

## Research Article

# Sharp Nonexistence Results for a Linear Elliptic Inequality Involving Hardy and Leray Potentials

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We deal with nonnegative distributional supersolutions for a class of linear elliptic equations involving inverse-square potentials and logarithmic weights. We prove sharp nonexistence results.

## 1. Introduction

In recent years, a great deal of work has been made to find necessary and sufficient conditions for the existence of distributional solutions to linear elliptic equations with singular weights. Most of the papers deal with weak solutions belonging to suitable Sobolev spaces. We quote for instance, [1–4] and references therein.

In the present paper, we focus our attention on a class of model elliptic inequalities involving singular weights and we adopt the weakest possible concept of solution, that is, that one of distributional solution.

Let  $N \geq 2$  be an integer,  $R \in (0, 1]$ , and let  $B_R$  be the ball in  $\mathbb{R}^N$  of radius  $R$  centered at 0. In the first part of the paper, we study nonnegative solutions to

$$-\Delta u - \frac{(N-2)^2}{4}|x|^{-2}u \geq \alpha|x|^{-2}|\log|x||^{-2}u \quad \text{in } \mathcal{D}'(B_R \setminus \{0\}), \quad (1.1)$$

where  $\alpha \in \mathbb{R}$  is a varying parameter. By a standard definition, a solution to (1.1) is a function  $u \in L^1_{\text{loc}}(B_R \setminus \{0\})$  such that

$$-\int_{B_R} u \Delta \varphi \, dx - \frac{(N-2)^2}{4} \int_{B_R} |x|^{-2} u \varphi \, dx \geq \alpha \int_{B_R} |x|^{-2} |\log|x||^{-2} u \varphi \, dx \quad (1.2)$$

for any nonnegative  $\varphi \in C_c^\infty(B_R \setminus \{0\})$ . Notice that the weights in (1.1) derive from the inequality

$$\int_{B_1} |\nabla u|^2 \, dx - \frac{(N-2)^2}{4} \int_{B_1} |x|^{-2} |u|^2 \, dx \geq \frac{1}{4} \int_{B_1} |x|^{-2} |\log|x||^{-2} |u|^2 \, dx, \quad (1.3)$$

which holds for any  $u \in C_c^\infty(B_1 \setminus \{0\})$ . It is well known that the constants  $(N-2)^2/4$  and  $1/4$  are sharp and not achieved (see, e.g., [5–8] and Appendix A). Inequality (1.3) was firstly proved by Leray [9] in the lower-dimensional case  $N = 2$ .

Due to the sharpness of the constants in (1.3), a necessary and sufficient condition for the existence of nontrivial and nonnegative solutions to (1.1) is that  $\alpha \leq 1/4$  (compare with Theorem B.2 in Appendix B and with Remark 2.6).

In case  $\alpha \leq 1/4$ , we provide necessary conditions on the parameter  $\alpha$  to have the existence of nontrivial solutions satisfying suitable integrability properties.

**Theorem 1.1.** *Let  $R \in (0, 1]$  and let  $u \geq 0$  be a distributional solution to (1.1). Assume that there exists  $\gamma \leq 1$  such that*

$$u \in L^2_{\text{loc}}\left(B_R; |x|^{-2} |\log|x||^{-2\gamma} \, dx\right), \quad \alpha \geq \frac{1}{4} - (1 - \gamma)^2. \quad (1.4)$$

*Then  $u = 0$  almost everywhere in  $B_R$ .*

We remark that Theorem 1.1 is sharp, in view of the explicit counterexample in Remark 2.6.

Let us point out some consequences of Theorem 1.1. We use the Hardy-Leray inequality (1.3) to introduce the space  $\widetilde{H}_0^1(B_1)$  as the closure of  $C_c^\infty(B_1 \setminus \{0\})$  with respect to the scalar product

$$\langle u, v \rangle = \int_{B_1} \nabla u \cdot \nabla v \, dx - \frac{(N-2)^2}{4} \int_{B_1} |x|^{-2} uv \, dx \quad (1.5)$$

(see, e.g., [3]). It turns out that  $\widetilde{H}_0^1(B_1)$  strictly contains the standard Sobolev space  $H_0^1(B_1)$ , unless  $N = 2$ .

Take  $\gamma = 1$  in Theorem 1.1. Then problem (1.1) has no nontrivial and nonnegative solutions  $u \in L^2_{\text{loc}}(B_R; |x|^{-2} |\log|x||^{-2} dx)$  if  $\alpha = 1/4$ . Therefore, if in the dual space  $\widetilde{H}_0^1(B_R)'$ , a function  $u \in \widetilde{H}_0^1(B_R)$ , solves

$$\begin{aligned} -\Delta u - \frac{(N-2)^2}{4} |x|^{-2} u &\geq \frac{1}{4} |x|^{-2} |\log|x||^{-2} u \quad \text{in } B_R, \\ u &\geq 0, \end{aligned} \quad (1.6)$$

then  $u = 0$  in  $B_R$ .

Next take  $\gamma = 0$  and  $\alpha \geq -3/4$ . From Theorem 1.1 it follows that problem (1.1) has no nontrivial and nonnegative solutions  $u \in L^2_{\text{loc}}(B_R; |x|^{-2} dx)$ . In particular, if  $N \geq 3$  and if  $u \in H_0^1(B_R) \hookrightarrow L^2(B_R; |x|^{-2} dx)$  is a weak solution to

$$\begin{aligned} -\Delta u - \frac{(N-2)^2}{4} |x|^{-2} u &\geq -\frac{3}{4} |x|^{-2} |\log|x||^{-2} u \quad \text{in } B_R \\ u &\geq 0, \end{aligned} \quad (1.7)$$

then  $u = 0$  in  $B_R$ . Thus Theorem 1.1 improves some of the nonexistence results in [2] and in [4].

The case of boundary singularities has been little studied. In Section 2, we prove sharp nonexistence results for inequalities in cone-like domains in  $\mathbb{R}^N$ ,  $N \geq 1$ , having a vertex at 0. A special case concerns linear problems in half-balls. For  $R > 0$ , we let  $B_R^+ = B_R \cap \mathbb{R}_+^N$ , where  $\mathbb{R}_+^N$  is any half-space. Notice that  $B_R^+ = (0, R)$  or  $B_R^+ = (-R, 0)$  if  $N = 1$ . A necessary and sufficient condition for the existence of nonnegative and nontrivial distributional solutions to

$$-\Delta u - \frac{N^2}{4} |x|^{-2} u \geq \alpha |x|^{-2} |\log|x||^{-2} u \quad \text{in } \mathfrak{D}'(B_R^+) \quad (1.8)$$

is that  $\alpha \leq 1/4$  (see Theorem B.3 and Remark 3.3), and the following result holds.

**Theorem 1.2.** *Let  $R \in (0, 1]$ ,  $N \geq 1$ , and let  $u \geq 0$  be a distributional solution to (1.8). Assume that there exists  $\gamma \leq 1$  such that*

$$u \in L^2(B_R^+; |x|^{-2} |\log|x||^{-2\gamma} dx), \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2. \quad (1.9)$$

Then  $u = 0$  almost everywhere in  $B_R^+$ .

The key step in our proofs consists in studying the ordinary differential inequality

$$\begin{aligned} -\psi'' &\geq \alpha s^{-2} \psi \quad \text{in } \mathfrak{D}'(a, \infty), \\ \psi &\geq 0, \end{aligned} \quad (1.10)$$

where  $a > 0$ . In our crucial Theorem 2.3, we prove a nonexistence result for (1.10), under suitable weighted integrability assumptions on  $\psi$ . Secondly, thanks to an “averaged Emden-Fowler transform”, we show that distributional solutions to problems of the form (1.1) and (1.8) give rise to solutions of (1.10); see Sections 2.2 and 3, respectively. Our main existence results readily follow from Theorem 2.3. A similar idea, but with a different functional change, was already used in [10] to obtain nonexistence results for a large class of superlinear problems.

