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Doctoral Thesis

Elliptic Genera in Gauged Linear Sigma Models

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to the memory
of my mother
Biba

List of Publications

This thesis is based on the following publications:

- Matteo Poggi. "Elliptic genus derivation of 4d holomorphic blocks". In: *Journal of High Energy Physics* 2018.3 (2018), pp. 1–16. arXiv: 1711.07499 [hep-th];
- Francesco Benini, Giulio Bonelli, Matteo Poggi, and Alessandro Tanzini. *Elliptic non-Abelian Donaldson-Thomas invariants of* $\mathbb{C}^3$. 2018. arXiv: 1807.08482 [hep-th].

Contents

1. Introduction

According to an old adage everything in Nature is, up to certain approximation, a harmonic oscillator. A curious person that wants to uncover some secrets of Nature, has therefore to learn how to deal with it[1]. Indeed, during the past century, theoretical particle physicists succeeded in mastering this fundamental ingredient and, building perturbation theory around it, they were able to formulate, in the context of Quantum Field Theory (QFT), the Standard Model (SM), which is one of the greatest scientific successes up to now because of its precise agreement with experimental data. Nonetheless SM cannot be the end of the exploration of particle physics, since there are some unexplained phenomena such as confinement, charge screening and the existence of a mass gap in quantum chromodynamics (QCD), that are genuinely non-perturbative and, despite being foundational for experimental observations, cannot be tackled with perturbation theory. More generally, especially[2] in dimension $d > 2$, the strongly coupled regime of a generic QFT, that is when the perturbation theory become unreliable, still lacks a satisfactory treatment.

The difficulties in studying the non-perturbative aspects of a generic QFT lead theorists to consider some special models enjoying supersymmetry (SUSY). Besides the potentially phenomenological interest in fact, SUSY turned out to be a sort of powerful "theoretical laboratory" in which one can learn many properties about strongly coupled systems and, hopefully, use the hints taken from them to try to have a better understanding of the non-SUSY cases. The fact that SUSY models are much easier to study than the non-SUSY ones relies on the rigid structure that this kind of symmetry impose: among all the possible couplings that a non-SUSY theory may have, this symmetry selects the "canonical" ones[3]. A point of strength of SUSY theories is the presence of Bogomol'nyi–Prasad–Sommerfield (BPS) operators: they accommodate in representations of SUSY algebra that are shorter than those of non-BPS operators; therefore, along the renormalization group flow, they do not mix with non-BPS operators. Moreover they are highly constrained by the SUSY algebra itself and their quantum corrections are restricted; these restrictions make feasible computations involving BPS operators. The general principle is the following: the more supersymmetric the theory is, the more analytic control we have of its quantities of interest. The price to pay is to restrict our study to less and less generic cases. For instance, we know that for a 4d $\mathcal{N} = 1$ non-linear sigma model (NLSM) all the information of our model is contained in the Kähler potential and in the superpotential. In line with [7, 8], a renowned result in the context of $\mathcal{N} = 1$ SQCD is the duality proposed by Seiberg [9].

[1]This is almost what S. Coleman said: http://media.physics.harvard.edu/video/?id=Sidney_Coleman_lecture_02.

[2]For 2d QFT, non-perturbative results are known since [3]. More example are subject matter of textbooks [4, 5].

[3]For instance in gauged sigma models every coupling is a geometrical object; the SUSY models have couplings which are described by a canonical choice of these objects [6].

Increasing the amount of SUSY to $\mathcal{N} = 2$, it has been found that one just needs a holomorphic function, called *prepotential*, to fully specify the NLSM. Seiberg and Witten [9, 10] were able to determine this superpotential for low energy but for all coupling. A microscopic derivation of such prepotential by means of instanton counting has been performed by Nekrasov [11] using *equivariant localization*; the semiclassical limit of Nekrasov partition function was found in [12]. For the interested reader we recommend also [13, 14] which treat the Nekrasov partition function from different points of view. Going further one arrives, for instance, at the celebrated $\mathcal{N} = 4$ Yang-Mills (SYM) which is believed to be superconformal and integrable and has been extensively studied in literature. In particular, SYM on S^4 [15] has been the first instance of application of *supersymmetric localization*: with this technique Pestun was able to compute partition function and expectation values of SUSY circular Wilson loops early conjectured in [16].

