Internal symmetries of the axisymmetric gravitational fields

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The group H of the internal symmetries of the axisymmetric field equations in general relativity is known to be isomorphic to SO(2,1), which is the double covering of the conformal group of the hyperbolic complex plane $\mathscr{H}$. The Ernst potential ξ can then be geometrically understood as a map $\xi : R^3/\mathrm{SO}(2)\to\mathscr{H}$. The fact that the hyperbolic plane is split into two connected components is used to introduce an algebraic invariant $n\in Z^+$ for every axisymmetric solution. It is shown that under reasonable hypotheses this invariant is related to the number of S^1 curves where the manifold is intrinsically singular.

INTRODUCTION

The axisymmetric field equations in general relativity contain a large amount of symmetries, which have been extensively discussed by several authors.[1] The main line of research in this field has been directed during the last few years towards the study of the infinite parameter group K, which combines both the coordinate group G and the internal symmetry group H.[2] Nevertheless, there are still some interesting results which can be derived from the study of the group H alone, as shown in the following.

The starting point of the present approach is to note that the most natural geometric interpretation of the Ernst equation is achieved considering the Ernst potential ξ as a map from $R^3/\mathrm{SO}(2)$ to the complex plane with the Poincaré metric. Because of the isomorphism $H\simeq\mathrm{SO}(2,1)$ and of the fact that SO(2,1) is a double covering of the conformal group of the hyperbolic plane, one can interpret the internal symmetries of the Ernst equation as isometries of the hyperbolic plane itself. This amounts to translating into elementary complex geometry the approach by Eris and Nutku.[3]

The map ξ is then studied, and it is shown that one can introduce an algebraic invariant, which classifies the asymptotically flat solutions according to their causal structure.

Finally the particular case in which ξ depends on a single real function is geometrically interpreted as the geodesic problem of the hyperbolic plane.

GEOMETRIC MEANING OF THE ERNST EQUATION

The axisymmetric stationary line element in canonical cylindrical coordinates reads[4]

$$ds = f^{-1}[e^{2\gamma}(dz^2 + d\rho^2) + \rho^2 d\phi^2] - f(dt - \omega d\phi)^2, \quad (1)$$

where f, ω, γ depend on ρ, z only. In this form the field equations for γ decouple, and the relevent problem reduces to two coupled equations for f, ω, which by means of the substitution

$$f = \frac{\xi\bar\xi - 1}{|\xi + 1|^2} \quad (2)$$

$$\nabla\omega = \phi \times \nabla\frac{\xi - \bar\xi}{|\xi + 1|^2} \quad (3)$$

(where $\hat\phi$ is the azimuthal versor pf R^3, and ∇ is the three-dimensional operator) can be transformed into the Ernst equation for the complex potential ξ[5]

$$(\xi\bar\xi - 1)\nabla^2\xi = 2\bar\xi\,\nabla\xi\cdot\nabla\xi. \quad (4)$$

Equation (4) can be derived from the Lagrangian density

$$L = \frac{\nabla\xi\cdot\nabla\bar\xi}{(\xi\bar\xi - 1)^2} = g(\nabla\xi,\nabla\bar\xi). \quad (5)$$

From Eq. (5) it is apparent that the bilinear operator $g(\ ,\ )$ coincides with the Poincaré metric for the complex hyperbolic plane $\mathscr{H}$ and the Ernst potential can be considered as the map

$$\xi: R^3/\mathrm{SO}(2)\to\mathscr{H}, \quad (6)$$

which in view of the field equation (4) must be extremal.

It is now obvious that the internal symmetries of the problem coincide with the isometries of the hyperbolic plane. These include a continous group (i.e., the conformal group $\mathscr{C}$)[6]

$$\xi\to e^{i\chi}\frac{\xi - p}{1 - \bar p\xi}, \quad 0\leqslant\chi<2\pi,\ p\bar p<1, \quad (7)$$

and the following discrete transformations:

$$\xi\to -\xi, \quad (8)$$

$$\xi\to\bar\xi, \quad (9)$$

$$\xi\to 1/\xi, \quad (10)$$

Equations (8) and (9) are reflections of $\mathscr{H}$, while Eq. (10) arises from the fact that the unit disk $\xi\bar\xi<1$ is an isometric copy of the domain $\xi\bar\xi>1$ under inversion. This explains the origin of the discrete symmetries discovered by Ernst while discussing his equation.[7]

The conformal group (7) coincides with the form of the group H given by Kinnersley.[8] Note that the isotropy subgroup of $\mathscr{C}$ at the origin ($p = 0$) is given by $\xi\to e^{i\chi}\xi$ and generates NUT transformations of the manifold.[9]

From the point of view of elementary group theory, the

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present interpretation of the Ernst equation amounts to using the well-known isomorphism $H \simeq \mathrm{SO}(2,1)$, and to noting that SO(2,1) is a double covering of the conformal group $\mathscr{C}$.

Incidentally one can emphasize that the Lagrangian (5) presents some formal analogies with the one given by Woo[10] for the σ nonlinear model. In that case, however, the gauge group is SO(3), which is compact, and therefore the conformal factor is the spherical one [i.e., $(\xi\bar{\xi}+1)^2$] instead of the hyperbolic one appearing in Eq. (5). Moreover, the base space for the σ nonlinear problem is R^2 instead of $R^3/\mathrm{SO}(2)$ as in the present case.

