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Quasi-periodic oscillations for wave equations under periodic forcing

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Analisi funzionale. — *Quasi-periodic oscillations for wave equations under periodic forcing.* Nota di MASSIMILIANO BERTI e MICHELA PROCESI, presentata (*) dal Socio A. Ambrosetti.

ABSTRACT. — Existence of quasi-periodic solutions with two frequencies of completely resonant, periodically *forced*, nonlinear wave equations with periodic spatial boundary conditions is established. We consider both the cases the forcing frequency is (Case A) a rational number and (Case B) an irrational number.

KEY WORDS: Nonlinear wave equation; Quasi-periodic solutions; Variational methods; Lyapunov-Schmidt reduction; Infinite dimensional Hamiltonian systems.

RIASSUNTO. — *Oscillazioni quasi periodiche per equazioni delle onde con forzante periodica.* Si dimostra l'esistenza di soluzioni quasi periodiche con due frequenze per una classe di equazioni delle onde non lineari completamente risonanti aventi un termine forzante periodico. Consideriamo che la frequenza forzante sia un numero razionale (Caso A), sia irrazionale (Caso B).

1. INTRODUCTION

We present in this *Note* the results of [7] concerning existence of small amplitude quasi-periodic solutions for completely resonant forced nonlinear wave equations like

$$\begin{cases} v_{tt} - v_{xx} + f(\omega_1 t, v) = 0 \\ v(t,x) = v(t, x+2\pi) \end{cases} \tag{1.1}$$

where the nonlinear forcing term

$$f(\omega_1 t, v) = a(\omega_1 t)v^{2d-1} + O(v^{2d}), \qquad d > 1,\ d \in \mathbb{N}^+$$

is $2\pi/\omega_1$-periodic in time and the forcing frequency is:

- *A)* $\omega_1 \in \mathbb{Q}$
- *B)* $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$.

Existence of periodic solutions for completely resonant forced wave equations was first proved in the pioneering paper [10] (with Dirichlet boundary conditions) if the forcing frequency is a rational number ($\omega_1 = 1$ in [10]). This requires to solve an infinite dimensional bifurcation equation which lacks compactness property; see [4] and references therein for other results. If the forcing frequency is an irrational number existence of periodic solutions has been proved in [8]: here the bifurcation equation is trivial but a «small divisors problem» appears.

To prove existence of small amplitude quasi-periodic solutions for completely resonant PDE's like (1.1) one has to deal both with a small divisor problem and with an infinite dimensional bifurcation equation. A major point is to understand from which periodic solutions of the linearized equation

$$v_{tt} - v_{xx} = 0\,,$$

(*) Nella seduta del 22 aprile 2005.

the quasi-periodic solutions branch-off: indeed such linear equation possesses only 2π-periodic solutions of the form $q_+(t+x)+ q_-(t-x)$.

For completely resonant autonomous PDE's, existence of quasi-periodic solutions with 2-frequencies have been recently obtained in [9] for the specific nonlinearities $f = u^3 + O(u^5)$. Here the bifurcation equation is solved by ODE methods.

In [7] we prove existence of quasi-periodic solutions with two frequencies ω_1, ω_2 for the completely resonant forced equation (1.1) in both the two cases A) and B).

The more interesting case is $\omega_1 \in \mathbb{Q}$ (case A) when the forcing frequency ω_1 enters in resonance with the linear frequency 1. To find out from which solutions of the linearized equation quasi-periodic solutions of (1.1) branch-off, requires to solve an infinite dimensional bifurcation equation which can not be solved in general by ODE techniques (it is a system of integro-differential equations). However, exploiting the variational nature of equation (1.1) like in [5], the bifurcation problem can be reduced to finding critical points of a suitable action functional which, in this case, possesses the infinite dimensional linking geometry [3].

On the other hand, we avoid the inherent small divisor problem by restricting the parameters to uncountable zero-measure sets as, *e.g.*, in [9, 5].