In Appendix A, we give a simple proof of the Hardy-Leray inequality for maps with support in cone-like domains that includes (1.3) and that motivates our interest in problem (1.8).

Appendix B deals in particular with the case  $\alpha > 1/4$ . The nonexistence Theorems B.2 and B.3 follow from an Allegretto-Piepenbrink type result (Lemma B.1).

In the last appendix, we point out some related results and some consequences of our main theorems.

*Notation 1.* We denote by  $\mathbb{R}_+$  the half real line  $(0, \infty)$ . For  $a > 0$ , we put  $I_a = (a, \infty)$ . We denote by  $|\Omega|$  the Lebesgue measure of the domain  $\Omega \subset \mathbb{R}^N$ . Let  $q \in [1, +\infty)$  and let  $\omega$  be a nonnegative measurable function on  $\Omega$ . The weighted Lebesgue space  $L^q(\Omega; \omega(x)dx)$  is the space of measurable maps  $u$  in  $\Omega$  with finite norm  $(\int_{\Omega} |u|^q \omega(x)dx)^{1/q}$ . For  $\omega \equiv 1$  we simply write  $L^q(\Omega)$ . We embed  $L^q(\Omega; \omega(x)dx)$  into  $L^q(\mathbb{R}^N; \omega(x)dx)$  via null extension.

## 2. Proof of Theorem 1.1

The proof consists of two steps. In the first one, we prove a nonexistence result for a class of linear ordinary differential inequalities that might have some interest in itself.

### 2.1. Nonexistence Results for Problem (1.10)

We start by fixing some terminologies. Let  $\mathfrak{D}^{1,2}(\mathbb{R}_+)$  be the Hilbert space obtained via the Hardy inequality

$$\int_0^{\infty} |v'|^2 ds \geq \frac{1}{4} \int_0^{\infty} s^{-2} |v|^2 ds, \quad v \in C_c^{\infty}(\mathbb{R}_+) \quad (2.1)$$

as the completion of  $C_c^{\infty}(\mathbb{R}_+)$  with respect to the scalar product

$$\langle v, w \rangle = \int_0^{\infty} v' w' ds. \quad (2.2)$$

Notice that  $\mathfrak{D}^{1,2}(\mathbb{R}_+) \hookrightarrow L^2(\mathbb{R}_+; s^{-2} ds)$  with a continuous embedding and moreover  $\mathfrak{D}^{1,2}(\mathbb{R}_+) \subset C^0(\mathbb{R}_+)$  by Sobolev embedding theorem. By Hölder inequality, the space  $L^2(\mathbb{R}_+; s^2 ds)$  is continuously embedded into the dual space  $\mathfrak{D}^{1,2}(\mathbb{R}_+)'$ .

Finally, for any  $a > 0$  we put  $I_a = (a, \infty)$  and

$$\mathfrak{D}^{1,2}(I_a) = \left\{ v \in \mathfrak{D}^{1,2}(\mathbb{R}_+) \mid v(a) = 0 \right\}. \quad (2.3)$$

We need two technical lemmata.

**Lemma 2.1.** *Let  $f \in L^2(I_a; s^2 ds)$  and  $v \in C^2(\mathbb{R}_+) \cap L^2(I_a; s^{-2} ds)$  be a function satisfying  $v(a) = 0$  and*

$$-v'' \leq f \quad \text{in } I_a. \tag{2.4}$$

Put  $v^+ := \max\{v, 0\}$ . Then  $v^+ \in \mathfrak{D}^{1,2}(I_a)$  and

$$\int_a^\infty |(v^+)'|^2 ds \leq \int_a^\infty f v^+ ds. \tag{2.5}$$

*Proof.* We first show that  $(v^+)' \in L^2(\mathbb{R})$  and that (2.5) holds. Let  $\eta \in C_c^\infty(\mathbb{R})$  be a cutoff function satisfying

$$0 \leq \eta \leq 1, \quad \eta(s) \equiv 1 \quad \text{for } |s| \leq 1, \quad \eta(s) \equiv 0 \quad \text{for } s \geq 2, \tag{2.6}$$

and put  $\eta_h(s) = \eta(s/h)$ . Then  $\eta_h v^+ \in \mathfrak{D}^{1,2}(I_a)$  and  $\eta_h v^+ \geq 0$ . Multiply (2.4) by  $\eta_h v^+$  and integrate by parts to get

$$\int_a^\infty \eta_h |(v^+)'|^2 ds - \frac{1}{2} \int_a^\infty \eta_h'' |v^+|^2 ds \leq \int_a^\infty \eta_h f v^+ ds. \tag{2.7}$$

Notice that for some constant  $c$  depending only on  $\eta$  it results that

$$\left| \int_a^\infty \eta_h'' |v^+|^2 ds \right| \leq c \int_h^{2h} s^{-2} |v^+|^2 ds \longrightarrow 0 \tag{2.8}$$

as  $h \rightarrow \infty$ , since  $v^+ \in L^2(I_a; s^{-2} ds)$ . Moreover,

$$\int_a^\infty \eta_h f v^+ ds \longrightarrow \int_a^\infty f v^+ ds \tag{2.9}$$

by Lebesgue theorem, as  $f v^+ \in L^1(I_a)$  by Hölder inequality. In conclusion, from (2.7) we infer that

$$\int_a^h |(v^+)'|^2 ds \leq \int_a^\infty f v^+ ds + o(1) \tag{2.10}$$

since  $\eta_h \equiv 1$  on  $(a, h)$ . By Fatou's Lemma, we get that  $(v^+) \in L^2(I_a)$  and (2.5) readily follows from (2.10). To prove that  $v^+ \in \mathfrak{D}^{1,2}(I_a)$ , it is enough to notice that  $\eta_h v^+ \rightarrow v^+$  in  $\mathfrak{D}^{1,2}(I_a)$ . Indeed,

$$\begin{aligned} \int_a^\infty |1 - \eta_h|^2 |(v^+)'|^2 &\leq \int_h^\infty |(v^+)'|^2 ds = o(1), \\ \int_a^\infty |\eta_h'|^2 |v^+|^2 ds &\leq c \int_h^\infty s^{-2} |v^+|^2 ds = o(1), \end{aligned} \quad (2.11)$$

as  $(v^+) \in L^2(I_a)$  and  $v^+ \in L^2(I_a; s^{-2} ds)$ .  $\square$

Through the paper, we let  $(\rho_n)$  be a standard mollifier sequence in  $\mathbb{R}$ , such that the support of  $\rho_n$  is contained in the interval  $(-1/n, 1/n)$ .

**Lemma 2.2.** *Let  $a > 0$  and  $\psi \in L^2(I_a; s^{-2} ds)$ . Then  $\rho_n \star \psi \in L^2(I_a; s^{-2} ds)$  and*

$$\rho_n \star \psi \rightarrow \psi \quad \text{in } L^2(I_a; s^{-2} ds), \quad (2.12)$$

$$g_n := \rho_n \star (s^{-2} \psi) - s^{-2} (\rho_n \star \psi) \rightarrow 0 \quad \text{in } L^2(I_a; s^2 ds). \quad (2.13)$$

*Proof.* We start by noticing that  $\rho_n \star \psi \rightarrow \psi$  almost everywhere. Then we use Hölder inequality to get

$$\begin{aligned} s^{-2} |(\rho_n \star \psi)(s)|^2 &= s^{-2} \left| \int \rho_n(s-t)^{1/2} \rho_n(s-t)^{1/2} \psi(t) dt \right|^2 \\ &\leq s^{-2} \left( \frac{1}{n} + s \right)^2 \int \rho_n(s-t) t^{-2} |\psi(t)|^2 dt \\ &\leq \left( 1 + \frac{1}{na} \right)^2 |(\rho_n \star (s^{-2} \psi^2))(s)| \end{aligned} \quad (2.14)$$

for any  $s > a > 0$ . Since  $s^{-2} \psi^2 \in L^1(I_a)$ , then  $\rho_n \star (s^{-2} \psi^2) \rightarrow s^{-2} \psi^2$  in  $L^1(I_a)$ . Thus  $s^{-1} (\rho_n \star \psi) \rightarrow s^{-1} \psi$  in  $L^2(I_a)$  by the (generalized) Lebesgue Theorem, and (2.12) follows.

