A first order system of differential equations for covariant σ models

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A first order system of differential equations is obtained for a covariant σ model defined on a Riemannian (or pseudo-Riemannian) manifold of arbitrary dimension n. In the case of compact groups, i.e., SO(n), the first order system coincides with the one yielded by topological arguments. Our considerations hold true also for SO(p,q), $p+q=n+1$, invariance groups. The scale invariance of the problem is discussed.

1. INTRODUCTION

It is well known that the solutions for the SO(3) nonlinear σ model found by Belavin and Polyakov through topological arguments can also be derived by means of purely local tools.[1-3] The same local approach yields a class of solutions even for the noncompact problem with the same dimensionality, i.e., for the SO(2,1) invariant problem defined by R^2. In fact, parametrizing the field variables in complex stereographic coordinates, the action becomes

$$S = \int \frac{\xi_z \bar{\xi}_{\bar{z}} + \xi_{\bar{z}} \bar{\xi}_z}{(1 + a\xi\bar{\xi})} dz \Lambda\, d\bar{z}, \tag{1}$$

where $a = +1$ for SO(3) and $a = -1$ for SO(2,1). Writing the field equations as the divergence of the energy–momentum tensor, one has the first order system

$$(1 + a\xi\bar{\xi})\xi_z \bar{\xi}_z = f(z), \tag{2}$$

where $f(z)$ is analytic. For the compact case ($a = 1$), regularity and the finite action condition amount to imposing that $f(z) = 0$ (see Ref. 2) and therefore the system (2) reduces to the Cauchy–Riemann equation for the field ξ. The noncompact case ($a = -1$) is equivalent to the problem of finding axisymmetric solutions of the vacuum Einstein equations which are of the Petrov type N.[4] In this case there is no physical reason for the action to be finite. Nevertheless, one can show that the only relevant solutions are still given by $\xi_{\bar{z}} = 0$ as in the compact case.

The aim of this paper is to show that, independently of the compactedness of the group and the dimension of the base space, it is possible to find a first order system (f.o.s) naturally associated to a certain class of σ models.

In the next section we obtain the general form of the f.o.s. for a generally covariant σ model defined on a Riemannian or pseudo-Riemannian manifold of arbitrary dimension. The third section is devoted to the analysis of our result in the physically interesting case of $n = 4$.

2. GENERALIZED σ MODELS

In this section we study a generally covariant σ model defined on a Riemannian (or pseudo-Riemannian) manifold M of arbitrary dimension n. The invariance group of the model is taken to be SO(p,q) with $p + q = n + 1$. Let Φ be a map

$$\Phi{:}M \to R^{p+q} \tag{3}$$

invariant under the action of SO(p,q) on R^{p+q}. The action of the problem is

$$S = \int * \, d\Phi^a \Lambda \, d\Phi_a = \int \lambda \tag{4}$$

with the constraint

$$\Phi^a \Phi_a = 1, \tag{5}$$

where the internal indices are saturated with the Killing form of SO(p,q), d is the exterior differential, Λ is the wedge product, and $*$ is the Hodge duality operator.

Since the action is a geometrical object on the manifold M, it is manifestly covariant under the general group GL(n,R). By varying the action (4) and taking into account the condition (5), one has the following field equations:

$$d * d\Phi_a + \Phi_a (d\Phi^b \Lambda * d\Phi_b) = 0. \tag{6}$$

In the case of compact invariance groups (i.e., when $q = 0$ and $\Phi{:}M \to S^n$) it is relevant to consider the following n form:

$$*\rho = \epsilon_{a_1 a_2 \cdots a_{n+1}} \Phi^{a_1} d\Phi^{a_2} \Lambda \cdots \Lambda \, d\Phi^{a_{n+1}}, \tag{7}$$

which upon integration on M gives the homotopy class of the map Φ:

$$Q = \int *\rho \in \Pi_n(M, S^n). \tag{8}$$

For instance, when $n = 2$ and $M = S^2$, one has that $Q \in \Pi_2(S^2) = Z$ is the "topological charge" introduced by Belavin and Polyakov. In this case one has also the remarkable inequality $S \geqslant Q$. The condition

$$* d\Phi_{b_1} = -\epsilon_{b_1 b_2 b_3} \Phi^{b_2} d\Phi^{b_3} \tag{9}$$

yields $S = Q$ and therefore it is sufficient for satisfying the field equations. It is also necessary if one restricts oneself to search for finite action solutions. Note that Eq. (9) yields $\lambda = *\rho$. In the noncompact case the topological argument given above does not apply any more. Nevertheless there is a local reason why the condition $\lambda = *\rho$, which yields the f.o.s.