1.1. *Main results.* We look for quasi-periodic solutions $v(t,x)$ of (1.1) of the form

$$\begin{cases} v(t,x) = u(\omega_1 t, \omega_2 t + x) \\ u(\varphi_1 + 2k_1\pi, \varphi_2 + 2k_2\pi) = u(\varphi_1, \varphi_2), \qquad \forall k_1, k_2 \in \mathbb{Z} \end{cases}$$

with frequencies

$$\omega = (\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$$

imposing the frequency $\omega_2 = 1+\varepsilon$ to be close to the linear frequency 1. Therefore we get

$$[\omega_1^2\partial^2_{\varphi_1} + (\omega_2^2 - 1)\partial^2_{\varphi_2} + 2\omega_1\omega_2\partial_{\varphi_1}\partial_{\varphi_2}]u(\varphi) + f(\varphi_1, u) = 0. \tag{1.2}$$

We assume that the forcing term $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$

$$f(\varphi_1, u) = a_{2d-1}(\varphi_1)u^{2d-1} + O(u^{2d}), \qquad d \in \mathbb{N}^+,\ d > 1$$

is analytic in u but has only finite regularity in φ_1. More precisely

- (H) $f(\varphi_1, u) := \sum_{k=2d-1}^{\infty} a_k(\varphi_1)u^k$, $d \in \mathbb{N}^+$, $d > 1$ and the coefficients $a_k(\varphi_1) \in H^1(\mathbb{T})$

verify, for some $r > 0$, $\sum_{k=2d-1}^{\infty} |a_k|_{H^1} r^k < \infty$. The function $f(\varphi_1, u)$ is not identically constant in φ_1.

We look for solutions u of (1.2) in the multiplicative Banach algebra

$$\mathcal{H}_{\sigma,s} := \left\{ u(\varphi) = \sum_{l \in \mathbb{Z}^2} \hat{u}_l e^{il\cdot\varphi} \ : \ \hat{u}_l^* = \hat{u}_{-l} \text{ and } |u|_{\sigma,s} := \sum_{l \in \mathbb{Z}^2} |\hat{u}_l| e^{|l_2|\sigma}[l_1]^s < +\infty \right\}$$

where $[l_1] := \max\{|l_1|, 1\}$ and $\sigma > 0$, $s \geq 0$.

In [7] we prove the following theorems.

THEOREM A. *Let* $\omega_1 = n/m \in \mathbb{Q}$. *Assume that* f *satisfies assumption* (H) *and* $a_{2d-1}(\varphi_1) \neq 0$, $\forall \varphi_1 \in \mathbb{T}$. *Let* $\mathcal{B}_\gamma$ *be the uncountable zero-measure Cantor set*

$$\mathcal{B}_\gamma := \left\{ \varepsilon \in (-\varepsilon_0, \varepsilon_0) : \quad |l_1 + \varepsilon l_2| > \frac{\gamma}{|l_2|}, \ \forall l_1, l_2 \in \mathbb{Z} \setminus \{0\} \right\}$$

where $0 < \gamma < 1/6$.

There exist constants $\overline{\sigma} > 0$, $\overline{s} > 2$, $\overline{\varepsilon} > 0$, $\overline{C} > 0$, *such that* $\forall \varepsilon \in \mathcal{B}_\gamma$, $|\varepsilon|\gamma^{-1} \leq \overline{\varepsilon}/m^2$, *there exists a classical solution* $u(\varepsilon, \varphi) \in \mathcal{H}_{\overline{\sigma}, \overline{s}}$ *of* (1.2) *with* $(\omega_1, \omega_2) = (n/m, 1 + \varepsilon)$ *satisfying*

$$\left| u(\varepsilon, \varphi) - |\varepsilon|^{\frac{1}{2(d-1)}} \overline{q}_\varepsilon(\varphi) \right|_{\overline{\sigma}, \overline{s}} \leq \overline{C} \frac{m^2 |\varepsilon|}{\gamma \omega_1^3} |\varepsilon|^{\frac{1}{2(d-1)}} \tag{1.3}$$

for an appropriate function $\overline{q}_\varepsilon \in \mathcal{H}_{\sigma,s} \setminus \{0\}$ *of the form* $\overline{q}_\varepsilon(\varphi) = \overline{q}_+(\varphi_2) + \overline{q}_-(2m\varphi_1 - n\varphi_2)$.

