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On generalisation of H-measures

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Abstract. In some applications it is useful to consider variants of H-measures different from those introduced in the classical or the parabolic case. We introduce the notion of *admissible* manifold and define variant H-measures on $\mathbf{R}^d \times P$ for any *admissible* manifold P. In the sequel we study one special variant, *fractional H-measures with orthogonality property*, where the corresponding manifold and projection curves are orthogonal, as it was the case with classical or parabolic H-measures, and prove the localisation principle. Finally, we present a simple application of the localisation principle.

1. Introduction

In various situations concerning partial differential equations one often encounters L^2 weakly converging sequences, which do not converge strongly. This lack of strong convergence can be measured by H-measures defined in Theorem 1 below. H-measures are (matrix) Radon measures defined on $\Omega\times\mathrm{S}^{d-1}$ (here S^{d-1} denotes the unit sphere in $\mathbf{R}^d$), for a domain $\Omega \subseteq \mathbf{R}^d$. In the sequel we shall consider only the case $\Omega = \mathbf{R}^d$ as functions defined only on Ω can be extended by zero to $\mathbf{R}^d$. H-measures were introduced independently by Luc Tartar [14] and Patrick Gérard [6] in the late 1980s and their existence is established by the following theorem.

Theorem 1. *If (u_n) is a sequence in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$ such that $\mathsf{u}_n \rightharpoonup \mathsf{0}$, then there exist a subsequence $(\mathsf{u}_{n'})$ and an $r \times r$ Hermitian complex matrix Radon measure $\boldsymbol{\mu}$ on $\mathbf{R}^d \times \mathrm{S}^{d-1}$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ one has:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 \mathsf{u}_{n'}) \otimes \mathcal{A}_\psi(\varphi_2 \mathsf{u}_{n'})\, d\mathbf{x} &= \langle \boldsymbol{\mu}, (\varphi_1\overline{\varphi_2}) \boxtimes \overline{\psi}\rangle \\ &= \int_{\mathbf{R}^d\times\mathrm{S}^{d-1}} \varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})}\, d\boldsymbol{\mu}(\mathbf{x},\boldsymbol{\xi})\,,\end{aligned} \tag{1}$$

where $\mathcal{F}(\mathcal{A}_\psi v)(\boldsymbol{\xi}) = \psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})\mathcal{F}v(\boldsymbol{\xi})$. ∎

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The complex matrix measure $\boldsymbol{\mu}$ defined by the previous theorem is called *the H-measure* associated to the subsequence $(\mathsf{u}_{n'})$.

Above, as well as in the rest of the paper, by $\otimes$ we denote the tensor product of vectors on $\mathbf{C}^r$, defined by $(\mathsf{a}\otimes\mathsf{b})\mathsf{v} = (\mathsf{v}\cdot\mathsf{b})\mathsf{a}$, where $\cdot$ stands for the (complex) scalar product ($\mathsf{a}\cdot\mathsf{b} := \sum_{i=1}^r a_i\bar{b}_i$), resulting in $[\mathsf{a}\otimes\mathsf{b}]_{ij} = a_i\bar{b}_j$, while $\boxtimes$ denotes the tensor product of functions in different variables. The Fourier transform we define as $\hat{\mathsf{u}}(\boldsymbol{\xi}) := \mathcal{F}\mathsf{u}(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}\mathsf{u}(\mathbf{x})\,d\mathbf{x}$, and its inverse by $(\mathsf{u})^\vee(\boldsymbol{\xi}) := \bar{\mathcal{F}}\mathsf{u}(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}\mathsf{u}(\mathbf{x})\,d\mathbf{x}$.

As it was mentioned above, the H-measure describes a deviation of sequence $\mathsf{u}_n \rightharpoonup \mathsf{0}$ from being strongly convergent in the sense that $\boldsymbol{\mu} = \mathbf{0}$ implies $\mathsf{u}_{n'} \longrightarrow \mathsf{0}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^r)$.

An important property, *the localisation principle*, of the H-measure defined by a sequence of functions satisfying differential constraints is given by the next theorem (cf. [14, Theorem 1.6]):

Theorem 2. *If* $\mathsf{u}_n \rightharpoonup \mathsf{0}$ *in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$*, with associated H-measure* $\boldsymbol{\mu}$*, and if* u_n *satisfies*

$$\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}_n) \longrightarrow \mathsf{0} \ \textit{strongly in}\ \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)\,,$$

where (matrix) coefficients $\mathbf{A}_k$ *are continuous on* $\mathbf{R}^d$*, then it holds*

$$\Big(\sum_{k=1}^d \xi_k\mathbf{A}_k(\mathbf{x})\Big)\boldsymbol{\mu} = \mathbf{0} \ \textit{on}\ \mathbf{R}^d\times\mathrm{S}^{d-1}\,.$$

∎

By the localisation principle the compensated compactness theory could be extended from constant coefficient differential relations to variable coefficients [14, 6].

From the previous theorems it is clear that H-measures are suitable for problems where all variables are *equal*, i.e. in the observed differential relation the ratio among all partial derivatives is the same. However, it has been proved (see [2]) that in the cases where that is not satisfied (e.g. parabolic equations), we get unsatisfactory results. For the parabolic equations, i.e. equations with the ratio 1:2 between the order of time and spatial derivatives, a new variant, *parabolic H-measures*, has been introduced [2] and successfully applied to small-amplitude homogenisation [2], to explicit formulæ and bounds in homogenisation [3], as well as to deriving differential relations for the microlocal energy densities, as a consequence of the *propagation principle* [4]. Further generalisations of the previous object are the *ultra-parabolic H-measures* [11, 12] allowing more than one variable of order 1, which are suitable for applications to ultra-parabolic equations.

Although the majority of important equations from mathematical physics can be treated by one of the mentioned variants (having ratios 1:1 or 1:2), recently the study of differential relations with fractional derivatives has become more popular, so the previous approach needs to be generalised to arbitrary ratios. In [9] a new variant was introduced and applied to the existence of weak solutions of fractional conservation laws. This concept was later applied to the velocity averaging results [8] and compensated compactness for the fractional differential relations [10].

In the next section we introduce the notion of an *admissible* manifold (which is slightly more general then the one given in [9]) and prove a variant of the first commutation lemma using it. The third section is devoted to an important example of *admissible* manifold where we supplement some arguments from [9]. In the following section we prove that a corresponding ellipsoid is orthogonal to the curves studied in the third section, and propose another variant of H-measures, *the fractional H-measure with orthogonality property* defined on that ellipsoid which we hope to be suitable for obtaining the propagation principle as it was done in [4]. In the last section we prove the localisation principle for fractional H-measures with orthogonality property, giving also a simple application.

2. Variants of the first commutation lemma

The proof of Theorem 1 relies on the next lemma (see [14, Lemma 1.7]).

Lemma 1. **(first commutation lemma)** *Let $a \in \mathrm{C}(\mathrm{S}^{d-1})$, $b \in \mathrm{C}_0(\mathbf{R}^d)$, and let $\mathcal{A}$ be the Fourier multiplier with symbol a, and B a multiplication operator, defined by:*

$$\mathcal{F}(\mathcal{A}u)(\xi) = a\Big(\frac{\xi}{|\xi|}\Big)\mathcal{F}u(\xi)\,,$$
$$Bu = bu\,.$$

Then their commutator $C = [\mathcal{A}, B] = \mathcal{A}B - B\mathcal{A}$ is a compact operator from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(\mathbf{R}^d)$. ∎

Following the proof of Lemma 1, we can notice that it relies on a basic fact that the diameter of projection (along rays from the origin) of a compact set K to the unit sphere S^{d-1} decreases as the distance of K from the origin increases. To be more precise, we have the following extension [15, Lemma 28.2]:

Lemma 2. *Let $b \in \mathrm{C}_0(\mathbf{R}^d)$ and let $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ satisfies*

$$\begin{aligned}&(\forall\, R > 0)(\forall\, \varepsilon > 0)(\exists\, r > 0)(\forall\, \xi, \boldsymbol{\eta} \in \mathbf{R}^d)\\ &\qquad |\xi|, |\boldsymbol{\eta}| > r \;\;\&\;\; |\xi - \boldsymbol{\eta}| \leqslant R \quad\Longrightarrow\quad |a(\xi) - a(\boldsymbol{\eta})| < \varepsilon\,.\end{aligned} \tag{2}$$

Furthermore, let a Fourier multiplier $\mathcal{A}$ with symbol a and a multiplication operator B be defined as in Lemma 1, with the exception that we now use $a(\xi)$ instead of $a(\xi/|\xi|)$. Then $C = \mathcal{A}B - B\mathcal{A}$ is a compact operator from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(\mathbf{R}^d)$. ∎

In some applications an imperfection of Theorem 1 is in the scaling along rays from the origin that makes H-measures appropriate only for the study of hyperbolic problems. Recently [2], the first commutation lemma was extended to the case of scaling along parabolas, allowing for applications of thus obtained (parabolic) H-measures to parabolic problems [3, 4].

With the goal of introducing conditions more intuitive than (2) and allowing more general scalings, we prove a variant of the first commutation lemma based on concrete curves and manifolds. This will be a generalisation of a variant given in [9], making a verification of further results given in that article easier (see the example of the next section).

In the next definition we are going to use an arbitrary metric d on $\mathbf{R}^d$, besides the standard one defined by the Euclidean norm $|\cdot|$, with the property that the inclusion $(\mathbf{R}^d, |\cdot|) \hookrightarrow (\mathbf{R}^d, d)$ uniformly preserves boundedness of sets:

$$(\forall\, R > 0)(\exists\, C_R > 0)(\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{R}^d) \quad |\mathbf{x} - \mathbf{y}| \leqslant R \quad\Longrightarrow\quad d(\mathbf{x}, \mathbf{y}) < C_R\,. \tag{3}$$

However, when we talk about continuity we always mean it with respect to the Euclidean metric on $\mathbf{R}^d$. We shall also use the notation $\mathbf{R}^d_* = \mathbf{R}^d \setminus \{0\}$.

