Quantum moduli space of Chern-Simons quivers, wrapped D6-branes and AdS$_4$/CFT$_3$

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ABSTRACT: We study the quantum moduli space of $\mathcal{N} = 2$ Chern-Simons quivers with generic ranks and CS levels, proving along the way exact formulas for the charges of bare monopole operators. We then derive $\mathcal{N} = 2$ Chern-Simons quiver theories dual to $AdS_4 \times Y^{p,q}(\mathbb{CP}^2)$ M-theory backgrounds, for the whole family of Sasaki-Einstein seven-manifolds and for any value of the torsion G_4 flux. The derivation of the gauge theories relies on the reduction to type IIA string theory, in which M2-branes become D2-branes while the conical geometry maps to RR flux and D6-branes wrapped on compact four-cycles. M5-branes on torsion cycles map to flux and wrapped D4-branes. The moduli space of the quiver is shown to contain the corresponding CY$_4$ cone and all its crepant resolutions.

Contents

1   Introduction

According to the AdS/CFT correspondence, M-theory on $AdS_4 \times Y_7$, with Y_7 a seven-dimensional manifold preserving some supersymmetry (SUSY), is dual to a three-dimensional superconformal field theory (3d SCFT) which describes the low energy dynamics of M2-branes on the cone $C(Y_7)$. Important progress [1–13] has been made in the last few years to write down such field theories for simple geometries, especially in the wake of the ABJM proposal [4]. Nevertheless how to provide an explicit description of the SCFT for a generic Y_7 remains an open problem. The cases which are best understood so far are those for which the M-theory setup has a dual type IIB brane description à la Hanany-Witten [14], from which one can read off the low energy theory [4, 8, 15, 16].

In this paper we tackle the problem of deriving the field theory for $C(Y_7)$ a Calabi-Yau (CY) four-fold, preserving $\mathcal{N} = 2$ SUSY in 3d. Notice that $\mathcal{N} = 2$ is the highest amount of SUSY for which the field theory can have matter in non-real representations, which we loosely call *chiral* like in 4d.

We follow the logic first articulated in [12] by Aganagic, who suggested to Kaluza-Klein (KK) reduce a CY_4 M-theory background to type IIA on a wisely chosen circle S^1, such that the type IIA background is a CY_3 fibered over $\mathbb{R}$ and with Ramond-Ramond (RR) fluxes. In the reduction M2-brane probes become D2-brane probes, and such configuration allows in principle to extract the low energy field theory thanks to the knowledge of D-brane theories at CY_3 singularities. It is exciting to discover how the 4d and 3d cases are linked in this way.

In practice the method is hampered by two difficulties: 1) knowing the field theory dual to D-branes at the tip of the CY_3; 2) being able to interpret whichever extra singularity, besides the D2-branes, the reduction to type IIA brings about. In particular in [12] the simplest case in which only D2-branes are present was considered. We will argue that this is only possible if the CY_3 does not have compact exceptional[1] 4-cycles, the reason being that 4-cycles do not enjoy flop transitions. Then the CY_3 is a generalized conifold, a dual Hanany-Witten brane construction exists, and the $\mathcal{N} = 2$ field theory one obtains is non-chiral.

The immediate generalization, which is the subject of this paper, is to allow D6-branes in type IIA. They arise whenever the M-theory circle shrinks on a codimension-four submanifold in the eight-dimensional cone. Examples with non-compact D6-branes have already been considered preserving both $\mathcal{N} = 3$ [16–18] and $\mathcal{N} = 2$ [19, 20] supersymmetry. Their effect is to add to the quiver chiral multiplets in (anti)fundamental representations and (non-Abelian) flavor symmetries; most $\mathcal{N} = 2$ instances are chiral theories. In this paper we observe that whenever the CY_3 has exceptional 4-cycles, a necessary condition

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[1]An exceptional cycle is one which does not exist in the base of the CY cone but that appears upon partial resolution of the conical singularity.

for the type IIA background to be geometric is that the singularity hides some compact D6-branes wrapped on those 4-cycles. The wrapped D6-branes contribute dynamical, as opposed to external, three-dimensional gauge bosons: they are fractional D2-branes that unbalance the gauge ranks of the quiver. Going back to M-theory, the construction allows us to describe four-folds with exceptional 6-cycles.

Compact D6-branes are similar to the fractional branes added by ABJ [7] to the ABJM theory [4], as both modify some ranks of the quiver gauge theory, yet they are very different in other respects. The fractional M2-branes of ABJ descend to — in the 4d language—"non-anomalous" fractional D2-branes, that is D4-branes wrapping non-exceptional 2-cycles in the CY_3. The ABJ example is $\mathbb{C}^4/\mathbb{Z}_k$: upon IIA reduction it gives the conifold, which has such a 2-cycle. These branes are related to "non-anomalous" baryonic symmetries, enjoy Seiberg-like dualities, and do not give quantum corrections to monopole operators. On the other hand compact D6-branes on exceptional 4-cycles and D4-branes on exceptional 2-cycles are "*anomalous*" fractional D2-branes: the quivers they give rise to would be anomalous in 4d, although they are perfectly well-defined in 3d. The addition of such branes changes gauge ranks and CS levels, induces quantum corrections to the charges of monopole operators and deform some relations in the chiral ring, making a classical analysis of the field theory inadequate.

To make our life easier, and to circumvent problem 1) above, we limit ourselves to toric CY_4 geometries in M-theory, which give toric CY_3 manifolds in type IIA upon suitable reduction. We do this to have full control on the geometry and on D-branes therein, however we believe that our construction is valid more generally. In this paper we discuss in details the simplest example: a family of conical toric CY_4 geometries, cones over the so-called $Y^{p,q}(\mathbb{CP}^2)$ [21, 22] (or simply[2] $Y^{p,q}$ in the following) Sasaki-Einstein seven-manifolds which are $S^3/\mathbb{Z}_p$ bundles over $\mathbb{CP}^2$ (a notable member is $Y^{2,3} = M^{3,2}$). The type IIA reduction gives $\mathbb{C}^3/\mathbb{Z}_3$ with p D6-branes wrapping the exceptional $\mathbb{CP}^2$. Metrics for these Sasaki-Einstein spaces are known [21, 22], but we will not need them. A broader discussion of more general geometries is left for a companion paper [23].

Field theories for M2-branes probing a subclass of $C(Y^{p,q})$ geometries have been proposed by Martelli and Sparks (MS) based on Chern-Simons (CS) quivers with equal ranks [24] (see also [9]). The MS field theories correctly reproduce the $C(Y^{p,q})$ geometries as a branch of their moduli space in the parameter range $q \in [p, 2p]$, to be contrasted with the wider range $q \in [0, 3p]$ for which the metrics are known. Moreover the partial resolutions of the $\mathbb{C}^2/\mathbb{Z}_p$ fiber are not present in their field theories. We will clarify the reason for these puzzling properties.

The reduction from M-theory to type IIA was also considered in [22], but performed only on the $AdS_4 \times Y^{p,q}$ near horizon background. In this case the degeneration locus of the M-theory circle action coincides with the tip of the cone, which is not part of the near horizon geometry after the backreaction of the stack of M2-branes placed there is taken into account: the D6-branes have disappeared. Resolving the CY_4 singularity in M-theory

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[2]In the literature the name $Y^{p,q}$ usually refers to some five-dimensional Sasaki-Einstein spaces. We hope not to create confusion.

blows up the 4-cycle wrapped by the D6-branes in the CY_3, making them visible even when the backreaction of regular branes is taken into account. On the other hand, a careful analysis of charges allows us to take into account the D6-branes even in the conformal near-horizon geometry.

We derive field theories for M2-branes probing $C(Y^{p,q})$ in the full parameter range $q \in [0, 3p]$: they have quiver diagrams as in [24] but unequal ranks for "anomalous" groups and different CS levels. This has some consequences. First, a classical analysis of the field theory is not adequate, for instance to find its moduli space. The chiral ring is generated by chiral fields appearing in the Lagrangian, plus some monopole operators. The monopoles acquire global charges at one-loop (in fact we prove, with localization techniques, that the charges are one-loop exact), and satisfy quantum F-term relations not directly ensuing from the superpotential. We collect the relevant field theory tools at the beginning of the paper. Second, the field theories that we propose contain in their moduli space all toric crepant resolutions of the CY_4.

The geometry $Y^{p,q}$ has an interesting homology group $H_3(Y^{p,q}, \mathbb{Z})$ which is a finite Abelian group of order $q(3p - q)$ [22]: M5-branes can be wrapped on its elements, giving rise to torsion G_4-flux in M-theory, and $B_2 \wedge F_2$ flux in type IIA. We find the full family of superconformal theories dual to these different $AdS_4 \times Y^{p,q}$ backgrounds, generalizing the torsionless case. The moduli spaces of the field theories dual to M-theory vacua with torsion flux generically do not account for all the partial resolutions of the conical geometry. This field-theoretic result qualitatively agrees with the observation that torsion G_4-fluxes may obstruct partial resolutions [25]. Among this family, we find two quiver CS theories (related by parity times charge conjugation) with equal ranks and levels as in [24], if (and only if) $q \in [p, 2p]$. The reasons for the restricted parameter space and the lack of some partial resolutions in the original proposal of [24] are now clear, and tied to the presence of torsion G_4-flux in the gravity dual. Similar puzzles are generic with the chiral CS quivers appearing in the literature: we expect similar phenomena to take place.

The paper is organized as follows. In section 2, which can be read independently of the rest of the paper, we present exact formulæ for the charges of chiral monopole operators in $\mathcal{N} = 2$ CS-matter theories, and we use them to compute the quantum chiral ring of CS quiver gauge theories. We also present a more conventional one-loop computation of the moduli space, following known results. In section 3 we discuss M-theory on the cone over $Y^{p,q}$, and its type IIA reduction to the resolved orbifold $\mathbb{C}^3/\mathbb{Z}_3$ fibered over $\mathbb{R}$. We discuss the D-brane Page charges present in type IIA, including the effect of the Freed-Witten anomaly, and explain how type IIA reproduces the finite group $H^4(Y^{p,q})$. In section 4 we review some useful facts about fractional branes on $\mathbb{C}^3/\mathbb{Z}_3$. In section 5 we use that information to derive the field theories describing M2-branes on $C(Y^{p,q})$. We check our proposals by computing the moduli space, which contains all partial resolutions of the CY_4 singularity. Several useful computations are relegated to the appendices. Moreover, in appendix B we analyze partial resolutions of the moduli spaces of the flavored ABJM theories of [19, 20].

2 Monopole operators and moduli space of $\mathcal{N} = 2$ quivers

A striking difference between 3d $\mathcal{N} = 2$ SCFTs and their $\mathcal{N} = 1$ four-dimensional cousins is the existence in three dimensions of local monopole operators. These can be seen as the dimensional reduction of 4d 't Hooft line operators along the line [26, 27]. They carry global charges under the topological (or magnetic) currents $J = *\operatorname{Tr} F$ associated to 3d photons. A subset of monopole operators transforms in short representations of the SUSY algebra: chiral multiplets. Therefore topological charges grade the chiral ring of 3d theories.

Monopole operators in 3d CFTs have been studied in the pioneering works [28–30], and among the more recent ones we emphasize [31–34]. The crucial point for us is that the charges of monopole operators can receive corrections at one-loop. It was recently shown, using localization techniques, that the one-loop result is exact for BPS monopole operators in theories with at least $\mathcal{N} = 3$ superconformal symmetry [33]. That analysis can be extended to $\mathcal{N} = 2$ superconformal theories, as we show in appendix C. The main difficulty is to determine the exact superconformal R-charge: within the set of R-symmetries of the theory, only one is in the supermultiplet of the stress tensor and determines the dimension of chiral primaries.[3] However for our purposes it suffices to consider any R-symmetry, with the superconformal R-symmetry being some linear combination of it with the other Abelian symmetries of the theory:

$$R_{\text{superconformal}} = R + \sum_n \alpha_n Q_n + \sum_i \beta_i H_i^{(T)} \,, \tag{2.1}$$

where Q_n denote Abelian flavor symmetries and $H_i^{(T)}$ topological symmetries. We assume that the IR R-symmetry is some combination of the UV symmetries.[4]

2.1 Charges of half-BPS monopoles

Classically and in radial quantization, a half-BPS monopole operator in a $\mathcal{N} = 2$ superconformal theory is a configuration

$$A_\mu dx^\mu = H\, B_i dx^i \,, \qquad \sigma = \frac{H}{2r} \,, \qquad \phi = 0 \tag{2.2}$$

with all other fields vanishing. Here H is the magnetic flux in the algebra $\mathfrak{g}$ of the gauge group $\mathcal{G}$, B_i is the Dirac monopole configuration of magnetic charge one, σ is the adjoint real scalar field in the $\mathcal{N} = 2$ vector multiplet (A_μ, λ, σ), r is the radius of the 2-sphere, and ϕ are complex scalars in chiral multiplets. H can always be gauge rotated to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Classically and in the absence of CS terms, monopoles are only charged under the topological symmetries.

Monopoles can acquire charges, both gauge and global, from CS terms. In the following we will distinguish between gauge, R-, flavor and topological symmetries, as well as manifest and hidden: gauge, R- and flavor symmetries are manifest in the Lagrangian, and commute

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[3]A method to determine the superconformal R-symmetry has recently been proposed in [35].

[4]A theory with such a property can be called a "good theory", similarly to the discussion of $\mathcal{N} = 4$ quivers of Gaiotto and Witten [36].

with each other; topological Abelian symmetries are not manifest — the only thing we see are the currents $J = *\text{Tr}\, F$. Hidden symmetries arise at the fixed point (possibly as enhancement of topological symmetries), but are not symmetries of the UV Lagrangian.

Consider a monopole with flux $H = n$ in an Abelian factor. Then a CS term $\frac{k}{4\pi} \int A \wedge F$ induces electric charge nk. Mixed Abelian CS terms $\sum_{ij} \frac{k_{ij}}{4\pi} \int A_i \wedge dA_j$ between dynamical and external (global) gauge fields induce charges under manifest global symmetries. Let A_f be an external gauge field associated to an Abelian symmetry, then $\frac{k_{g,f}}{2\pi} \int A_f \wedge F$ (where $k_{g,f}$ is a gauge-flavor CS term) induces global charge $nk_{g,f}$. This argument immediately suggests that a monopole cannot transform under simple (in the sense of simple group) manifest global symmetries, because we cannot write a mixed CS term with a simple group. Notice that we are talking about *bare* monopole operators: in the terminology of [28–30] they are the Fock vacuum in the fermionic Fock space of zero-modes, and so cannot form non-unidimensional representations. On the contrary the gauge-invariant monopoles of [28–30] are obtained by multiplication by fundamental fields, so that they can transform under simple manifest global symmetries.

In [27] it was shown that monopoles with flux in a generic gauge group $\mathcal{G}$, in the presence of a CS term at level k, transform in a non-trivial representation of the gauge group. The flux is specified by a homeomorphism $\eta : \text{U}(1) \to \mathcal{G}$ or equivalently by $H \equiv \eta\big(1 \in \mathfrak{u}(1)\big) \in \mathfrak{g}$ (constrained by Dirac quantization), up to conjugation. The monopole action transforms as

$$\delta S = k \,\text{Tr}(H\,\delta A)\Big|_{\text{monopole}} \tag{2.3}$$

under a gauge transformation δA, so that the monopole transforms in a representation whose highest weight is $k\,\text{Tr}(H \cdot).$[5]

Charges of monopole operators receive quantum corrections due to zero-modes, and for $\mathcal{N} = 2$ superconformal theories the one-loop answer is exact. In appendix C we discuss the formulæ for the quantum correction δq to any Abelian charge, in case of generic gauge group $\mathcal{G}$ and matter representations R_Φ:

$$\delta q = -\frac{1}{2} \sum_{\text{fermions } f} \sum_{\rho \in R_f} \big|\rho(H)\big|\, q_f \tag{2.4}$$

where f are all fermions in the theory and ρ are the weights of the representation. It is easy to see that there are no quantum corrections to topological charges and that the charge under any Cartan of a simple manifest global symmetry group is zero, confirming that monopoles do not transform under simple manifest global symmetries.

Let us specialize here to quiver theories, where matter is restricted to the adjoint, bifundamental or (anti)fundamental representation. We consider an $\mathcal{N} = 2$ CS quiver with gauge group $\mathcal{G} = \prod_{i=1}^{G} \text{U}(N_i)$, CS levels[6] k_i, bifundamental chiral superfields X_{ij}, fundamentals q_{im} and antifundamentals $\tilde{q}_{mi}$. Let F_i^+ (F_i^-) be the number of (anti)fundamentals

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[5]In the case of a U(N) gauge group that we will consider below we have $\mathfrak{g} \cong \mathfrak{g}^*$ and we can therefore write the weights simply as kH.

[6]In general U(1) and SU(N) in U(N) can have different CS levels. We will discuss this possibility in section 6 in relation to anomalies.

of the group $U(N_i)$. A monopole operator is characterized by its magnetic charges (GNO charges [26, 37])

$$H_i = (n_{i,1}, \ldots, n_{i,N_i}), \qquad i = 1, \ldots, G \qquad (2.5)$$

in the Cartan subalgebra $\mathfrak{h}$. Under any R-symmetry $U(1)_R$ (normalized such that gaugini have R-charge 1), the monopole of (2.5) acquires a charge

$$R = -\frac{1}{2} \sum_{X_{ij}} (R[X_{ij}] - 1) \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} |n_{i,k} - n_{j,l}| \; - \frac{1}{2} \sum_{i=1}^{G} \sum_{k=1}^{N_i} \sum_{l=1}^{N_i} |n_{i,k} - n_{i,l}|$$
$$- \frac{1}{2} \sum_i \left(\sum_{m=1}^{F_i^+} (R[q_{im}] - 1) + \sum_{m=1}^{F_i^-} (R[\tilde{q}_{mi}] - 1) \right) \sum_{k=1}^{N_i} |n_{i,k}| . \qquad (2.6)$$

The second line, due to flavors, is the correction studied in [19, 20]. Since any $U(1)$ flavor symmetry is the difference of two R-symmetries, under any non-R symmetry Q we have the induced charge

$$Q = -\frac{1}{2} \sum_{X_{ij}} Q[X_{ij}] \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} |n_{i,k} - n_{j,l}|$$
$$- \frac{1}{2} \sum_i \left(\sum_{m=1}^{F_i^+} Q[q_{im}] + \sum_{m=1}^{F_i^-} Q[\tilde{q}_{mi}] \right) \sum_{k=1}^{N_i} |n_{i,k}| . \qquad (2.7)$$

In this paper we will not consider flavors, so the second lines can be neglected.

Let us consider now gauge symmetries. In [31, 38] it was shown (see appendix C) that in doing localization, the integrand in the path-integral picks up a phase

$$e^{ib_0(a)} , \qquad \text{with} \qquad b_0(a) = -\frac{1}{2} \sum_\Phi \sum_{\rho \in R_\Phi} |\rho(H)| \, \rho(a) \qquad (2.8)$$

where $a \in \mathfrak{h}$ is the Cartan gauge field and the sum is over all chiral multiplets. Since the function $b_0(a)$ is linear, it is easy to work out the variation of the action with respect to A, from which we infer that the monopole transform in a representation whose highest weight is

$$w = k \, \mathrm{Tr}(H \cdot) - \frac{1}{2} \sum_\Phi \sum_{\rho \in R_\Phi} |\rho(H)| \, \rho . \qquad (2.9)$$

In the special case of a quiver theory, the induced electric charges $g_{i,k}$ under each $U(1)_{i,k}$ ($k_i = 1, \ldots, N_i$) in Cartan subgroup of $\prod_i U(N_i)$ are

$$g_{i,k} = k_i \, n_{i,k} \; + \; \delta g_{i,k} . \qquad (2.10)$$

Here δg receives contributions from bifundamental and (anti)fundamental matter, but not from adjoint matter nor from gaugini:

$$\delta g_{i,k} = -\frac{1}{2} \sum_{X_{ij}} \sum_{l=1}^{N_j} |n_{i,k} - n_{j,l}| \; + \frac{1}{2} \sum_{X_{ji}} \sum_{l=1}^{N_j} |n_{i,k} - n_{j,l}| \; - \; \frac{1}{2} \left(F_i^+ - F_i^- \right) |n_{i,k}| . \qquad (2.11)$$

We see that only "chiral matter" (matter in non-real representations) can induce gauge charges. The U(1) charges (2.10) define a weight for the gauge group $U(N_i)$,

$$w_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,N_i}) \,, \tag{2.12}$$

which is the highest weight (2.9) of the representation under which the monopole transforms. Whenever $\delta g_{i,k}$ are half-integer, the parity anomaly [39–41] forces k_i to be half-integer as well so that $g_{i,k}$ are always integers.

Finally, consider hidden symmetries whose Cartan currents are visible as topological currents $J = *\operatorname{Tr} F$: monopoles are by definition charged under them. Full representations of a hidden symmetry can be formed by different monopoles (as opposed to different states of a single monopole), therefore monopoles can transform under simple hidden global symmetries.[7] Notice that the distinction between manifest and hidden global symmetries is unphysical, but so is the distinction between monopoles and matter fields.

2.2 Toric quivers and diagonal monopoles

The knowledge of the quantum charges of monopole operators allows us in principle to work out all holomorphic gauge invariant operators of the theory. What we are interested in, however, is the chiral ring, defined through relations between the holomorphic operators:

$$A(\text{quiver}) = \frac{\mathbb{C}\,[\mathcal{O}_1, \mathcal{O}_2, \ldots]}{\mathcal{I}} \,. \tag{2.13}$$

In 4d quiver SCFTs, the chiral ring is generated by chiral fields in the Lagrangian and the ideal $\mathcal{I}$ can be read from the classical F-terms: $\mathcal{I} = (\partial W)$; more precisely $\mathcal{I}$ consists of all gauge invariant relations that follow from $\partial W = 0$, together with the so-called syzygies [42]. In 3d there are two differences: 1) the chiral ring is generated by chiral monopole operators besides chiral fundamental fields; 2) there might exist relations between monopole operators which are not easily derived from the superpotential. For this reason the chiral ring of a 3d quiver is much more complicated to analyze than that of its 4d parent. Since we do not know of honest field theory methods to compute the quantum chiral ring relations in general, we will keep the strategy followed in [19, 20]: an educated guess of the most important chiral ring relations between so-called diagonal monopole operators, based on their global charges.

Consider a quiver with ranks $N_i = \tilde{N} + M_i$, $\tilde{N} = \min\{N_i\}$, and no flavors. We focus on diagonal monopole operators T_H, which turn on fluxes

$$T_H : \qquad n_{i,k} \equiv n_{0,k} \quad \forall i \,, \quad \text{if } k \leq \tilde{N} \,, \qquad \text{and} \quad n_{i,k} = 0 \,, \quad \text{if } k > \tilde{N} \,. \tag{2.14}$$

We will be mainly interested in the simplest diagonal monopole operators

$$T, \tilde{T} : \qquad H = (\pm 1, 0, \ldots, 0) \,. \tag{2.15}$$

In the case of equal ranks, $M_i = 0$, the R-charges of T_H are

$$R[T_H] = -\frac{1}{2}\left(G + \sum_{\text{fields } X} (R[X] - 1)\right) \sum_{k=1}^{\tilde{N}} \sum_{l=1}^{\tilde{N}} |n_{0,k} - n_{0,l}| \,. \tag{2.16}$$

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[7]For $\mathcal{N} \geq 4$ SUSY theories these hidden symmetries have been studied in [33, 36].

The quantity in parenthesis automatically vanishes for *toric quivers*, also known as brane tilings (see [43] for a review). We will restrict the following analysis to such theories. A brane tiling is a bipartite graph, where each gauge group is represented by a face F_i ($i = 1, \ldots, G$), each bifundamental field by an edge X_{ij} between two faces, and each superpotential term W_α by a black/white vertex. We have the further constraint that the graph tiles a torus, which implies $G + V - E = 0$, where V and E are the number of vertices and edges respectively. We have $\sum_X R[X] = \frac{1}{2} \sum_{W_\alpha} \sum_{X \in W_\alpha} R[X] = \frac{1}{2} \sum_{W_\alpha} 2 = V$, so that the quantity in parenthesis equals $G + V - E = 0$. Similarly one can show that diagonal monopoles do not receive any quantum correction at all. The chiral ring is generated by chiral fields in the Lagrangian as well as T, $\tilde{T}$, subject to the classical relation $T\tilde{T} = 1$ [19]. Hence the classical analysis of the moduli space, as in [24], gives the correct result.

