The Malgrange form and Fredholm determinants

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The Malgrange Form and Fredholm Determinants

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Abstract. We consider the factorization problem of matrix symbols relative to a closed contour, i.e., a Riemann–Hilbert problem, where the symbol depends analytically on parameters. We show how to define a function τ which is locally analytic on the space of deformations and that is expressed as a Fredholm determinant of an operator of "integrable" type in the sense of Its–Izergin–Korepin–Slavnov. The construction is not unique and the non-uniqueness highlights the fact that the tau function is really the section of a line bundle.

Key words: Malgrange form; Fredholm determinants; tau function

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1 Introduction

We shall consider the following prototypical matrix Riemann–Hilbert problem (RHP) on the unit circle Σ (or any smooth closed simple contour):

$$\Gamma_+(z;\mathbf{t}) = \Gamma_-(z;\mathbf{t})M(z;\mathbf{t}), \qquad \forall\, z\in\Sigma, \qquad \Gamma(\infty)=\mathbf{1}. \tag{1.1}$$

Here $\mathbf{t}$ stands for a vector of parameters which we refer to as "deformation parameters". The assumptions are the following;

1. The matrix $M(z;\mathbf{t}) \in \mathrm{GL}_n(\mathbb{C})$ is jointly analytic for z in a fixed tubular neighbourhood $N(\Sigma)$ of Σ and $\mathbf{t}$ in an open connected domain $\mathcal{S}$, which we refer to as the "deformation space".
2. The index of $\det M(z;\mathbf{t})$ around Σ vanishes for all $\mathbf{t}\in\mathcal{S}$.
3. The partial indices are generically zero, i.e., the RHP (1.1) generically admits solution.

Let us remind the reader of some facts that can be extracted from [7]

– There exists a matrix function $Y_-(z)$ analytic and analytically invertible in $\mathrm{Ext}(\Sigma)\cup N(\Sigma)$ (and uniformly bounded) and similarly a matrix function $Y_+(z)$ analytic and analytically invertible in $\mathrm{Int}(\Sigma)\cup N(\Sigma)$ and n integers $k_1,\dots,k_n$ (called *partial indices*) such that

$$D(z)Y_+(z) = Y_-(z)M(z), \quad z\in\Sigma, \qquad D=\operatorname{diag}\left(z^{k_1},\dots,z^{k_n}\right).$$

– The RHP (1.1) is solvable if and only if all partial indices vanish, $k_j=0$, $\forall\, j=1,\dots,n$.

Note that since $\operatorname{ind}_\Sigma \det M = \sum\limits_{j=1}^n k_j$, the condition (2) in our assumptions is necessary for the solvability of (1.1).

We shall denote by $\mathbb{D}_\pm$ the interior $(+)$ and the exterior $(-)$ regions separated by Σ. Define $\mathcal{H}_+$ to be the space of functions that are in $L^2(\Sigma, |\mathrm{d}z|)$ and extend to analytic functions in the interior. We will use the notation $\vec{\mathcal{H}}_+ = \mathcal{H}_+ \otimes \mathbb{C}^r$ (i.e., vector-valued such functions). The vectors will be thought of as row-vectors. We also introduce the Cauchy projectors $C_\pm\colon L^2(\Sigma, |\mathrm{d}z|) \to \mathcal{H}_\pm$:

$$C_\pm[f](z) = \oint_\Sigma \frac{f(w)\mathrm{d}w}{(w-z)2i\pi}, \qquad z \in \mathbb{D}_\pm.$$

It is well known [14] that the RHP (1.1) is solvable if and only if the Toeplitz operator[1]

$$T_S\colon\ \vec{\mathcal{H}}_+ \to \vec{\mathcal{H}}_+,$$
$$T_S[\vec{f}] = C_+[\vec{f} S], \qquad S(z;\mathbf{t}) := M^{-1}(z;\mathbf{t})$$

is invertible, in which case the inverse is given by

$$T_S^{-1}[\vec{f}] = C_+\big[\vec{f}\Gamma_-^{-1}\big]\Gamma_+.$$

Moreover the operator is Fredholm and

$$\dim\ker(T_S) - \dim\operatorname{coker}(T_S) = \operatorname{ind}_\Sigma \det M = 0.$$

There is no reasonable way, however, to define a "determinant" of T_S as it stands. Such a function of $\mathbf{t}$ would desirably have the property that the RHP (1.1) is not solvable if and only if this putative determinant is zero.

While this is notoriously impossible in this naive form, we now propose a proxy for the notion of determinant, in terms of a simple Fredholm determinant.

The Malgrange one-form. As Malgrange explains [15] one can define a central extension on the loop group $\mathcal{G} := \{M\colon \Sigma \to \mathrm{GL}_n(\mathbb{C})\colon\ \operatorname{ind}_\Sigma \det M = 0\}$ given by $\widehat{\mathcal{G}} = \{(M,u) \in \mathcal{G} \times \mathbb{C}^\times\}$ with the group law[2]

$$(M,u)\cdot(\widetilde{M},\widetilde{u}) = \big(M\widetilde{M}, u\widetilde{u}c\big(M,\widetilde{M}\big)\big), \qquad c(M,\widetilde{M}) := \det_{\mathcal{H}_+}\big(T_{M^{-1}}T_{\widetilde{M}^{-1}}T^{-1}_{(M\widetilde{M})^{-1}}\big).$$