Definition 1. *A compact continuous manifold $P \subseteq \mathbf{R}^d$ is* admissible *if there exists a family of continuous functions (curves) $\varphi_{\boldsymbol{\nu}} : \mathbf{R}^+ \longrightarrow \mathbf{R}^d$, indexed by $\boldsymbol{\nu} \in P$, satisfying*

$$\varphi_{\boldsymbol{\nu}}(1) = \boldsymbol{\nu}\,, \tag{4}$$

and the following properties:

(i) $(\forall\, \xi \in \mathbf{R}^d_*)(\exists\,!\, s \in \mathbf{R}^+)(\exists\,!\, \boldsymbol{\nu} \in P) \quad \xi = \varphi_{\boldsymbol{\nu}}(s)$;

(ii) there is an increasing function $f : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$, $\lim_{t\to\infty} f(t) = \infty$, such that

$$\begin{aligned}&(\forall\, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in P)(\forall\, s_1, s_2 \in [1, \infty\rangle)\\ &\qquad d(\varphi_{\boldsymbol{\nu}_1}(s_1), \varphi_{\boldsymbol{\nu}_2}(s_2)) \geqslant f(\min\{s_1, s_2\})|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2|\,,\end{aligned}$$

for at least one metric d satisfying (3);

(iii) function $t_{\boldsymbol{\nu}}(s) := |\varphi_{\boldsymbol{\nu}}(s)|$ is strictly increasing and

$$(\forall\, s \in \mathbf{R}^+) \quad \sup_{\boldsymbol{\nu}\in P} t_{\boldsymbol{\nu}}(s) =: C_s < \infty\,.$$

In the terminology of [9, Definition 2.1], the condition (i) means that $\mathbf{R}^d_*$ admits a complete fibration along the family of curves $(\varphi_{\boldsymbol\nu}, \boldsymbol\nu \in P)$. The corresponding projection of an arbitrary point $\boldsymbol\xi \in \mathbf{R}^d_*$ we denote by $\pi_P(\boldsymbol\xi) \in P$, so $\pi_P : \mathbf{R}^d_* \longrightarrow P$ is well-defined surjection. Here we replaced S^{d-1} (which was used in [9]) by a more general compact continuous manifold P, since the appropriate choice of manifold significantly simplifies verification of the conditions from the previous definition (as can be seen in the succeeding section). However, the higher generality requires one additional assumption, (iii), which says that Euclidean norm of the points on curves $\varphi_{\boldsymbol\nu}$ increases as parameter s increases, but the growth rate has to be uniformly bounded. To conclude, let us just remark that $s > 1$ ($s < 1$) implies that $\varphi_{\boldsymbol\nu}(s)$ is outside (inside) of manifold P (as a consequence of (4) and (iii)).

In the next lemma we give a variant of the first commutation lemma for functions that behave well along the curves from Definition 1.

Lemma 3. (variant of the first commutation lemma) *Let P be an admissible manifold and $(\varphi_{\boldsymbol\nu}, \boldsymbol\nu \in P)$ the corresponding family of curves. Also, let $b \in \mathrm{C}_0(\mathbf{R}^d)$ and $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ satisfy*

$$(\exists\, a_\infty \in \mathrm{C}(P)) \quad \lim_{s\to\infty} a(\varphi_{\boldsymbol\nu}(s)) = a_\infty(\boldsymbol\nu),\ \text{uniformly in } \boldsymbol\nu \in P. \tag{5}$$

Furthermore, let the Fourier multiplier $\mathcal{A}$ with symbol a and a multiplication operator B be defined as in Lemma 2. Then $C = \mathcal{A}B - B\mathcal{A}$ is a compact operator from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{L}^2(\mathbf{R}^d)$.

Proof. By Lemma 2 it suffices to prove (2).

Let $R > 0$ and $\varepsilon > 0$ be arbitrary, and $\boldsymbol\xi_1, \boldsymbol\xi_2 \in \mathbf{R}^d_*$ such that $|\boldsymbol\xi_1 - \boldsymbol\xi_2| \leqslant R$. By ($i$) we can uniquely define $s_1, s_2, \boldsymbol\nu_1$ and $\boldsymbol\nu_2$ such that $\boldsymbol\xi_1 = \varphi_{\boldsymbol\nu_1}(s_1)$, $\boldsymbol\xi_2 = \varphi_{\boldsymbol\nu_2}(s_2)$.

The assumption on function a implies the existence of $r_1 > 0$ for which

$$s_1, s_2 > r_1 \quad \Longrightarrow \quad |a(\boldsymbol\xi_1) - a_\infty(\boldsymbol\nu_1)| < \frac{\varepsilon}{3} \ \ \&\ \ |a(\boldsymbol\xi_2) - a_\infty(\boldsymbol\nu_2)| < \frac{\varepsilon}{3}\,.$$

Further on, the compactness of P gives us that a_∞ is uniformly continuous, hence there exists $\delta > 0$ such that

$$|\boldsymbol\nu_1 - \boldsymbol\nu_2| < \delta \quad \Longrightarrow \quad |a_\infty(\boldsymbol\nu_1) - a_\infty(\boldsymbol\nu_2)| < \frac{\varepsilon}{3}\,.$$

Moreover, by (ii) and the properties of function f and metric d it follows that there exists $r_2 \geqslant 1$ such that

$$s_1, s_2 > r_2 \quad \Longrightarrow \quad |\boldsymbol\nu_1 - \boldsymbol\nu_2| \leqslant \frac{d(\boldsymbol\xi_1, \boldsymbol\xi_2)}{f(r_2)} \leqslant \frac{C_R}{f(r_2)} < \delta\,.$$

Thus, for $\boldsymbol\xi_1, \boldsymbol\xi_2 \in \mathbf{R}^d$ such that $|\boldsymbol\xi_1 - \boldsymbol\xi_2| \leqslant R$ and $s_1, s_2 > \tilde r := \max\{r_1, r_2\}$ we have

$$|a(\boldsymbol\xi_1) - a(\boldsymbol\xi_2)| \leqslant |a(\boldsymbol\xi_1) - a_\infty(\boldsymbol\nu_1)| + |a_\infty(\boldsymbol\nu_1) - a_\infty(\boldsymbol\nu_2)| + |a_\infty(\boldsymbol\nu_2) - a(\boldsymbol\xi_2)| < \varepsilon\,.$$

It remains to prove that $s_1, s_2 > \tilde r$ can be replaced by $|\boldsymbol\xi_1|, |\boldsymbol\xi_2| > r$ for some $r > 0$. This follows from the property (iii). Namely, we claim that

$$(\forall\, \tilde r > 0)(\exists\, r > 0)(\forall\, \boldsymbol\nu \in P)(\forall\, s > 0) \quad t_{\boldsymbol\nu}(s) > r \quad \Longrightarrow \quad s > \tilde r.$$

Assuming the contrary gives

$$(\exists\, \tilde r > 0)(\forall\, r > 0)(\exists\, \boldsymbol\nu \in P)(\exists\, s > 0) \quad s \leqslant \tilde r \ \ \&\ \ t_{\boldsymbol\nu}(s) > r\,,$$

which leads to a contradiction to the conditions (iii) as

$$r < t_{\boldsymbol\nu}(s) \leqslant t_{\boldsymbol\nu}(\tilde r) \leqslant \sup_{\boldsymbol\nu\in P} t_{\boldsymbol\nu}(\tilde r) = C_{\tilde r} < \infty\,,$$

and r is arbitrary. Therefore we get the claim. $\square$

It is trivial to see that $\psi \circ \pi_P$ satisfies (5) for $\psi \in \mathrm{C}(P)$, and π_P being projection along the family of curves (φ_ν). Actually, by (*iii*) it follows that (5) implies

$$\lim_{|\xi|\to\infty}\Big(a(\xi) - a_\infty(\pi_P(\xi))\Big) = 0\,.$$

In the special case when $P = \mathrm{S}^{d-1}$ and curves φ_ν are parametrised by the distance from the origin, the proof of the previous lemma can be found in [9, Lemma 2.2]. Moreover, if $\varphi_\nu(s) = s\nu$, $\nu \in \mathrm{S}^{d-1}$, are rays from the origin, the set of continuous functions satisfying (5) is equal to $\mathrm{C}(\mathrm{K}_\infty(\mathbf{R}^d))$ (see [15, Definition 32.5]).

As we have already remarked, having a new variant of the first commutation lemma enables us to define the corresponding variant of H-measure on $\mathbf{R}^d \times P$, whose construction follows the same steps as in [9], and here we will study some further properties allowing us to prove the localisation principle. However, in [4] it was pointed out how the choice of manifold is important for deriving additional results, such as the propagation principle. Hence, on one hand we want to choose a manifold P for which the verification of conditions (*i*)–(*iii*) is easy, and on the other hand we want to choose a manifold suitable for deriving further results. To make things simpler, we want to be able to define a variant H-measure also for some manifolds for which *admissibility* is not known. In the following condition we determine some manifolds with that property, which happen to be sufficient for our applications.

More precisely, if we are able to define H-measures on $\mathbf{R}^d \times P$, for an *admissible* manifold P, we would like to be able to define them also on $\mathbf{R}^d \times Q$, for a manifold Q for which *admissibility* is not known, if the following condition holds:

(*iv*) for each $\nu \in P$ the curve φ_ν intersects Q in a single point η, and the function $\nu \mapsto \eta$ is continuous.

Of course, for $Q = P$ the condition (*iv*) is trivially satisfied. Let us remark that Q satisfying (*iv*) is actually a generalisation of the notion of admissible manifold given in [9, Definition 2.3].

An immediate consequence of the previous condition is that restriction $\pi_P|_Q : Q \longrightarrow P$ is a bijection with continuous inverse which we denote by Φ. Since P is compact, we have also that $\Phi^{-1} = \pi_P|_Q$ is continuous, thus we finally get that Φ is a homeomorphism. The corresponding pull-back operator $\Phi^* : \mathrm{C}(Q) \longrightarrow \mathrm{C}(P)$, $\Phi^*\psi := \psi \circ \Phi$, is an isometric isomorphism of Banach spaces $\mathrm{C}(Q)$ and $\mathrm{C}(P)$, with its inverse given by $(\Phi^*)^{-1} = (\Phi^{-1})^*$.

By the means of the above homeomorphism between P and Q, the statement of the previous lemma is also valid for $a = \psi \circ \pi_Q$, where $\psi \in \mathrm{C}(Q)$ and $\pi_Q := \Phi \circ \pi_P$ is the natural projection on Q. Therefore, for $\psi \in \mathrm{C}(Q)$ we have $\psi \circ \pi_Q = (\Phi^*\psi) \circ \pi_P$.

Like in [9, Theorem 2.4], where a different notion of admissibility was used for manifolds, we have the following extension of H-measures.