If the ranks $N_i = \tilde{N} + M_i$ are unequal, the extra contribution to the R-charge of diagonal monopole operators is

$$\delta R[T_H] = -\frac{1}{2}\left(\sum_{\text{fields } X_{ij}} (R[X_{ij}] - 1)(M_i + M_j) + \sum_{i=1}^{G} 2M_i \right) \sum_{k=1}^{\tilde{N}} |n_{0,k}| \, . \tag{2.17}$$

For a toric quiver we have

$$\sum_{\text{fields } X_{ij}} M_i = \sum_{\text{fields } X_{ij}} M_j = \frac{1}{2} \sum_{W_\alpha} \sum_{X_{ij} \in W_\alpha} M_i = \frac{1}{2} \sum_{W_\alpha} \sum_{F_i \in W_\alpha} M_i = \frac{1}{2} \sum_{F_i} M_i \sum_{W_\alpha \in F_i} 1 \, , \tag{2.18}$$

where $F_i \in W_\alpha$ means the faces that touch the vertex W_α, and reciprocally for $W_\alpha \in F_i$. Denoting the number of edges (or equivalently vertices) around a face F_i by E_i, we can reshuffle (2.17) into

$$R[T_H] = \frac{1}{2}\left(\sum_i (E_i - 2)M_i - \sum_{\text{fields } X_{ij}} R[X_{ij}](M_i + M_j) \right) \sum_{k=1}^{\tilde{N}} |n_{0,k}| \, . \tag{2.19}$$

Remark that $E_i - 2 > 0$ (unless there are double bonds [44] in the brane tiling, a situation we will not consider); importantly, $E_i - 2$ is even.

Similarly we can compute the electric charge (2.10) under each $U(1)_{i,k}$ in the Cartan of the gauge group:

$$g_{i,k}[T_H] = k_i \, n_{0,k} - \frac{1}{2}\left(\sum_{X_{ij}} M_j - \sum_{X_{ji}} M_j \right) |n_{0,k}| \, , \tag{2.20}$$

for $k = 1, \ldots, \tilde{N}$, while $g_{i,k} = 0$ for $k > \tilde{N}$. Monopoles transform in a representation whose highest weight is $w = (g_{i,1}, \ldots, g_{i,N_i})$ and for the simplest monopoles T, $\tilde{T}$ (2.15) we have

$$w_i(T) = (g_i, 0, \ldots, 0) \, , \qquad w_i(\tilde{T}) = (\tilde{g}_i, 0, \ldots, 0) \, , \tag{2.21}$$

with

$$g_i = k_i - \frac{1}{2}\left(\sum_{X_{ij}} M_j - \sum_{X_{ji}} M_j \right) \, , \qquad \tilde{g}_i = -k_i - \frac{1}{2}\left(\sum_{X_{ij}} M_j - \sum_{X_{ji}} M_j \right) \, . \tag{2.22}$$

Thus the monopole operators T, $\tilde{T}$ transform in symmetric representations of the $U(N_i)$ gauge groups.

The quantum numbers of T, $\tilde{T}$ strongly constrain the possibilities for chiral ring relations involving $T\tilde{T}$. Generically we can have the gauge invariant relation (there could be several ways to contract gauge indices)

$$T\tilde{T} \prod_X X_{ij}^{(M_i+M_j)} = \prod_\alpha (W_\alpha)^{m_\alpha} \,, \tag{2.23}$$

with $\sum_\alpha m_\alpha = \frac{1}{2}\sum_i (E_i - 2)M_i$. Here W_α stands for the superpotential terms. Superpotential terms have the property of having R-charge 2, vanishing charges under non-R $U(1)$ symmetries, and of being gauge invariant. This is why they appear in the previous relation. Note that all superpotential terms of a toric quiver are equivalent in the chiral ring, therefore there is no ambiguity related to a choice of superpotential term. (2.23) can sometimes be simplified if a gauge invariant operator can be factored out.

The gauge invariant relations involving both T and $\tilde{T}$ do not all necessarily take the form (2.23). If a field appears in both sides of (2.23), we can replace it by another (elementary or composite) field charged under the same representation of the gauge group, if the latter exists. So doing we may find gauge invariant relations which transform covariantly under the global symmetries of the quiver gauge theory.

2.3 Moduli spaces of toric quivers from monopole operators

Our purpose is to apply the method explained above to compute the Coulomb branch of the moduli space of CS quivers, and eventually compare it with some CY_4 used in the M-theory background. In particular if we have a 3d CS quiver conjectured to describe the low energy dynamics of M2-branes on a CY_4, we expect the moduli space of the theory to contain $\tilde{N}$ symmetrized copies of it.[8]

One way of characterizing such geometric branch of the Coulomb moduli space is to compute the chiral ring of the quiver, and compare it with the coordinate ring of the CY_4. Let us consider the pseudo-Abelian case, $\tilde{N} = 1$. We collect the gauge invariant operators constructed out of $\{X_{ij}, T, \tilde{T}\}$ with at most one power of T or $\tilde{T}$, which we denote $\mathcal{O}^{(0)}$, $\mathcal{O}^{(\pm 1)}$ according to their magnetic charge, and construct

$$A(\text{quiver})_{\text{geom}} \equiv \frac{\mathbb{C}\left[\mathcal{O}^{(0)}, \mathcal{O}^{(+1)}, \mathcal{O}^{(-1)}\right]}{\left(\partial W, \ T\tilde{T}\mathcal{P}_1(X) - \mathcal{P}_2(X)\right)} \,, \tag{2.24}$$

where $T\tilde{T}\mathcal{P}_1(X) = \mathcal{P}_2(X)$ are the quantum F-term relations proposed in (2.23). We will find in examples that indeed $A(\text{quiver})_{\text{geom}} = A(CY_4)$. In the case of flavored quivers with equal ranks it was possible [19] to give a general proof of that, due to manipulations of the brane tiling techniques of [10, 45]. In the present case of quivers with unequal ranks it seems that such easy techniques are not available.

We will talk about the geometry in later sections, but we already anticipated in the introduction that to derive the field theories we KK reduce the CY_4 along a wisely-chosen

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[8]$\tilde{N}$ need not be equal to the total number N of M2-branes, but only to the number of mobile M2-branes. We will see in examples that generically $\tilde{N} \leq N$.

circle, such that we obtain a CY$_3$ fibered along $\mathbb{R}$, and then exploit the quiver that describes D-branes at the tip of the CY$_3$. Indeed in the field theory we can consider the subring

$$A(\mathrm{CY}_3) = \frac{\mathbb{C}\left[\mathcal{O}^{(0)}\right]}{(\partial W)} \tag{2.25}$$

which is precisely the coordinate ring of the aforementioned CY$_3$.

2.4 Moduli spaces from a semi-classical computation

We can approach the computation of the Coulomb branch of the moduli space in a more conventional way, by performing a semi-classical calculation that includes one-loop effects in the effective theory on the Coulomb branch. This approach follows the work of [46–49] on 3d $\mathcal{N} = 2$ Yang-Mills (YM) and Yang-Mills-Chern-Simons (YM-CS) theories, and gives results that perfectly match with the quantum chiral ring presented above. The advantage of the computation with monopoles is that it is one-loop exact;[9] the disadvantage is that it knows only the complex structure of the moduli space and not its Kähler structure. On the other hand the semi-classical computation probes the Kähler structure and so it is particularly suited to analyze partial resolutions (to be discussed in a specific example in section 5.3), even though it does not capture possible non-perturbative corrections to the metric on the moduli space.

Consider a 3d $\mathcal{N} = 2$ YM-CS quiver as in the previous sections, with gauge group $\prod_{i=1}^{G} \mathrm{U}(\tilde{N} + M_i)$, YM coupling constants e_i, and bifundamental fields X_{ij}. The classical YM-CS scalar potential reads (see e.g. [24] for a nice account):

$$V = \sum_{i=1}^{G} \frac{e_i^2}{2}\left(\mathcal{D}_i - \sigma_i \frac{k_i}{2\pi} - \xi_i^{\mathrm{bare}}\right)^2 + \sum_{\text{fields } X_{ij}} \left|\sigma_i X_{ij} - X_{ij}\sigma_j\right|^2 + \sum_{\text{fields } X_{ij}} \left|\frac{\partial W}{\partial X_{ij}}\right|^2 , \tag{2.26}$$

where σ_i are the Hermitian scalar fields in 3d $\mathcal{N} = 2$ vector multiplets, $\mathcal{D}_i$ are the 4d D-terms

$$\mathcal{D}_i \equiv \sum_{X_{ij}} X_{ij} X_{ij}^\dagger - \sum_{X_{ji}} X_{ji}^\dagger X_{ji} , \tag{2.27}$$

W is the superpotential and ξ_i^{bare} are possible bare Fayet-Iliopoulos (FI) terms, that for the moment we set to zero. Vanishing of V would lead to the equations

$$\mathcal{D}_i = \sigma_i \frac{k_i}{2\pi} , \qquad \sigma_i X_{ij} - X_{ij}\sigma_j = 0 \qquad \forall\, i,\, j\, , \qquad \partial_{X_{ij}} W = 0 . \tag{2.28}$$

Notice that the first set of equations become pure constraints in a purely CS theory (formally in the limit $e_i \to \infty$).

The semi-classical analysis of the moduli space goes as follows. First we choose a background for the Hermitian scalars in vector multiplets, diagonalized via gauge transformations:

$$\sigma_i = \mathrm{diag}(\sigma_{n_i}^{(i)}) , \qquad n_i = 1, \ldots, \tilde{N} + M_i . \tag{2.29}$$

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[9]There can be non-perturbative corrections to the superpotential on the moduli space, though.

These VEVs partially break the gauge group (generically to the maximal torus) and give an effective real mass to most of the components of bifundamentals X_{ij},

$$\delta M[(X_{ij})^{n_i}{}_{n_j}] = \sigma^{(i)}_{n_i} - \sigma^{(j)}_{n_j}, \qquad (2.30)$$

freezing the massive fields to vanishing VEV. Integrating out the massive chiral multiplets (which include fermions) generates CS interactions at one-loop, as reviewed in appendix A. We thus compute effective field-dependent CS levels k_i^{eff} and FI parameters ξ_i^{eff} for the unbroken gauge group, which depend on the VEVs (2.29). Finally, we look for SUSY vacua solving D-term (which contain the effective FI parameters) and F-term equations of the effective theory, and modding out by the unbroken gauge group.

The moduli space contains more directions: all photons in the effective theory which are not coupled to matter can be dualized to real periodic pseudoscalars — the *dual photons* — which, because of the CS couplings, shift under some gauge transformations. One can use those gauge transformations to gauge fix the dual photons, as in [4, 24] but here in the effective theory. As a result the space of solutions of F-term and σ-dependent D-term equations has to be modded out by a subgroup of the unbroken gauge group (possibly including a residual discrete gauge symmetry that depends on k_i^{eff}).

Let us consider the *geometric branch* of the moduli space, defined by

$$\sigma_i = \text{diag}(\sigma_1, \ldots, \sigma_{\tilde{N}}, 0, \ldots, 0) \qquad \forall \, i = 1, \ldots, G \,, \qquad (2.31)$$

in which all real scalars are equal. To begin with, we fix to zero the extra M_i components of σ_i, that in the following we will call $\tilde{\sigma}_m$ ($m = 1, \ldots, \max(M_i)$). The gauge group is generically broken as

$$\prod_{i=1}^{G} \text{U}(\tilde{N} + M_i) \quad \rightarrow \quad \left(\text{U}(1)^G\right)^{\tilde{N}} \prod_{i=1}^{G} \text{U}(M_i) \qquad (2.32)$$

and the allowed bifundamental VEVs generically are

$$X_{ij} = \begin{pmatrix} X_{ij}^{\text{diag}} & 0 \\ 0 & 0 \end{pmatrix} \,, \qquad (2.33)$$

where X_{ij}^{diag} are $\tilde{N} \times \tilde{N}$ diagonal matrices. Depending on the quiver and the ranks, additional diagonal entries of X_{ij} might acquire VEV, but we consider here VEVs which are always allowed by (2.31). In the CFTs of the next sections, diagonal (2.33) are the most general VEVs allowed by generic (2.31).

Each of the $\tilde{N}$ U(1)^G factors represents a copy of the Abelian quiver, under which only the corresponding eigenvalues of X_{ij}^{diag} are charged. Between each copy and the remaining $\prod_i \text{U}(M_i)$ gauge groups there can be chiral fields that get massive on the Coulomb branch and should be integrated out, shifting the CS levels. On the other hand, for generic (2.31) the $\tilde{N}$ Abelian quivers decouple. Since permutations of eigenvalues are a residual gauge symmetry, the geometric branch of the CFT is an $\tilde{N}$-symmetric product of a U(1)^G quiver moduli space with D- and F-term equations

$$\xi_i^{\text{eff}}(\sigma) = \sum_{X_{ij}} |x_{ij}|^2 - \sum_{X_{ji}} |x_{ji}|^2 \,, \qquad \partial_{x_{ij}} W(x) = 0 \,, \qquad (2.34)$$

where $x_{ij} \in \mathbb{C}$. The equations for $\prod_i \mathrm{U}(M_i)$ are trivially solved. The effective CS terms are $k_i^{\mathrm{eff}} = k_i + \frac{1}{2} \mathrm{sign}(\sigma)\big(\sum_{X_{ij}} M_j - \sum_{X_{ji}} M_j\big)$, while the effective FI parameters are

$$\xi_i^{\mathrm{eff}}(\sigma) = \frac{k_i}{2\pi}\sigma + \frac{1}{4\pi}\bigg(\sum_{X_{ij}} M_j - \sum_{X_{ji}} M_j\bigg)|\sigma| \, , \qquad (2.35)$$

which is a particular case of formula (A.10) and includes the classical and one-loop contribution. At fixed σ (2.34) describes the CY$_3$ associated to the quiver, as is well known, with resolution parameters corresponding to $\xi_i^{\mathrm{eff}}(\sigma)$. Together they describe a CY$_3$ fibered over a line $\mathbb{R} \cong \{\sigma\}$.

We can rewrite (2.35) more compactly using the adjacency matrix A_{ij} of the quiver: the integer $A_{ij} = -A_{ji}$ is the net number of arrows from node i to node j. We have

$$\xi_i^{\mathrm{eff}}(\sigma) = \frac{k_i}{2\pi}\sigma + \frac{1}{4\pi}|\sigma| \sum_{j=1}^{G} A_{ij} \, N_j \, . \qquad (2.36)$$

Note that since the quivers we consider have as many incoming as outgoing arrows at each node, $\sum_j A_{ij} N_j = \sum_j A_{ij} M_j$ for $N_j = \tilde{N} + M_j$.

Each Abelian quiver provides a dual photon as well: the diagonal U(1) vector in U(1)^G can be dualized to φ as in [4, 24, 47], and together with the CY$_3$ bundle over $\mathbb{R}$ they make the moduli space a four-fold (in fact a CY$_4$). The dual photon shifts under the topological symmetry associated to the diagonal magnetic flux (while matter fields are invariant), and this topological symmetry maps to a U(1)$_M$ isometry of the four-fold. The main difference with respect to [24] is that the effective theory is one-loop corrected with respect to the "bare quiver", and the regions at $\sigma \gtrless 0$ are not continuously related. Indeed the quantized effective CS levels jump at $\sigma = 0$, and this is possible only if the circle parametrized by the dual photon shrinks there.[10]

Finally let us briefly consider switching on bare FI parameters ξ_i^{bare} and the extra eigenvalues $\tilde{\sigma}_m$ in

$$\sigma_i = \mathrm{diag}(\sigma_1, \ldots, \sigma_{\tilde{N}}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{M_i}) \qquad \forall \, i = 1, \ldots, G \, . \qquad (2.37)$$

The ξ_i^{bare} simply enter in V and therefore in (2.35); the $\tilde{\sigma}_m$ effectively provide real masses and so deform (2.35) further:

$$\xi_i^{\mathrm{eff}} = \xi_i^{\mathrm{bare}} + \frac{k_1}{2\pi}\sigma + \frac{1}{4\pi}\bigg(\sum_{X_{ij}}\sum_{m=1}^{M_j} |\sigma - \tilde{\sigma}_m| - \sum_{X_{ji}}\sum_{m=1}^{M_j} |\sigma - \tilde{\sigma}_m|\bigg) \, . \qquad (2.38)$$

This time $\prod_i \mathrm{U}(M_i) \to \mathrm{U}(1)^{\sum_i M_i}$ and their equations are non-trivial. Depending on the CS levels and ranks of the quiver, it is not generically possible to solve the D-term equations for the massless fields. However when it is possible, they provide resolution parameters of the geometric moduli space, as we will see in section 5.3. It is clear from (2.38) that every

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[10]To be more precise, since the dual photon is not gauge invariant, we should say that the global circle action — that shifts the dual photon and leaves the other fields invariant — has a fixed point.

time σ crosses one of the $\tilde{\sigma}_m$, a real mass changes sign, the effective CS levels jump and the effective field theory changes. Therefore at $\sigma = \tilde{\sigma}_m$ the dual photon degenerates. Finally, each of the resolutions modes $\tilde{\sigma}_m$ is also related to a dual photon: together they describe complexified Kähler classes of the four-fold.

To conclude, let us stress how the two approaches, semi-classical and via monopoles, agree: The SCFT chiral ring analysis of sections 2.2–2.3 reproduces the coordinate ring of a complex variety (it will turn out to be a CY_4 in our examples), which by (2.25) contains as a subvariety the CY_3 associated to the quiver. On the other hand the semi-classical approach reproduces the data of a IIA background: it yields a foliation of a CY_3 along $\mathbb{R}$. The full four-fold moduli space is achieved with the inclusion of the dual photon.

3 From M-theory to type IIA

Consider M2-branes probing a CY_4 geometry, which is the cone over a Sasaki-Einstein seven-manifold Y_7. In [12] Aganagic proposed an elegant method to derive the 3d low energy field theory on the M2-branes: one KK reduces M-theory along a wisely-chosen circle in the CY_4 in such a way that the resulting type IIA background is a CY_3 fibration along a real line $\mathbb{R} = \{r_0\}$ with RR 2-form flux.[11] The analysis of [12] was restricted to a $U(1)_M$ group acting freely outside the CY_4 singularity, but we will generalize it allowing fixed points of $U(1)_M$ below. We will call the resulting type IIA manifold $X_7 \equiv CY_4/U(1)_M$, which can be represented as a fibration over $\mathbb{R}$ of the CY_3 given by the Kähler quotient $CY_3 = CY_4//U(1)_M$. If the CY_4 is a cone over a Sasaki-Einstein Y_7, X_7 can also be viewed as a cone over the six-manifold $M_6 = Y_7/U(1)_M$. M2-branes are mapped to D2-branes, so the problem boils down to finding the field theory dual to transverse D-branes probing a CY_3 with 2-form fluxes. Much about this problem is known, mainly from the study of D3-branes on three-folds: the low energy worldvolume theory on the branes is a quiver gauge theory. We will focus on toric CY_4's which descend to toric CY_3's, because it is easy to analyze their geometry and the field theory dual to toric CY three-folds with D-branes is always known [50]; however the same logic goes through the more general non-toric case.

In type IIA the CY_3 is fibered along $\mathbb{R} = \{r_0\}$, with its Kähler moduli being linear functions of r_0. When the CY_4 is conical, the CY_3 fiber over $r_0 = 0$ is conical rather than resolved. Since $U(1)_M$ is in general non-trivially fibered over its base, the reduction also introduces RR F_2 flux in IIA. In addition (see appendix E) a flat dynamically quantized NSNS potential B_2 may appear. D2-branes at the CY_3 singularity break up into mutually BPS states, called *fractional D2-branes*, which in the large volume limit can be thought of as D4-branes wrapped on 2-cycles or D6-branes wrapped on 4-cycles, possibly with worldvolume fluxes F. It is easy to see from the Wess-Zumino (WZ) action that F_2 and $\mathcal{F} = \hat{B}_2 + F$ (hat means pull-back) induce CS terms in the three-dimensional action. The 3d theory is then a quiver gauge theory with CS terms.

With $\mathcal{N} = 2$ supersymmetry (the one preserved by M2-branes on a CY_4) the CS term is contained in the superspace piece $k \int d^4\theta \, V \, \Sigma(V) \supset k\big(\operatorname{Tr} \omega_{CS}(A) + 2\operatorname{Tr} D\sigma\big)$, where k is the CS level, V the vector multiplet, Σ the linear multiplet containing its field strength, ω_{CS}

[11]This is a topological statement: the IIA metric on a CY_3 fiber at fixed r_0 needs not be Ricci-flat.

the CS form, σ the real scalar in the vector multiplet and D the D-term. As we vary σ, that we will see corresponds to r_0, we get an effective FI parameter linear in σ. Recalling that FI parameters in the gauge theory measure resolution parameters of the transverse CY_3 geometry, we see that whenever F_2 and B_2 are constant, the Kähler parameters of the CY_3 are linear functions of r_0. Supersymmetry relates the constant F_2 flux through a 2-cycle in the CY_3 to the first derivative with respect to r_0 of the Kähler parameter of the 2-cycle.

So far we have assumed as in [12] that the KK reduction of the M-theory CY_4 geometry is not singular: in that case the integrals of F_2 and B_2 on the CY_3 are constant along r_0. This need not be the case, and in fact generically the $U(1)_M$ action degenerates. The simplest degeneration that we can allow are sets of fixed points for the $U(1)_M$ action. Such KK monopoles lead to D6-branes wrapping 4-cycles of the CY_3: if the wrapped 4-cycle is noncompact the D6-brane is visible also in M_6 (as studied in [19, 20]); if instead it is a compact 4-cycle hidden at the CY_3 singularity over $r_0 = 0$ the D6-brane does not appear in M_6. However they become visible as soon as a partial resolution of the CY_4 blows up the exceptional 4-cycle in the CY_3. In any case it is simple to trace their presence in the reduction of the CY_4 to X_7 because the cohomology class of F_2 in the CY_3 jumps at their location on the base $\mathbb{R}$. The type IIA background involves piecewise constant (in r_0) RR 2-form fluxes through 2-cycles, equal by supersymmetry to the first derivatives in r_0 of the piecewise linear Kähler parameters of the same 2-cycles. This behavior is reflected in the dual field theory: the D6-branes add matter fields (from D2-D6 strings) with real masses dependent on σ, leading to jumps in the effective CS levels (first derivatives of effective FI parameters) whenever a real mass changes sign.

In fact whenever the CY_3 has exceptional 4-cycles (this is the case if the CY_4 has exceptional 6-cycles), a necessary condition for the type IIA background to be in a geometric phase of the CY_3 sigma model is that compact D6-branes be present. We make the latter requirement since we will rely on the DBI-WZ action of D-branes to deduce their worldvolume field theory. Holomorphic 2-cycles inside exceptional 4-cycles in the CY_3 cannot flop, therefore their Kähler parameters must remain non-negative to keep the CY_3 fiber in a geometric phase over the entire $\mathbb{R}$ line. Since they are piecewise linear on $\mathbb{R}$, a suitable number of D6-branes wrapping the exceptional 4-cycle are needed.

The discussion above was general, but exploiting toric geometry allows us to be very explicit. After performing an $SL(3,\mathbb{Z})$ transformation such that the toric $U(1)_M$ corresponds to the vertical direction in the 3d toric diagram of the CY_4, the 2d toric diagram of the CY_3 is the vertical projection of the latter. Each $U(1)_M$ KK monopole (D6-brane) is identified by a pair of adjacent vertically aligned points in the 3d toric diagram of the CY_4 [19], and wraps a toric divisor in the CY_3 associated to the point in the 2d toric diagram that the pair of points projects to.

In the following we will analyze these issues, thus generalizing [19] to compact D6-branes, in the simplest model: the family of CY_4 cones over the so-called $Y^{p,q}(\mathbb{CP}^2)$ 7-manifolds [21, 22]. Notable members are $Y^{2,3} \equiv M^{3,2} = M^{1,1,1}$ and $Y^{p,0} = S^7/\mathbb{Z}_{3p}$. The type IIA reduction gives $\mathbb{C}^3/\mathbb{Z}_3$, which has an exceptional $\mathbb{CP}^2$ at the tip. A discussion of more general geometries is left for a companion paper [23].
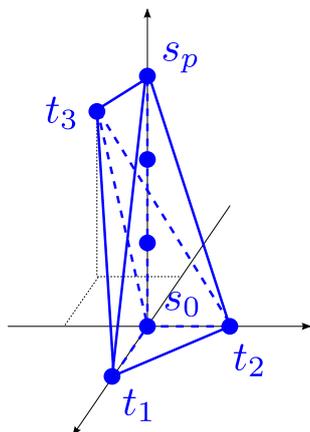
Figure 1. Toric diagram of $C(Y^{p,q})$ (drawn for $Y^{3,2}$).