As a consequence, equation (1.1) *admits the quasi-periodic solution* $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ *with two frequencies* $(\omega_1, \omega_2) = (n/m, 1 + \varepsilon)$ *and the map* $t \to v(\varepsilon, t, \cdot) \in H^{\overline{\sigma}}(\mathbb{T})$ *has the form* (1)

$$\left| v(\varepsilon, t, x) - |\varepsilon|^{\frac{1}{2(d-1)}} [\overline{q}_+(x + (1+\varepsilon)t) + \overline{q}_-((1-\varepsilon)\, nt - nx)] \right|_{H^{\overline{\sigma}}(\mathbb{T})} = O\left(\frac{m^2}{\gamma \omega_1^3} |\varepsilon|^{\frac{2d-1}{2(d-1)}} \right).$$

Remark that the bifurcation of quasi-periodic solutions u occurs both for $\omega_2 > 1$ and $\omega_2 < 1$; actually both $\mathcal{B}_\gamma \cap (0, \varepsilon_0)$ and $\mathcal{B}_\gamma \cap (-\varepsilon_0, 0)$ are uncountable $\forall \varepsilon_0 > 0$.

At the first order the quasi-periodic solution $v(\varepsilon, t, x)$ of equation (1.1) is the superposition of two waves traveling in opposite directions (in general, both components q_+, q_- are non trivial).

The bifurcation of quasi-periodic solutions looks quite different if ω_1 is irrational.

THEOREM B. *Let* $\omega_1 \in \mathbb{R} \setminus \mathbb{Q}$. *Assume that* f *satisfies assumption* (H), $\int_0^{2\pi} a_{2d-1}(\varphi_1) d\varphi_1 \neq 0$ *and* $f(\varphi_1, u) \in H^s(\mathbb{T})$, $s \geq 1$, *for all* u.

Let $\mathcal{C}_\gamma \subset D \equiv (-\varepsilon_0, \varepsilon_0) \times (1, 2)$ *be the uncountable zero-measure Cantor set*

$$\mathcal{C}_\gamma := \left\{ \begin{array}{ll} (\varepsilon, \omega_1) \in D : & \omega_1 \notin \mathbb{Q}, \quad \dfrac{\omega_1}{\omega_2} \notin \mathbb{Q}, \quad |\omega_1 l_1 + \varepsilon l_2| > \dfrac{\gamma}{|l_1| + |l_2|}, \\ & |\omega_1 l_1 + (2+\varepsilon) l_2| > \dfrac{\gamma}{|l_1| + |l_2|}, \ \forall\, l_1 \in \mathbb{Z} \setminus \{0\} \end{array} \right\}. \tag{1.4}$$

Fix any $0 < \overline{s} < s - 1/2$. *There exist positive constants* $\overline{\varepsilon}$, $\overline{C}$, $\overline{\sigma} > 0$, *such that,* $\forall (\varepsilon, \omega_1) \in \mathcal{C}_\gamma$ *with* $|\varepsilon|\gamma^{-1} < \overline{\varepsilon}$ *and* $\varepsilon \int_0^{2\pi} a_{2d-1}(\varphi_1)\, d\varphi_1 > 0$, *there exists a nontrivial solution*

(1) We denote $H^\sigma(\mathbb{T}) := \{u(\varphi) = \sum_{l \in \mathbf{Z}} \hat{u}_l e^{il\varphi} \ : \ |u|_{H^\sigma(\mathbb{T})} := \sum_{l \in \mathbb{Z}} |\hat{u}_l| e^{\sigma |l|} < +\infty\}$.

$u(\varepsilon, \varphi) \in \mathcal{H}_{\bar\sigma, \bar s}$ of equation (1.2) *with* $(\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$ *satisfying*

$$\Big| u(\varepsilon, \varphi) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar q_\varepsilon(\varphi_2) \Big|_{\bar\sigma, \bar s} \le \overline{C} \frac{|\varepsilon|}{\gamma} |\varepsilon|^{\frac{1}{2(d-1)}} \tag{1.5}$$

for some function $\bar q_\varepsilon(\varphi_2) \in H^{\bar\sigma}(\mathbb{T}) \setminus \{0\}$.