Supersymmetric localization is an infinite-dimensional version of equivariant localization. The rough idea of the latter is the following: if we have an ordinary (finite dimensional) integral over a manifold which is acted on by a Lie group, under certain conditions, the result of the integral receives contributions only from the fixed points of the action (the localization locus), therefore, instead of performing the integral over all the manifold, we have just to evaluate some function of the integrand at this points. These conditions were known by mathematicians since 80's; the main results are theorems by Duistermaat–Heckman [17], Berline–Vergne [18, 19] and Atiyah–Bott [20]. Successively the ideas behind these results were applied by Witten to Path Integral (PI) of SUSY field theories, in the context of topological ones [21, 22]. This is because, at that time, the only way to implement SUSY on a curved manifold was through topological twist[4]. A good review of these techniques is [23]. After two decades, starting from [15], localization enjoyed a revival since it was understood that a larger class of SUSY theories can be put on curved manifold without performing the topological twist. The power of SUSY localization relies in the fact that the PI of a SUSY theory is invariant under a certain class of deformations of the action and this allows us to deform it in the way it is more convenient to compute it *exactly*. Nonetheless, sometimes, to find the suitable deformation can require much effort; this is perhaps why, in the common lore, localization is regarded as an "art" rather than as a technique. Another important feature is that, once the localization locus has been found, the expectation value of any BPS operator[5] is easily obtained: we have just to evaluate this operator on the localization locus. Further progress were made after some years when a systematic way to build rigid SUSY on a given manifold was found [24] exploiting supergravity (SUGRA). Modern and compact introductions to the subject are [25, 26]; a less compact review is [27].

In all these either weakly- or strongly-coupled tales, a central rôle is played by the PI. Actually, it is possible to use it to spot another very interesting effect that will enter on the game: in general the space of fields configurations on which we path-integrate can have a non-trivial topology and therefore can be divided in sectors labeled by some discrete numbers. Thus, when one path-integrates, he has to sum over all these sectors. A very famous instance of this circumstance dates back to the 50's and is the Aharonov–Bohm effect. In the context of YM theory [28], the states we are summing over are called *instantons*. These, together

[4] Compactness of the manifold is an important ingredient: it is needed, for instance, to cure infra-red divergences of observables.

[5] Actually the SUSY charge that annihilates the inserted operator must be a subset of the localization.

with their lower-codimensional versions and with situations in which the configuration space has a non-trivial topology (vortices, monopoles, solitons, kinks, domain walls), are purely non-perturbative phenomena.

Instantons and instanton counting (for good review we recommend [29]) have been an important contact point between Physics and Mathematics: we recall the famous ADHM construction [30], from the mathematical side and the interpretation in terms of arrangements of D-brane as the physical counterpart [31–33]. Also important in this context is the work by Donaldson [34] in which he start the study of the topology of low-dimensional manifolds using non-linear classical field theory. Thus we see that mathematicians began to use physical ideas to solve their problems; in this direction, the introduction of SUSY and therefore of its mathematical structures, makes the subject of growing interest for the mathematical community: very soon several standard tools of QFT such as PI became of common use also among some mathematicians and, on the other hand, physicists started to use their native techniques to tackle mathematical problems. As a result the interplay between these two subjects became stronger and stronger[6]. A basic example of this interplay is the so-called "supersymmetric proof of the Atiyah–Singer theorem" [36]. Then Witten[7] began the exploration of Topological Quantum Field Theory using SUSY QFT to rederive Donaldson invariants. Successively, these invariants were extended to higher dimensional manifolds [37]: they are the *Donaldson–Thomas* (DT) invariants. From the string theory point of view they can be computed exploiting a brane construction [38, 39].

Also the study of brane dynamics has revealed, over the years, to be a constant source of delightful results both in Physics and Mathematics. It offers valuable insights into the non-perturbative dynamics of gauge and string theories, and it displays deep connections with enumerative geometry via BPS bound-state counting. Often brane systems provide a string theory realization of interesting moduli spaces, and SUSY localization allows us to perform the exact counting of BPS states in a variety of them. This philosophy has been applied successfully in many contexts. For instance, the S^2 partition functions [40, 41] of GLSMs capture geometric properties of the moduli spaces of genus-zero pseudo-holomorphic maps to the target, and represent a convenient way to extract Gromov–Witten invariants [42]. They show that suitable coordinates enjoy mutations of cluster algebras [43], as physically suggested by IR dualities [44]. Exact S^2 partition functions have been exploited in the study of D1/D5 brane systems in [45, 46] providing a direct link between quantum cohomology of Nakajima quiver varieties, quantum integrable systems of hydrodynamical type and higher-rank equivariant DT invariants $\mathbb{P}^1 \times \mathbb{C}^2$ [47, 48]. As another example, certain equivariant K-theories of vortex moduli spaces are conveniently captured by a twisted 3d index [49, 50]. Such an object is intimately related to black hole entropy in AdS_4 [51, 52], thus providing a sort of generalization of Gopakumar-Vafa invariants [53].

