t, y^\perp)$

$$y_1 \mapsto \mathcal{U}_1(t, (y_1, y^\perp))$$

is decreasing because $\bar{\mathsf{m}}_1 < 0$, and then classical results on the flow of monotone operators apply.

The fact that the level sets of $\mathcal{U}_1$ are $\delta_c$-Lipschitz in the coordinates $(y_1, y^\perp)$ implies that

$$\mathcal{U}_1\big(t, \gamma(t) + (\delta_1\ell + \delta_c(\ell - |y^\perp|)-, y^\perp)\big) \geq \mathcal{U}_1\big(t, \gamma(t) + ((\delta_1 + \delta_c)\ell-, 0)\big),$$

so that the solution starting from $\bar\ell_1 = \tau\ell > (\delta_1 + \delta_c)\ell$ satisfies

$$\ell^+_{1,\gamma}(t, y^\perp) \geq \delta_1\ell + \delta_c(\ell - |y^\perp|) \geq \delta_1$$

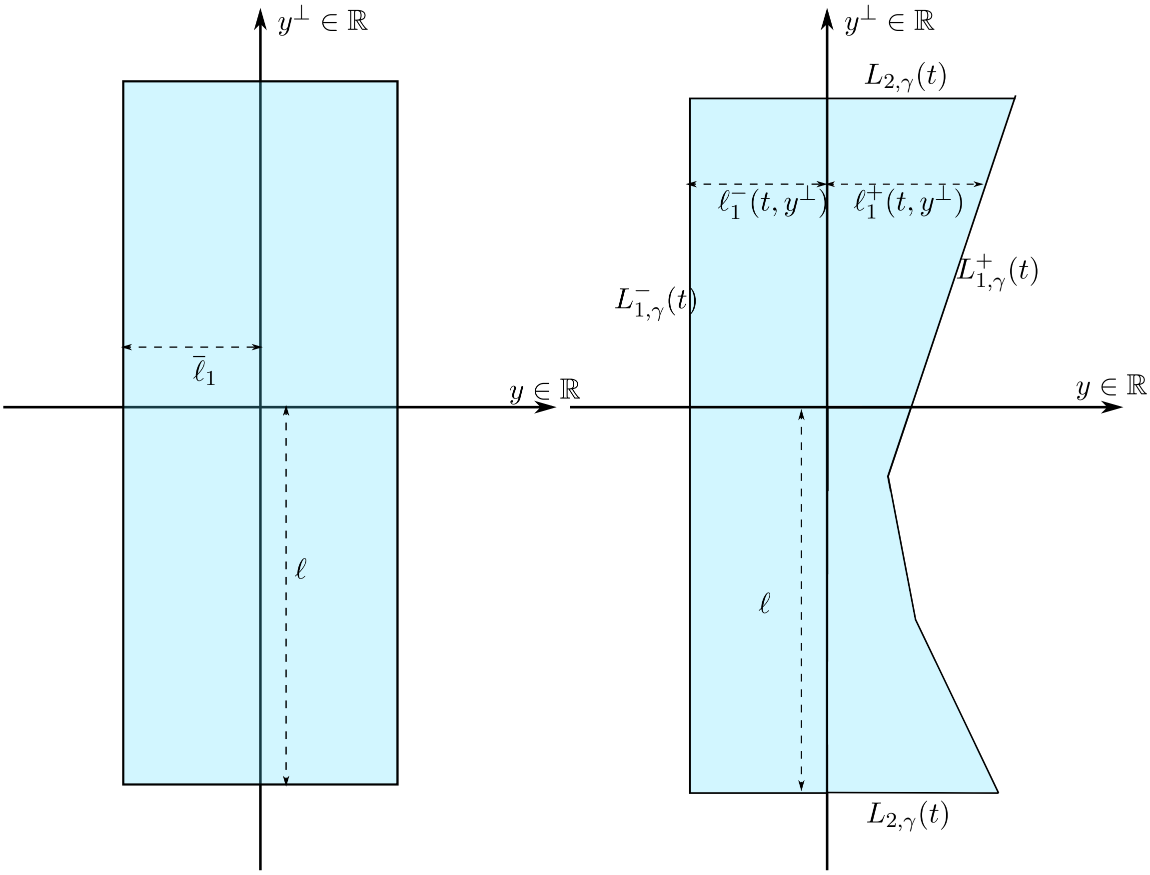
when $|y^\perp| < \ell$.

For $\bar y^\perp$ fixed, again from the $\delta_c$-Lipschitz regularity of the level sets of $\mathcal{U}$, it is easy to see that the cone

$$\big|y_1 - \ell^\pm_{1,\gamma}(t, \bar y^\perp)\big| \leq \delta_c |y^\perp - \bar y^\perp|$$

is invariant for the flow of the ODEs (8.11), so that for any fixed time $t$ it holds that $\ell^+_{1,\gamma}(t, y^\perp)$ is $\delta_c$-Lipschitz. □

**Case 2:** $\bar{\mathsf{m}}_1 > \tau$**.** Define the functions $\ell^\pm_{1,\gamma} : [t^-_\gamma, t^+_\gamma] \times B^{d-1}_\ell \to \mathbb{R}$ by solving the ODEs (8.11a) backward in time with final data $\ell^\pm_{1,\gamma}(t^+_\gamma, y^\perp) = \bar\ell_1$. As in Lemma 8.7, one can check that




**(a)** The base of the cylinder, i.e. the set $\bar{Q}$.

**(b)** The base of the cylinder at a time $t$, i.e. the set $Q(t)$.

**Figure 1.** Time sections of the cylinder of approximate flow in the singular, 2D case.

(1) $[t_\gamma^-, t_\gamma^+] \mapsto \ell_{1,\gamma}^\pm(t, y^\perp)$ is increasing;
(2) $B_r^{d-1}(0) \ni y^\perp \mapsto \ell_{1,\gamma}^\pm(t, y^\perp)$ is $\delta_c$-Lipschitz continuous;
(3) $\delta_1 \ell \leq \ell_{1,\gamma}^\pm(t, y^\perp) \leq \bar\ell_1$.

**Case 3:** $|\overline{\mathbf{m}}_1| < \tau$**.** In this case set $\ell_{1,\gamma}^\pm(t) = \bar\ell_1$ constant.

Define (see Figure 1b and 2b)

$$Q_\gamma^\ell(t) = Q_{\ell_{1,\gamma}^\pm, \ell} = Q_{\ell_{1,\gamma}^-, \ell_{1,\gamma}^+, \ell}(t) := \Big\{ y = (y_1, y^\perp) : -\ell_1^-(t, y^\perp) \leq y \leq \ell_1^+(t, y^\perp),\ |y^\perp| \leq \ell \Big\}.$$

For future reference we call

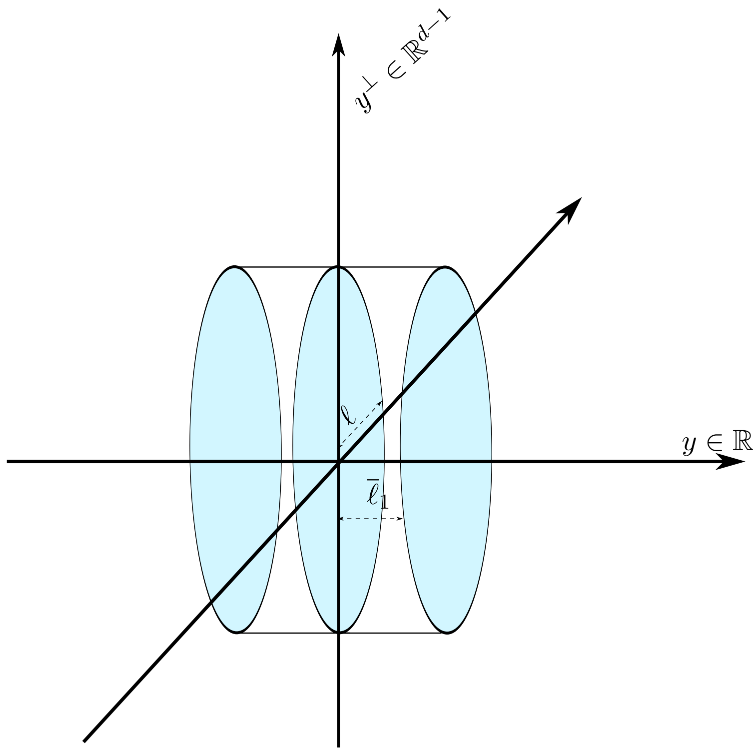
$$\bar{Q} := \Big\{ y = (y, y^\perp) : -\bar\ell_1 \leq y \leq \bar\ell_1,\ |y^\perp| \leq \ell \Big\},$$

see also Figure 1a and Figure 2a. Define the lateral sides of $Q_{\ell_{1,\gamma}^\pm, \ell}(t)$ as

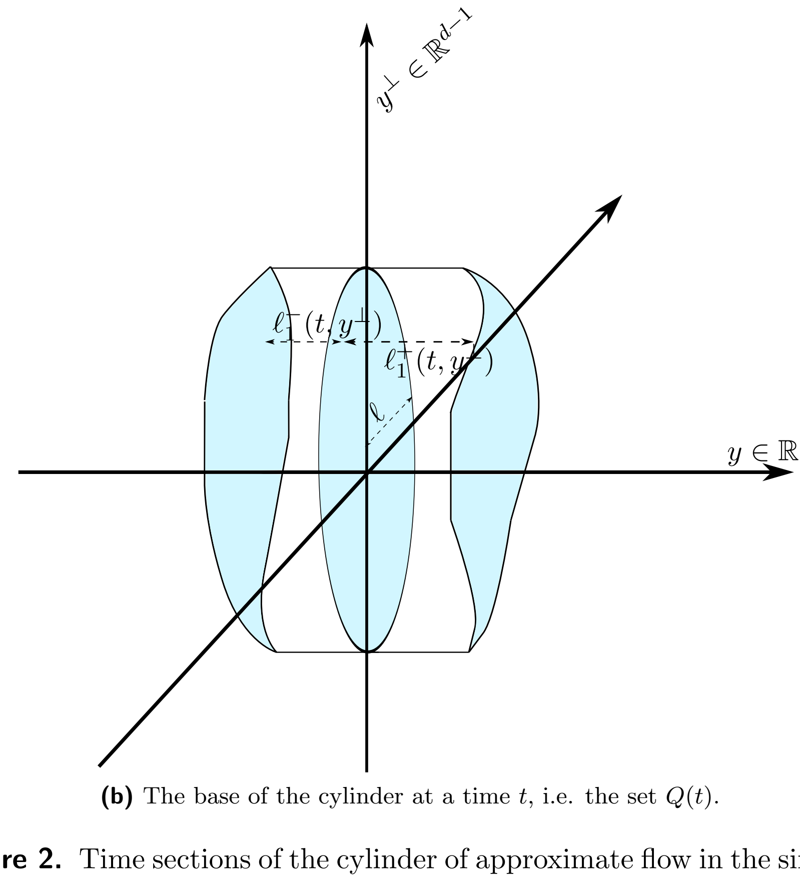
$$L_{1,\gamma}^\pm(t) := \pm \operatorname{Graph} \ell_{1,\gamma}^\pm(t)$$

and

$$L_{2,\gamma}(t) = \Big\{ (y_1, y^\perp), -\ell_{1,\gamma}^-(t, y^\perp) \leq y_1 \leq \ell_{1,\gamma}^+(t, y^\perp), |y^\perp| = \ell \Big\}.$$




**(a)** The base of the cylinder, i.e. the set $\bar{Q}$.




**(b)** The base of the cylinder at a time $t$, i.e. the set $Q(t)$.

**Figure 2.** Time sections of the cylinder of approximate flow in the singular, $d$-dimensional case.

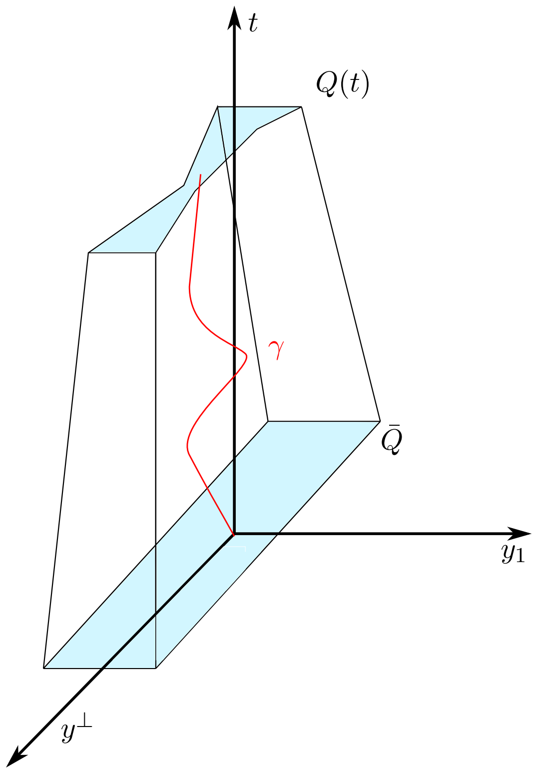



**Figure 3.** Evolution in time of the cylinder of approximate flow in the singular, 2D case.

After some standard computations, we have that the lateral inner flow across $Q_{\ell^{\pm}_{1,\gamma},\ell}$ is given by

$$\int_{t\in(t^-_\gamma,t^+_\gamma)} \big|\,\mathsf{Tr}^{\rm in}\big((1,\boldsymbol{b}),Q_{\ell^{\pm}_{1,\gamma},\ell}(t)\big)\cdot\mathbf{n}(t)\big|\,\mathscr{H}^d\llcorner_{\partial Q_{\ell^{\pm}_{1,\gamma},\ell}} = I_{2,\gamma}+I^+_{1,\gamma}+I^-_{1,\gamma},$$

where

$$I_{2,\gamma} := \int_{t^-_\gamma}^{t^+_\gamma}\int_{L_{2,\gamma}(t)} \big|\big(\dot\gamma(t)-\boldsymbol{b}^-\big)\cdot\boldsymbol{e}^\perp\big|\,\mathscr{H}^{d-1}dt \tag{8.12}$$

and

$$I^+_{1,\gamma} := \int_{t^-_\gamma}^{t^+_\gamma}\int_{L^+_{1,\gamma}(t)} \big|\big(\partial_t\ell^+_{1,\gamma}\boldsymbol{e}_1+\dot\gamma-\boldsymbol{b}^-\big)\cdot\mathbf{n}\big|\,\mathscr{H}^{d-1}dt, \tag{8.13a}$$

$$I^-_{1,\gamma} := \int_{t^-_\gamma}^{t^+_\gamma}\int_{L^-_{1,\gamma}(t)} \big|\big(-\partial_t\ell^-_{1,\gamma}\boldsymbol{e}_1-\dot\gamma+\boldsymbol{b}^-\big)\cdot\mathbf{n}\big|\,\mathscr{H}^{d-1}dt. \tag{8.13b}$$

To simplify notations we put an apex $-$ to denote the inner trace of $\boldsymbol{b}$ on the boundary of a Lipschitz set, and we recall that $\mathbf{n}=(1,-\nabla_{y^\perp}\ell^{\pm}_{1,\gamma}(t))/|(1,-\nabla_{y^\perp}\ell^{\pm}_{1,\gamma}(t))|$.

8.2.2.2. *Estimates on the flux.* The following lemmata will be proved in the next section.

Lemma 8.8 (Transversal flux). *For all $(\bar t,\bar x)\in K^\tau_{\delta_c,\bar r}$, $r<\bar r$ it holds*

$$\int \frac{1}{\mathscr{L}^d(\bar Q)} I_{2,\gamma}\,\eta(d\gamma)\le C_{d-1}\tau|D\boldsymbol{b}|\big(B^{d+1}_{r+2\ell}(\bar t,\bar x)\big).$$

Lemma 8.9 (Non-transversal flux). *For all $(\bar t,\bar x)\in K^\tau_{\delta_c,\bar r}$, $r<\bar r$ it holds*

$$\int \frac{1}{\mathscr{L}^d(\bar Q)} I^{\pm}_{1,\gamma}\,\eta(d\gamma)\le C_{d-1}\tau|D\boldsymbol{b}|\big(B^{d+1}_{r+2\ell}(\bar t,\bar x)\big).$$

From these results we deduce the following proposition.

Proposition 8.10. *For every point* $(\bar{t},\bar{x}) \in K^{\tau}_{\delta_c,\bar{r}}$ *and* $\varepsilon > 0$, *there exists a family of* $(1,\boldsymbol{b})$*-proper balls* $\{B^{d+1}_r(\bar{t},\bar{x})\}_r$, *with* $r < \bar{r}$ *having* 0 *as a Lebesgue point, such that Assumption 7.6 holds with constant*

$$\varpi_r(\bar{t},\bar{x}) \leq C_{d-1}\tau|D\boldsymbol{b}|\big(B^{d+1}_{r+\varepsilon}(\bar{t},\bar{x})\big).$$

Proof. First of all, by the regularity assumptions on $\boldsymbol{b}$, it follows that the lateral boundary of $Q_{\ell^{\pm}_{1,\gamma},\ell}$ is inner regular, so that Point (1) of Assumption 7.6 is verified. Moreover by construction Point (2) holds with constant $M = \delta_1$, being $\delta_1\ell \leq \ell^{\pm}_{1,\gamma}$. Finally for $2\ell < \varepsilon$ one applies the above Lemmata to recover Point (3). □

By the above proposition and Proposition 8.6 we thus conclude that

Theorem 8.11. *Assumptions 7.27 holds for a vector field of the form* $(1,\boldsymbol{b})$ *with* $\boldsymbol{b} \in L^1_{\mathrm{loc}}(\mathbb{R},\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d))$.

Proof. By choosing the local balls accordingly to Proposition 8.10 on $K^{\tau}_{\delta_c,\bar{r}}$ and according to Proposition 8.6 in the remaining points, one sees that the measure $\varpi^{\tau}$ can be taken to be

$$\varpi^{\tau} = C_{d-1}\tau|D\boldsymbol{b}|\llcorner_{K^{\mathrm{a.c.}}\cup K^{\tau}_{\delta_c,\bar{r}}} + C_d|D\boldsymbol{b}|\llcorner_{\mathbb{R}^{d+1}\setminus(K^{\mathrm{a.c.}}\cup K^{\tau}_{\delta_c,\bar{r}})},$$

where $K^{\mathrm{a.c.}}$ is a compact set made of Lebesgue points for $D^{\mathrm{a.c.}}\boldsymbol{b}$. In particular the measure $\varpi^{\tau}$ can be made arbitrarily small by letting first $\tau \to 0$ and then $\bar{r} \to 0$, so to have $K^{\mathrm{a.c.}} \cup K^{\tau}_{\delta_c,\bar{r}} \nearrow \mathbb{R}^{d+1}$. □

## 8.3. Flux estimates and proof of Lemmata 8.8 and 8.9

Here we prove the two lemmata that allow to control the boundary flux of $(1,\boldsymbol{b})$ on $Q_{\ell^{\pm}_{1,\gamma},\ell}$. We will just prove the case $\overline{\mathsf{m}}_1 < -\tau$, being the second case completely analogous by inverting time and the case $\ell^{\pm}_{1,\gamma} = \bar{\ell}_1$ a simple variation of the first situation.

Observe that for a given positive Borel function $f(x,y)$ it holds

$$\begin{aligned}\int_{\Gamma}\int_{L_{2,\gamma}(t)} f(\gamma(t),y)\,\mathscr{H}^{d-1}(dy)\eta(d\gamma) &\leq \int_{\Gamma}\int_{\bar{L}_2} f(\gamma(t),y)\,\mathscr{H}^{d-1}(dy)\eta(d\gamma) \\ &= \int_{(B^{d+1}_{\bar{r}}(\bar{t},\bar{x}))_t}\int_{\bar{L}_2} f(x,y)\,\mathscr{H}^{d-1}(dy)\mathscr{L}^d(dx),\end{aligned} \tag{8.14}$$

where we used the notation

$$\bar{L}_2 := \big\{(y_1,y^{\perp}), |y_1| \leq \bar{\ell}_1, |y^{\perp}| = \ell\big\},$$

and $(B^{d+1}_{\bar{r}}(\bar{t},\bar{x}))_t$ is the $t$-time section of the ball where $(\mathtt{p}_{t,x})_\sharp\eta$ is concentrated.

### 8.3.1. Proof of Lemma 8.8.

We recall that the quantity $I_{2,\gamma}$ was defined in (8.12) as

$$I_{2,\gamma} = \int_{t^-_{\gamma}}^{t^+_{\gamma}}\int_{L_{2,\gamma}(t)}\big|\big(\dot{\gamma}(t) - \boldsymbol{b}(t,\gamma(t)+y-)\big)\cdot\boldsymbol{e}^{\perp}\big|\,\mathscr{H}^{d-1}(dy)dt.$$

Since this quantity is defined for a curve $\gamma$ and then integrated in $\gamma$, by the a.c. of the projection of $\eta$ on $\{t\}\times\mathbb{R}^d$ we will consider $\boldsymbol{b}$ defined on suitable planes passing through $\gamma(t)$. We will also avoid putting the $-$ sign to remember that we are taking the inner trace: for this term indeed, begin the surface $L_{2,\gamma}$ a subset of $\gamma + \{|y_1| < \bar{\ell}_1\}\times B^{d-1}_{\ell}(0)$, one can assume that it is made of Lebesgue points.

PROOF OF LEMMA 8.8. Observe first that, for fixed $t$, adding and subtracting the term $\boldsymbol{b}(t,(\gamma_1(t)+y_1,\gamma^\perp(t)))$ and using the triangular inequality, we can write

$$\begin{aligned}
&\int_{L_{2,\gamma}(t)} \Big|\big(\dot\gamma(t)-\boldsymbol{b}(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp))\big)\cdot\boldsymbol{e}^\perp\Big|\,\mathscr{H}^{d-1}(dy)\\
&\qquad\le \int_{L_{2,\gamma}(t)} \Big|\Big[\boldsymbol{b}\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp)\big)-\boldsymbol{b}\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big)\Big]\cdot\boldsymbol{e}^\perp\Big|\,\mathscr{H}^{d-1}(dy)\\
&\qquad\quad+\int_{L_{2,\gamma}(t)} \Big|\Big[\boldsymbol{b}\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big)-\dot\gamma(t)\Big]\cdot\boldsymbol{e}^\perp\Big|\,\mathscr{H}^{d-1}(dy).
\end{aligned}\tag{8.15}$$

Integrating (8.15) in $\eta$ and dividing by $\mathscr{L}^d(\bar Q)=2\omega_{d-1}\ell^{d-1}\bar\ell_1$, we have that

$$\begin{aligned}
&\frac{1}{\mathscr{L}^d(\bar Q)}\int_\Gamma\int_{L_{2,\gamma}(t)} \big|\big(\dot\gamma(t)-\boldsymbol{b}(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp))\big)\cdot\boldsymbol{e}^\perp\big|\,\mathscr{H}^{d-1}(dy)\eta(d\gamma)\\
&\le \frac{1}{\mathscr{L}^d(\bar Q)}\int_\Gamma\int_{L_{2,\gamma}(t)} \Big|\boldsymbol{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp)\big)\\
&\qquad\qquad\qquad -\boldsymbol{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big)\Big|\,\mathscr{H}^{d-1}(dy)\eta(d\gamma)\\
&\quad+\frac{1}{\mathscr{L}^d(\bar Q)}\int_\Gamma\int_{L_{2,\gamma}(t)} \Big|\boldsymbol{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big)-(\dot\gamma(t))^\perp\Big|\,\mathscr{H}^{d-1}(dy)\eta(d\gamma)\\
&=: S_2^{\mathrm{BV}}(t)+S_2^{\mathrm{av}}(t).
\end{aligned}$$

We now proceed to estimate the two terms separately.