Theorem 3. *Let P be an admissible manifold. If (u_n) is a sequence in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$ such that $\mathsf{u}_n \rightharpoonup 0$, then there exist a subsequence $(\mathsf{u}_{n'})$ and an $r \times r$ Hermitian matrix Radon measure $\boldsymbol{\mu}_P$ on $\mathbf{R}^d \times P$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}(P)$ one has:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 \mathsf{u}_{n'}) \otimes \mathcal{A}_{\psi\circ\pi_P}(\varphi_2 \mathsf{u}_{n'})\, d\mathbf{x} &= \langle \boldsymbol{\mu}_P, (\varphi_1\overline{\varphi_2}) \boxtimes \overline{\psi}\rangle \\ &= \int_{\mathbf{R}^d\times P} \varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})\psi(\xi)}\, d\boldsymbol{\mu}_P(\mathbf{x},\xi)\,.\end{aligned}$$

∎

Moreover, we could state the previous theorem for Q satisfying (*iv*) as well. Namely, let $(\mathsf{u}_{n'})$ be a subsequence as in the statement of the previous theorem. Then for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}(Q)$ we have

$$\begin{aligned}\lim_{n'}\int_{\mathbf{R}^d}(\varphi_1\mathsf{u}_{n'})\otimes\mathcal{A}_{\psi\circ\pi_Q}(\varphi_2\mathsf{u}_{n'})\,d\mathbf{x} &= \lim_{n'}\int_{\mathbf{R}^d}(\varphi_1\mathsf{u}_{n'})\otimes\mathcal{A}_{(\Phi^*\psi)\circ\pi_P}(\varphi_2\mathsf{u}_{n'})\,d\mathbf{x}\\ &= \left\langle\boldsymbol{\mu}_P,(\varphi_1\overline{\varphi_2})\boxtimes\overline{\Phi^*\psi}\right\rangle\\ &= \left\langle\Phi_*\boldsymbol{\mu}_P,(\varphi_1\overline{\varphi_2})\boxtimes\overline{\psi}\right\rangle,\end{aligned}\tag{6}$$

where Φ_* is the push-forward operator in $\boldsymbol{\xi}$.

By $\boldsymbol{\mu}_Q := \Phi_*\boldsymbol{\mu}_P$ we define a variant H-measure on Q which is justified since the above limit is satisfied.

This allows us to define variant H-measures corresponding to the same family of curves, but defined on different spaces (manifolds) in $\boldsymbol{\xi}$ as long as (*iv*) is satisfied.

3. An example

The main application in [9] was provided by a variant of H-measure defined on manifold

$$P = \left\{\boldsymbol{\xi}\in\mathbf{R}^d : |\boldsymbol{\xi}|_{\boldsymbol{\alpha}} = 1\right\},\tag{7}$$

where $|\boldsymbol{\xi}|_{\boldsymbol{\alpha}} := \sum_{k=1}^d |\xi_k|^{\alpha_k}$ and $\boldsymbol{\alpha}\in\langle 0,1]^d$ (smoother variants of this manifold were considered in [8, 10]). The existence of such a variant H-measure relies on the fact that P, with respect to the corresponding family of curves

$$\varphi_{\boldsymbol{\nu}}(s) = \mathsf{diag}\,\{s^{\frac{1}{\alpha_1}},\dots,s^{\frac{1}{\alpha_d}}\}\boldsymbol{\nu}\,,\quad s>0\,,\quad \boldsymbol{\nu}\in P\,,\tag{8}$$

is admissible. That proved to be a non-trivial task when expressed in terms of [9, Definition 2.3], so here we present different approach, by proving that P is admissible in terms of Definition 1.

In the sequel we shall often have estimates depending on the minimal value of the components of $\boldsymbol{\alpha}$, so for simplicity let us denote $\alpha_{\min} := \min_k \alpha_k \in \langle 0,1]$.

It is clear that each curve $\varphi_{\boldsymbol{\nu}}$ satisfies (4). Let us also check the properties (*i*)–(*iii*).

For an arbitrary $\boldsymbol{\xi}\in\mathbf{R}^d_*$ we first prove that there exist a unique $\boldsymbol{\nu}\in P$ and an $s\in\mathbf{R}^+$ such that $\boldsymbol{\xi} = \varphi_{\boldsymbol{\nu}}(s)$. From $\xi_k = \nu_k s^{\frac{1}{\alpha_k}}$, by taking the power α_k and summation we get

$$s = \sum_{k=1}^d |\xi_k|^{\alpha_k} = |\boldsymbol{\xi}|_{\boldsymbol{\alpha}}\,.$$

Hence, by inserting back into the initial formula we obtain

$$\nu_k = \xi_k|\boldsymbol{\xi}|_{\boldsymbol{\alpha}}^{-\frac{1}{\alpha_k}}\,,$$

and the property (*i*) is thus proved. The projection π_P is then well defined and given by

$$\pi_P(\boldsymbol{\xi}) = \left(\xi_1|\boldsymbol{\xi}|_{\boldsymbol{\alpha}}^{-\frac{1}{\alpha_1}},\dots,\xi_d|\boldsymbol{\xi}|_{\boldsymbol{\alpha}}^{-\frac{1}{\alpha_d}}\right),$$

which is smooth on $\mathbf{R}^d_*$ and constant along corresponding curves, i.e. $\pi_P(\lambda^{\frac{1}{\alpha_1}}\xi_1,\dots,\lambda^{\frac{1}{\alpha_d}}\xi_d) = \pi_P(\boldsymbol{\xi})$ for $\lambda\in\mathbf{R}^+$, as the result of $|(\lambda^{\frac{1}{\alpha_1}}\xi_1,\dots,\lambda^{\frac{1}{\alpha_d}}\xi_d)|_{\boldsymbol{\alpha}} = \lambda|\boldsymbol{\xi}|_{\boldsymbol{\alpha}}$.

Since at least one component of $\boldsymbol{\nu}$ is non-trivial, $0<s_1<s_2$ implies

$$|\varphi_{\boldsymbol{\nu}}(s_1)|^2 = \sum_{k=1}^d s_1^{\frac{2}{\alpha_k}}\nu_k^2 < \sum_{k=1}^d s_2^{\frac{2}{\alpha_k}}\nu_k^2 = |\varphi_{\boldsymbol{\nu}}(s_2)|^2\,,$$

thus, the mapping $s\mapsto|\varphi_\nu(s)|$ is strictly increasing. Moreover, from the estimate

$$|\varphi_{\boldsymbol{\nu}}(s)| \leqslant \sum_{k=1}^d s^{\frac{1}{\alpha_k}}|\nu_k| \leqslant \max\{1,s^{\frac{1}{\alpha_{\min}}}\}\sum_{k=1}^d|\nu_k|^{\alpha_k} = \max\{1,s^{\frac{1}{\alpha_{\min}}}\}\,,$$

where we have used that $|v_k| \leqslant 1$ in the second inequality, the verification of property (*iii*) is completed.

The proof of property (*ii*) requires some additional effort, so we present it in the form of a lemma.

Lemma 4. *There exists a constant $C > 0$ such that for any $\xi, \eta \in \mathbf{R}^d_*$ we have*

$$|\xi_0 - \eta_0| \leqslant C \frac{|\xi - \eta|_\alpha}{|\xi|_\alpha + |\eta|_\alpha},$$

where ξ_0 and η_0 are the projections on manifold P of points ξ and η.

Proof. Let us take $\xi, \eta \in \mathbf{R}^d_*$ such that $\frac{1}{q} := |\xi|_\alpha \geqslant |\eta|_\alpha =: \frac{1}{p}$. Therefore, we have $\xi_0 = (q^{\frac{1}{\alpha_1}}\xi_1, \dots, q^{\frac{1}{\alpha_d}}\xi_d)$ and $\eta_0 = (p^{\frac{1}{\alpha_1}}\eta_1, \dots, p^{\frac{1}{\alpha_d}}\eta_d)$. Moreover, let us denote by η' the intersection of the manifold given by $\{\xi \in \mathbf{R}^d : |\xi|_\alpha = \frac{q}{p}\}$ (which is contained inside P since $\frac{q}{p} \leqslant 1$) and the corresponding curve through η, $\operatorname{diag}\{s^{\frac{1}{\alpha_1}}, \dots, s^{\frac{1}{\alpha_d}}\}\eta$, which implies (see Figure 1)

$$\eta' = (\eta'_1, \dots, \eta'_d) = (q^{\frac{1}{\alpha_1}}\eta_1, \dots, q^{\frac{1}{\alpha_d}}\eta_d).$$

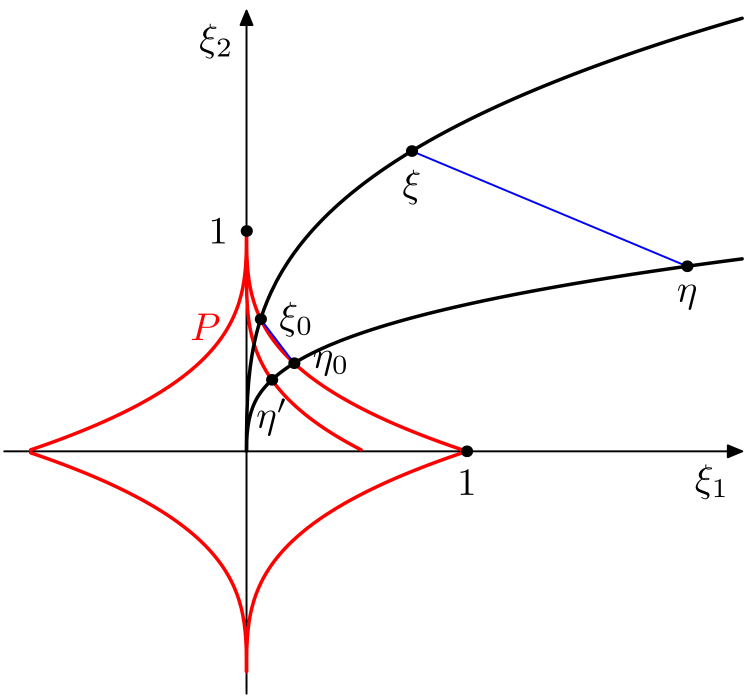

Figure 1: The manifolds and curves with $\alpha_1 = \dfrac{1}{3}$, $\alpha_2 = 1$.

The triangle inequality for Euclidean metric gives us

$$|\xi_0 - \eta_0| \leqslant |\xi_0 - \eta'| + |\eta' - \eta_0|.$$

Since both ξ_0 and η' are contained inside P, their components are less or equal to 1, and so $|\xi_0 - \eta'|_\infty \leqslant 2$. By Lemma 5 below we get the bound

$$|\xi_0 - \eta'| \leqslant 2^{1-\alpha_{\min}} |\xi_0 - \eta'|_\alpha = 2^{1-\alpha_{\min}} q |\xi - \eta|_\alpha.$$

For the second term we have

$$|\boldsymbol{\eta}' - \boldsymbol{\eta}_0| = \sqrt{\sum_k \Big|\eta_k\Big(q^{\frac{1}{\alpha_k}} - p^{\frac{1}{\alpha_k}}\Big)\Big|^2}\,.$$

The function $x \mapsto x^{\frac{1}{\alpha}}$, $\alpha > 0$, is smooth for $x > 0$, so we can apply the Lagrange mean value theorem to the right hand side and since $p \geqslant q$ we get

$$\begin{aligned}
|\boldsymbol{\eta}' - \boldsymbol{\eta}_0| &\leqslant \sqrt{\sum_k \eta_k^2 \frac{1}{\alpha_k^2} p^{\frac{2-2\alpha_k}{\alpha_k}} (p-q)^2} \\
&\leqslant \frac{p-q}{\alpha_{\min}} \sqrt{\sum_k \eta_k^2 |\boldsymbol{\eta}|_{\boldsymbol{\alpha}}^{\frac{2\alpha_k - 2}{\alpha_k}}} \\
&\leqslant \frac{p-q}{\alpha_{\min}} \sqrt{\sum_k \eta_k^2 |\eta_k|^{2\alpha_k - 2}} \leqslant \frac{p-q}{\alpha_{\min}} |\boldsymbol{\eta}|_{\boldsymbol{\alpha}} = \frac{p-q}{p\alpha_{\min}}\,,
\end{aligned}$$

where in the third inequality we have used that $|\boldsymbol{\eta}|_{\boldsymbol{\alpha}} \geqslant |\eta_k|^{\alpha_k}$, and in the last one the equivalence of standard norms $|\cdot|$ and $|\cdot|_1$ in $\mathbf{R}^d$.