3.1 The geometry of $Y^{p,q}(\mathbb{CP}^2)$

We consider, as a specific example, M2-branes in M-theory probing a family of toric CY four-folds which are cones over the Sasaki-Einstein (SE) seven-manifolds $Y^{p,q}(\mathbb{CP}^2)$ introduced in [21, 22]. $Y^{p,q}$ are $S^3/\mathbb{Z}_p$ bundles over $\mathbb{CP}^2$, parameterized by the integer q. The SE$_7$ metrics are known [21, 22], but we will not use them. The manifolds are smooth for $0 < q < 3p$; if $\gcd(p,q) = k > 1$, then $Y^{p,q} = Y^{p/k,q/k}/\mathbb{Z}_k$ where the orbifold group acts freely. When $q = 0, 3p$, $Y^{p,q} = Y^{1,q/p}/\mathbb{Z}_p$ but this time the resulting cone has a complex codimension-three line of orbifold singularities which locally looks like $\mathbb{C}^3/\mathbb{Z}_3$. Replacing $q \to 3p - q$ gives an identical geometry [24]. It corresponds to a change of orientation of the M-theory circle, as we explain in section 3.6 below.

The cone over $Y^{p,q}$ is a toric CY$_4$ and its toric diagram has five external points [22]

$$t_1 = (1,0,0) \quad t_2 = (0,1,0) \quad t_3 = (-1,-1,q) \quad s_0 = (0,0,0) \quad s_p = (0,0,p) \tag{3.1}$$

as in figure 1. Recall that the 3d toric diagram is defined up to SL$(3,\mathbb{Z})$ transformations. We have already arranged the diagram such that the Kähler reduction along U$(1)_M$ we are interested in corresponds to a vertical projection.

In general a toric CY$_4$, including all its toric crepant resolutions,[12] can be realized as the moduli space of a supersymmetric Abelian gauged linear sigma model (GLSM), quotiented by a finite Abelian group Λ. Calling $w_A \in \mathbb{Z}^3$ the points of the 3d toric diagram, the GLSM is obtained by assigning a complex field to each vector and assigning a U(1) gauge symmetry to each linear relation among $v_A = (1, w_A) \in \mathbb{Z}^4$. The Abelian group is then $\Lambda = \mathbb{Z}^4/\text{Span}_{\mathbb{Z}}\{v_A\}$. In our examples Λ is trivial.

---

[12] Remark that in complex dimension 2 crepant resolutions always exist and are unique, in dimension 3 always exist but need not be unique, in dimension 4 and bigger need not exist. The simplest example is $\mathbb{C}^4/\mathbb{Z}_k$, which does not have crepant resolutions. Whenever $\Lambda = \{1\}$, the toric CY can be completely smoothed out by toric crepant resolutions.

For the cones over $Y^{p,q}$, the GLSM can be written (excluding the last line, that will be relevant for the reduction to type IIA later):

CY$_4$	t_1	t_2	t_3	s_0	s_1	s_2	$\ldots$	s_{p-2}	s_{p-1}	s_p	FI
$\mathcal{C}$	1	1	1	$q-3$	$-q$	0	$\ldots$	0	0	0	ξ^c
$\mathcal{C}_2$	0	0	0	1	-2	1	$\ldots$	0	0	0	ξ_2
$\mathcal{C}_3$	0	0	0	0	1	-2	$\ldots$	0	0	0	ξ_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{C}_{p-1}$	0	0	0	0	0	0	$\ldots$	-2	1	0	ξ_{p-1}
$\mathcal{C}_p$	0	0	0	0	0	0	$\ldots$	1	-2	1	ξ_p
U(1)$_M$	0	0	0	1	-1	0	$\ldots$	0	0	0	r_0

$$(3.2)$$

The first line denotes the fields. The second line describes their charges under a U(1) subgroup, and the subset $\{t_1, t_2, t_3\}$ describes $\mathbb{CP}^2$. The following $p-1$ lines describe the GLSM for a $\mathbb{C}^2/\mathbb{Z}_p$ singularity, fibered over $\mathbb{CP}^2$. The last column includes FI parameters of the GLSM: they control resolutions of the geometry, and $\xi_{2,\ldots,p}$ should be non-negative to keep the GLSM in a geometric phase. By abuse of notation we have identified the p U(1) gauge groups with holomorphic 2-cycles $\mathcal{C}_a$.

In section 5 it will be useful to have an algebraic description of the CY$_4$ as a non-complete intersection. From the GLSM we construct the following gauge invariants:

$$a_{(3)} = t_{i_1} t_{i_2} t_{i_3} s_0 s_1 \ldots s_p \qquad\qquad b_{(q)} = t_{i_1} \ldots t_{i_q} s_1 s_2^2 \ldots s_p^p$$
$$c_{(3p-q)} = t_{i_1} \ldots t_{i_{3p-q}} s_0^p s_1^{p-1} \ldots s_{p-1}$$

$$(3.3)$$

where the subscripts indicate the number of symmetrized t_i. They identically satisfy the relations

$$b_{(q)}\, c_{(3p-q)} = a_{(3)}^p \,, \tag{3.4}$$

together with total symmetrization of SU(3) indices in their products.

The topology of $Y^{p,q}$ was discussed in [22]. The homology groups are

$Y^{p,q}$	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7
	$\mathbb{Z}$	$\mathbb{Z}_{\gcd(p,q)}$	$\mathbb{Z}$	Γ	0	$\mathbb{Z} \oplus \mathbb{Z}_{\gcd(p,q)}$	0	$\mathbb{Z}$

$$(3.5)$$

The most interesting one is the third homology group, which is torsion (we refer to our companion paper [23] for a detailed discussion):

$$H_3(Y^{p,q}, \mathbb{Z}) \cong H^4(Y^{p,q}, \mathbb{Z}) \cong \Gamma \,. \tag{3.6}$$

Γ is a finite Abelian group of dimension[13] $q(3p-q)$:

$$\Gamma = \mathbb{Z}^2 \, / \, \langle (3q, q), (q, p) \rangle \,. \tag{3.7}$$

Given the $Y^{p,q}$ manifold with its SE metric, the $\mathcal{N} = 2$ supersymmetric AdS$_4$ solution of 11d supergravity is

$$ds^2 = R^2 \left(\frac{1}{4}\, ds^2(AdS_4) + ds^2(Y^{p,q}) \right)$$
$$G_4 = \frac{3}{8} R^3\, d\text{vol}(AdS_4) \,. \tag{3.8}$$

---

[13]If $q = 0$ or $3p$, CY$_4 = (\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{C})/\mathbb{Z}_p$ and the torsion group is $\Gamma = \mathbb{Z}_p$.

There are N units of M2-brane charge on $Y^{p,q}$, where N is related to the radius R by:

$$\frac{1}{(2\pi l_p)^6} \int_{Y_7} *G_4 = N = \frac{6R^6}{(2\pi l_p)^6} \text{Vol}(Y^{p,q}) .$$ (3.9)

Turning on some torsion G_4 flux (which can be represented as a flat C_3 connection, see appendix E) on $Y^{p,q}$ does not affect the supergravity equations of motion, hence there is a distinct M-theory background for each element

$$[G_4] \in \Gamma .$$ (3.10)

We will discuss in section 5 the family of field theories dual to such backgrounds, for each torsion flux (3.10). In terms of the CY_4, such a solution corresponds to N M2-branes at the tip, together with M5-branes wrapped on the relevant torsion cycle in $H_3 \cong \Gamma$.

3.2 The geometric reduction of $C(Y^{p,q})$ to type IIA

We proceed to KK reduce M-theory on the CY_4 to type IIA on the seven-manifold $X_7 = CY_4/U(1)_M$. The reduction is performed along an isometry circle $U(1)_M$, chosen such that the Kähler quotient $CY_4//U(1)_M$ is a CY_3. In terms of the GLSM description, this happens if the charges of the fields under $U(1)_M$ sum up to zero [12].

The last line in (3.2) defines our choice of $U(1)_M$ symmetry acting on the M-theory circle. As we anticipated, the symmetry $U(1)_M$ acts as the vector $s_0 - s_1$ on the 3d toric diagram (as subgroup of the $U(1)^4$ toric symmetry it is associated to the vector $(0,0,0,-1)$ in $\mathbb{Z}^4$). It corresponds to the vertical direction in figure 1. Including the last line in the GLSM (3.2) yields a toric CY three-fold which is the Kähler quotient $CY_3 = CY_4//U(1)_M$. Its 2d toric diagram is the vertical projection of the 3d toric diagram of the four-fold.

However the type IIA geometry consists of a simple quotient, as opposed to a Kähler quotient, by $U(1)_M$. Therefore we keep the moment map r_0 as an unconstrained real field. The type IIA geometry then involves a fibration of the previous CY_3 over the real line, parametrized by the moment map r_0 [12]. To obtain the precise form of the fibration, let us rearrange the charges in (3.2). First define

$$\zeta_0 = -\infty , \qquad \zeta_1 = 0 , \qquad \zeta_a = \sum_{b=2}^{a} \xi_b \quad (a = 2, \ldots, p) , \qquad \zeta_{p+1} = +\infty .$$ (3.11)

Given the constraint $\xi_a \geq 0$, they satisfy $\zeta_a \leq \zeta_{a+1}$. Then rewrite the GLSM, including the last line, as

CY_3	t_1	t_2	t_3	s_0	s_1	s_2	$\cdots$	s_{p-2}	s_{p-1}	s_p	FI
$\mathcal{C}$	1	1	1	$q-3$	$-q$	0	$\cdots$	0	0	0	ξ^c
	0	0	0	1	-1	0	$\cdots$	0	0	0	$r_0 - \zeta_1$
	0	0	0	0	1	-1	$\cdots$	0	0	0	$r_0 - \zeta_2$
	0	0	0	0	0	1	$\cdots$	0	0	0	$r_0 - \zeta_3$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	0	0	0	0	$\cdots$	1	-1	0	$r_0 - \zeta_{p-1}$
	0	0	0	0	0	0	$\cdots$	0	1	-1	$r_0 - \zeta_p$

(3.12)

This is a redundant description of the CY$_3$, as all but one of the $s_{a=0,...,p}$ coordinates can be eliminated. Which one of the s_a is unconstrained depends on the value of r_0: given a GLSM with two fields, charges $(1, -1)$ and FI ξ, the variable that can be eliminated is the one with charge sign(ξ). Therefore the unconstrained variable is

$$t_0 = s_a \quad \text{if} \quad \zeta_a \le r_0 \le \zeta_{a+1} \ . \tag{3.13}$$

We can rewrite the CY$_3$ GLSM as

$$\frac{\text{CY}_3 \begin{array}{cccc} t_1 & t_2 & t_3 & t_0 \end{array} \ \bigg| \ \text{FI}}{\mathcal{C} \ \begin{array}{cccc} 1 & 1 & 1 & -3 \end{array} \bigg| \chi(r_0)} \tag{3.14}$$

which describes a resolved $\mathbb{C}^3/\mathbb{Z}_3$ with a blown-up $\mathbb{CP}^2$ of size $\chi(r_0)$. The holomorphic 2-cycle $\mathcal{C}$ is the hyperplane $\mathbb{CP}^1 \subset \mathbb{CP}^2$, and it is exceptional. The FI parameter is

$$\chi(r_0) = \xi^c - q\,(r_0 - \zeta_1) + 3 \sum_{b=1}^{p} (r_0 - \zeta_b)\,\Theta(r_0 - \zeta_b) \ , \tag{3.15}$$

where $\Theta(x)$ is the Heaviside step function. The Kähler parameter $\chi(r_0)$ is continuous in r_0, while its first derivative jumps by 3 at $r_0 = \zeta_{a=1,...,p}$ (where the unconstrained coordinate jumps from s_{a-1} to s_a). As we will shortly see, this is due to the presence of a $\overline{\text{D6}}$-brane wrapping the toric divisor $\mathbb{CP}^2$ in $\mathbb{C}^3/\mathbb{Z}_3$.[14]

If the CY$_4$ is conical, that is $\xi^c = \zeta_a = 0$ for all a, the resolution parameter is

$$\chi(r_0) = \big[-q + 3p\,\Theta(r_0) \big]\, r_0 \ . \tag{3.16}$$

We can read the intersection numbers of $\mathbb{C}^3/\mathbb{Z}_3$ from (3.14). There is a compact toric divisor D_0, which is the exceptional blown-up $\mathbb{CP}^2$, and noncompact divisors D_i $(i = 1, 2, 3)$, subject to linear equivalences $D_1 = D_2 = D_3$ and $D_0 = -3D_1$. Abusing notation we use D_α for the cohomology class Poincaré dual to the toric divisor $D_\alpha = \{t_\alpha = 0\}$. The compact 2-cycle $\mathcal{C}$ is the $\mathbb{CP}^1$ inside $\mathbb{CP}^2$. The intersections are:

$$\mathcal{C} \cdot D_0 = -3\mathcal{C} \cdot D_i = -3 \ , \qquad D_0 \cdot D_0 \cdot D_0 = 9 \ , \qquad D_0 \cdot D_0 = -3\mathcal{C} \ . \tag{3.17}$$

3.3 RR F_2 flux and D6-branes

Since the M-theory circle is non-trivially fibered, the U(1)$_M$ connection gives a RR 2-form field strength in type IIA. In the CY$_4$ the fibration is encoded in a vielbein [12]

$$\mathbb{Re} \left(d\theta_0 + i \sum_{j=1}^{3} q_j \frac{dt_j}{t_j} + i \sum_{a=0}^{p} b_a \frac{ds_a}{s_a} \right) , \tag{3.18}$$

from which we read off the U(1)$_M$ connection. Its curvature is a 2-cocycle

$$[F_2] = \sum_{i=1}^{3} q_i D_i + \sum_{a=0}^{p} b_a D_{(s_a)} \tag{3.19}$$

---

[14]We explain why they are $\overline{\text{D6}}$-branes rather than D6 in section 4.

where $[\cdot]$ stands for cohomology class. The integers q_i, b_a are fixed by gauge invariance of (3.18): $0 = q_1 + q_2 + q_3 + (q-3)s_0 - qs_1 = b_a - 2b_{a+1} + b_{a+2}$ for all $a = 2, \ldots, p$ and $1 = b_0 - b_1$. The system is solved if the coefficient in front of each divisor class equals minus the value of the vertical coordinate of the corresponding point in the 3d toric diagram. The solution is unique in cohomology, i.e. up to linear equivalences:

$$[F_2] = -qD_3 - \sum_{a=1}^{p} a\, D_{(s_a)} \,. \tag{3.20}$$

This expression is still in terms of the redundant GLSM for the CY_3. In terms of the reduced GLSM in (3.14), in the window $\zeta_a \leq r_0 \leq \zeta_{a+1}$ where the unconstrained coordinate is $t_0 = s_a$, we have[15] $[F_2] = -qD_3 - aD_0$. Therefore we can write the general expression

$$[F_2] = -qD_3 - D_0 \sum_{a=1}^{p} \Theta(r_0 - \zeta_a) \,. \tag{3.21}$$

The flux $[F_2]$ jumps by $-D_0$ at $r_0 = \zeta_{a=1,\ldots,p}$: such discontinuity is due to a magnetic source for F_2—a $\overline{\text{D6}}$-brane wrapping $\mathbb{CP}^2$ at $r_0 = \zeta_a$. The $\mathbb{C}^2/\mathbb{Z}_p$ Kähler parameters ξ_a are thus the separations between p $\overline{\text{D6}}$-branes on $\mathbb{CP}^2$ along r_0. When the CY_4 is conical, that is $\xi^c = \xi_a = 0$, the type IIA background has p coincident $\overline{\text{D6}}$-branes wrapping the collapsed $\mathbb{CP}^2$ at $r_0 = 0$.

The flux of F_2 on the holomorphic 2-cycle $\mathcal{C} = \mathbb{CP}^1$ can be obtained from its intersection numbers with the toric divisors:

$$\int_{\mathcal{C} \text{ at } r_0} F_2 = -q + 3\sum_{a=1}^{p} \Theta(r_0 - \zeta_a) = \chi'(r_0) \,. \tag{3.22}$$

The equality between 2-form fluxes and derivatives of the Kähler parameters is a consequence of supersymmetry.

3.4 The manifold M_6

The manifold M_6 which appears in the type IIA supergravity background $AdS_4 \times_w M_6$, is defined as $Y^{p,q}/\text{U}(1)_M$ [22]. Recall that $Y^{p,q}$ is an $S^3/\mathbb{Z}_p$ bundle over $\mathbb{CP}^2$. S^3 has $\text{SU}(2) \times \text{SU}(2)_{\text{base}}$ symmetry, while the $\mathbb{Z}_p$ quotient breaks the first factor to $\text{U}(1)_M$. The circle $\text{U}(1)_M$ is the fiber in the Hopf fibration $S^1 \hookrightarrow S^3/\mathbb{Z}_p \to S^2$ of the lens space. Therefore M_6 is an S^2 bundle over $\mathbb{CP}^2$,

$$S^2 \hookrightarrow M_6 \to \mathbb{CP}^2 \,. \tag{3.23}$$

On the other hand $\text{SU}(2)_{\text{base}}$ acts on the base of the Hopf fibration. In the fibration over $\mathbb{CP}^2$, $S^3/\mathbb{Z}_p$ is twisted by a $\text{U}(1)_{\text{twist}} \subset \text{SU}(2)_{\text{base}}$. As a result, in M_6 the S^2 fiber is twisted on $\mathbb{CP}^2$ by $\text{U}(1)_{\text{twist}}$. This has two fixed points: the north and the south pole.

---

[15]Given the GLSM with two fields, charges $(1, -1)$ and FI ξ, the toric divisor corresponding to the variable with charge $\text{sign}(\xi)$ is empty.

The homology of M_6 is

$$
\begin{array}{c|ccccccc}
M_6 & H_0 & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \\
\hline
& \mathbb{Z} & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}
\end{array}
\tag{3.24}
$$

Since the poles of S^2 are invariant under $\mathrm{U}(1)_{\mathrm{twist}}$, one can construct push forward maps $\sigma_{N,S} : \mathbb{CP}^2 \to M_6$ which uplift $\mathbb{CP}^2$ to global sections of the bundle [22]. We define the following representatives of $H_4(M_6)$:

$$
S^2 \hookrightarrow D \to \mathbb{CP}^1 \,, \qquad D^+ = \sigma_N \mathbb{CP}^2 \,, \qquad D^- = \sigma_S \mathbb{CP}^2 \,.
\tag{3.25}
$$

Here D is the restriction of the S^2 bundle to the hyperplane $\mathbb{CP}^1$ in the base $\mathbb{CP}^2$. The 4-cycles (D, D^-, D^+) form an over-complete but convenient basis for $H_4(M_6)$. Similarly we define the 2-cycles

$$
\mathcal{C}_0 \cong S^2 \,, \qquad \mathcal{C}^+ = \sigma_N \mathbb{CP}^1 \,, \qquad \mathcal{C}^- = \sigma_S \mathbb{CP}^1 \,,
\tag{3.26}
$$

where $\mathcal{C}_0$ is the S^2 fiber over any point in $\mathbb{CP}^2$. One can show that in homology

$$
D^+ = D^- + 3D \,, \qquad \mathcal{C}^+ = \mathcal{C}^- + 3\mathcal{C}_0 \,.
\tag{3.27}
$$

The intersections between 2-cycles and 4-cycles are easily worked out from their definitions (recall that $\mathbb{CP}^1 \cdot \mathbb{CP}^1 = 1$ in $\mathbb{CP}^2$) and the linear relations above:

$$
\begin{array}{c|ccc|ccc}
& \mathcal{C}_0 & \mathcal{C}^+ & \mathcal{C}^- & D & D^+ & D^- \\
\hline
D & 0 & 1 & 1 & \mathcal{C}_0 & \mathcal{C}^+ & \mathcal{C}^- \\
D^+ & 1 & 3 & 0 & \mathcal{C}^+ & 3\mathcal{C}^+ & 0 \\
D^- & 1 & 0 & -3 & \mathcal{C}^- & 0 & -3\mathcal{C}^-
\end{array}
\tag{3.28}
$$

We have seen that if the original CY_4 is a cone, the type IIA manifold X_7 can be sliced in two different ways: either as a $\mathrm{CY}_3 = \mathrm{CY}_4/\!/\mathrm{U}(1)_M$ fibration along a real line parametrized by r_0, or as a real cone over $M_6 = Y_7/\mathrm{U}(1)_M$. While the latter is manifest in the type IIA supergravity solution, the former is more useful to identify the dual field theory. In general cycles of M_6 are not cycles of the CY_3 and viceversa. However — in the conical case — the north pole (south pole) of the S^2 fiber of M_6 is the resolved tip of $\mathbb{C}^3/\mathbb{Z}_3$ at $r_0 > 0$ ($r_0 < 0$). We have:

$$
\begin{aligned}
\mathcal{C}^+ = \mathbb{CP}^1 \,, \quad D^+ = \mathbb{CP}^2 \qquad \text{at } r_0 > 0 \\
\mathcal{C}^- = \mathbb{CP}^1 \,, \quad D^- = \mathbb{CP}^2 \qquad \text{at } r_0 < 0
\end{aligned}
\tag{3.29}
$$

where $\mathbb{CP}^1$ and $\mathbb{CP}^2$ are at the tip of the resolved $\mathbb{C}^3/\mathbb{Z}_3$. If the CY_4 itself is resolved, one has to be careful that M_6 degenerates (some of its cycles shrink) at the scale of the resolutions, but the map is still valid for $|r_0|$ big enough. These are the only cycles which are common to the CY_3 and M_6. The relation (3.29) can be understood by noting that topologically M_6 is the projectivisation of the anti-canonical bundle over $\mathbb{CP}^2$ [21]; see [23] for a more detailed discussion.

3.5 Page charges and Freed-Witten anomaly

We move to consider the charges present in the type IIA supergravity solution. We want to study D-brane charges that are conserved and quantized. The most useful notion is that of Page charges [51]. In the context of AdS/CFT, they have first been applied to 4d theories in [52–56] to study duality cascades, and then in [57–59] to 3d theories. Page charges have the advantage that they are sourced only by branes and the worldvolume flux F on them, not by $\hat{B}_2$ or background fluxes. On the other hand they are not gauge-invariant: they transform under large gauge transformations of the B-field.

The most interesting charge is the IIA reduction of the torsion flux (3.10). Since G_4 vanishes in real cohomology, the M-theory flat connection C_3 descends in type IIA to a flat B-field B_2. The gauge invariant RR 4-form flux vanishes ($F_4 = 0$). Nevertheless the Page D4-charge is non-vanishing, and quantized:

$$[G_4] \ \to \ [B_2 \wedge F_2] \neq 0 \ . \tag{3.30}$$

We explicitly show this for $Y^{p,q}$ in appendix E. Gauge transformations of the Page charge are crucial to reproduce the periodicity (3.7) of the torsion group Γ. The following analysis generalizes the one in [57].

Let us start with the D6-charge. On M_6 the RR 2-form is [22]

$$[F_2] = p \, D^+ - q \, D \ . \tag{3.31}$$

It will be convenient to use the homology basis $\{-\mathcal{C}_0, \mathcal{C}^+\}$ and its dual basis $\{-D^-, D\}$. Accordingly, from the intersection numbers the D6-charges are:

$$Q_{6;0} \equiv \int_{-\mathcal{C}_0} F_2 = -p \ , \qquad Q_{6;+} \equiv \int_{\mathcal{C}^+} F_2 = 3p - q \ , \qquad Q_{6;-} \equiv \int_{\mathcal{C}^-} F_2 = -q \ . \tag{3.32}$$

We have included the third linearly dependent charge for completeness.

Next we consider the Page D4-charge. Let us parameterize the flat B-field as

$$[B_2] = -b_0 \, D^- + b^+ \, D \tag{3.33}$$

where the coefficients have been chosen in such a way that

$$b_0 \equiv \int_{-\mathcal{C}_0} B_2 \ , \qquad b^+ \equiv \int_{\mathcal{C}^+} B_2 \ , \qquad b^- \equiv \int_{\mathcal{C}^-} B_2 = b^+ + 3b_0 \ . \tag{3.34}$$

Since $F_4 = F_0 = 0$, the Page D4-charge is simply $\int F_2 \wedge B_2$ which depends on p, q and b_0, b^+. The potentials b_0, b^+ are not kinematically quantized in type IIA (although they are periodically identified). However Page D4-charges are quantized. We define them as integrals on the dual basis $\{-D^-, D\}$ and parameterize them by two integers

$$(n_0 \, , \, n_1) \in \mathbb{Z}^2 \ . \tag{3.35}$$

We get (see below for an anomalous correction):

$$
\begin{aligned}
Q_{4;-} &= \int_{-D^-} F_2 \wedge B_2 = q(3b_0 + b^+) \equiv n_0 \\
Q_{4;D} &= \int_{D} F_2 \wedge B_2 = qb_0 + pb^+ \equiv n_1 \\
Q_{4;+} &= \int_{-D^+} F_2 \wedge B_2 = (3p - q)b^+ \ .
\end{aligned}
\tag{3.36}
$$

These relations can be inverted, to express the B-field in terms of the integers (n_0, n_1):

$$
b_0 = \frac{p\, n_0 - q\, n_1}{q(3p - q)} \ , \qquad b^+ = \frac{3n_1 - n_0}{3p - q} \ , \qquad b^- = \frac{n_0}{q} \ , \tag{3.37}
$$

from which the dynamical quantization of the B-field follows. In this discussion we just assumed that Page D4-charges are integers: in fact due to Freed-Witten anomalies the quantization is shifted by half-integers, as we show below.

