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Localization techniques in quantum field theories

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Preface

Localization techniques in quantum field theories

Abstract

This is the foreword to the special issue on localization techniques in quantum field theory. The summary of individual chapters is given and their interrelation is discussed.

1. Summary

This is the summary of the special issue 'Localization techniques in quantum field theories' which contains 17 individual reviews[*]. The focus of the collection is on the localization technique and its applications in supersymmetric gauge theories. Although the ideas of equivariant localization in quantum field theory go back 30 years, here we concentrate on the recent surge in the subject during the last ten years. This subject develops rapidly and thus it is impossible to have a fully satisfactory overview of the field. This collection is about two and a half years in the making, and during this period some important new results have been obtained, and it was difficult to incorporate all of them. However, we think that it is important to provide an overview and an introduction to this quickly developing subject. This is important both for young researchers, who have just entered the field, and to established scientists as well. We have tried to do our best to review the main results during the last ten years.

The issue has two types of chapter: some concentrate on the localization calculation in different dimensions by itself, and others concentrate on the major applications of the localization result. Obviously, such a separation is sometimes artificial. The chapters are ordered roughly according to the dimensions of the corresponding supersymmetric theories. First, we try to review the localization calculation in a given dimension, and then we move to the discussion of the major applications.

The issue covers the localization calculations for the supersymmetric theories in dimensions 2, 3, 4 and 5. The collection discusses the applications of these calculations for theories living up to dimension 6, and for string/M theories. We have to apologize in advance for omitting from the review the new and important calculations which have appeared during the last couple of years.

This collection is intended to be a single issue where the different articles cover the different but related topics within a certain focused scope. Some contributions depend on results presented in a different article, but the dependency is not a simple linear order.

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[*] The full preprint version of the volume containing 17 chapters is available at https://arxiv.org/src/1608.02952/anc/LocQFT.pdf or http://pestun.ihes.fr/pages/LocalizationReview/LocQFT.pdf.

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The whole issue, when published, can be cited as

Pestun V and Zabzine M (ed) 2017
'Localization techniques in quantum field theories' *J. Phys. A: Math. Theor.* **50** 440301

The arXiv preprint version can be accessed from the arXiv summary entry which lists all authors and links to all 17 individual contributions; the corresponding citation is

`arXiv:1608.02952`

An individual contribution can be cited by its article number, for example

Pufu S 2017 The F-theorem and F-maximization, chapter 8 in Pestun V and Zabzine M (ed)
'Localization techniques in quantum field theories' *J. Phys. A: Math. Theor.* **50** 443008

and accessed on arXiv and cited using the arXiv number

`arXiv:1608.02960`.

2. Individual reviews

Below we summarize the content of each individual contribution/review:

Contribution [1]: 'Introduction to localization in quantum field theory' (Vasily Pestun and Maxim Zabzine)

This is the introduction to the whole volume, outlining its scope and reviewing the field as a whole. It discusses concisely the history of equivariant localization both in finite and infinite dimensional setups. The derivation of the finite dimensional Berline–Vergne–Atiyah–Bott formula is given in terms of supergeometry. This derivation is formally generalized to the infinite dimensional setup in the context of supersymmetric gauge theories. The result for supersymmetric theories on spheres is presented in a uniform fashion over different dimensions, and the related index theorem calculations are reviewed. The applications of localization techniques are listed and briefly discussed.

Contribution [2]: 'Review of localization in geometry' (Vasily Pestun)

This review is a short summary of the mathematical aspects of the Berline–Vergne–Atiyah–Bott formula and Atiyah–Singer index theory. These tools are routinely used throughout the issue. This contribution reviews the definition of equivariant cohomology, and its Weyl and Cartan models. The standard characteristic classes and their equivariant versions are reviewed. The equivariant integration is discussed and the mathematical derivation of the Berline–Vergne–Atiyah–Bott formula is explained. The Atiyah-Singer index theorems and their equivariant versions are briefly reviewed.