The operator in the determinant is of the form $\mathrm{Id}_{\mathcal{H}_+} +$ (trace class) and hence the Fredholm determinant is well defined. This group law is only valid for pairs M, $\widetilde{M}$ for which the inverse of $T^{-1}_{(M\widetilde{M})^{-1}}$ exists. The left-invariant Maurer–Cartan form of this central extension is then given by $\big(S^{-1}\delta S, \frac{\mathrm{d}u}{u} + \widehat{\omega}_M\big)$, where

$$\widehat{\omega}_M := \mathrm{Tr}_{\mathcal{H}_+}\big(T_S^{-1}\circ T_{\delta S} - T_{S^{-1}\delta S}\big), \qquad S := M^{-1},$$

and this can be written as the following integral

$$\widehat{\omega}_M = \oint_\Sigma \mathrm{Tr}\big(\Gamma_+^{-1}\Gamma_+' M^{-1}\delta M\big)\frac{\mathrm{d}z}{2i\pi}. \tag{1.2}$$

Here, and below, δ denotes the exterior total differentiation in the deformation space $\mathcal{S}$:

$$\delta = \sum \delta t_j \frac{\partial}{\partial t_j}.$$

[1]Due to our choices of symbols, the matrix symbol of the relevant Toeplitz operator is M^{-1}. We apologize for the inconvenience.

[2]We are being a bit cavalier in this description; we invite the reader to read pp. 1373–1374 in [15].

The Malgrange form is a *logarithmic* form in the sense that it has only simple poles on a co-dimension 1 analytic submanifold of the deformation space $\mathcal{S}$ and with positive integer Poincaré residue along it; this manifold is precisely the exceptional "divisor" $(\Theta) \subset \mathcal{S}$ (the *Malgrange divisor*) where the RHP (1.1) becomes non solvable, i.e., where some partial indices of the Birkhoff factorization become non-zero.

Closely related to (1.2) is the following one-form, which we still name after Malgrange:

$$\omega_M := \oint_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}\Gamma_-' \delta M M^{-1}\right)\frac{\mathrm{d}z}{2i\pi}. \tag{1.3}$$

It is also a logarithmic form with the same pole-divisor; indeed one verifies that

$$\widehat{\omega}_M - \omega_M = \oint_\Sigma \mathrm{Tr}\left(M' M^{-1}\delta M M^{-1}\right)\frac{\mathrm{d}z}{2i\pi},$$

which is an analytic form of the deformation parameters $\mathbf{t} \in \mathcal{S}$. In [2] the one-form (1.3) was posited as an object of interest for general Riemann–Hilbert problems (not necessarily on closed contours) and its exterior derivative computed (with an important correction in [3], which is however irrelevant in the present context). It was computed (but the computation can be traced back to Malgrange himself in this case) that

$$\delta\omega_M = \frac{1}{2}\oint_\Sigma \mathrm{Tr}\left(\Xi(z)\wedge\frac{\mathrm{d}}{\mathrm{d}z}\Xi(z)\right)\frac{\mathrm{d}z}{2i\pi}, \qquad \Xi(z;\mathbf{t}) := \delta M(z;\mathbf{t})M(z;\mathbf{t})^{-1}. \tag{1.4}$$

It appears from (1.4) that this two form $\delta\omega_M$ is not only closed, but also smooth on the whole of $\mathcal{S}$, including the Malgrange-divisor. As such, it defines a line-bundle $\mathcal{L}$ over $\mathcal{S}$ by the usual construction: one covers $\mathcal{S}$ by appropriate open sets U_α where $\delta\omega_M = \delta\theta_\alpha$; on the overlap $U_\alpha \cap U_\beta$ the form $\theta_\alpha - \theta_\beta$ is also exact and one defines then the transition functions by $g_{\alpha\beta}(\mathbf{t}) = \exp\left(\int \theta_\alpha - \theta_\beta\right)$. Then a section of this line bundle is provided by the collection of functions $\tau_\alpha \colon U_\alpha \to \mathbb{C}$ such that

$$\tau_\alpha(\mathbf{t}) = \exp\left[\int(\omega_M - \theta_\alpha)\right].$$

Since ω_M is a logarithmic form and each θ_α is analytic in the respective U_α, the functions $\tau_\alpha(\mathbf{t})$ have zero of finite order precisely on the Malgrange divisor $(\Theta) \subset \mathcal{S}$ (under appropriate transversality assumptions, the order of the zero is the dimension of $\mathrm{Ker}\, T_S$).

Our goal is to provide an explicit construction of the τ_α's in terms of Fredholm determinants of simple operators of the Its–Izergin–Korepin–Slavnov "integrable" type [13]. Their definition is recalled in due time.

2 Construction of the Fredholm determinants

The construction carried out below is not unique, and also only local in the deformation space $\mathcal{S}$; this is however not only not a problem, but rather an interesting feature, as we will illustrate in the case of $\mathrm{SL}_2(\mathbb{C})$. The non-uniqueness is precisely a consequence of the fact that we are trying to compute a section of the aforementioned line bundle.

Preparatory step. The assumption that $M(z;\mathbf{t}) \in \mathrm{GL}_n$ can be replaced without loss of generality with $M(z;\mathbf{t}) \in \mathrm{SL}_n$; this is so because of the assumption on the index of $\det M$. Indeed we can solve the scalar problem $y_+(z) = y_-(z)\det M(z)$, $y(\infty) = 1$ and then define a new RHP where $\widetilde{\Gamma}_\pm(z) := \Gamma_\pm \,\mathrm{diag}\left(y_\pm^{-1}, 1, \dots\right)$ and hence the new matrix jump for $\widetilde{\Gamma}$ is $\widetilde{M}(z) = \mathrm{diag}(y_-(z), 1, \dots)M(z)\,\mathrm{diag}\left(y_+^{-1}(z), 1, \dots\right)$ with $\det\widetilde{M} \equiv 1$. For this reason, from here on we assume $M \in \mathrm{SL}_n(\mathbb{C})$.

We define an *elementary* matrix (for our purposes) to be a matrix of the form $\mathbf{1} + cE_{jk}$, with $j \neq k$, where E_{jk} denotes the (j,k)-unit matrix.