The periodicity (3.7) is realized as the following large gauge transformations:

$$
\begin{aligned}
\delta(n_0,\, n_1) &= (3q, q) & \longleftrightarrow & \qquad \delta(b_0,\, b^+) = (1, 0) \\
\delta(n_0,\, n_1) &= (q, p) & \longleftrightarrow & \qquad \delta(b_0,\, b^+) = (0, 1)
\end{aligned}
\tag{3.38}
$$

which generate $\mathbb{Z}^2$ in the B-field space.

Freed-Witten anomaly. We can engineer the F_2 flux (3.31) on M_6 by wrapping $p\ \overline{D6}$-branes around D^+ and q D6-branes around D, at a finite radius on AdS_4, and letting them fall towards the origin. The 4-cycles D^+ and D both suffer from the Freed-Witten (FW) anomaly [60]. The anomaly on a D6-brane is canceled by a half-integrally quantized worldvolume flux on all 2-cycles on which the second Stiefel-Whitney class is non-vanishing (i.e. the first Chern class is odd). In fact we find that we can cancel the anomaly on any D6-brane using the pull-back of a single bulk 2-form, that we loosely call F.

Consider a D6-brane wrapped on $(-D^+)$. At finite radius it is a domain wall which increases p by one:

$$
[F_2] \to [F_2] + D^+ \ , \qquad \leftrightarrow \qquad (p, q) \to (p+1, q) \ . \tag{3.39}
$$

The first Chern class of D^+ is $c_1(D^+)|_{D^+} = 3D \cdot D^+$, which is odd. Let us take the bulk 2-form F as $F = \frac{1}{2}D$. Its pull-back on the D6-brane is

$$
F\Big|_{-D^+} = -\frac{1}{2}D \cdot D^+ = -\frac{1}{2}\mathcal{C}^+ \ , \tag{3.40}
$$

which cancels the FW anomaly. It also shifts the Page D4-charges according to

$$
\delta Q_{4;-} = 0 \ , \qquad \delta Q_{4;+} = \frac{3}{2} \ , \qquad \delta Q_{4;D} = -\frac{1}{2} \ . \tag{3.41}
$$

Similarly, a D6-brane wrapped on D increases q by one:

$$
[F_2] \to [F_2] - D \ , \qquad \leftrightarrow \qquad (p, q) \to (p, q+1) \ . \tag{3.42}
$$

The first Chern class is $c_1(D)|_D = (-D + 2D^+) \cdot D$, so that the anomaly is canceled by the same $F = \frac{1}{2}D$ as before. The pull-back on the D6-brane is $F\big|_D = \frac{1}{2}D \cdot D = \frac{1}{2}\mathcal{C}_0$, which gives rise to a shift of Page D4-charges $\delta Q_{4;\,-} = \delta Q_{4;\,+} = -\frac{1}{2}$ and $\delta Q_{4;\,D} = 0$. We conclude that for the $Y^{p,q}$ geometry the Page D4-charges (3.36) are shifted to

$$Q_{4;\,-} = n_0 - \frac{q}{2}\,, \qquad Q_{4;\,D} = n_1 - \frac{p}{2}\,, \qquad Q_{4;\,+} = n_0 - 3n_1 - \frac{1}{2}(q - 3p)\,. \qquad (3.43)$$

The correct B-field periods are therefore

$$b_0 = \frac{p\,n_0 - q\,n_1}{q(3p - q)}\,, \qquad b^+ = -\frac{1}{2} + \frac{3n_1 - n_0}{3p - q}\,, \qquad b^- = -\frac{1}{2} + \frac{n_0}{q}\,, \qquad (3.44)$$

In particular, we have $B_2 = -\frac{1}{2}D$ at the torsionless point $n_0 = n_1 = 0$, such that $\mathcal{F} = \hat{B}_2 + F = 0$ on the D6-branes.

Page D2-charge. The Page D2-charge of our background with $F_4 = F_0 = 0$ reads

$$Q_2 = \int_{M_6} \left(*F_4 - \frac{1}{2}B_2 \wedge B_2 \wedge F_2 \right)\,. \qquad (3.45)$$

Remark that $Q_2 = Q_2^{\mathrm{Maxwell}} + \tilde{Q}_2$, where we defined

$$\tilde{Q}_2 \equiv -\frac{1}{2}\int_{M_6} B_2 \wedge B_2 \wedge F_2\,. \qquad (3.46)$$

At the torsionless point $(n_0 = n_1 = 0)$ $Q_2^{\mathrm{Maxwell}} = N$. Note that there are higher curvature corrections similarly to [61], but we neglect such contributions in this paper since they do not affect our line of argument. The Page D2-charge is therefore

$$Q_2 = N - \frac{p}{8}\,. \qquad (3.47)$$

As we move in the torsion group Γ, the value of $\tilde{Q}_2$ changes according to (3.44):

$$\tilde{Q}_2 = -\frac{p}{2}(b^+)^2 - \frac{q}{2}b_0(3b_0 + 2b^+) = -\frac{p}{8} - \frac{pn_0^2 + qn_1(3n_1 - 2n_0 - 3p + q)}{2q(3p - q)}\,. \qquad (3.48)$$

Since Page charges are not sourced by B_2, the D2-charge Q_2 is invariant as we vary B_2 continuously, while the Maxwell D2-charge changes accordingly. On the other hand Q_2 shifts by integers under the large gauge transformations (3.38). We have:

$$\begin{aligned}
\delta(b_0, b^+) = (1, 0) \quad &: \quad \delta Q_2 \equiv Q_2[n_0 + 3q, n_1 + q] - Q_2[n_0, n_1] \quad = -n_0 - q \\
\delta(b_0, b^+) = (0, 1) \quad &: \quad \delta Q_2 \equiv Q_2[n_0 + q, n_1 + p] - Q_2[n_0, n_1] \quad = -n_1\,.
\end{aligned} \qquad (3.49)$$

3.6 Remarks on parity and fundamental domain for Γ

There are two interesting parity transformations in supergravity. First, there is the $\mathrm{U}(1)_M$-parity which changes sign to the M-theory circle coordinate. In type IIA it corresponds to

$$F_2 \to -F_2\,, \qquad B_2 \to -B_2\,, \qquad \text{with} \quad D^+ \leftrightarrow -D^-\,. \qquad (3.50)$$
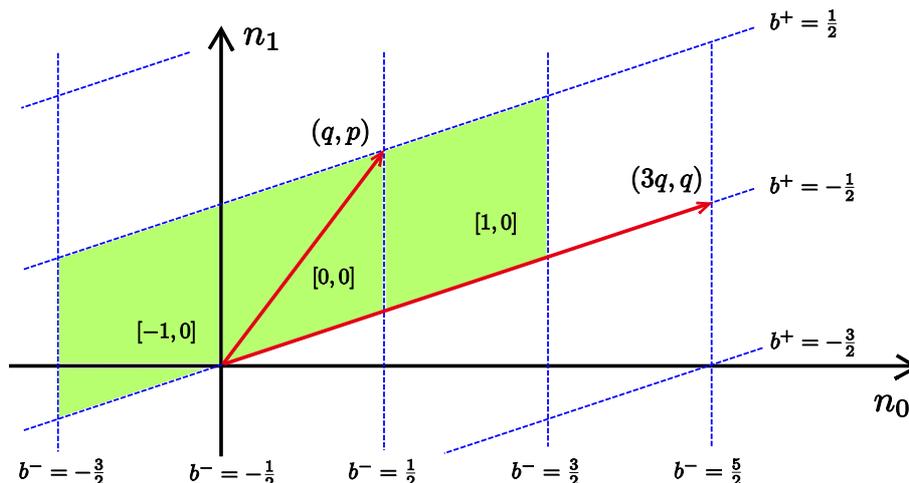
Figure 2. The (n_0, n_1) plane. The torsion group Γ is obtained by quotienting this $\mathbb{Z}^2$ lattice by the sublattice generated by the periodicity vectors (q, p) and $(3q, q)$ shown in red. The shaded area is a choice of fundamental domain, where the opposite sides are identified according to the red vectors (for instance the lower boundary of the area $[0, 0]$ is identified with the upper boundary of the area $[1, 0]$). The three parallelograms denoted $[-1, 0]$, $[0, 0]$ and $[1, 0]$ correspond to the three windows needed to cover Γ once, as will be explained in section 5. The dashed blue lines correspond to loci where either b^- or b^+ is half integer, as indicated.

In the $\mathrm{CY}_3 \times \mathbb{R}$ description, it also corresponds to $r_0 \to -r_0$. One easily checks that this M-theory parity can be undone by the following change of parameters:

$$(p, q) \to (p, 3p - q), \qquad (n_0, n_1) \to (3n_1 - n_0, n_1). \tag{3.51}$$

Indeed the space $Y^{p,3p-q}$ is identical to $Y^{p,q}$ [24].

Second, there is the usual parity in M-theory, which changes sign to one spatial coordinate in $\mathbb{R}^{1,2}$ and to C_3. In type IIA it corresponds to parity in $\mathbb{R}^{1,2}$ and

$$B_2 \to -B_2, \tag{3.52}$$

while F_2 is invariant. D6-charges (p, q) are left invariant, while the D4-charges (3.43) change sign. This corresponds to the change of parameters

$$n_0 \to q - n_0, \qquad n_1 \to p - n_1. \tag{3.53}$$

In the field theory this operation corresponds to parity times charge conjugation (CP).

For each $Y^{p,q}$ there are only few type IIA backgrounds invariant under M-theory parity, among the family indexed by $(n_0, n_1) \in \Gamma$. One is the torsionless background: $n_0 = n_1 = 0$. The B-field is $b^+ = b^- = -\frac{1}{2}$ and the parity transformation (3.52) gives an identical background up to a large gauge transformation.[16] Another one, which exists only if q

---

[16]The parity transformed theory is at $(n_0, n_1) = (q, p)$. It is identified with the theory at $(n_0, n_1) = (0, 0)$ by a large gauge transformation, with vanishing shift of D2-charge (3.49).

and p are even, is at $(n_0, n_1) = \left(\frac{q}{2}, \frac{p}{2}\right)$ and therefore $b^+ = b^- = 0$. A third one is at $(n_0, n_1) = \left(2q, \frac{p+q}{2}\right)$ corresponding to $(b_0, b^+) = \left(\frac{1}{2}, 0\right)$. It is also mapped to itself by (3.53) together with a large gauge transformation that does not shift the D2-charge.

The torsion group Γ is represented in figure 2. The shaded area is a convenient choice of fundamental domain. The full plane is divided into "windows" by vertical dashed lines, corresponding to values of n_0 for which b^- is half-integral, and diagonal dashed lines, corresponding to b^+ being half-integral. As explained below, those are marginal stability walls in the Kähler moduli space of the resolved $\mathbb{C}^3/\mathbb{Z}_3$ geometry. In section 5 we will call window $[n, m]$ the one where $b^- \in \left[n-\frac{1}{2},\, n+\frac{1}{2}\right]$ and $b^+ \in \left[m-\frac{1}{2},\, m+\frac{1}{2}\right]$. The fundamental domain is divided into three windows. Under the parity transformation (3.53), window $[0, 0]$ is mapped to itself, whilst windows $[1, 0]$ and $[-1, 0]$ are exchanged.

3.7 Limiting geometry at $q = 0$ or $3p$

When $q = 0$ (or equivalently $q = 3p$) the $Y^{p,q}$ geometry has additional orbifold singularities. Consider for definiteness the case $q = 0$. The CY_4 geometry is an orbifold of flat space, $(\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{C})/\mathbb{Z}_p$, with

$$(z_1,\ z_2,\ z_3,\ z_4) \sim \left(e^{-\frac{2\pi i}{3p}} z_1,\ e^{-\frac{2\pi i}{3p}} z_2,\ e^{-\frac{2\pi i}{3p}} z_3,\ e^{\frac{2\pi i}{p}} z_4\right), \qquad (3.54)$$

as one can show from the toric diagram (3.1). The M_6 manifold that appears in type IIA has the topology of the weighted projective space $W\mathbb{CP}^3_{[1,1,1,3]}$ [22]. The $CY_3 \times \mathbb{R}$ background has a $\mathbb{C}^3/\mathbb{Z}_3$ singularity along $r_0 \leq 0$, with vanishing RR fluxes on the blown down $\mathbb{CP}^2$:

$$\int_{\mathbb{CP}^1 \text{ at } r_0 < 0} F_2 = 0\,, \qquad\qquad \int_{\mathbb{CP}^2 \text{ at } r_0 < 0} F_2 \wedge B_2 = 0\,, \qquad (3.55)$$

and B-field $b^- = -\frac{1}{2}$. The manifold M_6 has a single 2-cycle in homology, which we still denote $\mathcal{C}^+$, and a dual 4-cycle D, with $D \cdot \mathcal{C}^+ = 3$. $W\mathbb{CP}^3_{[1,1,1,3]}$ has an isolated $\mathbb{Z}_3$ singularity, where an exceptional 2-cycle $\mathcal{C}_0$ lives — if we blow up that 2-cycle we recover the same topology as the M_6 for $q > 0$ [22], hence the terminology.

In the orbifold (3.54) the $\mathbb{Z}_p$ subgroup acts freely and gives rise to a torsion group $\Gamma = \mathbb{Z}_p$ in the seven-dimensional base. This is reproduced by the IIA geometry, where the Page D4-charge and B-field periods are

$$Q_{4;\,D} = \int_D F_2 \wedge B_2 = n_1 - \frac{p}{2}\,, \qquad b^+ = \int_{\mathcal{C}^+} B_2 = \frac{n_1}{p} - \frac{1}{2}\,. \qquad (3.56)$$

In this last formula we included the shift due to the Freed-Witten anomaly, which arises as in the $q > 0$ case.

4 Fractional branes on $\mathbb{C}^3/\mathbb{Z}_3$

D-branes on the orbifold $\mathbb{C}_3/\mathbb{Z}_3$ have been studied in detail in the literature [62–65]. At the singularity, the point-like D-brane (a D2-brane in our context) is a marginal bound state of three fractional branes, which we denote by $E_1^\vee$, $E_2^\vee$, $E_3^\vee$. The dynamics of the

fractional branes near the singularity is well described by the quiver of figure 3 below. Each fractional brane corresponds to one node of the quiver. The $\mathbb{C}^3/\mathbb{Z}_3$ orbifold admits a crepant resolution to $\mathcal{O}_{\mathbb{CP}^2}(-3)$, the canonical bundle over $\mathbb{CP}^2$. As we resolve the singularity, fractional branes are best seen as D-branes wrapping holomorphic cycles (B-branes). In this section we review in some details how to translate between the orbifold basis and the B-brane basis, or in other words between the quiver description and the CY_3 geometry.

The non-compact CY_3 $\mathcal{O}_{\mathbb{CP}^2}(-3)$ has a Kähler moduli space of complex dimension one, with Kähler parameter

$$t \equiv \int_H (B_2 + iJ) \equiv b + i\chi \,, \tag{4.1}$$

where H is the hyperplane curve $\mathbb{CP}^1 \subset \mathbb{CP}^2$, J is the Kähler form and B_2 is the B-field. We are interested in D-branes which are wrapped on the compact cycles $\mathbb{CP}^2$, H and on a point, corresponding to D6-, D4- and D2-branes, respectively. The brane charge we consider is the Chern character of the B-brane, which we write as a vector:

$$ch(E^\vee) = (Q_6, Q_4, Q_2) = (r, c_1, ch_2) \,, \tag{4.2}$$

where r, c_1 and ch_2 are the rank, the first and the second Chern characters of the vector bundle. Therefore, a D6-brane with worldvolume flux $F = nH$ would have $ch(D6) = (1, n, \frac{1}{2}n^2)$. This is however too naive, because of the Freed-Witten anomaly [60]. The manifold $\mathbb{CP}^2$ is not spin but only spin^c, and in order to define the spin^c bundle we need to turn on half-integral flux F on the D6-brane wrapped on $\mathbb{CP}^2$. With the minimal choice $F = \frac{1}{2}H$, the D6-brane has charge

$$ch(D6) = \left(1, \frac{1}{2}, \frac{1}{8}\right) \,, \tag{4.3}$$

while the naive D6-brane with Chern character $(1, 0, 0)$ does not exist in the physical spectrum. On the other hand, we have $ch(D4) = (0, 1, 0)$ and $ch(D2) = (0, 0, 1)$ by definition. An important quantity characterizing a D-brane state $E^\vee$ is its central charge Z, which tells us which half of the 8 supercharges of the CY_3 background is preserved by $E^\vee$. We have

$$Z(E^\vee) = ch(E^\vee) \cdot \Pi \,, \qquad \text{with} \quad \Pi = (\Pi_6, \Pi_4, \Pi_2) \,, \tag{4.4}$$

where Π_6, Π_4 and Π_2 are so-called *periods* associated to the states with Chern characters $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. At large volume ($\chi \to \infty$), the central charge is given explicitly by

$$Z(E^\vee) = \int_{\mathbb{CP}^2} e^{B_2 + iJ} \, ch(E^\vee) \sqrt{\frac{\hat{\text{A}}(T\mathbb{CP}^2)}{\hat{\text{A}}(N\mathbb{CP}^2)}} \; + \; \mathcal{O}(e^{2\pi it}) \,, \tag{4.5}$$

where the last term includes α' corrections. In that limit we have

$$\Pi_6 = \frac{1}{2}t^2 + \frac{1}{8} + \mathcal{O}(e^{2\pi it}) \,, \qquad \Pi_4 = t \,, \qquad \Pi_2 = 1 \,. \tag{4.6}$$

Remark that, for $\chi \gg b$, we have $\Pi_6 \sim -\frac{1}{2}\chi^2 < 0$ while $\Pi_2 = 1 > 0$. Therefore, the D2-and D6-brane on $\mathbb{CP}^2$ cannot be mutually BPS at large volume;[17] instead the D2-brane can be mutually BPS with the $\overline{D6}$-brane (for $b = -\frac{1}{2}$).

Type II string theory is invariant under the large gauge transformation

$$B_2 \to B_2 + H , \qquad\qquad F \to F - H , \qquad\qquad (4.7)$$

where F is the worldvolume flux on any D-brane. The action (4.7) on the charges of any state $E^\vee$ and on the periods Π is given by

$$ch(E^\vee) \to ch(E^\vee) M_\infty^{-1} , \qquad \Pi \to M_\infty \Pi \qquad \text{with} \qquad M_\infty = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} , \qquad (4.8)$$

so that the central charge $Z(E^\vee)$ is indeed invariant. While the transformation M_∞ corresponds to $t \to t + 1$ in (4.6), it should also be true for the exact periods Π. As one can easily check, this implies that $\Pi_4 = t$ and $\Pi_2 = 1$ *exactly*; therefore only Π_6 is subject to α' corrections.

4.1 Exact periods and fractional branes

Because of α' corrections, at small Kähler volume it is better to consider the mirror geometry where the periods Π can be computed exactly from the Picard-Fuchs equation. The quantum moduli space can be visualized as a Riemann sphere with complex coordinate w and $z = 1/w$, and with three singularities at $w = 0$, $w = 1$ and $w = \infty$. The point $w = \infty$ is the large volume limit. Near $z = 1/w = 0$, we have [64, 67]

$$t(z) = \frac{1}{2\pi i} \ln\left(-\frac{z}{27} \right) + \mathcal{O}(z) . \qquad\qquad (4.9)$$

The relation $t(z)$ between the parameters t and z is called the mirror map. We see that the transformation $t \to t + 1$ corresponds to the logarithmic monodromy of $t(z)$ near the large volume point $z = 0$. Let us define the period

$$t_6 \equiv \Pi_6 + \frac{1}{2}t + \frac{1}{8} , \qquad\qquad (4.10)$$

which from the formulæ above is the central charge of the D6-brane: $Z(D6) = t_6$. We have the exact periods

$$t_6(w) = \frac{1}{3} + \frac{1}{4\pi^2} \left(\omega^2 G_1(w) + \omega G_2(w) \right)$$
$$t(w) = \frac{\omega^2 - \omega}{4\pi^2} \left(G_1(w) - G_2(w) \right) , \qquad\qquad (4.11)$$

in terms of the special functions G_1, G_2 defined in appendix D, and $\omega = e^{\frac{2\pi i}{3}}$. We choose the branch cuts to lie on the positive real axis in the complex plane, from $w = 0$ to $w = 1$,

---

[17]This corresponds to the fact that mutual BPS-ness requires the worldvolume flux $\mathcal{F} = \hat{B}_2 + F$ on the D6 to be anti-self-dual [66], so that $\int \mathcal{F} \wedge \mathcal{F} \leq 0$. Notice also that the flux $F = nH$ as well as $\hat{B}_2$ can only be self-dual ($H^2 = 1$), therefore a single $\overline{D6}$ can be BPS with a D2 only if $\mathcal{F} = 0$. Multiple $\overline{D6}$-branes can have anti-self-dual non-Abelian (instanton-like) flux.

and from $z = 1$ to $z = 0$. The point $w = 0$ is called the *orbifold point* in Kähler moduli space. One can check that as we circle $w \to e^{-2\pi i}w$, we have the monodromy:

$$t_6 \to -2t_6 + t + 1 \,, \qquad t \to -3t_6 + t + 1 \,, \qquad 1 \to 1 \,. \qquad (4.12)$$

The corresponding action on the charges and on Π is

$$ch(E^\vee) \to ch(E^\vee) M_o^{-1} \,, \qquad \Pi \to M_o \Pi \qquad \text{with} \qquad M_o = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{5}{16} \\ -3 & -\frac{1}{2} & \frac{5}{8} \\ 0 & 0 & 1 \end{pmatrix} \,. \qquad (4.13)$$

This monodromy generates a $\mathbb{Z}_3$ group, since $(M_o)^3 = 1$. This corresponds to the $\mathbb{Z}_3$ symmetry of the orbifold string theory. At the orbifold point $w = 0$ the D2-brane fractionates into three fractional branes $E_i^\vee$ ($i = 1, 2, 3$), which are rotated by this $\mathbb{Z}_3$. We have $Z(D2) = \sum_{i=1}^3 Z(E_i^\vee) = 1$, so we must have $Z(E_i^\vee) = \frac{1}{3}$ at $w = 0$. Since $t_6(0) = \frac{1}{3}$, the D6-brane is actually one of the fractional branes, and we can obtain the other two by the action of M_o on t_6. We have:

$$
\begin{aligned}
E_1^\vee : & & t_6 \,, & & ch(E_1^\vee) = \left(1, \frac{1}{2}, \frac{1}{8}\right) & \\
E_2^\vee : & & -2t_6 + t + 1 \,, & & ch(E_2^\vee) = \left(-2, 0, \frac{3}{4}\right) & \qquad (4.14) \\
E_3^\vee : & & t_6 - t \,, & & ch(E_3^\vee) = \left(1, -\frac{1}{2}, \frac{1}{8}\right) \,. &
\end{aligned}
$$

This is the map we need. We summarize the relation between the brane charges $\boldsymbol{Q}_{\text{source}} = (Q_6, Q_4, Q_2)$ and the ranks $\boldsymbol{N} = (N_1, N_2, N_3)$ in the $\mathbb{C}^3/\mathbb{Z}_3$ quiver in a matrix $Q^\vee$, which we call a *dictionary*:

$$\boldsymbol{Q}_{\text{source}} = \boldsymbol{N} \, Q^\vee \,, \qquad Q^\vee \equiv Q^\vee[0] = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ -2 & 0 & \frac{3}{4} \\ 1 & -\frac{1}{2} & \frac{1}{8} \end{pmatrix} \,. \qquad (4.15)$$

The reason for the notation $Q^\vee[n]$, with $n \in \mathbb{Z}$, is explained below.

