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On the body of supermanifolds

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The problem of constructing the body of a G^∞ manifold is considered. It is shown that any such manifold is foliated, and the body is defined to be the space of the leaves of this foliation. Under certain regularity conditions on the foliation, the body is a smooth finite-dimensional real manifold.

I. INTRODUCTION

In supersymmetric field theories and supergravity one extends space-time to a "superspace," where four anticommuting coordinates appear, as well as the usual commuting ones. Superspace was introduced as a somewhat heuristic tool, which proved to be effective in handling very complex field theories, where one deals with commuting (bosonic) as well as anticommuting (fermionic) fields in supersymmetry.

Setting up such theories in a proper geometric framework was a bit of a problem, because one was forced to work either on a space (like superspace) where no proper differential calculus was established, or on a "supermanifold" (like those of Konstant and Batchelor) where all the fields are commuting (see, e.g., Ref. 1). The definition by Rogers[2] of G^∞ manifolds seems able to bypass both these shortcomings in physical application, because these are actually Banach manifolds, and the natural fields on them are Grassmann valued. So, anticommuting variables and fields can be treated on the same ground as the commuting ones.

After the introduction of G^∞ manifolds, some work has been devoted to the study of their relations with ordinary real differentiable manifolds. To understand these relations is crucial in view of possible applications to supersymmetric field theories and supergravity. Indeed the physical meaning of such theories can be understood only in terms of representation of the Poincaré group, that is, after the theory has been suitably reduced on ordinary space-time. It is therefore important to inquire to what extent G^∞ manifolds provide extension of space-time. Also from the purely mathematical point of view, it seems natural to inquire about the relations between the category of G^∞ manifolds and that of C^∞ manifolds.

This question was already considered by Rogers,[2] by introducing the notion of the "body" of a G^∞ manifold. This definition was stated in terms of local coordinates. After the work by Jadczyk and Pilch,[3] Percacci and Marchetti[4] and Hoyos *et al.*[5] came back to the problem, showing that the local definition by Rogers did not extend globally, unless the G^∞ structure was quite peculiar.

In this paper we came back to the problem of defining the body of a G^∞ manifold. Our approach is independent of charts, and it is based on the fact that any G^∞ manifold is foliated (as shown in Sec. II). Then the body arises as the quotient space of the G^∞ manifold by this foliation, which always exists as a topological space both in the finite- and the infinite-dimensional case. However, as is usual when taking the quotient by a foliation, the body does not admit a manifold structure, even at a topological level. A simple example of this phenomenon, relevant for the present case, is given in Sec. III. Finally we show that, under suitable regularity conditions on the foliation, a G^∞ manifold admits a smooth differentiable structure on its body. Examples of regular manifolds are the ρ manifolds of Ref. 5.

To avoid a long list of notation and definitions, we adopt the notation of Yadczyk and Pilch.[3] In particular, Q will usually denote a Banach–Grassmann algebra, and it is infinite dimensional over the reals. When we speak of finite-dimensional G^∞ manifolds, we mean a manifold which is finite dimensional over the reals; so in this cases Q will stand for a Grassmann algebra with L odd generators (i.e., we identify Q with B_L, according to the notations of Ref. 2).

II. FOLIATION AND EQUIVALENCE RELATIONS ON A G^∞ MANIFOLD

In this section we show that any G^∞ manifold $\tilde{X}$ is foliated. The basic fact is that one can define an involutive subbundle Σ of the tangent bundle $T\tilde{X}$, by considering tangent vectors whose components in any chart have vanishing real parts.

To be definite, let $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)$ be a G^∞ atlas for $\tilde{X}$, with coordinate maps $\tilde{\varphi}_\alpha : \tilde{U}_\alpha \to \tilde{A}_\alpha \subset Q^{m,n}$, and consider the map ϵ: $Q^{m,n} \to R^m$ gotten by taking the real parts. The map f_α: $\tilde{U}_\alpha \to \epsilon(Q_\alpha) \subset R^m$, given by $f_\alpha = \epsilon \cdot \tilde{\varphi}_\alpha$ is clearly a submersion. Its differential df_α: $T\tilde{U}_\alpha \to R^m$ has a closed kernel. Then we set $\Sigma_\alpha = \ker df_\alpha$.