Lemma 2.1. *Any matrix $M \in \mathrm{SL}_n(\mathbb{C})$ can be written as a product of elementary matrices. The entries of the factorization are rational in the entries of M with denominators that are monomials in a suitable set of $n-1$ nested minors of M.*

Proof. We recall that given any matrix $M \in \mathrm{SL}_n$, there is a permutation Π of the columns such that the principal minors (the determinants of the top left square submatrices) do not vanish, and hence we can write it as

$$M = LDU\Pi = \widehat{M}\Pi,$$

where L, U are lower/upper triangular matrices with unit on the diagonal and $D = \mathrm{diag}(x_1, \dots, x_n)$ is a diagonal matrix (see for example [8, Vol. 1, Chapter II]). Denote $q_\ell = \det \left[\widehat{M}_{j,k}\right]_{j,k\leq \ell}$ the principal minors; these are the nested minors of the original matrix M alluded to in the statement. The matrices L, U are rational in the entries of M and with denominators that are monomials in the q_ℓ's.

Now, both L, U can clearly be written as products of elementary matrices whose coefficients are polynomials in the entries of L, U (respectively) and so it remains to show that we can write D as product of elementary matrices.

To this end we observe the 'LULU' identity (there is a similar 'ULUL' identity)

$$\begin{bmatrix} x & 0 \\ 0 & \frac{1}{x} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1-x}{x} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{x} \\ 0 & 1 \end{bmatrix}.$$

As $D \in \mathrm{SL}_n$, we can represent it in terms of product of embedded SL_2 matrices using the root decomposition of SL_n:

$$D = \mathrm{diag}\left(x_1, \frac{1}{x_1}, 1, \dots, 1\right) \mathrm{diag}\left(1, x_2 x_1, \frac{1}{x_1 x_2}, 1, \dots, 1\right) \\ \times \mathrm{diag}\left(1, 1, x_1 x_2 x_3, \frac{1}{x_1 x_2 x_3}, 1, \dots, 1\right) \cdots,$$

and then embed the LULU identity for each factor.

Finally, also permutation matrices can be written as product of elementary matrices embedding appropriately the simple identity;

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

This concludes the proof. ■

Let now $M(z)$ be an SL_n matrix valued function, analytic in a tubular neighbourhood $N(\Sigma)$ of Σ, and let $q_\ell(z)$, $\ell = 1, \dots, n-1$ be the nested minors alluded to in the Lemma so that they are not identically zero. Since the entries are analytic in $N(\Sigma)$ we can slightly deform the contour Σ to a contour $\widetilde{\Sigma}$ that avoids all zeroes of every principal minor $q_\ell(z)$. The resulting RHP is "equivalent" to the original in the sense that the solvability of one implies the solvability of the other. Note also that this deformation can be done in a piecewise constant way locally with respect to $\mathbf{t} \in \mathcal{S}$. Thus we have

$$M(z) = F_1(z) \cdots F_R(z), \qquad F_\nu(z) = \mathbf{1} + a_\nu(z) E_{j_\nu, k_\nu}, \qquad \nu = 1, \dots, R.$$

Corresponding to this factorization we can define an equivalent RHP with jumps on R contours $\Sigma_1, \dots, \Sigma_R$, with $\Sigma_1 = \Sigma$ and Σ_{j+1} in the interior of Σ_j and all of them in the joint domain of analyticity of the scalar functions $a_\nu(z)$ (see Fig. 1) which may have poles only at the zeroes

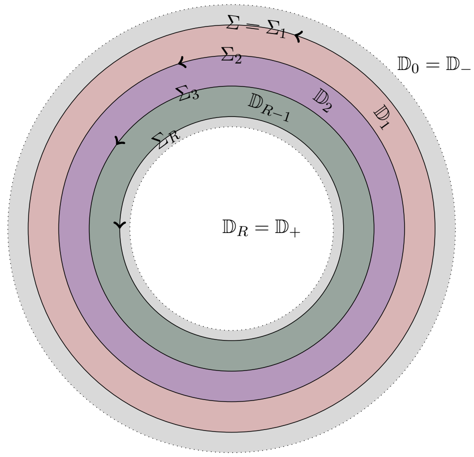

Figure 1. An illustration of the splitting of the jump matrix into elementary jumps. The matrix $M(z;\mathbf{t})$ is analytic in the shaded regions.

of the principal minors $q_\ell(z)$ of $M(z)$. This is accomplished by "extending" the matrix $\Gamma_-(z)$ to the annular regions $\mathbb{D}_0 = \mathbb{D}_-$ and $\mathbb{D}_j = \mathrm{Int}(\Sigma_j) \cap \mathrm{Ext}(\Sigma_{j+1})$ as

$$\begin{aligned} &\Theta_0(z) := \Gamma_-(z), \qquad \forall\, z \in \mathbb{D}_0, \\ &\Theta_\nu(z) := \Gamma_-(z) \overrightarrow{\prod_{\ell=1}^{\nu}} F_\ell(z), \qquad \forall\, z \in \mathbb{D}_\nu. \end{aligned} \tag{2.1}$$

By doing so we obtain the following relations

$$\Theta_\nu(z) = \Theta_{\nu-1}(z) F_\nu(z), \qquad \forall\, z \in \Sigma_\nu, \qquad \nu = 1, \dots, R.$$

The piecewise analytic matrix function $\Theta(z)$ whose restriction to $\mathbb{D}_\nu$ coincides with the matrices Θ_ν (2.1), satisfies a final RHP

$$\Theta_+(z) = \Theta_-(z) F_\nu(z), \qquad \forall\, z \in \Sigma_\nu, \qquad \Theta(\infty) = \mathbf{1}. \tag{2.2}$$