4.2 Monodromies in Kähler moduli space and dictionaries at any b

In the derivation of the fractional brane states (4.14) we assumed that $b \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. This corresponds to a choice of sheet in the mirror map (4.9) (with the cut at z real and positive). More generally, let us consider

$$b \in \left[n - \frac{1}{2}, \, n + \frac{1}{2}\right] \qquad n \in \mathbb{Z} \,. \qquad (4.16)$$

In such window, the relevant fractional branes (objects with $Z = \frac{1}{3}$ at the orbifold point) are not (4.14) anymore, but they can be obtained from (4.14) by a large volume monodromy — in other words, by an active M_∞ transformation. For later convenience, we accompany this M_∞ action by a $\mathbb{Z}_3$ rotation of the fractional branes. We define the dictionary $Q^\vee[n]$ by:

$$Q^\vee[n] = (M_{o, \text{frac}})^n \, Q^\vee \, (M_\infty)^{-n} \,, \qquad (4.17)$$
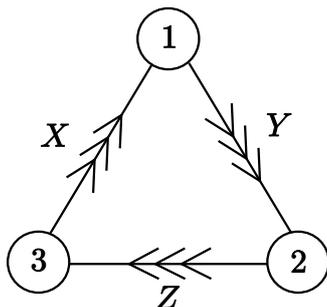
Figure 3. Quiver diagram of the theory for M2-branes on the real cone over $Y^{p,q}(\mathbb{CP}^2)$, the same quiver for D3-branes at $\mathbb{C}^3/\mathbb{Z}_3$.

with

$$M_{o,\,\mathrm{frac}} = Q^\vee M_o (Q^\vee)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \tag{4.18}$$

In particular, we have

$$Q^\vee[-1] = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ 1 & \frac{3}{2} & \frac{9}{8} \\ -2 & -2 & -\frac{1}{4} \end{pmatrix} , \qquad Q^\vee[1] = \begin{pmatrix} -2 & 2 & -\frac{1}{4} \\ 1 & -\frac{3}{2} & \frac{9}{8} \\ 1 & -\frac{1}{2} & \frac{1}{8} \end{pmatrix} , \tag{4.19}$$

corresponding to the dictionaries valid for $b \in \left[-\frac{3}{2}, -\frac{1}{2} \right]$ and $b \in \left[\frac{1}{2}, \frac{3}{2} \right]$, respectively.

Physically, as we cross some $b = n + \frac{1}{2}$, $n \in \mathbb{Z}$ locus in Kähler moduli space we have some decay and recombination of the fractional branes. Consider for instance the case $b = -\frac{1}{2}$. This occurs on the branch cut that runs from $z = 1$ to $z = 0$. The point $z = 1$ (with $b = -\frac{1}{2}$) is called a *conifold point*: $t_6(1) = 0$ and the first fractional brane $E_1^\vee$ in (4.14) becomes tensionless. At this point and anywhere on the $b = -\frac{1}{2}$ marginal stability wall, we have a recombination

$$E_2^\vee + E_3^\vee \;\to\; (E_2^\vee + 3E_1^\vee) + (E_3^\vee - 3E_1^\vee) . \tag{4.20}$$

While at $b > -\frac{1}{2}$ the states on the left in (4.20) are the lightest ones, at $b < -\frac{1}{2}$ the lightest are those on the right. So (4.20) is a marginal recombination at $b = -\frac{1}{2}$, but becomes a true decay as we cross the wall.

5 Derivation of the CS quivers dual to $Y^{p,q}$

Our next task is to identify the low energy field theory living on the branes at the singularity. We argued that it is a Chern-Simons quiver gauge theory, whose quiver and superpotential describe point-like D-branes at the tip of $\mathbb{C}^3/\mathbb{Z}_3$. The quiver is in figure 3. The superpotential is

$$W = \epsilon^{ijk} \operatorname{Tr}(X_i Y_j Z_k) , \tag{5.1}$$

and i, j, k are indices of the SU(3) global symmetry. There are two things to determine: 1) given the Page charges measured on M_6, we should find the number of fractional branes of each kind at the singularity, and then the gauge ranks in the quiver making use of the dictionaries (4.17); 2) from the same Page charges we should find the fluxes on cycles at the singularity, and then the CS terms induced by the Wess-Zumino action. Notice that indeed there are 5 parameters on both sides of the duality:[18] in supergravity the Page charges $(Q_{6;0}, Q_{6;+}, Q_{4;-}, Q_{4;D}, Q_2)$; in field theory the ranks and CS levels $(N_1, N_2, N_3, k_1, k_2, -k_1 - k_2)$.

The counting of branes is more easily done on $\text{CY}_3 \times \mathbb{R}$. Let us define Page charges on $\mathbb{C}^3/\mathbb{Z}_3$ as functions of r_0:

$$Q_6(r_0) = \int_{\mathbb{CP}^1 \text{ at } r_0} F_2 \qquad\qquad Q_4(r_0) = -\int_{\mathbb{CP}^2 \text{ at } r_0} F_2 \wedge B_2$$
$$Q_2(r_0) = \int_{\mathbb{C}^3/\mathbb{Z}_3 \text{ at } r_0} F_6 - \frac{1}{2} F_2 \wedge B_2 \wedge B_2 \qquad\qquad b(r_0) = \int_{\mathbb{CP}^1 \text{ at } r_0} B_2 \tag{5.2}$$

where the last one is the B-field rather than a Page charge. From the intersection numbers on $\mathbb{C}^3/\mathbb{Z}_3$ in (3.17) they must satisfy $Q_4(r_0) = -Q_6(r_0) \, b(r_0)$. We detect the presence of branes at r_0 by the jump of Page charges: from the intersection numbers we get $\delta Q_6 = -3$ for a D6-brane on $\mathbb{CP}^2$ (without worldvolume flux, regardless of its actual existence), $\delta Q_4 = -3$ for a D4-brane on $\mathbb{CP}^1$, and $\delta Q_2 = 1$ for a D2-brane, where $\delta Q_p(r_0) \equiv Q_p(r_0 + \epsilon) - Q_p(r_0 - \epsilon)$. The relation between branes and gauge ranks is in (4.15): $\boldsymbol{N} = \boldsymbol{Q}_{\text{source}} \, Q^{\vee -1}$, where now $\boldsymbol{Q}_{\text{source}} = \left(-\frac{1}{3}\delta Q_6, -\frac{1}{3}\delta Q_4, \delta Q_2\right)$. Exploiting the relation (3.29) between cycles on $\mathbb{C}^3/\mathbb{Z}_3$ and on M_6, together with $\delta Q_2 = Q_2$, we find:

$$\boldsymbol{N} = \left(-\frac{Q_{6;+} - Q_{6;-}}{3} \,\Big|\, -\frac{Q_{4;+} - Q_{4;-}}{3} \,\Big|\, Q_2\right) Q^{\vee -1}. \tag{5.3}$$

Notice also that $b(r_0 > 0) = b^+$ and $b(r_0 < 0) = b^-$.

Which dictionary should we use in (5.3)? The dictionary $Q^\vee[0]$ is applicable only if $b(r_0)$ is within the range $\left[-\frac{1}{2}, \frac{1}{2}\right]$ for any r_0. Since $b(r_0)$ jumps when crossing a bunch of branes (as Page charges do), $Q^\vee[0]$ is applicable only for $-\frac{1}{2} \leq b^-, b^+ \leq \frac{1}{2}$. These constraints draw a window in the space of torsion:

$$\text{window } [0,0]: \qquad 0 \leq n_0 \leq q, \qquad 0 \leq 3n_1 - n_0 \leq 3p - q \qquad \Rightarrow \qquad 0 \leq n_1 \leq p \tag{5.4}$$

where we used (3.44). This window covers only one third of the fundamental domain of torsion, as is clear from figure 2. We define window $[n, m]$ as a subset of the (n_0, n_1)-plane where $b^- \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ and $b^+ \in \left[m - \frac{1}{2}, m + \frac{1}{2}\right]$.

In window $[n, n]$, where both b^+ and b^- are in the range $\left[n - \frac{1}{2}, n + \frac{1}{2}\right]$, we have a different set of BPS states of minimal tension and therefore a different dictionary $Q^\vee[n]$. The ranks are obtained with the same formula (5.3). Window $[n, n]$ is the image of window

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[18]We have set $F_0 = 0$ in IIA, because F_0 does not have a M-theory lift [68], and correspondingly we will find $k_1 + k_2 + k_3 = 0$.

$[0, 0]$ under a large gauge transformation of the B-field $\delta(b^+, b_0) = (n, 0)$, which indeed shifts the Page charges as $\delta(n_0, n_1) = (nq, np)$ and

$$\delta N = -n \, n_1 - \frac{n(n-1)}{2} p \, . \tag{5.5}$$

The resulting theory is exactly the same theory as before the large gauge transformation, up to a cyclic rotation of the ranks. However the windows $[n, n]$—being images of $[0, 0]$—do not help us covering new pieces of the torsion domain.

In windows $[n, m]$ with $n \neq m$, b^+ and b^- fall in different ranges. In other words the CY_3 is in different Kähler domains on the two sides of $r_0 = 0$: the branes sit exactly on top of a Kähler wall and no dictionary is straightforwardly applicable. As we cross a bunch of branes moving along r_0, Page charges jump and B_2 jumps as well, according to $Q_4(r_0) = -Q_6(r_0) \, b(r_0)$.

Let us consider window $[1, 0]$, where $\frac{1}{2} \leq b^- \leq \frac{3}{2}$ and $-\frac{1}{2} \leq b^+ \leq \frac{1}{2}$. Since we cannot use neither $Q^\vee[0]$ nor $Q^\vee[1]$, we split the branes along r_0 in two groups in such a way that $b(r_0) = \frac{1}{2}$ in the middle:[19] this requires $Q_6(r_{\text{mid}}) = -2Q_4(r_{\text{mid}})$. Neglecting the detail of how we actually split the branes, we will simply impose that $Q_4(r_{\text{mid}}) = \rho$, $Q_6(r_{\text{mid}}) = -2\rho$ and $Q_2(r_{\text{mid}}) = \eta$ in terms of undetermined parameters ρ, η. We can now use the dictionary $Q^\vee[1]$ with the bunch at $r < r_{\text{mid}}$ and $Q^\vee[0]$ with the bunch at $r > r_{\text{mid}}$. We use (5.3) for each bunch of branes separately:

$$\boldsymbol{N} = \left(-\frac{Q_{6;+} - (-2\rho)}{3} \,\bigg|\, -\frac{Q_{4;+} - \rho}{3} \,\bigg|\, Q_2 - \eta \right) Q^\vee[0]^{-1}$$
$$+ \left(-\frac{(-2\rho) - Q_{6;-}}{3} \,\bigg|\, -\frac{\rho - Q_{4;-}}{3} \,\bigg|\, \eta \right) Q^\vee[1]^{-1} \, .$$

In fact the unknowns ρ, η cancel out: this just follows from the fact that at $b = \frac{1}{2}$ the set of three states described by $Q^\vee[0]$ is only marginally unstable to decay to the set of $Q^\vee[1]$, and viceversa. In other words at $b = \frac{1}{2}$, $Q^\vee[0]$ and $Q^\vee[1]$ produce exactly the same theory. Running a similar argument with multiple splittings we arrive to a general formula for the ranks in window $[n, m]$:

$$\boldsymbol{N} = \left(\frac{Q_{6;-}}{3} \,\bigg|\, \frac{Q_{4;-}}{3} \,\bigg|\, Q_2 \right) Q^\vee[n]^{-1} - \left(\frac{Q_{6;+}}{3} \,\bigg|\, \frac{Q_{4;+}}{3} \,\bigg|\, 0 \right) Q^\vee[m]^{-1} \, . \tag{5.6}$$

Chern-Simons levels are induced by the background fluxes [12]. For a D6-brane with Chern character $ch(E^\vee) = (1, F, ch_2)$, the Wess-Zumino action produces CS level

$$k(E^\vee) = \int_{\mathbb{CP}^2} e^{F+B} \wedge \sum_q dC_q = \int_{\mathbb{CP}^2} e^F \wedge \sum_p F_p^{\text{Page}} \, , \tag{5.7}$$

which depends on the Page charges again. Let $\boldsymbol{k} = (k_1, k_2, k_3)$ be the vector of CS levels of the quiver. If we place a probe D6-brane bound state at r_0, we get on it:

$$\boldsymbol{k} = \left(-Q_4(r_0) \,\big|\, Q_6(r_0) \,\big|\, 0 \right) Q^{\vee\mathsf{T}} \, , \tag{5.8}$$

_____

[19]In general we cannot do that supersymmetrically, but it is good enough to engineer the field theory: we will release the branes and all will fall at $r_0 = 0$, the tip of the cone.

$$
\begin{array}{l}
\text{Window } [-1,0]: \qquad -q \le n_0 \le 0 \qquad\qquad 0 \le 3n_1 - n_0 \le 3p - q \\[4pt]
\mathrm{U}(N + n_1 - p - n_0)_{-n_0 + \frac{3}{2}n_1} \times \mathrm{U}(N)_{\frac{1}{2}n_0 - 3n_1 + \frac{3}{2}p - q} \times \mathrm{U}(N - n_1)_{\frac{1}{2}n_0 + \frac{3}{2}n_1 - \frac{3}{2}p + q} \\[12pt]
\text{Window } [0,0]: \qquad 0 \le n_0 \le q \qquad\qquad 0 \le 3n_1 - n_0 \le 3p - q \\[4pt]
\mathrm{U}(N + n_1 - p)_{-n_0 + \frac{3}{2}n_1} \times \mathrm{U}(N)_{2n_0 - 3n_1 + \frac{3}{2}p - q} \times \mathrm{U}(N - n_1)_{-n_0 + \frac{3}{2}n_1 - \frac{3}{2}p + q} \\[12pt]
\text{Window } [1,0]: \qquad q \le n_0 \le 2q \qquad\qquad 0 \le 3n_1 - n_0 \le 3p - q \\[4pt]
\mathrm{U}(N + n_1 - p)_{\frac{1}{2}n_0 + \frac{3}{2}n_1 - \frac{3}{2}q} \times \mathrm{U}(N)_{\frac{1}{2}n_0 - 3n_1 + \frac{3}{2}p + \frac{1}{2}q} \times \mathrm{U}(N - n_1 + n_0 - q)_{-n_0 + \frac{3}{2}n_1 - \frac{3}{2}p + q}
\end{array}
$$

Table 1. Theories for $Y^{p,q}$ in a fundamental domain of the torsion group.

because the worldvolume flux F and its square are contained in $Q^{\vee}$. The last entry is taken to be zero because we are not allowing a Romans mass F_0 and its D8-charge (the generalization is left for future work). Remark that, by construction, $\sum_i k_i = 0$.

There are two subtleties, however. First, since the branes source fluxes, we should compute the CS levels induced by the background flux removing the fields produced by the branes themselves. This is easily accomplished by taking the average of the fluxes on the two sides of the branes. Given the linearity of (5.8), this is equivalent to computing the CS levels for probes on the two sides of a bunch of branes, and taking the average. Second, when we are in a window $[n, m]$ with $n \ne m$, we should again split the bunch of branes into groups, with $b(r_0) \in \mathbb{Z} + \frac{1}{2}$ between them, and use different dictionaries for each group. All undetermined parameters cancel out, and we arrive at the formula:

$$
\boldsymbol{k} = \frac{1}{2}\Big[\big(-Q_{4;-}\,\big|\,Q_{6;-}\,\big|\,0\big)\,Q^{\vee}[n]^{\mathsf{T}} + \big(-Q_{4;+}\,\big|\,Q_{6;+}\,\big|\,0\big)\,Q^{\vee}[m]^{\mathsf{T}}\Big]. \tag{5.9}
$$

We repeat for convenience the Page charges of the background:

$$
\begin{array}{lll}
Q_{6;0} = -p\,, & Q_{6;+} = 3p - q\,, & Q_{6;-} = -q \\[6pt]
Q_{4;D} = n_1 - \dfrac{p}{2}\,, & Q_{4;+} = n_0 - 3n_1 + \dfrac{3p - q}{2}\,, & Q_{4;-} = n_0 - \dfrac{q}{2} \\[6pt]
Q_2 = N - \dfrac{p}{8}\,.
\end{array} \tag{5.10}
$$

We have thus found formulæ for the theories in all windows.

The possible half-integral CS levels that can appear after taking the average are precisely adequate to cancel parity anomalies in the $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2) \times \mathrm{SU}(N_3)$ gauge subgroup. The parity anomaly in $\mathrm{U}(1)^3$ is not canceled: we discuss possible solutions in section 6.

Each window covers one third of the fundamental domain of torsion, see figure 2. We choose windows $[-1, 0]$, $[0, 0]$ and $[1, 0]$ to cover it all. The three set of theories with their respective domains of validity are summarized in table 1. Window $[0,0]$ is delimited, in the (n_0, n_1) plane, by $(0, 0)$, $(0, p - \frac{q}{3})$, $(q, \frac{q}{3})$, (q, p). The two corners $(n_0, n_1) = (0, 0) \cong (q, p)$ are identified, and correspond to the CP-invariant theories of $p\,\overline{\mathrm{D6}}$'s. The center $(\frac{q}{2}, \frac{p}{2})$, present

if p, q are even, corresponds to the CP-invariant theory of $\frac{p}{2}$ $2\overline{D6}$ bound states. As explained in section 3.6, the parity symmetry of M-theory acts in type IIA as $B_2 \to -B_2$, which means

$$n_0 \to q - n_0 , \qquad n_1 \to p - n_1 , \qquad (5.11)$$

together with a parity operation in $\mathbb{R}^{1,2}$. In field theory it corresponds to parity times charge conjugation.[20] This operation leaves window $[0,0]$ invariant, whilst exchanges windows $[1,0]$ and $[-1,0]$.

Let us conclude with a series of remarks. First, the theories nicely glue on the borders of the torsion domain: On the common edges of the windows, the theories coincide. On edges which are identified (by large gauge transformations), the theories coincide up to a shift of N, which agrees with the shift of the Page D2-charge (see section 3.5):

$$\begin{aligned}
Q_2(n_0 + q,\, n_1 + p) - Q_2(n_0, n_1) &= -n_1 \\
Q_2(n_0 + 3q,\, n_1 + q) - Q_2(n_0, n_1) &= -n_0 - q \\
Q_2(n_0 + 2q,\, n_1 + q - p) - Q_2(n_0, n_1) &= n_1 - n_0 - p .
\end{aligned} \qquad (5.12)$$

Second, one can check — although it is not manifest — that the map from (N, p, q, n_0, n_1) to the set of field theories $(N_1, N_2, N_3, k_1, k_2, -k_1 - k_2)$ (modulo cyclic permutations of the gauge groups) is injective (up to identifications of (n_0, n_1)) and surjective: every possible rank and CS assignment[21] to the quiver in figure 3 is realized by one and only one background (N, p, q, n_0, n_1).

In particular we do not find the phenomenon of supersymmetry breaking as in the ABJ case [7]. In the example of ABJ, the two theories at the boundaries of the torsion domain are identified by a Seiberg-like duality, and theories beyond that are conjectured to break SUSY. The theory on $\mathbb{C}^3/\mathbb{Z}_3$ does not enjoy Seiberg-like dualities, so we cannot expect the same mechanism to be at work. Indeed theories at the boundaries of the torsion domain are just trivially identical and, on the other hand, all possible theories arise from some $Y^{p,q}$ with some torsion.

We can further speculate that while "non-anomalous" fractional branes (the nomenclature is taken from 4d) enjoy Seiberg-like dualities and suffer SUSY breaking, "anomalous" fractional branes are free of both. Whilst $\mathbb{C}^3/\mathbb{Z}_3$ has only the latter, in examples with both types of branes — like $S^3/\mathbb{Z}_p$ bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1$ [23]—we should expect mixed phenomena.

5.1 Moduli space without resolutions

As a first check of the theories in table 1, let us compute the geometric branch of the moduli space of those CFTs. As explained in section 2, it can be done either by considering monopole operators (as in [19, 20]) or with a semiclassical computation. Let us adopt the computation with monopoles.

---

[20]This agrees with the sign flip of B_2 under a change of orientation of the string worldsheet, which acts as charge conjugation for the open string degrees of freedom.

[21]With the constraint $k_1 + k_2 + k_3 = 0$, which could be relaxed by considering IIA backgrounds with F_0 flux [69–73].

We start with the theory in window $[0,0]$. On the Coulomb branch the gauge group is broken to $\left(\mathrm{U}(1)^G\right)^{\tilde{N}} \prod_{i=1}^G \mathrm{U}(M_i)$. Let us consider monopoles of the subgroup $\mathrm{U}(1)^G$: one copy of the Abelian quiver. The R-, gauge, flavor and topological charges of the two simple monopoles T and $\tilde{T}$ are:

	X	Y	Z	T	$\tilde{T}$
R	$2 - R_Y - R_Z$	R_Y	R_Z	$p - \frac{3}{2} n_1 R_Y - \frac{3}{2}(p - n_1) R_Z$	$p - \frac{3}{2} n_1 R_Y - \frac{3}{2}(p - n_1) R_Z$
g_1	-1	1	0	$-n_0$	$n_0 - 3n_1$
g_2	0	-1	1	$2n_0 - q$	$-2n_0 + 6n_1 - 3p + q$
g_3	1	0	-1	$-n_0 + q$	$n_0 - 3n_1 + 3p - q$
$\mathrm{SU}(3)$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{U}(1)_M$	0	0	0	1	-1

Note that we have used an R-symmetry which is not mixed with topological symmetries. Such charges allow us to write down the following formal gauge invariants:

$$a_{(3)} = XYZ \,, \qquad b_{(q)} = TY^{n_0} Z^{q - n_0} \,, \qquad c_{(3p-q)} = \tilde{T} Y^{3n_1 - n_0} Z^{3p - q - 3n_1 + n_0} \qquad (5.13)$$

where the subscripts refer to the number of SU(3) indices, and the following formal relations:

$$T\tilde{T} Y^{3n_1} Z^{3p - 3n_1} = (XYZ)^p \,. \qquad (5.14)$$

In appendix C.1 we give some evidence for the relation above from a counting of fermion zero-modes. The quantum relations in the chiral ring join the classical F-term relations coming from the superpotential:

$$X_{[i} Y_{j]} = Y_{[i} Z_{j]} = Z_{[i} X_{j]} = 0 \qquad (5.15)$$

which imply that SU(3) indices are always symmetrized.

The expressions above are only formal because some exponents could be negative: in that case one should multiply the expressions by extra powers of the gauge invariant (XYZ) to eliminate negative powers. However the boundaries of window $[0,0]$ in the (n_0, n_1)-plane are precisely such that all powers are non-negative. The gauge invariants above satisfy the relations (3.4)

$$b_{(q)} c_{(3p-q)} = a_{(3)}^p \,, \qquad (5.16)$$

together with symmetrization of SU(3) indices in products, which provide the algebraic description of $C(Y^{p,q})$. We have found that the moduli space contains a copy of $C(Y^{p,q})$, corresponding to the motion of one M2-brane on the CY$_4$. In fact the theories in window $[0,0]$ contain $\tilde{N}$ symmetrized copies of the Abelian quiver, where $\tilde{N} = N - \max(n_1, p - n_1)$. Therefore the Coulomb branch contains the symmetrized product $\mathrm{Sym}_{\tilde{N}} C(Y^{p,q})$, describing the motion of $\tilde{N}$ M2-branes on the CY$_4$. The extra piece of the theory will be analyzed in section 5.3, and shown to describe resolutions of the CY$_4$.

The same analysis can be repeated for the other windows, the only changes being in the charges of monopole operators and in the bifundamentals appearing in the gauge invariants $b_{(q)}$ and $c_{(3p-q)}$. The final result is the same: in each window the Coulomb branch

contains the symmetric product of a number of copies of $C(Y^{p,q})$. The number of copies equals the smallest gauge rank. The boundaries of the windows are precisely such that the geometric branch is $C(Y^{p,q})$ inside and on the boundaries, but not outside.

This can also be understood in the semiclassical approach by looking at the Kähler (FI) parameter space of the quiver theory, which is divided into three chambers characterized by different linear relations between FI parameters and Kähler parameters of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ (see for instance [74]). As σ spans $\mathbb{R}$, the effective FI parameters of the CS theories draw two rays joined at the origin in the Kähler cone. The rays are in the directions of the effective CS levels, i.e. the gauge charges of bare monopole operators which determine the dependence of $b_{(q)}$ and $c_{(3p-q)}$ on bifundamentals. The two rays lie either inside the same chamber (for window $[0,0]$ and its images), or inside different chambers (for windows $[0,\pm 1]$ and their images), with a wall crossing at $\sigma = 0$. The latter possibility corresponds to the wall in the Kähler moduli space of the CY$_3$ crossed at $r_0 = 0$ by probe D2-branes mobile along $\mathbb{R}$ in the type IIA setup. A window thus corresponds to a pair of Kähler chambers, both in geometry and in field theory. A dictionary between geometry and field theory is associated to each chamber, and the change of dictionaries as we change windows precisely accounts for the modifications of the field theory needed to reproduce the same $C(Y^{p,q})$ geometric branch.

5.2 Special cases

Among the $Y^{p,q}$ theories, there are a few cases deserving special attention.

CP-invariant theories. Here we discuss the three CP-invariant theories, corresponding to particular values of (n_0, n_1) as described in section 3.6.