Contribution [3]: 'Supersymmetric localization in two dimensions' (Francesco Benini and Bruno Le Floch)

This review concentrates on the localization techniques for 2d supersymmetric gauge theories and on the major applications of 2d localization results. The main example is the calculation of the partition function for $\mathcal{N} = (2, 2)$ gauge theory on S^2. Two different approaches are presented, the Coulomb branch localization (when the result is written as an integral over the Cartan subalgebra) and the Higgs branch localization (when the answer is written as a sum). Briefly $\mathcal{N} = (2, 2)$ gauge theories on other curved backgrounds are discussed, and the calculation for the hemisphere is presented. The important calculation of the partition function for $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (2, 0)$ theories on the torus is presented, this quantity is known as the elliptic genus. The result is written in terms of the Jeffrey–Kirwan residue, which is a higher dimensional analog of the residue operation. The mathematical aspects of the Jeffrey–Kirwan residue operation are briefly explained. As the main application of the localization calculation in 2d, some dualities are discussed; in particular mirror symmetry and Seiberg duality.

Contribution [4]: 'Gromov–Witten invariants and localization' (David Morrison)

This review concentrates on an important application of 2d localization calculation, see contribution [3]. It provides a pedagogical introduction to the relation between the genus 0 Gromov–Witten invariants (counting of holomorphic maps) and the localization of 2d gauged linear sigma models. The relation is based on the conjecture which connects the partition function of $\mathcal{N} = (2, 2)$ gauge theories on S^2 with the Zamolodchikov metric on the conformal manifold of the theory. This relation allows the deduction of the Gromov–Witten invariants on the Calabi–Yau manifold from the partition function on S^2 of the corresponding linear sigma model. This contribution explains this conjecture and reviews the main step of the calculation.

Contribution [5]: 'An Introduction to supersymmetric field theories in curved space' (Thomas Dumitrescu)

This review addresses the problem of defining rigid supersymmetric theories on curved backgrounds. The systematic approach to this problem is based on the Festuccia–Seiberg work on organizing the background fields into off-shell supergravity multiplets. The contribution concentrates in detail on two major examples, $\mathcal{N} = 1$ supersymmetric theories in 4d and $\mathcal{N} = 2$ supersymmetric theories in 3d. The full classification of supersymmetric theories on curved backgrounds can be given for the theories with four or fewer supersymmetry in four or fewer dimensions.

Contribution [6]: 'Localization on three-dimensional manifolds' (Brian Willett)

This review provides an introduction to the localization technique for 3d supersymmetric gauge theories. The 3d $\mathcal{N} = 2$ supersymmetric theories are introduced and their formulation on curved space is briefly discussed. This is closely related to contribution [5]. The calculation of the partition function on S^3 is presented in great detail with the final result presented as an integral over the Cartan sublagebra of the Lie algebra of the gauge group. The calculation on the lens spaces, on $S^2 \times S^1$ and different applications of these calculations, are also discussed. The dualities between different gauge theories are briefly discussed. The factorization of the result into holomorphic blocks is also considered, and in this context the Higgs branch localization is discussed.

Contribution [7]: 'Localization at large N in Chern–Simons-matter theories' (Marcos Mariño)

The result of the localization calculation in 3d is given in terms of matrix integrals, see contribution [6]. These matrix integrals are complicated and it is not easy to extract information from this answer. This review is devoted to the study of 3d matrix models and extracting physical information from them. It concentrates on the famous ABJM model which plays a crucial role in the AdS/CFT correspondence. The M-theory expansion for the ABJM model is discussed in detail and the relation to topological strings is presented.

Contribution [8]: 'The F-Theorem and F-Maximization' (Silviu Pufu)

The partition function on S^3 for $\mathcal{N} = 2$ supersymmetric gauge theories is written as matrix integrals which depend on the different parameters of the theory, see contribution [6]. This review studies the properties of the free energy (minus the logarithm of the sphere partition function), which is regarded as the measure of the degrees of freedom in the theory. In particular this contribution states and explains the F-theorem and F-maximization principles for 3d theories. The F-theorem is a 3d analogue of the Zamolodchikov's c-theorem in 2d and the a-theorem in 4d. For 3d theories the F-theorem makes a precise statement about the idea that the number of degrees of freedom decreases along the RG flow.