This type of RHP is of the general type of "integrable kernels" and its solvability can be determined by computing the Fredholm determinant of an integral operator of $L^2(\bigsqcup \Sigma_\nu, |\mathrm{d}z|) \simeq \bigoplus_{\nu=1}^{R} L^2(\Sigma_\nu, |\mathrm{d}z|)$ with kernel (we use the same symbol for the operator and its kernel)

$$K(z,w) = \frac{\vec{f}^T(z)\vec{g}(w)}{2i\pi(w-z)}, \qquad \vec{f}(z) = \sum_{\nu=1}^{R} \mathbf{e}_{j_\nu} \chi_\nu(z) a_\nu(z), \qquad \vec{g}(z) = \sum_{\nu=1}^{R} \mathbf{e}_{k_\nu} \chi_\nu(z), \tag{2.3}$$

where $\chi_\nu(z)$ is the projector (indicator function) on the component $L^2(\Sigma_\nu, |\mathrm{d}z|)$. Indeed, as explained in [4, 10, 13], the Fredholm determinant $\det(\mathrm{Id} - K)$ is zero if and only if the RHP (2.2) is non-solvable and moreover the resolvent operator $R = K(\mathrm{Id} - K)^{-1}$ of K has kernel

$$R(z,w) = \frac{\vec{f}^t(z)\Theta^t(z)(\Theta^t)^{-1}(w)\vec{g}(w)}{z-w}.$$

Theorem 2.2. *The RHP (2.2) and hence (1.1) is solvable if and only if* $\tau := \det(\mathrm{Id} - K) \neq 0$.

Proposition 2.3 (see, e.g., [4, Theorem 2.1])**.** *Let ∂ be any deformation of the functions $a_\nu(z)$, then*

$$\partial \ln \tau = \sum_{\nu=1}^{R} \oint_{\Sigma_\nu} \mathrm{Tr}\left(\Theta_-^{-1}\Theta_-'(z)\partial F_\nu F_\nu^{-1}\right)\frac{\mathrm{d}z}{2i\pi} = \sum_{\nu=1}^{R} \oint_{\Sigma_\nu} \left(\Theta_{\nu-1}^{-1}\Theta_{\nu-1}'(z)\right)_{k_\nu, j_\nu} \partial a_\nu(z) \frac{\mathrm{d}z}{2i\pi}.$$

3 The SL_2 case

We would like to express the Malgrange one-form directly in terms of the τ function (Fredholm determinant). Rather than obscuring the simple idea with the general case, we consider in detail the SL_2 case. Let $M(z;\mathbf{t})$ be analytic in $(z,\mathbf{t}) \in N(\Sigma) \times \mathcal{S}$ and with values in $\mathrm{SL}_2(\mathbb{C})$. Using the general scheme above, we have the following factorizations

$$
\begin{aligned}
&(1) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1+c-a}{a} & 1 \end{bmatrix}}_{F_1(z)} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{F_2(z)} \underbrace{\begin{bmatrix} 1 & 0 \\ a-1 & 1 \end{bmatrix}}_{F_3(z)} \underbrace{\begin{bmatrix} 1 & \frac{b-1}{a} \\ 0 & 1 \end{bmatrix}}_{F_4(z)}, \qquad a \not\equiv 0,\\
&(2) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{1+b-d}{d} \\ 0 & 1 \end{bmatrix}}_{F_1(z)} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{F_2(z)} \underbrace{\begin{bmatrix} 1 & d-1 \\ 0 & 1 \end{bmatrix}}_{F_3(z)} \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{c-1}{d} & 1 \end{bmatrix}}_{F_4(z)}, \qquad d \not\equiv 0,\\
&(3) \quad \begin{bmatrix} a & b \\ -\frac{1}{b} & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & b-ab \\ 0 & 1 \end{bmatrix}}_{F_1(z)} \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{1}{b} & 1 \end{bmatrix}}_{F_2(z)} \underbrace{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}}_{F_3(z)},\\
&(4) \quad \begin{bmatrix} 0 & b \\ -\frac{1}{b} & d \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{d-1}{b} \\ 0 & 1 \end{bmatrix}}_{F_1(z)} \underbrace{\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}}_{F_2(z)} \underbrace{\begin{bmatrix} 1 & -\frac{1}{b} \\ 0 & 1 \end{bmatrix}}_{F_3(z)},\\
&(5) \quad \begin{bmatrix} 0 & b \\ -\frac{1}{b} & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}}_{F_1(z)} \underbrace{\begin{bmatrix} 1 & -\frac{1}{b} \\ 0 & 1 \end{bmatrix}}_{F_2(z)} \underbrace{\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}}_{F_3(z)}.
\end{aligned} \tag{3.1}
$$

3.1 Fredholm determinants for different factorizations

Each of the factorization (3.1) leads to an integrable operator of the form (2.3) and hence to a corresponding Fredholm determinant; we now establish their mutual relationships.

There are two types of questions that we address here

1. How are the Fredholm determinants associated with the different factorizations $(1,2)$ in (3.1) related to each other?
2. For a fixed factorization, how does the Fredholm determinant depend on the choice of contour Σ (within the analyticity domain $N(\Sigma)$).