To prove (2.13), we first argue as before to check that

$$s^2 \left| \int \rho_n(s-t) t^{-2} \psi(t) dt \right|^2 \leq \left( 1 - \frac{1}{na} \right)^{-2} |(\rho_n \star (s^{-2} \psi^2))(s)| \quad (2.15)$$

for any  $s > a > 0$ . Thus  $\rho_n \star (s^{-2} \psi)$  converges to  $s^{-2} \psi$  in  $L^2(I_a; s^2 ds)$  by Lebesgue's Theorem. In addition,  $s^{-2} (\rho_n \star \psi) \rightarrow s^{-2} \psi$  in  $L^2(I_a; s^2 ds)$  by (2.12). Thus  $g_n \rightarrow 0$  in  $L^2(I_a; s^2 ds)$  and the Lemma is completely proved.  $\square$

The following result for solutions to (1.10) is a crucial step in the proofs of our main theorems.

**Theorem 2.3.** Let  $a > 0$  and let  $\psi$  be a distributional solution to (1.10). Assume that there exists  $\gamma \leq 1$  such that

$$\psi \in L^2(I_a; s^{-2\gamma} ds), \quad \alpha \geq \frac{1}{4} - (1 - \gamma)^2. \quad (2.16)$$

Then  $\psi = 0$  almost everywhere in  $I_a$ .

*Proof.* We start by noticing that  $L^2(I_a; s^{-2\gamma} ds) \hookrightarrow L^2(I_a; s^{-2} ds)$  with a continuous embedding for any  $\gamma < 1$ . In addition, we point out that we can assume

$$\alpha = \frac{1}{4} - (1 - \gamma)^2. \quad (2.17)$$

Let  $\rho_n$  be a standard sequence of mollifiers, and let

$$\psi_n = \rho_n \star \psi, \quad g_n = \rho_n \star (s^{-2}\psi) - s^{-2}(\rho_n \star \psi). \quad (2.18)$$

Then  $\psi_n \rightarrow \psi$  in  $L^2(I_a; s^{-2\gamma} ds)$  and almost everywhere, and  $g_n \rightarrow 0$  in  $L^2(I_a; s^2 ds)$  by Lemma 2.2. Moreover,  $\psi_n \in C^\infty(\bar{I}_a)$  is a nonnegative solution to

$$-\psi_n'' \geq \alpha s^{-2}\psi_n + \alpha g_n \quad \text{in } \mathfrak{D}'(I_a). \quad (2.19)$$

We assume by contradiction that  $\psi \neq 0$ . We let  $s_0 \in I_a$  such that  $\varepsilon_n := \psi_n(s_0) \rightarrow \psi(s_0) > 0$ . Up to a scaling and after replacing  $g_n$  with  $s_0^2 g_n$ , we may assume that  $s_0 = 1$ . We will show that

$$\varepsilon_n := \psi_n(1) \rightarrow \psi(1) > 0 \quad (2.20)$$

leads to a contradiction. We fix a parameter

$$\delta > \frac{1}{2} - \gamma \geq -\frac{1}{2} \quad (2.21)$$

and for  $n$  large we put

$$\varphi_{\delta,n}(s) := \varepsilon_n s^{-\delta} \in L^2(I_1; s^{-2\gamma} ds). \quad (2.22)$$

Clearly,  $\varphi_{\delta,n} \in C^\infty(\mathbb{R}_+)$  and one easily verifies that  $(\varphi_{\delta,n})_n$  is a bounded sequence in  $L^2(I_1; s^{-2\gamma} ds)$  by (2.20) and (2.21). Finally, we define

$$v_{\delta,n} = \varphi_{\delta,n} - \psi_n = \varepsilon_n s^{-\delta} - \psi_n, \quad (2.23)$$

so that  $v_{\delta,n} \in L^2(I_1; s^{-2\gamma} ds)$  and  $v_{\delta,n}(1) = 0$ . In addition,  $v_{\delta,n}$  solves

$$-v_{\delta,n}'' \leq \alpha s^{-2}v_{\delta,n} - c_\delta \varepsilon_n s^{-2-\delta} - \alpha g_n \quad \text{in } I_1, \quad (2.24)$$

where  $c_\delta := \delta(\delta + 1) + \alpha = \delta(\delta + 1) + (1/4) - (1 - \gamma)^2$ . Notice that  $c_\delta > 0$  and that all the terms in the right-hand side of (2.24) belong to  $L^2(I_1; s^2 ds)$ , by (2.21). Thus Lemma 2.1 gives  $v_{\delta,n}^+ \in \mathfrak{D}^{1,2}(I_1)$  and

$$\int_1^\infty \left| (v_{\delta,n}^+)' \right|^2 ds \leq \alpha \int_1^\infty s^{-2} |v_{\delta,n}^+|^2 ds - c_\delta \varepsilon_n \int_1^\infty s^{-2-\delta} v_{\delta,n}^+ ds + o(1), \quad (2.25)$$

since  $v_{\delta,n}^+$  is bounded in  $L^2(I_1; s^{-2} ds)$  and  $g_n \rightarrow 0$  in  $L^2(I_1; s^2 ds)$ . By (2.17) and Hardy's inequality (2.1), we conclude that

$$(1 - \gamma)^2 \int_1^\infty s^{-2} |v_{\delta,n}^+|^2 ds + c_\delta \varepsilon_n \int_1^\infty s^{-2-\delta} v_{\delta,n}^+ ds = o(1). \quad (2.26)$$

Thus, for any fixed  $\delta$  we get that  $v_{\delta,n}^+ \rightarrow 0$  almost everywhere in  $I_1$  as  $n \rightarrow \infty$ , since  $\varepsilon_n c_\delta$  is bounded away from 0 by (2.20). Finally, we notice that

$$\varphi_n = \varphi_{\delta,n} - v_{\delta,n} \geq \varepsilon_n s^{-\delta} - v_{\delta,n}^+. \quad (2.27)$$

Since  $\varphi_n \rightarrow \varphi$  and  $v_{\delta,n}^+ \rightarrow 0$  almost everywhere in  $I_1$ , and since  $\varepsilon_n \rightarrow \varphi(1) > 0$ , we infer that

$$\varphi \geq \varphi(1) s^{-\delta} \quad \text{almost everywhere in } I_1. \quad (2.28)$$

This conclusion contradicts the assumption  $\varphi \in L^2(I_1; s^{-2\gamma} ds)$ , as  $\delta > 1/2 - \gamma$  was arbitrarily chosen. Thus (2.20) cannot hold and the proof is complete.  $\square$

*Remark 2.4.* If  $\alpha > 1/4$ , then every nonnegative solution  $\varphi \in L^1_{\text{loc}}(I_a)$  to problem (1.10) vanishes. This is an immediate consequence of Lemma B.1 in Appendix B and the sharpness of the constant  $1/4$  in the Hardy inequality (2.1).