Clearly this local definition extends to a global one, because a tangent vector v at p belongs to $\Sigma_\alpha|_p$ if and only if its components in the chart $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)$ have vanishing real parts, a property which is clearly independent of charts. We call such a vector of type σ. More computatively if v^A are the

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components of v in a chart $(\tilde{U}, \tilde{\varphi})$ around p, its components $v^{A'}$ in another chart $(\tilde{U}', \tilde{\varphi}')$ around p are given by $v^{A'} = (d\tilde{\psi})^{A'}_{A} v^{A}$, where $\tilde{\psi} = \tilde{\varphi}' \circ \tilde{\varphi}^{-1}$ is the coordinate transformation between the two charts. Since $\tilde{\psi}$ is G^{∞}, one has that $\epsilon(v^{A'}) = \epsilon(d\tilde{\psi})^{A'}_{A} \epsilon(v^{A})$. Now $\epsilon(d\tilde{\psi})^{A'}_{A}$ is invertible, and hence, being of type σ, is independent of charts. Accordingly two local distributions Σ_α and Σ_β agree at any $p \in U_\alpha \cap U_\beta$, i.e., $\Sigma_\alpha|_p = \Sigma_\beta|_p$.

From the construction above it is clear that Σ is integrable, its leaves being locally given by the equation $f_\alpha = \epsilon \cdot \tilde{\varphi}_\alpha = \text{const}$. Then we give the following definition.

Definition: Two points $p,q \in \tilde{X}$ are equivalent $(p \sim q)$ if they belong to the same connected maximal integral manifold (leaf) of Σ.

It is apparent that this is an equivalence relation independent of charts, which refines the definitions previously attempted in the literature.[2,4] To make contact with these, we notice that if p,q belong to the same chart $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)$, then $f_\alpha(p) = f_\alpha(q)$ (i.e., having the same real coordinates) implies that $p \sim q$. The converse is not true in general, because the intersection of a leaf of Σ with $\tilde{U}_\alpha$ may be not connected. Hence two points in $\tilde{U}_\alpha$, belonging to different connected components, may very well be equivalent, without having the same real coordinates.

One may argue that the present equivalence relation is in some sense unnatural, in that it seems better to start with the usual local relation[2,4,5] defined as follows. Whenever $p,q \in \tilde{U}_\alpha$, one sets $p \underset{\text{loc}}{\sim} q$ if and only if $f_\alpha(p) = f_\alpha(q)$. The trouble with this local relation is that (i) it is not independent of charts, and (ii) its global extension is not trivial. As to (i), it is sufficient to notice that if p,q belong to two disconnected components of the intersection of two charts $U_\alpha \cap U_\beta$, it may happen that $f_\alpha(p) = f_\alpha(q)$ but $f_\beta(p) \neq f_\beta(q)$. If $Q = B_L$ is finite dimensional, one can overcome this difficulty by taking a suitable refinement of the G^{∞} atlas of $\tilde{X}$, as shown in Appendix A. Another possible way out is to assume that $\tilde{X}$ has a special G^{∞} structure, i.e., that the images $\varphi_\alpha(U_\alpha \cap U_\beta) \subset Q^{m,n}$ are ϵ connected,[4] or that the manifold is as in Ref. 5.

Even when the relation $\underset{\text{loc}}{\sim}$ is suitably treated to yield independence of charts, one faces the fact that it is reflexive and symmetric, but fails, in general, to be transitive. To get, in any case, an equivalence relation, one follows the standard prescription of considering the transitive closure of the subset $R_{\text{loc}} = \{p,q / p \underset{\text{loc}}{\sim} q\}$, i.e., the minimal $R \subset \tilde{X} \times \tilde{X}$ which contains R_{loc} and is an equivalence relation $\underset{R}{\sim}$. In other words, one has that $p \underset{R}{\sim} q$ if and only if there exists a finite sequence $p_i, q_i (1 < i < N)$ of points such that $p_1 = p$, $q_N = q$, and $p_i \underset{\text{loc}}{\sim} q_i$. In this form this equivalence has been introduced in Ref. 5.