By Lemma 5, $|\cdot|_{\boldsymbol{\alpha}}$ satisfies the triangle inequality so we have

$$p - q = pq(|\boldsymbol{\xi}|_{\boldsymbol{\alpha}} - |\boldsymbol{\eta}|_{\boldsymbol{\alpha}}) \leqslant pq|\boldsymbol{\xi} - \boldsymbol{\eta}|_{\boldsymbol{\alpha}}\,,$$

and lastly

$$|\boldsymbol{\eta}' - \boldsymbol{\eta}_0| \leqslant \frac{1}{\alpha_{\min}} q|\boldsymbol{\xi} - \boldsymbol{\eta}|_{\boldsymbol{\alpha}}\,.$$

Hence, under the starting assumption $q \leqslant p$ we have

$$|\boldsymbol{\xi}_0 - \boldsymbol{\eta}_0| \leqslant \Big(2^{1-\alpha_{\min}} + \frac{1}{\alpha_{\min}}\Big) q|\boldsymbol{\xi} - \boldsymbol{\eta}|_{\boldsymbol{\alpha}}\,.$$

Finally, since $2qp \geqslant \min\{q, p\}(q + p)$, we get the required estimate

$$|\boldsymbol{\xi}_0 - \boldsymbol{\eta}_0| \leqslant 2\Big(2^{1-\alpha_{\min}} + \frac{1}{\alpha_{\min}}\Big) \frac{|\boldsymbol{\xi} - \boldsymbol{\eta}|_{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|_{\boldsymbol{\alpha}} + |\boldsymbol{\eta}|_{\boldsymbol{\alpha}}}\,.$$

□

Lemma 5. *For $\boldsymbol{\alpha} \in \langle 0, 1]^d$, by $d_{\boldsymbol{\alpha}}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{k=1}^d |\xi_k - \eta_k|^{\alpha_k}$ is given a metric on $\mathbf{R}^d$ satisfying property (3).*
Moreover for every compact $K \subseteq \mathbf{R}^d$ there exist C_K such that $|\boldsymbol{\xi}| \leqslant C_K|\boldsymbol{\xi}|_{\boldsymbol{\alpha}}$, for $\boldsymbol{\xi} \in K$.

Proof. Only the triangle inequality requires special attention; to this end, we need to check that for non-negative a, b we have $(a + b)^{\alpha_k} \leqslant a^{\alpha_k} + b^{\alpha_k}$, or equivalently

$$\Big(\frac{a}{b} + 1\Big)^{\alpha_k} \leqslant \Big(\frac{a}{b}\Big)^{\alpha_k} + 1,$$

where, without loss of generality, we assumed $b \neq 0$.

The last inequality follows from the fact that the function $f(x) = (x + 1)^{\alpha_k} - x^{\alpha_k} - 1$ is non-increasing on the interval $[0, \infty\rangle$. Indeed, since $f'(x) = \alpha_k((x + 1)^{\alpha_k - 1} - x^{\alpha_k - 1})$ and $\alpha_k - 1 \leqslant 0$, it follows $f'(x) \leqslant 0$ on $\langle 0, \infty\rangle$, so f is non-increasing. Therefore, we have that $d_{\boldsymbol{\alpha}}$ is a metric on $\mathbf{R}^d$.

Since for any $\boldsymbol{\xi} \in \mathbf{R}^d$ we have $|\boldsymbol{\xi}|_{\boldsymbol{\alpha}} \leqslant \sum_{k=1}^d |\boldsymbol{\xi}|^{\alpha_k}$, it follows that $d_{\boldsymbol{\alpha}}$ satisfies (3).

Finally, for compact $K \subseteq \mathbf{R}^d$ there exists $R \geqslant 1$ such that $|\boldsymbol{\xi}|_\infty \leqslant R$ for $\boldsymbol{\xi} \in K$, thus

$$|\boldsymbol{\xi}| \leqslant |\boldsymbol{\xi}|_1 = \sum_{k=1}^d |\xi_k|^{1-\alpha_k} |\xi_k|^{\alpha_k} \leqslant R^{1-\alpha_{\min}} |\boldsymbol{\xi}|_{\boldsymbol{\alpha}}\,,$$

which completes the proof. □

Since P is an admissible manifold, we can define the corresponding variant H-measure on $\mathbf{R}^d \times P$ by Theorem 3. This variant was the essential tool in [9] for the application to fractional conservation laws.

At the end, let us show another useful property of the variant H-measure by the following example. We want to calculate an H-measure $\boldsymbol{\mu}_P$ (for P from this section) and a classical H-measure $\boldsymbol{\mu}$ corresponding to the sequence of oscillations

$$u_n = e^{2\pi i \mathsf{k}\cdot(n^{\beta_1}x_1,\dots,n^{\beta_d}x_d)} = e^{2\pi i (n^{\beta_1}k_1,\dots,n^{\beta_d}k_d)\cdot\mathbf{x}}\,,$$

where $\mathsf{k} \in \mathbf{R}^d$ and for the simplicity

$$\beta_1 = \ldots = \beta_p > \beta_{p+1} \geqslant \ldots \geqslant \beta_{q-1} > \beta_q = \ldots = \beta_d =: \beta_{\min} > 0\,,$$

for $p, q \in \{0, \dots, d\}$. We actually want to see which choice of $\boldsymbol{\alpha}$ in (7) and (8) gives us the most information contained in $\boldsymbol{\mu}_P$. A simple calculation shows that $\widehat{\varphi u_n}(\boldsymbol{\xi}) = \hat{\varphi}(\boldsymbol{\xi} - (n^{\beta_1}k_1, \dots, n^{\beta_d}k_d))$ (for $\varphi \in C_0(\mathbf{R}^d)$) and then by the Plancherel theorem and a linear change of variables, the first integral in Theorem 3 can be expressed as

$$\int_{\mathbf{R}^d} \hat{\varphi}_1(\boldsymbol{\xi})\overline{\hat{\varphi}_2(\boldsymbol{\xi})\psi(\pi_P(\boldsymbol{\xi} + (n^{\beta_1}k_1, \dots, n^{\beta_d}k_d)))}\, d\boldsymbol{\xi}\,.$$

For the argument of function ψ we have

$$\begin{aligned}\Big[\pi_P(\boldsymbol{\xi} + (n^{\beta_1}k_1, \dots, n^{\beta_d}k_d))\Big]_i &= \frac{\xi_i + n^{\beta_i}k_i}{\Big(\sum_{j=1}^d |\xi_j + n^{\beta_j}k_j|^{\alpha_j}\Big)^{1/\alpha_i}} \\ &= \frac{\frac{\xi_i}{n^{\beta_i}} + k_i}{\Big(\sum_{j=1}^d \Big|\frac{\xi_j}{n^{(\alpha_i\beta_i)/\alpha_j}} + n^{\beta_j - \frac{\alpha_i\beta_i}{\alpha_j}}k_j\Big|^{\alpha_j}\Big)^{1/\alpha_i}}\,,\end{aligned}$$

and the limit of this expression (as $n \to \infty$) is always non-trivial only if $\alpha_i\beta_i$ is constant, hence, we take $\alpha_i = \beta_{\min}/\beta_i \leqslant 1$. Under this condition, the limit of the above expression is $k_i|\mathsf{k}|_{\boldsymbol{\alpha}}^{-\frac{1}{\alpha_i}} = [\pi_P(\mathsf{k})]_i$, and by the Lebesgue dominated convergence theorem the above integral converges to $\overline{\psi(\pi_P(\mathsf{k}))}\int_{\mathbf{R}^d}\hat{\varphi}_1(\boldsymbol{\xi})\overline{\hat{\varphi}_2(\boldsymbol{\xi})}\,d\boldsymbol{\xi}$, which gives us

$$\boldsymbol{\mu}_P = \lambda \boxtimes \delta_{\pi_P(\mathsf{k})}\,,$$

where λ is the Lebesgue measure on $\mathbf{R}^d$ and δ is a Dirac mass. This result is satisfactory since for any choice of $\boldsymbol{\beta}$ we can choose $\boldsymbol{\alpha}$ such that the corresponding variant H-measure contains a complete information about the direction of oscillations.

On the other hand, the corresponding classical H-measure reads

$$\boldsymbol{\mu} = \lambda \boxtimes \delta_{\frac{(k_1,\dots,k_p,0,\dots,0)}{|(k_1,\dots,k_p,0,\dots,0)|}}\,,$$

from which we do not have any information about variables $\xi_{p+1}, \dots, \xi_d$, i.e. about oscillations of the observed sequence in the directions $x_{p+1}, \dots, x_d$.

So far we have tried to systematise the ideas used by Nenad Antonić, Martin Lazar and Evgenij Jurjevič Panov (see [4], [12]) when defining parabolic and ultra-parabolic H-measures. In [15] and [8] it was shown that certain results can be obtained directly from Lemma 2. However, those results require the smoothness of the used manifold, and explicit formulas for projections on the manifold.

In the sequel we shall deal with another variant associated to the same family of curves, but defined on a different manifold.

4. Fractional H-measures with orthogonality property

We study the same family of curves as in (8), but we want to reparameterise these curves so that manifold P, given by (7), is replaced by one with better features.

This family of curves has been used in all known variants of H-measures and H-distributions. In the classical case we have $\alpha_1 = \alpha_2 = \cdots = \alpha_d = 1$ (the rays from the origin), while the parabolic scaling is given by $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \cdots = \alpha_d = 1$. Actually, the coefficients α_k determine the ratio of orders of derivatives involved in the partial differential equation under study, so the constraint $\alpha_k \in \langle 0, 1]$ does not reduce the generality, but assures the differentiability of the curves.