As a consequence, equation (1.1) *admits the non-trivial quasi-periodic solution* $v(\varepsilon, t, x) := u(\varepsilon, \omega_1 t, x + \omega_2 t)$ *with two frequencies* $(\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$ *and the map* $t \to v(\varepsilon, t, \cdot) \in H^{\bar\sigma}(\mathbb{T})$ *has the form*

$$\Big| v(\varepsilon, t, x) - |\varepsilon|^{\frac{1}{2(d-1)}} \bar q_\varepsilon(x + (1+\varepsilon)t) \Big|_{H^{\bar\sigma}(\mathbb{T})} = O\Big(\gamma^{-1} |\varepsilon|^{\frac{2d-1}{2(d-1)}}\Big).$$

REMARK 1. Imposing in the definition of $\mathcal{C}_\gamma$ the condition $\omega_1/\omega_2 = \omega_1/(1+\varepsilon) \in \mathbb{Q}$ we obtain, by Theorem B, the existence of periodic solutions of equation (1.1). They are reminiscent, in this completely resonant context, of the Birkhoff-Lewis periodic orbits with large minimal period accumulating at the origin, see [2].

In Theorem B, existence of quasi-periodic solutions could follow by other hypotheses on f. However the hypothesis that the leading term in the nonlinearity f is an odd power of u is not of technical nature. Actually, the following non-existence result holds:

THEOREM C (Non-existence). *Let* $f(\varphi_1, u) = a(\varphi_1) u^D$ *with* D *even and* $\int_0^{2\pi} a(\varphi_1)\, d\varphi_1 \neq 0$. *$\forall R > 0$ there exists $\varepsilon_0 > 0$ such that $\forall \sigma \ge 0$, $\bar s > s - \frac{1}{2}$, $\forall (\varepsilon, \omega_1) \in \mathcal{C}_\gamma$ with $|\varepsilon| < \varepsilon_0$, equation* (1.2) *does not possess solutions $u \in \mathcal{H}_{\sigma, \bar s}$ in the ball $|u|_{\sigma, \bar s} \le R |\varepsilon|^{1/(D-1)}$.*

Theorem C also highlights that the existence of periodic solutions of nonlinear wave equations in the case of even powers, see [5], is due to the boundary effects imposed by the Dirichlet conditions.

2. SKETCH OF THE PROOF OF THEOREM A

We sketch now the proof of (the more difficult) Theorem A. To simplify notations we assume $\omega_1 = 1$, $\varepsilon > 0$ and $a_{2d-1}(\varphi_1) > 0$.

Instead of looking for solutions of equation (1.2) in a shrinking neighborhood of 0, it is a convenient devise to perform the rescaling

$$u \to \delta u \qquad \text{with} \qquad \delta := |\varepsilon|^{1/2(d-1)}$$

enhancing the relation between the amplitude δ and the frequency $\omega_2 = 1 + \varepsilon$. We obtain the equation

$$\mathcal{L}_\varepsilon u + \varepsilon f(\varphi_1, u, \delta) = 0 \tag{2.1}$$

where

$$\mathcal{L}_\varepsilon := \Big[\partial^2_{\varphi_1} + 2\partial_{\varphi_1}\partial_{\varphi_2}\Big] + \varepsilon\Big[(2+\varepsilon)\partial^2_{\varphi_2} + 2\partial_{\varphi_1}\partial_{\varphi_2}\Big] \equiv L_0 + \varepsilon L_1$$

and

$$f(\varphi_1, u, \delta) := \frac{f(\varphi_1, \delta u)}{\delta^{2(d-1)}} = \Big(a_{2d-1}(\varphi_1)u^{2d-1} + \delta a_{2d}(\varphi_1)u^{2d} + \dots\Big).$$