One is the torsionless theory dual to the M-theory background with $(n_0, n_1) = (0,0)$. In type IIA the B-field is $(b^+, b^-, b_0) = \left(-\frac{1}{2}, -\frac{1}{2}, 0\right)$, which under M-theory parity is mapped to itself up to a large gauge transformation. The CP-invariant gauge theory is

$$\mathrm{U}(N-p)_0 \times \mathrm{U}(N)_{\frac{3}{2}p-q} \times \mathrm{U}(N)_{-\frac{3}{2}p+q}. \tag{5.17}$$

Since the M-theory background does not have torsion, there are no obstructions to a complete crepant resolution of the CY$_4$ by blowing up $\mathbb{CP}^2$ and the $(p-1)$ $\mathbb{CP}^1$'s at the tip of the $\mathbb{C}^2/\mathbb{Z}_p$ fiber. In type IIA such resolutions correspond to blowing up $\mathbb{CP}^2$ in the CY$_3$ at $r_0 = 0$, and to separate along r_0 the p $\overline{\mathrm{D6}}$-branes wrapping $\mathbb{CP}^2$. At least when the volume of $\mathbb{CP}^2$ is large, we can use the brane action. In our conventions only $\overline{\mathrm{D6}}$-branes can be mutually SUSY with D2-branes (which appear in the background). The SUSY condition for one $\overline{\mathrm{D6}}$-brane is that its worldvolume flux $\mathcal{F}$ is a $(1,1)$-form and $*_4\mathcal{F} = -\mathcal{F}$ [75]. The only $(1,1)$-form on $\mathbb{CP}^2$ is the Kähler form, which however is self-dual, therefore the only solution is $\mathcal{F} = \hat{B}_2 + F = 0$. In view of the Freed-Witten anomaly on $\mathbb{CP}^2$, this requires $b^+ = b^- = -\frac{1}{2}$ (up to large gauge transformations). Therefore the B-field has the right value to allow a complete SUSY separation of the $\overline{\mathrm{D6}}$-branes. Comparing with $Q^\vee$ in (4.15) we see that the actual BPS bound-state of D6-charge -1 is made of one $\overline{\mathrm{D6}}$ and one D2 (last two rows of $Q^\vee$), and it corresponds to the last two groups in field theory. This suggests that out of the N D2-branes, p are stuck on the $\overline{\mathrm{D6}}$-branes. The field theory computation of section 5.3 confirms this.

We already saw that the field theory moduli space indeed contains $N - p$ symmetrized copies of $C(Y^{p,q})$, describing the $N - p$ free D2-branes. We will see in section 5.3 that it also contains all p resolution parameters, related to the adjoint scalars of the last two groups.

The second theory is the one for $(n_0, n_1) = \left(\frac{q}{2}, \frac{p}{2}\right)$ (it only exists if p, q are even). It is also dual to a parity-invariant M-theory background. In type IIA the B-field vanishes: $b^+ = b^- = b_0 = 0$. The CP-invariant gauge theory is

$$U\left(N - \frac{p}{2}\right)_{\frac{3}{4}p - \frac{1}{2}q} \times \mathrm{U}(N)_0 \times U\left(N - \frac{p}{2}\right)_{-\frac{3}{4}p + \frac{1}{2}q}. \tag{5.18}$$

The number of geometric resolution parameters allowed by the torsion flux is easily understood in type IIA. First we can resolve $\mathbb{CP}^2$. With $B_2 = 0$, the minimal large volume object mutually SUSY with D2-branes is the bound state[22] of two $\overline{\mathrm{D6}}$-branes with a rigid SU(2) instanton (second row of $Q^\vee$ in (4.15)), and it corresponds to the second gauge group. In fact the background is made of $p/2$ of them (a total of p $\overline{\mathrm{D6}}$-branes), which can be separated along r_0. We obtain that, due to torsion, only $\frac{p}{2} - 1$ 2-cycles can be blown-up. This agrees with the field theory moduli space.

The third CP-invariant theory is at $(n_0, n_1) = \left(2q, \frac{q+p}{2}\right)$ (it exists if $q + p$ is even), on the boundary of window $[1, 0]$, corresponding to $(b^-, b^+) = \left(\frac{3}{2}, 0\right)$. It is

$$U\left(N + \frac{q - p}{2}\right)_{\frac{3}{4}p + \frac{1}{4}q} \times \mathrm{U}(N)_0 \times U\left(N + \frac{q - p}{2}\right)_{-\frac{3}{4}p - \frac{1}{4}q}. \tag{5.19}$$

Another special case: quiver with three equal ranks. Inspection of table 1 reveals that this is only possible in window $[1, 0]$ with $(n_0, n_1) = (p + q, p)$:

$$\mathrm{U}(N)_{2p-q} \times \mathrm{U}(N)_{-p+q} \times \mathrm{U}(N)_{-p},$$

and in window $[-1, 0]$ with $(n_0, n_1) = (-p, 0)$ which gives the CP-transformed theory. Because of the borders of the windows, they actually contain those points only if $p \leq q \leq 2p$.

The theories above are the ones proposed in [24] for M2-branes at the tip of the cone over $Y^{p,q}$: we have therefore found a stringy derivation of their proposal. Let us add two comments. First, the authors already noticed that their proposals can only realize $C(Y^{p,q})$ if $p \leq q \leq 2p$: here we understand why. Second, the theories with equal ranks can only describe one resolution parameter of the CY$_4$, the blow-up of $\mathbb{CP}^2$, while they cannot describe any resolution of the $\mathbb{C}^2/\mathbb{Z}_p$ fiber. The reason is that they are dual to $AdS_4 \times Y^{p,q}$ backgrounds with torsion G_4 flux, and such torsion obstructs the resolutions. In type IIA, SUSY does not allow the bound state of all D6-branes to be broken in smaller pieces.

Field theory for the limiting case $q = 0, 3p$. In the case $q = 0$, the IIA background has a non-isolated $\mathbb{C}^3/\mathbb{Z}_3$ singularity, as discussed in section 3.7. The field theory derivation nevertheless runs similarly, giving

$$\mathrm{U}(N + n_1 - p)_{\frac{3}{2}n_1} \times \mathrm{U}(N)_{-3n_1 + \frac{3}{2}p} \times \mathrm{U}(N - n_1)_{\frac{3}{2}n_1 - \frac{3}{2}p},$$

---

[22]On $\mathbb{CP}^2$ there is an exceptional stable holomorphic rank 2 bundle V with $c_1 = -1$ and $c_2 = 1$, which does not have moduli and is member of a discrete family [76]. We can put it on two $\overline{\mathrm{D6}}$-branes, forming a bound state, but the Freed-Witten anomaly requires to combine it with $-\frac{1}{2}$ units of Abelian flux (per brane). The resulting bundle has $c_1 = 0$, $ch_2 = \frac{3}{4}$, takes values in SU(2) only, is anti-self-dual and does not have moduli.

where $n_1 \in \mathbb{Z}_p$ lies in the torsion group of the $(\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{C})/\mathbb{Z}_p$ orbifold. This corresponds to the $q = n_0 = 0$ limit of general theory for $Y^{p,q}$ in window $[0,0]$. Similarly, the $q = 3p$ theory is the $\mathrm{U}(1)_M$-parity dual to the $q = 0$ case.

5.3 Moduli space with resolutions

In section 5.1 we computed the chiral ring of our theories using monopoles, and that describes the complex structure of the moduli space. However there can be resolutions, i.e. Kähler deformations, of the CY_4, and to see them the semi-classical approach is more powerful.

To begin with, consider the CP-invariant torsionless theory (5.17):

$$\mathrm{U}(N-p)_0 \times \mathrm{U}(N)_{\frac{3}{2}p-q} \times \mathrm{U}(N)_{-\frac{3}{2}p+q} \,. \tag{5.20}$$

Referring to section 2.4 and to figure 3, the classical D-term equations are[23]

$$\xi_1 = YY^\dagger - X^\dagger X \,, \qquad \xi_2 + \left(\frac{3}{2}p - q\right)\sigma_2 = ZZ^\dagger - Y^\dagger Y$$

$$\xi_3 - \left(\frac{3}{2}p - q\right)\sigma_3 = XX^\dagger - Z^\dagger Z \,, \tag{5.21}$$

where the SU(3) index has been suppressed and $\xi_{1,2,3}$ are bare FI terms. Gauge rotations can be used to diagonalize σ_i, and the gauge group is generically broken to a bunch of $\mathrm{U}(1)'s$. The geometric branch consists of diagonalized configurations:

$$X = \begin{pmatrix} x \\ 0 \end{pmatrix} \,, \quad Y = \begin{pmatrix} y & 0 \end{pmatrix} \,, \quad Z = \begin{pmatrix} z & 0 \\ 0 & \tilde{z} \end{pmatrix} \,, \quad \sigma_1 = \sigma \,, \quad \sigma_2 = \sigma_3 = \begin{pmatrix} \sigma & 0 \\ 0 & \tilde{\sigma} \end{pmatrix} \,, \tag{5.22}$$

where x, y, z, σ are diagonal matrices of size $N - p$, while $\tilde{z}, \tilde{\sigma}$ are diagonal matrices of size p. The equations admit solutions only for $\xi_1 = 0$ and $\xi_2 = -\xi_3 \equiv \xi$, which we then impose.

The classical moduli space is corrected at one-loop. Following section 2.4, let us focus on one of the untilded directions and its coupling to the tilded directions:

$$\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \mathrm{U}(1)_3 \times \prod_{a=1}^p \mathrm{U}(1)_{2a} \times \mathrm{U}(1)_{3a} \,, \tag{5.23}$$

the so-called pseudo-Abelian quiver. We can use the residual Weyl group and a field redefinition in the decoupled totally diagonal $\mathrm{U}(1)$ to fix

$$0 = \tilde{\sigma}_1 \leq \tilde{\sigma}_2 \leq \cdots \leq \tilde{\sigma}_p \tag{5.24}$$

and introduce the index $0 \leq I \leq p$ such that $\tilde{\sigma}_I \leq \sigma \leq \tilde{\sigma}_{I+1}$ (where formally $\tilde{\sigma}_0 = -\infty$, $\tilde{\sigma}_{p+1} = +\infty$). The effective CS terms are

$$k_{11}^{\mathrm{eff}} = 0 \,, \qquad k_{22}^{\mathrm{eff}} = -k_{33}^{\mathrm{eff}} = 3I - q \,, \qquad k_{2a,2a}^{\mathrm{eff}} = -k_{3a,3a}^{\mathrm{eff}} = 3a - \frac{3}{2} - q$$

$$k_{1,2a}^{\mathrm{eff}} = k_{2,3a}^{\mathrm{eff}} = -k_{1,3a}^{\mathrm{eff}} = -k_{3,2a}^{\mathrm{eff}} = \begin{cases} -\frac{3}{2} \,, & a \leq I \\ \frac{3}{2} \,, & I+1 \leq a \end{cases} \,, \qquad k_{2a,3b} = \begin{cases} -\frac{3}{2} \,, & b < a \\ \frac{3}{2} \,, & a < b \end{cases} \,. \tag{5.25}$$

---

[23]From now on we neglect factors of 2π in the effective FI parameters.

From these we compute the effective FI terms:

$$\xi_1^{\mathrm{eff}} = 0$$
$$\xi_2^{\mathrm{eff}} = -\xi_3^{\mathrm{eff}} = \xi + (3I - q)\,\sigma + \frac{3}{2}\sum_a \tilde{\sigma}_a - 3\sum_{a \leq I} \tilde{\sigma}_a \qquad (5.26)$$
$$\xi_{2a}^{\mathrm{eff}} = -\xi_{3a}^{\mathrm{eff}} = \xi + (3a - q)\,\tilde{\sigma}_a + \frac{3}{2}\sum_b \tilde{\sigma}_b - 3\sum_{b \leq a} \tilde{\sigma}_b \ .$$

Along the untilded direction, the one-loop corrected D-term equations are (recall that SU(3) indices are implicit):

$$|y|^2 - |x|^2 = 0 \ , \qquad |z|^2 - |y|^2 = \xi + (3I - q)\,\sigma + \frac{3}{2}\sum_a \tilde{\sigma}_a - 3\sum_{a \leq I} \tilde{\sigma}_a \ . \qquad (5.27)$$

They describe a resolved $\mathbb{C}^3/\mathbb{Z}_3$. For $\sigma \to \pm\infty$, ξ_2^{eff} is positive: we are in a chamber where $x = y = 0$ is the tip, and there $\mathbb{CP}^2$ has size ξ_2^{eff}. For ξ large enough, the size is always non-negative and the theory is in a geometric phase. We can rewrite the size $\chi(\sigma)$ of $\mathbb{CP}^2$ as

$$\chi(\sigma) = \left(\xi + \frac{3}{2}\sum_{a=1}^p \tilde{\sigma}_a\right) - q\,(\sigma - \tilde{\sigma}_1) + 3\sum_{a=1}^p (\sigma - \tilde{\sigma}_a)\,\Theta(\sigma - \tilde{\sigma}_a) \qquad (5.28)$$

in terms of the Heaviside step function $\Theta(x)$.

Comparing with (3.15) we see that the semi-classical computation has *exactly* reproduced the type IIA manifold $X_7 = \mathrm{CY}_4/\mathrm{U}(1)_M$, upon the identification

$$\sigma = r_0 \ , \qquad \xi + \frac{3}{2}\sum_{a=1}^p \tilde{\sigma}_a = \xi^c \ , \qquad \tilde{\sigma}_a = \zeta_a \ . \qquad (5.29)$$

The M-theory geometry is reproduced by gauge fixing the dual photon, as explained in section 2.4. Thus the field theory moduli space contains all p resolutions of the CY_4:[24] ξ controls the resolution of $\mathbb{CP}^2$, while $\tilde{\sigma}_{2,\ldots,p}$ control the resolutions of the $\mathbb{C}^2/\mathbb{Z}_p$ fiber. From the point of view of type IIA, $\tilde{\sigma}_a$ are the positions along r_0 of p $\overline{\mathrm{D6}}$-branes. The one-loop corrections arise from integrating out massive D2-D6 strings.

Along the a-th tilded direction, the quiver is $\mathrm{U}(1)_{2a} \times \mathrm{U}(1)_{3a}$ with 3 bifundamental fields z_α. The one-loop corrected D-term equation is

$$|\tilde{z}|^2 = \chi(\tilde{\sigma}_a) \ . \qquad (5.30)$$

This describes a $\mathbb{CP}^2$ of size $\chi(\tilde{\sigma}_a)$. There is also a dual photon, whose gauge fixing provides an extra $\mathrm{U}(1)_J$ direction. Indeed $\tilde{\sigma}_a$ controls the Kähler class of a resolved 2-cycle in the CY_4: supersymmetry pairs it up with a pseudoscalar b_a—acted upon by $\mathrm{U}(1)_J$—into a complexified Kähler class. We expect b_a to correspond in M-theory to C_6 integrated on a 6-cycle, and in type IIA to C_5 integrated on the 5-chain between two $\overline{\mathrm{D6}}$-branes.

---

[24]More precisely, $\tilde{\sigma}_a$ are moduli of the theory and should correspond to normalizable deformations of the CY_4; ξ is a parameter in field theory and should correspond to a non-normalizable mode in supergravity.

The natural geometric interpretation of the $\mathbb{CP}^2$ is as the moduli of a D2-brane confined on the $\overline{\text{D6}}$-brane at $r_0 = \tilde{\sigma}_a$. Indeed we observed in section 5.2 that the bound states at $b = -\frac{1}{2}$ are D2-$\overline{\text{D6}}$, and supersymmetry does not allow to separated them. This suggests that in M-theory p M2-branes are confined to the tip. We leave a precise understanding of the responsible mechanism for the future.

A similar analysis can be performed on the CP-invariant theories (5.18),

$$U\left(N - \frac{p}{2}\right)_{\frac{3}{4}p - \frac{1}{2}q} \times \text{U}(N)_0 \times U\left(N - \frac{p}{2}\right)_{-\frac{3}{4}p + \frac{1}{2}q} . \tag{5.31}$$

(Recall that p, q must be even). The geometric branch is given by diagonalized fields

$$X = \begin{pmatrix} x \end{pmatrix} , \quad Y = \begin{pmatrix} y & 0 \end{pmatrix} , \quad Z = \begin{pmatrix} z \\ 0 \end{pmatrix} , \quad \sigma_1 = \sigma_3 = \sigma , \quad \sigma_2 = \begin{pmatrix} \sigma & 0 \\ 0 & \tilde{\sigma} \end{pmatrix} , \tag{5.32}$$

where x, y, z, σ are diagonal matrices of size $N - \frac{p}{2}$, while $\tilde{\sigma}$ is a diagonal matrix of size $\frac{p}{2}$. The classical F- and D-term equations admit solutions only for $\xi_1 = -\xi_3 = \xi$ and $\xi_2 = 0$. We use gauge rotations to set $0 = \tilde{\sigma}_1 \leq \cdots \leq \tilde{\sigma}_{p/2}$. The one-loop corrected D-term equations along an untilded direction are

$$|z|^2 - |y|^2 = 0 , \qquad |y|^2 - |x|^2 = \xi + \left(3I - \frac{q}{2}\right)\sigma + \frac{3}{2}\sum_{a=1}^{p/2}\tilde{\sigma}_a - 3\sum_{a\leq I}\tilde{\sigma}_a . \tag{5.33}$$

We are in a chamber where the tip of the geometry is at $x = 0$: the classical F-term equations impose that the vectors y_α and z_α point in the same direction in $\mathbb{CP}^2$, which then has size

$$\chi(\sigma) = \left(2\xi - 3\sum_{a=1}^{p/2}\tilde{\sigma}_a\right) - q\left(\sigma - \tilde{\sigma}_1\right) + 3\sum_{a=1}^{p/2}2(\sigma - \tilde{\sigma}_a)\,\Theta(\sigma - \tilde{\sigma}_a) . \tag{5.34}$$

Along the tilded directions we get trivial equations, and the only extra modulus is the dual photon that with $\tilde{\sigma}_a$ forms a complexified Kähler class.

Comparing with (3.15) we recognize the type IIA manifold X_7 with $\frac{p}{2}$ resolutions: ξ controls the resolution of $\mathbb{CP}^2$, while $\tilde{\sigma}_{2,\ldots,p/2}$ control half of the resolutions of $\mathbb{C}^2/\mathbb{Z}_p$. From the point of view of type IIA, the minimal object which is SUSY in the background is the bound-state of $2\overline{\text{D6}}$ with non-Abelian gauge bundle, and $\tilde{\sigma}_a$ are the positions of those $\frac{p}{2}$ bound-states along r_0. Supersymmetry prevents us to completely break the p $\overline{\text{D6}}$-branes apart. From the point of view of M-theory the mechanism should be similar, even though a better understanding along the lines of [25] would be welcome: the torsion flux should not be compatible with a full resolution of the geometry and SUSY. Notice that in this case we do not get extra degrees of freedom on the $\overline{\text{D6}}$-branes.

Let us comment on the case of a generic theory in table 1, leaving the details for future work. From the field theory point of view, resolutions are possible only if the one-loop corrected D-term equations admit solutions with non-trivial $\xi_{1,2,3}$ or $\tilde{\sigma}$. In fact generically these resolutions are obstructed — some resolutions open up only for special values of (n_0, n_1).

From the point of view of type IIA, this has a simple interpretation. Partial resolutions are possible only when the B-field is such that either or both of the following conditions are satisfied: 1) the bunch of all $\overline{\text{D6}}$-branes with worldvolume fluxes is SUSY at large volume; 2) the bunch can be broken in smaller bound states, all of which are SUSY at large volume.

6 Canceling parity anomalies: off-diagonal CS terms

The CS quiver theories we considered in the previous section still suffer from a so-called parity anomaly, making them inconsistent. Cancelation of all the $\mathbb{Z}_2$ anomalies in a CS quiver requires the conditions [40, 41, 47]

$$
\begin{aligned}
k_i + \frac{1}{2} \sum_j A_{ij} N_j &\in \mathbb{Z} \\
\Lambda_{ij} - \frac{1}{2} A_{ij} &\in \mathbb{Z} \,,
\end{aligned}
\tag{6.1}
$$

where A_{ij} is the adjacency matrix of the quiver and Λ_{ij} are Chern-Simons coupling between U(1) photons of two different gauge groups of the quiver, that we introduce below. The first line insures the cancelation of anomalies for each U$(N)_i$ factor in the gauge group. Our models satisfy this condition. The second line is a condition for the cancelation of $\mathbb{Z}_2$ anomalies for the Abelian part of the gauge group. Since we have considered $\Lambda_{ij} = 0$ so far, and because $A_{ij} = \pm 3$ for $i \neq j$, the last condition in (6.1) is *not* satisfied by our $\mathbb{C}^3/\mathbb{Z}_3$ quiver; this problem was also noticed in [77].

The anomaly can be canceled in different ways. One way is to add some appropriate CS couplings to the Lagrangian. In general this would affect the moduli space, and spoil the nice matching with the M-theory CY$_4$. However, we show below that the anomaly cancelation is possible even without affecting the moduli space. Another way to cancel the anomaly could be to constraint the Abelian field strengths. A further way could be to consider gauge group SU$(N_1) \times$ SU$(N_2) \times$ SU$(N_3) \times$ U(1), the last factor being the diagonal subgroup of the original U$(1)^3$. Unfortunately we were not able to derive what mechanism string theory realizes.

6.1 Off-diagonal CS terms and moduli space

Since U$(N) \cong$ SU$(N) \times$ U$(1)/\mathbb{Z}_N$, the most general CS term we can write for a U(N) gauge field A is

$$
\mathcal{L}_{\text{CS}} = \frac{k}{4\pi} \int \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{\Lambda}{4\pi} \int \text{Tr}\, A \wedge d\, \text{Tr}\, A \,.
\tag{6.2}
$$

While the first term is the U(N) CS term we considered so far, the second term is a correction for the U(1) part only. If we have multiple gauge fields A_i of U(N_i), the most general CS interaction is

$$
\mathcal{L}_{\text{CS}} = \sum_i \frac{k_i}{4\pi} \int \text{Tr} \left(A_i \wedge dA_i + \frac{2}{3} A_i \wedge A_i \wedge A_i \right) + \sum_{ij} \frac{\Lambda_{ij}}{4\pi} \int \text{Tr}\, A_i \wedge d\, \text{Tr}\, A_j \,.
\tag{6.3}
$$

Λ_{ij} is a symmetric matrix, that contains off-diagonal CS interactions between the Abelian gauge fields (such mixing is not possible with non-Abelian gauge fields). Consider a vacuum in which the gauge group $\prod_j U(N_j)$ is broken to its maximal torus: let $A_{i,m}$ be the photon in the $U(1)_{i,m} \in U(N_i)$ (with $m = 1, \dots, N_i$). The CS term (6.3) becomes:

$$\mathcal{L}_{\mathrm{CS}} = \sum_{i,m} \frac{k_i + \Lambda_{ii}}{4\pi} \int A_{i,m} \wedge dA_{i,m} + \sum_i \sum_{m \neq n} \frac{\Lambda_{ii}}{4\pi} \int A_{i,m} \wedge dA_{i,n}$$
$$+ \sum_{i \neq j} \sum_{m,l} \frac{\Lambda_{ij}}{4\pi} \int A_{i,m} \wedge dA_{j,l} . \quad (6.4)$$

To these Abelian CS couplings the quantization condition applies. Since k_i already satisfy the first condition in (6.2), the parity anomaly is canceled if

$$\Lambda_{ij} - \frac{1}{2} A_{ij} \in \mathbb{Z} \qquad \forall\, i, j . \quad (6.5)$$

If the extra CS terms Λ_{ij} are generic, they affect the theory in an important way. In particular they modify the gauge charges of the monopole operators T, $\tilde{T}$, ruining the match we found with the dual geometry. In order for that not to happen, the following conditions should be met:

$$\sum_{j=1}^{3} \Lambda_{ij} = 0 \qquad i = 1, 2, 3 . \quad (6.6)$$

In that case the chiral ring is unmodified. Therefore we can cancel the $\mathbb{Z}_2$ anomaly by choosing appropriate Λ_{ij} CS levels for the $U(1)$ factors in the gauge group, and there is a 3-parameter family of solutions.

The semi-classical computation of section 5.3 is only slightly affected. Consider for instance the torsionless theory $n_0 = n_1 = 0$, one can show that the effective FI terms (see (5.26)) are modified to

$$\xi_1^{\mathrm{eff}} = -\Lambda_{11} \sum_a \tilde{\sigma}_a$$
$$\xi_2^{\mathrm{eff}} = \xi - \Lambda_{12} \sum_a \tilde{\sigma}_a + (3I - q)\sigma + \frac{3}{2} \sum_a \tilde{\sigma}_a - 3 \sum_{a \leq I} \tilde{\sigma}_a \quad (6.7)$$
$$\xi_3^{\mathrm{eff}} = -\xi - \Lambda_{13} \sum_a \tilde{\sigma}_a - (3I - q)\sigma - \frac{3}{2} \sum_a \tilde{\sigma}_a + 3 \sum_{a \leq I} \tilde{\sigma}_a .$$

In the derivation of the above we have used the constraints (6.6). Since we need $\xi_1^{\mathrm{eff}} = 0$ and $\xi_2^{\mathrm{eff}} = -\xi_3^{\mathrm{eff}}$ in order to have SUSY solutions, we can either introduce a bare ξ_1^{bare} or simply take $\Lambda_{11} = 0$. Assuming the latter, the final result (5.28) becomes

$$\chi(\sigma) = \left(\xi + \left(\frac{3}{2} - \Lambda_{12} \right) \sum_a \tilde{\sigma}_a \right) - (\sigma - \tilde{\sigma}_1)\, q + 3 \sum_a (\sigma - \tilde{\sigma}_a)\, \Theta(\sigma - \tilde{\sigma}_a) . \quad (6.8)$$

Remark that for $\Lambda_{12} = \frac{3}{2}$ the map (5.29) between field theory and geometric parameters is diagonal. A convenient choice in this case is

$$(\Lambda_{ij}) = \begin{pmatrix} 0 & \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & -3 & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} & 0 \end{pmatrix} . \quad (6.9)$$

While we have shown that from the 3d point of view the $\mathbb{Z}_2$ anomaly can be canceled at no cost by a small modification of the Chern-Simons interactions, we cannot *a priori* exclude that string theory chooses a more exotic method of anomaly cancelation.