Consider the cases $(3.1)_{(\rho)}$, $\rho = 1,\dots,5$. We compute the logarithmic derivative of the corresponding Fredholm determinant using Proposition 2.3

$$
\partial \ln \tau_{(\rho)} = \sum_{\nu=1}^{R} \oint_{\Sigma_\nu} \mathrm{Tr}\left(\Theta_-^{-1}\Theta_-' \partial F_\nu F_\nu^{-1}\right)\frac{\mathrm{d}z}{2i\pi}. \tag{3.2}
$$

By using the relationship (2.1) between Γ_- (extended to an analytic function on $\mathrm{Ext}(\Sigma)\cup N(\Sigma)$), we can re-express it in terms of the Malgrange one-form of the original problem (1.1); we use the fact that we can deform the contours back to $\Sigma = \Sigma_1$ by Cauchy's theorem. Plugging (2.1) appropriately in (3.2) and using Leibnitz rule, after a short computation we obtain

$$
\begin{aligned}
\delta \ln \tau_{(\rho)} = {}& \oint_{\Sigma} \mathrm{Tr}\left(\Gamma_-^{-1}\Gamma_-' \partial M M^{-1}\right)\frac{\mathrm{d}z}{2i\pi} \\
&+ \oint_{\Sigma} \mathrm{Tr}\left(F_1^{-1}F_1'\partial F_2 F_2^{-1} + F_{12}^{-1}F_{12}'\partial F_3 F_3^{-1} + F_{123}^{-1}F_{123}'\partial F_4 F_4^{-1}\right)\frac{\mathrm{d}z}{2i\pi} =: \omega_M + \theta_{(\rho)},
\end{aligned} \tag{3.3}
$$

where $F_{1\dots k} = F_1F_2F_3\cdots F_k$ (if the factorization has only three term, then we set $F_4 \equiv \mathbf{1}$).

The cases (3,4,5). The last cases $(3.1)_{(3,4,5)}$ lead essentially to a RHP with a triangular jump; it suffices to re-define Γ by $\Gamma\left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]$ for $z\in \mathrm{Int}(\Sigma)$. If the index of b is zero, $\mathrm{ind}_\Sigma b=0$,then the solution can be written explicitly in closed form and it is interesting to compute the Fredholm determinant associated to our factorization of the matrix. We will show

Proposition 3.1. *In the cases* $(3.1)_{(3,4,5)}$ *and under the additional assumption that* $\mathrm{ind}_\Sigma b=0$*, the* τ *function given by* $\det(\mathrm{Id}_{L^2(\cup\Sigma_j)}-K)$ *and* K *as in* (2.3)*, equals the constant in the strong Szegö formula, given by* [1, 5]

$$\tau=\exp\left[\sum_{j>0} j\beta_{-j}\beta_j\right]=\det_{\mathcal{H}_+} T_bT_{b^{-1}},$$

where now the Toeplitz operator is for the scalar symbol $b(z)$ *and* $\mathcal{H}_+$ *is the Hardy space of scalar functions analytic in* $\mathbb{D}_+$ *and* β_j *are the coefficients of* $\ln b(z)$ *in the Laurent expansion centered at the origin*

$$\beta(z):=\ln b(z)=\sum_{j\in\mathbb{Z}}\beta_j z^j. \tag{3.4}$$

The same applies in the case that the jump is triangular ($b\equiv 0$ *and/or* $c\equiv 0$) *under the assumption* $\mathrm{ind}_\Sigma a=0$ *and replacing* b *with* a *in the above formulas.*

Proof. From a direct computation of the term $\theta_{(\rho)}$ in (3.3), we find

$$\theta_{_{(3,4,5)}}=\oint_\Sigma\left(\delta\beta\frac{\mathrm{d}}{\mathrm{d}z}\beta\right)\frac{\mathrm{d}z}{2i\pi},\qquad \beta(z):=\ln b(z)$$

and the solution of the RHP is explicit,

$$\Gamma(z)=\begin{cases}\begin{bmatrix}\mathrm{e}^{B(z)} & -\mathrm{e}^{-B(z)}\displaystyle\oint_\Sigma\frac{a(w)b(w)\mathrm{e}^{2B_-(w)}}{w-z}\frac{\mathrm{d}w}{2i\pi}\\ 0 & \mathrm{e}^{-B(z)}\end{bmatrix}, & z\in\mathbb{D}_-,\\[2ex] \begin{bmatrix}\mathrm{e}^{B(z)} & -\mathrm{e}^{-B(z)}\displaystyle\oint_\Sigma\frac{a(w)b(w)\mathrm{e}^{2B_-(w)}}{w-z}\frac{\mathrm{d}w}{2i\pi}\\ 0 & \mathrm{e}^{-B(z)}\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, & z\in\mathbb{D}_+,\end{cases}$$
$$B(z):=\oint_\Sigma\frac{\beta(w)}{w-z}\frac{\mathrm{d}w}{2i\pi}.$$

Thus we can write explicitly the Malgrange form:

$$\begin{aligned}\omega_M&=\oint_\Sigma\mathrm{Tr}\left(\Gamma_-^{-1}\Gamma_-'\delta MM^{-1}\right)\frac{\mathrm{d}z}{2i\pi}=\oint_\Sigma\mathrm{Tr}\left(\sigma_3\oint_\Sigma\frac{\beta(w)}{(w-z_-)^2}\frac{\mathrm{d}w}{2i\pi}\delta\beta(z)\sigma_3\right)\frac{\mathrm{d}z}{2i\pi}\\ &=2\oint_{\Sigma_-}\oint_\Sigma\frac{\beta(w)\delta(\beta(z))}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}\frac{\mathrm{d}z}{2i\pi}.\end{aligned}$$

In this integral, z is integrated on a slightly "larger" contour Σ_-. The Fredholm determinant satisfies

$$\delta\ln\tau_{(3,4,5)}=\omega_M+\theta_{_{(3,4,5)}}=2\oint_{\Sigma_-}\oint_\Sigma\frac{\beta(w)\delta(\beta(z))}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}\frac{\mathrm{d}z}{2i\pi}+\oint_\Sigma\left[\beta(\delta\beta)'\right]\frac{\mathrm{d}w}{2i\pi}.$$

We now show that

$$\delta\ln\tau=\delta\oint_{\Sigma_-}\oint_\Sigma\frac{\beta(w)\beta(z)}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}\frac{\mathrm{d}z}{2i\pi}=\oint_{\Sigma_-}\oint_\Sigma\frac{\delta\beta(w)\beta(z)+\beta(w)\delta\beta(z)}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}\frac{\mathrm{d}z}{2i\pi}.$$