Notice that the existence of $\underset{R}{\sim}$ depends crucially on being independent of charts. In this case, we can show that the two equivalence $\underset{R}{\sim}$ and $\sim$ are actually the same.

Proposition: Two points $p,q \in \tilde{X}$ are R equivalent if and only if they belong to the same leaf of Σ.

Proof: Any curve $p(t) \subset \tilde{X}$ of R-equivalent points has tangent vectors of type σ. Hence $\forall t, p(t)$ belongs to the same leaf of Σ. Conversely if p,q belong to the same leaf of Σ, then there exists a compact curve $p(t)$ of type σ connecting them. Then we can choose points $p_i = p(t_i)$ and $q_i = p(t'_i)$ such that $p_i \underset{\text{loc}}{\sim} q_i$ and $p_0 = p$, $q_N = q$.

Remark: As already mentioned, the proof above requires that $\underset{R}{\sim}$ be independent of charts. If not, we stress that $\underset{R}{\sim}$ cannot be consistently defined, while our equivalence relation $\sim$ exists in any case.

III. THE BODY OF A G^{∞} MANIFOLD

Definition: The body X of a G^{∞} manifold $\tilde{X}$ is the space of the leaves of Σ on $\tilde{X}$.

When R equivalence exists on $\tilde{X}$, one can as well say that $X = \tilde{X} / \underset{R}{\sim}$ is the set of R-equivalence classes of points in $\tilde{X}$.

We give $\tilde{X}/\sim$ the quotient topology, thus yielding that the canonical projection $\pi: \tilde{X} \to X$ is continuous and open. The question is now if X can be given a manifold structure. As is well known, the answer to this question for a generic foliated manifold is negative. In any case, to build a manifold structure on the space of leaves, one has at least to assume that the foliation was regular (see, e.g., Ref. 6).

Thanks to the properties of G^{∞} manifolds, we can say a bit more in the present case. First notice that "concrete" G^{∞} manifolds are built gluing together charts, and giving them the topology which makes the coordinate maps homeomorphisms. Now, around any $p \in \tilde{X}$ one can define a cubic and flat coordinate patch $(\tilde{U}_p, \tilde{\varphi}_p)$ centered at p as follows.

Let $(\tilde{U}, \tilde{\varphi})$ be a chart containing p; we set $\tilde{\varphi}_p = \tilde{\varphi} - \tilde{\varphi}(p)$ so that $\tilde{\varphi}_p(p) = 0 \in Q^{m,n}$. If $(x^1,...,x^m) = \epsilon \cdot \tilde{\varphi}_p(q) \in R^m$ denote the real coordinates of $q \in \tilde{U}$, we consider a cube $c \subset R^m$, of width $2a$, given by $|x^i| < a$. Then let $\tilde{U}_p = \tilde{\varphi}_p^{-1}[\epsilon^{-1}(c) \cap \tilde{\varphi}_p(\tilde{U})]$. From Sec. II it follows that the leaves of Σ in $\tilde{U}_p$ are parametrized by the real coordinates $(x^1,...,x^m) = \tilde{\varphi}_p(q) \in \tilde{U}_p$, that is, the coordinate patch $(\tilde{U}_p, \tilde{\varphi}_p)$ is "flat."

The trouble here is that the correspondence between leaves of Σ in $\tilde{U}_p$ and the real coordinates $(x^1,...,x^m)$ is not a bijection, that is in general one has no maps φ_p making the following diagram commutative:

$$\begin{array}{ccc} \tilde{U}_p & \xrightarrow{\ \tilde{\varphi}_p\ } & \tilde{A} \subset Q^{m,n} \\ \pi \downarrow & & \downarrow \epsilon \\ U_p & \underset{\varphi_p}{- - \to} & A \subset R^m . \end{array} \qquad (3.1)$$

If, on the contrary, for any p one has a patch $(\tilde{U}_p, \tilde{\varphi}_p)$ and a map φ_p such that the diagram above commutes, we say that the Σ foliation is regular. A G^{∞} manifold whose Σ foliation is regular will be called regular itself.