*Remark 2.5.* Consider the characteristic equation of the ordinary differential equation (1.10):

$$\delta(\delta + 1) + \alpha = 0. \quad (2.29)$$

For  $\alpha \leq 1/4$ , let

$$\delta_\alpha = \frac{\sqrt{1 - 4\alpha} - 1}{2} \quad (2.30)$$

be the largest root of the above equation. Then it is not difficult to see that the proof of Theorem 2.3 highlights that

$$\varphi(s) \geq cs^{-\delta_\alpha} \quad \text{in } I_a, \quad (2.31)$$

for some constant  $c > 0$ . Moreover, one can easily verify that the function  $s \mapsto s^{-\delta_\alpha}$  belongs to  $L^2(I_a; s^{-2\gamma} ds)$  if and only if  $\alpha < (1/4) - (1 - \gamma)^2$ .



## 2.2. Conclusion of the Proof

We will show that any nonnegative distributional solution  $u$  to problem (1.1) gives rise to a function  $\varphi$  solving (1.10), and such that  $\varphi = 0$  if and only if  $u = 0$ . To this aim, we introduce the Emden-Fowler transform  $u \mapsto Tu$  by letting

$$u(x) = |x|^{(2-N)/2} (Tu) \left( |\log|x||, \frac{x}{|x|} \right). \quad (2.32)$$

By change of variable formula, for any  $R' \in (0, R)$ , it results that

$$\int_{B_{R'}} |x|^{-2} |\log|x||^{-2\gamma} |u|^2 dx = \int_{|\log R'|}^{\infty} \int_{\mathbb{S}^{N-1}} s^{-2\gamma} |Tu|^2 ds d\sigma, \quad (2.33)$$

so that  $Tu \in L^2(I_a \times \mathbb{S}^{N-1}; s^{-2\gamma} ds d\sigma)$  for any  $a > a_R := |\log R|$ . Now, for an arbitrary  $\varphi \in C_c^\infty(I_{a_R})$  we define the radially symmetric function  $\tilde{\varphi} \in C_c^\infty(B_R)$  by setting

$$\tilde{\varphi}(x) = |x|^{(2-N)/2} \varphi(|\log|x||), \quad (2.34)$$

so that  $\varphi = T\tilde{\varphi}$ . By direct computations, we get

$$\begin{aligned} \int_{B_R} u \left( \Delta \tilde{\varphi} + \frac{(N-2)^2}{4} |x|^{-2} \tilde{\varphi} \right) dx &= \int_{a_R}^{\infty} \varphi'' \int_{\mathbb{S}^{N-1}} Tu d\sigma ds, \\ \int_{B_R} |x|^{-2} |\log|x||^{-2} u \tilde{\varphi} dx &= \int_{a_R}^{\infty} s^{-2\gamma} \varphi \int_{\mathbb{S}^{N-1}} Tu d\sigma ds. \end{aligned} \quad (2.35)$$

Thus we are led to introduce the function  $\psi$  defined in  $I_{a_R}$  by setting

$$\psi(s) = \int_{\mathbb{S}^{N-1}} (Tu)(s, \sigma) d\sigma. \quad (2.36)$$

We notice that  $\psi \in L^2(I_a; s^{-2\gamma} ds)$  for any  $a > a_R$ , since

$$\int_a^\infty s^{-2\gamma} |\psi|^2 ds \leq |\mathbb{S}^{N-1}| \int_a^\infty \int_{\mathbb{S}^{N-1}} s^{-2\gamma} |Tu|^2 ds d\sigma \quad (2.37)$$

by Hölder inequality. Moreover, from (2.35) it immediately follows that  $\psi \geq 0$  is a distributional solution to

$$-\psi'' \geq \alpha s^{-2} \psi \quad \text{in } \mathfrak{D}'(I_{a_R}). \quad (2.38)$$

By Theorem 2.3, we infer that  $\psi = 0$  in  $I_{a_R}$ , and hence  $u = 0$  in  $B_R$ . The proof of Theorem 1.1 is complete.

*Remark 2.6.* The assumptions on the integrability of  $u$  in Theorem 1.1 are sharp. If  $\alpha > 1/4$ , use the results in Appendix B. For  $\alpha \leq 1/4$ , let  $\delta_\alpha$  be defined in (2.30) and notice that the function  $u_\alpha \in L^1_{\text{loc}}(B_1)$  defined by

$$u_\alpha(x) = |x|^{(2-N)/2} |\log|x||^{-\delta_\alpha} \quad (2.39)$$

solves

$$-\Delta u_\alpha - \frac{(N-2)^2}{4} |x|^{-2} u_\alpha = \alpha |x|^{-2} |\log|x||^{-2} u_\alpha \quad \text{in } \mathfrak{D}'(B_1 \setminus \{0\}). \quad (2.40)$$

Moreover, if  $\gamma \leq 1$  then

$$u_\alpha \in L^2_{\text{loc}}(B_1; |x|^{-2} |\log|x||^{-2\gamma} dx) \quad \text{iff } \alpha < \frac{1}{4} - (1-\gamma)^2. \quad (2.41)$$

Finally we notice that, by Remark 2.5, for every solution  $u \in L^1_{\text{loc}}(B_R \setminus \{0\})$ , there exists a constant  $c > 0$  such that

$$\int_{\mathbb{S}^{N-1}} u(r\sigma) d\sigma \geq cu_\alpha(r\sigma) \quad \text{for a.e. } r \in (0, R). \quad (2.42)$$

### 3. Cone-Like Domains

Let  $N \geq 2$ . To any Lipschitz domain  $\Sigma \subset \mathbb{S}^{N-1}$ , we associate the *cone*

$$\mathcal{C}_\Sigma := \{r\sigma \in \mathbb{R}^N \mid \sigma \in \Sigma, r > 0\}. \quad (3.1)$$

For any given  $R > 0$ , we introduce also the *cone-like domain*

$$\mathcal{C}_\Sigma^R := \mathcal{C}_\Sigma \cap B_R = \{r\sigma \in \mathbb{R}^N \mid r \in (0, R), \sigma \in \Sigma\}. \quad (3.2)$$

Notice that  $\mathcal{C}_{\mathbb{S}^{N-1}} = \mathbb{R}^N \setminus \{0\}$  and  $\mathcal{C}_{\mathbb{S}^{N-1}}^R = B_R \setminus \{0\}$ . If  $\Sigma$  is an half-sphere  $\mathbb{S}_+^{N-1}$ , the  $\mathcal{C}_{\mathbb{S}_+^{N-1}}$  is an half-space  $\mathbb{R}_+^N$  and  $\mathcal{C}_{\mathbb{S}_+^{N-1}}^R$  is a half-ball  $B_{R'}^+$ , as in Theorem 1.2.

Assume that  $\Sigma$  is properly contained in  $\mathbb{S}^{N-1}$ . Then we let  $\lambda_1(\Sigma) > 0$  be the principal eigenvalue of the Laplace operator on  $\Sigma$ . If  $\Sigma = \mathbb{S}^{N-1}$ , we put  $\lambda_1(\mathbb{S}^{N-1}) = 0$ .