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$$*d\Phi_{b_1} = -\epsilon_{b_1 b_2 b_3 \cdots b_{n+1}} \Phi^{b_2} d\Phi^{b_3}\Lambda\cdots\Lambda\, d\Phi^{b_{n+1}}, \quad (10)$$

is still relevant from the mathematical point of view, since every solution (if any) of the system (10) is also a solution of the field equations (6). Of course, the converse may not be true. The proof is given in two steps. First, note that for a field satisfying condition (10) one has the identity

$$d\Phi^{b_3}\Lambda\cdots\Lambda\, d\Phi^{b_{n+1}} = \frac{(-1)^q}{(n-1)!}\epsilon^{a_1 a_2 b_3 \cdots b_{n+1}} \Phi_{a_1} *d\Phi_{a_2}, \quad (11)$$

where q is the number of negative eigenvalues of the Killing form of SO(p,q). The second step amounts to substituting Eq. (10) into the field equations. The Laplacian of Φ reads

$$d* d\Phi_{b_1} = -\epsilon_{b_1 b_2 \cdots b_{n+1}} d\Phi^{b_2}\Lambda\cdots\Lambda\, d\Phi^{b_{n+1}}. \quad (12)$$

Using Eq. (11), one has that

$$\begin{aligned} d*d\Phi_{b_1} &= -\delta^{a_1 a_2}_{b_1 b_2}\Phi_{a_1} d\Phi^{b_2}\Lambda*d\Phi_{a_2} \\ &= -\Phi_{b_1}(d\Phi^{b}\Lambda*d\Phi_b), \end{aligned} \quad (13)$$

showing that the field equations are satisfied.

Let us end this section noting that the f.o.s. (10) is scale invariant whenever the action (4) is. It is well known that this is equivalent to the nondimensionality of the constraint (5), which in turn implies a constraint on the dimension n of the basic space. In fact, assuming that the background covariant metric tensor has dimension $n-2$ as in general relativity, one has

$$\begin{cases} \dim[d\Phi^a] = \dim[\Phi^a], \\ \dim[*d\Phi^a] = \left(\frac{n^2}{2} - 3n + 4\right) + \dim[\Phi^a]. \end{cases} \quad (14)$$

Accordingly, if $\dim[\Phi^a] = 0$, the action is nondimensional only for $n = 2,4$.

3. CLOSING REMARKS

The construction given above naturally introduces a f.o.s. of differential equations as a sufficient condition for the solution of the second order system (6). It would be very interesting to ascertain the hypotheses restricting the class of fields for which Eq. (10) is also a necessary condition for the solution of the field equations. Note that for $n = 2$ solutions of Eq (10) have been already found both for the compact[1] and the noncompact case.[5] From the physical point of view, however, the only interesting case is a four dimensional covariant problem, with group SO(5) or SO(4,1). De Alfaro, Fubini, and Furlan[6] first studied such models, coupling the Φ field with the gravitational field in a generally invariant fashion. In their approach the gravitational field is a dynamical variable coupled to the energy–momentum tensor of the Φ's through the Einstein equations. Here the metric field is considered as a background field, and no corrections to the curvature due to the presence of the Φ's is introduced.

As an example, for the SO(5) σ model on the instanton background given by[6]

$$ds^2 = \frac{\alpha^2}{(\alpha + \chi^2)^2}\delta_{\mu\nu}\, dx^\mu\, dx^\nu,$$

one finds that

$$\Phi_\mu = \frac{2\alpha\chi_\mu}{\alpha^2 + \chi^2}, \quad \Phi_5 = \frac{\alpha^2 - \chi^2}{\alpha^2 + \chi^2}$$

satisfies the f.o.s.

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