To see that regularity is missing in general consider the following example.

Example: We construct a torus over $Q_1^{1,1}$. If $x + \theta y \in Q_1$

with $\theta^2 = 0$, then we have coordinates $(x,y\theta) \in Q_1^{1,1}$. Consider the intersection of the two strips

(I) $\alpha(x-1) \leqslant y \leqslant \alpha(x+1)$, $(\alpha \in R^+)$

(II) $-\alpha(x+1) \leqslant y \leqslant -\alpha(x-1)$

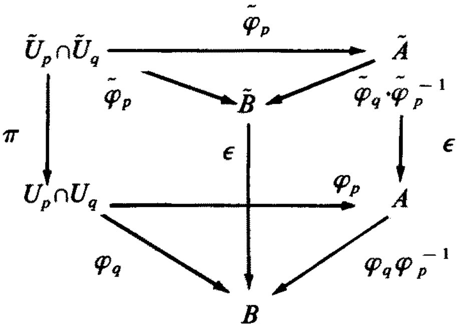
and identify the boundaries of (I) by $(a+b\theta) \rightarrow (a+1,(b+\alpha)\theta)$ and of (II) by $(a,b\theta) \rightarrow (a-1,(b+\alpha)\theta)$. It is clear that this operation is G^∞, and that the resulting torus $\tilde{X}$ has a G^∞ structure. Now if α is not rational, the leaves of Σ are dense on $\tilde{X}$, and, therefore, the Σ foliation is not regular. From this example we see that regularity is by no means a local property. In other words the existence of a map φ in diagram (3.1) crucially depends on the global behavior of the leaves of Σ. Since φ_p is lacking, one has no coordinates on $\tilde{X}/\sim$, that is, the body of $\tilde{X}$ is not even a topological manifold.

Although regularity will be difficult to check in a generic case, one can give sufficient conditions. It is easy to show that the ρ supermanifolds of Ref. 5 have regular foliation. Conversely if the foliation was regular, than the flat coordinate charts are ϵ connected and, the diagram being commutative, it yields a ρ supermanifold structure on $\tilde{X}$. Examples of regular supermanifolds are the G^∞ extension of any ordinary C^∞ space-time constructed by Bonora, Pasti, and Tonin.[7]

Whenever $\tilde{X}$ is regular, its body is obviously a topological manifold. We can also prove the following theorem.

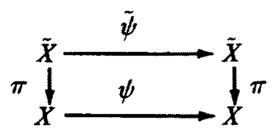
Theorem: Let $\tilde{X}$ be a regular G^∞ manifold. Then its body X is a C^∞ manifold.

Proof: Since $\tilde{X}$ is regular, one has an atlas $\{(\tilde{U}_p,\tilde{\varphi}_p)\}$ and bijections φ_p making the diagram (3.1) commute. Then one has bijections $\varphi_q\varphi_p^{-1}:A\rightarrow B$, $A,B\subset\mathbb{R}^n$ such that the diagram



commutes. Now the transition functions $\varphi_p\cdot\varphi_q^{-1}$ are clearly local homeomorphisms. They are also C^∞ diffeomorphisms. Indeed we can represent them by $\varphi_p\cdot\varphi_q^{-1} = \epsilon\cdot\tilde{\varphi}_p\cdot\tilde{\varphi}_q^{-1}\cdot\sigma$, where σ: $A\rightarrow\tilde{A}$ is a C^∞ section of $\tilde{A}\rightarrow\epsilon(\tilde{A}) = A$. Then $\varphi_p\cdot\varphi_q^{-1}$ arises as a composition of C^∞ maps, and hence it is C^∞. The same applies to the inverse $\varphi_q\cdot\varphi_p^{-1}$. Hence X is a C^∞ manifold.

Next, by similar arguments, one proves that if $\tilde{\psi}:\tilde{X}\rightarrow\tilde{X}$ is a G^∞ diffeomorphism, then there exists a unique $\psi:X\rightarrow X$ which is a C^∞ diffeomorphism and such that the diagram



is commutative. So the body $X = \pi(\tilde{X})$ is unique up to diffeomorphisms. More precisely one may say that π is a functor from the category of regular G^∞ manifolds with G^∞ diffeomorphisms to the category of C^∞ manifolds with C^∞ diffeomorphisms.