Equation (2.1) is the Euler-Lagrange equation of the *Lagrangian action functional* $\Psi_\varepsilon \in C^1(\mathcal{H}_{\sigma,s}, \mathbb{R})$ defined by

$$\Psi_\varepsilon(u) := \int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1}u)^2 + (\partial_{\varphi_1}u)(\partial_{\varphi_2}u) + \frac{\varepsilon(2+\varepsilon)}{2}(\partial_{\varphi_2}u)^2 + \varepsilon(\partial_{\varphi_1}u)(\partial_{\varphi_2}u) - \varepsilon F(\varphi_1, u, \delta)$$

$$\equiv \Psi_0(u) + \varepsilon\Gamma(u, \delta)$$

where $F(\varphi_1, u, \delta) := \int_0^u f(\varphi_1, \xi, \delta)d\xi$ and

$$\Psi_0(u) := \int_{\mathbb{T}^2} \frac{1}{2}(\partial_{\varphi_1}u)^2 + (\partial_{\varphi_1}u)(\partial_{\varphi_2}u)$$

$$\Gamma(u, \delta) := \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2}u)^2 + (\partial_{\varphi_1}u)(\partial_{\varphi_2}u) - F(\varphi_1, u, \delta)\,.$$

To find critical points of Ψ_ε we perform a variational Lyapunov-Schmidt reduction inspired to [5], see also [1].

The unperturbed functional $\Psi_0 : \mathcal{H}_{\sigma,s} \to \mathbb{R}$ possesses an infinite dimensional linear space Q of critical points which are the solutions q of the equation

$$L_0 q = \partial_{\varphi_1}\Big(\partial_{\varphi_1} + 2\partial_{\varphi_2}\Big)q = 0\ .$$

The space Q can be written as

$$Q = \Big\{q = \sum_{l\in\mathbb{Z}^2} \hat{q}_l e^{il\cdot\varphi} \in \mathcal{H}_{\sigma,s} \ \big|\ \hat{q}_l = 0 \ \text{ for } \ l_1(l_1 + 2l_2) \neq 0\Big\}$$

and, in view of the variational linking argument used for the bifurcation equation, we split Q as

$$Q = Q_+ \oplus Q_0 \oplus Q_-$$

where (²)

$$Q_+ := \Big\{q \in Q \ :\ \hat{q}_l = 0 \ \text{ for } \ l_1 = 0\Big\} = \Big\{q_+ := q_+(\varphi_2) \in H_0^\sigma(\mathbb{T})\Big\}$$

$$Q_0 := \Big\{q_0 \in \mathbb{R}\Big\}$$

$$Q_- := \Big\{q \in Q \ :\ \hat{q}_l = 0 \ \text{ for } \ l_1 + 2l_2 = 0\Big\} = \Big\{q_- := q_-(2\varphi_1 - \varphi_2),\ q_-(\,\cdot\,) \in H_0^{\sigma,s}(\mathbb{T})\Big\}.$$

(²) $H_0^\sigma(\mathbb{T})$ denotes the functions of $H^\sigma(\mathbb{T})$ with zero average. $H^{\sigma,s}(\mathbb{T}) := \{u(\varphi) = \sum_{l\in\mathbb{Z}} \hat{u}_l e^{il\varphi} \ :\ u_l^* = u_{-l},\ |u|_{H^{\sigma,s}(\mathbb{T})} := \sum_{l\in\mathbb{Z}} |\hat{u}_l| e^{\sigma|l|}[l]^s < +\infty\}$ and $H_0^{\sigma,s}(\mathbb{T})$ its functions with zero average.

We shall also use in Q the H^1-norm $|\cdot|_{H^1}$ which is the natural norm for applying variational methods to the bifurcation equation.

In order to prove analyticity (in φ_2) of the solutions and to highlight the compactness of the problem we perform a *finite* dimensional Lyapunov-Schmidt reduction [6], decomposing

$$\mathcal{H}_{\sigma,s} = Q_1 \oplus Q_2 \oplus P$$

where

$$Q_1 := Q_1(N) := \Big\{ q = \sum_{|l|\le N} \hat{q}_l e^{il\cdot\varphi} \in Q \Big\}, \ Q_2 := Q_2(N) := \Big\{ q = \sum_{|l|> N} \hat{q}_l e^{il\cdot\varphi} \in Q \Big\}$$

and

$$P := \Big\{ p = \sum_{l\in\mathbb{Z}^2} \hat{p}_l e^{il\cdot\varphi} \in \mathcal{H}_{\sigma,s} \mid \hat{p}_l = 0 \ \text{ for } \ l_1(2l_2+l_1)=0 \Big\} .$$