It has been noticed in [11, 12], that

$$\mu(\mathcal{C}_\Sigma) := \inf_{\substack{u \in C_c^\infty(\mathcal{C}_\Sigma) \\ u \neq 0}} \frac{\int_{\mathcal{C}_\Sigma} |\nabla u|^2 dx}{\int_{\mathcal{C}_\Sigma} |x|^{-2} |u|^2 dx} = \frac{(N-2)^2}{4} + \lambda_1(\Sigma). \quad (3.3)$$

The infimum  $\mu(\mathcal{C})$  is the best constant in the Hardy inequality for maps having compact support in  $\mathcal{C}_\Sigma$ . In particular, for any half-space  $\mathbb{R}_+^N$ , it holds that

$$\mu(\mathbb{R}_+^N) = \frac{N^2}{4}. \tag{3.4}$$

The aim of this section is to study the elliptic inequality

$$-\Delta u - \mu(\mathcal{C}_\Sigma)|x|^{-2}u \geq \alpha|x|^{-2}|\log|x||^{-2}u \quad \text{in } \mathfrak{D}'(\mathcal{C}_\Sigma^R). \tag{3.5}$$

Notice that (3.5) reduces to (1.1) if  $\Sigma = \mathbb{S}^{N-1}$ . Problem (3.5) is related to an improved Hardy inequality for maps supported in cone-like domains which will be discussed in Appendix A.

**Theorem 3.1.** *Let  $\Sigma$  be a Lipschitz domain properly contained in  $\mathbb{S}^{N-1}$ ,  $R \in (0, 1]$ , and let  $u \geq 0$  be a distributional solution to (3.5). Assume that there exists  $\gamma \leq 1$  such that*

$$u \in L^2(\mathcal{C}_{\Sigma}^R; |x|^{-2}|\log|x||^{-2\gamma} dx), \quad \alpha \geq \frac{1}{4} - (1 - \gamma)^2. \tag{3.6}$$

Then  $u = 0$  almost everywhere in  $\mathcal{C}_{\Sigma}^R$ .

*Proof.* Let  $\Phi \in C^2(\Sigma) \cap C(\bar{\Sigma})$  be the positive eigenfunction of the Laplace-Beltrami operator  $-\Delta_\sigma$  in  $\Sigma$  defined by

$$\begin{aligned} -\Delta_\sigma \Phi &= \lambda_1(\Sigma)\Phi \quad \text{in } \Sigma, \\ \Phi &= 0 \quad \text{on } \partial\Sigma. \end{aligned} \tag{3.7}$$

Let  $u \in L^2(\mathcal{C}_{\Sigma}^R; |x|^{-2}|\log|x||^{-2\gamma} dx)$  be as in the statement, and put  $a_R = |\log R|$ . We let  $Tu \in L^2(I_{a_R} \times \Sigma; s^{-2\gamma} ds d\sigma)$  be the Emden-Fowler transform, as in (2.32). We further let  $\psi \in L^2(I_{a_R}; s^{-2\gamma} ds)$  defined as

$$\psi(s) = \int_{\Sigma} (Tu)(s, \sigma)\Phi(\sigma)d\sigma. \tag{3.8}$$

Next, for  $\varphi \in C_c^\infty(I_{a_R})$  being an arbitrary nonnegative test function, we put

$$\tilde{\varphi}(x) = |x|^{(2-N)/2}\varphi(|\log|x||)\Phi\left(\frac{x}{|x|}\right). \tag{3.9}$$

In essence, our aim is to test (3.5) with  $\tilde{\varphi}$  to prove that  $\varphi$  satisfies (1.10) in  $I_{a_R}$ . To be more rigorous, we use a density argument to approximate  $\Phi$  in  $W^{2,2}(\Sigma) \cap H_0^1(\Sigma)$  by a sequence

of smooth maps  $\Phi_n \in C_c^\infty(\Sigma)$ . Then we define  $\tilde{\varphi}_n$  accordingly with (3.9), in such a way that  $T\tilde{\varphi}_n = \varphi\Phi_n$ . By direct computation, we get

$$\begin{aligned} \int_{\mathcal{C}_\Sigma^R} u \left( \Delta\tilde{\varphi}_n + \frac{(N-2)^2}{4}|x|^{-2}\tilde{\varphi}_n \right) dx &= \int_{a_R}^\infty \int_\Sigma (Tu)\varphi''\Phi_n d\sigma ds \\ &\quad + \int_{a_R}^\infty \int_\Sigma (Tu)\varphi\Delta_\sigma\Phi_n d\sigma ds, \\ \lambda_1(\Sigma) \int_{\mathcal{C}_\Sigma^R} |x|^{-2}u\tilde{\varphi}_n dx &= \lambda_1(\Sigma) \int_{a_R}^\infty \int_\Sigma (Tu)\varphi\Phi_n d\sigma ds, \\ \int_{\mathcal{C}_\Sigma^R} |x|^{-2}|\log|x||^{-2}u\tilde{\varphi}_n dx &= \int_{a_R}^\infty \int_\Sigma s^{-2}(Tu)\varphi\Phi_n d\sigma ds. \end{aligned} \tag{3.10}$$

Since  $\tilde{\varphi}_n \in C_c^\infty(\mathcal{C}_\Sigma^R)$  is an admissible test function for (3.5), using also (3.3) we get

$$\begin{aligned} - \int_{a_R}^\infty \int_\Sigma (Tu)\varphi''\Phi_n d\sigma ds &\geq \alpha \int_{a_R}^\infty \int_\Sigma s^{-2}(Tu)\varphi\Phi_n d\sigma ds \\ &\quad - \int_{a_R}^\infty \int_\Sigma (Tu)\varphi(\Delta_\sigma\Phi_n + \lambda_1(\Sigma)\Phi_n) d\sigma ds. \end{aligned} \tag{3.11}$$

Since  $\Phi_n \rightarrow \Phi$  and  $\Delta_\sigma\Phi_n + \lambda_1(\Sigma)\Phi_n \rightarrow 0$  in  $L^2(\Sigma)$ , we conclude that

$$- \int_{a_R}^\infty \varphi''\varphi ds \geq \alpha \int_{a_R}^\infty s^{-2}\varphi\varphi ds. \tag{3.12}$$

By the arbitrariness of  $\varphi$ , we can conclude that  $\varphi$  is a distributional solution to (1.10). Theorem 2.3 applies to give  $\varphi \equiv 0$ , that is,  $u \equiv 0$  in  $\mathcal{C}_\Sigma^R$ .  $\square$

The next result extends Theorem 3.1 to cover the case  $N = 1$ . Notice that  $\mathbb{R}_+ = (0, \infty)$  is a cone and  $(0, 1)$  is a cone-like domain in  $\mathbb{R}$ .

**Theorem 3.2.** *Let  $R \in (0, 1]$  and let  $u \geq 0$  be a distributional solution to*

$$-u'' - \frac{1}{4}t^{-2}u \geq \alpha t^{-2}|\log t|^{-2}u \quad \text{in } \mathfrak{D}'(0, R). \tag{3.13}$$

*Assume that there exists  $\gamma \leq 1$  such that*

$$u \in L^2\left((0, R); t^{-2}|\log t|^{-2\gamma} dt\right), \quad \alpha \geq \frac{1}{4} - (1 - \gamma)^2. \tag{3.14}$$

*Then  $u = 0$  almost everywhere in  $(0, R)$ .*

*Proof.* Write  $u(t) = t^{1/2}\psi(|\log t|) = t^{1/2}\psi(s)$  for a function  $\psi \in L^2(I_{a_R}; s^{-2\gamma} ds)$  and then notice that  $\psi$  is a distributional solution to

$$-\psi'' \geq \alpha s^{-2}\psi \quad \text{in } \mathfrak{D}'(I_{a_R}). \tag{3.15}$$

The conclusion readily follows from Theorem 2.3. □

*Remark 3.3.* If  $\alpha > 1/4$ , then every nonnegative solution  $u \in L^1_{\text{loc}}(\mathbb{C}^R_\Sigma)$  to problem (3.5) vanishes by Theorem B.3.