*Step 1. Estimate of the term $S_2^{\mathrm{BV}}(t)$.* By (8.14) we have

$$\begin{aligned}
S_2^{\mathrm{BV}}(t) &= \frac{1}{\mathscr{L}^d(\bar Q)}\int_\Gamma\int_{L_{2,\gamma}(t)} \Big|\boldsymbol{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp)\big)\\
&\qquad\qquad\qquad -\boldsymbol{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big)\Big|\,\mathscr{H}^{d-1}(dy)\eta(d\gamma)\\
&\le \frac{1}{\mathscr{L}^d(\bar Q)}\int_{(B^{d+1}_{\bar r}(\bar t,\bar x))_t}\int_{\bar L_2} \Big|\boldsymbol{b}^\perp\big(t,(x_1+y_1,x^\perp+y^\perp)\big)\\
&\qquad\qquad\qquad -\boldsymbol{b}^\perp\big(t,(x_1+y_1,x^\perp)\big)\Big|\,\mathscr{H}^{d-1}(dy)\mathscr{L}^d(dx).
\end{aligned}$$

By Fubini and the one dimensional slicing of BV functions [Zie89, Theorem 5.3.5], we deduce

$$\begin{aligned}
S_2^{\mathrm{BV}}(t) &\le \frac{1}{\mathscr{L}^d(\bar Q)}\int_{\bar L_2}\bigg[\int_{(B^{d+1}_{\bar r}(\bar t,\bar x))_t} \big|D^\perp\boldsymbol{b}_t^\perp\big|\big(x_1+y_1,(x^\perp,x^\perp+y^\perp)\big)\,\mathscr{L}^d(dx)\bigg]\mathscr{H}^d(dy)\\
&\le \frac{1}{\mathscr{L}^d(\bar Q)}\int_{\bar L_2}\ell\big|D^\perp\boldsymbol{b}_t^\perp\big|\big((B^{d+1}_{\bar r+\ell+\bar\ell_1}(\bar t,\bar x))_t\big)\,\mathscr{H}^d(dy)\\
&\le \frac{1}{2\omega_{d-1}\ell^{d-1}\bar\ell_1}\cdot 2(d-1)\omega_{d-1}\ell^{d-2}\bar\ell_1\cdot\ell\big|D^\perp\boldsymbol{b}_t^\perp\big|\big((B^{d+1}_{\bar r+(1+\tau)\ell}(\bar t,\bar x))_t\big)\\
&\le C_{d-1}\big|D^\perp\boldsymbol{b}_t^\perp\big|\big((B^{d+1}_{\bar r+2\ell}(\bar t,\bar x))_t\big).
\end{aligned}$$

Finally, integrating in time and using (8.9), we obtain

$$\int S_2^{\mathrm{BV}}(t)\,\mathscr{L}^1(dt)\le C_{d-1}\big|D^\perp\boldsymbol{b}^\perp\big|\big(B^{d+1}_{\bar r+2\ell}(\bar t,\bar x)\big)\le C_{d-1}\tau|D\boldsymbol{b}|\big(B^{d+1}_{\bar r+2\ell}(\bar t,\bar x)\big).\tag{8.16}$$

*Step 2. Estimate of the term $S_2^{\mathrm{av}}(t)$.* We have using again (8.14)

$$S_2^{\mathrm{av}}(t) = \frac{1}{\mathscr{L}^d(\bar{Q})} \int_\Gamma \int_{L_{2,\gamma}(t)} \Big| \boldsymbol{b}^\perp\big(t, \big(\gamma_1(t) + y_1, \gamma^\perp(t)\big)\big) - (\dot\gamma(t))^\perp \Big| \mathscr{H}^{d-1}(dy)\eta(d\gamma)$$
$$\leq \frac{1}{\mathscr{L}^d(\bar{Q})} \int_{(B_r^{d+1}(\bar t,\bar x))_t} \int_{\bar L_2} \Big| \boldsymbol{b}^\perp\big(t, \big(x_1 + y_1, x^\perp\big)\big) - \boldsymbol{b}^\perp(t,x) \Big| \mathscr{H}^{d-1}(dy)\mathscr{L}^d(dx),$$

and arguing as before, using Fubini and the one dimensional slicing of BV functions, we obtain

$$\begin{aligned}
S_2^{\mathrm{av}}(t) &\leq \frac{1}{\mathscr{L}^d(\bar{Q})} \int_{\bar L_2} \int_{(B_r^{d+1}(\bar t,\bar x))_t} \Big| \boldsymbol{b}^\perp\big(t, \big(x_1 + y_1, x^\perp\big)\big) - \boldsymbol{b}^\perp(t,x) \Big] \cdot e^\perp \Big| \mathscr{L}^d(dx)\mathscr{H}^{d-1}(dy) \\
&\leq \frac{1}{\mathscr{L}^d(\bar{Q})} \int_{\bar L_2} \bar\ell_1 |D_1 \boldsymbol{b}_t|\big((B^{d+1}_{\bar r+\bar\ell_1}(\bar t,\bar x))_t\big) \,\mathscr{H}^{d-1}(dy) \\
&\leq \frac{1}{2\omega_{d-1}\ell^{d-1}\bar\ell_1} \cdot 2(d-1)\omega_{d-1}\ell^{d-2}\bar\ell_1 \cdot \bar\ell_1 |D_1 \boldsymbol{b}_t|\big((B^{d+1}_{r+\bar\ell_1}(\bar t,\bar x))_t\big) \\
&\leq C_{d-1}\frac{\bar\ell_1}{\ell} |D_1 \boldsymbol{b}_t|\big((B^{d+1}_{r+\bar\ell_1}(\bar t,\bar x))_t\big) \\
&\leq C_{d-1}\tau |D_1 \boldsymbol{b}_t|\big((B^{d+1}_{r+\bar\ell_1}(\bar t,\bar x))_t\big).
\end{aligned}$$

Integrating in time we obtain

$$\int S_2^{\mathrm{av}}(t)\,\mathscr{L}^1(dt) \leq C_{d-1}\tau |D\boldsymbol{b}|\big(B^{d+1}_{r+\bar\ell_1}(\bar t,\bar x)\big). \tag{8.17}$$

Summing up (8.16) and (8.17) we finally deduce, for $\tau \ll 1$,

$$\begin{aligned}
&\frac{1}{\mathscr{L}^d(\bar{Q})} \int_\Gamma I_{2,\gamma}\,\eta(d\gamma) \\
=&\frac{1}{\mathscr{L}^d(\bar{Q})} \int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \int_{L_{2,\gamma}(t)} \big| \big(\dot\gamma(t) - \boldsymbol{b}(t, x+y)\big) \cdot \boldsymbol{e}^\perp \big| \mathscr{H}^{d-1}(dy)\mathscr{L}^1(dt)\eta(d\gamma) \\
\leq& \int S_2^{\mathrm{BV}}(t)\,\mathscr{L}^1(dt) + \int S_2^{\mathrm{av}}(t)\,\mathscr{L}^1(dt) \\
\leq& C_{d-1}\tau |D\boldsymbol{b}|\big(B^{d+1}_{r+2\ell}(\bar t,\bar x)\big).
\end{aligned}$$

which is the claim. □

**8.3.2. Proof of Lemma 8.9.** The proof of Lemma 8.9 depends heavily on the shape of the cylinders, which cancel the effect of the divergence thanks to the choice of $\ell^\pm_{1,\gamma}(t, y^\perp)$. The goal is to show Lemma 8.9, i.e.

$$\frac{1}{\mathscr{L}^d(\bar{Q})} \int I^\pm_{1,\gamma}\,\eta(d\gamma) \leq C_{d-1}\tau |D\boldsymbol{b}|\big(B_{r+2\ell}\big).$$

We will prove only the estimate for $I^{1,+}_\gamma$ being the other case identical.

Proof of Lemma 8.9 for $I^+_{1,\gamma}$. Recall that the quantity $I^+_{1,\gamma}$ was defined in (8.13) as

$$
\begin{aligned}
I^+_{1,\gamma} &= \int_{\bar t-\bar r/2}^{\bar t+\bar r/2}\int_{L^+_{1,\gamma}(t)} \big|\big(\dot\ell^+_{1,\gamma}\boldsymbol{e}_1+\dot\gamma-\boldsymbol{b}^-\big)\cdot\mathbf{n}\big|\,\mathscr{H}^{d-1}(dy)dt\\
&\le \int_{\bar t-\bar r/2}^{\bar t+\bar r/2}\int_{|y^\perp|<\ell} \big|\dot\ell^+_{1,\gamma}+\dot\gamma_1-\boldsymbol{b}_1\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t),y^\perp)\big)\big|\,\mathscr{L}^{d-1}(dy^\perp)dt\\
&\quad+\int_{\bar t-\bar r/2}^{\bar t+\bar r/2}\int_{|y^\perp|<\ell} \big|\big(\dot\gamma^\perp-\boldsymbol{b}^\perp\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t),y^\perp)\big)\big)\cdot\nabla_{y^\perp}\ell^+_{1,\gamma}(t,y^\perp)\big|\,\mathscr{L}^{d-1}(dy^\perp)dt\\
&\le \int_{\bar t-\bar r/2}^{\bar t+\bar r/2}\int_{|y^\perp|<\ell} \Big|\boldsymbol{b}_1\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t),y^\perp)\big)\\
&\qquad\qquad -\boldsymbol{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot\ell^+_1(t)\Big|\,\mathscr{L}^{d-1}(dy^\perp)dt\\
&\quad+\int_{\bar t-\bar r/2}^{\bar t+\bar r/2}\int_{|y^\perp|<\ell} \big|\boldsymbol{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot\gamma_1(t)\big|\,\mathscr{L}^{d-1}(dy^\perp)dt\\
&\quad+\int_{\bar t-\bar r/2}^{\bar t+\bar r/2}\int_{|y^\perp|<\ell} \big|\big(\dot\gamma^\perp-\boldsymbol{b}^\perp\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t),y^\perp)\big)\big)\cdot\nabla_{y^\perp}\ell^+_{1,\gamma}(t,y^\perp)\big|\,\mathscr{L}^{d-1}(dy^\perp)dt.
\end{aligned}
$$

Integrating at a fixed time $t$ the above equation in $\gamma$ and dividing by the area of $\bar Q$, we have that

$$
\begin{aligned}
&\frac{1}{\mathscr{L}^d(\bar Q)}\int_\Gamma\int_{|y^\perp|<\ell}\big|\big(\dot\ell^+_{1,\gamma}\boldsymbol{e}_1+\dot\gamma-\boldsymbol{b}^-\big)\cdot\mathbf{n}\big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\le \frac{1}{\mathscr{L}^d(\bar Q)\delta_1\ell}\int_\Gamma\int_{|y^\perp|<\ell}\Big|\boldsymbol{b}_1\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)\\
&\qquad\qquad -\boldsymbol{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot\ell^+_1(t)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\quad+\frac{1}{\mathscr{L}^d(\bar Q)\delta_1\ell}\int_\Gamma\int_{|y^\perp|<\ell}\Big|\boldsymbol{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot\gamma_1(t)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\quad+\frac{1}{\mathscr{L}^d(\bar Q)\delta_1\ell}\int_\Gamma\int_{|y^\perp|<\ell}\Big|\big(\boldsymbol{b}^\perp\big(t,\gamma(t)\big)-\boldsymbol{b}^\perp\big(t,\gamma(t)\\
&\qquad\qquad +(\ell^+_{1,\gamma}(t),y^\perp)\big)\big)\cdot\nabla_{y^\perp}\ell^+_{1,\gamma}(t,y^\perp)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&=: S^{\mathrm{RL}}_1(t)+S^{\mathrm{av}}_1(t)+S^{\mathrm{tr}}_1(t).
\end{aligned}
$$

We now proceed to estimate the terms separately.

*Step 1. Estimate of the term $S^{\mathrm{av}}_1(t)$.* We have

$$
\begin{aligned}
S^{\mathrm{av}}_1(t) &= \frac{1}{\mathscr{L}^d(\bar Q)}\int_\Gamma\int_{|y^\perp|\le\ell}\Big|\boldsymbol{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot\gamma_1(t)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&= \frac{1}{2\bar\ell_1}\int_{(B^{d+1}_{\bar r}(\bar t,\bar x))_t}\int_{|y^\perp|\le\ell}\Big|\boldsymbol{b}_1\big(t,x+((\delta_1+\delta_c)\ell,0)\big)-\boldsymbol{b}_1(t,x)\big)\Big|\,\mathscr{L}^d(dx)\\
&\le \frac{(\delta_1+\delta_c)\ell}{2\bar\ell_1}|D\boldsymbol{b}_t|\big((B^{d+1}_{\bar r+\ell}(\bar t,\bar x))_t\big)\\
&\le \tau|D\boldsymbol{b}_t|\big((B^{d+1}_{\bar r+\ell}(\bar t,\bar x))_t\big)
\end{aligned}
$$

by Fubini and the one dimensional slicing of BV.

*Step 2: estimate of the term $S_1^{\mathrm{RL}}(t)$.* By the definition of $\ell^+_{1,\gamma}(t,y^\perp)$ through the ODE (8.11b) we obtain that $S_1^{\mathrm{RL}}(t)$ can be estimated by

$$\begin{aligned}
&\frac{1}{\mathscr{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|\boldsymbol{b}_1\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)\\
&\qquad\qquad\qquad -\boldsymbol{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot{\ell}^+_1(t,y^\perp)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&=\frac{1}{\mathscr{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)\\
&\qquad\qquad\qquad -(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\le\frac{1}{\mathscr{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)\\
&\qquad\qquad\qquad -(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,y^\perp)\big)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\quad+\frac{1}{\mathscr{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+(\delta_1+\delta_c,y^\perp)\big)\\
&\qquad\qquad\qquad -(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\le\frac{1}{\mathscr{L}^d(\bar{Q})}\int_\Gamma|D\boldsymbol{b}-D\mathcal{U}|\big(\gamma(t)+\{|y^\perp|<\ell,\delta_1\ell<y_1<\bar{\ell}_1\}\big)\eta(d\gamma)\\
&\quad+\frac{1}{\mathscr{L}^d(\bar{Q})}\int_{(B^{d+1}_{\bar{r}}(\bar{t},\bar{x}))_t}\int_{|y^\perp|<\ell}\Big|(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,x+(\delta_1+\delta_c,y^\perp)\big)\\
&\qquad\qquad\qquad -(\boldsymbol{b}_1-\mathcal{U}_1)\big(t,x+((\delta_1+\delta_c)\ell,0)\big)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\mathscr{L}^d(dx)\\
&\le\frac{1}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot 2\omega_{d-1}\ell^{d-1}\bar{\ell}_1\cdot|D\boldsymbol{b}_t-D\mathcal{U}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\\
&\quad+\frac{1}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot 2\omega_{d-1}\ell^{d-1}\ell\cdot|D^\perp\boldsymbol{b}_t-D^\perp\mathcal{U}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\\
&\le C_{d-1}\left(\tau+\frac{\ell\tau^2}{\bar{\ell}_1}\right)|D\boldsymbol{b}_t|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\le C_{d-1}\tau|D\boldsymbol{b}_t|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big),
\end{aligned}$$

where we applied (8.10) and (8.9) to control the normal derivative.

*Step 3: estimate of the term $S_1^{\mathrm{tr}}(t)$.* For the last term, recalling that $y^\perp\mapsto\ell^+_{1,\gamma}$ is $\delta_c$-Lipschitz by Lemma 8.7, we have that $S_1^{\mathrm{tr}}(t)$ can be estimated by

$$\begin{aligned}
&\frac{1}{\mathscr{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|\le\ell}\big|\big(\dot{\gamma}^\perp-\boldsymbol{b}^\perp\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)\big)\cdot\nabla_{y^\perp}\ell^+_{1,\gamma}(t,y^\perp)\big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\le\frac{\delta_c}{\mathscr{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|\le\ell}\Big|\boldsymbol{b}^\perp\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)-\boldsymbol{b}^\perp\big(t,\gamma(t)\big)\Big|\,\mathscr{L}^{d-1}(dy^\perp)\eta(d\gamma).
\end{aligned}$$

Again enlarging the set $Q_{\ell^\pm_{1,\gamma},\ell}(t)$ to $\bar{Q}$ we obtain

$$\begin{aligned}
S_1^{\mathrm{tr}}(t)&\le\frac{\delta_c}{\mathscr{L}^d(\bar{Q})}\int_{|y^\perp|\le\ell}\big(\ell|D^\perp\boldsymbol{b}|+\bar{\ell}_1|D_1\boldsymbol{b}|\big)\big((B^{d+1}_{r+\ell+\ell_1}(\bar{t},\bar{x}))_t\big)\,\mathscr{L}^{d-1}(dy^\perp)\\
&\le\frac{\delta_c}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot\omega_{d-1}\ell^{d-1}\cdot(\tau+1)\ell|D\boldsymbol{b}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\\
&\le C_{d-1}\tau|D\boldsymbol{b}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big),
\end{aligned}$$

by the choice of $\delta_c\le\tau^2$.

Integrating in time and summing up the three terms we conclude the proof of Lemma 8.9. □

CHAPTER 9

# Forward untangling and vector fields with weak $L^p$ bounds on the gradient

ABSTRACT. We collect in this chapter some results of a work in progress with S. Bianchini [BB17c] where we study possible refinements of the concept of untangling. In particular, in Section 9.1 we discuss the case in which one has only a control of the amount of trajectories which cross *forward in time*: a suitable local condition can be given and a related *forward untangling functional* can be studied, obtaining results similar to the ones presented in Chapter 7.

In particular, with this strategy we recover some results already present in the literature: in Section 9.2 we show that if one has a vector field with suitable bounds on its incremental quotients, then the Lagrangian representation is forward untangled (in particular, we recover in our setting the results of [BC13]). Finally, in Section 9.3, we show a quantitative stability estimate for the (Regular Lagrangian) Flow associated to this class of vector fields.

## 9.1. Forward untangling

Consider a proper set $\Omega \subset \mathbb{R}^{d+1}$, and let $\Omega^\varepsilon$ be the pertubed set constructed in Theorem 6.15. For convenience, in the first part of this section we will drop the index $\varepsilon$ and refer to $\Omega^\varepsilon$ directly as $\Omega$. Recall that the set $S_1$ is defined in (6.12), so that essentially all inflow and outflow of $\rho(1, \boldsymbol{b})$ are occurring on open sets which are contained in finitely many time-flat hyperplanes $\{t = t_i\}$. We can assume without loss of generality that $\mathtt{p}_t(S_1) \subset \{\{t = t_i\}$ is locally proper$\}$. Define now

$$\eta^{\text{in}} := \int_{S_1} \eta_z^{\text{in}} \rho(z)\, \mathscr{H}^d(dz) = \eta \llcorner_{\{\text{Graph}\, \gamma \cap S_1 \neq \emptyset\}},$$

according to Remark 3.7.

We give the following

DEFINITION 9.1. A Lagrangian representation $\eta$ of $\rho(1, \boldsymbol{b})$, with $\operatorname{div}(\rho(1, \boldsymbol{b})) = \mu$, is said to be *forward untangled* if the following condition holds true: $\eta$ is concentrated on a set $\Delta^{\text{for}} \subset \Gamma$ made up of trajectories such that for every $\gamma, \gamma' \in \Delta^{\text{for}} \times \Delta^{\text{for}}$ the following implication holds:

$$\text{if there exists } t \in [\max\{t_\gamma^-, t_{\gamma'}^-\}, \min\{t_\gamma^+, t_{\gamma'}^+\}] \text{ such that } \gamma(t) = \gamma'(t) \text{ then}$$
$$\text{Graph}\gamma\llcorner_{[t, \min\{t_\gamma^+, t_{\gamma'}^+\}]} \cap \text{Graph}\,\gamma'\llcorner_{[t, \min\{t_\gamma^+, t_{\gamma'}^+\}]} \text{ coincides with}$$
$$\text{Graph}\gamma\llcorner_{[t, \min\{t_\gamma^+, t_{\gamma'}^+\}]} \text{ and with } \text{Graph}\,\gamma'\llcorner_{[t, \min\{t_\gamma^+, t_{\gamma'}^+\}]}.$$

This means that the trajectories can bifurcate only in the "past".

**9.1.1. Local theory of forward untangling.** We begin by pointing out a necessary condition for a Lagrangian representation to be forward untangled.

PROPOSITION 9.2. *Let $\eta$ be a forward untangled Lagrangian representation and let $\Omega$ be a perturbed proper set. Then, given $\varpi > 0$, for every $R > 0$ there exists $r > 0$ such that*

$$\int \frac{1}{\sigma(B^d_r(\gamma(t^-_\gamma)))}\eta^{\text{in}}\left(\left\{\gamma' : \begin{array}{l}\gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \\ \operatorname{Graph}\gamma'\llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \operatorname{Graph}\gamma + B^d_R(0)\end{array}\right\}\right)\eta^{\text{in}}(d\gamma) \le \varpi.$$

PROOF. By the forward untangling, it follows that

$$\eta^{\text{in}} = \int \delta_{\gamma_z}\,\rho(z)\mathscr{H}^d(dz)$$

i.e. for any $z \in S_1$ only the curve $\gamma_z$ enters in $\Omega$. By Lusin's Theorem, for every $\delta > 0$, we can find a compact set $K_\delta \subset S_1$ with

$$\int_{K^c_\delta} \rho(z)\,\mathscr{H}^d(dz) < \delta \quad \text{and} \quad K_\delta \ni z \mapsto \gamma_z \text{ continuous w.r.t. } C^0\text{-topology.} \tag{9.1}$$

By the uniform continuity on compact sets, for every $R > 0$ there exists $r > 0$ such that

$$z, z' \in K_\delta : \ \gamma_{z'}(t^-_{\gamma'}) \in \gamma_z(t^-_\gamma) + B^d_r(0) \Rightarrow \operatorname{Graph}\gamma_{z'}\llcorner_{[t^-_\gamma, t^+_\gamma]} \subset \operatorname{Graph}\gamma_z + B^d_R(0). \tag{9.2}$$

Observe that we can write

$$\begin{aligned}&\int \frac{1}{\sigma(B^d_r(\gamma(t^-_\gamma)))}\eta^{\text{in}}\left(\left\{\gamma' : \begin{array}{l}\gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \\ \operatorname{Graph}\gamma'\llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \operatorname{Graph}\gamma + B^d_R(0)\end{array}\right\}\right)\eta^{\text{in}}(d\gamma) \\ &= \int \left\{\frac{1}{\sigma(B^d_r(z))}\int \mathbb{1}_{\{z' : |z'-z| < r, \operatorname{Graph}\gamma_{z'}\llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \operatorname{Graph}\gamma_z + B^d_R(0)\}}\,\rho(z')\mathscr{H}^d(dz')\right\}\rho(z)\mathscr{H}^d(dz).\end{aligned}$$

Now we split the integral in $z$ in two terms, one on the compact set $K_\delta$ and the other in the complement. For simplicity, denote by $\mathcal{A}_R := \{z' \in S_1 : \operatorname{Graph}\gamma_{z'}\llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \operatorname{Graph}\gamma_z + B^d_R(0)\}$. Then we have

$$\begin{aligned}&\int \left\{\frac{1}{\sigma(B^d_r(z))}\int \mathbb{1}_{\{z' : |z'-z| < r, \operatorname{Graph}\gamma_{z'}\llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \operatorname{Graph}\gamma_z + B^d_R(0)\}}\,\rho(z')\mathscr{H}^d(dz')\right\}\rho(z)\mathscr{H}^d(dz) \\ &= \int \left\{\frac{1}{\sigma(B^d_r(z))}\int \mathbb{1}_{B_r(z)\cap\mathcal{A}_R}(z')\,\rho(z')\mathscr{H}^d(dz')\right\}\rho(z)\mathscr{H}^d(dz) \\ &= \int \left\{\frac{1}{\sigma(B^d_r(z))}\int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R}(z')\,\rho(z')\mathscr{H}^d(dz')\right\}\rho(z)\mathscr{H}^d(dz) \\ &\le \int_{K^c_\delta} \Bigg\{\underbrace{\frac{1}{\sigma(B^d_r(z))}\int_{B^d_r(z)} \rho(z')\mathscr{H}^d(dz')}_{\le 1}\Bigg\}\rho(z)\mathscr{H}^d(dz) \\ &\qquad + \int_{K_\delta} \left\{\frac{1}{\sigma(B^d_r(z))}\int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R}(z')\rho(z')\mathscr{H}^d(dz')\right\}\rho(z)\mathscr{H}^d(dz) \\ &\overset{(9.1)}{<} \delta + \int_{K_\delta} \left\{\frac{1}{\sigma(B^d_r(z))}\int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R}(z')\rho(z')\mathscr{H}^d(dz')\right\}\rho(z)\mathscr{H}^d(dz),\end{aligned}$$

For the second integral we notice that the contribution of $z' \in B_r(z) \cap K_\delta$ is zero, in view of (9.2). Hence by Fubini Theorem

$$\begin{aligned}
&\int_{K_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R}(z')\rho(z')\mathscr{H}^d(dz') \right\} \rho(z)\mathscr{H}^d(dz) \\
&= \int_{K_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z) \cap K^c_\delta} \mathbb{1}_{\mathcal{A}_R}(z')\rho(z')\mathscr{H}^d(dz') \right\} \rho(z)\mathscr{H}^d(dz) \\
&\leq \int_{|z-z'|<r, z' \in K^c_\delta} \frac{1}{\sigma(B^d_r(z))} \rho(z)\rho(z')\, \mathscr{H}^{2d}(dz dz') \\
&= \int_{K^c_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z')} \rho(z)\mathscr{H}^d(dz) \right\} \rho(z')\, \mathscr{H}^d(dz') \\
&\overset{(9.1)}{<} \delta.
\end{aligned}$$

The proof is concluded by taking $\delta$ so that $2\delta \leq \varpi$. □

We now turn to prove the converse, which is more delicate and thus we will split the proof in several Lemmata. We will denote now by $\eta$ a Lagrangian representation of $\operatorname{div}(\rho(1, \boldsymbol{b})) = \mu$ in $\Omega$ (which can be taken as the restriction of a Lagrangian representation in $\mathbb{R}^{d+1}$, in view of Theorem 6.30).