Indeed, the exchange of order of integration of one of the addenda (and relabeling the variables) yields the other term plus the residue on the diagonal,

$$\oint_{\Sigma_-}\oint_{\Sigma}\frac{\delta\beta(w)\beta(z)+\beta(w)\delta\beta(z)}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}\frac{\mathrm{d}z}{2i\pi}$$
$$=\oint_{\Sigma_-}\oint_{\Sigma}\frac{2\beta(w)\delta\beta(z)}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}\frac{\mathrm{d}z}{2i\pi}+\oint_{\Sigma}\beta(z)(\delta\beta)'\frac{\mathrm{d}z}{2i\pi}.$$

Therefore, in conclusion, we have (we fix the overall constant of τ by requiring it to be 1 for $b\equiv 1$)

$$\ln\tau=\oint_{\Sigma_-}\oint_{\Sigma}\frac{\ln b(w)\ln b(z)}{(w-z)^2}\frac{\mathrm{d}w}{2i\pi}\frac{\mathrm{d}z}{2i\pi}. \tag{3.5}$$

If we write a Laurent expansion of $\ln b$ (3.4) the formula (3.5) gives the explicit expression

$$\ln\tau=\sum_{j>0}j\beta_{-j}\beta_j,$$

which is also the formula for the second Szegö limit theorem for the limit of the Toeplitz determinants of the symbol $b(z)$, and it is known to to be the Fredholm determinant of an operator [1, 5]

$$\tau=\det_{\mathcal{H}_+}T_bT_{b^{-1}},$$

where now the Toeplitz operator is for the scalar symbol $b(z)$ and $\mathcal{H}_+$ is the Hardy space of scalar functions analytic in $\mathbb{D}_+$. ■

Remark 3.2. The index assumption is only necessary for the case $(3.1)_{(5)}$ or $(3.1)_{(1)}$ when $b\equiv 0\equiv c$, because in these situations the RHP separates into two scalar problems. However, the assumption $\mathrm{ind}_\Sigma b=0$ for cases (3,4) or $\mathrm{ind}_\Sigma a=0$ for the triangular case is not necessary as we now show (in the latter form). Consider the RHP

$$Y_+=Y_-\begin{bmatrix}1 & \mu(z)\\ 0 & 1\end{bmatrix},\quad z\in\Sigma,\qquad Y(z)=\big(\mathbf{1}+\mathcal{O}(z^{-1})\big)z^{n\sigma_3},\quad z\to\infty.$$

By defining

$$\Gamma(z)=\begin{cases}Y(z), & z\in\mathrm{Int}(\Sigma),\\ Y(z)z^{-n\sigma_3}, & z\in\mathrm{Ext}(\Sigma),\end{cases}$$

we are lead to the RHP in standard form

$$\Gamma_+=\Gamma_-\begin{bmatrix}z^n & z^n\mu(z)\\ 0 & z^{-n}\end{bmatrix},\quad z\in\Sigma,\qquad \Gamma(\infty)=\mathbf{1},$$

which is case (1) with $c\equiv 0$ (or essentially cases (3,4) up to a multiplication by piecewise constant matrices). Even if $a(z)=z^n$ and thus $\mathrm{ind}_\Sigma a=n$, this problem is still generically solvable in terms of appropriate "orthogonal polynomials" $\{p_\ell(z)\}_{\ell\in\mathbb{N}}$ defined by the "orthogonality" property

$$\oint_\Sigma p_\ell(z)p_k(z)\mu(z)\mathrm{d}z=h_\ell\delta_{\ell k}.$$

Then the solution of the Y-problem is written as

$$Y(z) = \begin{bmatrix} p_n(z) & \displaystyle\oint_\Sigma \frac{p_n(w)\mu(w)\mathrm{d}w}{(w-z)2i\pi} \\ \dfrac{-2i\pi p_{n-1}(z)}{h_{n-1}} & \displaystyle\oint_\Sigma \frac{-p_{n-1}(w)\mu(w)\mathrm{d}w}{(w-z)h_{n-1}} \end{bmatrix},$$

and the solvability depends only on the condition [6]

$$\det\left[\oint_\Sigma z^{\ell+j-2}\mu(z)\mathrm{d}z\right]_{j,\ell=1}^n \neq 0.$$

The cases (1,2). A straightforward computation using the explicit expression $(3.1)_{(1)}$ yields

$$\begin{aligned} \partial \ln \tau_{_{(1)}} &= \oint_\Sigma \Big[\mathrm{Tr}\left(\Gamma_-^{-1}\Gamma_-'\partial M M^{-1}\right) + \partial \ln a (\ln a)'(1+bc) \\ &\qquad - c\partial b(\ln a)' + c'\partial b - c' b\partial(\ln a)\Big]\frac{\mathrm{d}z}{2i\pi} \\ &= \oint_\Sigma \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial M M^{-1}\right)\frac{\mathrm{d}z}{2i\pi} + \underbrace{\oint_\Sigma \left(\frac{(a)'}{a}(d\partial a - c\partial b) + ac'\partial\left(\frac{b}{a}\right)\right)\frac{\mathrm{d}z}{2i\pi}}_{\theta_{_{(1)}}}. \end{aligned} \tag{3.6}$$

Since $\partial \ln \tau$ is a closed differential, the exterior derivative of $\theta_{_{(1)}}$ must be opposite to the one of the first term, which is given by (1.4). Let us verify this directly; to this end we compute the exterior derivative of $\theta_{_{(1)}}$. A straightforward computation yields

$$\begin{aligned} \delta\theta_{_{(1)}} = \oint_\Sigma \Bigg(&\delta b\wedge(\delta c)' + \frac{\delta a \wedge (\delta a)'}{a^2}(1+bc) + \delta a \wedge \left(\frac{\delta c}{a}\right)' b \\ &- \delta a\wedge \delta b\frac{(c)'}{a} - \delta b\wedge(\delta a)'\frac{c}{a} - \delta b\wedge\delta c\frac{a'}{a}\Bigg)\frac{\mathrm{d}z}{2i\pi}. \end{aligned}$$