In [4] it was noticed that for transport properties of H-measures it is important to have in Definition 1 a manifold P perpendicular to the scaling curves. We show that the manifold perpendicular to curves of the form (8) is an ellipsoid.

Let $F(\xi_1, \ldots, \xi_d) = C$ be the equation of such a manifold. For the orthogonality, for each $k = 1, \ldots, d$ we must have

$$\partial_k F(\nu_1, \ldots, \nu_d) = c\, \dot{\xi}_k(1) = \frac{c}{\alpha_k} \nu_k \,.$$

After integration we get that F is of the form

$$F(\xi_1, \ldots, \xi_d) = \frac{c}{2\alpha_1}\xi_1^2 + \frac{c}{2\alpha_2}\xi_2^2 + \cdots + \frac{c}{2\alpha_d}\xi_d^2 \,.$$

Since constants c and C are arbitrary, manifold Q given by

$$\frac{\xi_1^2}{\alpha_1} + \frac{\xi_2^2}{\alpha_2} + \cdots + \frac{\xi_d^2}{\alpha_d} = \frac{1}{\alpha_{\min}} \tag{9}$$

has the desired property. The constants have been chosen in such a way that Q depends only on the ratio of coefficients α_k and not on their exact value, that Q coincides with S^{d-1} in the classical case and with the ellipsoid given in [4, Section 2] in the parabolic case. In particular, if we have $\alpha_1 = \cdots = \alpha_d = \alpha \in \langle 0, 1]$ then $Q = \mathrm{S}^{d-1}$, thus the measure given by Theorem 4 is a (classical) H-measure.

However, this choice has also some disadvantages. In general, we cannot obtain an explicit formula for projections to the manifold Q given by (9) along curves (8), which would make the verification of assumptions of Lemma 3 easier. Namely, for an arbitrary $\boldsymbol{\xi} \in \mathbf{R}^d_*$, from $\xi_k = \nu_k s^{\frac{1}{\alpha_k}}$ we want to calculate s and $\boldsymbol{\nu} \in Q$ depending on $\boldsymbol{\xi}$ only. After dividing $\xi_k = \nu_k s^{\frac{1}{\alpha_k}}$ by $s^{\frac{1}{\alpha_k}}$ and taking the square, it follows

$$\frac{\xi_k^2}{s^{\frac{2}{\alpha_k}}} = \nu_k^2 \,,$$

while after dividing by α_k and summing we get

$$\sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s^{\frac{2}{\alpha_k}}} = \sum_{k=1}^{d} \frac{\nu_k^2}{\alpha_k} = \frac{1}{\alpha_{\min}} \,, \tag{10}$$

which is an equation than can be explicitly solved in s only in certain cases. So, the projections are given by

$$\nu_k = \frac{\xi_k}{s(\boldsymbol{\xi})^{\frac{1}{\alpha_k}}} \,, \tag{11}$$

where $s(\boldsymbol{\xi})$ is a solution of (10).

In the parabolic case ($\alpha_1 = \frac{1}{2}$, $\alpha_2 = \cdots = \alpha_d = 1$), equation (10) is in fact biquadratic and the corresponding formula (11) can be used effectively (see [4]). Let us now try to investigate the solutions of (10) in the general case.

Since the left-hand side of (10) is strictly decreasing and continuous in $s \in \mathbf{R}^+$, the equation has a unique positive real solution. This shows that projections are well-defined, although implicitly, by above formulæ; besides, we also have $s(\boldsymbol{\nu}) = 1$.

Function $s(\cdot)$ is implicitly defined by

$$G(\boldsymbol{\xi}, s) = \sum_{k=1}^{d} \frac{1}{\alpha_k} \xi_k^2 s^{-\frac{2}{\alpha_k}} - \frac{1}{\alpha_{\min}} = 0\,,$$

and as we have, for arbitrary $\boldsymbol{\xi} \in \mathbf{R}^d_*$ and $s \in \mathbf{R}^+$,

$$\partial_s G(\boldsymbol{\xi}, s) = -\sum_{k=1}^{d} \frac{2}{\alpha_k^2} \xi_k^2 s^{-\frac{2}{\alpha_k}-1} \neq 0\,,$$

by the implicit function theorem it follows that $s(\cdot)$ is a C^∞ function on $\mathbf{R}^d_*$. In particular, (11) defines a continuous projection from $\mathbf{R}^d_*$ to Q. Hence, its restriction to P given in the previous section is also continuous, which together with admissibility of P gives that Q satisfies property (*iv*) from the second section. Therefore, based on the remark after Theorem 3, we can state the following result on existence of *fractional H-measures with orthogonality property:*

Theorem 4. *Let Q be a closed compact surface given by (9) and let π_Q be the projection to the manifold Q along projection curves (8) given by (11).*

If (u_n) is a sequence in $L^2(\mathbf{R}^d; \mathbf{C}^r)$ such that $\mathsf{u}_n \rightharpoonup 0$, then there exist a subsequence $(\mathsf{u}_{n'})$ and an $r \times r$ Hermitian matrix Radon measure $\boldsymbol{\mu}$ on $\mathbf{R}^d \times Q$ such that for any $\varphi_1, \varphi_2 \in C_0(\mathbf{R}^d)$ and $\psi \in C(Q)$ one has:

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 \mathsf{u}_{n'}) \otimes \mathcal{A}_{\psi\circ\pi_Q}(\varphi_2 \mathsf{u}_{n'})\, d\mathbf{x} &= \langle \boldsymbol{\mu}, (\varphi_1 \overline{\varphi_2}) \boxtimes \overline{\psi} \rangle \\ &= \int_{\mathbf{R}^d\times Q} \varphi_1(\mathbf{x}) \overline{\varphi_2(\mathbf{x}) \psi(\boldsymbol{\xi})}\, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi})\,.\end{aligned}$$

∎

Similarly as in the previous section (using property (12) below), or using the push-forward technique from the section 2, we can get that the fractional H-measure with orthogonality property $\boldsymbol{\mu}$ corresponding to the sequence of oscillations

$$u_n = e^{2\pi i \mathsf{k}\cdot(n^{c/\alpha_1} x_1, \dots, n^{c/\alpha_d} x_d)}\,,$$

where c is an arbitrary constant, is

$$\boldsymbol{\mu} = \lambda \boxtimes \delta_{\pi_Q(\mathsf{k})}\,.$$

Although we already have the existence of associated variant H-measure, for the completeness and further analysis we shall need some additional properties of s, such as the triangle inequality.

By (11) we have that s has certain anisotropic homogeneity property

$$s\Big(\lambda^{\frac{1}{\alpha_1}} \xi_1, \dots, \lambda^{\frac{1}{\alpha_d}} \xi_d\Big) = \lambda s(\boldsymbol{\xi}), \quad \lambda \in \mathbf{R}^+\,, \tag{12}$$

so the whole branch of the *coordinate curve* given by (8) is projected to the same point on Q, and s takes constant value $\lambda \in \mathbf{R}^+$ on each ellipsoid

$$\sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k \lambda^{\frac{2}{\alpha_k}}} = \frac{1}{\alpha_{\min}}\,,$$

showing that $s(\boldsymbol{\xi})$ measures certain distance of $\boldsymbol{\xi}$ from the origin. In particular, (12) implies that s can be extended by continuity to the origin by zero. Therefore, we claim that $d_s(\boldsymbol{\xi}, \boldsymbol{\eta}) := s(\boldsymbol{\xi} - \boldsymbol{\eta})$ defines a metric on $\mathbf{R}^d$.

The only non-trivial fact to be checked is the triangle inequality. We want to prove that for arbitrary $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbf{R}^d$ we have

$$s(\boldsymbol{\xi} + \boldsymbol{\eta}) \leqslant s(\boldsymbol{\xi}) + s(\boldsymbol{\eta})\,. \tag{13}$$

In doing so we need properties of s given in the following lemma.

Lemma 6.
a) $(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d_*) \quad s(\boldsymbol{\xi}) = s(|\xi_1|, \dots, |\xi_d|)$ *,*
b) $|\eta_k| \geqslant |\xi_k|,\ k = 1, \dots, d \quad \Longrightarrow \quad s(\boldsymbol{\eta}) \geqslant s(\boldsymbol{\xi})$*.*

Proof. a) By the uniqueness of s and the identity

$$\frac{1}{\alpha_{\min}} = \sum_{k=1}^{d} \frac{|\xi_k|^2}{\alpha_k s(|\xi_1|, \dots, |\xi_d|)^{2/\alpha_k}} = \sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s(|\xi_1|, \dots, |\xi_d|)^{2/\alpha_k}}\,,$$

it follows that $s(\boldsymbol{\xi}) = s(|\xi_1|, \dots, |\xi_d|)$.
b) From

$$\frac{1}{\alpha_{\min}} = \sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s(\boldsymbol{\xi})^{2/\alpha_k}} = \sum_{k=1}^{d} \frac{\eta_k^2}{\alpha_k s(\boldsymbol{\eta})^{2/\alpha_k}} \geqslant \sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s(\boldsymbol{\eta})^{2/\alpha_k}}$$

we obtain

$$\sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k} \left(\frac{1}{s(\boldsymbol{\xi})^{\frac{2}{\alpha_k}}} - \frac{1}{s(\boldsymbol{\eta})^{\frac{2}{\alpha_k}}} \right) \geqslant 0\,.$$

The terms in parentheses are of the same sign, so it easily follows that $s(\boldsymbol{\eta}) \geqslant s(\boldsymbol{\xi})$. $\square$