Proposition 9.3. *Let $\eta$ be a Lagrangian representation in a perturbed proper set $\Omega \subset \mathbb{R}^{d+1}$. Let $\varpi > 0$ and assume that for all $R > 0$ there exists $r = r(R) > 0$ such that*

$$\int_\Gamma \frac{1}{\sigma(B^d_r(\gamma(t^-_\gamma)))} \eta^{\text{in}} \left( \left\{ \gamma' \in \Gamma : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \\ \operatorname{Graph} \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \operatorname{Graph} \gamma + \operatorname{clos} B^d_R(0) \end{array} \right\} \right) \eta^{\text{in}}(d\gamma) \leq \varpi.$$

*Then there exists a set of trajectories $U$ such that $\eta^{\text{in}} \llcorner_U$ is forward untangled and*

$$\eta^{\text{in}}(U^c) \leq \inf_{C \geq 1} \left\{ \left(2 + \frac{3C}{4}\right) \varpi + \frac{\mu^-(\Omega)}{C} \right\}.$$

We begin by proving the following Lemma, which shows how the piece of information contained in the hypothesis of Proposition 9.3 can be passed to the limit:

Lemma 9.4. *In the setting of Proposition 9.3, it holds*

$$\int \eta^{\text{in}}_z \otimes \eta^{\text{in}}_z \left( \left\{ (\gamma, \gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma), \\ \operatorname{Graph} \gamma \not\subseteq \operatorname{Graph} \gamma',\ \operatorname{Graph} \gamma' \not\subseteq \operatorname{Graph} \gamma \end{array} \right\} \right) \rho(z)\, \mathscr{H}^d(dz) \leq \varpi. \tag{9.3}$$

Proof. The proof is completely analogous to Proposition 7.2 therefore we will only sketch it. We begin noticing that for fixed $\bar{R} \geq R$ we have

$$\begin{aligned}
&\eta^{\text{in}} \Big( \Big\{ \gamma' : \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \operatorname{Graph} \gamma' \not\subseteq \operatorname{Graph} \gamma + \operatorname{clos} B^d_R(0) \Big\} \Big) \\
&\geq \eta^{\text{in}} \Big( \Big\{ \gamma' : \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \operatorname{Graph} \gamma' \not\subseteq \operatorname{Graph} \gamma + \operatorname{clos} B^d_{\bar{R}}(0) \Big\} \Big).
\end{aligned}$$

By keeping $\bar{R}$ fixed and sending $R \searrow 0$, we obtain a family of $\{r_n\}_{n \in \mathbb{N}}$ such that

$$\int \frac{1}{\sigma(B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\text{in}} \left( \left\{ \gamma' : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_{r_n}(0), \\ \operatorname{Graph} \gamma' \not\subseteq \operatorname{Graph} \gamma + \operatorname{clos} B^d_{\bar{R}}(0) \end{array} \right\} \right) \eta^{\text{in}}(d\gamma) \leq \varpi.$$

We now let $r_n \to 0$ and we make use of the following facts as in Proposition 7.2:

(1) the set
$$\Big\{ \gamma' : \operatorname{Graph} \gamma' \not\subseteq \operatorname{Graph} \gamma + \operatorname{clos} B^d_R(0) \Big\}$$
is open in $\Gamma$;

(2) for a.e. $z$ (in view of the equivalence between Lebesgue points w.r.t. $\eta$ and w.r.t. $\rho\mathscr{H}^d$) it holds

$$\fint_{B^d_r(z)} \eta_{z'} \mathscr{H}^d(dz') \rightharpoonup \eta_z, \qquad \text{as measures on } \Gamma.$$

At this point one uses the l.s.c. of the weak convergence on open sets and Fatou's Lemma to obtain

$$\begin{aligned}
\varpi &\geq \liminf_n \int \frac{1}{\sigma(B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\text{in}} \left( \left\{ \gamma' : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_{r_n}(0), \\ \text{Graph}\,\gamma' \not\subseteq \text{Graph}\,\gamma + \text{clos}\, B^d_{\bar{R}}(0) \end{array} \right\} \right) \eta^{\text{in}}(d\gamma) \\
&\geq \int \liminf_n \left\{ \fint_{B^d_{r_n}(\gamma(t^-_\gamma))} \eta^{\text{in}}_{z'} \Big( \Big\{ \gamma' : \text{Graph}\,\gamma' \not\subseteq \text{Graph}\,\gamma + \text{clos}\, B^d_{\bar{R}}(0) \Big\} \Big) \rho(z') \mathscr{H}^d(dz') \right\} \eta^{\text{in}}(d\gamma) \\
&\geq \int \eta^{\text{in}}_z \otimes \eta^{\text{in}}_z \Big( \Big\{ (\gamma, \gamma') : \max_t \text{dist}\, \big(\gamma(t), \gamma'(t)\big) > \bar{R} \Big\} \Big) \rho(z)\, \mathscr{H}^d(dz).
\end{aligned}$$

Finally, we send $\bar{R} \to 0$ and we use the Monotone Convergence Theorem, so that

$$\int \eta^{\text{in}}_z \otimes \eta^{\text{in}}_z \left( \left\{ (\gamma, \gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma), \\ \text{Graph}\,\gamma \not\subseteq \text{Graph}\,\gamma', \\ \text{Graph}\,\gamma' \not\subseteq \text{Graph}\,\gamma \end{array} \right\} \right) \rho(z)\, \mathscr{H}^d(dz) \leq \varpi,$$

which is what we wanted to prove. □

We now show a simple inequality which will be very useful to conclude the argument of the proof of Proposition 9.3.

Lemma 9.5. *There exists positive constants $D_0, D_1 > 0$ such that*

$$1 - \alpha \leq D_0(1-\alpha) \max\big\{1-\alpha, \alpha - \beta\big\} + \frac{\beta}{D_1}, \qquad \forall \alpha, \beta \in \mathbb{R} :\ 0 \leq \beta \leq \alpha \leq 1.$$

Proof. We split into two cases:

- if
$$1 - \alpha \geq \alpha - \beta \quad \Leftrightarrow \quad \beta \geq 2\alpha - 1,$$
then the inequality becomes
$$D_0(1-\alpha)^2 + \frac{\beta}{D_1} - (1-\alpha) \geq 0,$$
for all $0 \leq \alpha \leq 1$. Since in the current regime
$$D_0(1-\alpha)^2 + \frac{\beta}{D_1} - (1-\alpha) D_0(1-\alpha)^2 + \frac{2\alpha-1}{D_1} - (1-\alpha) := g(\alpha),$$
to conclude it is sufficient to show that there exist $D_0, D_1$ so that $g(\alpha) \geq 0$ for every $\alpha \in [0,1]$. Optimizing in $\alpha$, we find that the minimum of $g(\alpha)$ is
$$\min_{[0,1]} g = -\frac{1}{4D_0}\left(1 + \frac{2}{D_1}\right)^2 + \frac{1}{D_1}$$
and choosing $D_0 \geq \frac{3}{4} D_1$ and $D_1 \geq 1$ we see that $\min g(\alpha) \geq 0$ and this concludes the proof in this case.
- Let us now consider the other case, i.e. it holds
$$1 - \alpha \leq \alpha - \beta \quad \Leftrightarrow \quad \beta \leq 2\alpha - 1.$$
From the constraint $\beta \geq 0$, we deduce $\alpha \geq \frac{1}{2}$. So the inequality we want to prove becomes
$$D_0(1-\alpha)(\alpha - \beta) + \frac{\beta}{D_1} - (1-\alpha) \geq 0$$

for all $\alpha \in \left[\frac{1}{2}, 1\right]$, i.e.

$$\left(\frac{1}{D_1} - D_0(1-\alpha)\right)\beta + (1-\alpha)(D_0\alpha - 1) \geq 0.$$

This expression is linear in $\beta$: if $\left(\frac{1}{D_1} - D_0(1-\alpha)\right) \geq 0$ the minimum is achived in $\beta = 0$ and its value is

$$(1-\alpha)(D_0\alpha - 1)$$

which is easily seen to be non-negative for every $\alpha \in \left[\frac{1}{2}, 1\right]$ as soon as $D_0 \geq 2$. On the other hand, if $\left(\frac{1}{D_1} - D_0(1-\alpha)\right) \leq 0$ the minimum is achieved in $\beta = 2\alpha - 1$ and this was treated in the case above.

In conclusion, picking any $D_0 \geq \frac{3}{4}D_1 + 2$ and $D_1 \geq 1$, the inequality is satisfied and this concludes the proof. □

We are eventually ready to prove Proposition 9.3.

Proof (of Proposition 9.3). To begin, let us define a strict order relation on the set $\Gamma$. We consider the set

$$\mathcal{R} := \left\{(\gamma, \gamma') \in \Gamma^2 : \operatorname{Graph}\gamma \subsetneq \operatorname{Graph}\gamma'\right\}.$$

It is immediate to check the relation $\mathcal{R}$ is irreflexive, transitive and antisymmetric (as the strict set inclusion is), so it is a (partial) strict order on $\Gamma$. We will write $\gamma \prec \gamma'$ meaning $(\gamma, \gamma') \in \mathcal{R}$ and we will write $\preccurlyeq$ for the associated (partial, non strict) order relation, i.e.

$$\gamma \preccurlyeq \gamma' \Leftrightarrow \gamma \prec \gamma' \vee \operatorname{Graph}\gamma = \operatorname{Graph}\gamma'.$$

Notice that, in this language, we can rephrase the conclusion of Lemma 9.4, namely (9.3), by saying that

$$\begin{aligned} \varpi &\geq \int \eta_z^{\text{in}} \otimes \eta_z^{\text{in}} \left( \left\{ (\gamma, \gamma') : \begin{array}{c} \gamma'(t_{\gamma'}^-) = \gamma(t_\gamma^-), \\ \operatorname{Graph}\gamma \not\subseteq \operatorname{Graph}\gamma', \\ \operatorname{Graph}\gamma' \not\subseteq \operatorname{Graph}\gamma \end{array} \right\} \right) \rho(z)\, \mathscr{H}^d(dz) \\ &\geq \int \eta_z^{\text{in}} \otimes \eta_z^{\text{in}} \Big( \Big\{ (\gamma, \gamma') : \gamma'(t_{\gamma'}^-) = \gamma(t_\gamma^-),\, (\gamma, \gamma') \in \Gamma^2 \setminus \big(\mathcal{R} \cup \mathcal{R}^T\big) \Big\} \Big) \rho(z)\, \mathscr{H}^d(dz) \end{aligned} \tag{9.4}$$

where we have used the notation $\mathcal{R}^T$ to denote the set $\{(\gamma, \gamma') : (\gamma', \gamma) \in \mathcal{R}\}$.

For $z \in S_1$ let us now define

$$a_z := \sup_\gamma \eta_z^{\text{in}}\big(\{\gamma' : \gamma' \preccurlyeq \gamma\}\big).$$

Thus, for $z \in S_1$, for every $\varepsilon > 0$, by definition of supremum, there exists $\gamma_z$ such that, having set $A_z := \{\gamma' : \gamma' \preccurlyeq \gamma_z\}$, it holds

$$\eta_z^{\text{in}}\big(A_z\big) \geq a_z - \varepsilon.$$

Set then $B_z := \{\gamma' : \gamma' \prec \gamma_z\}$ and

$$b_z := \eta_z^{\text{in}}\big(B_z\big).$$

Clearly, $b_z \leq a_z$ for $\rho\mathscr{H}^d$-a.e. $z \in S_1$; furthermore, we emphasize that $B_z$ is the set of curves whose graph is contained in the almost-maximizer $\gamma_z$ but are different from it: in view of this, these curves must have a final point inside the domain $\Omega$, so that the following bound holds:

$$\begin{aligned}\mu^-(\Omega) &= \int_\Gamma \delta_{(t_\gamma^+,\gamma(t_\gamma^+))}(\Omega)\eta(d\gamma)\\ &= \int\int_\Gamma \delta_{(t_\gamma^+,\gamma(t_\gamma^+))}(\Omega)\eta_z^{\rm in}(d\gamma)\rho(z)\mathscr{H}^d(dz)\\ &\geq \int\int_{B_z} \delta_{(t_\gamma^+,\gamma(t_\gamma^+))}(\Omega)\eta_z^{\rm in}(d\gamma)\rho(z)\mathscr{H}^d(dz)\\ &= \int \eta_z^{\rm in}(B_z)\rho(z)\mathscr{H}^d(dz) = \int b_z\rho(z)\mathscr{H}^d(dz).\end{aligned}$$

Notice now that for every $\gamma$, $\eta_z^{\rm in}\big(\{\gamma' : \gamma' \prec \gamma\}\big) \leq a_z$, being $a_z$ the supremum. Furthermore, if $\gamma \notin A_z$, it holds

$$\{\gamma' \in \Gamma : \gamma' \prec \gamma\} \subset (A_z)^c \cup B_z. \tag{9.5}$$

Let us show (9.5) by proving the reverse inclusion for the complementary sets: if $\gamma' \in A_z \setminus B_z$, then necessarily $\gamma' = \gamma_z$: if it were $\gamma' \prec \gamma$ then we would have $\gamma_z \prec \gamma$ and this, together with $\gamma \notin A_z$, would contradict the fact that $a_z$ is the supremum. Having shown (9.5), we deduce in particular

$$\eta_z^{\rm in}\big(\{\gamma' \in \Gamma : \gamma' \prec \gamma\}\big) \leq \eta_z^{\rm in}\big((A_z)^c\big) + \eta_z^{\rm in}\big(B_z\big) \leq 1 - a_z + \varepsilon + b_z.$$

We can summarize what we have just proved by writing that

$$\eta_z^{\rm in}\big(\{\gamma' \in \Gamma : \gamma' \prec \gamma\}\big) \leq \min\{a_z, 1 - a_z + b_z + \varepsilon\}$$

which readily implies

$$\eta_z^{\rm in}\big(\{\gamma' \in \Gamma : \gamma' \not\prec \gamma\}\big) \geq 1 - \min\{a_z, 1 - a_z + b_z + \varepsilon\} = \max\{1 - a_z, a_z - b_z - \varepsilon\}. \tag{9.6}$$

In particular, taking into account (9.4) we finally have by Fubini

$$\begin{aligned}\varpi &\geq \int \eta_z^{\rm in}\otimes\eta_z^{\rm in}\Big(\Big\{(\gamma,\gamma') : \gamma'(t_{\gamma'}^-) = \gamma(t_\gamma^-),\, (\gamma,\gamma') \in \Gamma^2 \setminus \big(\mathcal{R}\cup\mathcal{R}^T\big)\Big\}\Big)\rho(z)\,\mathscr{H}^d(dz)\\ &= 2\int\Big[\int \eta_z^{\rm in}\Big(\Big\{\gamma' : \gamma'(t_{\gamma'}^-) = \gamma(t_\gamma^-),\, \gamma' \not\prec \gamma\Big\}\Big)\eta_z^{\rm in}(d\gamma)\Big]\rho(z)\,\mathscr{H}^d(dz)\\ &\geq 2\int\Big[\int_{\Gamma\setminus A_z} \eta_z^{\rm in}\Big(\Big\{\gamma' : \gamma'(t_{\gamma'}^-) = \gamma(t_\gamma^-),\, \gamma' \not\prec \gamma\Big\}\Big)\eta_z^{\rm in}(d\gamma)\Big]\rho(z)\,\mathscr{H}^d(dz)\\ &\overset{(9.6)}{\geq} 2\int\int_{\Gamma\setminus A_z} \max\{1 - a_z, a_z - b_z - \varepsilon\}\eta_z^{\rm in}(d\gamma)\,\rho(z)\,\mathscr{H}^d(dz)\\ &\geq 2\int \eta_z^{\rm in}\big(\Gamma\setminus A_z\big)\max\{1 - a_z, a_z - b_z - \varepsilon\}\,\rho(z)\,\mathscr{H}^d(dz)\\ &= 2\int (1 - a_z)\max\{1 - a_z, a_z - b_z - \varepsilon\}\,\rho(z)\,\mathscr{H}^d(dz)\end{aligned}$$

On the other hand by Lemma 9.5 with $\alpha = a_z$ and $\beta = b_z + \varepsilon$ we have

$$\eta_z^{\rm in}(\Gamma\setminus A_z) = 1 - a_z \leq D_0(1 - a_z)\max\big\{1 - a_z, a_z - b_z - \varepsilon\big\} + \frac{b_z + \varepsilon}{D_1},$$

so that,

$$\int \eta_z^{\rm in}(\Gamma\setminus A_z)\rho(z)\mathscr{H}^d(dz) \leq D_0\int (1 - a_z)\max\big\{1 - a_z, a_z - b_z\big\} + \int \frac{b_z + \varepsilon}{D_1}\rho(z)\mathscr{H}^d(dz)$$

and, letting $\varepsilon \to 0$,

$$\int \eta^{\text{in}}_z(\Gamma \setminus A_z)\rho(z)\mathscr{H}^d(dz) \leq \frac{D_0}{2}\varpi + \frac{\mu^-(\Omega)}{D_1} \leq \frac{1}{2}\left(2 + \frac{3D_1}{4}\right)\varpi + \frac{\mu^-(\Omega)}{D_1} \tag{9.7}$$

and this yields the desired conclusions: indeed, setting $U := \bigcup_{z\in S_1} A_z$, from (9.7) we have that

$$\eta^{\text{in}}(U^c) \leq \inf_{C\geq 1}\left\{\frac{1}{2}\left(2 + \frac{3C}{4}\right)\varpi + \frac{\mu^-(\Omega)}{C}\right\}$$

and $\eta^{\text{in}}\llcorner_U$ is forward untangled by construction. □

**9.1.2. Subadditivity of untangling functional.** We now want to study how the local pieces of information contained in Propositions 9.3 can be glued in a global one. Roughly speaking, we consider here the case in which the quantity $\varpi$ is (the mass of) a measure: we will show that a suitable functional (the *forward untangling functional*) is subadditive and this allows to compare it with a measure. We begin by giving the following

DEFINITION 9.6. Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set. The *forward untangling functional for a Lagrangian representation* $\eta$ is defined as

$$\mathfrak{f}^{\text{for}}(\Omega) := \inf\Big\{(\mathtt{R}_\Omega)_\sharp\eta^{\text{in}}(N) : \Gamma \setminus N \subset \Delta^{\text{for}}\Big\}. \tag{9.8}$$

In other words, the forward untangling functional applied on a proper set $\Omega$ gives the amount of curves we have to removed (from the ones seen by $(\mathtt{R}_\Omega)_\sharp\eta^{\text{in}}$) so that the remaining ones are disjoint in the future. We now show the following remarkable property of the forward untangling functional:

PROPOSITION 9.7. *The functional* $\mathfrak{f}^{\text{for}}$ *defined in* (9.8) *is subadditive on the class of proper sets. More precisely, if* $\mathrm{U}, \mathrm{V} \subset \mathbb{R}^{d+1}$ *are proper sets whose union* $\Omega := \mathrm{U} \cup \mathrm{V}$ *is proper, then*

$$\mathfrak{f}^{\text{for}}(\Omega) \leq \mathfrak{f}^{\text{for}}(\mathrm{U}) + \mathfrak{f}^{\text{for}}(\mathrm{V}).$$

PROOF. By definition, for every $\varepsilon > 0$ there exists a set $N(\mathrm{U}) \subset \Gamma(\mathrm{U})$ such that

$$\mathfrak{f}^{\text{for}}(\mathrm{U}) \leq (\mathtt{R}_\mathrm{U})_\sharp\eta^{\text{in}}(N(\mathrm{U})) + \varepsilon$$

and

$$\big(\Gamma(\mathrm{U}) \setminus N(\mathrm{U})\big) \subset \Delta^{\text{for}}.$$

Let $N(\mathrm{V})$ be an analogous set for V. Set

$$N := \big\{\gamma \in \Gamma(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{U}\gamma \in N(\mathrm{U})\right)\big\} \cup \big\{\gamma \in \Gamma(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{V}\gamma \in N(\mathrm{V})\right)\big\}.$$

By Proposition 6.32

$$\begin{aligned}\eta^{\text{in}}(N) &\leq \eta^{\text{in}}\Big(\big\{\gamma \in \Gamma(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{U}(\gamma) \in N(\mathrm{U})\right)\big\}\Big) + \eta^{\text{in}}\Big(\big\{\gamma \in \Gamma(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{V}(\gamma) \in N(\mathrm{V})\right)\big\}\Big)\\ &\leq (\mathtt{R}_\mathrm{U})_\sharp\eta^{\text{in}}(N(\mathrm{U})) + (\mathtt{R}_\mathrm{V})_\sharp\eta(N(\mathrm{V}))\\ &\leq \mathfrak{f}^{\text{for}}(\mathrm{U}) + \mathfrak{f}^{\text{for}}(\mathrm{V}) + 2\varepsilon.\end{aligned}$$

Being $\varepsilon$ arbitrary we thus obtain that $\eta^{\text{in}}(N) \leq \mathfrak{f}^{\text{for}}(\mathrm{U}) + \mathfrak{f}^{\text{for}}(\mathrm{V})$ so that, in order to conclude, it remains to show that $\Gamma(\Omega) \setminus N \subset \Delta^{\text{for}}$. To do this, observe that

$$\mathtt{R}_\mathrm{U}(\Gamma(\Omega)) \subset \Gamma(\mathrm{U}),$$

and the same for V. Hence, if $\mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega} \neq \emptyset$ then they must coincide forward in time either in $\mathrm{clos}\,\mathrm{U}$ or $\mathrm{clos}\,\mathrm{V}$ and hence in $\mathrm{clos}\,\mathrm{U} \cup \mathrm{clos}\,\mathrm{V} = \mathrm{clos}\,\Omega$. □

We are thus led to consider the following

ASSUMPTION 9.8. There exist $\tau > 0$ and a non-negative measure $\varpi^\tau$ of mass $\tau$ such that for some $C \geq 1$, for all $(t,x) \in \Omega$ there exists a family of proper balls $\{B_r^{d+1}(t,x)\}_r$ such that it holds

$$\mathfrak{f}^{\mathrm{for}}(B_r^{d+1}(t,x)) \leq \left(2 + \frac{3C}{4}\right)\varpi^\tau(B_r^{d+1}(t,x)) + \frac{\mu^-(B_r^{d+1}(t,x))}{C}. \tag{9.9}$$

For future reference let us define the measure

$$\zeta_\tau^{C,\mathrm{for}} := \left(1 + \frac{3C}{4}\right)\varpi^\tau + \frac{\mu^-}{C}.$$

By means of a standard covering argument we have the following

PROPOSITION 9.9. *If Assumption (9.8) holds in a proper set $\Omega$ with compact closure, then*

$$\mathfrak{f}^{\mathrm{for}}(\Omega) \leq C_d \zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega), \tag{9.10}$$

*where $C_d$ is a dimensional constant.*

PROOF. By Vitali Covering Theorem, for any $\varepsilon > 0$, we can cover the compact set $\operatorname{clos}\Omega$ with finitely many proper balls $B_i$ such that (9.9) holds and

$$\sum_i \zeta_C^{\tau,\mathrm{for}}(B_i) \leq C_d \zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega) + \varepsilon.$$

Thanks to the subadditivity (and the monotonicity) of $\mathfrak{f}^{\mathrm{for}}$ we can thus write

$$\mathfrak{f}^{\mathrm{for}}(\Omega) \leq \mathfrak{f}^{\mathrm{for}}\left(\bigcup_i B_i\right) \leq \sum_i \mathfrak{f}^{\mathrm{for}}(B_i) \leq C_d \zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega) + \varepsilon$$

and sending $\varepsilon \to 0$ we obtain (9.10). □

We finally show that the validity of Assumption 9.8 is enough, thanks to the subadditivity proved in Proposition 9.7, to have that $\eta$ is forward untangled.

COROLLARY 9.10. *Suppose there exist sequences $\tau_i \searrow 0$ and $C_i \nearrow +\infty$ such that Assumption 9.8 holds for $\tau_i, C_i$ and moreover*

$$C_i \tau_i \to 0.$$

*Then $\eta$ is forward untangled.*

PROOF. It is enough to observe that under the assumptions above $\|\zeta_{C_i}^{\tau_i,\mathrm{for}}\| \to 0$. □

## 9.2. Vector fields with weak $L^p$ bounds on the gradient

We now want to consider an interesting setting where the forward untangling method described in the Section 9.1 applies. To do this, we need first to recall the definition of *weak Lebesgue spaces*, as they will be used later on.

**9.2.1. Weak Lebesgue spaces.** We begin by giving the following

DEFINITION 9.11. Let $u \colon \Omega \to \mathbb{R}$ be a measurable function defined on an open set $\Omega \subset \mathbb{R}^d$. For any $1 \leq p < \infty$ we set

$$|||u|||^p_{M^p(\Omega)} := \sup_{\lambda > 0} \left\{ \lambda^p \mathscr{L}^d\big(\{x \in \Omega : |u(x)| > \lambda\}\big) \right\} \tag{9.11}$$

and we define the *weak Lebesgue space* $M^p(\Omega)$ as the space consisting of all measurable functions $u \colon \Omega \to \mathbb{R}$ with $|||u|||_{M^p(\Omega)} < +\infty$. By convention, we simply set $M^\infty(\Omega) = L^\infty(\Omega)$.

We explicitly notice that $|||\cdot|||_{M^p}$ is not subadditive, hence $M^p(\Omega)$ is not a Banach space. Nevertheless, it holds

$$|||u+v|||_{M^p(\Omega)} \le C_p\Big(|||u|||_{M^p(\Omega)} + |||v|||_{M^p(\Omega)}\Big),$$

for some constant $C_p > 0$. From the very easy fact that

$$\begin{aligned}\lambda^p \mathscr{L}^d\big(\{x\in\Omega : |u(x)|>\lambda\}\big) &= \lambda^p \int_{\{|u|>\lambda\}} 1\,dx \\ &= \int_{\{|u|>\lambda\}} \lambda^p\,dx \\ &\le \int_{\{|u|>\lambda\}} |u(x)|^p\,dx \le \|u\|^p_{L^p(\Omega)},\end{aligned}$$

we deduce that the inclusion $L^p(\Omega) \subset M^p(\Omega)$ holds, with $|||u|||_{M^p(\Omega)} \le \|u\|_{L^p(\Omega)}$. The inclusion is strict, as the function $1/x$ belongs to $M^1(0,1)$ but it is not $L^1(0,1)$.

**9.2.2. An interpolation lemma.** We now present a lemma which shows that we can interpolate between $M^1$ and $M^p$, for $p>1$, obtaining a bound on the $L^1$ norm, depending logarithmically on the $M^p$ norm.