Projecting equation (2.1) onto the closed subspaces Q_i $(i=1,2)$ and P, setting $u = q_1 + q_2 + p$ with $q_i \in Q_i$ and $p \in P$, we obtain

$$\begin{cases} L_1[q_1] + \Pi_{Q_1}[f(\varphi_1, q_1+q_2+p,\delta)] = 0 & (Q_1) \\ L_1[q_2] + \Pi_{Q_2}[f(\varphi_1, q_1+q_2+p,\delta)] = 0 & (Q_2) \\ \mathcal{L}_\varepsilon[p] + \varepsilon \Pi_P[f(\varphi_1, q_1+q_2+p,\delta)] = 0 & (P) \end{cases}$$

where $\Pi_{Q_i} : \mathcal{H}_{\sigma,s} \to Q_i$ are the projectors onto Q_i and $\Pi_P : \mathcal{H}_{\sigma,s} \to P$ is the projector onto P.

Step 1 (Solution of the (Q_2)-(P)-equations). For $\varepsilon \in \mathcal{B}_\gamma$, $\mathcal{L}_\varepsilon$ restricted to P has a bounded inverse satisfying $|\mathcal{L}_\varepsilon^{-1}[h]|_{\sigma,s} \le |h|_{\sigma,s}\gamma^{-1}$. Moreover, also the operator $L_1 : Q_2 \to Q_2$ is invertible and $|L_1^{-1}[h]|_{\sigma,s} \le |h|_{\sigma,s} N^{-2}$.

Fixed points of the nonlinear operator $\mathcal{G} : Q_2 \oplus P \to Q_2 \oplus P$ defined by

$$\mathcal{G}(q_2, p; q_1) := \Big(-L_1^{-1}\Pi_{Q_2} f(\varphi_1, q_1+q_2+p,\delta), \ -\varepsilon\mathcal{L}_\varepsilon^{-1}\Pi_P f(\varphi_1, q_1+q_2+p,\delta) \Big)$$

are solutions of the (Q_2)-(P)-equations.

Using the Contraction Mapping Theorem, we can prove that $\forall R > 0$ there exist an integer $N_0(R) \in \mathbb{N}^+$ and positive constants $\varepsilon_0(R) > 0$, $C_0(R) > 0$ such that:

$$\forall |q_1|_{H^1} \le 2R\,, \ \ \forall \varepsilon \in B_\gamma, \ |\varepsilon|\gamma^{-1} \le \varepsilon_0(R)\,, \ \ \forall N \ge N_0(R) \ : \ 0 \le \sigma N \le 1\,,$$

there exists a unique solution

$$(q_2(q_1), p(q_1)) := (q_2(\varepsilon, N, q_1), p(\varepsilon, N, q_1)) \in Q_2 \oplus P$$

of the (Q_2)-(P)-equations which satisfies $|q_2(\varepsilon, N, q_1)|_{\sigma,s} \le C_0(R) N^{-2}$ and $|p(\varepsilon, N, q_1)|_{\sigma,s} \le C_0(R)|\varepsilon|\gamma^{-1}$.

Step 2 (Solution of the (Q_1)-equation). There remains to solve the finite dimensional (Q_1)-equation

$$L_1[q_1] + \Pi_{Q_1} f(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta) = 0\,. \tag{2.2}$$

Actually (see [5, 1]) the bifurcation equation (2.2) is the Euler-Lagrange equation of the reduced Lagrangian action functional

$$\Phi_{\varepsilon,N} : B_{2R} \subset Q_1 \to \mathbb{R}, \qquad \Phi_{\varepsilon,N}(q_1) := \Psi_\varepsilon(q_1 + q_2(q_1) + p(q_1))$$

which can be written as

$$\Phi_{\varepsilon,N}(q_1) = \text{const} + \varepsilon(\Gamma(q_1) + \mathcal{R}_{\varepsilon,N}(q_1))$$

where

$$\Gamma(q_1) := \int_{\mathbb{T}^2} \frac{(2+\varepsilon)}{2}(\partial_{\varphi_2} q_1)^2 + (\partial_{\varphi_1} q_1)(\partial_{\varphi_2} q_1) - a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d}$$

