THE ASTROPHYSICAL JOURNAL, 227:596–599, 1979 January 15

SOME PROPERTIES AND PROBLEMS OF ACCRETION DISKS ABOUT KERR NAKED SINGULARITIES

C. REINA AND A. TREVES
Istituto di Fisica dell'Università, Milano, Italy
Received 1978 February 13; accepted 1978 July 28

ABSTRACT

Some properties of accretion disks about Kerr naked singularities are deduced by studying circular orbits of test particles. It is found that for $a/M \gg 1$ the inner radius of the disk and the corresponding energy increase with a/M. For $1 < a/M < (32/27)^{1/2}$ there is a pathology related to the definition of positive and negative energy states. For $a/M = (32/27)^{1/2}$ the entire mass-energy of the test particle can be extracted.

Subject headings: black holes — relativity — stars: accretion

I. INTRODUCTION

The occurrence of singularities is a common feature of many physical theories, and general relativity is no exception. In the case of black holes the singularity is surrounded by an event horizon, and in the region of spacetime external to this surface the physics is not directly affected by the pathologies connected with the singularity itself. However, solutions of the Einstein equations are known without horizons (naked singularities), and up to now no general argument has been given which definitely excludes their actual existence (cosmic censor, Penrose 1969).

Although the basic characteristics of black holes have been clear since the late 1930s, astrophysical applications appeared only recently, when it was realized that accretion by a black hole in a binary system could be a powerful source of energy, possibly explaining the emission of some X-ray sources. In particular the interaction of the accretion disk with the hole must be treated in accordance with general relativity. For instance, one can deduce that the inner extension of an accretion disk around a static black hole must be $r = 6GM/c^2$ since that is the radius of the last stable orbit, and by the same argument the shear stress is taken null at that radius (see, e.g., Pringle and Rees 1972; Shakura and Sunyaev 1973).

It is therefore natural, although unconventional, to inquire if by the same methods applied to black holes it is possible to deduce the appearance of naked singularities interacting with an external medium. The problem is by no means simple to handle, even in principle, because of the existence of closed timelike curves which violate causality, as shown by Carter (1968). However, we shall neglect the astrophysical implication of this possibility, and limit ourselves to sketching the structure of disk accretion by examining the stable circular orbits of a test particle in a Kerr background.

The study of equatorial orbits is a direct generalization of the black hole case. However, one finds that for singularities with angular momentum parameter a in the interval $1 < a/M < (32/27)^{1/2}$ there is a region where positive and negative energy states are mixed and a classical interpretation of the energy of the test particle becomes inadequate. Another interesting feature is that the radius of the last stable orbit is an increasing function of a/M, and therefore one can derive that, for extreme naked singularities ($a/M > 1$), the energy produced in the disk becomes negligible.

II. ORBITS

Orbits in a Kerr manifold are easily studied since three constants of the motion can be derived: the angular momentum L, the total energy E, and the Carter constant Q (Carter 1968). The literature on orbits about black holes is extensive (e.g., Bardeen, Press, and Teukolsky 1972; Rees, Ruffini, and Wheeler 1975, and references therein), and some contributions consider specifically the naked singularity case (De Felice 1974). We refer in particular to the paper by Bardeen, Press, and Teukolsky (1972), whose notation is followed here.

The conditions giving circular orbits for a particle of mass μ, which are the most relevant in the study of accretion disks, read

$$\boldsymbol{p}\cdot\boldsymbol{e}_r = 0\,, \quad \dot{\boldsymbol{p}}\cdot\boldsymbol{e}_r = 0\,, \tag{1}$$

where p is the four-momentum of the particle, $\dot{p}$ its derivative with respect to an affine parameter, and e_r is a unit vector in the radial direction for an observer at rest at infinity. The first of the conditions (1) is equivalent to saying that the observer does not see any radial momentum for the particle, and the second one ensures that the radial momentum will remain null during the whole history of the particle itself.