LEMMA 9.12 ([BC13, Lemma 2.2]). *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set and let $u\colon \Omega \to [0,\infty)$ be a non negative, measurable function (supported on $\Omega$). Then, for every $1<p<\infty$, it holds*

$$\|u\|_{L^1(\Omega)} \le \frac{p}{p-1}|||u|||_{M^1(\Omega)}\left[1+\log\left(\mathscr{L}^d(\Omega)^{1-1/p}\frac{|||u|||_{M^p(\Omega)}}{|||u|||_{M^1(\Omega)}}\right)\right]. \tag{9.12}$$

*For $p=\infty$ we have*

$$\|u\|_{L^1(\Omega)} \le |||u|||_{M^1(\Omega)}\left[1+\log\left(\mathscr{L}^d(\Omega)\frac{\|u\|_{L^\infty(\Omega)}}{|||u|||_{M^1(\Omega)}}\right)\right]. \tag{9.13}$$

PROOF. Let for $\lambda \ge 0$

$$m(\lambda) := \mathscr{L}^d(\{u>\lambda\}\cap\Omega),$$

so that we can write

$$\|u\|_{L^1(\Omega)} = \int_0^{+\infty} m(\lambda)d\lambda.$$

By the very definition we have that

$$m(\lambda) \le \mathcal{L}^d(\Omega), \qquad m(\lambda) \le \frac{\|u\|^p_{M^p(\Omega)}}{\lambda^p},$$

hence

$$\|u\|_{M^1} = \sup \lambda m(\lambda) = \big(\sup \lambda^p m(\lambda)^p\big)^{1/p} \le \big(\sup \lambda^p m(\lambda)|\Omega|^{p-1}\big)^{1/p} \le \|u\|_{M^p}\mathscr{L}^d(\Omega)^{1-1/p}.$$

In other words,

$$\frac{|||u|||_{M^1(\Omega)}}{\mathscr{L}^d(\Omega)} \le |||u|||_{M^p(\Omega)}\mathscr{L}^d(\Omega)^{-1/p}$$

which can be written, by easy algebraic manipulations, as

$$\alpha := \frac{|||u|||_{M^1(\Omega)}}{\mathscr{L}^d(\Omega)} \le \left(\frac{|||u|||^p_{M^p(\Omega)}}{|||u|||_{M^1(\Omega)}}\right)^{\frac{1}{p-1}} =: \beta.$$

Hence we can split

$$\begin{aligned}\int_0^{+\infty} m(\lambda)d\lambda &= \int_0^{\alpha} m(\lambda)d\lambda + \int_\alpha^{\beta} m(\lambda)d\lambda + \int_\beta^{+\infty} m(\lambda)d\lambda \\ &\le |||u|||_{M^1(\Omega)} + \int_\alpha^\beta \frac{|||u|||_{M^1(\Omega)}}{\lambda}d\lambda + \int_\beta^{+\infty}\frac{|||u|||^p_{M^p(\Omega)}}{\lambda^p}d\lambda \\ &= |||u|||_{M^1} + \frac{p}{p-1}|||u|||_{M^1}\log\left(\mathscr{L}^d(\Omega)^{1-1/p}\frac{|||u|||_{M^p}}{|||u|||_{M^1}}\right) + \frac{1}{p-1}|||u|||_{M^1},\end{aligned}$$

which gives (9.12). For (9.13), it is enough to notice that for $\lambda > \|u\|_\infty$ the set $\{u > \lambda\} = \emptyset$ thus the RHS of (9.11) is less than $\|u\|^p_{L^\infty}\mathscr{L}^d(\Omega)$ just by estimating the two factors separately, i.e.

$$|||u|||_{M^p(\Omega)} \le \|u\|_{L^\infty}\mathscr{L}^d(\Omega)^{1/p}.$$

Plugging this into (9.12), taking into account the monotonicity of log, and sending $p \to +\infty$ we obtain (9.13) and the proof is thus completed. □

**9.2.3. The setting. Estimate on difference quotients.** We are ready to describe precisely the setting in which we are going to work in this section.

We shall always consider in the following vector fields

$$\boldsymbol{b}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$$

satisfying the following assumption.

ASSUMPTION 9.13. It holds:

(R1) there exists a function $M \in M^1((0,T)\times\mathbb{R}^d)$ such that

$$\big|\boldsymbol{b}(t,y) - \boldsymbol{b}(t,x)\big| \le |x-y|\big(M(t,y) + M(t,x)\big).$$

(R2) for every $\varepsilon > 0$ there exists functions $M_1, M_2\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}$ so that one can decompose the function $M$ of (R1) into

$$M = M_1 + M_2,$$

with

$$|||M_1|||_{M^1} \le \varepsilon, \qquad \|M_2\|_{L^1} \le C_\varepsilon.$$

We explicitly remind that $M \in M^1((0,T)\times\mathbb{R}^d)$ means that

$$|||M|||_{M^1} = \sup_\lambda \lambda\mathscr{L}^d(M > \lambda) < +\infty.$$

Before going on, we would like to notice that the validity of Assumption (R1) is indeed enough to conclude that $\boldsymbol{b} \in L^p_{\mathrm{loc}}$ for suitable exponents $p \ge 1$. We have the following

LEMMA 9.14 ($L^p$ embedding). *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable, non-negative function and assume that it holds*

$$|f(x) - f(y)| \le |x-y|\big(M(x) + M(y)\big), \qquad \forall x,y \in \mathbb{R}^d$$

*where $M \in M^q(\mathbb{R}^d)$ for some $q \in [1,d)$. Then $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for every $p \in [1, s(q))$, where $s(q) := \frac{qd}{d-q}$ is the exponent of the Sobolev embedding $W^{1,q}$ in dimension $d$.*

PROOF. Without loss of generality, we can assume that $f(0) = 0$ and that $M(0) = 1$. We localize everything in the unit ball, so that $x, y \in B^d_1(0)$. We set

$$C := \max\{|||M|||_{M^q(\mathbb{R}^d)}, 1\}$$

and, for $\lambda > 0$,

$$S_\lambda := \{x \in B^d_1 : M(x) > \lambda\}$$

so that, by assumption, it holds

$$|f(x) - f(y)| \le |x-y|\big(M(x) + M(y)\big), \qquad \forall x,y \in \mathbb{R}^d$$

and, for some $q \geq 1$

$$|S_\lambda| \leq \frac{C}{\lambda^q}. \tag{9.14}$$

By triangular inequality, it follows that for every $x \in S_1^c$

$$|f(x)| \leq |f(0)| + |x - 0|(M(0) + M(x)) \leq 1 + C \leq 2.$$

In a similar way, for $n \in \mathbb{N}$, if $x \in S_{2^n}^c$ and $y \in S_{2^{n-1}}^c$ we have

$$|f(x)| \leq |f(y)| + |x-y|\big(2^n + 2^{n-1}\big) \leq \sup_{y \in S_{2^{n-1}}^c} |f(y)| + |x-y|\big(2^n + 2^{n-1}\big).$$

Optimizing in $y$ in the RHS we see that

$$|f(x)| \leq \sup_{y \in S_{2^{n-1}}^c} |f(y)| + \operatorname{dist}(x, S_{2^{n-1}}^c)\big(2^n + 2^{n-1}\big).$$

Now observe that since (9.14) holds, the maximum radius of a ball contanined in $S_{2^{n-1}}^c$ is $r_{\max} = (c_d 2^{(1-n)q})^{1/d}$. Hence we get

$$\begin{aligned}
|f(x)| &\leq \sup_{y \in S_{2^{n-1}}^c} |f(y)| + \operatorname{dist}(x, S_{2^{n-1}}^c)\big(2^n + 2^{n-1}\big) \\
&\leq \sup_{y \in S_{2^{n-1}}^c} |f(y)| + \big(c_d 2^{q(1-n)}\big)^{1/d}\big(2^n + 2^{n-1}\big) \\
&= \sup_{y \in S_{2^{n-1}}^c} |f(y)| + \frac{3c_d^{1/d}}{2} 2^{q/d + n(1-q/d)}.
\end{aligned}$$

Iterating we get that

$$\sup_{x \in S_{2^n}^c \cap S_{2^{n-1}}} |f(x)| \leq 2 + \sum_{k=1}^{n} \frac{3c^{1/d} 2^{q/d}}{2} 2^{k(1-q/d)} = 2 + C_{d,q} 2^{n(1-q/d)}$$

where we have taken into account $q < d$ and denoted by $C_{d,q}$ a positive constant which depends only on $d, q$. In particular, we have

$$\sup_{x \in S_{2^n}^c \cap S_{2^{n-1}}} |f(x)|^p \leq C_{d,p,q}(2^p + 2^{np(1-q/d)}).$$

Thus computing the $L^p$-norm we finally obtain by Dominated Convergence Theorem

$$\begin{aligned}
\|f\|_p^p &= \int_{B_1^d} |f(x)|^p \, dx = \\
&= \int_{B_1^d} \sum_{n=0}^{\infty} \mathbb{1}_{S_{2^n}^c \cap S_{2^{n-1}}}(x) |f(x)|^p \, dx \\
&\leq \int_{B_1^d} \sum_{n=0}^{\infty} \mathbb{1}_{S_{2^n}^c \cap S_{2^{n-1}}}(x) C_{d,p,q}(2^p + 2^{np(1-q/d)}) \, dx \\
&\leq \sum_{n=0}^{\infty} C_{d,p,q}(2^p + 2^{np(1-q/d)}) \mathscr{L}^d\Big(S_{2^n}^c \cap S_{2^{n-1}}\Big) \\
&\leq \sum_{n=0}^{\infty} C_{d,p,q}(2^p + 2^{np(1-q/d)}) 2^{-q(n-1)} \\
&= C_{p,q,d}\left(1 + \sum_{n=0}^{\infty} 2^{n(p - q(1+1/d))}\right).
\end{aligned}$$

Notice that if $p < s(q)$ the series in the RHS converges and this concludes the proof. $\square$

Let us assume $\boldsymbol{b}$ is nearly incompressible: let $\rho\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}$ be a non-negative, bounded solution to the continuity equation $\partial_t\rho+\mathrm{div}(\rho\boldsymbol{b}) = 0$ in the sense of distributions on $(0,T)\times\mathbb{R}^d$. Accordingly, let $\eta$ be a Lagrangian representation of $\rho(1,\boldsymbol{b})$ in the whole space $(0,T)\times\mathbb{R}^d$. We will assume $\|\rho\|_\infty \le 1$.

REMARK 9.15. One can consider also the more general case $\mathrm{div}_{t,x}(\rho(1,\boldsymbol{b})) = \mu \in \mathscr{M}$ by means of the usual localization methods of Chapter 6 already exploited in Chapter 7. For the sake of simplicity we present here the argument in the case described above. ♠

We now state the main result of this section. In order to keep the notation contained, we will write

$$\eta \otimes \frac{\eta}{\sigma(B^d_r(\gamma(0)))}(A) := \int \frac{1}{\sigma(B^d_r(\gamma(0)))}\eta^{\mathrm{in}}(A_\gamma)\,\eta^{\mathrm{in}}(d\gamma)$$

for any set $A\subset\Gamma^2$, where we recall $A_\gamma = \{\gamma' : (\gamma,\gamma')\in A\}$ denotes the $\gamma$-section of $A$.

PROPOSITION 9.16. *Let $\boldsymbol{b}$ be a vector field satisfying (R1) and (R2) and let $\rho\ge 0$ be such that $\partial_t\rho + \mathrm{div}_x(\rho\boldsymbol{b}) = 0$ in the sense of distributions. Then for every $\varpi>0$ and for every $R>0$ there exists $r = r(\varpi,R)>0$ such that*

$$\eta\otimes\frac{\eta}{\sigma(B^d_r(\gamma(0)))}\left(\Big\{(\gamma,\gamma')\in\Gamma\times\Gamma : |\gamma(0)-\gamma'(0)|\le r, |\gamma(T)-\gamma'(T)|\ge R\Big\}\right) < \varpi,$$

*where we recall $\sigma(B^d_r(\gamma(0))) := \int_{B^d_r(\gamma(0))}\rho(0,x)\,dx$ is the amount of curves which start from the ball of radius $r$ around $\gamma(0)$.*

As an immediate corollary, taking into account Proposition 9.9, we thus deduce

COROLLARY 9.17. *Let $\boldsymbol{b}$ be a vector field satisfying (R1) and (R2) and let $\rho\ge0$ be such that $\partial_t\rho+\mathrm{div}_x(\rho\boldsymbol{b}) = 0$ in the sense of distributions. Then every Lagrangian representation $\eta$ of $\rho(1,\boldsymbol{b})$ is forward untangled.*

We now turn to the proof of 9.16.

PROOF. We split the proof in several steps.

*Step 1. Decomposition.* In view of Assumption (R2) for all $\varepsilon>0$ there exists functions $M_1, M_2\colon (0,T)\times\mathbb{R}^d\to\mathbb{R}$ so that one can write

$$M = M_1 + M_2, \qquad \text{with } |||M_1|||_{M^1}\le\varepsilon \text{ and } \qquad \|M_2\|_{L^1}\le C_\varepsilon.$$

Let $\bar M>0$ be a positive real number, large enough to be chosen later. Define then the set

$$A := \big\{(t,\gamma,\gamma') : M_1(t,\gamma(t)) < \bar M,\ M_1(t,\gamma'(t))\le\bar M\big\}.$$

*Step 2. Duhamel formula.* In this step we are going to estimate from above the measure of the set

$$\Big\{(\gamma,\gamma')\in\Gamma\times\Gamma : |\gamma(0)-\gamma'(0)|\le r, |\gamma(T)-\gamma'(T)|\ge R\Big\}. \tag{9.15}$$

To do this, we begin by noticing that for every $\gamma,\gamma'\in\Gamma$ we have

$$\begin{aligned}\frac{d}{dt}|\gamma(t)-\gamma'(t)| &= \frac{\gamma(t)-\gamma'(t)}{|\gamma(t)-\gamma'(t)|}\cdot\Big[\boldsymbol{b}\big(t,\gamma(t)\big)-\boldsymbol{b}\big(t,\gamma'(t)\big)\Big]\\ &= \frac{\gamma(t)-\gamma'(t)}{|\gamma(t)-\gamma'(t)|}\cdot\Big[\boldsymbol{b}\big(t,\gamma(t)\big)-\boldsymbol{b}\big(t,\gamma'(t)\big)\Big]\big(\mathbb{1}_A(t,\gamma,\gamma')+\mathbb{1}_{A^c}(t,\gamma,\gamma')\big)\\ &\le |\gamma(t)-\gamma'(t)|\big(M(t,\gamma)+M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\\ &\quad+|\boldsymbol{b}(t,\gamma(t))-\boldsymbol{b}(t,\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma,\gamma').\end{aligned}$$

Hence, by comparison principle and standard calculus formula, we deduce that

$$\begin{aligned}
&|\gamma(T)-\gamma'(T)| \\
&\le \mathcal{W}(\gamma,\gamma')\Big[r+\int_0^T \underbrace{e^{-\int_0^t \left(M(t,\gamma)+M(t,\gamma')\right)d\tau}}_{\le 1}|\boldsymbol{b}(t,\gamma(t))-\boldsymbol{b}(t,\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma,\gamma')\,dt\Big] \\
&\le \mathcal{W}(\gamma,\gamma')\Big[r+\int_0^T |\boldsymbol{b}(t,\gamma(t))-\boldsymbol{b}(t,\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma,\gamma')\,dt\Big].
\end{aligned}$$

where

$$\mathcal{W}(\gamma,\gamma'):=\exp\left(\int_0^T \big(M(t,\gamma)+M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\,dt\right)$$

In particular, taking logarithms we deduce that we can estimate from above the measure of (9.15) in the following way:

$$\begin{aligned}
&\Big\{(\gamma,\gamma'):|\gamma-\gamma'|(0)\le r, |\gamma-\gamma'|(T)\ge R\Big\} \\
&\subseteq \Big\{(\gamma,\gamma'):\mathcal{W}(\gamma,\gamma')\Big[r+\int_0^T |\boldsymbol{b}(t,\gamma(t))-\boldsymbol{b}(t,\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma,\gamma')\,dt\Big]\ge R\Big\} \\
&\subseteq \Big\{(\gamma,\gamma'):\mathcal{W}_1(\gamma,\gamma')+\mathcal{W}_2(\gamma,\gamma')+\mathcal{W}_3(\gamma,\gamma')\ge \log R\Big\},
\end{aligned}$$

where we have set

$$\mathcal{W}_1(\gamma,\gamma'):=\int_0^T \big(M_1(t,\gamma)+M_1(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\,dt,$$

$$\mathcal{W}_2(\gamma,\gamma'):=\int_0^T \big(M_2(t,\gamma)+M_2(t,\gamma')\big)\,dt,$$

$$\mathcal{W}_3(\gamma,\gamma'):=\log\Big[r+\int_0^T |\boldsymbol{b}(t,\gamma(t))-\boldsymbol{b}(t,\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma,\gamma')\,dt\Big].$$

We have thus reduced the problem to estimating from above by $\varpi$ the quantity

$$\eta\otimes\frac{\eta}{\sigma(B_r^d(\gamma(0)))}\left(\Big\{(\gamma,\gamma'):\mathcal{W}_1(\gamma,\gamma')+\mathcal{W}_2(\gamma,\gamma')+\mathcal{W}_3(\gamma,\gamma')\ge \log R\Big\}\right).$$

*Step 3. Separate estimates of the terms.* We now proceed to estimate separately the three terms above. We will use essentially Chebyshev inequality and the interpolation Lemma 9.12.

*Step 3.1. Estimate of $\mathcal{W}_1$.* Fix $c_1>0$ to be chosen later. Being the integrand non-negative, we clearly have

$$\begin{aligned}
\Big\{(\gamma,\gamma'):\mathcal{W}_1(\gamma,\gamma')\ge 2c_1\Big\}\subseteq &\Big\{(\gamma,\gamma'):\int_0^T M_1(t,\gamma)\mathbb{1}_A(t,\gamma,\gamma')dt\ge c_1\Big\} \\
&\bigcup\Big\{(\gamma,\gamma'):\int_0^T M_1(t,\gamma')\mathbb{1}_A(t,\gamma,\gamma')dt\ge c_1\Big\}
\end{aligned}\tag{9.16}$$

so that it is enough tho estimate only one of the two terms. We have now by projection and Chebyshev inequality

$$\begin{aligned}
&\eta\otimes\frac{\eta}{\sigma(B_r^d(\gamma(0)))}\left(\left\{(\gamma,\gamma'):\int_0^T M_1(t,\gamma)\mathbb{1}_A(t,\gamma,\gamma')\,dt\geq c_1\right\}\right)\\
&\leq\frac{1}{c_1}\int\frac{1}{\sigma(B_r^d(\gamma(0)))}\int\int_0^T M_1(t,\gamma)\mathbb{1}_A(t,\gamma,\gamma')d\eta(\gamma')\,d\eta(\gamma)\\
&\leq\frac{1}{c_1}\int\frac{1}{\sigma(B_r^d(\gamma(0)))}\int\int_0^T M_1(t,\gamma)\mathbb{1}_{\{M_1(t,\gamma)\leq\bar M\}}(t,\gamma)d\eta(\gamma')\,d\eta(\gamma)\\
&\leq\frac{1}{c_1}\int\int_0^T M_1(t,\gamma)\mathbb{1}_{\{M_1(t,\gamma)\leq\bar M\}}(t,\gamma)\,d\eta(\gamma)
\end{aligned}$$

because we are integrating only in the set of curves $\gamma'$ such that $\gamma'(0)\in B_r(\gamma(0))$. Finally, by projection

$$\begin{aligned}
&\eta\otimes\frac{\eta}{\sigma(B_r^d(\gamma(0)))}\left(\left\{(\gamma,\gamma'):\int_0^T M_1(t,\gamma)\mathbb{1}_A(t,\gamma,\gamma')\,dt\geq c_1\right\}\right)\\
&\leq\frac{1}{c_1}\int_\Gamma\int_0^T M_1(t,\gamma)\mathbb{1}_{\{M_1(t,\gamma)\leq\bar M\}}(t,\gamma)\,d\eta(\gamma)\\
&\leq\frac{1}{c_1}\iint_{(0,T)\times\mathbb{R}^d} M_1(t,x)\mathbb{1}_{\{M_1\leq\bar M\}}(t,x)dtdx\\
&\leq\frac{1}{c_1}|||M_1|||_{M^1}\left[1+\log\left(\frac{\bar MC_d}{|||M_1|||_{M^1}}\right)\right]\\
&\leq\frac{\varepsilon}{a}\left[1+\log\left(\frac{\bar MC_d}{\varepsilon}\right)\right]
\end{aligned}$$

for some dimensional constant $C_d$, where we have used Lemma 9.12 on the function $M_1\mathbb{1}_{\{M_1\leq\bar M\}}$ and the fact that $|||M_1|||_{M^1}<\varepsilon$ (and the trivial fact that the function $s\mapsto s(1-\log s+\log C_d)$ is increasing in a right neighbourhood of $s=0$).

In particular, if we choose $c_1:=\frac{6\varepsilon}{\varpi}\left[1+\log\left(\frac{\bar MC_d}{\varepsilon}\right)\right]$ taking into account (9.16) we have that

$$\eta\otimes\frac{\eta}{\sigma(B_r^d(\gamma(0)))}\left(\left\{(\gamma,\gamma'):\int_0^T\big(M_1(\gamma)+M_1(\gamma')\big)\mathbb{1}_A\,dt\geq\frac{12\varepsilon}{\varpi}\left[1+\log\left(\frac{\bar MC_d}{\varepsilon}\right)\right]\right\}\right)\leq\frac{\varpi}{3}.$$

*Step 3.2. Estimate of $\mathcal{W}_2$.* Arguing again as above splitting the set into two parts and then using Chebyshev inequality, we easily deduce that for $c_2>0$ it holds

$$\eta\otimes\frac{\eta}{\sigma(B_r^d(\gamma(0)))}\left(\left\{(\gamma,\gamma'):\int_0^T\big(M_2(\gamma)+M_2(\gamma')\big)\geq 2c_2\right\}\right)\leq\frac{2C_\varepsilon}{c_2}$$

so that if we choose $c_2:=\frac{6C_\varepsilon}{\varpi}$ we get

$$\eta\otimes\frac{\eta}{\sigma(B_r^d(\gamma(0)))}\left(\left\{(\gamma,\gamma'):\int_0^T\big(M_2(\gamma)+M_2(\gamma')\big)\geq\frac{12C_\varepsilon}{\varpi}\right\}\right)\leq\frac{\varpi}{3}$$

*Step 3.3. Estimate of $\mathcal{W}_3$.* Finally, in order to estimate the third term $\mathcal{W}_3(\gamma,\gamma')$ we use once more Chebyshev and Hölder inequalities: taking into account the monotonicity of the log, it suffices to estimate for $c_3 > 0$

$$
\begin{aligned}
&\eta \otimes \frac{\eta}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T |\boldsymbol{b}(\gamma) - \boldsymbol{b}(\gamma')| \mathbb{1}_{A^c}\, dt \geq c_3\right\}\right)\\
&\leq \frac{1}{c_3}\int \frac{1}{\sigma(B^d_r(\gamma(0)))}\int\int_0^T |\boldsymbol{b}(\gamma) - \boldsymbol{b}(\gamma')|\mathbb{1}_{A^c}\, dt\, d\eta(\gamma')\, d\eta(\gamma)\\
&\leq \frac{2}{c_3}\|\boldsymbol{b}\|_{L^p}(\mathscr{L}^{d+1}((t,x) : M_1(t,x) > \bar{M}))^{1/q}\\
&\leq \frac{2}{c_3}\|\boldsymbol{b}\|_{L^p}\left(\frac{|||M_1|||_{M^1}}{\bar{M}}\right)^{1/q}\\
&\leq \frac{2}{c_3}\|\boldsymbol{b}\|_{L^p}\left(\frac{\varepsilon}{\bar{M}}\right)^{1/q},
\end{aligned}
$$

where $q \in (d, +\infty]$ is the conjugate exponent to $p \in [1, \frac{d}{d-1})$. Thus, if we take $c_3 := \frac{6}{\varpi}\|\boldsymbol{b}\|_{L^p}\left(\frac{\varepsilon}{\bar{M}}\right)^{1/q}$ we get

$$
\eta \otimes \frac{\eta}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T |\boldsymbol{b}(\gamma) - \boldsymbol{b}(\gamma')|\mathbb{1}_{A^c}\, dt > \frac{6}{\varpi}\|\boldsymbol{b}\|_{L^p}\left(\frac{\varepsilon}{\bar{M}}\right)^{1/q}\right\}\right) \leq \frac{\varpi}{3}.
$$

*Step 4. Conclusion of the argument.* From Step 3, we deduce that

$$
\eta \otimes \frac{\eta}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \mathcal{W}_1(\gamma,\gamma') + \mathcal{W}_2(\gamma,\gamma') + \mathcal{W}_3(\gamma,\gamma') \geq c_1 + c_2 + c_3\right\}\right) \leq \varpi.
$$

To conclude, we have to show that up to this $\varpi$-set of orbits, in the remaining set we can tune the parameters to that it holds

$$
c_1 + c_2 + c_3 = \frac{12\varepsilon}{\varpi}\left[1 + \log\left(\frac{\bar{M}C}{\varepsilon}\right)\right] + \frac{12C_\varepsilon}{\varpi} + \log\left[r + \frac{6}{\varpi}\|\boldsymbol{b}\|_{L^p}\left(\frac{\varepsilon}{\bar{M}}\right)^{1/q}\right] \leq \log R.
$$

Let us choose $r := c_3$ so that have to study the inequality

$$
\frac{12\varepsilon}{\varpi}\left[1 + \log\left(\frac{\bar{M}C}{\varepsilon}\right)\right] + \frac{12C_\varepsilon}{\varpi} + \log\left[\frac{12}{\varpi}\|\boldsymbol{b}\|_{L^p}\left(\frac{\varepsilon}{\bar{M}}\right)^{1/q}\right] \leq \log R.
$$

i.e.

$$
\frac{12\varepsilon}{\varpi} + \frac{12\varepsilon}{\varpi}\log\bar{M} + \frac{12\varepsilon}{\varpi}\log\left(\frac{C}{\varepsilon}\right) + \frac{12C_\varepsilon}{\varpi} + \log\left[\frac{12\|\boldsymbol{b}\|_{L^p}\varepsilon^{1/q}}{\varpi}\right] - \frac{1}{q}\log\bar{M} \leq \log R.
$$

Collecting the terms in $\bar{M}$ we have

$$
\left(\frac{12\varepsilon}{\varpi} - \frac{1}{q}\right)\log\bar{M} + \frac{12(C_\varepsilon + \varepsilon)}{\varpi} + \frac{12\varepsilon}{\varpi}\log\left(\frac{C}{\varepsilon}\right) + \log\left[\frac{12\varepsilon^{1/q}\|\boldsymbol{b}\|_{L^p}}{\varpi}\right] \leq \log R.
$$

i.e.

$$
\left(\frac{12\varepsilon}{\varpi} - \frac{1}{q}\right)\log\bar{M} + C(\varepsilon,\varpi,\|\boldsymbol{b}\|_{L^p}) \leq \log R,
$$

for some ininfluent constant $C(\varepsilon,\varpi,\|\boldsymbol{b}\|_{L^p})$. In particular, by choosing $\varepsilon < \frac{\varpi}{12q}$ the leading term is negative, thus the relation above holds taking simply

$$
\bar{M} \geq \frac{C'(\varepsilon,\varpi,\|\boldsymbol{b}\|_{L^p})}{R^{\left(\frac{1}{q} - \frac{12\varepsilon}{\varpi}\right)^{-1}}} \tag{9.17}
$$

and this finishes the proof. $\square$

We conclude this section with a couple of remarks.