7 Conclusions

We have discussed a stringy derivation for the low energy theory of M2-branes at the tip of a class of conical Calabi-Yau four-folds called $C(Y^{p,q}(\mathbb{CP}^2))$. We have found that the theories are quiver Chern-Simons theories, with quivers already known in the literature but with shifted ranks and levels. Such theories would be anomalous in 4d. In 3d they are well-defined, and the would-be anomalies appear instead as quantum corrections to the charges of monopole operators. As a result, the chiral rings of the theories receive one-loop corrections. We have shown with localization techniques that in $\mathcal{N} = 2$ superconformal theories the charges of chiral monopoles are one-loop exact; in some sense it is the analogue of the one-loop exactness of 4d anomalies. Even though what we have discussed is a specific example, we believe that similar considerations apply to more general classes of geometries. In a companion paper [23] we will initiate the systematic study of $\mathbb{C}^2/\mathbb{Z}_p$ bundles over del Pezzo surfaces, finding similar results.

There are a number of open questions to be stressed.

We have explicitly shown that a torsion flux $G_4 \in H^4(Y^{p,q}, \mathbb{Z})$ can be described by a flat C_3 connection and descends in the type IIA reduction of [22] to a B-field, such that the Page D4-charge $F_2 \wedge B_2$ is quantized. On the other hand, the Freed-Witten anomaly on D6-branes implies a semi-integral shift of the Page charge. One could then expect a similar mechanism to take place in M-theory, possibly along the lines of [78].

In the case of the torsionless theory of section 5.2, analyzing the moduli space we have found indications that each $\overline{\text{D6}}$-brane has a D2-brane stuck on it. At the orbifold point this follows from the knowledge of the mutually BPS states, described by the quiver. However from the point of view of M-theory the reason is less clear. An observation is that one could expect non-perturbative corrections, both in field theory and M-theory, leading to a runaway behavior and decompactification [46, 47, 79]. In [80] it was argued from supergravity that those corrections can be present rather generally when the CY$_4$ has exceptional six-cycles. While EM5-branes wrapped on 6-cycles can make the geometry unstable [79], the results of [80] suggests that when M2-branes are at the location of an exceptional 6-cycle we can blow it up without instability. In our field theory analysis we did not include possible non-perturbative corrections. It will be interesting to do so and compare field theory to M-theory expectations.

We have stressed that our field theories still have a $\mathbb{Z}_2$ anomaly that needs to be cured. One possibility is that extra CS terms are present for the U(1)^3 central gauge subgroup, and this would not spoil the moduli space. These extra CS terms, not considered in the literature so far, could be at the origin of the difficulties to reproduce the $N^{3/2}$ behavior of the partition function on S^3, but for now this is pure speculation.

Finally, it would be interesting to include a Romans mass (D8-charge) in the analysis [72, 73]. From our derivation of the CS levels in section 5 it is already obvious that the

D8-charge F_0 maps to $\sum_i k_i \neq 0$, as already proposed in [68, 72, 81–83]. A subtlety here is that for $F_0 \neq 0$ one has to include gravitational corrections, which conspire to keep the CS levels properly quantized.

Acknowledgments

A One-loop shift of Chern-Simons levels and effective FI terms

Three-dimensional theories do not suffer from chiral anomalies, however they can suffer from $\mathbb{Z}_2$ valued *parity anomalies* [40, 41] which depend on the matter content. Parity anomalies can always be canceled by half-integral Chern-Simons levels (so that the theory eventually breaks parity). One way to understand this is to start with a non-anomalous theory and integrate out some fermions such that the would-be low energy theory is naively anomalous. In fact a one-loop diagram provides a half-integral shift of the Chern-Simons level, which keeps the theory consistent.

Consider an Abelian theory with gauge group $\mathrm{U}(1)^G$. It can have generic Chern-Simons interactions

$$\sum_{ij}^{G} \frac{k_{ij}}{4\pi} \int A_i \wedge dA_j \, . \tag{A.1}$$

Let the theory have fermions f of electric charges $Q_i(f)$ and real masses M_f. Integrating out these fields (for $M_f \neq 0$) shifts the CS levels by [46, 47]

$$k_{ij} \rightarrow k_{ij} + \frac{1}{2} \sum_f Q_i(f) Q_j(f) \, \mathrm{sign}(M_f) \, , \tag{A.2}$$

where the sum is over all fermions f in the theory and a regularization of $\mathrm{sign}(x)$ which vanishes if $x = 0$ is implied. Invariance of the partition function under large gauge transformations then requires the quantization of bare Chern-Simons levels k_{ij} [47]

$$k_{ij} + \frac{1}{2} \sum_f Q_i(f) Q_j(f) \, \in \, \mathbb{Z} \, . \tag{A.3}$$

Consider in particular an $\mathcal{N} = 2$ theory: real masses have a bare contribution m_f and a contribution from the adjoint real scalars σ_i in vector multiplets:

$$M_f = m_f + \sum_i Q_i(f)\,\sigma_i\,. \tag{A.4}$$

In fact the bare masses m_f can be considered as expectation values of adjoint real scalars in external vector multiplets.

Consider now an $\mathcal{N} = 2$ quiver theory of generic gauge ranks N_i (some gauge fields could be external) with all fields X_{ij} in bifundamental or adjoint representations. When the theory is Higgsed to an Abelian one through generic VEV's for σ_i,

$$\sigma_i = (\sigma^{(i)})_{n_i} \qquad n_i = 1, \ldots, N_i\,, \tag{A.5}$$

and recall that they can always be diagonalized by gauge transformations, then chiral fields X_{ij} split into $N_i \times N_j$ components which have real masses

$$M[(X_{ij})^{n_i}{}_{n_j}] = \sigma^{(i)}_{n_i} - \sigma^{(j)}_{n_j}\,. \tag{A.6}$$

Integrating them out, the CS levels are shifted by

$$\begin{aligned}
\delta k_{n_i n_i'} &= \delta_{n_i n_i'} \frac{1}{2} \sum_j \sum_{n_j} \eta_{ij}\, \mathrm{sign}(\sigma^{(i)}_{n_i} - \sigma^{(j)}_{n_j}) \\
\delta k_{n_i n_j} &= -\frac{1}{2} \eta_{ij}\, \mathrm{sign}(\sigma^{(i)}_{n_i} - \sigma^{(j)}_{n_j}) \qquad \text{for } i \neq j
\end{aligned} \tag{A.7}$$

where we denoted by η_{ij} the net number of chiral fields between $\mathrm{U}(N_i)$ and $\mathrm{U}(N_j)$:

$$\eta_{ij} = \#(X_{ij}) - \#(X_{ji})\,. \tag{A.8}$$

Notice that $\eta_{ii} = 0$.

The presence of CS terms determines an effective shift of the bare FI parameters along the Coulomb branch [48], and the effective IR shift of the CS levels determines an extra one-loop contribution:

$$\xi^{\text{eff}}_{n_i} = \xi^{\text{bare}}_i + \sum_{j=1}^{G} \sum_{n_j=1}^{N_j} k^{\text{eff}}_{n_i n_j} \sigma^{(j)}_{n_j}\,. \tag{A.9}$$

Plugging in the actual effective CS levels corrected at one-loop, we obtain:

$$\xi^{\text{eff}}_{n_i} = \xi^{\text{bare}}_i + \sum_{j=1}^{G} \sum_{n_j=1}^{N_j} k^{\text{bare}}_{n_i n_j} \sigma^{(j)}_{n_j} + \frac{1}{2} \sum_{j=1}^{G} \sum_{n_j=1}^{N_j} \eta_{ij} \left| \sigma^{(i)}_{n_i} - \sigma^{(j)}_{n_j} \right|\,. \tag{A.10}$$

A.1 Monopole operators and dual photon

Consider a free Maxwell theory. In Euclidean signature, the dual photon φ is defined by

$$\partial_\mu \varphi = -\frac{2\pi i}{e^2} \epsilon_{\mu\nu\rho} F^{\mu\nu}\,. \tag{A.11}$$

Due to flux quantization, it is periodic of period 2π. In a $\mathcal{N} = 2$ supersymmetric theory, φ and σ pair into a chiral superfield of lowest component

$$\Phi = \frac{2\pi}{e^2}\sigma + i\varphi \ . \tag{A.12}$$

The monopole operator of flux n is defined by

$$T^{(n)} = \exp(n\Phi) \ . \tag{A.13}$$

This operator corresponds to the Borokhov-Kapustin-Wu definition of a monopole operator [28]: it is an operator which inserts n units of magnetic flux at $x = 0$. We can see it explicitly in the dual photon formulation of the Maxwell theory. We have

$$\langle T(0)^{(n)} \rangle = \int \mathcal{D}\varphi \, \mathcal{D}\sigma \exp \int d^3x \left[-\frac{1}{2e^2}(\partial\sigma)^2 - \frac{e^2}{8\pi^2}(\partial\varphi)^2 + n\left(\frac{2\pi}{e^2}\sigma + i\varphi\right)\delta^3(x) \right] \ . \tag{A.14}$$

This insertion changes the perturbative saddle point from $\sigma = \varphi = 0$ to

$$\sigma = \frac{n}{2r} \ , \qquad\qquad \varphi = \frac{2\pi i}{e^2}\frac{n}{2r} \ . \tag{A.15}$$

Such a singularity in the dual photon field corresponds to n units of magnetic flux at $r = 0$, as we can see by using the relation (A.11). This discussion is closely related to the discussion of line operators given by Kapustin in [26].

B Flavored ABJM: resolved geometry and field theory

In [19] (see also [20]) field theories on M2-branes probing a family of toric CY_4 cones whose 3d toric diagrams project to that of the conifold were derived reducing M-theory to type IIA. There is a 5-parameter family of such toric CY_4 singularities and of dual flavored ABJM quiver gauge theories. These dualities have been thoroughly checked in [84], where the partition function of the CFTs on S^3 and the superconformal R-charges of gauge invariant operators were computed and matched with data of the dual M-theory backgrounds.

Here we explicitly show how real masses in the flavored ABJM models of [19, 20] correspond to partial resolutions of the CY_4 geometry. First we sketch the chiral ring of the theories with monopoles, then we compute the geometric moduli space semiclassically, as in section 2.4, in the presence of bare FI parameters and real masses for flavors which map to resolution parameters of the CY_4, similarly to section 5.3. Finally we show how to see the resolutions using the monopole operators of the effective theories. Some remarks have already appeared in [85], which mainly focused on the dual type IIB brane configuration to study fractional M2-branes.

For comparison with section 3, we consider a 2-parameter subfamily of CY_4 singularities, whose toric diagrams take the form (see figure 4a)

$$a_i = (0,0,i) \ , \qquad b_0 = (1,0,0) \ , \qquad c_h = (1,1,h) \ , \qquad d_0 = (0,1,0) \tag{B.1}$$

(a) Toric diagram.
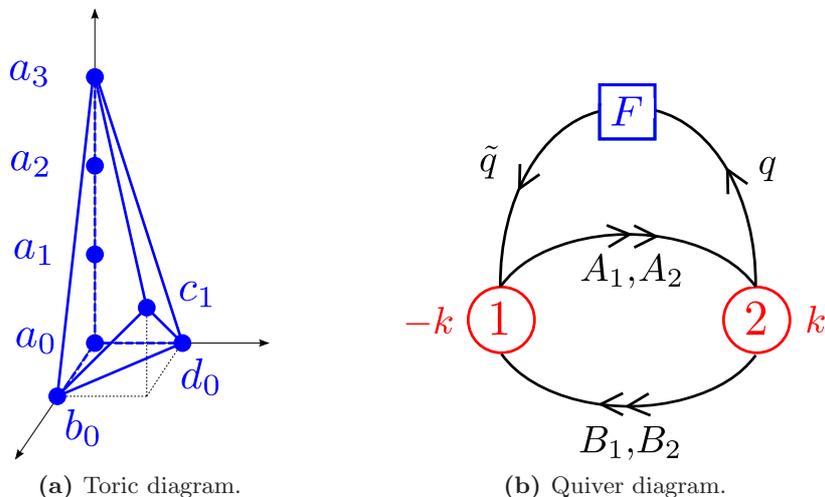
(b) Quiver diagram.

Figure 4. Toric diagram of a toric CY_4 singularity leading to ABJM with p flavors, for $p = 3$ and $h = 1$, and quiver diagram of the flavored ABJM theory.

with $i = 0, 1, \ldots, p$ running over the single vertical column. The charges of the GLSM for the CY_4, including the global $U(1)_M$ charges in the last line, are

$$
\begin{array}{c|ccccccccccc|c}
CY_4 & a_0 & a_1 & a_2 & \cdots & a_{p-2} & a_{p-1} & a_p & b_0 & c_h & d_0 & FI \\
\hline
 & h+1 & -h & 0 & \cdots & 0 & 0 & 0 & -1 & 1 & -1 & \xi^c \\
 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \xi_2 \\
 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \xi_3 \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 & 0 & 0 & 0 & \xi_{p-1} \\
 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 & 0 & 0 & \xi_p \\
\hline
U(1)_M & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & r_0
\end{array}
\tag{B.2}
$$

with $\xi_a \geq 0$ for all $a = 2, \ldots, p$. The minimal GLSM for the fibered conifold $\mathcal{C}$ in IIA is

$$
\begin{array}{cccc|c}
a & b & c & d & FI \\
\hline
1 & -1 & 1 & -1 & \chi(r_0)
\end{array}
\tag{B.3}
$$

where

$$
\chi(r_0) = \xi^c - h(r_0 - \zeta_1) - \sum_{a=1}^{p} (r_0 - \zeta_a)\Theta(r_0 - \zeta_a)
\tag{B.4}
$$

and the parameters ζ_a are defined as in (3.11). The RR 2-form flux through the 2-cycle of the resolved $\mathcal{C}$ is $\chi'(r_0)$, and there is a D6-brane along the divisor D_a of the resolved conifold over $r_0 = \zeta_a$, for each $a = 1, \ldots, p$.

The type IIA background allows us to derive the field theory on N 2-brane probes [19]: an ABJM $U(N)_{-k} \times U(N)_k$ quiver with p pairs of flavors $\tilde{q}^a$, q_a as in figure 4b and superpotential

$$
W = \epsilon^{ij}\epsilon^{rs} \operatorname{Tr}(A_i B_r A_j B_s) + \sum_{a=1}^{p} \tilde{q}^a A_1 q_a \,.
\tag{B.5}
$$

The CS level k is related to the integers h and p appearing in the CY_4 geometry by $k = h + \frac{p}{2}$, in agreement with the quantization law (A.3). The complete map between geometric and field theory data will be provided below.

The flavor symmetry group of the field theory is $SU(p)$ rather than $U(p)$, since the action of the overall $U(1)$ in $U(p)$ coincides with that of the overall diagonal gauge $U(1)$. Its rank $p-1$ equals the number of independent real masses for flavors. However, to compare with (B.4), we find it convenient to work in terms of the flavor group $U(p)$ of the ungauged theory and introduce p real masses $\{m_a\}$ as the eigenvalues of the corresponding background hermitian scalar. One of them can be absorbed by a shift of the real scalar σ in the overall diagonal gauge $U(1)$.

Following the approach of section 2.3, the geometric moduli space of the Abelian gauge theory, where the flavor fields do not acquire a VEV, was computed in [19, 20] for the superconformal theory with vanishing bare FI parameters and real masses. The gauge charges of bifundamentals and unit flux diagonal monopole operators are

$$
\begin{array}{c|cccc}
 & A_j & B_r & T & \tilde{T} \\
\hline
U(1)_{-h-\frac{p}{2}} & 1 & -1 & -h & h+p \\
U(1)_{h+\frac{p}{2}} & -1 & 1 & h & -h-p
\end{array}
\tag{B.6}
$$

and the quantum corrected F-term relation involving monopole operators is $T\tilde{T} = A_1^p$. It is then easy to prove that the geometric moduli space (2.24) of the Abelian gauge theory reproduces the CY_4 cone geometry probed by the M2-brane in M-theory [19].

We now perform the semiclassical analysis of these flavored ABJM theories, in the general case where real masses m_a and bare FI parameters $\xi_1 = -\xi_2$ are allowed. As explained in section 2.4, it suffices to consider the Abelian gauge theory. By a Weyl transformation in the flavor group and a shift of the real scalar σ in the gauge vector multiplet, we order the real masses as follows

$$
0 = m_1 \leq m_2 \leq \cdots \leq m_p \, .
\tag{B.7}
$$

We first compute the effective CS levels

$$
k_1^{\text{eff}}(\sigma) = -k_2^{\text{eff}}(\sigma) = -\left(h + \frac{p}{2}\right) - \frac{1}{2} \sum_{a=1}^{p} \text{sign}(\sigma - m_a)
\tag{B.8}
$$

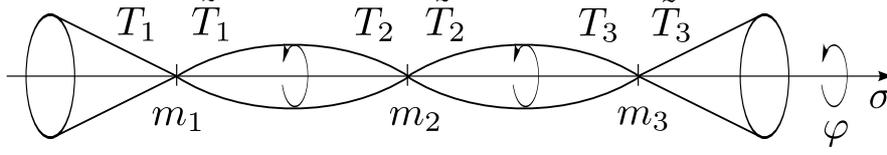and then the effective FI parameters of the two gauge groups

$$
\xi_1^{\text{eff}}(\sigma) = -\xi_2^{\text{eff}}(\sigma) = \xi_1 + \left(\frac{h}{p} + \frac{1}{2}\right) \sum_{a=1}^{p} m_a - \left(h + \frac{p}{2}\right)\sigma - \frac{1}{2} \sum_{a=1}^{p} |\sigma - m_a| \, .
\tag{B.9}
$$

Rewriting (B.9) as

$$
\xi_1^{\text{eff}}(\sigma) = \xi_1 + \frac{h}{p} \sum_{a=1}^{p} m_a - h\,(\sigma - m_1) - \sum_{a=1}^{p} (\sigma - m_a)\Theta(\sigma - m_a) \, ,
\tag{B.10}
$$

we compare it to the resolution parameter (B.4) of the fibered conifold in IIA to find the precise map between parameters in field theory and geometry:

$$
\sigma = r_0 \, , \qquad m_a = \zeta_a \, , \qquad \xi_1 + \frac{h}{p} \sum_a m_a = \xi^c \, , \qquad \xi_1^{\text{eff}} = \chi \, .
\tag{B.11}
$$

Figure 5. Topology of the complex dimension parametrized by bare monopole operators in flavored ABJM for $p = 3$ flavors with non-vanishing real masses.

Finally, we can introduce the dual photon φ in the one-loop field theory analysis to complexify the real scalar σ. Exponentiating the resulting complex scalars provides BPS diagonal monopole operators of flux one (section A.1). We are now considering a more general situation compared to section 2.4, as we allow real masses and FI parameters. A crucial insight on the degeneration loci of the $U(1)_M$ circle acting on the dual photon is provided by the quantized effective CS level (B.8), which jumps whenever σ equals one of the real masses m_a. Such a jump is possible only if the $U(1)_M$ circle shrinks there. As a result, the topology of the complex direction parametrized by bare monopole operators is the one shown in figure 5: a chain of $p - 1$ $\mathbb{CP}^1$'s is generated, whose sizes equal the differences of adjacent real masses $m_{a+1} - m_a$.

This topology arises from the resolution of the $\mathbb{C}^2/\mathbb{Z}_p$ singularity parametrized by the bare monopole operators T, $\tilde{T}$ and the flavored bifundamental A_1 subject to the quantum relation $T\tilde{T} = A_1^p$, as we now explain (see for instance [86] for a review of this resolution). The resolved singularity is a smooth variety covered by p patches U_a parametrized by $(T_a, \tilde{T}_a)$, $a = 1, \ldots, p$, with covering maps

$$(T_a, \tilde{T}_a) \longmapsto \begin{cases} T = T_a^a \tilde{T}_a^{a-1} \\ \tilde{T} = T_a^{p-a} \tilde{T}_a^{p+1-a} \\ A_1 = T_a \tilde{T}_a \end{cases} \tag{B.12}$$

and transitions given by

$$\tilde{T}_a T_{a+1} = 1 , \qquad T_a \tilde{T}_a = T_{a+1} \tilde{T}_{a+1} . \tag{B.13}$$

In particular the monopole operators of the CFT can be expressed as $T = T_1$ and $\tilde{T} = \tilde{T}_p$, which describe the noncompact cones in figure 5. The remaining T_a and $\tilde{T}_a$ describe the $p-1$ blown up $\mathbb{CP}^1$'s, since $\tilde{T}_a T_{a+1} = 1$. The gauge charges of T_a and $\tilde{T}_a$ follow from (B.12) and (B.13),

$$Q_i[T_a] = Q_i[T_1] - (a - 1)Q_i[A_1]$$
$$Q_i[\tilde{T}_a] = Q_i[\tilde{T}_p] - (p - a)Q_i[A_1] . \tag{B.14}$$

Specializing to the first gauge group we find $Q_1[T_a] = -h - a + 1$ and $Q_1[\tilde{T}_a] = h + a$, in agreement with the charges induced on diagonal monopoles of flux one by the effective CS levels (B.8) in the corresponding range of σ. Finally note that

$$T_a \tilde{T}_b = A_1^{b-a+1} , \qquad b \geq a . \tag{B.15}$$

Summarizing, pairs of unit flux monopole operators $(T_a, \tilde{T}_{a+1})$ appear when real mass differences $m_{a+1} - m_a$ are turned on. Pairs $(T_a, \tilde{T}_a)$ disappear when the corresponding real masses m_a are sent to infinity, and the holomorphic relations between surviving monopole operators take the form (B.15). By this method we can describe, at the level of the geometric moduli space and the chiral ring, real mass deformations of flavored ABJM models which correspond to partial or complete resolutions of the $\mathbb{C}^2/\mathbb{Z}_p$ fiber in the CY_4 geometry.

C Charges of monopoles and localization

In this appendix we show with localization methods that the monopole charges used in section 2 are one-loop exact. We will essentially borrow the ideas in [28, 29, 33], and integrate them with the results in [31, 38], whose conventions we follow.

In any 3d conformal theory there is a one-to-one correspondence between local operators on $\mathbb{R}^3$ and states on $S^2 \times \mathbb{R}$ (we will consider the Euclidean theory). Therefore to determine the charges of a monopole operator at the origin of $\mathbb{R}^3$, we compute the charges on $S^2 \times \mathbb{R}$ of a state carrying some magnetic flux on S^2.

We are interested in half-BPS monopole operators (described by chiral multiplets) in an $\mathcal{N} = 2$ superconformal theory, because to them we can apply localization methods. Localization allows us to exactly compute the path-integral of a supersymmetric theory, with the insertion of $\mathcal{Q}$-closed operators and with $\mathcal{Q}$-invariant boundary conditions ($\mathcal{Q}$ being a supercharge). The operators we insert are the charges we want to measure, the boundary conditions are those of the monopole state.