Writing the equation $\boldsymbol{p}\cdot\boldsymbol{p} = -\mu^2$ in terms of the constants of motion, one gets the two equations

$$V(r) = 0\,, \tag{2}$$

$$\frac{dV}{dr} = 0\,, \tag{3}$$

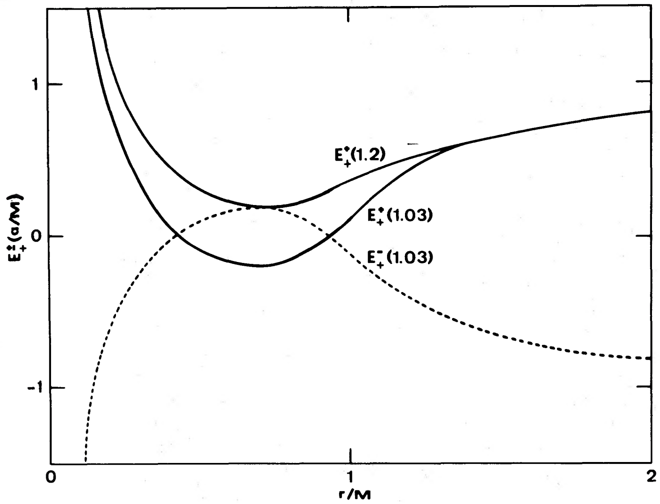

FIG. 1.—Energy of a particle of mass $\mu = 1$ in a circular orbit as a function of radial distance for different values of a/M. Note that for $a/M = 1.03$ there is a region of space where $E^- > E^+$.

where

$$V = T^2 - \Delta[\mu^2 r^2 + (L - aE)^2]\,, \tag{4}$$

and

$$T = E(r^2 + a^2) - La \tag{5}$$

for equatorial orbits.

Boyer-Lindquist (1967) coordinates are used, $\Delta = r^2 - 2Mr + a^2$, and M is the mass and a is the density of angular momentum of the naked singularity. It is apparent that equations (2) and (3) are invariant under the exchange $E \to -E$, $L \to -L$. Therefore, one can generalize the solutions of the system (2), (3) given by Bardeen, Press, and Teukolsky (1972) by considering also negative energy states. The need for the generalization will be clear in the following. We have then

$$E_{\pm}{}^{+}/\mu = \frac{r^{3/2} - 2Mr^{1/2} \pm aM^{1/2}}{r^{3/4}(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2})^{1/2}}\,, \tag{6}$$

$$L_{\pm}{}^{+}/\mu = \frac{\pm M^{1/2}(r^2 \mp 2aM^{1/2}r^{1/2} + a^2)}{r^{3/4}(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2})^{1/2}}\,, \tag{7}$$

$$E_{\pm}{}^{-} = -E_{\pm}{}^{+}\,, \tag{8}$$

$$L_{\pm}{}^{-} = -L_{\pm}{}^{+}\,. \tag{9}$$

The subscript plus or minus signs correspond to corotating or counterrotating particles at infinity.

The condition for the existence of a circular orbit is that E and L be real; this is always satisfied by choosing the plus sign (i.e., corotating orbits) and

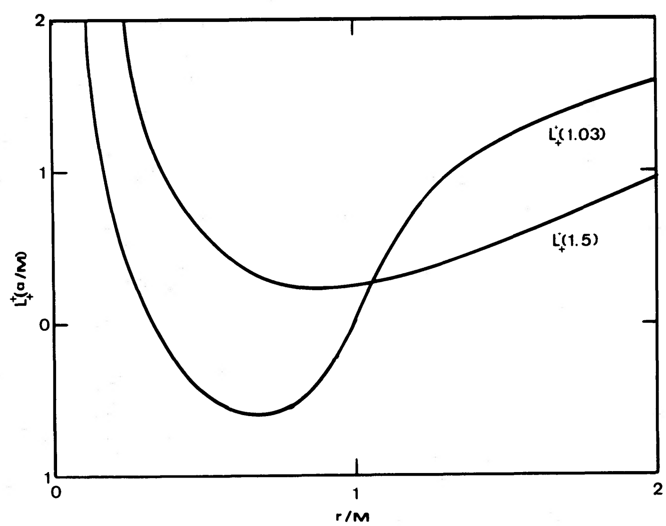

FIG. 2.—Angular momentum of a particle of mass $\mu = 1$ in a circular orbit for different values of a/M

taking the naked singularity case. Counterrotating circular orbits do not exist for small values of r as in the black hole case.