Remark 9.18. The above proof actually gives something more at a Lagrangian level. Indeed, it says that the Regular Lagrangian flow associated to $\boldsymbol{b}$ has a Hölder-Lusin property, meaning that for every $\varpi > 0$, up to a set of orbits of $\eta$-measure at most $\varpi$, the flow is Hölder continuous with an exponent that depends on $q$ and blows up as $\varpi \to 0$. To prove this assertion, observe that, in the proof above, we have chosen $r := c_3$ i.e.

$$r = \frac{6}{\varpi}\|\boldsymbol{b}\|_{L^p}\left(\frac{\varepsilon}{\bar{M}}\right)^{1/q}.$$

Plugging into this equality the choice of $\bar{M}$ given by (9.17) we obtain

$$r = C''(\varepsilon, \varpi, \|\boldsymbol{b}\|_{L^p}, q) R^{\left(1-\frac{12q\varepsilon}{\varpi}\right)^{-1}} \qquad \text{i.e.} \qquad R = C'''(\varepsilon, \varpi, \|\boldsymbol{b}\|_{L^p}, q) r^{1-\frac{12q\varepsilon}{\varpi}}.$$

♠

Remark 9.19. We would like to point out that a relevant case which fits in the setting considered here is the one of velocity fields $\boldsymbol{b}$ whose gradient is given by a singular integral of an $L^1$ function, i.e. $\nabla \boldsymbol{b} = K * \omega$ for some singular integral kernel $K$. This case has been extensively considered in [BC13] and, even more recently, in [CNSS17]. In particular, in the paper [BC13], besides some growth conditions on $\boldsymbol{b}$, the authors assumed the following:

(R1') For every $i, j = 1, \dots, d$ we have

$$\partial_j \boldsymbol{b} = \sum_{k=1}^{m} \boldsymbol{S}_{jk} \boldsymbol{g}_{jk} \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{R}^d) \tag{9.18}$$

in which $\boldsymbol{S}_{jk}$ is a vector consisting of $d$ singular integral operators, and for every $j = 1, \dots, d$ and every $k = 1, \dots, m$ we have $\boldsymbol{g}_{jk} \in L^1\big((0,T)\times\mathbb{R}^d; \mathbb{R}^d\big)$.

In [BC13, Prop. 4.2] the authors obtain a precise estimate on the different quotients of a function $\boldsymbol{b}$ for which (9.18) holds: in particular, this shows that condition (R1') implies the validity of (R1). Furthermore, (R1') readily implies (R2), as one can check via a simple equi-integrability argument (see [BC13, Lemma 5.8]): here plays a crucial role the fact that $\boldsymbol{g}_{jk}$ are $L^1$ functions (and not measures). ♠

## 9.3. Continuous dependence on the vector field

We now turn to consider the problem of continuous dependence of the flow as a function of the vector field in $L^p$.

Let us preliminarly observe, that, as a consequence of Proposition 9.16, any Lagrangian representation $\eta$ associated to a nearly incompressible vector field $\rho(1, \boldsymbol{b})$, with $\boldsymbol{b}$ satisfying (R1) and (R2), is forward untangled, and thus define the (regular Lagrangian) flow of $\boldsymbol{b}$:

$$\eta = \int_{\mathbb{R}^d} \eta_x \, dx, \qquad \eta_x = \delta_{\gamma_x(\cdot)} \qquad \text{for a.e. } x \in \mathbb{R}^d. \tag{9.19}$$

We will adopt from now onwards the standard notation $\boldsymbol{X}(t,x) = \gamma_x(t)$, being $\gamma_x$ defined in (9.19).

We want now to prove the following proposition:

Proposition 9.20. *Let $\boldsymbol{b}_1 : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a nearly incompressibile vector field satisfying (R1) and (R2). Let $\boldsymbol{b}_2 : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be another nearly incompressible vector field with density $\rho_2$. Then for every $\varpi > 0$ there exists $R = R(\varpi, \|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}) > 0$ such that*

$$\eta^2\big(\big\{\gamma' \in \Gamma : \gamma \notin \boldsymbol{X}^1([0,T]\times\mathbb{R}^d) + B^d_R(0)\big\}\big) \le \varpi, \tag{9.20}$$

*and $R \to 0$ as $\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1} \to 0$ (for fixed $\varpi$), where $\boldsymbol{X}^1 = \boldsymbol{X}^1(t,x)$ is the (regular Lagrangian) flow associated to $\boldsymbol{b}_1$ in the sense of (9.19) and $\eta^2$ is a Lagrangian representation of $\rho_2(1, \boldsymbol{b}_2)$.*

Proof. Notice that it is enough to prove that for every $\varpi > 0$ and for every $r > 0$ there exists $R > 0$ so that

$$\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') \in \Gamma \times \Gamma : |\gamma(0) - \gamma'(0)| \le r,\ |\gamma - \gamma'|(T) \ge R\right\}\right) \le \varpi. \quad (9.21)$$

Indeed, passing to the limit in $r \to 0$ in (9.21) we obtain (9.20).

We now split again the proof into several steps.

*Step 1. Decomposition.* As before, notice that Assumption (R1) implies $\boldsymbol{b}_1 \in L^p_{\mathrm{loc}}$ for $p \in \big[1, s(1)\big)$. Furthermore, in view of (R2), we may write

$$M = M_1 + M_2, \qquad \text{with} \qquad |||M_1|||_{M^1} \le \varepsilon \qquad \text{and} \qquad \|M_2\|_{L^1} \le C_\varepsilon.$$

Let $\bar{M} > 0$ be a positive real number, large enough to be chosen later. Define then the set

$$A := \big\{(t,\gamma,\gamma') : M_1(t,\gamma(t)) < \bar{M},\ M_1(t,\gamma'(t)) \le \bar{M}\big\}.$$

*Step 2. Reduction to Duhamel formula.* For every $\gamma \in \Gamma_1$ and $\gamma' \in \Gamma_2$ we thus have

$$\begin{aligned}
&\frac{d}{dt}|\gamma(t) - \gamma'(t)| = \frac{\gamma(t) - \gamma'(t)}{|\gamma(t) - \gamma'(t)|} \cdot \big[\boldsymbol{b}_1(t,\gamma(t)) - \boldsymbol{b}_2(t,\gamma'(t))\big] \\
&= \frac{\gamma(t) - \gamma'(t)}{|\gamma(t) - \gamma'(t)|} \cdot \big[\boldsymbol{b}_1(t,\gamma(t)) - \boldsymbol{b}_1(t,\gamma'(t)) + \boldsymbol{b}_1(t,\gamma'(t)) - \boldsymbol{b}_2(t,\gamma'(t))\big](\mathbb{1}_A + \mathbb{1}_{A^c})(t,\gamma,\gamma') \\
&\le |\gamma(t) - \gamma'(t)|\big(M(t,\gamma) + M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma') + \mathcal{E}(t,\gamma(t),\gamma'(t))
\end{aligned}$$

where we have set $\mathcal{E} := \mathcal{E}_1 + \mathcal{E}_2$, being

$$\mathcal{E}_1(t,\gamma(t),\gamma'(t)) := |\boldsymbol{b}_1(t,\gamma(t)) - \boldsymbol{b}_1(t,\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma,\gamma')$$

and

$$\mathcal{E}_2(t,\gamma(t),\gamma'(t)) := |\boldsymbol{b}_1(t,\gamma'(t)) - \boldsymbol{b}_2(t,\gamma'(t))|.$$

Hence, by comparison principle and standard calculus formula, we deduce that

$$\begin{aligned}
|\gamma(T) - \gamma'(T)| &\le e^{\int_0^T \big(M(t,\gamma)+M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\,dt}\left[r + \int_0^T e^{-\int_0^t \big(M(t,\gamma)+M(t,\gamma')\big)d\tau}\mathcal{E}(t,\gamma,\gamma')\,dt\right] \\
&\le e^{\int_0^T \big(M(t,\gamma)+M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\,dt}\left[r + \int_0^T \mathcal{E}(t,\gamma,\gamma')\,dt\right].
\end{aligned}$$

In particular, taking logarithms we deduce that

$$\begin{aligned}
&\left\{(\gamma,\gamma') : |\gamma(0) - \gamma'(0)| \le r, |\gamma - \gamma'|(T) \ge R\right\} \\
&\subseteq \left\{(\gamma,\gamma') : e^{\int_0^T \big(M(t,\gamma)+M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\,dt}\left[r + \int_0^T \mathcal{E}(t,\gamma,\gamma')\,dt\right] \ge R\right\} \\
&\subseteq \left\{(\gamma,\gamma') : \int_0^T \big(M(t,\gamma) + M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\,dt + \log\left[r + \int_0^T \mathcal{E}(t,\gamma,\gamma')\,dt\right] \ge \log R\right\}
\end{aligned}$$

We thus have reduced the problem to estimating from above by $\varpi$ the $\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}$ measure of the set

$$\left\{(\gamma,\gamma') : \int_0^T \big(M(t,\gamma) + M(t,\gamma')\big)\mathbb{1}_A(t,\gamma,\gamma')\,dt + \log\left[r + \int_0^T \mathcal{E}(t,\gamma,\gamma')\,dt\right] \ge \log R\right\}.$$

*Step 3. Separate estimates of the terms.* We now proceed to estimate separately the three terms above. We will use essentially Chebyshev inequality and the interpolation Lemma 9.12.

*Step 3.1. Estimate of the first term.* Arguing exactly in the same way as in Proposition 9.16, by Chebyshev inequality and Lemma 9.12, we have for $c_1 > 0$

$$\begin{aligned}
&\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T M_1(t,\gamma)\mathbb{1}_A(t,\gamma,\gamma')\,dt \geq c_1\right\}\right)\\
&\leq \frac{1}{c_1}\int_\Gamma\int_0^T M_1(t,\gamma)\mathbb{1}_{\{M_1(t,\gamma)\leq \bar M\}}(t,\gamma)\,d\eta_1(\gamma)\\
&\leq \frac{1}{c_1}\iint_{(0,T)\times\mathbb{R}^d} M_1(t,x)\mathbb{1}_{\{M_1\leq \bar M\}}(t,x)dtdx\\
&\leq \frac{1}{c_1}|||M_1|||_{M^1}\left[1+\log\left(\frac{\bar M C_d}{|||M_1|||_{M^1}}\right)\right]\\
&\leq \frac{\varepsilon}{c_1}\left[1+\log\left(\frac{\bar M C_d}{\varepsilon}\right)\right]
\end{aligned}$$

for some dimensional constant $C_d$. In particular, if we choose $c_1 := \frac{6\varepsilon}{\varpi}\left[1+\log\left(\frac{\bar M C_d}{\varepsilon}\right)\right]$ we have that

$$\eta_1\otimes\frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T (M_1(\gamma)+M_1(\gamma'))\mathbb{1}_A\,dt \geq \frac{12\varepsilon}{\varpi}\left[1+\log\left(\frac{\bar M C_d}{\varepsilon}\right)\right]\right\}\right) \leq \frac{\varpi}{3}.$$

*Step 3.2. Estimate of the second term.* Again arguing in the same way as in Proposition 9.16, by Chebyshev inequality, we easily deduce that for $c_2 > 0$ it holds

$$\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T (M_2(\gamma)+M_2(\gamma')) \geq 2c_2\right\}\right) \leq \frac{2C_\varepsilon}{c_2}$$

so that if we choose $c_2 := \frac{6C_\varepsilon}{\varpi}$ we get

$$\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T (M_2(\gamma)+M_2(\gamma')) \geq \frac{12C_\varepsilon}{\varpi}\right\}\right) \leq \frac{\varpi}{3}$$

*Step 3.3. First part of estimate of the third term: $\mathcal{E}_1$.* We use once more Chebyshev and Hölder inequalities: taking into account the monotonicity of the log, it suffices to estimate for $c_3 > 0$

$$\begin{aligned}
&\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T |\boldsymbol{b}_1(t,\gamma(t)) - \boldsymbol{b}_1(\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma(t),\gamma'(t)\,dt \geq c_3\right\}\right)\\
&\leq \frac{1}{c_3}\int_{\Gamma_1}\frac{1}{\sigma(B^d_r(\gamma(0)))}\int_{\Gamma_2}\int_0^T |\boldsymbol{b}_1(t,\gamma(t)) - \boldsymbol{b}_1(t,\gamma'(t))|\mathbb{1}_{A^c}(t,\gamma(t),\gamma'(t))\,dt\,d\eta_2(\gamma')\,d\eta_1(\gamma)\\
&\leq \frac{2}{c_3}\|\boldsymbol{b}_1\|_{L^p}(\mathscr{L}^{d+1}((t,x) : M_1(t,x) > \bar M))^{1/q}\\
&\leq \frac{2}{c_3}\|\boldsymbol{b}_1\|_{L^p}\left(\frac{|||M_1|||_{M^1}}{\bar M}\right)^{1/q}\\
&\leq \frac{2}{c_3}\|\boldsymbol{b}_1\|_{L^p}\left(\frac{\varepsilon}{\bar M}\right)^{1/q}
\end{aligned}$$

so that if we take $c_3 := \frac{12}{\varpi}\|\boldsymbol{b}_1\|_{L^p}\left(\frac{\varepsilon}{\bar M}\right)^{1/q}$ we get (dropping for simplicity the argument of $\mathbb{1}_{A^c}$)

$$\eta\otimes\frac{\eta}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma'):\int_0^T |\boldsymbol{b}(t,\gamma(t)) - \boldsymbol{b}(t,\gamma'(t))|\mathbb{1}_{A^c}\,dt > \frac{12\|\boldsymbol{b}_1\|_{L^p}}{\varpi}\left(\frac{\varepsilon}{\bar M}\right)^{1/q}\right\}\right) \leq \frac{\varpi}{6}.$$

*Step 3.4. Second part of estimate of the third term: $\mathcal{E}_2$.* Finally, by Chebyshev again

$$\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T |\boldsymbol{b}_1(t,\gamma'(t)) - \boldsymbol{b}_2(t,\gamma'(t))|\,dt > c_4\right\}\right) \leq \frac{L'}{c_4}\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}.$$

So that taking $c_4 = \frac{6L'}{\varpi}\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}$ we have

$$\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') : \int_0^T |\boldsymbol{b}_1(t,\gamma'(t)) - \boldsymbol{b}_2(t,\gamma'(t))|\, dt > \frac{6L'}{\varpi}\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}\right\}\right) \le \frac{\varpi}{6}$$

*Step 4. Conclusion of the argument.* Taking into account Step 3, we throw away a $\varpi$-set of orbits and we show that, in the remaining set, we can tune the parameters so that it holds

$$\frac{12\varepsilon}{\varpi}\left[1 + \log\left(\frac{\bar{M}C}{\varepsilon}\right)\right] + \frac{12C_\varepsilon}{\varpi} + \log\left[r + \frac{12}{\varpi}\|\boldsymbol{b}_1\|_{L^p}\left(\frac{\varepsilon}{\bar{M}}\right)^{1/q} + \frac{6L'}{\varpi}\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}\right] \le \log R. \tag{9.22}$$

In particular, we choose

$$\bar{M} := \frac{\varepsilon\|\boldsymbol{b}_1\|^q_{L^p}}{(6L'\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1})^q}$$

so that (9.22) becomes

$$\frac{\varepsilon}{\varpi}\log\left(\frac{C\|\boldsymbol{b}_1\|^q_{L^p}}{(L'\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^p})^q}\right) + \frac{4C_\varepsilon + \varepsilon}{\varpi} + \log\left[r + \frac{78L'}{\varpi}\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}\right] \le \log R.$$

In particular, taking

$$R_r(\varpi, \|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^p}) := \exp\left[\frac{C_1(\varepsilon, \|\boldsymbol{b}_1\|_{L^1}, L', q)}{\varpi} - \log\varpi + \log\left(r + \frac{78L'}{\varpi}\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}\right)\right]$$

we have obtained that

$$\eta_1 \otimes \frac{\eta_2}{\sigma(B^d_r(\gamma(0)))}\left(\left\{(\gamma,\gamma') \in \Gamma\times\Gamma : |\gamma(0) - \gamma'(0)| \le r,\ |\gamma - \gamma'|(T) \ge R_r\right\}\right) \le \varpi.$$

Sending $r \to 0$, and setting

$$\begin{aligned} R(\varpi, \|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}) &:= \exp\left(\frac{C_1(\varepsilon, \|\boldsymbol{b}_1\|_{L^p}, L', q)}{\varpi} - \log\varpi + \left(1 - \frac{q\varepsilon}{\varpi}\right)\log\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|_{L^1}\right) \\ &\simeq \frac{C}{\varpi}\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|^{1-\frac{q\varepsilon}{\varpi}}_{L^1} \end{aligned}$$

we thus obtain (9.20) and this concludes the proof. □

# Part 3

# A Lagrangian approach for multidimensional conservation laws

CHAPTER 10

# Lagrangian representations for multidimensional conservation laws

ABSTRACT. In this chapter, we introduce a notion of Lagrangian representation for entropy solutions to scalar conservation laws in several space dimensions

$$\begin{cases} \partial_t u + \mathrm{div}_x(\boldsymbol{f}(u)) = 0 & (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0,\cdot) = \bar{u}(\cdot) \end{cases}$$

The construction is based on the transport-collapse method introduced by Brenier in [Bre84]: after stating and proving some preliminaries results in Section 10.1, we show in Section 10.2 how a Lagrangian representation can be defined. As usual, we will first deal with the case $\bar{u}$ is of class BV and then, by exploiting compactness properties of the representation, for generic initial datum $\bar{u} \in L^\infty(\mathbb{R}^d)$. As a first application of this tool, we show in Section 10.3, that if the solution $u$ is continuous, then it is hypograph is given by the set

$$\big\{(t,x,h) : h \leq \bar{u}(x - \boldsymbol{f}'(h)t)\big\},$$

i.e. it is the translation of each level set of $\bar{u}$ by its characteristic speed. As a consequence, we obtain that the entropy-dissipation measure associated to a continuous solution vanishes.

## 10.1. Introduction and preliminaries

In a series of papers [BM14, BY15, BM16], various notions of Lagrangian representation for the entropy solution $u$ to a scalar conservation law in one space dimension

$$\partial_t u + \partial_x f(u) = 0$$

have been introduced. The basic idea is to use the wavefront tracking and observe that the wavefronts trajectories generates a flow $\boldsymbol{X} = \boldsymbol{X}(t,y)$ which is Lipschitz in times and monotone in $y$: this compactness allows to pass to the limit as the initial data is BV, and using the notion of admissible boundary, even for $L^\infty$ or measure valued entropy solutions [BM17]. A series of works culminating in [BM15] extends the Lagrangian representation also to systems of conservation laws.
An important application is the proof of the structure of $L^\infty$ solutions, and as a consequence the fact that the entropy dissipation is concentrated (see [BM17]).

Aim of this chapter is to present the results of [BBM], where we propose a suitable notion of Lagrangian representation for the multidimensional scalar equation

$$\partial_t u + \mathrm{div}_x\, \boldsymbol{f}(u) = 0, \qquad \boldsymbol{f} \colon \mathbb{R} \to \mathbb{R}^d \text{ smooth} \tag{10.1}$$

for non-negative solutions $u \geq 0$. The key step is always to find an a priori compactness estimate and an approximating scheme exploiting this compactness: in this situation, the transport collapse method introduced by Brenier [Bre84].
This approximation method is based on the interpretation of the evolution of the solution as the action of two operators:

**Transport map:** a translation of each level set of $u$ by the transport map

$$\mathrm{hyp}\ u(t) := \big\{(x,h) : h \leq u(t,x)\big\} \mapsto \mathsf{Tr}(s, \mathrm{hyp}\ u(t)) := \big\{(x,h) : h \leq u(t, x - f(h)s)\big\};$$

**Collapse operator:** the monotone mapping of each $x$ section of a generic set $E \subset \mathbb{R}^d \times [0,+\infty)$ into an interval with the same measure,

$$(E,x,h) \mapsto \mathsf{C}(E,x,h) := \big(x, \mathscr{H}^1((\{x\} \times [0,h]) \cap E)\big).$$

The image is clearly an hypograph of a function.

The transport collapse method is then the standard operator splitting approximation applied to the two operators $\mathsf{Tr}$, $\mathsf{C}$: the solution $u(t)$ to (10.1) is the limit of approximate solutions $u_n$ defined for $t \in 2^{-n}\mathbb{R}$ by

$$\operatorname{Graph} u_n([2^n t]2^{-n}) = \big(\mathsf{C}(\mathsf{Tr}(2^{-n},\cdot), \|u\|_\infty)\big)^{[2^n t]} \mathrm{hyp}\ u_0, \tag{10.2}$$

where $[\cdot]$ is the integer part of a real number. The composition $\mathsf{C}(\mathsf{Tr}(2^{-n},\cdot),\|u\|_\infty)$ means that given a set, one first translates the level set according to the characteristic speed for a time $2^{-n}$, and then find the total length on the vertical line at each pont $x \in \mathbb{R}^d$. Observe indeed that the projection operator $\mathsf{C}$ assign the new position of each point in a set $E \subset \mathbb{R}^{d+1}$, and does not just yield a function. A more detailed description is given in Section 10.2.3.

The natural compactness appears when interpreting the transport collapse method as a map acting on the whole hypograph of a function, i.e. assigning to every initial point $(x,h) \in \mathrm{hyp}\ u_0$ a trajectory $(\gamma^1(t),\gamma^2(t)) \in \mathbb{R}^{d+1}$. Indeed, by inspection of (10.2), the curve $t \mapsto \gamma^1(t)$ is uniformly Lipschitz, with Lipschitz constant bounded by $\|\boldsymbol{f}'\|_\infty$, while the second trajectory $t \mapsto \gamma^2(t)$ is decreasing in time.

The set of trajectories described above are clearly compact in the set of $L^1_{\mathrm{loc}}([0,+\infty),\mathbb{R}^{d+1})$ functions, so that one can apply standard compactness results to prove that there exists a finite measure $\omega$ such that

(1) it is concentrated on the solutions to the "characteristic ODE"

$$\dot\gamma^1 = \boldsymbol{f}'(\gamma^2), \quad \dot\gamma^2 \le 0,$$

(2) its push-forward $p_\sharp(\mathscr{L}^1 \times \omega)$ is the measure $\mathscr{L}^{d+2} \llcorner_{\mathrm{hyp}\ u}$, where

$$p(t,\gamma^1,\gamma^2) = (t,\gamma^1(t),\gamma^2(t)).$$

We can think the measure $\omega$ as a continuous version of the transport collapse operator splitting method, and following the nomenclature used in the one dimensional case, we call the measure $\omega$ a *Lagrangian representation* of the entropy solution $u(t)$.

**10.1.1. Notations for this chapter.** In the following, if $f\colon X \to [0,+\infty)$ is a non-negative function defined on some set $X$, we will denote its *hypograph* by

$$\mathrm{hyp}\ f := \{(x,h) \in X \times [0,+\infty) : 0 \le h \le f(x)\}.$$

Conversely, if $U \subset X \times [0,+\infty)$ we will use the notation

$$\mathrm{hyp}^{-1}(U) = f \tag{10.3}$$

to indicate that the set $U$ is the hypograph of the function $f$.

Recall also that there are natural "projection" operators defined on the space of curves, namely the *evaluation map* at time $t > 0$

$$\begin{aligned} e_t\colon \Gamma &\to \mathbb{R}^{d+1} \\ \gamma &\mapsto \gamma(t) \end{aligned} \tag{10.4}$$

and

$$\begin{aligned} p\colon (0,+\infty)\times\Gamma &\to (0,+\infty)\times\mathbb{R}^{d+1} \\ (t,\gamma) &\mapsto \big(t,\gamma(t)\big). \end{aligned} \tag{10.5}$$

Usually, the curves we will consider are not necessarily continuous, but they enjoy BV regularity. Accordingly, for the derivative we will write

$$D_t\gamma = \widetilde{D}_t\gamma + D^{\mathrm{j}}_t\gamma \tag{10.6}$$

where $\widetilde{D}_t\gamma$ is the *continuous* (or diffuse) part and $D^{\mathrm{j}}_t\gamma$ is the *jump* part.

The *essential interior* of a set $\Omega \subset \mathbb{R}^d$, $\operatorname{ess\,Int}(\Omega)$, is the set of points $x \in \mathbb{R}^d$ for which there exists a Lebesgue negligible set $N$ such that $x \in \operatorname{Int}(\Omega \cup N)$, being Int the standard topological interior.

**10.1.2. Preliminaries.** We collect here two preliminary results we will need in the rest of this chapter.

LEMMA 10.1. *Let $I = [a, b] \subset \mathbb{R}$ be a closed interval in $\mathbb{R}$. Let $(D_n)_n$ be an increasing sequence of finite sets $D_1 \subset D_2 \subset \ldots \subset I$ such that their union*

$$D := \bigcup_n D_n$$

*is dense in $I$. Let moreover $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps $f_n : I \to X$ where $(X, d)$ is a complete metric space. Assume that:*

(1) *$a \in D_1$;*
(2) *there exists a compact set $K \subset X$ such that for every $n, m \in \mathbb{N}$ with $n \leq m$ and for every $q \in D_n$, $f_m(q) \in K$;*
(3) *there exists a constant $C > 0$ such that for every $n, m \in \mathbb{N}$ with $n \leq m$, for every $q \in D_n$ and for every $x \in I$ with $q < x$, it holds*

$$d\big(f_m(q), f_m(x)\big) \leq C(x - q).$$

*Then there exist a subsequence $(n_k)_k$ and a $C$-Lipschitz function $f : I \to X$ such that*

$$f_{n_k} \to f \qquad \textit{uniformly on } I \textit{ as } k \to +\infty.$$

PROOF. By Condition (2) and the standard diagonal argument there exists a subsequence $f_{n_k}$, that we will denote by $f_k$, which converges pointwise in $D$. Therefore, for every $q \in D$, the sequence $(f_k(q))_{k\in\mathbb{N}}$ is a Cauchy sequence in $X$. Since $D_n$ is finite for every $n \in \mathbb{N}$, the convergence is uniform on each $D_n$. In particular for every $n \in \mathbb{N}$, there exists $N_n : [0, +\infty) \to \mathbb{N}$ such that for every $\varepsilon > 0$, for every $l, m \geq N_n(\varepsilon)$ and for every $q \in D_n$, it holds $d(f_l(q), f_m(q)) \leq \varepsilon$.