The exterior derivative of ω_M is given by (1.4), in which we can insert the explicit expression of $M(z;\mathbf{t})$; after a somewhat lengthy but straightforward computation we find that

$$\delta\omega_M + \delta\theta_{_{(1)}} = -\frac{1}{2}\oint_\Sigma \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{c}{a}\delta a\wedge\delta b - \frac{b}{a}\delta a\wedge\delta c + \delta b\wedge\delta c\right)\frac{\mathrm{d}z}{2i\pi} = 0,$$

thus confirming that the differential $\delta\ln\tau_{_{(1)}}$ is indeed closed (of course it must be, since the tau function is a Fredholm determinant!). The case (2) is analogous with the replacements $b \leftrightarrow c$ and $a \leftrightarrow d$.

3.1.1 Determinants for different choices of Σ

The factorizations $(3.1)_{(1,2)}$ require that we deform the contour Σ so that $a(z)$ (or $d(z)$) does not have any zero on Σ. This leads to completely equivalent RHPs of the form (1.1) but not entirely equivalent RHP when expressed in the form (2.2).

Note that $(3.1)_{(3,4,5)}$ do not suffer from this ambiguity, because $b(z)$ cannot have any zeroes in $N(\Sigma)$ since we assumed analyticity of the jump matrix $M(z;\mathbf{t})$.

Consider the factorization $(3.1)_{(1)}$ (with similar considerations applying to the other factorization); in general $a(z;\mathbf{t})$ has zeroes in its domain of analyticity, and their positions depend on $\mathbf{t}$. Therefore it may be necessary, when considering the dependence on $\mathbf{t}$, to move the contour so that certain zeroes are to the left or to the right of it because a zero may sweep across $N(\Sigma)$ as we vary $\mathbf{t}\in\mathcal{S}$.

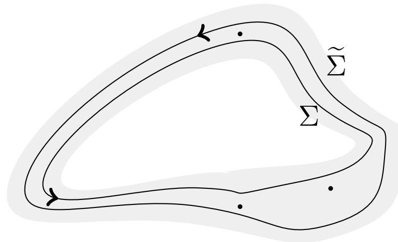

Figure 2. The two contours and the zeroes of $a(z;\mathbf{t})$ within the enclosed region. The shaded area is $N(\Sigma)$ where $M(z;\mathbf{t})$ is analytic (uniformly w.r.t. $\mathbf{t}\in\mathcal{S}$).

So, let Σ, $\widetilde{\Sigma}$ be two contours in the common domain of analyticity of $M(z;\mathbf{t})$ and such that $a(z;\mathbf{t})$ has no zeroes on either one and $\Sigma\subset\mathrm{Int}(\widetilde{\Sigma})$ (see Fig. 2).

Denote with $\tau_{(1)}$ and $\widetilde{\tau}_{(1)}$ the corresponding Fredholm determinants of the operators defined as described; following the same steps as above. Our goal is to show that

Theorem 3.3. *The ratio of the two Fredholm determinants is given by*

$$\widetilde{\tau}_{(1)}=\tau_{(1)}\prod_{\substack{v\in\mathrm{Int}(\widetilde{\Sigma})\cap\mathrm{Ext}(\Sigma):\\ a(v)=0}}(c(v(\mathbf{t});\mathbf{t}))^{-\mathrm{ord}_v(a)}.$$

Note that the evaluation of c at the zeroes of a cannot vanish because $\det M\equiv 1$.

Proof. From the formula (3.6) we get

$$\partial\ln\frac{\widetilde{\tau}_{(1)}}{\tau_{(1)}}=\left(\oint_{\widetilde{\Sigma}}-\oint_{\Sigma}\right)\mathrm{Tr}\left(\Gamma_-^{-1}\Gamma_-'\partial MM^{-1}\right)\frac{\mathrm{d}z}{2i\pi}$$
$$+\left(\oint_{\widetilde{\Sigma}}-\oint_{\Sigma}\right)\left(\frac{(a)'}{a}(d\partial a-c\partial b)+ac'\partial\left(\frac{b}{a}\right)\right)\frac{\mathrm{d}z}{2i\pi}.$$

Here Γ_- means the analytic extension of the solution Γ to the region $\mathrm{Ext}(\Sigma)\cup N(\Sigma)$. Since the integrand of the first term is holomorphic in the region bounded by $\widetilde{\Sigma}$ and Σ, it yields a zero contribution by the Cauchy's theorem and we are left only with the second term, which is computable by the residue theorem;

$$\partial\ln\frac{\widetilde{\tau}_{(1)}}{\tau_{(1)}}=\sum_{\substack{v\in\mathrm{Int}(\widetilde{\Sigma})\cap\mathrm{Ext}(\Sigma):\\ a(v)=0}}\operatorname*{res}_{z=v}\left(\frac{(a)'}{a}(d\partial a-c\partial b)+c'a\partial\left(\frac{b}{a}\right)\right).$$