In case  $\alpha \leq 1/4$ , the assumptions on  $\alpha$  and on the integrability of  $u$  in Theorems 3.1 and 3.2 are sharp. Fix  $\alpha \leq 1/4$ , let  $\delta_\alpha$  be defined in (2.30) and define the function

$$u_\alpha(r\sigma) = r^{(2-N)/2} |\log r|^{-\delta_\alpha} \Phi(\sigma). \tag{3.16}$$

Here  $\Phi$  solves (3.7) if  $N \geq 2$ . If  $N = 1$ , we agree that  $\sigma = 1$  and  $\Phi \equiv 1$ . By direct computations, one has that  $u_\alpha$  solves (3.5). Moreover, if  $\gamma \leq 1$  and  $R \in (0, 1)$  then  $u_\alpha \in L^2(\mathbb{C}^R_\Sigma; |x|^{-2} |\log|x||^{-2\gamma} dx)$  if and only if  $\alpha < (1/4) - (1 - \gamma)^2$ .

*Remark 3.4.* Nonexistence results for linear inequalities involving the differential operator  $-\Delta - \mu(\mathbb{C}_\sigma)|x|^{-2}$  were already obtained in [12].

## Appendices

### A. Hardy-Leray Inequalities on Cone-Like Domains

In this appendix, we give a simple proof of an improved Hardy inequality for mappings having support in a cone-like domain. We recall that for  $\Sigma \subset \mathbb{S}^{N-1}$  we have set  $\mathbb{C}^1_\Sigma = \{r\sigma \mid r \in (0, 1), \sigma \in \Sigma\}$  and that  $\mu(\mathbb{C}_\Sigma) = (N - 2)^2/4 + \lambda_1(\Sigma)$ .

**Proposition A.1.** *Let  $\Sigma$  be a domain in  $\mathbb{S}^{N-1}$ , with  $N \geq 2$ . Then*

$$\int_{\mathbb{C}^1_\Sigma} |\nabla u|^2 dx - \mu(\mathbb{C}_\Sigma) \int_{\mathbb{C}^1_\Sigma} |x|^{-2} |u|^2 \geq \frac{1}{4} \int_{\mathbb{C}^1_\Sigma} |x|^{-2} |\log|x||^{-2} |u|^2 dx \tag{A.1}$$

for any  $u \in C^\infty_c(\mathbb{C}^1_\Sigma)$ .

*Proof.* We start by fixing an arbitrary function  $v \in C^\infty_c(\mathbb{R}_+ \times \Sigma)$ . We apply the Hardy inequality to the function  $v(\cdot, \sigma) \in C^\infty_c(\mathbb{R}_+)$ , for any fixed  $\sigma \in \Sigma$ , and then we integrate over  $\Sigma$  to get

$$\int_0^\infty \int_\Sigma |v_s|^2 ds d\sigma \geq \frac{1}{4} \int_0^\infty \int_\Sigma s^{-2} |v|^2 ds d\sigma. \tag{A.2}$$

In addition, notice that  $v(s, \cdot) \in C_c^\infty(\Sigma)$  for any  $s \in \mathbb{R}_+$ . Thus, the Poincaré inequality for maps in  $\Sigma$  plainly implies

$$\int_0^\infty \int_\Sigma |\nabla_\sigma v|^2 ds d\sigma - \lambda_1(\Sigma) \int_0^\infty \int_\Sigma |v|^2 ds d\sigma \geq 0. \quad (\text{A.3})$$

Adding these two inequalities, we conclude that

$$\int_0^\infty \int_\Sigma [ |v_s|^2 + |\nabla_\sigma v|^2 ] ds d\sigma - \lambda_1(\Sigma) \int_0^\infty \int_\Sigma |v|^2 ds d\sigma \geq \frac{1}{4} \int_0^\infty \int_\Sigma s^{-2} |v|^2 ds d\sigma \quad (\text{A.4})$$

for any  $v \in C_c^\infty(\mathbb{R}_+ \times \Sigma)$ . We use once more the Emden-Fowler transform  $T$  in (2.32) by letting  $v := Tu \in C_c^\infty(\mathbb{R}_+ \times \Sigma)$  for  $u \in C_c^\infty(\mathcal{C}_\Sigma^1)$ . Since

$$\int_{\mathcal{C}_\Sigma^1} \left[ |\nabla u|^2 - \frac{(N-2)^2}{4} |x|^{-2} |u|^2 \right] dx = \int_0^\infty \int_{\mathbb{S}^{N-1}} [ |v_s|^2 + |\nabla_\sigma v|^2 ] ds d\sigma, \quad (\text{A.5})$$

then (2.33) readily leads to the conclusion.  $\square$

*Remark A.2.* The arguments we have used to prove Proposition A.1 and the fact that the best constant in the Hardy inequality for maps in  $C_c^\infty(\mathbb{R}_+)$  is not achieved show that the constants in inequality (A.1) are sharp, and not achieved.

*Remark A.3.* Notice that for  $N \geq 1$ , we have  $\mathcal{C}_{\mathbb{S}^{N-1}} = \mathbb{R}^N \setminus \{0\}$  and  $\mu(\mathcal{C}_{\mathbb{S}^{N-1}}) = (N-2)^2/4$ . Thus (A.1) gives (1.3) for  $u \in C_c^\infty(B_1 \setminus \{0\})$ .

In the next proposition, we extend the inequality (A.1) to cover the case  $N = 1$ .

**Proposition A.4.** *It holds that*

$$\int_0^1 |u'|^2 dt - \frac{1}{4} \int_0^1 t^{-2} |u|^2 dt \geq \frac{1}{4} \int_0^1 t^{-2} |\log t|^{-2} |u|^2 dt \quad (\text{A.6})$$

for any  $u \in C_c^\infty(0, 1)$ . The constants are sharp, and not achieved.

*Proof.* Write  $u(t) = t^{1/2} \psi(|\log t|) = t^{1/2} \psi(s)$  for a function  $\psi \in C_c^\infty(\mathbb{R}_+)$  and then apply the Hardy inequality to  $\psi$ .  $\square$

Next, let  $\theta \in \mathbb{R}$  be a given parameter and let  $\Sigma$  be a Lipschitz domain in  $\mathbb{S}^{N-1}$ , with  $N \geq 2$ . For an arbitrary  $u \in C_c^\infty(\mathcal{C}_\Sigma^1)$ , we put  $v = |x|^{-\theta/2} u$ . Then the Hardy-Leray inequality (A.1) and integration by parts plainly imply that

$$\int_{\mathcal{C}_\Sigma^1} |x|^\theta |\nabla v|^2 dx - \mu(\mathcal{C}_\Sigma; \theta) \int_{\mathcal{C}_\Sigma^1} |x|^{\theta-2} |v|^2 \geq \frac{1}{4} \int_{\mathcal{C}_\Sigma^1} |x|^{\theta-2} |\log|x||^{-2} |v|^2 dx \quad (\text{A.7})$$

for any  $v \in C_c^\infty(C_\Sigma^1)$ , where

$$\mu(C_\Sigma; \theta) := \frac{(N - 2 + \theta)^2}{4} + \lambda_1(\Sigma). \tag{A.8}$$

It is well known that

$$\frac{(N - 2 + \theta)^2}{4} = \inf_{\substack{u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \\ u \neq 0}} \frac{\int_{B_1} |x|^\theta |\nabla u|^2 dx}{\int_{B_1} |x|^{\theta-2} |u|^2 dx} \tag{A.9}$$

is the Hardy constant relative to the operator  $L_\theta v = -\operatorname{div}(|x|^\theta \nabla v)$ . For the case  $N = 1$ , one can obtain in a similar way the inequality

$$\int_0^1 t^\theta |v'|^2 dt - \frac{(\theta - 1)^2}{4} \int_0^1 t^{\theta-2} |v|^2 dt \geq \frac{1}{4} \int_0^1 t^{\theta-2} |\log t|^{-2} |v|^2 dt \tag{A.10}$$

which holds for any  $\theta \in \mathbb{R}$  and for any  $v \in C_c^\infty(0, 1)$ .