Contribution [9]: 'Perturbative and nonperturbative aspects of complex Chern–Simons theory' (Tudor Dimofte)

This review discusses another important application for the localization calculation in 3d. It starts by briefly reviewing some basic facts about the complex Chern–Simons theory; the main interest is the Chern–Simons theory for $SL(N,\mathbb{C})$. There is a short discussion of the 3d/3d correspondence, which states that the partition function of the complex Chern–Simons theory on M is the same as the partition function of a specific supersymmetric gauge theory (whose field content depends on M) on the lens space. The contribution finishes with a discussion of the quantum modularity conjecture.

Contribution [10]: '$\mathcal{N}=2$ SUSY gauge theories on $\mathbf{S}^4$' (Kazuo Hosomichi)

This review gives a detailed exposition of the calculation of the partition function and other supersymmetric observables for $\mathcal{N}=2$ supersymmetric gauge theories on S^4, both round and squashed. Using off-shell supergravity, the construction of $\mathcal{N}=2$ supersymmetric theories on squashed S^4 is presented. The localization calculation is performed and the determinants are explicitly evaluated using index theorems (review in contribution [2]). The inclusion of supersymmetric observables (Wilson loops, 't Hooft operators and surface operators) into the localization calculation on S^4 is discussed.

Contribution [11]: 'Localization and AdS/CFT Correspondence' (Konstantin Zarembo)

One of the major applications of the localization calculation on S^4 (see contribution [10]) is the application to AdS/CFT. This review is devoted to the study of the matrix models which appear in the calculation on S^4 and its application to the AdS/CFT correspondence. Localization offers a unique laboratory for the AdS/CFT correspondence, since we are able to explore the supersymmetric gauge theory in non-perturbative domain. Using holography the localization computation can be compared to string theory and supergravity calculations.

Contribution [12]: 'A brief review of the 2d/4d correspondences' (Yuji Tachikawa)

From the perspective of the $\mathcal{N}=(0,2)$ self-dual 6d theory, this contribution explains the 2d/4d correspondence (AGT), considering the 6d theory on a product of 2d and 4d manifold. This correspondence relates the 4d computations for supersymmetric gauge theories of class $\mathcal{S}$, obtained by compactification of the 6d theory on the 2d manifold, to 2d computations in 2d theory obtained by compactification of 6d theory on the 4d manifold. The contribution starts by reviewing basic facts about 2d q-deformed Yang–Mills theory and the Liouville theory. The main building block of rank 1 theories considered in the review is the trifundamental multiplet coupled with $SU(2)$ gauge fields. The partition function on $S^1 \times S^3$ for such a 4d theory is computed in 2d by q-deformed Yang–Mills and the partition function on S^4 is computed in 2d by the Liouville theory.

Contribution [13]: 'The supersymmetric index in four dimensions' (Leonardo Rastelli and Shlomo Razamat)

This review studies the partition function on $S^3 \times S^1$ for $\mathcal{N} = 1$ superymmetric theories in 4d, also known as the 4d supersymmetric index. It starts by defining the supersymmetric index and reviewing combinatorial tools to compute it in theories with a Lagrangian description. After illustrating some basic properties of the index in the simple setting of supersymmetric sigma models, the discussion turns to the index of supersymmetric gauge theories, emphasizing physical applications. The index contains useful information about the spectrum of shortened multiplets, and how to extract this information is discussed in some detail. The most important application of the index, as a powerful tool for checking non-perturbative dualities between supersymmetric gauge theories, is illustrated in several examples. The last part of the chapter considers several interesting limits of the supersymmetric index.

Contribution [14]: 'Review of localization for 5d supersymmetric gauge theories' (Jian Qiu and Maxim Zabzine)

This contribution provides the introduction to localization calculation for $\mathcal{N} = 1$ supersymmetric gauge theories on toric Sasaki–Einstein manifolds, for example on a five-sphere S^5. It starts by recalling basic facts about supersymmetry and supersymmetric gauge theories in flat 5d space. Then the construction of the supersymmetric gauge theory on the Sasaki–Einstein manifolds is explicitly given. Using the field redefinition, the supersymmetry transformations are rewritten in terms of differential forms, thus making geometrical aspects of the localization more transparent. For toric Sasaki–Einstein manifolds the localization calculation can be carried out completely, the calculation of determinants is given and the full partition function is conjectured. The review ends with comments about deducing the flat space results from the curved result.