Flavor charges commute with the supercharges and do not pose any problem. On the contrary the R-charge does not commute with the supercharges, and cannot be directly computed with localization by inserting it in the path-integral. Likewise on $S^2 \times \mathbb{R}$ the supercharges do not commute with the Hamiltonian H that generates translations on $\mathbb{R}$ (on $\mathbb{R}^3$, H generates dilations and the supercharges have dimension $1/2$):

$$[R, Q_\alpha] = Q_\alpha , \quad [R, \bar{Q}_\alpha] = -\bar{Q}_\alpha , \quad [H, Q_\alpha] = \frac{1}{2} Q_\alpha , \quad [H, \bar{Q}_\alpha] = \frac{1}{2} \bar{Q}_\alpha . \qquad \text{(C.1)}$$

Here both H and R are Hermitian operators. Notice that in radial quantization Q_α and $\bar{Q}_\alpha$ are no longer Hermitian conjugate to each other: their Hermitian conjugates are the special supercharges $S_\alpha = Q_\alpha^\dagger$ and $\bar{S}_\alpha = \bar{Q}_\alpha^\dagger$, of dimension $-1/2$. Therefore

$$[R, S_\alpha] = -S_\alpha , \quad [R, \bar{S}_\alpha] = \bar{S}_\alpha , \quad [H, S_\alpha] = -\frac{1}{2} S_\alpha , \quad [H, \bar{S}_\alpha] = -\frac{1}{2} \bar{S}_\alpha . \qquad \text{(C.2)}$$

To find the eigenvalues of H and R on the monopole background, we select a particular supercharge $\mathcal{Q}$ (specified below) and its conjugate $\mathcal{Q}^\dagger$, which preserve the monopole background and satisfy the algebra [87]

$$\{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = 0 , \qquad [\mathcal{Q}, H + J_3] = 0 , \qquad \{\mathcal{Q}, \mathcal{Q}^\dagger\} = H - R - J_3 \qquad \text{(C.3)}$$

where J_i are the generators of rotations on S^2. Notice that the last relation implies the BPS energy bound $E \geq r + j_3$ (for the eigenvalues of H, R, J_3), while it also implies that

$(H - R - J_3)$ commutes with $\mathcal{Q}$ and $\mathcal{Q}^\dagger$. Therefore we can compute them localizing the path integral with respect to $\mathcal{Q}$. We are interested in rotationally invariant monopole states, $J_3 = 0$, so the computation of $(H + J_3)$ gives us the dimension E of the monopole operator. The charge $(H - R - J_3)$ simply vanishes, because on scalars in short representations $E = r$.

To compute the energy E of a monopole state we will need the dimensions Δ_Φ of fundamental fields. As a consequence, we can only compute the R-charge of a monopole operator as a function of the R-charges of fundamental fields. The values of the charges at the IR fixed point could be computed with an extremization method [35]. However having them as functions will be good enough to our purposes of determining quantum relations in the chiral ring.

Let us review the computation in [38]. The $\mathcal{N} = 2$ superconformal algebra has 8 supercharges; in Euclidean signature they are parametrized by the independent spinors ϵ and $\bar{\epsilon}$. On a conformally flat background the spinor ϵ satisfies the Killing equation $D_\mu \epsilon = \gamma_\mu \kappa$, where κ is an arbitrary spinor, and it is parameter for the global supercharges Q and S. Similarly $\bar{\epsilon}$ satisfies $D_\mu \bar{\epsilon} = \gamma_\mu \bar{\kappa}$ and parametrizes $\bar{Q}$ and $\bar{S}$.

The supercharge $\mathcal{Q}$ we will use for localization is constructed out of $\bar{Q}$ and $\bar{S}$. Therefore let us look at the anti-holomorphic transformations of fields. For a vector multiplet $V = (A_\mu, \sigma, D, \lambda)$:

$$\delta\sigma = \bar{\epsilon}\lambda \,, \qquad \delta A_\mu = -i\,\bar{\epsilon}\gamma_\mu\lambda \,, \qquad \delta\lambda = 0$$

$$\delta D = i\,\bar{\epsilon}\gamma^\mu D_\mu\lambda + i\,\bar{\epsilon}[\sigma,\lambda] + \frac{i}{3}\,D_\mu\bar{\epsilon}\gamma^\mu\lambda \tag{C.4}$$

$$\delta\bar{\lambda} = -\frac{i}{2}\,\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} - \gamma^\mu\bar{\epsilon}\,D_\mu\sigma + i\,D\bar{\epsilon} - \frac{2}{3}\,\gamma^\mu D_\mu\bar{\epsilon}\,\sigma \,.$$

The transformations for a chiral multiplet $\Phi = (\phi, \psi, F)$ with Weyl weight Δ_Φ are:

$$\delta\phi^\dagger = \sqrt{2}\,\bar{\epsilon}\bar{\psi} \,, \qquad \delta\phi = 0 \,, \qquad \delta\bar{\psi} = \sqrt{2}\,i\,\bar{\epsilon}F^\dagger \,, \qquad \delta F^\dagger = 0$$

$$\delta\psi = \sqrt{2}\,\bar{\epsilon}\sigma\phi - \sqrt{2}\,\gamma^\mu\bar{\epsilon}\,D_\mu\phi - \frac{2\sqrt{2}}{3}\,\Delta_\Phi\phi\,\gamma^\mu D_\mu\bar{\epsilon} \tag{C.5}$$

$$\delta F = \sqrt{2}\,i\,\bar{\epsilon}\gamma^\mu D_\mu\psi + \sqrt{2}\,i\,\bar{\epsilon}\sigma\psi + 2i\,\bar{\epsilon}\bar{\lambda}\,\phi + \frac{2\sqrt{2}\,i}{3}\left(\Delta_\Phi - \frac{1}{2}\right)D_\mu\bar{\epsilon}\gamma^\mu\psi \,.$$

$\mathcal{N} = 2$ superconformal symmetry forces the superconformal R-charge of a chiral multiplet to be equal to Δ_Φ.

We deform the theory by some δ_1-exact piece:

$$S_{\text{def}}^{\text{vector}} = \delta_1\delta_2 \int d^3x\,\sqrt{g}\,\text{tr}\left(-\frac{t}{2}\bar{\lambda}\lambda\right)$$

$$S_{\text{def}}^{\text{chiral}} = \delta_1\delta_2 \int d^3x\,\sqrt{g}\left(-\frac{it}{2}\,\phi^\dagger F\right) \tag{C.6}$$

where δ_i are SUSY variations with respect to two linearly independent anti-holomorphic spinors. On $S^2 \times \mathbb{R}$ we will take as conformally Killing spinors the two solutions $\bar{\epsilon}_{1,2}$ of

$$D_\mu\bar{\epsilon} = \frac{1}{2r}\gamma_\mu\gamma_3\bar{\epsilon} \,. \tag{C.7}$$

These form a doublet of SO(3) rotations of S^2, and we take the third component j_3 such that $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ have j_3 eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Therefore our charge $\mathcal{Q}$ corresponds to $\bar{\epsilon}_1$. With that choice of $\bar{\epsilon}_{1,2}$, the deformation action becomes

$$
\begin{aligned}
S_{\text{def}}^{\text{vector}} &= t \int d^3x \, \sqrt{g} \, \text{tr} \left[V_\mu V^\mu + D^2 - 2\bar{\lambda}\gamma^\mu D_\mu \lambda - 2\bar{\lambda}[\sigma, \lambda] - \frac{1}{r} \bar{\lambda}\gamma_3 \lambda \right] \\
S_{\text{def}}^{\text{chiral}} &= t \int d^3x \, \sqrt{g} \left[\frac{1 - 2\Delta_\Phi}{r} \left(\phi^\dagger D_3 \phi + \frac{1}{2} \bar{\psi}\gamma_3 \psi \right) + \frac{\Delta_\Phi(1 - \Delta_\Phi)^2}{r} \phi^\dagger \phi \right]
\end{aligned}
\tag{C.8}
$$

where the vector V_μ is defined as

$$
V_1 = F_{23} - D_1\sigma \,, \qquad V_2 = F_{31} - D_2\sigma \,, \qquad V_3 = F_{12} - D_3\sigma - \frac{1}{r}\sigma \,. \tag{C.9}
$$

The path-integral localizes around $V_\mu = 0$, $\phi = 0$. Among the solutions, we find BPS monopole states:

$$
A_\mu dx^\mu = H \, B_i dx^i \,, \qquad\qquad \sigma = \frac{H}{2r} \tag{C.10}
$$

with all other fields vanishing, where B_i is the Dirac monopole configuration with unit magnetic charge, and up to gauge fixing H takes value in the Cartan part $\mathfrak{h}$ of the gauge Lie algebra $\mathfrak{g}$. Notice that due to the boundary conditions on $\mathbb{R}$ the classical background has no zero-modes.

The localization is carried out in [31] for the vector multiplet, and in [38] for the chiral multiplet. We do not need their full result, only the charges, because everything else cancels out in the normalization of the monopole state. After a suitable regularization, their result is

$$
\begin{aligned}
E &= \frac{1}{2} \sum_\Phi (1 - \Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(H)| - \frac{1}{2} \sum_{\alpha \in \mathcal{G}} |\alpha(H)| \\
q_i &= -\frac{1}{2} \sum_\Phi F_i(\Phi) \sum_{\rho \in R_\Phi} |\rho(H)| \\
b_0(a) &= -\frac{1}{2} \sum_\Phi \sum_{\rho \in R_\Phi} |\rho(H)| \, \rho(a) \,,
\end{aligned}
\tag{C.11}
$$

where ρ are weights of the representation R_Φ, α are roots of the gauge group $\mathcal{G}$, F_i are the flavor charges of the fields Φ_i, E is the dimension (equal to the superconformal R-charge) and q_i are global charges. The linear function $b_0(a)$ on $\mathfrak{h}$ appears in the path-integral as a phase $e^{ib_0(a)}$, function of the Cartan gauge field $a \in \mathfrak{h}$. In fact it is an element of $\mathfrak{h}^*$. The topological charges are not quantum corrected because no fundamental fields are charged under them.

Flavor and gauge charges are the result we were looking for. The dimension $E = r$, where r is the superconformal R-symmetry, still depends on the unknowns $\Delta_\Phi = r_\Phi$. However we can rewrite the formulae above as a sum over fermions (gaugini and in chiral multiplets):

$$
\delta q = -\frac{1}{2} \sum_{\text{fermions } f} \sum_{\rho \in R_f} |\rho(H)| \, q_f \,. \tag{C.12}
$$

This is true for both superconformal R-charge, flavor charges and gauge charges (δ means quantum correction: classically monopoles can have gauge charges). Since the formula is linear in the charges, it is correct for any R-charge, even those which are not in the stress-tensor multiplet (they are combinations of the superconformal R-symmetry and of flavor symmetries).

C.1   Quantum F-term relations

In section 2 we showed how to obtain the chiral ring of $\mathcal{N} = 2$ CS-matter theories: first compute the (one-loop exact) charges of monopole operators, then construct quantum F-term relations compatible with those charges. We constructed the shortest gauge-invariant relations compatible with the charges. However it is also crucial that the coefficient in those relations is not zero, otherwise the chiral ring is different. As partial evidence, we show here that the counting of fermionic zero-modes is compatible with a non-vanishing coefficient, following an argument in [29].

Let us consider, for simplicity, window $[0,0]$ of the $Y^{p,q}(\mathbb{CP}^2)$ quiver with the extra restriction $n_1 \leq p/2$ (similar considerations apply in all other windows). The quantum F-term relation reads schematically

$$T\tilde{T}Y^{3n_1}Z^{3p-3n_1} \sim (XYZ)^p \tag{C.13}$$

and we want to check whether the coefficient is non-vanishing. To do that, we should compute the amplitude

$$\langle \tilde{T}^\dagger | Z^{\dagger p} Y^{\dagger p} X^{\dagger p} Y^{3n_1} Z^{3p-3n_1} | T \rangle \tag{C.14}$$

in radial quantization. To be precise, such amplitude is not gauge invariant, because we are using bare monopoles. To make it gauge-invariant we should either connect the bare monopoles with the undaggered operators by Wilson lines, or move the undaggered operators to $\pm\infty$ in Euclidean time (essentially using the gauge invariant monopoles $b_{(q)}$ and $c_{(3p-q)}$ in place of the bare monopoles T and $\tilde{T}$). We can think of adding the Wilson lines, and take the limit at the end. The amplitude has a chance to be non-vanishing if the operators inserted absorb all fermionic zero-modes on the monopole background.

We essentially already computed the number of zero-modes when computing the monopole charges. We have the following number of fermionic zero-modes:

$$3(p - 2n_1) \text{ for } \psi_X \qquad\qquad 2(p - n1) \text{ for } \lambda_2$$
$$3(p - n_1) \text{ for } \psi_Y \qquad\qquad 2(p - 2n_1) \text{ for } \lambda_3$$
$$3(2p - 3n_1) \text{ for } \psi_Z \ .$$

These very zero-modes are responsible for the charges of bare monopoles.

The operators we have inserted are only scalars, however they are coupled to matter fermions by terms from the superpotential:

$$V \supset \partial_i \partial_j W \, \psi_i \psi_j \qquad \Rightarrow \qquad X^\dagger \sim \psi_Y \psi_Z \ , \qquad Y^\dagger \sim \psi_Z \psi_X \ , \qquad Z^\dagger \sim \psi_X \psi_Y \tag{C.15}$$

and to gaugini by terms from the D-term kinetic terms:

$$\int d^4\theta \, \Phi^\dagger e^V \Phi \ \rightarrow \ \phi^\dagger \, \lambda \, \psi_\phi \tag{C.16}$$

so that we get

$$X \sim \lambda_3 \psi_X \sim \lambda_2 \psi_X \,, \qquad Y \sim \lambda_1 \psi_Y \sim \lambda_2 \psi_Y \,, \qquad Z \sim \lambda_2 \psi_Z \sim \lambda_3 \psi_Z \,. \qquad (C.17)$$

If we just substitute these relations, we can soak up the following zero-modes:

$$\psi_X^{2p} \psi_Y^{2p+3n_1} \psi_Z^{5p-3n_1} \lambda_2^{p+3n_1} \lambda_3^{2p-3n_1}$$

which is not quite enough. However there are other terms in the Lagrangian that couple scalar fields: for instance the scalar potential contains

$$V \subset XYY^\dagger X^\dagger \qquad \Rightarrow \qquad X^\dagger \sim X^\dagger Y Y^\dagger \sim \psi_Y \psi_Z (\lambda_3 \psi_X \psi_Y \psi_Z)$$

and so on. Using these other relations all the fermionic zero-modes can be soaked up.

D Mirror geometry and exact periods

For completeness, we write down in this appendix the exact central charges for the B-branes on $\mathcal{O}_{\mathbb{CP}^2}(-3)$, as was found in [64].[25] The periods are solutions of Picard-Fuchs equations in the mirror CY$_3$. The mirror of $\mathcal{O}_{\mathbb{CP}^2}(-3)$ can be described as a double fibration over a plane $\mathbb{C} \cong \{W\}$:

$$W = P(x,y) = x + y + \frac{1}{xy} - \psi \,, \qquad W = uv \,. \qquad (D.1)$$

The parameter ψ is a complex structure modulus of the mirror geometry, corresponding to the single Kähler modulus of $\mathcal{O}_{\mathbb{CP}^2}(-3)$. The first equation in (D.1) describes a Riemann surface of genus one, fibered over the plane, which degenerates at 3 critical points on the $\{W\}$ plane, namely at

$$W + \psi = 2\omega^n + \omega^{-2n} \,, \qquad \text{where} \qquad x = y = \omega^n \,; \qquad n = 0, 1, 2 \,. \qquad (D.2)$$

ω is the third rood of unity. A good coordinate on the complex moduli space of the curve is

$$z = \frac{27}{\psi^3} \qquad (D.3)$$

The factor of 27 is there for convenience, such that $z = 1$ is the conifold point in moduli space (where one of the fractional branes becomes tensionless). The Picard-Fuchs equation is [67]

$$\left(\theta_z^3 - z \left(\theta_z + \frac{2}{3} \right) \left(\theta_z + \frac{1}{3} \right) \theta_z \right) \Pi = 0 \,, \qquad (D.4)$$

with $\theta_z \equiv z \partial_z$. Obviously, the constant period $\Pi = 1$ is a solution. This equation is a Meijer equation; a basis for the non-trivial solutions is given by two Meijer G-functions; see e.g. [89]. Instead of the large volume point $z = 0$, we are rather interested in the form of

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[25]See also [88] for an interesting discussion.

Π near the orbifold point $w = 1/z = 0$. We can write the two relevant Meijer G-functions in terms of generalized hypergeometric functions:

$$
\begin{aligned}
G_1(w) &= \frac{9}{2} \frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})} (-w)^{\frac{2}{3}} \, {}_3F_2 \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{4}{3}, \frac{5}{3}; w \right) , \\
G_2(w) &= 3 \frac{\Gamma(\frac{1}{3})^2}{\Gamma(\frac{2}{3})} (-w)^{\frac{1}{3}} \, {}_3F_2 \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{4}{3}, \frac{5}{3}; w \right) .
\end{aligned}
\tag{D.5}
$$

These functions are defined on the whole $\{w\}$-plane by analytic continuation; we choose the cuts to lie on the real positive axis, from $w = 0$ to $w = 1$, and from $w = 1$ to $w = \infty$. One can prove that the exact periods for the D4-brane on $\mathbb{CP}^1 \subset \mathbb{CP}^2$ and for the D6-brane on $\mathbb{CP}^2$ are:

$$
\begin{aligned}
Z(D4) &= t(w) = \frac{\omega^2 - \omega}{4\pi^2} \left(G_1(w) - G_2(w) \right) , \\
Z(D6) &= t_6(w) = \frac{1}{3} + \frac{1}{4\pi^2} \left(\omega^2 G_1(w) + \omega G_2(w) \right) ,
\end{aligned}
\tag{D.6}
$$

where $\omega = e^{\frac{2\pi i}{3}}$.

E Torsion flux, B-field and Page charges

In this appendix we want to explicitly show, at least for the case of $Y^{p,q}(\mathbb{CP}^2)$, that an M-theory torsion flux $G_4 \in H^4(Y^{p,q}, \mathbb{Z}) = \Gamma$ is in fact represented by a flat connection C_3 (well-defined only patch by patch), which descends in type IIA to a vanishing F_4 and a non-trivial B-field B_2, constrained in such a way that the Page D4-charge $F_2 \wedge B_2$ is quantized. Moreover C_3 in M-theory and B_2 in type IIA are pure gauge precisely on the kernel of the projection from $\mathbb{Z}^2$ to Γ. The discussion here is classical, so it is not corrected by the Freed-Witten anomaly which requires a deeper analysis.

The metric of $Y^{p,q}(\mathbb{CP}^2)$ is [21, 22]:

$$
\begin{aligned}
ds^2(Y^{p,q}) &= \rho^2 d\tilde{s}^2(\mathbb{CP}^2) + \frac{d\rho^2}{\mathrm{U}(\rho)} + q(\rho)(d\psi + A)^2 + w(\rho)\ell^2 \left[d\gamma + \ell^{-1} f(\rho)(d\psi + A) \right]^2 \\
ds^2(\mathbb{CP}^2) &= \frac{3}{4} \left[d\beta^2 + \sin^2 \beta \, \sigma_3^2 + \sin^2 \frac{\beta}{2} (d\theta^2 + \sin^2 \theta \, d\varphi^2) \right] \\
A &= 3 \sin^2 \frac{\beta}{2} \sigma_3 \qquad\qquad \sigma_3 = \frac{d\chi - \cos\theta \, d\varphi}{2} \\
\tilde{J} &= \frac{3}{4} \sin\beta \, d\beta \wedge \sigma_3 + \frac{3}{4} \sin^2 \frac{\beta}{2} \sin\theta \, d\theta \wedge d\varphi .
\end{aligned}
\tag{E.1}
$$

The range of coordinates is $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, $\chi \in [0, 4\pi)$, $\beta \in [0, \pi]$, $\rho \in [\rho_1, \rho_2]$, $\psi \in [0, 2\pi)$, $\gamma \in [0, 2\pi)$ and intervals with an open end are compactified. The functions $\mathrm{U}(\rho)$, $q(\rho)$, $w(\rho)$ and $f(\rho)$ are defined in [21, 22] and depend on a parameter: U and q have simple roots at $\rho_{1,2}$ with $0 < \rho_1 < \rho_2$, while w is positive in $[\rho_1, \rho_2]$. $\tilde{J}$ is the Kähler form of $\mathbb{CP}^2$ and satisfies $2\tilde{J} = dA$. (ρ, ψ) describe an S^2 bundle over $\mathbb{CP}^2$, that is M_6, and γ describes a U(1) bundle over M_6. The metric is normalized such that $\widetilde{\mathrm{Ric}} = 2\tilde{g}$ and

Ric $= 6g$. The constant ℓ is chosen in such a way that

$$C_1 \equiv \ell^{-1} f(\rho)(d\psi + A) \,, \quad \int_{\mathcal{C}_0} \frac{dC_1}{2\pi} = f(\rho_2) - f(\rho_1) \equiv p \,, \quad \int_{\mathcal{C}^+} \frac{dC_1}{2\pi} = f(\rho_2) \equiv 3p - q \quad \text{(E.2)}$$

that is the U(1) bundle over M_6 is well-defined. $\mathcal{C}_0$ is the S^2 fiber of M_6 and $\mathcal{C}^+$ is $\mathbb{CP}^1 \subset \mathbb{CP}^2$ at $\rho = \rho_2$. In fact $F_2 = dC_1$, living on M_6, becomes the RR 2-form in type IIA.

We add torsion flux $G_4 \in \Gamma$ on $Y^{p,q}$ in M-theory, where $\Gamma = \mathbb{Z}^2/\langle(3q,q), (q,p)\rangle$. We represent it with a non-trivial flat connection C_3, which will be expressed in terms of a closed 2-form B_2 on M_6: in fact $H^2(M_6, \mathbb{R}) = \mathbb{R}^2$ is generated by dC_1 and $\tilde{J}$. We cannot use $[d\gamma + C_1] \wedge B_2$ because it is not closed, nor can we use $d\gamma \wedge B_2$ because $d\gamma$ is singular at 4 points on M_6: $\beta = \pi$, $\rho = \rho_{2,1}$, $\theta = 0, \pi$. Thus we cover $Y^{p,q}$ with four patches, each including one singular point and not the other three, and write:

$$\begin{aligned}
C_3\big|_{NN} &= \left[d\gamma + \ell^{-1} f(\rho_2)\left(d\psi + 3\frac{d\chi - d\varphi}{2}\right)\right] \wedge B_2 \\
C_3\big|_{NS} &= \left[d\gamma + \ell^{-1} f(\rho_2)\left(d\psi + 3\frac{d\chi + d\varphi}{2}\right)\right] \wedge B_2 \\
C_3\big|_{SN} &= \left[d\gamma + \ell^{-1} f(\rho_1)\left(d\psi + 3\frac{d\chi - d\varphi}{2}\right)\right] \wedge B_2 \\
C_3\big|_{SS} &= \left[d\gamma + \ell^{-1} f(\rho_1)\left(d\psi + 3\frac{d\chi + d\varphi}{2}\right)\right] \wedge B_2
\end{aligned} \quad \text{(E.3)}$$

where the first N/S refers to $\rho = \rho_{2,1}$, the second N/S to $\theta = 0, \pi$, respectively. These are constructed in such a way to coincide with $[d\gamma + C_1] \wedge B_2$ at the singular points, yet being closed.

The connections smoothly glue together if and only if the transition functions are well-defined. For instance, $C_3|_{NN} - C_3|_{NS} = -3\ell^{-1} f(\rho_2) d\varphi \wedge B_2 \equiv d\lambda_2$, and the transition function λ_2 is well-defined only if $\lambda_2(\varphi = 2\pi) - \lambda_2(\varphi = 0) \in 2\pi H^2(M_6, \mathbb{Z})$. We can rewrite the condition as

$$H^2(M_6, \mathbb{Z}) \ni \int_\varphi \frac{d\lambda_2}{2\pi} = \int_\varphi \frac{C_3|_{NN} - C_3|_{NS}}{2\pi} \,. \quad \text{(E.4)}$$

By construction, $C_3|_{NN} - C_3|_{NS} = [d\gamma + C_1] \wedge B_2|_{NN} - [d\gamma + C_1] \wedge B_2|_{NS} = \int_\theta dC_1 \wedge B_2$ where the integral goes from one singular point to another one. Substituting in the previous formula we get that $F_2 \wedge B_2$ must be quantized on D^+. The analysis of the other transition functions work similarly, and we get the quantization condition

$$F_2 \wedge B_2 \in H^4(M_6, \mathbb{Z}) \,. \quad \text{(E.5)}$$

So the M-theory flat connection C_3 is well-defined whenever $F_2 \wedge B_2$ is quantized on M_6. On the other hand, the reduction of C_3 gives a B-field B_2 in type IIA, and $F_4 = 0$. Thus the condition is that the Page D4-charge is quantized in type IIA.

Finally C_3 could be pure gauge. This happens if all its periods are trivial: $\int C_3 \in 2\pi\mathbb{Z}$. Now $H_3(Y^{p,q}, \mathbb{Z})$ is generated by U(1) bundles over $H_2(M_6, \mathbb{Z})$, so C_3 is trivial whenever

$$H^2(M_6, \mathbb{Z}) \ni \int_\gamma \frac{C_3|_{**}}{2\pi} = \int_\gamma \frac{[d\gamma + C_1] \wedge B_2}{2\pi} = B_2 \,. \quad \text{(E.6)}$$

So C_3 is pure gauge in M-theory whenever B_2 is pure gauge in type IIA. The resulting torsion group is Γ, as shown in section 3.5.

Summarizing, we showed that, given the ansatz (E.3) for a flat C_3 connection, the set of bundles we obtain is precisely Γ; moreover the quantization of $F_2 \wedge B_2$ in IIA comes from the quantization of G_4 and gauge transformations of B_2 come from transformations of C_3.

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