Now we prove that actually the sequence $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to the sup-norm. Fix $\varepsilon > 0$. Then by Condition (1), the monotonicity of the sequence $(D_n)_{n\in\mathbb{N}}$ and the density of $D \subset I$ there exists $\bar{n}$ such that for every $x \in I$ there exists $q \in D_{\bar{n}}$ such that $0 < x - q < \varepsilon$. Then for every $l, m \geq \bar{n} \vee N_{\bar{n}}(\varepsilon)$, it holds

$$\begin{aligned} d(f_l(x), f_m(x)) &\leq d(f_l(x), f_l(q)) + d(f_l(q), f_m(q)) + d(f_m(q), f_m(x)) \\ &\leq C(x - q) + \varepsilon + C(x - q) \\ &\leq (2C + 1)\varepsilon. \end{aligned}$$

Therefore the sequence $f_k$ converges uniformly to a function $f$. Now we check that $f$ is $C$-Lipschitz. For every $x, y \in I$ with $x < y$ and for every $q \in D$ with $q < x$, it holds

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(q)) + d(f(q), f(y)) \\ &\leq C(x - q + y - q). \end{aligned}$$

Letting $q \to x$ from below we get that $f$ is $C$-Lipschitz and this concludes the proof. □

We will also need the following standard result in the theory of sets of finite perimeter.

LEMMA 10.2. *Let $E \subset \mathbb{R}^d$ be a set of finite measure and of finite perimeter and let $v \in \mathbb{R}^d$ with $|v| = 1$. Then for every $\bar{t} \geq 0$ if $E_{\bar{t}v} := \{x + \bar{t}v : x \in E\}$ it holds*

$$\mathscr{L}^d(E \,\Delta\, E_{\bar{t}v}) \leq 2\bar{t}\operatorname{Per}(E).$$

PROOF. By Anzellotti-Giaquinta Theorem [**AFP00**, Theorem 3.9] there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty \cap W^{1,1}(\mathbb{R}^d)$ such that $u_n \to \mathbb{1}_E$ in $L^1(\mathbb{R}^d)$ and $Du^n \rightharpoonup D\mathbb{1}_E$ in duality with continuous, bounded functions over $\mathbb{R}^d$ and $\|Du^n\| \to \|D\mathbb{1}_E\|$. We want to compute

$$\mathscr{L}^d(E\,\Delta\,E_{tv}) = 2\int_{\mathbb{R}^d}(1-\mathbb{1}_E(x))\mathbb{1}_{E_{tv}}(x)\,dx.$$

Now we set

$$g_n(t) := \int_{E^c} u^n(x-tv)\,dx, \qquad g(t) := \int_{E^c} \mathbb{1}_{E_{tv}}(x)\,dx.$$

For $\phi \in C_c^\infty((0,+\infty))$ we have

$$-\langle D_t g_n, \phi\rangle = \int_0^{+\infty}\int_{E^c} u^n(x-tv)\,\phi'(t)\,dx\,dt = \int_{E^c}\int_0^{+\infty} \nabla u^n(x-tv)\cdot v\,\phi(t)\,dt\,dx.$$

This shows that

$$D_t g_n = -\int_{E^c} \nabla u^n(x-tv)\cdot v\,dx.$$

In particular,

$$|D_t g_n| \le \int_{E^c} |\nabla u^n(x-tv)\cdot v|\,dx \le \|Du^n\|.$$

We thus have

$$g_n(\bar t) - g_n(0) \le \int_0^{\bar t} \|Du^n\| dt = \bar t\|Du^n\|.$$

By observing that $g_n \to g$ pointwise and using that $\|Du^n\| \to \|D\mathbb{1}_E\| = \operatorname{Per} E$, we conclude the proof. □

## 10.2. Lagrangian representation

We consider *scalar multidimensional conservation laws*, i.e. first order partial differential equations of the form

$$\partial_t u + \operatorname{div}_x(\boldsymbol{f}(u)) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{R}^d, \tag{10.7}$$

where $u\colon (0,+\infty)\times\mathbb{R}^d \to \mathbb{R}$ is a scalar function and $\boldsymbol{f}\colon \mathbb{R}\to\mathbb{R}^d$ is a smooth map, called the *flux function.*

### 10.2.1. Definition and properties of the Lagrangian representation.

Since we only consider $L^\infty$ solutions, up to a translation in the flux $\boldsymbol{f}$, we can assume $u \ge 0$. We denote by

$$\Gamma := \Big\{\gamma = (\gamma^1,\gamma^2)\colon (0,+\infty) \to \mathbb{R}^d\times[0,+\infty) : \gamma^1 \text{ is continuous and } \gamma^2 \text{ is decreasing}\Big\}$$

equipped with the product of the uniform convergence on compact sets topology and of the $L^1_{\mathrm{loc}}$-topology.

DEFINITION 10.3. A *Lagrangian representation* of a solution $u$ to (10.7) is a measure $\omega \in \mathscr{M}^+(\Gamma)$ such that:

(1) it holds

$$p_\sharp(\mathscr{L}^1\times\omega) = \mathscr{L}^{d+2}\llcorner_{\mathrm{hyp}\;u}, \tag{10.8}$$

where we recall $p$ is the projection map defined in (10.5);

(2) $\omega$ is concentrated on the set of curves $\gamma = (\gamma^1,\gamma^2)\in\Gamma$ such that

$$\begin{cases}\dot\gamma^1(t) = \boldsymbol{f}'(\gamma^2(t)) & \mathscr{L}^1\text{-a.e. } t\in[0,+\infty),\\ \dot\gamma^2 \le 0 \text{ in the sense of distributions.}\end{cases} \tag{10.9}$$

The following lemma shows that the condition expressed in (10.8) is equivalent to its pointwise version.

LEMMA 10.4. *Assume that $t \mapsto u(t)$ is strongly continuous in $L^1$. Then in Definition 10.3, Condition (1) can be replaced with the following:*

(1') *for every $t > 0$, it holds*

$$e_{t\sharp}\omega = \mathscr{L}^{d+1}\llcorner_{\text{hyp } u(t)}, \tag{10.10}$$

*where we recall $e_t$ is the evaluation map defined in (10.4).*

PROOF. Condition (1') clearly implies (1). On the other hand, by Fubini, Condition (1) gives that (10.10) for $\mathscr{L}^1$-a.e. $t$. By exploiting the $L^1$-continuity in time of $u$, we now show that (10.10) holds indeed for every $t \in [0, +\infty)$. To do this, we write $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ and we fix $\bar{t}$; we take as test function the following

$$\varphi(t, x, h) = \phi(x, h)\psi_\delta(t)$$

where $\phi \colon \mathbb{R}^{d+1} \to \mathbb{R}$ is arbitrary, $\psi_\delta \colon [0, +\infty) \to \mathbb{R}$ is a non negative smooth function, with $\operatorname{supp}\psi_\delta \subset (\bar{t}, \bar{t} + \delta)$ and $\int_{\mathbb{R}^+} \psi_\delta = 1$. Taking the limit as $\delta \to 0^+$ of (10.8) tested against $\varphi$, we have

$$\int_{\mathbb{R}^{d+1}} \phi(x, h)\, d\mathscr{L}^{d+1}\llcorner_{\text{hyp } u(\bar{t})} = \int_\Gamma \phi(\gamma(\bar{t}+))\, d\omega$$

where $\gamma(\bar{t}+)$ denotes the right limit (which exists because $\gamma^1$ is continuous and $\gamma^2$ is decreasing). Similarly, on the left side, we get

$$\int_{\mathbb{R}^{d+1}} \phi(x, h) d\mathscr{L}^{d+1}\llcorner_{\text{hyp } u(\bar{t})} = \int_\Gamma \phi(\gamma(\bar{t}-))\, d\omega$$

thus, in particular,

$$0 = \int_\Gamma \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}-)\big) - \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}+)\big)\, d\omega.$$

Let us fix a compact set $K \subset \mathbb{R}^d$ and choose $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\partial_h\phi \geq 1$ in $K \times (0, \|u\|_\infty)$ and $\partial_h \phi \geq 0$ in $\mathbb{R}^d \times (0, \|u\|_\infty)$: being $\gamma^2$ decreasing, we have

$$\begin{aligned}
0 &= \int_\Gamma \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}-)\big) - \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}+)\big)\, d\omega \\
&\geq \int_{\Gamma \setminus \Gamma_K} \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}-)\big) - \phi\big(\gamma^1(\bar{t}), \gamma^2(\bar{t}+)\big)\, d\omega + \int_{\Gamma_K} \big(\gamma^2(\bar{t}-) - \gamma^2(\bar{t}+)\big)\, d\omega \\
&\geq \int_{\Gamma_K} |\gamma^2(\bar{t}-) - \gamma^2(\bar{t}+)|\, d\omega,
\end{aligned}$$

where $\Gamma_K \subset \Gamma$ is the set of curves such that $\gamma^1(\bar{t}) \in K$. This shows that for every $t \in (0, +\infty)$, $\omega$-a.e. $\gamma$ is continuous in $t$: in particular, we have $(e_t)_\sharp\omega = \mathscr{L}^{d+1}\llcorner_{\text{hyp } u(t)}$ for every $t$. ☐

We now present the following proposition, which says that Conditions (1), (2) in Definition 10.3 imply that $u$ is an entropy solution to (10.7).

PROPOSITION 10.5. *Let $\omega \in \mathscr{M}^+(\Gamma)$ be a non-negative measure on the space of curves and assume there exists a non-negative, bounded function $u \colon (0, +\infty) \times \mathbb{R}^d \to [0, +\infty)$ such that Conditions (1), (2) of Definition 10.3 hold. Then $u$ is an entropy solution to (10.7).*

PROOF. Let $(\eta, \boldsymbol{q})$ be an entropy-entropy flux pair with $\eta$ convex (w.l.o.g. $\eta(0) = 0, \boldsymbol{q}(0) = 0$). Using the elementary identities

$$u(t, x) = \int_0^{+\infty} \mathbb{1}_{[0, u(t,x)]}(h)\, dh$$

and

$$\eta(u(t, x)) = \int_0^{+\infty} \mathbb{1}_{[0,u(t,x)]}(h)\eta'(h)\, dh, \qquad \boldsymbol{q}(u(t, x)) = \int_0^{+\infty} \mathbb{1}_{[0,u(t,x)]}(h)\boldsymbol{q}'(h)\, dh$$

and recalling that $\boldsymbol{q}' = \eta' \boldsymbol{f}'$, for any non-negative test function $\phi \in C^1_c([0,+\infty) \times \mathbb{R}^d)$ we can write

$$\begin{aligned}
&-\langle \eta(u)_t + \mathrm{div}_x(\boldsymbol{q}(u)), \phi \rangle \\
&= \int_{\mathbb{R}^d} \int_0^{+\infty} \eta(u(t,x))\phi_t(t,x) + \boldsymbol{q}(u(t,x)) \cdot \nabla_x \phi(t,x)\, dt\, dx \\
&= \int_{\mathbb{R}^d} \int_0^{+\infty} \left[ \int_0^{+\infty} \mathbb{1}_{[0,u(t,x)]}(h)\eta'(h)\phi_t(t,x) + \boldsymbol{q}'(h) \cdot \nabla_x \phi(t,x)\, dh \right] dt\, dx \\
&= \int_{\mathbb{R}^d} \int_0^{+\infty} \int_0^{+\infty} \mathbb{1}_{[0,u(t,x)]}(h)\eta'(h)\big(\phi_t(t,x) + \boldsymbol{f}'(h) \cdot \nabla_x \phi(t,x)\big)\, dh\, dt\, dx \\
&= \int_{\mathbb{R}^{d+2}} \eta'(h)\big(\phi_t(t,x) + \boldsymbol{f}'(h) \cdot \nabla_x \phi(t,x)\big)\, d\big(\mathscr{L}^{d+2} \llcorner \mathrm{hyp}\ u\big).
\end{aligned}$$

By Condition (1) we have $p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2} \llcorner \mathrm{hyp}\ u$, so that

$$\begin{aligned}
-\langle \eta(u)_t + \mathrm{div}_x(\boldsymbol{q}(u)), \phi \rangle &= \int_{\mathbb{R}^{d+2}} \eta'(h)\big(\phi_t(t,x) + \boldsymbol{f}'(h) \cdot \nabla_x \phi(t,x)\big)\, d\big(\mathscr{L}^{d+2} \llcorner \mathrm{hyp}\ u\big) \\
&= \int_\Gamma \int_0^{+\infty} \eta'(\gamma^2(t)) \Big( \phi_t(t,\gamma^1(t)) + \boldsymbol{f}'(\gamma^2(t)) \cdot \nabla_x \phi(t,\gamma^1(t)\Big)\, dt\, d\omega.
\end{aligned}$$

Moreover, let us define for a.e. $t \in (0,+\infty)$ and for $\omega$-a.e. $\gamma$ the function

$$g_\gamma(t) := \eta'\big(\gamma^2(t)\big). \tag{10.11}$$

Recall that $\eta$ is convex and that for $\omega$-a.e. $\gamma$ the function $\gamma^2$ is decreasing by Condition (2); thus we have that $g_\gamma$ is decreasing for $\omega$-a.e. $\gamma$. Hence it holds $g'_\gamma \leq 0$ in the sense of distributions. By Fubini Theorem, we finally have

$$\begin{aligned}
-\langle \eta(u)_t + \mathrm{div}_x(\boldsymbol{q}(u)), \phi \rangle &= \int_\Gamma \int_0^{+\infty} \eta'(\gamma^2(t)) \Big( \phi_t\big(t,\gamma^1(t)\big) + \boldsymbol{f}'\big(\gamma^2(t)\big) \cdot \nabla_x \phi\big(t,\gamma^1(t)\big)\Big)\, dt\, d\omega \\
&= \int_\Gamma \int_0^{+\infty} \eta'(\gamma^2(t)) \Big( \phi_t\big(t,\gamma^1(t)\big) + \dot{\gamma}^1(t) \cdot \nabla_x \phi\big(t,\gamma^1(t)\big)\Big)\, dt\, d\omega \\
&= \int_\Gamma \int_0^{+\infty} \eta'(\gamma^2(t)) \frac{d}{dt} \phi(t,\gamma^1(t))\, dt\, d\omega \\
&= \int_\Gamma \int_0^{+\infty} g_\gamma(t) \phi'_\gamma(t)\, dt\, d\omega \geq 0
\end{aligned} \tag{10.12}$$

where the last inequality comes from the distributional definition of derivative for the function $g_\gamma$, being $\phi_\gamma(t) := \phi(t,\gamma^1(t))$ an admisible, non-negative test function. Thus we have established that, for any convex entropy $\eta$, it holds in the sense of distributions

$$\eta(u)_t + \mathrm{div}_x(\boldsymbol{q}(u)) \leq 0. \tag{10.13}$$

In particular, by taking $\eta(s) = \pm s$ and repeating the computation above, we get

$$u_t + \mathrm{div}_x(\boldsymbol{f}(u)) = 0. \tag{10.14}$$

Having established the two conditions (10.13) and (10.14), we have that $u$ is by definition an entropy solution to (10.7), hence the proof is complete. □

This proof shows also how the dissipation measure can be decomposed along the characteristic curves. Since this fact will be useful, we fix some notation and explicit this decomposition.

Let $\eta$ be a convex entropy and set

$$\mu^\eta_\gamma = (\mathbb{I},\gamma)_\sharp \Big( (\eta' \circ \gamma^2)\widetilde{D}\gamma^2 \Big) + \eta''(h)\mathscr{H}^1 \llcorner_{\{(t,x,h):\gamma^1(t)=x, h\in(\gamma^2(t+),\gamma^2(t-))\}}.$$

Accordingly define

$$\nu^\eta := \int_\Gamma \mu^\eta_\gamma \, d\omega. \tag{10.15}$$

LEMMA 10.6. *It holds*

$$(\pi_{t,x})_\sharp \nu^\eta = \mu^\eta,$$

*where the map* $\pi_{t,x} \colon \mathbb{R}^d \times [0,+\infty) \times [0,+\infty) \ni (t,x,h) \mapsto (t,x) \in \mathbb{R}^d \times [0,+\infty)$ *is the projection on the* $t, x$ *variables.*

PROOF. By definition we immediately get

$$(\pi_{t,x})_\sharp(\mu_\gamma) = (\mathbb{I}, \gamma^1)_\sharp(D_t g_\gamma), \tag{10.16}$$

where $g_\gamma$ is defined in (10.11). Including (10.16) in (10.12) we get

$$\begin{aligned}
\langle \eta(u)_t + \mathrm{div}_x(\boldsymbol{q}(u)), \phi \rangle &= -\int_\Gamma \int_0^{+\infty} g_\gamma(t)\phi'_\gamma(t)\, dt\, d\omega \\
&= \int_\Gamma \int_{[0,+\infty)\times\mathbb{R}^d} \phi \, d((\pi_{t,x})_\sharp \mu_\gamma) d\omega \\
&= \int_{[0,+\infty)\times\mathbb{R}^d} \phi \, d((\pi_{t,x})_\sharp \nu^\eta),
\end{aligned}$$

where in the last inequality we used the definition of $\nu^\eta$ (10.15) and the relation

$$\int_\Gamma (\pi_{t,x})_\sharp \mu^\eta_\gamma \, d\omega = (\pi_{t,x})_\sharp \left( \int_\Gamma \mu^\eta_\gamma \, d\omega \right). \qquad \square$$

In the following we will write $\bar\nu$ to denote $\nu^\eta$ in the particular case in which the entropy $\eta(h) = \frac{h^2}{2}$.

PROPOSITION 10.7. *The dissipation* $\bar\nu$ *in the essential interior of* hyp $u$ *is zero.*

PROOF. Let $\psi : \mathbb{R}^d \times [0,+\infty) \to [0,+\infty)$ be a smooth function such that for every $t \in (t_1, t_2)$, $\operatorname{supp}\psi \subset \operatorname{ess\,Int}(\text{hyp } u(t), \mathbb{R}^d \times [0,+\infty))$, then

$$t \mapsto \int_{R^{d+1}} \psi(x,h)\, d(e_t)_\sharp \omega$$

is constant. Take $(\bar t, \bar x, \bar w)$ in the essential interior of hyp $u$. Take $\psi(x,w) = \psi_1(x)\psi_2(w)$, where

$$\psi_1(x) = \sigma(|x - \bar x|), \qquad \partial_h \psi_2 < 0 \text{ in } [0, \bar h) \qquad \text{and} \qquad \psi_2(h) = 0 \text{ for } h > \bar h,$$

where $\sigma$ is smooth and nonnegative and $\sigma > 0$ in $[0, r)$, where $r \ll 1$. For every $\phi \in C^1_c((t_1,t_2))$, it holds

$$\begin{aligned}
0 &= -\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \phi'(t)\psi(x,h) d(e_t)_\sharp \omega \, dt \\
&= \int_\Gamma \int_{(t_1,t_2)} \phi(t) d(D_t(\psi \circ \gamma)) d\omega \\
&= \int_\Gamma \int_{(t_1,t_2)} \phi(t) \nabla\psi(\gamma(t)) d(\widetilde{D}_t \gamma) + \int_\Gamma \sum_i \phi(t_i) \left( \psi(\gamma(t_i^+)) - \psi(\gamma(t_i^-)) \right) d\omega,
\end{aligned}$$

by Volpert chain rule, where $\widetilde{D}_t\gamma$ is the continuous part of the derivative defined in (10.6). For every $\phi \geq 0$, and using the assumptions on $\psi$

$$\begin{aligned}
\int_\Gamma \int_{(t_1,t_2)} \phi(t) \nabla\psi(\gamma(t)) d(\widetilde{D}_t\gamma)\, d\omega = &\int_\Gamma \int_{t_1}^{t_2} \phi(t) \nabla_x \psi(\gamma(t)) \cdot \boldsymbol{f}'(\gamma^2(t)) dt\, d\omega \\
&+ \int_\Gamma \int_{t_1}^{t_2} \phi(t) \partial_h \psi(\gamma(t)) d(\widetilde{D}_t \gamma^2),\, d\omega,
\end{aligned}$$

by splitting horizontal and vertical components. We prove that the horizontal contribution is zero:

$$\begin{aligned}\int_\Gamma \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(\gamma(t)) \cdot \boldsymbol{f}'(\gamma^2(t))dtd\omega &= \int_{\mathbb{R}^{d+1}} \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(x,h) \cdot \boldsymbol{f}'(h)dtd\mathscr{L}^{d+1}\llcorner_{\mathrm{hyp}\ u(t)} \\ &= \int_{t_1}^{t_2} \phi(t) \int_0^{+\infty} \boldsymbol{f}'(h) \cdot \int_{B_r(\bar{x})} \nabla_x\psi(x,h)d\mathscr{L}^d\, dh\, dt \\ &= 0.\end{aligned}$$

We conclude that

$$\begin{aligned}0 &= -\int_{t_1}^{t_2}\int \phi'(t)\psi(x,h)\, d(e_t)_\sharp\omega\, dt \\ &= \int_\Gamma\int_{t_1}^{t_2} \phi(t)\partial_h\psi(\gamma(t))\, d(\widetilde{D}_t\gamma^2) + \int_\Gamma \sum_i \phi(t_i)\left(\psi(\gamma(t_i^+)) - \psi(\gamma(t_i^-))\right) d\omega \\ &= \int_{\mathbb{R}^{d+2}} \phi(t)\partial_h\psi\, d\bar{\nu}.\end{aligned}$$

By arbitrariness of $\phi, \psi$ (or by using $\bar{\nu} \leq 0$) we get $\bar{\nu} = 0$ in the interior of the hypograph. □

**10.2.2. Compactness and stability of Lagrangian representations.** We now turn to analyze stability properties that, in particular, will be useful in the construction of Lagrangian representations. In the following proposition, we show how the compactness of approximate solutions translates into tightness of the corresponding Lagrangian measures and how Condition (1) and Condition (2) pass to the limit.

Actually, we present the result in the more general framework in which the push forward of the measure $\mathscr{L}^1 \times \omega$ through the evaluation map $p$ is merely the Lebesgue measure $\mathscr{L}^{d+2}$ restricted to a set $U$, and not necessarily an hypograph. This allows more freedom in the construction of approximate solutions (e.g. Brenier's Transport-Collapse scheme will fit in this setting).

PROPOSITION 10.8 (Compactness and stability). *Let $(\omega^n)_{n\in\mathbb{N}} \subset \mathscr{M}^+(\Gamma)$ be a sequence of finite measures such that Condition (2) in Definition 10.3 holds. Assume that*

$$p_\sharp(\mathscr{L}^1 \times \omega^n) = \mathscr{L}^{d+2}\llcorner_{U^n}$$

*for some set $U^n \subset \mathbb{R}^{d+2}$ and assume that there exists $M > 0$ such that $U^n \subset (0,+\infty) \times \mathbb{R}^d \times [0,M]$ for every $n \in \mathbb{N}$. Assume furthermore that*

$$\mathbb{1}_{U^n} \to \mathbb{1}_U \quad \text{in } L^1(\mathbb{R}^{d+2}),$$

*for some set $U \subset \mathbb{R}^{d+2}$. Then $(\omega^n)_{n\in\mathbb{N}}$ is tight, every limit point $\omega$ satisfies Condition (2) in Definition 10.3 and it holds*

$$p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2}\llcorner_U.$$

PROOF. Since $\omega^n$ satisfies Condition (2) in Definition 10.3, we have that

$$\operatorname{supp}\omega^n \subset \mathrm{Lip}((0,+\infty),\mathbb{R}^d) \times \mathcal{D}$$

with local uniform bounds, hence $(\omega^n)_n$ is locally tight. Using a diagonal argument, we construct a measure $\omega$ which is the limit of $\omega^n$. We now show that

$$p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2}\llcorner_U.$$

where $p$ is the evaluation map defined in (10.5). Indeed, let $\varphi = \varphi(t,x,h)$ be a test function; we get

$$\begin{aligned}
\int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\, dp_\sharp(\mathscr{L}^1\times\omega)(t,x,h) &= \int_\Gamma\int_{\mathbb{R}^+} \varphi(t,\gamma(t))\,dt\,d\omega \\
&= \int_\Gamma \Phi(\gamma)d\omega(\gamma) \\
&= \lim_n \int_\Gamma \Phi(\gamma)d\omega^n(\gamma) \\
&= \lim_n \int_\Gamma\int_{\mathbb{R}^+} \varphi(t,\gamma(t))\,dt\,d\omega^n \\
&= \lim_n \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\,dp_\sharp(\mathscr{L}^1\times\omega^n) \\
&= \lim_n \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\,d(\mathscr{L}^{d+2}\llcorner_{U^n}) \\
&= \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,x,h)\,d(\mathscr{L}^{d+2}\llcorner_{U}),
\end{aligned}$$

where we have used in the second line the continuous function

$$\Phi(\gamma) := \int_0^{+\infty} \phi(t,\gamma(t))\,dt. \qquad \square$$

We conclude this paragraph by pointing out the following corollary, whose proof can be obtained particularizing Proposition 10.8 in the case where $U^n$ are hypographs of entropy solutions.