Contribution [15]: 'Matrix models for 5d super Yang–Mills' (Joseph Minahan)

The result of 5d localization calculation is given in terms of complicated matrix models, see contribution [14]. This review studies the resulting matrix models. The basic properties of the matrix models are described and the 't Hooft limit is analyzed for $\mathcal{N} = 1^*$ theory (a vector multiplet plus a hypermultiplet in the adjoint representation). For large 't Hooft coupling the free energy behaves as N^3 for $U(N)$ gauge theory and the corresponding supergravity analysis is performed. This analysis support the idea that the non-perturbative completion of 5d theory is the 6d $\mathcal{N} = (2, 0)$ superconformal field theory.

Contribution [16]: 'Holomorphic blocks and the 5d AGT correspondence' (Sara Pasquetti)

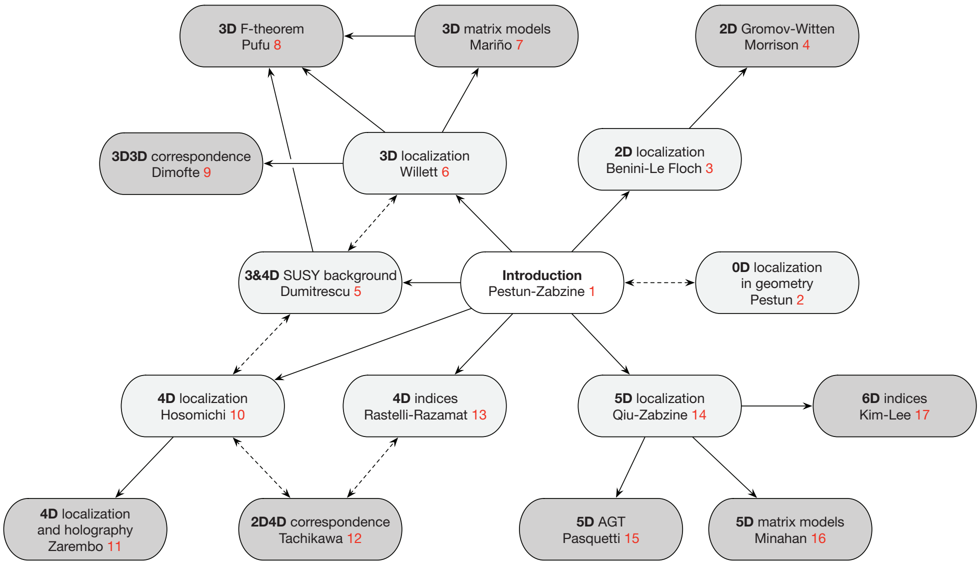
This review further studies partition functions from 2d to 5d. In particular it concentrates on the idea that the partition function on a compact manifold can be built up from basic blocks, so-called holomorphic blocks. The main point is that the geometric decomposition of the compact manifold should have its counterpart in the appropriate decomposition of the partition function. These factorization properties are reviewed in different dimensions. The rest of the contribution concentrates on a 5d version of the AGT correspondence.

Contribution [17]: 'Indices for 6 dimensional superconformal field theories' (Seok Kim and Kimyeong Lee)

This contribution deals with the 6d (2,0) superconformal field theory. This theory cannot be accessed directly, but it is related to many other supersymmetric gauge theories, e.g. it is believed to be the UV-completion of maximally supersymmetric 5d gauge theory. The relation between 5d partition function and 6d supersymmetric index is discussed in details in this review.

3. Volume structure

The different contributions are related to each other and the relation is not a simple linear one, which can be shown by their ordering in the volume. Below we provide a graphical relation between the different contributions. This diagram[23] gives the general idea.



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[23] Special thanks to Yuji Tachikawa for the final design of the diagram.

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