TOPOLOGIC AND ALGEBRAIC INVARIANTS

Although one could impose boundary conditions on ξ in order to compactify its domain, the hyperbolic plane is not compact, and therefore it seems irrelevant to investigate the homotopy classes of the map ξ.

There is, however, an interesting invariant, which is related to the algebraic structure of the map ξ. These, in fact, can be classified according to the number of jumps between the two connected components into which the complex plane is split by the Poincaré metric, i.e., according to the number n of rotational bisurfaces in R^3 where $\xi\bar{\xi}=1$. This number is independent of the coordinates chosen in $R^3/\mathrm{SO}(2)$, although it is not invariant with respect to the general group of transformations of the metric (3). Note in fact that the surfaces identified by the equations $\xi\bar{\xi}=1$ may contain coordinate singularities, the elimination of which will require transformations involving the asymptotically timelike coordinate t.

For instance, in the case of the Schwarzschild and Kerr solutions, for which $\xi_s = x$, $\xi_k = px + iqy, (p^2+q^2=1)$, respectively, in prolate spheroidal coordinates, it turns out that $n=2$.

Note that as the condition $\xi\bar{\xi}=1$ is invariant under the action of the conformal group $\mathscr{C}$, also the number n is invariant under its action. This means, for instance, that the NUT generalization of a given field does not change the number n. The interior of the unit disk of the hyperbolic plane is related to the "ergosphere" regions of M, where $f<0$, the unit circle itself (expect the point $\xi=-1$) being the domain into which ξ maps the "ergosurface." At the point $\xi=-1, f$ diverges, showing that $\xi=-1$ is the image of the intrinsic singularities of M.

A simple interpretation of the meaning of the number n can be obtained under few hypotheses on the map ξ. Choose prolate spheroidal coordinates x, y or $R^3/\mathrm{SO}(2)$, and assume that

(A) the gravitational field described by ξ is asymptotically flat. In particular, $\lim_{x\to\infty} |\xi| \neq 1$.

(B) reflecting the space time with respect to the equatorial plane (i.e. $y \to -y$) the angular momentum of the gravitational field changes sign, i.e., $\xi \to \bar{\xi}$. Therefore ξ is real on the equatorial plane ($y=0$).

(C) ξ is an odd function of x on the equatorial plane.[11]

Then the number m_+ of solutions of the equation $\xi=1$ is equal to the number m_- of solutions of $\xi=-1$. Obviously on the equatorial plane $m_+ + m_- = n$, and therefore $m_- = n/2$. Since on the equatorial plane ξ is a function of x alone, there will be $n/2$ distinct values $x_1,\ldots, x_{n/2}$, where $\xi=-1$. These points actually represent trajectories of the axisymmetric group, which topologically are S^1 curves (of which one can possibly degenerate to a point) along which $f\to\infty$ and therefore the manifold is singular.

Therefore one can conclude that under the hypotheses (A),(B),(C) the number of ring singularities of the space time described by ξ is exactly $n/2$.

"GEODESIC SOLUTIONS"

In the special case when $\xi=\xi(\tau)$ depends on one real function $\tau: R^3/\mathrm{SO}(2)\to R$, the present approach yields a nice geometrical interpretation. Note first that $\xi(\tau)$ is a curve in the hyperbolic plane. The Ernst equation reads

$$(\xi\bar{\xi}-1)\frac{d^2\xi}{d\tau^2} + 2\bar{\xi}\frac{d\xi}{d\tau}^2 = -(\xi\bar{\xi}-1)\frac{d\xi}{d\tau}\frac{\nabla^2\tau}{\nabla\tau\cdot\nabla\tau} \tag{11}$$

and coincides with the geodesic equation on the hyperbolic plane if $\nabla^2\tau=0$, with τ as affine parameter.

If τ is not harmonic, one can introduce a new function $\alpha(\tau)$, in terms of which Eq. (14) becomes

$$(\xi\bar{\xi}-1)\frac{d\xi}{d\alpha}\alpha'' + \frac{d^2\xi}{d\alpha^2}\alpha'^2 + 2\bar{\xi}\frac{d\xi}{d\alpha}^2\alpha'^2 = -(\xi\bar{\xi}-1)\frac{d\xi}{d\alpha}\alpha'\frac{\nabla^2\tau}{\nabla\tau\cdot\nabla\tau}.$$

Choosing α such that

$$\frac{\alpha''}{\alpha'} = \frac{\nabla^2\tau}{\nabla\tau\cdot\nabla\tau}, \tag{12}$$

one has

$$(\xi\bar{\xi}-1)\frac{d^2\xi}{d\alpha^2} + 2\bar{\xi}\frac{d\xi}{d\alpha}^2 = 0,$$

which is again the geodesic equation on the hyperbolic plane. From Eq. (12) one has that $\nabla^2\alpha=0$, and hence α must be harmonic. Therefore, one can conclude that the geodesics of the hyperbolic plane depending on an affine parameter, which is a harmonic function defined on $R^3/\mathrm{SO}(2)$, correspond one to one to the solution of the Ernst equation depending on a single real function. These include the Weyl[12] and Papapetrou[13] solutions.

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