In Figures 1 and 2 we give the functions $E_+{}^{\pm}$, $L_+{}^{+}$ for various values of a/M. For $a/M \leq (32/27)^{1/2}$, $E_+{}^{\pm}$ becomes null and there is a region where $E_+{}^{-} > E_+{}^{+}$. Note that in the special case $a/M = 1$ the functions E^+ are discontinuous at $r/M = 1$; however, the discontinuity is hidden from an external observer by the horizon. In fact, for $a/M = 1$ equation (6) reads

$$\frac{E_{\pm}{}^{+}}{\mu} = \frac{(r^{1/2} \mp M^{1/2})(r \pm M^{1/2} r^{1/2} - M)}{|r^{1/2} \mp M^{1/2}| r^{3/4} (r^{1/2} \pm 2M^{1/2})^{1/2}}, \quad (6a)$$

to be compared with equation (2.14) of Bardeen, Press, and Teukolsky (1972) which is valid only for $r/M \geq 1$. It is apparent that the discontinuity at $r/M = 1$ equals $2/\sqrt{3}$. For any a/M, E^+ is diverging for $r = 0$, and asymptotically increasing. The location of the minima and their values are given as a function of a/M in Figure 3.

Equation (6) indicates that for $a < (27/16)^{1/2}$ the angular momentum of the test particle changes sign before the last stable orbit is reached, which means that the corotating or counterrotating nature of a test particle should be defined at infinity.

Photon circular orbits are possible only for values of r which make null the denominator of equation (6). For $a/M < 1$ this occurs for both rotating and counterrotating orbits at a radius given by (Bardeen, Press, and Teukolsky 1972)

$$r = r_{\rm ph} \equiv 2M\{1 + [\tfrac{2}{3} \cos^{-1}(\mp a/M)]\}. \quad (10)$$

For $a/M > 1$, one finds that photon orbits exist only in the counterrotating case and are located at

$$r = r_{\rm ph} = (aM^{1/2})^{2/3}\{[1 + (1 - M^2/a^2)]^{1/3} + [1 - (1 - M^2/a^2]^{1/3}\}^2. \quad (11)$$

For $a \to M$, one has $r_{\rm ph} \to 4M$; and for $a/M \to \infty$, $r \to (2aM^{1/2})^{2/3}$. Therefore, for counterrotating orbits $r_{\rm ph}$ is a continuous function at $a/M = 1$. Note that within this approach it is impossible to examine the special case $a/M = 1$ for corotating orbits, since, as indicated above, function (6) is discontinuous at $r = 1$.

The condition of stability of an orbit, $V'' < 0$, gives for both signs of the energy

$$r > r_{\rm ms}$$

where r is the radius of the orbit and the "marginal stability radius" $r_{\rm ms}$ turns out to be (Bardeen, Press, and Teukolsky 1972)

$$r_{\rm ms} = M[3 + Z_2 \mp (3 - Z_1)(3 + Z_1 + 2Z_2)^{1/2}],$$

$$Z_1 = 1 + (1 - a^2/M^2)^{1/3}(1 + a/M)^{1/3} + (1 - a/M)^{1/3},$$

$$Z_2 = (3a^2/M^2 + Z_1{}^2)^{1/2}.$$

The values of $r_{\rm ms}$ and the corresponding values of $E_+{}^{\pm}$ obtained from equation (6) are given in Figure 3 versus a/M. For $a/M \sim \infty$, $r_{\rm ms}/M \sim 3^{1/2}a/M$. The function $r_{\rm ms}$ is continuous at $a/M = 1$, while $E_+{}^{+}$ and $E_+{}^{-}$ are discontinuous. The discontinuity derives from that described in equation (6a).

III. DISCUSSION

A self-consistent model of disk accretion requires that the hydrodynamics equation including viscosity be solved on the background spacetime, with a suitable inner boundary condition (see, e.g., Novikov and Thorne 1973).

On the other hand, as in the black hole case, the study of the orbits can help one understand the energetics of the disk even if this approach ignores the transfer of energy and angular momentum from the singularity to the disk itself, which is described by an inner boundary