### B. A General Necessary Condition

In this appendix, we show in particular that a necessary condition for the existence of nontrivial and nonnegative solutions to (1.1) and (3.5) is that  $\alpha \leq 1/4$ . We need the following general lemma, which naturally fits into the classical Allegretto-Piepenbrink theory (see for instance, [13, 14]).

**Lemma B.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 1$ . Let  $a \in L_{\text{loc}}^\infty(\Omega)$  and  $a > 0$  in  $\Omega$ . Assume that  $u \in L_{\text{loc}}^1(\Omega)$  is a nonnegative, nontrivial solution to*

$$-\Delta u \geq a(x)u \quad \mathfrak{D}'(\Omega). \tag{B.1}$$

Then

$$\int_\Omega |\nabla \phi|^2 dx \geq \int_\Omega a(x) |\phi|^2 dx, \quad \text{for any } \phi \in C_c^\infty(\Omega). \tag{B.2}$$

*Proof.* Let  $A \subset \Omega$  be a measurable set such that  $|A| > 0$  and  $u > 0$  in  $A$ . Fix any function  $\phi \in C_c^\infty(\Omega)$  and choose a domain  $\tilde{\Omega} \subset\subset \Omega$  such that  $|\tilde{\Omega} \cap A| > 0$  and  $\phi \in C_c^\infty(\tilde{\Omega})$ . For any integer  $k$  large enough, put  $f_k = \min\{a(x)u, k\} \in L^\infty(\tilde{\Omega})$ . Let  $v_k \in H_0^1(\tilde{\Omega})$  be the unique solution to

$$\begin{aligned} -\Delta v_k &= f_k && \text{in } \tilde{\Omega}, \\ v_k &= 0 && \text{on } \partial\tilde{\Omega}. \end{aligned} \tag{B.3}$$

Notice that  $v_k \in C^{1,\beta}(\tilde{\Omega})$  for any  $\beta \in (0, 1)$ . Since the function  $f_k$  is nonnegative and nontrivial, then  $v_k \geq 0$ . Actually it turns out that  $v_k^{-1} \in L_{\text{loc}}^\infty(\tilde{\Omega})$  by the Harnack inequality. Finally, a convolution argument and the maximum principle plainly give

$$u \geq v_k > 0 \quad \text{almost everywhere in } \tilde{\Omega}. \quad (\text{B.4})$$

Since  $v_k^{-1}\phi \in L^\infty(\tilde{\Omega})$ , then we can use  $v_k^{-1}\phi^2$  as test function for (B.3) to get

$$\int_{\Omega} \nabla v_k \cdot \nabla (v_k^{-1}\phi^2) dx = \int_{\Omega} f_k v_k^{-1}\phi^2 dx \geq \int_{\Omega} f_k u^{-1}\phi^2 dx \quad (\text{B.5})$$

by (B.4). Since  $\nabla v_k \cdot \nabla (v_k^{-1}\phi^2) = |\nabla\phi|^2 - |v_k \nabla(v_k^{-1}\phi)|^2 \leq |\nabla\phi|^2$ , we readily infer

$$\int_{\Omega} |\nabla\phi|^2 dx \geq \int_{\Omega} f_k u^{-1}\phi^2 dx \quad (\text{B.6})$$

and Fatou's Lemma implies that

$$\int_{\Omega} |\nabla\phi|^2 dx \geq \int_{\Omega} a(x) \phi^2 dx. \quad (\text{B.7})$$

The conclusion readily follows.  $\square$

The sharpness of the constants in (1.3) (compare with Remark A.2) and Lemma B.1 plainly imply the following result.

**Theorem B.2.** *Let  $N \geq 1$ ,  $R \in (0, 1]$ , and  $c, \alpha \geq 0$ . Let  $u \in L_{\text{loc}}^1(B_R \setminus \{0\})$  be a nonnegative distributional solution to*

$$-\Delta u - c|x|^{-2}u \geq \alpha|x|^{-2}|\log|x||^{-2}u \quad \text{in } \mathfrak{D}'(B_R \setminus \{0\}). \quad (\text{B.8})$$

(i) *If  $c > (N - 2)^2/4$ , then  $u \equiv 0$ .*

(ii) *If  $c = (N - 2)^2/4$  and  $\alpha > 1/4$ , then  $u \equiv 0$ .*

We notice that proposition (i) in Theorem B.2 was already proved in [15] (see also [16]).

Finally, from Remark A.2 and Lemma B.1, we obtain the next nonexistence result.

**Theorem B.3.** *Let  $\Sigma$  be a domain properly contained in  $\mathbb{S}^{N-1}$ ,  $R \in (0, 1]$ , and  $c, \alpha \geq 0$ . Let  $u \in L_{\text{loc}}^1(C_{\Sigma}^R)$  be a nonnegative distributional solution to*

$$-\Delta u - c|x|^{-2}u \geq \alpha|x|^{-2}|\log|x||^{-2}u \quad \text{in } \mathfrak{D}'(C_{\Sigma}^R). \quad (\text{B.9})$$



- (i) If  $c > \mu(\mathcal{C}_\Sigma)$ , then  $u \equiv 0$ .
- (ii) If  $c = \mu(\mathcal{C}_\Sigma)$  and  $\alpha > 1/4$ , then  $u \equiv 0$ .

*Remark B.4.* It would be of interest to know if the sign assumption on the coefficient  $a$  in Lemma B.1 can be weakened.

### C. Extensions

In this appendix, we state some nonexistence theorems that can be proved by using a suitable functional change  $u \mapsto \psi$  and Theorem 2.3. We shall also point out some corollaries of our main results.

#### C.1. The $k$ -Improved Weights

We define a sequence of radii  $R_k \rightarrow 0$  by setting  $R_1 = 1$  and  $R_k = e^{-1/R_{k-1}}$ . Then we use induction again to define two sequences of radially symmetric weights  $X_k(x) \equiv X_k(|x|)$  and  $z_k$  in  $B_{R_k}$  by setting  $X_1(|x|) = |\log|x||^{-1}$  for  $|x| < 1 = R_1$  and

$$X_{k+1}(|x|) = X_k\left(|\log|x||^{-1}\right), \quad z_k(x) = |x|^{-1} \prod_{i=1}^k X_i(|x|) \tag{C.1}$$

for all  $x \in B_{R_k} \setminus \{0\}$ . It can be proved by induction that  $z_k$  is well defined on  $B_{R_k}$  and  $z_k \in L^2_{\text{loc}}(B_{R_k})$ . We are interested in distributional solutions to

$$-\Delta u - \frac{(N-2)^2}{4} |x|^{-2} u \geq \alpha z_k^2 u \quad \mathfrak{D}'(B_R \setminus \{0\}) \tag{C.2}$$

for  $R \in (0, R_k]$ . The next result includes Theorem 1.1 by taking  $k = 1$ .

**Theorem C.1.** *Let  $k \geq 1$ ,  $R \in (0, R_k]$  and let  $u \geq 0$  be a distributional solution to (C.2). Assume that there exists  $\gamma \leq 1$  such that*

$$u \in L^2_{\text{loc}}\left(B_R; z_k^2 X_k^{2(\gamma-1)} dx\right), \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2. \tag{C.3}$$

Then  $u = 0$  almost everywhere in  $B_R$ .