COROLLARY 10.9. *Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of uniformly bounded entropy solutions to (10.7) and assume it is given a sequence $(\omega^n)_{n\in\mathbb{N}}$ of corresponding Lagrangian representations. If $u^n \to u$ locally in $L^1$, then $(\omega^n)_{n\in\mathbb{N}}$ is tight and every limit point $\omega$ is a Lagrangian representation of $u$.*

**10.2.3. Existence of Lagrangian representations for initial data in $L^\infty$.** The compactness properties stated in Corollary 10.9 and standard approximation results imply that, in order to prove the existence of Lagrangian representations for solutions with initial data in $L^\infty$, it is enough to construct them for solutions with bounded variation. In order to do this, we exploit a numerical scheme which was proposed by Brenier in [Bre84] and is called "transport-collapse". We consider the initial value problem

$$\begin{cases} \partial_t u + \operatorname{div}_x(\boldsymbol{f}(u)) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0(\cdot) \end{cases} \tag{10.17}$$

with $u_0 \in L^\infty(\mathbb{R}^d)\cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and we denote by $u$ the entropy solution to (10.17). As before, we assume that $u \geq 0$.

We define the following *transport* map

$$\begin{aligned}
\mathsf{Tr}\colon [0,+\infty)\times\mathbb{R}^d\times[0,+\infty) &\to \mathbb{R}^d\times[0,+\infty) \\
(t,x,h) &\mapsto (x+t\boldsymbol{f}'(h),h),
\end{aligned}$$

which moves a point in $\mathbb{R}^d\times[0,+\infty)$ with the characteristic speed. Observe that, in general, if $v = v(x)$ is a function of $x$ then, for $t > 0$, the image

$$\mathsf{Tr}(t,\operatorname{hyp} v) := \bigcup_{(x,h)\in\operatorname{hyp} v} \mathsf{Tr}(t,x,h) \subset \mathbb{R}^d\times[0,+\infty)$$

is not necessarily an hypograph.

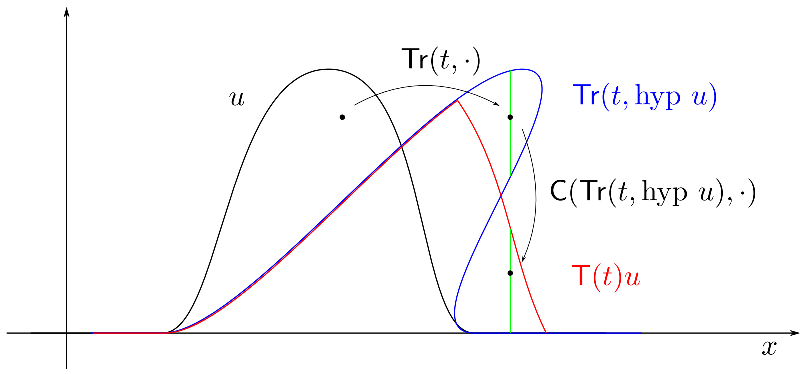



**Figure 1.** Picture of the transport collapse scheme.

Then we introduce the *collapse* operator: we first define the set

$$X := \Big\{(E, x, h) \in \mathcal{P}(\mathbb{R}^d \times [0, +\infty)) \times \mathbb{R}^d \times [0, +\infty) : (x, h) \in E\Big\},$$

where we recall $\mathcal{P}$ denotes the power set and then

$$\begin{aligned} \mathsf{C}\colon X &\mapsto \mathbb{R}^d \times [0, +\infty) \\ (E, x, h) &\mapsto \big(x, \mathscr{H}^1((\{x\} \times [0, h]) \cap E)\big) \end{aligned}$$

where $\mathcal{H}^1$ is the (outer) 1-dimensional Hausdorff measure. The collapse operator moves points vertically in the negative direction. Moreover the image of a set is always an hypograph (possibly taking value $+\infty$) and $\mathsf{C}(E, \cdot, \cdot)$ is the identity if and only if $E$ is an hypograph.

We now set

$$Y := \Big\{(v, x, h) \in L^\infty_+(\mathbb{R}^d) \times \mathbb{R}^d \times [0, +\infty) : (x, h) \in \text{hyp } v\Big\}.$$

We define the transport-collapse map at time $t > 0$ in the following way:

$$\begin{aligned} \mathsf{TC}_t\colon Y &\to \mathbb{R}^d \times [0, +\infty) \\ (v, x, h) &\mapsto \mathsf{C}(\mathsf{Tr}(t, \text{hyp } v), \mathsf{Tr}(t, x, h)) \end{aligned}$$

Remark 10.10. The contruction above is only a Lagrangian rephrase of the Transport-Collapse scheme proposed by Brenier in [**Bre84**]. There, the author defines the Transport-Collapse operator as the family of operators $\{\mathsf{T}(t)\}_{t>0}$ on $L^1(\mathbb{R}^d)$ whose restriction to the space of non-negative, integrable functions $L^1_+(\mathbb{R}^d)$ is

$$\begin{aligned} \mathsf{T}(t)\colon L^1_+(\mathbb{R}^d) &\to L^1_+(\mathbb{R}^d) \\ v &\mapsto (\mathsf{T}(t)v)(x) := \int_{\mathbb{R}} jv(x - t\boldsymbol{f}'(h), h)\, dh \end{aligned}$$

where

$$jv(x, h) := \mathbb{1}_{\text{hyp } v}(x, h) = \begin{cases} 1 & \text{if } 0 < h < v(x), \\ 0 & \text{else.} \end{cases}$$

The link between the two formulations is the following:

$$\text{hyp } (\mathsf{T}(t)v) = \mathsf{TC}_t(v, \text{hyp } v).$$

On the other hand, the map $\mathsf{TC}_t$ chooses the image of *each* point in the hypograph and not only the image of the whole hypograph (see Figure 1) . ♠

We are now in position to define an approximating sequence $(\mathsf{TC}^n_t)$ of the Kruzkov semigroup. We define first them inductively for $t \in 2^{-n}\mathbb{N}$:

$$\begin{cases} \mathsf{TC}^n_0(v,x,h) = (x,h), \\ \mathsf{TC}^n_{(k+1)\cdot 2^{-n}}(v,x,h) = \mathsf{TC}_{2^{-n}}\big(\text{hyp}^{-1}(\mathsf{TC}^n_{k\cdot 2^{-n}}(v, \text{hyp } v)), \mathsf{TC}^n_{k\cdot 2^{-n}}(v,x,h)\big), \end{cases}$$

where $\text{hyp}^{-1}(\cdot)$ is defined in (10.3).

For the intermediate times $t = s + k \cdot 2^{-n}$, with $s \in (0, 2^{-n})$, we set

$$\mathsf{TC}^n_t := \mathsf{Tr}(s) \circ \big(\mathsf{TC}^n_{k\cdot 2^{-n}}\big).$$

Taking now $u_0 \in L^\infty(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$, we define accordingly for every $(x,h) \in \text{hyp } u_0$ and for every $t > 0$,

$$\gamma^n_{(x,h)}(t) := \mathsf{TC}^n_t(u_0, x, h)$$

and we set

$$\omega^n := \int_{\text{hyp } u_0} \delta_{\gamma^n_{(x,h)}}\, dx\, dh. \tag{10.18}$$

Since the transport collapse scheme is measure preserving, there exists $U^n \subset [0,+\infty) \times \mathbb{R}^d \times [0,+\infty)$ such that

$$(e_t)_\sharp \omega^n = \mathscr{L}^d \llcorner_{U^n(t)}, \tag{10.19}$$

where

$$U^n(t) := \big\{(x,h) \in \mathbb{R}^d \times [0,+\infty) : (t,x,h) \in U\big\}.$$

10.2.3.1. *Total variation along Transport-Collapse.* A crucial property in [Bre84] is that the total variation decreases along the Transport-Collapse scheme. This is indeed stated and proved in the following lemma and we present the proof for the sake of completeness.

Lemma 10.11. *For every $t \geq 0$ and $u \in L^1_+(\mathbb{R}^d)$ it holds*

$$\text{Tot.Var.}(\mathsf{T}(t)u) \leq \text{Tot.Var.}(u).$$

Proof. For every $t \geq 0$, for any test vector field $\Phi \in C^1_c(\mathbb{R}^d; \mathbb{R}^d)$, with $\|\Phi\|_\infty \leq 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathsf{T}(t)u)(x) \operatorname{div} \Phi(x)\, dx &= \int_{\mathbb{R}^d} \int_0^{+\infty} ju(x - t\boldsymbol{f}'(h), h) \operatorname{div} \Phi(x)\, dh\, dx \\ &= \int_{\mathbb{R}^d} \int_0^{+\infty} ju(x,h) \operatorname{div} \Psi_h(x)\, dh\, dx \\ &\leq \int_0^{+\infty} \text{Tot.Var.}(ju(\cdot,h))\, dh, \end{aligned}$$

where we have set $\Psi_h(x) = \Phi(x + t\boldsymbol{f}'(h))$ and the last inequality holds by definition of total variation (together with the trivial fact that $\|\Psi_h\|_\infty \leq 1$). Finally, by Coarea formula V, we have

$$\int_0^{+\infty} \text{Tot.Var.}(ju(\cdot,h))dh = \text{Tot.Var.}(u).$$

Being $\Phi$ arbitrary, the proof is complete. □

10.2.3.2. *Passage to the limit of Transport-Collapse.* In this section we give an alternative proof of the fact that the iterated Transport-Collapse scheme converges to the Kruzkov semigroup, based on the Lagrangian representation. As a byproduct, we obtain the existence of Lagrangian representations for BV initial data and, as already noticed, this suffices for the general $L^\infty$ case.

Let us also fix $D_n := \{\frac{k}{2^n} : k \in \mathbb{N}_{\geq 0}\}$ so that for every $\bar{t} \in D_n$ there exists $u^n(\bar{t}) \in L^\infty(\mathbb{R}^d)$ such that

$$U^n(\bar{t}) = \text{hyp } u^n(\bar{t}).$$

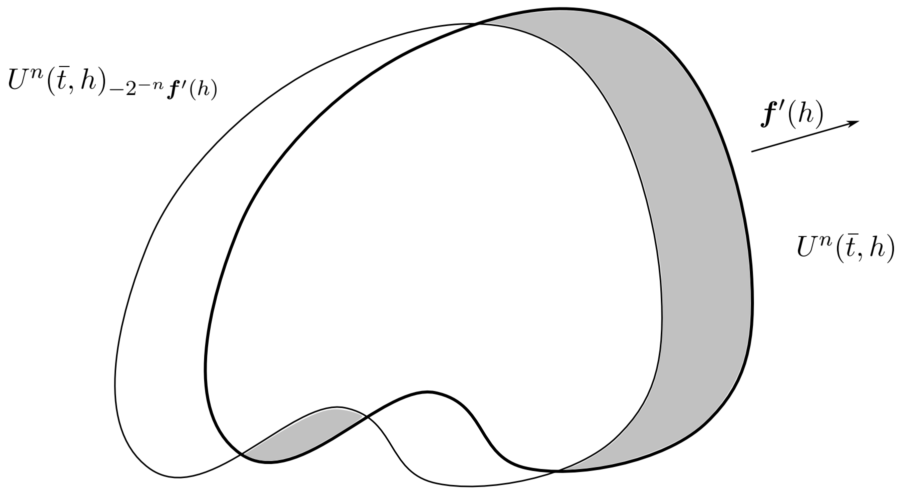



**Figure 2.** The set in grey is $U^n(\bar t, h) \cap \big(U^n(\bar t, h)_{-2^{-n}\boldsymbol{f}'(h)}\big)^c$.

The key point to prove the compactness of the family $(U^n)_{n\in\mathbb{N}}$ is contained in the following lemma.

LEMMA 10.12. *Let $\bar n \in \mathbb{N}$ and $\bar t \in D_{\bar n}$. Then for every $t > \bar t$ and for every $n \geq \bar n$, it holds*

$$\|(e_t)_\sharp\omega^n - (e_{\bar t})_\sharp\omega^n\|_{\mathscr{M}} = \mathscr{L}^{d+1}\big(U^n(t)\,\Delta\,U^n(\bar t)\big) \leq 2\|f'\|_\infty\big(t-\bar t\big)\text{Tot.Var.}(u_0). \tag{10.20}$$

PROOF. Let us now write $t - \bar t = k\cdot 2^{-n} + s$ for $s \in [0, 2^{-n})$. For $j = 0, \dots, k-1$ set

$$I_j := [t_{j,n}, t_{j+1,n}], \qquad \text{where} \qquad t_{j,n} := \bar t + j2^{-n}.$$

Observe that it holds

$$\mathscr{L}^{d+1}\big(U^n(t)\,\Delta\,U^n(\bar t)\big) = 2\omega^n\left(\{\gamma : \gamma(\bar t) \in U^n(\bar t), \gamma(t) \notin U^n(\bar t)\}\right).$$

Being $U(\bar t)$ the hypograph of $u^n(\bar t)$, for every $j = 0, \dots, k-1$ and $\gamma \in \operatorname{supp}\omega^n$,

$$\gamma(t_{j,n}-) \in U^n(\bar t) \qquad \Longrightarrow \qquad \gamma(t_{j,n}+) \in U^n(\bar t). \tag{10.21}$$

For any $j = 0, \dots, k-1$ we set

$$\mathcal{G}_{j,n} := \Big\{\gamma \in \operatorname{supp}\omega^n : \gamma(t_{j,n}+) \in U^n(\bar t), \gamma(t_{j+1,n}-) \notin U^n(\bar t)\Big\}.$$

Finally, if $s = 0$ we set $\mathcal{G}_k = \emptyset$ and if $s > 0$,

$$\mathcal{G}_{k,n} := \Big\{\gamma \in \operatorname{supp}\omega^n : \gamma(t_{k,n}+) \in U^n(\bar t), \gamma(t) \notin U^n(\bar t)\Big\}.$$

By (10.21), it holds

$$\big\{\gamma : \gamma(\bar t) \in U^n(\bar t), \gamma(t) \notin U^n(\bar t)\big\} \subset \bigcup_{j=0}^{k} \mathcal{G}_{j,n}.$$

Let us fix $j = 0, \dots, k-1$. By (10.19) and definition of $\omega^n$,

$$\begin{aligned}\omega^n\left(\mathcal{G}_{j,n}\right) &= \mathscr{L}^{d+1}\left(\big\{(x,h) \in U^n(\bar t) \cap U^n(t_{j,n}) : (x + \boldsymbol{f}'(h)2^{-n}, h) \notin U^n(\bar t)\big\}\right) \\ &= \int_0^{\|u_0\|_\infty} \mathscr{L}^d\left(\big\{x \in U^n(\bar t, h) \cap U^n(t_{j,n}, h) : x + \boldsymbol{f}'(h)2^{-n} \notin U^n(\bar t, h)\big\}\right) dh,\end{aligned} \tag{10.22}$$

where we have set $U(t,h) := \{x : (t,x,h) \in U\}$ and used Fubini theorem. Now we observe that

$$\Big\{x \in U^n(\bar t, h) \cap U^n(t_{j,n}, h) : x + \boldsymbol{f}'(h)2^{-n} \notin U^n(\bar t, h)\Big\} \subset U^n(\bar t, h) \cap (U^n(\bar t, h)_{-2^{-n}\boldsymbol{f}'(h)})^c,$$

where we recall that $E_{\boldsymbol{v}} := E + \boldsymbol{v}$ (see Figure 2). Since

$$\mathscr{L}^d\left(U^n(\bar{t},h) \cap (U^n(\bar{t},h)_{-2^{-n}\boldsymbol{f}'(h)})^c\right) = \frac{1}{2}\mathscr{L}^d\big(U^n(\bar{t},h)\Delta(U^n(\bar{t},h)_{-2^{-n}\boldsymbol{f}'(h)})\big),$$

by applying Lemma 10.2, we have

$$\mathscr{L}^d\big(U^n(\bar{t},h)\Delta(U^n(\bar{t},h)_{-2^{-n}\boldsymbol{f}'(h)})\big) \leq 2\|\boldsymbol{f}'\|_\infty 2^{-n}\operatorname{Per}(U^n(\bar{t},h)).$$

Taking into account (10.22), and using Coarea formula for functions of bounded variation V we get

$$\begin{aligned}\omega^n\left(\mathcal{G}_{j,n}\right) &\leq \int_0^{\|u_0\|_\infty} \|\boldsymbol{f}'\|_\infty 2^{-n}\operatorname{Per}(U^n(\bar{t},h))\,dh \\ &= 2^{-n}\|\boldsymbol{f}'\|_\infty \text{Tot.Var.}(u^n(\bar{t})) \\ &\leq 2^{-n}\|\boldsymbol{f}'\|_\infty \text{Tot.Var.}(u_0),\end{aligned}$$

where the last inequality follows by Lemma 10.11. Similarly we can prove that

$$\omega^n\left(\mathcal{G}_{k,n}\right) \leq s\|\boldsymbol{f}'\|_\infty \text{Tot.Var.}(u_0),$$

therefore summing over $j = 0, \dots, k$ we get

$$\begin{aligned}\mathscr{L}^{d+1}\big(U^n(t)\,\Delta\,U^n(\bar{t})\big) &\leq 2\sum_{j=0}^{k}\omega^n\left(\mathcal{G}_{j,n}\right) \\ &\leq 2((2^{-n}k+s)\|\boldsymbol{f}'\|_\infty \text{Tot.Var.}(u_0) \\ &= 2(t-\bar{t})\|\boldsymbol{f}'\|_\infty \text{Tot.Var.}(u_0).\end{aligned}$$ □

We now combine the estimate (10.20) together with Lemma 10.1 to deduce the existence of a Lagrangian representation for BV solutions.

PROPOSITION 10.13. *The sequence $(\omega^n)_{n\in\mathbb{N}}$ constructed in (10.18) is tight and every limit point $\omega$ is a Lagrangian representation of the entropy solution to (10.17).*

PROOF. As in the proof of Proposition 10.8, the tightness of the family follows from Condition (2) in Definition 10.3 together with uniform bounds. Let $\omega$ be any limit point.

We now want to apply Lemma 10.1: set $I = [0, T]$ and let $D_n := \{\frac{k}{2^n} : k = 0, \dots, 2^n T\}$. Let then $X := L^1(\mathbb{R}^{d+1})$ and accordingly define

$$\begin{aligned} f_n\colon I &\to L^1(\mathbb{R}^{d+1}) \\ t &\mapsto \mathbb{1}_{\operatorname{supp}(e_t)_\sharp\omega^n}(\cdot)\end{aligned}$$

Condition (1) is trivially satisfied; let us verify Assumption (2). For any $n \in \mathbb{N}$, for every $t \in D_n$ and every $m > n$ we have $(e_t)_\sharp\omega^m$ is concentrated on the hypograph of some function $u^m(t)$. By Lemma 10.11 the functions $(u^m(t))_{m\geq n}$ have uniformly bounded total variation, hence they are compact in $L^1(\mathbb{R}^d)$ and therefore the hypographs are compact in $L^1(\mathbb{R}^{d+1})$. To verify condition (3), it is enough to apply Lemma 10.12.

Thus we obtain a Lipschitz function $f\colon I \to L^1(\mathbb{R}^{d+1})$; since $f(t)$ is the characteristic function of an hypograph for every $t \in D$, by continuity, there exists

$$u \in \operatorname{Lip}([0,T];\operatorname{BV}(\mathbb{R}^d)) \quad \text{such that} \quad f(t) = \mathbb{1}_{\text{hyp } u(t)}$$

for every $t \in [0, T]$. Thanks to Proposition 10.8 we obtain that

$$(e_t)_\sharp\omega = \mathscr{L}^{d+1}\llcorner_{\text{hyp } u(t)}$$

for every $t \geq 0$. Finally, a direct application of Proposition 10.5 shows that the function $u$ is the entropy solution to (10.17) and concludes the proof. □

The compactness and stability properties of Lagrangian representations stated in Corollary 10.9, together with standard approximation results, yield immediately the following

THEOREM 10.14. *Let $u$ be the entropy solution to the initial value problem* (10.17) *with $u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a Lagrangian representation of $u$.*

### 10.3. The case of continuous solutions

In this section we prove that if $u$ is a continuous entropy solution of (10.7) then for every entropy-entropy flux pair $(\eta, \boldsymbol{q})$ with $\eta \in C^1(\mathbb{R})$, the dissipation measure $\mu$ vanishes, namely

$$\mu = \eta(u)_t + \operatorname{div}(\boldsymbol{q}(u)) = 0.$$

Denote the jump part of $\bar{\nu}$ by

$$\nu^j := \int_\Gamma \mu^j_\gamma d\omega, \qquad \text{where} \quad \mu^j_\gamma = \mathscr{H}^1 \llcorner_{\{(t,x,h):\gamma^1(t)=x, h\in(\gamma^2(t+),\gamma^2(t-))\}}.$$

As an intermediate step we prove that $\nu^j = 0$, which is equivalent, by definition, to the fact that $\omega$ is concentrated on continuous curves.

LEMMA 10.15. *Let $u\colon [0,T) \times \mathbb{R}^d \to \mathbb{R}$ be a continuous solution of* (10.7) *and let $\omega$ be a Lagrangian representation of $u$. Then $\omega$ is concentrated on* continuous *characteristic curves.*

PROOF. Since the solution $u$ is continuous, for every $(t,x,h) \in [0,+\infty)\times\mathbb{R}^d\times(0,+\infty)$ such that $h < u(t,x)$, it holds $(t,x,h) \in \operatorname{Int}(\operatorname{hyp} u)$. Hence for every $\gamma \in \operatorname{supp}\omega$,

$$\mu^j_\gamma = \mu^j_\gamma \llcorner_{\operatorname{Int}(\operatorname{hyp}\ u)}.$$

Therefore

$$\nu^j = \nu^j \llcorner_{\operatorname{Int}(\operatorname{hyp}\ u)} = 0,$$

by Proposition 10.7. This concludes the proof of this lemma. □

In the following proposition we show that for continuous solutions the hypograph at time $t$ is the translation of hyp $u_0$ along segments with characteristic speed.

PROPOSITION 10.16. *Let $u\colon [0,T)\times\mathbb{R}^d \to \mathbb{R}$ be a continuous entropy solution of* (10.7). *Then*

$$\bar{\omega} = \int_{\operatorname{hyp}\ u_0} \delta_{\bar{\gamma}_{x,h}}\, dx\, dh,$$

*where*

$$\bar{\gamma}_{x,h}(t) = (x + t\boldsymbol{f}'(h), h), \qquad t \in [0,T)$$

*is a Lagrangian representation of $u$.*

PROOF. To begin we notice that there exists a set $E$ with $\mathscr{L}^{d+2}(\operatorname{hyp} u \setminus E) = 0$ such that for every $z = (t,x,h) \in E$ there exists a curve $\gamma_z\colon [0,\bar{t}] \to \mathbb{R}^d \times [0,+\infty)$ with the following properties:

(1) $\gamma_z(t) = (x, w)$;
(2) $\gamma_z$ is a continuous characteristic curve;
(3) $\gamma_z\big([0,\bar{t}]\big) \subset \operatorname{hyp} u$;
(4) $\gamma^2_z$ is constant on the connected components of $\gamma_z^{-1}(\operatorname{Int}(\operatorname{hyp} u))$.

In fact, (1) follows from the definition of Lagrangian representation and (2) follows from Lemma 10.15. From the definition of Lagrangian representation $\omega$ is concentrated on curves that lie in hyp $u$ for $\mathscr{L}^1$-a.e. $t \in [0,T]$. By continuity of $u$, we thus get (3). Finally (4) follows by Proposition 10.7. Let $\bar{t} > 0$ and for every $(x,h) \in \text{hyp}\, u(\bar{t})$ we consider the function

$$\begin{aligned}\sigma_{(x,h)} \colon [0,\bar{t}] &\to \mathbb{R}^d \times [0,+\infty)\\ t &\mapsto \big(x - (\bar{t}-t)\boldsymbol{f}'(h), h\big).\end{aligned}$$

We first prove that for every $(x,h) \in \text{hyp}\, u(\bar{t})$ the segments

$$\sigma_{(x,h)}\big([0,\bar{t}]\big) \subset \text{hyp}\, u.$$

Fix $\varepsilon > 0$ and let us construct by iteration a curve contained in the hypograph which approximates the segment. By uniform continuity of $u$ there exists $\delta \in (0,1)$ such that

$$|(t,x)-(t',x')| \le \delta \quad \Rightarrow \quad |u(t,x)-u(t',x')| \le \varepsilon.$$

Let $\varepsilon' < \delta\varepsilon$ and fix $(t_1,x_1) \in [0,+\infty)\times\mathbb{R}$ and $\bar{h} > 0$ such that $(t_1,x_1,\bar{h}) \in \text{hyp}\, u$. For $k \ge 1$ we define by recursion the points $\tilde{z}_k$, $t_k$ and $x_k$ in the following way:

$$\tilde{z}_k = (\tilde{t}_k, \tilde{x}_k, \tilde{h}_k) \in B_{\varepsilon'}((t_k, x_k, \bar{h}-\varepsilon)) \cap E, \tag{10.23}$$

with $\tilde{t}_k < t_k$ and

$$t_{k+1} := \inf\{t \in [0,\tilde{t}_k] : \gamma_{\tilde{z}_k}(t) < \bar{h}+\varepsilon\}, \qquad x_{k+1} := \gamma^1_{\tilde{z}_k}(t_{k+1}+).$$

The procedure ends when $t_{k+1} = 0$. The existence of points $\tilde{z}_k$ is ensured by the fact that $E$ has full measure. We now prove that the procedure ends in finitely many steps. Since for every $k \ge 0$, $\gamma^2_{\tilde{z}_k}$ is constant on each connected component of $\gamma^{-1}_{\tilde{z}_k}(\text{Int}(\text{hyp}\, u))$ and $\gamma^2_{\tilde{z}_k}(\tilde{t}_k) < u(\tilde{t}_k,\tilde{x}_k) - \varepsilon$, by the uniform continuity of $u$

$$\tilde{t}_k - t_{k+1} \ge \frac{\delta}{\|\boldsymbol{f}'\|_\infty} \wedge \tilde{t}_k,$$

therefore the number of steps $N$ after which the procedure ends is bounded by

$$N \le 1 + \frac{\|\boldsymbol{f}'\|_\infty \bar{t}}{\delta}. \tag{10.24}$$

We now prove the following claim, which states that $\gamma_{\tilde{z}_k}$ approximates $\sigma_{(\bar{x},\bar{h})}$ in $(t_{k+1},\tilde{t}_k)$.