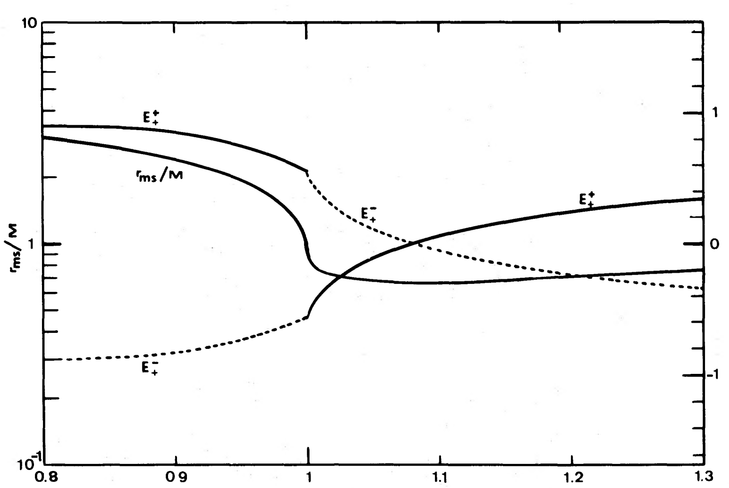

FIG. 3.—Marginal stability radius for different values of a/M. The corresponding positive and negative energies (*dotted lines*) are also reported.

condition that, at the present stage, is completely unknown.

Since we are interested in the accretion problem, we postulate that at large radii there is a mechanism (e.g., viscous stress) which disperses energy, and transfers angular momentum outward and mass inward while keeping the flow on quasi-geodesic circular orbits.

Let us now follow a test particle starting from a circular orbit at $r \gg M$. Independent of a/M and for large values of r, the orbit shrinks and E and L decrease, until the last stable circular orbit is reached. This radius corresponds to the minimum of the curve shown in Figure 1. Within this radius circular orbits are unstable and lie on the maxima of the effective potentials. This radius and the energy of the last stable orbit increase as a/M increases (see Fig. 3), indicating that the disk becomes less extended and luminous.

The energetics of the disk for $a/m \gtrsim 1$ requires a detailed discussion. In fact, for $a/m < (32/27)^{1/2}$ there is a region where the energy E^+ becomes negative. This could suggest that one can extract more energy than the mass of the particle, as in a generalized ergosphere, at the expense of the background field.

However, where $E^+ < 0$ one has that $E^- > 0$, and therefore there is an ambiguity in the definition of the energy of the orbit. Moreover, the occurrence of regions of spacetime where $E_+{}^- > E_+{}^+$ poses serious problems of interpretation. In quantum theory the only physically meaningful states are those which satisfy the condition $E > E^+$ or $E > E^-$. Here we have an example where the two conditions are satisfied at the same time. Martellini and Treves (1977) have shown that the occurrence of such regions is strictly connected with the existence of causality-violating sets in the sense discussed by Carter (1968, 1978). Because of the existence of timelike closed geodesics, any distinction between particles and antiparticles becomes impossible.

For $a/m > (32/27)^{1/2}$ these pathologies are not present and the efficiency of energy extraction can be directly computed. Note that for $a/M = (32/27)^{1/2}$ the entire mass of the particle is released.

This discussion of the energetics refers only to an equatorial disk. It is possible that the largest energy release occurs within r_{ms}, or that everything is dominated by a spontaneous emission from the singularity itself.

Of course, if there are astrophysical objects giving rise to exterior gravitational fields of the type discussed here, both the uncertainty of the inner boundary condition for the disk and the pathologies for $a/m < (32/27)^{1/2}$ are possibly removed by the presence of the source of the field, but the external disk would still have the characteristics given here.

We understand that Dr. F. De Felice is working on a subject similar to that treated here.

REFERENCES

Bardeen, J. M., Press, W. H., and Teukolsky, S. 1972, *Ap. J.*, **178**, 347.
Boyer, R. H., and Lindquist, R. W. 1967, *J. Math. Phys.*, **8**, 265.
Carter, B. 1968, *Phys. Rev.*, **174**, 1559.
———. 1978, *Gen. Rel. Grav.*, in press.
De Felice, F. 1974, *Astr. Ap.*, **34**, 15.
Martellini, M., and Treves, A. 1977, *Phys. Rev. D*, **16**, 3415.
Novikov, I., and Thorne, K. S. 1973, in *Black Holes*, ed. C. and B. de Witt (London: Gordon & Breach).
Penrose, R. 1969, *Rivista del Nuovo Cimento*, series 1, page 1.
Pringle, J. E., and Rees, M. J. 1972, *Astr. Ap.*, **21**, 1.
Rees, M., Ruffini, R., and Wheeler, J. A. 1975, *Black Holes, Gravitational Waves, and Cosmology* (London: Gordon & Breach).
Shakura, N. I., and Sunyaev, R. A. 1973, *Astr. Ap.*, **24**, 337.

CESARE REINA and ALDO TREVES: Istituto di Fisica, Via Celoria, 16, 20133 Milano, Italy