We shall first prove (13) under additional assumption that $\boldsymbol{\xi}, \boldsymbol{\eta} \in (\mathbf{R}^+_0)^d$, as in this case we know from Lemma 6 that $s(\boldsymbol{\xi}) \leqslant s(\boldsymbol{\xi} + \boldsymbol{\eta})$ and $s(\boldsymbol{\eta}) \leqslant s(\boldsymbol{\xi} + \boldsymbol{\eta})$. By using the classical triangle inequality for the Euclidean norm we get

$$\begin{aligned}
\frac{1}{\sqrt{\alpha_{\min}}} &= \sqrt{\sum_{k=1}^{d} \frac{(\xi_k + \eta_k)^2}{\alpha_k s(\boldsymbol{\xi} + \boldsymbol{\eta})^{2/\alpha_k}}} \leqslant \sqrt{\sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s(\boldsymbol{\xi} + \boldsymbol{\eta})^{2/\alpha_k}}} + \sqrt{\sum_{k=1}^{d} \frac{\eta_k^2}{\alpha_k s(\boldsymbol{\xi} + \boldsymbol{\eta})^{2/\alpha_k}}} \\
&= \sqrt{\sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s(\boldsymbol{\xi})^{2/\alpha_k}} \Big(\frac{s(\boldsymbol{\xi})}{s(\boldsymbol{\xi} + \boldsymbol{\eta})} \Big)^{\frac{2}{\alpha_k}}} + \sqrt{\sum_{k=1}^{d} \frac{\eta_k^2}{\alpha_k s(\boldsymbol{\eta})^{2/\alpha_k}} \Big(\frac{s(\boldsymbol{\eta})}{s(\boldsymbol{\xi} + \boldsymbol{\eta})} \Big)^{\frac{2}{\alpha_k}}} \\
&\leqslant \sqrt{\sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s(\boldsymbol{\xi})^{2/\alpha_k}} \Big(\frac{s(\boldsymbol{\xi})}{s(\boldsymbol{\xi} + \boldsymbol{\eta})} \Big)^{2}} + \sqrt{\sum_{k=1}^{d} \frac{\eta_k^2}{\alpha_k s(\boldsymbol{\eta})^{2/\alpha_k}} \Big(\frac{s(\boldsymbol{\eta})}{s(\boldsymbol{\xi} + \boldsymbol{\eta})} \Big)^{2}} \\
&= \frac{s(\boldsymbol{\xi})}{s(\boldsymbol{\xi} + \boldsymbol{\eta})} \sqrt{\sum_{k=1}^{d} \frac{\xi_k^2}{\alpha_k s(\boldsymbol{\xi})^{2/\alpha_k}}} + \frac{s(\boldsymbol{\eta})}{s(\boldsymbol{\xi} + \boldsymbol{\eta})} \sqrt{\sum_{k=1}^{d} \frac{\eta_k^2}{\alpha_k s(\boldsymbol{\eta})^{2/\alpha_k}}} \\
&= \frac{1}{\sqrt{\alpha_{\min}}} \frac{s(\boldsymbol{\xi}) + s(\boldsymbol{\eta})}{s(\boldsymbol{\xi} + \boldsymbol{\eta})}\,,
\end{aligned}$$

which is the desired triangle inequality.

For general $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbf{R}^d$ we now again use Lemma 6 to obtain

$$\begin{aligned}
s(\boldsymbol{\xi} + \boldsymbol{\eta}) &= s(|\xi_1 + \eta_1|, \dots, |\xi_d + \eta_d|) \leqslant s(|\xi_1| + |\eta_1|, \dots, |\xi_d| + |\eta_d|) \\
&\leqslant s(|\xi_1|, \dots, |\xi_d|) + s(|\eta_1|, \dots, |\eta_d|) = s(\boldsymbol{\xi}) + s(\boldsymbol{\eta})\,.
\end{aligned}$$

Using these properties of function s, together with Lemma 7 below, we could check that manifold (9) is admissible, which is not essential as we have already proved the existence of the corresponding variant H-measure.

Let us first study further properties on the special example.

Example 1. We are now going to solve (10) in the special case $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \cdots = \alpha_d = 1$, trying to get an idea on how to estimate the solution in the general case, and with the aim of proving the localisation principle for H-measures introduced by Theorem 4. We are actually interested in finding a unique positive solution $s(\boldsymbol{\xi})$.

We denote $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) = (\xi_1, \boldsymbol{\xi}')$, and get the equation (10) in the form

$$\frac{3\xi_1^2}{s^6} + \frac{|\boldsymbol{\xi}'|^2}{s^2} = 3\,,$$

or equivalently

$$s^6 - \frac{|\boldsymbol{\xi}'|^2}{3}s^4 - \xi_1^2 = 0\,.$$

Taking $t = s^2$, $a = \frac{|\boldsymbol{\xi}'|^2}{3}$, $b = \xi_1^2$ this is the third order algebraic equation, $t^3 - at^2 - b = 0$, which by substitution $z = t - \frac{a}{3}$ gets the form

$$z^3 - \frac{a^2}{3}z - \frac{2}{27}a^3 - b = 0\,,$$

suitable for application of Cardano's formula, which gives

$$z = \sqrt[3]{\frac{a^3}{27} + \frac{b}{2} + \sqrt{\frac{a^3 b}{27} + \frac{b^2}{4}}} + \sqrt[3]{\frac{a^3}{27} + \frac{b}{2} - \sqrt{\frac{a^3 b}{27} + \frac{b^2}{4}}}\,.$$

Taking

$$A = \frac{|\boldsymbol{\xi}'|^6}{729} + \frac{\xi_1^2}{2} \quad \text{and} \quad B = \sqrt{\frac{|\boldsymbol{\xi}'|^6 \xi_1^2}{729} + \frac{\xi_1^4}{4}}$$

one finally gets

$$s(\boldsymbol{\xi}) = \sqrt{\frac{|\boldsymbol{\xi}'|^2}{9} + \sqrt[3]{A+B} + \sqrt[3]{A-B}}\,.$$

We already proved that the equation (10) has a unique positive solution and this very solution we get by taking positive determination of square roots and real determination of cubic roots in the last formula.

Although we get the explicit formula, it is complicated and it seems more interesting to get some estimate on the solution. Therefore, we shall prove the following claim:

There are constants $C_1, C_2 > 0$ *such that the function* $s(\cdot)$ *given by the last formula satisfies*

$$C_1 \sqrt[6]{\xi_1^2 + |\boldsymbol{\xi}'|^6} \leqslant s(\boldsymbol{\xi}) \leqslant C_2 \sqrt[6]{\xi_1^2 + |\boldsymbol{\xi}'|^6}\,,$$

for an arbitrary $\boldsymbol{\xi} \in \mathbf{R}^d_*$.

It is easy to check that $A \geqslant B$ and it follows

$$(\sqrt[3]{A+B} + \sqrt[3]{A-B})^3 = A + B + 3\sqrt[3]{(A+B)^2(A-B)} + 3\sqrt[3]{(A+B)(A-B)^2} + A - B \geqslant 2A\,,$$

i.e. it holds

$$s(\boldsymbol{\xi}) \geqslant \sqrt{\frac{|\boldsymbol{\xi}'|^2}{9} + \sqrt[3]{2A}} \geqslant \sqrt[6]{2A} \geqslant \frac{\sqrt[6]{2}}{3}\sqrt[6]{\xi_1^2 + |\boldsymbol{\xi}'|^6}\,.$$

For the converse inequality we use inequality of arithmetic and cubic means to obtain

$$\frac{\sqrt[3]{A+B} + \sqrt[3]{A-B}}{2} \leqslant \sqrt[3]{\frac{A+B+A-B}{2}} = \sqrt[3]{A}\,,$$

and it follows

$$s(\boldsymbol{\xi}) \leqslant \sqrt{\frac{|\boldsymbol{\xi}'|^2}{9} + 2\sqrt[3]{A}} \leqslant \sqrt{3\sqrt[3]{A}} \leqslant \frac{\sqrt{3}}{\sqrt[6]{2}}\sqrt[6]{\xi_1^2 + |\boldsymbol{\xi}'|^6}\,.$$

∎

The example gives us an idea on how to obtain an estimate in the general case.

Lemma 7. *There are constants $C_1, C_2 > 0$ such that the solution $s(\boldsymbol{\xi})$ of (10) satisfies an estimate*

$$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d_*) \quad C_1|\boldsymbol{\xi}|_{\boldsymbol{\alpha}} \leqslant s(\boldsymbol{\xi}) \leqslant C_2|\boldsymbol{\xi}|_{\boldsymbol{\alpha}} \,.$$

Proof. Denoting $\eta_k = |\xi_k|^{\alpha_k}$ and $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)$, the desired estimate for $s = s(\boldsymbol{\xi})$ is equivalent to $C_1|\boldsymbol{\eta}|_1 \leqslant s \leqslant C_2|\boldsymbol{\eta}|_1$, so (10) reads

$$\sum_{k=1}^d \frac{1}{\alpha_k}\Big(\frac{\eta_k}{s}\Big)^{\frac{2}{\alpha_k}} = \frac{1}{\alpha_{\min}} \,.$$

Therefore, for any k we have

$$\frac{1}{\alpha_k}\Big(\frac{\eta_k}{s}\Big)^{\frac{2}{\alpha_k}} \leqslant \frac{1}{\alpha_{\min}} \,,$$

which gives

$$s \geqslant \Big(\frac{\alpha_{\min}}{\alpha_k}\Big)^{\frac{\alpha_k}{2}} \eta_k \,.$$

Since $\min_k\Big\{\alpha_k^{-\frac{\alpha_k}{2}}\Big\} \geqslant 1$, we can take $C_1 = \frac{\sqrt{\alpha_{\min}}}{d}$.

For the converse, notice that from the above it follows

$$\Big(\frac{\alpha_{\min}}{\alpha_k}\Big)^{\frac{\alpha_k}{2}} \frac{\eta_k}{s} \leqslant 1 \,,$$

so again by (10) we get

$$\frac{1}{\alpha_{\min}} = \sum_{k=1}^d \frac{1}{\alpha_k}\frac{\alpha_k}{\alpha_{\min}}\bigg(\Big(\frac{\alpha_{\min}}{\alpha_k}\Big)^{\frac{\alpha_k}{2}}\frac{\eta_k}{s}\bigg)^{\frac{2}{\alpha_k}} \leqslant \sum_{k=1}^d \frac{1}{\alpha_{\min}}\bigg(\Big(\frac{\alpha_{\min}}{\alpha_k}\Big)^{\frac{\alpha_k}{2}}\frac{\eta_k}{s}\bigg)^2 \leqslant \frac{1}{\alpha_{\min}}\sum_{k=1}^d \Big(\frac{\eta_k}{s}\Big)^2 \,,$$

which implies $s \leqslant |\boldsymbol{\eta}| \leqslant |\boldsymbol{\eta}|_1$, i.e. we can take $C_2 = 1$. □

5. Localisation principle

Let us first define some function spaces and investigate their basic properties. In the literature, we can find them as generalised Sobolev spaces. We mainly follow ideas and results presented in [4].

We shall use spaces consisting of tempered distributions u such that $k\hat{u} \in \mathrm{L}^2(\mathbf{R}^d)$, for some weight function k. These spaces are described in [7, Chapter 10.1] and denoted by $B_{2,k}$ there. More precisely, Hörmander's condition on the weight function $k > 0$ is

$$k(\boldsymbol{\xi} + \boldsymbol{\eta}) \leqslant (1 + C|\boldsymbol{\xi}|)^N k(\boldsymbol{\eta}) \,,$$

for some positive constants C and N. However, the condition

$$k(\boldsymbol{\xi} + \boldsymbol{\eta}) \leqslant C(1 + |\boldsymbol{\xi}|)^N k(\boldsymbol{\eta}) \tag{14}$$

could be used as well, together with additional assumption that k is continuous [7, p. 6]. In the sequel we recall the results from [7] needed for our purposes.