We are assuming that $M(z;\mathbf{t})$ is analytic and also that $\det M\equiv 1$. Now note that a zero $v(\mathbf{t})$ of $a(z;\mathbf{t})$ in general depends on $\mathbf{t}$; suppose that $a(z;\mathbf{t})=(z-v(\mathbf{t}))^k(C_k(\mathbf{t})+\mathcal{O}(z-v(\mathbf{t})))$; then

$$\frac{\partial a(z;\mathbf{t})}{a(z;\mathbf{t})}=-\frac{k\partial v(\mathbf{t})}{z-v(\mathbf{t})}+\mathcal{O}(1),$$

and hence the residue evaluation at $z=v(\mathbf{t})$ yields (we use $0=a'\partial v+\partial a|_v$ and $\partial(ad)|_v=d\partial a|_v=\partial(bc)|_v$)

$$\operatorname*{res}_{z=v}\left(\frac{(a)'}{a}(d\partial a-c\partial b)+ac'\partial\left(\frac{b}{a}\right)\right)=k\left(-da'\big|_v\partial v-c\partial b\big|_v\right)+kbc'\partial v$$
$$=k\left(d\partial a\big|_v-c\partial b\big|_v\right)+kbc'\partial v=k\left(\partial(bc)\big|_v-c\partial b\big|_v\right)+kbc'\partial v$$
$$=kb\partial c\big|_v+kbc'\big|_v\partial v\overset{bc|_v=-1}{=}-k\partial\ln c(v(\mathbf{t});\mathbf{t})$$

(note that $c(v(\mathbf{t});\mathbf{t})$ cannot be zero because $\det M\equiv 1$ and $z=v$ is already a zero of a). ■

3.1.2 Determinants for different factorizations

The tau functions (Fredholm determinants) defined thus far should be understood as defining a section of a line bundle over the *loop group* space; this is the line bundle associated to the two fom $\delta\omega_M$ (1.4).

This simply means that on the intersection of the open sets where the factorizations (1), (2) in (3.1) can be made, we have

$$\delta \ln \frac{\tau_{(1)}}{\tau_{(2)}} = \delta \ln \left(\Upsilon_{(1,2)}\right),$$

where

$$\delta \ln \left(\Upsilon_{(1,2)}\right) := \delta\left(\theta_{(1)} - \theta_{(2)}\right).$$

After a short computation we obtain

$$\delta \ln \left(\Upsilon_{(1,2)}\right) = \oint_\Sigma \left(\frac{c'\delta b - b'\delta c}{1+bc}\right)\frac{\mathrm{d}z}{2i\pi} = \oint_\Sigma \left(\delta \ln b \frac{\mathrm{d}}{\mathrm{d}z}\ln(ad) - \delta\ln(ad)\frac{\mathrm{d}}{\mathrm{d}z}\ln b\right)\frac{\mathrm{d}z}{2i\pi}.$$

Observe the last expression; in principle the functions $b(z)$, $a(z)d(z)$ may have nonzero index around Σ, but in any case the functions $\delta \ln b$ and $(\ln(ad))'$ are single-valued because the increments of the logarithms are integer multiples of $2i\pi$ and hence locally constant in the space of deformations. To write explicitly the transition function (or rather a representative of the same cocycle class) we need to choose a point $z_0 \in \mathrm{Int}(\Sigma)$; we choose $z = 0$ without loss of generality. Let $K = \mathrm{ind}_\Sigma b$, $L = \mathrm{ind}_\Sigma(ad)$; then we can rewrite the above expression as follows

$$\begin{aligned}\delta \ln \left(\Upsilon_{(1,2)}\right) = \oint_\Sigma \Bigg[&\delta \ln\left(\frac{b}{z^K}\right)\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{ad}{z^L}\right) - \delta\ln\left(\frac{ad}{z^L}\right)\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{b}{z^K}\right)\\ &- \delta\ln\left(\frac{ad}{z^L}\right)\frac{K}{z} + \delta\ln\left(\frac{b}{z^K}\right)\frac{L}{z}\Bigg]\frac{\mathrm{d}z}{2i\pi},\end{aligned}$$

and after integration by parts (which is now possible since all functions involved are single-valued in $N(\Sigma)$)

$$\delta \ln \left(\Upsilon_{(1,2)}\right) = \delta \oint_\Sigma \left[\ln\left(\frac{b}{z^K}\right)\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{ad}{z^L}\right) - \ln\left(\frac{ad}{z^L}\right)\frac{K}{z} + \ln\left(\frac{b}{z^K}\right)\frac{L}{z}\right]\frac{\mathrm{d}z}{2i\pi},$$

so that the transition function admits the explicit expression

$$\Upsilon_{(1,2)} = \exp\left[\oint_\Sigma \left[\ln\left(\frac{b}{z^K}\right)\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{ad}{z^L}\right) - \ln\left(\frac{ad}{z^L}\right)\frac{K}{z} + \ln\left(\frac{b}{z^K}\right)\frac{L}{z}\right]\frac{\mathrm{d}z}{2i\pi}\right].$$

4 Conclusion

We conclude this short note with a few comments.

First of all the choice of Σ as a single closed contour is not necessary; we can have a disjoint union of closed contours or an unbounded contour as long as $M(z)$ converges to the identity sufficiently fast as $|z| \to \infty$ or near the endpoints. The considerations extend with trivial modifications. Our approach is similar in spirit to the approach used in [9] to express the tau function of a general isomonodromic system with Fuchsian singularities in terms of an appropriate Fredholm determinant.

Much less clear to the writer is how to handle the case where Σ contains intersections; in this case we should stipulate a local "no-monodromy" condition at the intersection points as explained in [2, 3]. The obstacle is not the issue of factorization but the fact that the resulting RHP of the IIKS type leads to an operator K (2.3) which is not of trace-class (and not even Hilbert–Schmidt, which would be sufficient in order to construct a Hilbert–Carleman determinant).

Nonetheless, the two form $\delta\omega_M$ defines a line bundle as explained and therefore it is possible to compute the "transition functions" of the line-bundle; this is precisely what is accomplished (in different setting and in different terminology) in recent works [11, 12] and the transition functions can be expressed in terms of explicit expressions, analogously to what we have shown here in this general but simplified setting.

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