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We heard also from the referee that Boyer and Gitler[8] discussed the present problem very much on the same line as ours.

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APPENDIX A: THE EXISTENCE OF A "GOOD" SUPERATLAS

In this appendix we show the existence, in the finite-dimensional case, of a superatlas in which the $\underset{R}{\sim}$ relation is chart independent.

The existence of a "good" atlas is a consequence of a well-known result in the theory of ordinary differentiable manifold: Let X be a paracompact differentiable manifold of dim n. Then every open covering $\{V_\alpha\}$ of X has an open refinement $\{V_i\}$ such that (i) each V_i has compact closure; (ii) $\{V_i\}$ is locally finite; and (iii) any nonempty finite intersection of the V_i's is diffeomorphic to an open ball of R^n. Now, if we have a G^∞ supermanifold $\tilde{X}$ modeled on $B_L^{m,n}$ with a given superatlas $\{\tilde{V}_\alpha,\tilde{\varphi}_\alpha\}$, we can consider it as a real C^∞ manifold of dim $N = 2^{L-1}(m+n)$. In fact, every G^∞ manifold is a Banach manifold C^∞, and every G^∞ map between supermanifolds is also a C^∞ map, and hence there exists a forgetful functor

F:G^∞ supermanifolds$\rightarrow C^\infty$ manifolds.

The identification of $B_L^{m,n}$ with $R^{2^{L-1}(m+n)}$ is as follows: We take a basis of B_L, $\{\beta_\mu\}$, and set $Z^A = Z^{A\mu}\beta_\mu$. We then define a map f:$\tilde{X}\rightarrow F\tilde{X}$, which, on the underlying topological spaces, is the identity and on suitable atlases $\{\tilde{V}_\alpha,\tilde{\varphi}_\alpha\}$ of $\tilde{X}$ and $\{\tilde{V}_\alpha,\psi_\alpha\}$ of $F\tilde{X}$ has the representation

$$\tilde{\varphi}_\alpha\cdot f\cdot\psi_\alpha^{-1}(Z^A) = Z^{A\mu}.$$

Now we can apply the proposition above to the manifold $F\tilde{X}$ getting the "good" covering $\{\tilde{V}_i\}$. We can then transfer the sets $\{\tilde{V}_i\}$ on $\tilde{X}$ and we have "good" superatlas $\{\tilde{V}_i,\tilde{\varphi}_\alpha|_{V_i} = \tilde{\varphi}_i\}$, where α corresponds to a $\tilde{U}_\alpha$ of the original superatlas containing $\tilde{V}_i$. In fact, as $\tilde{V}_i\cap\tilde{V}_j$ is connected and the $\tilde{\varphi}_i$'s are homeomorphisms, $\tilde{\varphi}_i(\tilde{V}_i\cap\tilde{V}_j)$ and $\tilde{\varphi}_j(\tilde{V}_i\cap\tilde{V}_j)$ are connected open sets in $B_L^{m,n}$.

We recall now the following proposition (see Rogers[2]).

Proposition: Let U be open and connected in $B_L^{m,n}$ and let $f\in G^\infty(U)$. Then there exists a unique $f\in G^\infty(\epsilon^{-1}(\epsilon(U)))$ such that $f'|_U = f$. It follows from the properties of Taylor series and the fact that ϵ is an algebra homomorphism that if $x,y\in U$ (U open and connected in $B_L^{m,n}$) with $\epsilon(x) = \epsilon(y)$, then

$\epsilon f(x) = \epsilon f(y)$. We have then proved that the transition function $\tilde{\psi}_{ij}$ of the "good" atlas are "body preserving," i.e., if $x,y \in \tilde{\varphi}_j(V_i \cap V_j)$ are such that $\epsilon(x) = \epsilon(y)$ then $\epsilon\tilde{\psi}_{ij}(x) = \epsilon\tilde{\psi}_{ij}(y)$. So the relation $\sim_R$ is chart independent.

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