As for the classical space H^s, for $s \in \mathbf{R}$ and $\boldsymbol{\alpha} \in [0, 1]^d$ we define the anisotropic Sobolev space

$$\mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d) := \{u \in \mathcal{S}' : \hat{u} \text{ is a function, } k^s_{\boldsymbol{\alpha}}\hat{u} \in \mathrm{L}^2(\mathbf{R}^d)\} \,,$$

where $k_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) := 1 + |\boldsymbol{\xi}|_{\boldsymbol{\alpha}}$.

To justify the above notation, let us check that introduced spaces depend only on the product $s\boldsymbol{\alpha}$. If $s\boldsymbol{\alpha} = \mathsf{0}$, then $k^s_{\boldsymbol{\alpha}}$ is a constant. Suppose now that $s\boldsymbol{\alpha} = t\boldsymbol{\beta} \neq \mathsf{0}$. It is sufficient to prove the equivalence of weighted norms in $\mathrm{L}^2(\mathbf{R}^d)$ with the corresponding weights $k^s_{\boldsymbol{\alpha}}$ and $k^t_{\boldsymbol{\beta}}$. It is clear that s and t have the same sign, so let us first assume $s \geqslant t > 0$. Since $\frac{s}{t} \geqslant 1$ and all norms on $\mathbf{R}^{1+d}$ are equivalent, there exist $C_1, C_2 > 0$ such that for any $\boldsymbol{\xi} \in \mathbf{R}^d$ we have

$$C_1 k_{\boldsymbol{\beta}}^{\frac{t}{s}}(\boldsymbol{\xi}) = C_1\Big(1 + \sum_{k=1}^{d} |\xi_k|^{\alpha_k \frac{s}{t}}\Big)^{\frac{t}{s}} \leqslant k_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \leqslant C_2\Big(1 + \sum_{k=1}^{d} |\xi_k|^{\alpha_k \frac{s}{t}}\Big)^{\frac{t}{s}} = C_2 k_{\boldsymbol{\beta}}^{\frac{t}{s}}(\boldsymbol{\xi}),$$

from which we get the claim. The case $s \leqslant t < 0$ follows in the same manner.

In the next lemma we show that $k^s_{\boldsymbol{\alpha}}$ satisfies condition (14), and present some needed results which immediately follows by [7].

Lemma 8. *If positive functions k_1, k_2 satisfy condition (14), then, for any $s \in \mathbf{R}$, k_1^s and $k_1 + k_2$, as well as function $k^s_{\boldsymbol{\alpha}}$ given above satisfy the same condition.*

Proof. For $s \geqslant 0$ the first assertion is trivial, while for $s < 0$ we first substitute $\boldsymbol{\eta}$ with $\boldsymbol{\xi} + \boldsymbol{\eta}$ and $\boldsymbol{\xi}$ with $-\boldsymbol{\xi}$, and then take the power $-s$.

If k_1 and k_2 satisfy (14) with constants $C_1, N_1 > 0$ and $C_2, N_2 > 0$, then the same condition holds for $k_1 + k_2$ with constants $C = \max\{C_1, C_2\}$ and $N = \max\{N_1, N_2\}$.

To show that $k^s_{\boldsymbol{\alpha}}$ satisfies condition (14), it is now enough to check that the same holds for the function

$$k_0(\boldsymbol{\xi}) = \frac{1}{d} + |\xi_k|^{\alpha_k},$$

which follows from the estimate

$$\begin{aligned} k_0(\boldsymbol{\xi} + \boldsymbol{\eta}) &= \frac{1}{d} + |\xi_k + \eta_k|^{\alpha_k} \leqslant \frac{1}{d} + (|\xi_k| + |\eta_k|)^{\alpha_k} \leqslant \frac{1}{d} + |\xi_k|^{\alpha_k} + |\eta_k|^{\alpha_k} \\ &\leqslant (1 + d|\xi_k|^{\alpha_k})(\frac{1}{d} + |\eta_k|^{\alpha_k}) \leqslant (1 + d + d|\xi_k|)k_0(\boldsymbol{\eta}) \leqslant (1 + d)(1 + |\boldsymbol{\xi}|)k_0(\boldsymbol{\eta}), \end{aligned}$$

where the second inequality is proved in the proof of Lemma 5, except for the case $\alpha_k = 0$ which trivially follows. □

When equipped with the inner product

$$\langle\, u \mid v \,\rangle_{\mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d)} := \langle\, k^s_{\boldsymbol{\alpha}}\, \hat{u} \mid k^s_{\boldsymbol{\alpha}}\, \hat{v} \,\rangle_{\mathrm{L}^2(\mathbf{R}^d)},$$

$\mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d)$ becomes a Hilbert space.

The following lemma is a consequence of [7, Teorem 10.1.7].

Lemma 9. *The embeddings $\mathcal{S} \hookrightarrow \mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}'$ are dense and continuous. Moreover, $\mathrm{C}^\infty_c(\mathbf{R}^d)$ is dense in $\mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d)$.* ∎

Without loss of generality we can assume that for $0 \leqslant m \leqslant d$ we have $0 < \alpha_1, \alpha_2, \dots, \alpha_m < 1$, and $\alpha_{m+1} = \cdots = \alpha_d = 1$, which is the case considered in Lemma 10 and Theorem 5 below. We also use the notation $\mathbf{x} = (\bar{\mathbf{x}}, \mathbf{x}')$, $\bar{\mathbf{x}} = (x_1, \dots, x_m)$, $\mathbf{x}' = (x_{m+1}, \dots, x_d)$, and analogously for other vector variables.

The next lemma follows from [7, Teorem 10.1.10].

Lemma 10. *For any compact set $K \subseteq \mathbf{R}^d$ and any $r < s$ the embedding*

$$\mathrm{H}^{-r(\mathsf{0},\mathsf{1})}(\mathbf{R}^d) \cap \mathcal{E}'(K) \hookrightarrow \mathrm{H}^{-s\boldsymbol{\alpha}}(\mathbf{R}^d)$$

is compact, where $\mathsf{0} \in \mathbf{R}^m$, $\mathsf{1} = (1, \dots, 1) \in \mathbf{R}^{d-m}$, and $s > 0$. ∎

For the proof of localisation principle we also need the following lemma, whose proof in the scalar case and for $\alpha_i \in \mathbf{N}$ can be found in [3, Lemma 3], while more general situation is treated in [1, Lemma 9].

Lemma 11. *Let* f *be a measurable vector function on* $\mathbf{R}^d$*, and* h *be a continuous scalar function of the form* $h(\mathbf{x}) = \sum_{i=1}^{d} |2\pi x^i|^{\alpha_i}, \alpha_i \in \mathbf{R}^+$*. Furthermore, we assume that* (u_n) *is a sequence of vector functions with supports contained in a fixed compact set, such that* $\mathsf{u}_n \rightharpoonup \mathsf{0}$ *in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$*, and*

$$\frac{\mathsf{f}}{(1+h)^\beta} \cdot \hat{\mathsf{u}}_n \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$$

for some constant $\beta \in \mathbf{R}^+$*. If* $h^{-\beta}\mathsf{f} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^r)$*, then it also holds*

$$\frac{\mathsf{f}}{h^\beta} \cdot \hat{\mathsf{u}}_n \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)\,.$$

∎

As the space $\mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d)$ is *semilocal* (see [7, loc. cit.]), then the smallest *local space* containing it is

$$\mathrm{H}^{s\boldsymbol{\alpha}}_{\mathrm{loc}}(\mathbf{R}^d) := \left\{ u \in \mathcal{D}'(\mathbf{R}^d) : (\forall\, \varphi \in \mathrm{C}^\infty_c(\mathbf{R}^d)) \quad \varphi u \in \mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d) \right\}.$$

This space is endowed with the weakest topology in which all maps $u \mapsto \varphi u$ are continuous.

We also recall the definition of fractional derivative of order α: ∂_k^α is a pseudodifferential operator with symbol $(2\pi i \xi_k)^\alpha$, i.e.

$$\partial_k^\alpha u = \bar{\mathcal{F}}\left((2\pi i \xi_k)^\alpha\, \hat{u}(\xi)\right)\,,$$

where we consider the principle branch of complex logarithm.

This operator is well-defined on the union of Sobolev spaces $\mathrm{H}^{-\infty}(\mathbf{R}^d) = \bigcup_{s\in\mathbf{R}} \mathrm{H}^s(\mathbf{R}^d)$.

Theorem 5. (localisation principle) *Let* (u_n) *be a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$ *such that* $\mathsf{u}_n \rightharpoonup \mathsf{0}$*, let for any* $n \in \mathbf{N}$ *and* $\mathbf{x}' \in \mathbf{R}^{d-m}$ *supports of* $u_n(\cdot, \mathbf{x}')$ *be contained in a fixed compact set in* $\mathbf{R}^m$*, and let for* $l \in \mathbf{N}$

$$\sum_{|\bar{\gamma}|=l} \partial_1^{\alpha_1\gamma_1} \cdots \partial_m^{\alpha_m\gamma_m}(\mathbf{A}^{\bar{\gamma}}\mathsf{u}_n) + \sum_{|\gamma'|=l} \partial_{\mathbf{x}'}^{\gamma'}(\mathbf{A}^{\gamma'}\mathsf{u}_n) \longrightarrow \mathsf{0} \ \ \textit{strongly in} \ \ \mathrm{H}^{-l\boldsymbol{\alpha}}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^q)\,, \tag{15}$$

where $\mathbf{A}^{\bar{\gamma}}, \mathbf{A}^{\gamma'} \in \mathrm{C}_b(\mathbf{R}^d;\mathrm{M}_{q\times r}(\mathbf{C}))$*, for some* $q \in \mathbf{N}$*, with* $\bar{\gamma} \in \mathbf{N}_0^m$ *and* $\gamma' \in \mathbf{N}_0^{d-m}$*.*

Then, for the associated H-measure $\boldsymbol{\mu}$ *defined by Theorem 4, it holds*

$$\left(\sum_{|\bar{\gamma}|=l} \prod_{k=1}^{m} (2\pi i\, \xi_k)^{\alpha_k\gamma_k}\mathbf{A}^{\bar{\gamma}} + \sum_{|\gamma'|=l} (2\pi i\, \boldsymbol{\xi}')^{\gamma'}\mathbf{A}^{\gamma'}\right)\boldsymbol{\mu} = \mathbf{0}\,.$$

Proof. Let us first show that an analogue to relation (15) holds for any localised sequence $(\phi\mathsf{u}_n)$, where $\phi \in \mathrm{C}^\infty_c(\mathbf{R}^d)$. First, we take $\phi \in \mathrm{C}^\infty_c(\mathbf{R}^{d-m})$ (a function not depending on $\bar{\mathbf{x}}$), for which we have

$$\begin{aligned}
\sum_{|\bar{\gamma}|=l} \partial_1^{\alpha_1\gamma_1} \cdots \partial_m^{\alpha_m\gamma_m}(\mathbf{A}^{\bar{\gamma}}\phi\mathsf{u}_n) &+ \sum_{|\gamma'|=l} \partial_{\mathbf{x}'}^{\gamma'}(\mathbf{A}^{\gamma'}\phi\mathsf{u}_n) \\
&= \phi\Big(\sum_{|\bar{\gamma}|=l} \partial_1^{\alpha_1\gamma_1} \cdots \partial_m^{\alpha_m\gamma_m}(\mathbf{A}^{\bar{\gamma}}\mathsf{u}_n) + \sum_{|\gamma'|=l} \partial_{\mathbf{x}'}^{\gamma'}(\mathbf{A}^{\gamma'}\mathsf{u}_n)\Big) \\
&\quad + \sum_{|\gamma'|=l}\ \sum_{1\leqslant|\boldsymbol{\delta}|\leqslant l} \binom{\gamma'}{\boldsymbol{\delta}} \partial_{\mathbf{x}'}^{\boldsymbol{\delta}}\phi\, \partial_{\mathbf{x}'}^{(\gamma'-\boldsymbol{\delta})}(\mathbf{A}^{\gamma'}\mathsf{u}_n)\,.
\end{aligned}$$

By Lemma 10, the terms under the last summation converge strongly to zero in $\mathrm{H}^{-l\alpha}(\mathbf{R}^d;\mathbf{C}^q)$. The remaining terms converge in the same space by the assumptions of the theorem, which proves the claim.