is the «Poincaré-Melnikov» function [1] and the remainder

$$\mathcal{R}_{\varepsilon,N}(q_1) := \int_{\mathbb{T}^2} F(\varphi_1, q_1, \delta = 0) - F(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta) + \frac{1}{2} f(\varphi_1, q_1 + q_2(q_1) + p(q_1), \delta)(q_2(q_1) + p(q_1))$$

satisfies $|\mathcal{R}_{\varepsilon,N}(q_1)| \le C_1(R)(\delta + |\varepsilon|\gamma^{-1} + N^{-2})$ for some $C_1(R) \ge C_0(R)$.

The problem of finding non-trivial solutions of the (Q_1)-equation reduces to find non-trivial critical points in B_{2R} of the rescaled functional (still denoted $\Phi_{\varepsilon,N}$)

$$\Phi_{\varepsilon,N}(q_1) = \Big(\mathcal{A}(q_1) - \int_{\mathbb{T}^2} a_{2d-1}(\varphi_1)\frac{q_1^{2d}}{2d}\Big) + \mathcal{R}_{\varepsilon,N}(q_1)$$

where the quadratic form $\mathcal{A}(q) := \int_{\mathbb{T}^2} \frac{1}{2}(2+\varepsilon)(\partial_{\varphi_2} q)^2 + (\partial_{\varphi_1} q)(\partial_{\varphi_2} q)$ is positive definite on Q_+, negative definite on Q_- and zero-definite on Q_0. For $q_1 = q_+ + q_0 + q_- \in Q_1$,

$$\mathcal{A}(q_1) = \frac{\alpha_+}{2}|q_+|^2_{H^1} - \frac{\alpha_-}{2}|q_-|^2_{H^1}$$

for suitable positive constants α_+, α_-, bounded away from 0 by constants independent of ε.

The geometry of $\Phi_{\varepsilon,N}$ suggests to look for critical points of «linking type». However we can not directly apply the linking theorem because $\Phi_{\varepsilon,N}$ is defined only in B_{2R}. To overcome this difficulty we extend $\Phi_{\varepsilon,N}$ to the whole space in such a way that the extension $\widetilde{\Phi}_{\varepsilon,N}$ coincides with $\Gamma(q_1)$ outside B_{2R} and so verifies the global hypothesis of the linking theorem.

The final step is to prove (exploiting the homogeneity of the term $\int a_{2d-1} q_1^{2d}$) that

$$\exists\, R_* > 0 \ \text{ independent on } \ R, \varepsilon, N, \gamma$$

and functions $0 < \varepsilon_1(R) \le \varepsilon_0(R)$, $N_1(R) \ge N_0(R)$ such that $\forall |\varepsilon|\gamma^{-1} \le \varepsilon_1(R)$, $N \ge N_1(R)$ the functional

$$\widetilde{\Phi}_{\varepsilon,N} \text{ possesses a non trivial critical point } \ \overline{q}_1 \in Q_1 \text{ with } |\overline{q}_1|_{H_1} \le R_*\,.$$

Fix $\bar{R} := R_* + 1$, take $|\varepsilon|\gamma^{-1} \le \overline{\varepsilon} := \varepsilon(\bar{R})$ and $0 < \sigma < 1/N_2(\bar{R})$. The function

$$u(\varepsilon, \varphi) := |\varepsilon|^{1/2(d-1)}[\overline{q}_1 + q_2(\varepsilon, N_2(\bar{R}), \overline{q}_1) + p(\varepsilon, N_2(\bar{R}), \overline{q}_1)]$$

is the solution of equation (1.2) in Theorem A.

REMARK 2. In the proof of Theorem B the bifurcation equation could be solved through variational methods as for Theorem A. However there is a simpler technique available. The bifurcation equation reduces, in the limit $\varepsilon \to 0$, to a superquadratic Hamiltonian system with one degree of freedom. We prove existence of a non-degenerate, analytic solution by direct phase-space analysis. Therefore it can be continued by the Implicit function Theorem to an analytic solution of the complete bifurcation equation for ε small.

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