*Proof.* We start by introducing the  $k$ th Emden-Fowler transform  $u \mapsto T_k u$ ,

$$u(x) = z_k(|x|)^{-1/2} |x|^{(1-N)/2} X_k(|x|)^{1/2} (T_k u) \left( X_k(|x|)^{-1}, \frac{x}{|x|} \right). \tag{C.4}$$

Notice that for any  $R < R_k$  it results that

$$\int_{B_R} z_k^2 X_k^{2(\gamma-1)} |u|^2 dx = \int_{X_k(R)^{-1}}^\infty s^{-2\gamma} \int_{\mathbb{S}^{N-1}} |T_k u|^2 ds d\sigma, \tag{C.5}$$

so that  $T_k u \in L^2(I_a \times \mathbb{S}^{N-1}; s^{-2\gamma} ds d\sigma)$  for any  $a > X_k(R)^{-1}$ . This can be easily checked by noticing that  $X'_k = z_k X_k$ . Next we set

$$\varphi_u(s) := \int_{\mathbb{S}^{N-1}} (T_k u)(s, \sigma) d\sigma. \tag{C.6}$$

By (C.5), we have that  $\varphi \in L^2(I_a; s^{-2\gamma} ds)$  for any  $a > X_k(R)^{-1}$ . Thanks to Theorem 2.3, to conclude the proof, it suffices to show that  $\varphi$  is a distributional solution to  $-\varphi'' \geq \alpha s^{-2}\varphi$  in the interval  $I_{\tilde{a}}$ , where  $\tilde{a} = X_k(R)^{-1}$ . To this end, fix any test function  $\varphi \in C^\infty(I_{\tilde{a}})$  and define the radially symmetric mapping  $\tilde{\varphi} \in C_c^\infty(B_R \setminus \{0\})$  such that  $T_k \tilde{\varphi} = \varphi$ . By direct computation, one can prove that

$$\Delta \tilde{\varphi} + \frac{(N-2)^2}{4} |x|^{-2} \tilde{\varphi} = \omega \tilde{\varphi} + |x|^{(1-N)/2} z_k^{3/2} X_k^{-3/2} \varphi'' \left( X_k(|x|)^{-1} \right), \tag{C.7}$$

where  $\omega \equiv 0$  if  $k = 1$ , and

$$\omega = \frac{1}{2} \left[ \left( \sum_{i=1}^{k-1} z_i \right)^2 - \frac{1}{2} \sum_{i=1}^{k-1} z_i^2 \right] \tag{C.8}$$

if  $k \geq 2$ . Since  $\omega \geq 0$ , then

$$\int_{B_R} u \left( \Delta \tilde{\varphi} + \frac{(N-2)^2}{4} |x|^{-2} \tilde{\varphi} \right) dx \geq \int_{\tilde{a}}^\infty \varphi \varphi'' ds \tag{C.9}$$

provided that  $\varphi$  is nonnegative. In addition, it results that

$$\int_{B_R} z_k^2 u \tilde{\varphi} dx = \int_{\tilde{a}}^\infty s^{-2} \varphi \varphi ds. \tag{C.10}$$

Since  $\varphi$  was arbitrarily chosen, the conclusion readily follows. □

By similar arguments as above and in Section 2, we can prove a nonexistence result of positive solutions to the problem

$$-\Delta u - \mu(C_\Sigma) |x|^{-2} u \geq \alpha z_k^2 u \quad \mathfrak{D}'(C_\Sigma^R), \tag{C.11}$$

where  $C_\Sigma$  is a Lipschitz proper cone in  $\mathbb{R}^N$ ,  $N \geq 1$ , and  $C_\Sigma^R = C_\Sigma \cap B_R$ . We shall skip the proof of the following result.

**Theorem C.2.** *Let  $k \geq 1$ ,  $R \in (0, R_k]$ , and let  $u \geq 0$  be a distributional solution to (C.11). Assume that there exists  $\gamma \leq 1$  such that*

$$u \in L^2\left(\mathcal{C}_\Sigma^R; z_k^2 X_k^{2(\gamma-1)} dx\right), \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2. \quad (\text{C.12})$$

Then  $u = 0$  almost everywhere in  $\mathcal{C}_\Sigma^R$ .

Some related improved Hardy inequalities involving the weight  $z_k$  and which motivate the interest of problems (C.2) and (C.11) can be found in [5, 7, 8] and also [6].

## C.2. Exterior Cone-Like Domains

The Kelvin transform

$$u(x) \mapsto |x|^{2-N} u\left(\frac{x}{|x|^2}\right) \quad (\text{C.13})$$

can be used to get nonexistence results for exterior domains in  $\mathbb{R}^N$ .

Let  $\Sigma$  be a domain in  $\mathbb{S}^{N-1}$ ,  $N \geq 2$ , and let  $\mathcal{C}_\Sigma$  be the cone defined in Section 2. We recall that  $\mu(\mathcal{C}_\Sigma) = (N-2)^2/4 + \lambda_1(\Sigma)$ . Since the inequality in (1.1) is invariant with respect to the Kelvin transform, then Theorems 1.1 and 3.1 readily lead to the following nonexistence result.

**Theorem C.3.** *Let  $\Sigma$  be a Lipschitz domain in  $\mathbb{S}^{N-1}$ , with  $N \geq 2$ . Let  $R > 1$ ,  $\alpha \in \mathbb{R}$ , and let  $u \geq 0$  be a distributional solution to*

$$-\Delta u - \mu(\mathcal{C}_\Sigma)|x|^{-2}u \geq \alpha|x|^{-2}|\log|x||^{-2}u \quad \text{in } \mathfrak{D}'(\mathcal{C}_\Sigma \setminus \bar{B}_R). \quad (\text{C.14})$$

Assume that there exists  $\gamma \leq 1$  such that

$$u \in L^2\left(\mathcal{C}_\Sigma \setminus \bar{B}_R; |x|^{-2}|\log|x||^{-2\gamma} dx\right), \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2. \quad (\text{C.15})$$

Then  $u = 0$  almost everywhere in  $\mathcal{C}_\Sigma \setminus \bar{B}_R$ .

A similar statement holds in case  $N = 1$  for ordinary differential inequalities in unbounded intervals  $(R, 0)$  with  $R > 0$ , and for problems involving the weight  $z_k^2$ .

## C.3. Degenerate Elliptic Operators

Let  $\theta \in \mathbb{R}$  be a given real parameter. We notice that  $u$  is a distributional solution to (3.5) if and only if  $v = |x|^{-\theta/2}u$  is a distributional solution to

$$-\operatorname{div}\left(|x|^\theta \nabla v\right) - \mu(\mathcal{C}_\Sigma; \theta)|x|^{\theta-2}v \geq \frac{1}{4}|x|^{\theta-2}|\log|x||^{-2}v \quad \text{in } \mathfrak{D}'(\mathcal{C}_\Sigma^R), \quad (\text{C.16})$$

where  $\mu(C_\Sigma; \theta)$  is defined in Remark A.2. Therefore, Theorems 1.1 and 3.1 imply the following nonexistence result for linear inequalities involving the weighted Laplace operator  $L_\theta v = -\operatorname{div}(|x|^\theta \nabla v)$ .

**Theorem C.4.** *Let  $\Sigma$  be a Lipschitz domain in  $\mathbb{S}^{N-1}$ . Let  $\theta \in \mathbb{R}$ ,  $R \in (0, 1]$ ,  $\alpha \in \mathbb{R}$ , and let  $v \geq 0$  be a distributional solution to (C.16). Assume that there exists  $\gamma \leq 1$  such that*

$$v \in L^2\left(C_\Sigma^R; |x|^{\theta-2} |\log|x||^{-2\gamma} dx\right), \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2. \quad (\text{C.17})$$

Then  $v = 0$  almost everywhere in  $C_\Sigma^R$ .

A nonexistence result for the operator  $-\operatorname{div}(|x|^\theta \nabla v)$  similar to Theorem C.3 or to Theorem C.1 can be obtained from Theorem C.4, via suitable functional changes.

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