For an arbitrary $\phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, we can take a function $\varphi \in \mathrm{C}_c^\infty(\mathbf{R}^{d-m})$, not depending on $\bar{\mathbf{x}}$, such that $\phi = \phi\varphi$. We get

$$\begin{aligned}
&\Big\|\sum_{|\bar\gamma|=l}\partial_1^{\alpha_1\gamma_1}\cdots\partial_m^{\alpha_m\gamma_m}(\mathbf{A}^{\bar\gamma}\phi\mathsf{u}_n)+\sum_{|\gamma'|=l}\partial_{\mathbf{x}'}^{\gamma'}(\mathbf{A}^{\gamma'}\phi\mathsf{u}_n)\Big\|_{\mathrm{H}^{-l\alpha}(\mathbf{R}^d;\mathbf{C}^q)}\\
&\qquad\leqslant\Big\|\sum_{|\bar\gamma|=l}\frac{\prod_{k=1}^m(2\pi i\xi_k)^{\alpha_k\gamma_k}}{(\|\xi\|_\alpha)^l}\widehat{\mathbf{A}^{\bar\gamma}\phi\mathsf{u}_n}+\sum_{|\gamma'|=l}\frac{(2\pi i\xi')^{\gamma'}}{(\|\xi\|_\alpha)^l}\widehat{\mathbf{A}^{\gamma'}\phi\mathsf{u}_n}\Big\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^q)}\\
&\qquad=\Big\|\sum_{|\bar\gamma|=l}P_{\bar\gamma}(\phi\mathbf{A}^{\bar\gamma}\varphi\mathsf{u}_n)+\sum_{|\gamma'|=l}P_{\gamma'}(\phi\mathbf{A}^{\gamma'}\varphi\mathsf{u}_n)\Big\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^q)}\\
&\qquad\leqslant\Big\|\phi\Big(\sum_{|\bar\gamma|=l}P_{\bar\gamma}(\mathbf{A}^{\bar\gamma}\varphi\mathsf{u}_n)+\sum_{|\gamma'|=l}P_{\gamma'}(\mathbf{A}^{\gamma'}\varphi\mathsf{u}_n)\Big)\Big\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^q)}\\
&\qquad\qquad+\Big\|\sum_{|\bar\gamma|=l}[P_{\bar\gamma},M_\phi](\mathbf{A}^{\bar\gamma}\varphi\mathsf{u}_n)+\sum_{|\gamma'|=l}[P_{\gamma'},M_\phi](\mathbf{A}^{\gamma'}\varphi\mathsf{u}_n)\Big\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^q)},
\end{aligned}$$

where $P_{\gamma'}$ and $P_{\bar\gamma}$ denote Fourier multiplier operators associated to symbols $p_{\gamma'}(\xi)=\frac{(2\pi i\xi')^{\gamma'}}{(\|\xi\|_\alpha)^l}$ and $p_{\bar\gamma}(\xi)=\frac{\prod_{k=1}^m(2\pi i\xi_k)^{\alpha_k\gamma_k}}{(\|\xi\|_\alpha)^l}$, M_ϕ denotes a multiplication with ϕ, while $[P,M_\phi]=PM_\phi-M_\phi P$. By Lemma 3 and our example of admissible manifold, commutators $[P_{\bar\gamma},M_\phi]$ and $[P_{\gamma'},M_\phi]$ are compact on $\mathrm{L}^2(\mathbf{R}^d)$, giving that the last term in the expression above converges to 0. According to the first part of the proof, the sequence (φu_n) satisfies the analogous relation to (15) and by Lemma 11 the sequence of functions $\sum_{|\bar\gamma|=l}P_{\bar\gamma}(\mathbf{A}^{\bar\gamma}\varphi\mathsf{u}_n)+\sum_{|\gamma'|=l}P_{\gamma'}(\mathbf{A}^{\gamma'}\varphi\mathsf{u}_n)$ converges strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^q)$. Thus we have shown that $\sum_{|\bar\gamma|=l}\partial_1^{\alpha_1\gamma_1}\cdots\partial_m^{\alpha_m\gamma_m}(\mathbf{A}^{\bar\gamma}\phi\mathsf{u}_n)+\sum_{|\gamma'|=l}\partial_{\mathbf{x}'}^{\gamma'}(\mathbf{A}^{\gamma'}\phi\mathsf{u}_n)\longrightarrow 0$ in $\mathrm{H}^{-l\alpha}(\mathbf{R}^d;\mathbf{C}^q)$, and by the above calculation and Lemma 7 it also holds

$$s^{-l}(\xi)\Big(\sum_{|\bar\gamma|=l}\prod_{k=1}^m(2\pi i\xi_k)^{\alpha_k\gamma_k}\widehat{\mathbf{A}^{\bar\gamma}\phi\mathsf{u}_n}+\sum_{|\gamma'|=l}(2\pi i\xi')^{\gamma'}\widehat{\mathbf{A}^{\gamma'}\phi\mathsf{u}_n}\Big)\longrightarrow 0 \quad\text{in}\quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^q).$$

After multiplying the above sequence by $\psi\circ\pi_Q$, where $\psi\in\mathrm{C}(Q)$, and forming a tensor product with $\widehat{\phi\mathsf{u}_n}$, by Theorem 4 we have

$$\begin{aligned}
\mathbf{0}&=\lim_n\int_{\mathbf{R}^d}(\psi\circ\pi_Q)\Big(\sum_{|\bar\gamma|=l}\frac{\prod_{k=1}^m(2\pi i\xi_k)^{\alpha_k\gamma_k}}{s(\xi)^l}\widehat{\mathbf{A}^{\bar\gamma}\phi\mathsf{u}_n}+\sum_{|\gamma'|=l}\frac{(2\pi i\xi')^{\gamma'}}{s(\xi)^l}\widehat{\mathbf{A}^{\gamma'}\phi\mathsf{u}_n}\Big)\otimes\big(\widehat{\phi\mathsf{u}_n}\big)d\xi\\
&=\Big\langle\frac{1}{s(\xi)^l}\Big(\sum_{|\bar\gamma|=l}\prod_{k=1}^m(2\pi i\xi_k)^{\alpha_k\gamma_k}\mathbf{A}^{\bar\gamma}+\sum_{|\gamma'|=l}(2\pi i\xi')^{\gamma'}\mathbf{A}^{\gamma'}\Big)\boldsymbol{\mu},\,|\phi|^2\boxtimes\psi\Big\rangle.
\end{aligned}$$

As $s(\xi)=1$ on the support of measure $\boldsymbol{\mu}$, the claim follows. □

Let us remark that in [1, 4] the localisation principle was stated for $\boldsymbol{\mu}^\top$ since there a sesquilinear dual product was used, while here we work with a bilinear dual product.

Remark. By the previous theorem in the case $\alpha_1=\cdots=\alpha_d=\alpha\in\langle 0,1\rangle$, we obtain a generalisation of the known localisation principle for H-measures to the case of differential relations with fractional derivatives.

∎

Remark. The supports of u_n have to be contained in a fixed compact (in variable $\bar{\mathbf{x}}$), as fractional derivatives are not local, and we lack an appropriate variant of the Leibniz product rule for them.

This assumption can be substituted by the requirement that coefficients $\mathbf{A}^{\bar\gamma}$ and $\mathbf{A}^{\gamma'}$ have compact support in $\bar{\mathbf{x}}$.

∎

Remark. For some further studies on fractional H-measures with orthogonality property, including the development of the propagation principle, we refer to [5]. ∎

Recently, Evgenij Jurjevič Panov introduced an abstract concept of H-measures defined on a spectrum of general algebra of test symbols allowing more general pseudodifferential operators in (15) [13, Theorem 3.1], which one might use in the proof of the localisation principle.

At the end, we give a simple application of the localisation principle.

Corollary 1. *Let $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$ has supports contained in a fixed compact, and let u_n satisfies the equation*

$$\sum_{k=1}^{d} \partial_k^\alpha u_n + cu_n = f_n \,,$$

for some $\alpha \in \langle 0, 1\rangle$ and $c \in \mathrm{C}(\mathbf{R}^d)$. If f_n converges (strongly) to zero in $\mathrm{H}^{-(\alpha,\dots,\alpha)}(\mathbf{R}^d)$, then the H-measure μ associated to the (sub)sequence $(u_{n'})$ is trivial.

Proof. In this case, the measure from Theorem 4 and (classical) H-measure coincide.

According to Lemma 10, cu_n converges to zero in $\mathrm{H}^{-(\alpha,\dots,\alpha)}(\mathbf{R}^d)$, and a direct application of Theorem 5 gives

$$\left(\sum_{k=1}^{d} (i\xi_k)^\alpha\right)\mu = 0\,.$$

If $\xi_k > 0$, then $\mathsf{Re}\,(i\xi_k)^\alpha = \xi_k^\alpha \cos\frac{\alpha\pi}{2} > 0$, and if $\xi_k < 0$, then $\mathsf{Re}\,(i\xi_k)^\alpha = (-\xi_k)^\alpha \cos\frac{-\alpha\pi}{2} > 0$. Hence, the expression inside parentheses above is equal to zero if and only if $(\xi_1, \dots, \xi_d) = \mathbf{0}$. As $Q = \mathrm{S}^{d-1}$ does not contain the origin, we conclude that $\mu = 0$, which implies $u_{n'} \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$. □

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