**Claim.** There exists $C > 0$ independent of $\varepsilon$ such that for every $t \in [0,\bar{t}]$ there exists $k = 1,\dots,N$ and $s \in (t_{k+1},\tilde{t}_k)$ for which

$$|(s,\gamma_{\tilde{z}_k}(s)) - (t,\sigma_{(\bar{x},\bar{h})}(t))| < C\varepsilon. \tag{10.25}$$

First we observe that for every $k = 1,\dots,N$ and for every $s \in (t_{k+1},\tilde{t}_k)$ it holds

$$|\gamma^2_{\tilde{z}_k}(s) - \bar{h}| < 2\varepsilon. \tag{10.26}$$

The estimate for the first components follows by (10.26) and (10.9): for every $k = 1,\dots,N$,

$$\begin{aligned}|\gamma^1_{\tilde{z}_k}(t_{k+1}) - \sigma^1_{(\bar{x},\bar{h})}(t_{k+1})| &= \left|\gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k) - \int_{t_{k+1}}^{\tilde{t}_k} \big(\dot{\gamma}^1_{\tilde{z}_k}(t) - \boldsymbol{f}'(\bar{h})\big)\,dt\right|\\ &\le |\gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k)| + 2\varepsilon(\tilde{t}_k - t_{k+1})\|\boldsymbol{f}''\|_\infty.\end{aligned} \tag{10.27}$$

Moreover, by (10.23),

$$\begin{aligned}|\gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k)| &\le |\gamma^1_{\tilde{z}_k}(\tilde{t}_k) - \gamma^1_{\tilde{z}_k}(t_k)| + |\gamma^1_{\tilde{z}_k}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(t_k)| + |\sigma^1_{(\bar{x},\bar{h})}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(\tilde{t}_k)|\\ &\le 2\|\boldsymbol{f}'\|_\infty\varepsilon' + |\gamma^1_{\tilde{z}_k}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(t_k)|.\end{aligned} \tag{10.28}$$

By (10.27) and (10.28), it follows that for every $k = 1, \dots, N-1$ it holds

$$|\gamma^1_{\tilde{z}_k}(t_{k+1}) - \sigma^1_{(\bar{x},\bar{h})}(t_{k+1})| \le |\gamma^1_{\tilde{z}_k}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(t_k)| + 2\varepsilon(\tilde{t}_k - t_{k+1})\|\boldsymbol{f}''\|_\infty + 2\|\boldsymbol{f}'\|_\infty \varepsilon'. \tag{10.29}$$

For every $t \in [0, \bar{t}]$ let $\bar{k} = 1, \dots, N-1$ and $s \in (t_{\bar{k}+1}, \tilde{t}_{\bar{k}})$ be such that $|s - t| < \varepsilon'$. Then, iterating (10.29) for $k = \bar{k}, \dots, N-1$ and by (10.24), we have

$$\begin{aligned} |\gamma^1_{\tilde{z}_k}(s) - \sigma^1_{\bar{z}}(t)| &\le |\gamma^1_{\tilde{z}_k}(s) - \sigma^1_{\bar{z}}(s)| + |\sigma^1_{\bar{z}}(s) - \sigma^1_{\bar{z}}(t)| \\ &\le 2\varepsilon\|\boldsymbol{f}''\|_\infty(\bar{t} - s) + 2(N - \bar{k})\varepsilon'\|\boldsymbol{f}'\|_\infty + \|\boldsymbol{f}'\|_\infty |t - s| \\ &\le 2\varepsilon\|\boldsymbol{f}''\|_\infty T + 2\varepsilon'\|\boldsymbol{f}'\|_\infty + 2\varepsilon\|\boldsymbol{f}'\|^2_\infty \bar{t} + \|\boldsymbol{f}'\|_\infty \varepsilon' \\ &\le C\varepsilon, \end{aligned} \tag{10.30}$$

where $C = 2\|\boldsymbol{f}''\|_\infty T + 2\|\boldsymbol{f}'\|_\infty + 2\|\boldsymbol{f}'\|^2_\infty T + \|\boldsymbol{f}'\|_\infty$. The estimates (10.26) and (10.30) prove (10.25). Since hyp $u$ is closed, letting $\varepsilon \to 0$ we obtain that for every $(\bar{x}, \bar{h}) \in$ hyp $u(\bar{t})$, the segment

$$\sigma_{(\bar{x},\bar{h})}([0, \bar{t}]) \subset \text{hyp } u.$$

Let

$$\tilde{\omega} = \int_{\text{hyp } u(\bar{t})} \delta_{\sigma_{x,h}} dx dh.$$

Since the translations are area-preserving, for every $t \in [0, \bar{t}]$, there exists $U(t) \subset [0, +\infty) \times \mathbb{R}^d$ such that

$$(e_t)_\sharp \tilde{\omega} = \mathscr{L}^{d+1} \llcorner_{U(t)}$$

and

$$\mathscr{L}^{d+1}(U(t)) = \int_{\mathbb{R}^d} u(\bar{t}, x) dx. \tag{10.31}$$

Since we proved that for every $t \in [0, \bar{t}]$ it holds $U(t) \subset$ hyp $u(t)$, (10.31) implies that $U(t) =$ hyp $u(t)$. This proves that $\tilde{\omega} = \bar{\omega}$ and it is a Lagrangian representation of $u$. □

We are finally ready to prove Theorem C.

THEOREM 10.17. *Let $u$ be a continuous bounded entropy solution in $[0, T) \times \mathbb{R}^d$ to (10.7). Then for every $(t, x) \in [0, T) \times \mathbb{R}^d$, it holds*

$$u(t, x) = u_0(x - \boldsymbol{f}'(u(t, x))t). \tag{10.32}$$

*Moreover for every $\eta \colon \mathbb{R} \to \mathbb{R}$, $\boldsymbol{q} \colon \mathbb{R} \to \mathbb{R}^d$ Lipschitz such that $\boldsymbol{q}' = \eta' \boldsymbol{f}'$ a.e. with respect to $\mathscr{L}^1$, it holds*

$$\eta(u)_t + \operatorname{div}_x \boldsymbol{q}(u) = 0 \quad \text{in } \mathscr{D}'((0, T) \times \mathbb{R}^d). \tag{10.33}$$

PROOF. The validity of (10.32) is an immediate consequence of Proposition (10.16). Concerning the second claim, if $\eta$ is a convex $C^2$ entropy, then (10.33) follows by Lemma 10.6 and Proposition 10.16, since $\mu^\eta_\gamma = 0$ for every $\gamma \in \operatorname{supp} \omega$. If $\eta$ is $C^2$, then there exist $\eta_1, \eta_2$ of class $C^2$ and convex such that $\eta = \eta_1 - \eta_2$ and thus it is enough to apply the previous result to both $\eta_1$ and $\eta_2$. Finally, in order to prove that (10.33) holds for Lipschitz $(\eta, \boldsymbol{q})$, we consider a sequence $(\eta^n)_{n \in \mathbb{N}}$ such that $\eta^n \to \eta$ uniformly on $\mathbb{R}$ and $(\eta^n)' \to \eta'$ in $L^1_{\text{loc}}(\mathbb{R})$ with the associated $\boldsymbol{q}^n$ such that $\boldsymbol{q}^n(0) = \boldsymbol{q}(0)$. We have that $\boldsymbol{q}^n \to \boldsymbol{q}$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ and hence, for every test function $\phi \in C^\infty_c([0, T) \times \mathbb{R}^d)$,

$$\begin{aligned} -\langle \eta(u)_t + \operatorname{div}_x \boldsymbol{q}(u), \phi \rangle &= \int_0^T \int_{\mathbb{R}^d} \phi_t \eta(u) + \boldsymbol{q}(u) \cdot \nabla \phi \, dx \, dt \\ &= \lim_{n \to +\infty} \int_0^T \int_{\mathbb{R}^d} \phi_t \eta^n(u) + \boldsymbol{q}^n(u) \cdot \nabla \phi \, dx \, dt = 0, \end{aligned}$$

and this completes the proof. □

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# Glossary

$A$**:** generic set. xxii
$A \Subset B$**:** set $A$ whose compact closure is contained in $B$. xxii
$A^\star$**:** union of conncted components having positive lenght of a set $A \subset \mathbb{R}^2$. xxvii
$\mathsf{Adm}(\mu_i)$**:** sets of admissible transference plans. xxxii
$A_\gamma^\ell$**:** set of intersecting curves. 119
$A^\pm$**:** subset of $\partial\Omega$ where trajectories are exiting or entering, respectively. 114
$\fint_A f(x)\,\mu(dx)$**:** average integral on the sets $A$. xxv
$A(x)$**:** $x$ section of $A \subset X \times Y$. xxii

$B$**:** generic vector field in $\mathbb{R}^{d+1}$. xxiii
$\boldsymbol{b} = (b_i)_{i=1}^d$**:** vector. xxiii
$B_r^d(x)$**:** balls of radius $r$ centered at $x \in \mathbb{R}^d$. xxii
$\mathcal{O}(f)$**:** notation for constant of the order $f$. xxiii
$\mathscr{B}(X)$**:** $\sigma$-algebra of Borel sets of the space $X$. xxiv
$\mathrm{Fr}\, A$**:** frontier of a set A. xxii
$\partial\Omega$**:** frontier of a set in $\mathbb{R}^d$. xxii
$\mathrm{BV}(\Omega; \mathbb{R}^m)$**:** set of $\mathbb{R}^m$-valued functions of bounded variation in $\Omega \subset \mathbb{R}^d$. xxviii
$\mathrm{BD}(\Omega; \mathbb{R}^d)$**:** set of functions of bounded deformation in $\Omega \subset \mathbb{R}^d$. xxviii
$\boldsymbol{b}_t$**:** equivalent to $\boldsymbol{b}(t)$ for time dependent vector fields. xxiii
$\mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$**:** space of $\mathbb{R}^m$-valued locally BV functions in $\mathbb{R}^d$. xxviii

$C$**:** generic constant. xxiii
$C_d$**:** dimensional constant. xxiii
$C(X,Y)$**:** space of continuous functions over $X$. xxiii
$\mathbb{1}_A$**:** characteristic function of the set $A$. xxiii
$C^k(\mathbb{R}^d)$**:** space of functions on $\mathbb{R}^d$ with continuous derivatives up to order $k$. xxiii
$\mathrm{clos}(A, B)$**:** relative closure of the set $A$ in $B$. xxii
$\mathrm{clos}\, A$**:** closure of the set $A$. xxii
$C_c^\infty(\Omega)$**:** compactly supported smooth functions defined in the open set $\Omega \subset \mathbb{R}^d$. xxiii
$\mathrm{Conn}(A)$**:** set of conncted components of a set $A \subset \mathbb{R}^2$. xxvii
$\mathrm{Conn}^\star(A)$**:** set of conncted components having positive lenght of a set $A \subset \mathbb{R}^2$. xxvii
$*$**:** convolution in $\mathbb{R}^d$. xxiii
$\mathrm{Cyl}_{t,x}^{r,L}$**:** $\rho(1, \boldsymbol{b})$-proper cylinder. 97

$D_{\boldsymbol{e}} f$**:** directional derivative of $f$ along $\mathbf{e}$. xxiii
$D^{\mathrm{a.c.}}\boldsymbol{b}$**:** absolutely continuous part of $D\boldsymbol{b}$. xxix
$D^{\mathrm{c}}\boldsymbol{b}$**:** Cantor part of $D^{\mathrm{s}}\boldsymbol{b}$. xxix

$\delta_x$**:** Dirac mass at $x$. xxiv
$\delta_1$**:** maximal shrinking coefficient of an approximate cylinder of flow. 147
$Df$**:** differential of the function $f$. xxiii
$\mu = \int_Y \mu_y \, \nu(dy)$**:** disintegration of $\mu$ w.r.t. the measure $\nu$ and the function $f\colon X \to Y$ with $|f_\sharp\mu| \ll \nu$. xxvi
$\mathrm{dist}(x, E)$**:** distance of the point $x \in X$ from the set $E \subset X$ in a metric space $X$. xxii
$\mathscr{D}'(\Omega)$**:** space of distributions over the open set $\Omega \subset \mathbb{R}^d$. xxiii
$\operatorname{div} \boldsymbol{b}$**:** divergence of the vector field $\mathbf{b}$. xxiii
$D^{\mathrm{j}}\boldsymbol{b}$**:** jump part of $D^{\mathrm{s}}\boldsymbol{b}$. xxix
$\mathcal{D}(f)$**:** domain of the function $f$. xxiii
$D^{\mathrm{s}}\boldsymbol{b}$**:** singular part of $D\boldsymbol{b}$. xxix

$\boldsymbol{e}$**:** unit vector. xxiii
$E_h$**:** level set $H^{-1}(h)$ of a (Lipschitz) function $H\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. xxvii
$E\boldsymbol{b}$**:** symmetric part of the derivative $D\boldsymbol{b}$. xxviii
$E_{\mathfrak{a}}$**:** equivalence classes of $\sim$. 134
$E^\ell_\gamma$**:** set of curves not contained in $\operatorname{supp} \phi^\ell_\gamma$. 118
$\bar{\ell}_1$**:** starting shape of the approximate cylinder of flow in the BV case. 147
$\ell^\pm_{1,\gamma}$**:** evolution of the $\mathbf{e}_1$-boundary of the approximate cylinder. 147
$\eta$**:** Lagrangian representation. 32
$\eta^{\mathrm{cr}}$**:** restriction fo $\eta$ to $\Gamma^{\mathrm{Cr}}$. 124
$\eta^i_\Omega$**:** push forward of $\eta$ by $\mathtt{R}^i_\Omega$. 106
$\eta^\Xi$**:** restriction of $\eta$ to $\Xi$. 126

$\phi^{\delta,\pm}$**:** inner/outer distance functions from a set. 94
$\Phi^\ell_{\mathrm{enter}}(\gamma)$**:** functional computing intersecting curves across $\phi^\ell_\gamma$. 119
$\Phi^\ell_{\mathrm{exit}}(\gamma)$**:** functional computing the curves exiting the cylinder $\phi^\ell_\gamma$. 118
$\phi^\ell_\gamma$**:** approximate cylinder of flow. 116
$\mathfrak{f}^{\mathrm{in}}(\Omega)$**:** untangling functional for $\eta^{\mathrm{in}}$. 128
$\mathfrak{f}^{\mathrm{out}}(\Omega)$**:** untangling functional for $\eta^{\mathrm{out}}$. 128
$\mathfrak{A}$**:** suitable set of indexes. 134
$f \llcorner_A$**:** restriction of the function $f$ to the set $A$. xxiv
$f^r_x$**:** rescaled $f$ about $x \in \mathbb{R}^d$. xxv
$f_\sharp\mu$**:** push-forward of the measure $\mu$ through $f$. xxv
$f(\bar{x}\pm)$**:** right left limit of a 1d function at $\bar{x}$. xxviii

$\gamma$**:** curve define in an interval of time. 32
$\gamma \sim \gamma'$**:** equivalent relation among untangled trajectories. 134
$g \circ f$**:** composition of two functions. xxiii
$\operatorname{Graph} f$**:** graph of the function $f$. xxiii
$\operatorname{Graph} \gamma$**:** Graph of the a.c. curve $\gamma$ in the closed interval of definition. 33

$H\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$**:** Lipschitz function $H\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}$. xxvii
$\widehat{\mathtt{f}}$**:** quotient map for $\{E_{\mathfrak{a}}\}_{\mathfrak{a}}$. 134

$\mathscr{H}^{d-1}$**:** $(d-1)$-dimensional Hausdorff measure. xxiii

$I_{2,\gamma}$**:** flow across $L_{2,\gamma}$. 148
$\mathbb{I}$**:** identity function. xxiii
$I_\gamma = (t_\gamma^-, t_\gamma^+)$**:** interval of definition of the curve $\gamma$. 32
$I_{1,\gamma}^-$**:** flow across $L_{1,\gamma}^-$. 150
int $A$**:** interior of the set $A$. xxii
$\int_{\mathbb{R}^d} f(x)\,dx$**:** integral of a Borel function $f$ w.r.t. $\mathscr{L}^d$. xxiv
$\int_X f(x)\,\mu(dx)$**:** integral of a Borel function $f$ w.r.t. $\mu$. xxiv
$I_{1,\gamma}^+$**:** flow across $L_{1,\gamma}^+$. 148

$J$**:** jump set of a $BV$ function. xxix

$K_{\bar r}^{\varepsilon,\varepsilon'} \subset K^\varepsilon$**:** compact subset of $\partial\Omega$ defined in Lemma 6.13. 100
$K^n$**:** projection of $\mathcal{K}^n$. 131
$K_{\delta_c,\bar r}^\tau$**:** compact set with suitable local covering. 146
$K^{\tau,\pm}$**:** compact sets where the untangling functionals are controlled. 130, 132

$L$**:** scale constant. xxiii
$L^p(X,\mu)$**:** space of functions whose modulus is $\mu$-integrable at the $p$-th power. xxiv
$\bar L_2$**:** lateral boundary of $\bar Q$ with normal $\mathbf{e}_1^\perp$. 151
$L_{2,\gamma}$**:** lateral boundary of $Q_{\ell_{1,\gamma}^\pm,\ell}$ with normal $\mathbf{e}_1^\perp$. 148
$\langle f,\psi\rangle$**:** distribution $f$ evaluated on $\psi$. xxiii
$\mathscr{L}^d$**:** Lebesgue measure in $\mathbb{R}^d$. xxiii
$L_{1,\gamma}^\pm$**:** lateral boundary of $Q_{\ell_{1,\gamma}^\pm,\ell}$ given by the graph of $\ell_{1,\gamma}^\pm$. 148

$\mathscr{M}(X)$**:** set of Radon measures on $X$. xxv
$\mathsf{m}$**:** direction of the variation in the rank-one property. xxix
$\mathcal{K}^n$**:** compact subset of $\Gamma$ of trajectories with existence interval $\geq 2^{1-n}$. 131
$\mathscr{M}_b(X)$**:** set of finite Radon measures on $X$. xxv
$\bar{\mathsf{m}}$**:** direction of the variation in the rank-one property at the point $(\bar t,\bar x)$. 146
$\mathscr{M}^+(X)$**:** set of positive Radon measures on $X$. xxv
$\mathtt{M}$**:** deformation factor. 116, 123
$\mu^\beta$**:** measure $\mathrm{div}(\beta(\rho)(1,\boldsymbol{b}))$. 138
$\mu_{\mathfrak{a}}^\beta$**:** disitegration of the measure $\mathrm{div}(\beta(\rho)(1,\boldsymbol{b}))$. 138
$|\mu|$**:** total variation measure of $\mu$. xxiv
$\mu_x^r$**:** rescaled $\mu$ about $x\in\mathbb{R}^d$. xxv
$\mu\llcorner_A$**:** restriction of the measure $\mu$ to the set $A$. xxiv
$M(x)$**:** matrix derivative of the absolutely continuous part of a BV vector field. xxix

$N$**:** negligible set w.r.t. some measure. xxiv
$\mathsf{n}$**:** normal vector in the rank-one property. xxix
$\bar{\mathsf{n}}$**:** normal vector to the rank-one property at the point $(\bar t,\bar x)$. 146
$\|\cdot\|$**:** norm in a generic Banach space. xxii
$|\cdot|$**:** norm in $\mathbb{R}^d$. xxii

**$\nu^{\text{a.c.}}$:** absolutely continuous part of $\nu$. xxvi
**$\nu \ll \mu$:** $\nu$ is absolute continuous w.r.t. $\mu$. xxiv
**$\nu^{\perp}$:** orthogonal component of $\nu$ w.r.t. to another given measure. xxv

**$\Omega$:** generic open set. xxii
**$\omega_d$:** volume of the unit ball in $\mathbb{R}^d$. xxiv
**$\Omega^{\varepsilon}$:** perturbation of a proper set constructed in Theorem 6.15. 101
**$\mu \perp \nu$:** orthogonal measures. xxiv

**$\partial\Omega_1^{\varepsilon}$:** subset of $\partial\Omega$ defined in Theorem 6.15. 101
**$\partial\Omega_2^{\varepsilon}$:** subset of $\partial\Omega$ defined in Theorem 6.15. 102
**$\partial^{\star}F$:** reduced boundary of the set of finite perimeter $F$. xxx
**$\partial_{x_i} f$:** spatial partial derivative along the $i$-th direction. xxiii
**$\partial_t f_t$:** time partial derivative. xxiii
**$\pi$:** transference plan. xxxii
**$\mathtt{p}_X$:** projection on the space $X$. xxii

**$\bar{Q}$:** base of the cylinder $Q_{\ell^{\pm}_{1,\gamma},\ell}$. 148
**$Q^{\ell}_{\gamma}$:** cylinders of approximate flow. 123
**$Q_{\ell^{-}_{1,\gamma},\ell^{+}_{1,\gamma},\ell}$:** approximate cylnder with shape determined by $\ell^{\pm}_{1,\gamma}, \ell$. 147

**$\frac{d\nu}{d\mu}$:** Radon-Nikodym derivative of $\nu$ w.r.t. $\mu \geq 0$. xxvi
**$\mathcal{R}(f)$:** range of the function $f$. xxiii
**$\mathbb{R}^d$:** $d$-dimensional real vector space. xxii
**$\mathrm{Fr}(A, B)$:** relative frontier of $A$ in $B$. xxii
**$\mathrm{int}(A, B)$:** relative interior of the set $A$ in $B$. xxii
**$\rho^i_{\Omega}$:** evaluation of the measure $\eta^i_{\Omega}$. 106
**$\mathring{W}$:** trajectories with good intersetion properties in the open graph. 131
**$\mathtt{R}^i_{\Omega}$:** $i$-th restriction operator. 106
**$\rho^{\mathrm{cr}}$:** $((t, x)$-evaluation of $\eta^{\mathrm{Cr}}$. 124
**$\mathtt{R}_{\Omega}$:** restriction operator. 106
**$(\mathtt{R}_{\Omega})_{\sharp}\eta^{\mathrm{out}}$:** restrition of $(\mathtt{R}_{\Omega})_{\sharp}\eta$ to the exiting trajectories. 128

**$S_1$:** subset of $\partial(\Omega^{\varepsilon} \setminus \Omega)$ defined in Theorem 6.15. 103
**$S_2$:** partition of the set $\partial(\Omega^{\varepsilon} \setminus \Omega)$, Theorem 6.15. 103
**$S_3^-$:** partition of the set $\partial(\Omega^{\varepsilon} \setminus \Omega)$, Theorem 6.15. 103
**$S_3^+$:** partition of the set $\partial(\Omega^{\varepsilon} \setminus \Omega)$, Theorem 6.15. 103
**$S_4$:** partition of the set $\partial(\Omega^{\varepsilon} \setminus \Omega)$, Theorem 6.15. 103
**$\mathbb{S}^d$:** unit sphere of dimension $d$. xxii
**$\sigma(f(t))$:** evaluation of the function $f$ w.r.t. the measure $\rho(t)\mathscr{L}^d$. 117
**$o(f)$:** notation for constant infinitesimal w.r.t. $f$. xxiii
**$\mathcal{S}$:** sets of curves with the same initial point. 126
**$\mathrm{supp}\, f$:** support of a function $f$. xxiii
**$\mathrm{supp}\, \mu$:** support of a measure $\mu$. xxiv

# Acknowledgements

It is a pleasure to deeply thank my advisor, prof. Stefano Bianchini, for his wise guidance, for his generosity of ideas and for his enthusiasm: thank you for being for me, for the last four years, much more than a simple Master and PhD advisor and thanks to your family for your kind hospitality.

Studying at SISSA, I have had the good fortune of meeting, talking to and attending the courses of several professors, which I warmly acknowledge. A special mention is for Guido De Philippis for many inspiring discussions and for being always available and willing to discuss about Mathematics.

I thank prof. Gianluca Crippa, for the interest with whom he followed my PhD studies and for being, through his book, his papers and several enlightening discussions, an excellent reference on the research topic this thesis deals with. I am grateful to Simone Di Marino for stimulating discussions we had in Siena, Lyon and Paris.

I am deeply indebted with my dear friend Nikolay A. Gusev: thanks for being such a valuable collaborator and thanks to his family for the warm hospitality. Большое спасибо, Николай, Анатолий Николаевич и Мила! I am also grateful to prof. Paolo Caldiroli, to prof. Anna Capietto and to prof. Luca Lussardi for their experienced mentorship and for their goodwill towards me.

My utmost gratitude goes to my friends and mates met here at SISSA, among which I cannot forget Stefano Modena, Enrico Pasqualetto and Danka Lučic, Ivan Beschastnyi, Gabriele Bogo, Emanuele Tasso, Maicol Caponi and Flavia Santarcangelo. A special thought is for Giordano Cotti, for being always present since the very beginning of our adventure at SISSA five years ago. I believe I cannot find appropriate words to thank my friend and colleague Elio Marconi: thank you for your strong support and for sharing with me so many adventures in many different places around the world. These years would not have been the same without you.

I warmly thank M° Luca Delle Donne, for his friendship, which is so dear to me, and for the serenity he has shared with me for the latest months: thanks for the music and for the enjoyable discussions about literature, philosophy and art.

Finally, without the help, the constant support and the strong love of my family, this thesis would simply not have seen the light. Thanks to my mother Anna, to my father Sergio, to my brother Alberto for their help and also to D. for being such an inspiring, funny and lovable friend. Thanks to Marco and Daniela, for always making me feel beloved and at home; thanks to all the members of their family, in particular to Valentina, Elio, Adriano, Gino and Giovanna, for their true love which I appreciate so much.

And, of course, thanks to you, Giulia: I often think to how much you have helped me, how much you have encouraged me and I honestly wonder how you managed to do that. Thanks for letting our dream come true and for nurturing it every single day. Along with my deepest gratitude and love, this thesis is for you.

Trieste, September 2017