Let us now sketch how all the ingredients we introduced go together in this thesis. We

[6] The possible bafflement of both kinds of scientists is well expressed in the artwork on the cover of [35] due to Dijkgraaf.

[7] Once Atiyah said of Witten: "*Although he is definitely a physicist (as his list of publications clearly shows) his command of mathematics is rivaled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems* [...] *He has made a profound impact on contemporary mathematics. In his hands physics is once again providing a rich source of inspiration and insight in mathematics.*"

start with a certain arrangement of branes, say N Dq branes and k Dp branes[8] with $p < q$. Packing all the Dq branes together, their worldvolume is described, at low energy, by a U(N) SYM in $q+1$ dimensions, with a certain amount of SUSY. The presence of the Dp branes has various effects: first, thanks to it, one can consider three types of strings: Dq-Dq, Dq-Dp and Dp-Dp; however if we take the directions of the Dq's transverse to the Dp's not compactified, the Dq branes are much heavier, so we can neglect the effect of the Dq-Dq strings. Second, it lowers the amount of SUSY dependently on $q-p$, however it is possible, turning on a B-field along the Dq branes, to preserve some supercharges (see [54]). Third, scalars fields in the theory represent the fluctuations of the Dp branes in directions parallel to the Dq's and transverse to them. Correspondingly these scalars, together with their fermionic partners, fit in multiplets of SUSY in p dimensions in some representation of U(k). From the point of view of the Dp branes, we remain with a U(k) GLSM in $p+1$ dimensions which describes their dynamics. In particular, considering the Higgs branch of the moduli space of this model (this means that the Dp branes sit on the Dq's) one can recover the ADMH-like equations describing the non-perturbative object. Moreover, a careful analysis of the worldvolume theory of the Dq branes, allows us to compute quantum number of the instantonic objects under consideration. The reader interested in brane construction of non-perturbative objects can see [31–33] for instantons, [55] for monopoles, [56] for vortices and [57, 58]. All them are reviewed in [59]. The aforesaid GLSM, contains much geometrical information about the moduli space described by ADHM-like equations. We will focus on situation in which $p = 1$ and the two directions are compactified. The resulting GLSM will be defined on T^2 and will have a certain amount of SUSY. It is here that localization comes into the game: it is possible compute exactly the elliptic genus (see below) for these theories [60, 61]. This quantity, interesting both in Mathematics and in Physics, can be used to compute elliptic Donaldson–Thomas invariants.

Outline of the thesis. In chap. 2 we introduce two-dimensional SUSY in flat space. The first part is devoted to the theories with four supercharges, $\mathcal{N} = (2,2)$ which are dimensional reduction of the $\mathcal{N} = 1$ in 4d; in the second part we focus the theories with $\mathcal{N} = (0,2)$. In both cases we use a construction in terms of the appropriate superspace. In the first part, chiral and vector multiplet are build together with action for them (comprehending possible interaction terms). In the second part the same is done for chiral, Fermi and vector multiplets; moreover we also explain how it is possible to write $\mathcal{N} = (2,2)$ actions in terms of those with $\mathcal{N} = (0,2)$, which will be the main characters of the next chapter. All notation and conventions are specified in app. A.

In chap. 3 we introduce the *Elliptic Genus*, starting from mathematical definition and underlining that it can be thought as a generalization of Witten index. Then we introduce its "physical" definition, in terms of a partition function with the insertion of some operators. It is essentially an elliptic version of the Witten index refined with some chemical potentials associated to some flavor symmetries (or also R-symmetry in the $\mathcal{N} = (2,2)$ case). In the second part of the chapter, we introduce briefly supersymmetric localization and we apply this technique to compute the elliptic genus of a theory with $\mathcal{N} = (0,2)$ SUSY with chiral, Fermi and vector multiplet in arbitrary representation of the rank-one gauge group. This has been done for the first time in [60] where a detailed evaluation was made for $\mathcal{N} = (2,2)$. Here we spell out the details of the computation for the case $\mathcal{N} = (0,2)$, also using a shortcut that

[8] Of course if $p = q \mod 2 = 0$ we are in a IIA setup while in the other case we are in the IIB.

appeared the first time in [50]. The result is roughly that given the gauge group and matter content of the theory one has to write a matrix model and then integrate it. This boils down to take the residue at some poles selected by a regulator. Then we state the result for gauge group with generic rank as appeared in [61]. The higher complexity of generic ranks reflects in the so-called *Jeffrey–Kirwan residue* [62], which was originally proposed in the context of equivariant localization for non-Abelian symplectic quotients. In our situation, we can consider it as a generalization of the regulator we found in the rank-one analysis. This regularization procedure has two cases: the non-degenerate or *regular* one, and the degenerate or *singular* one: while the former is simple to apply in concrete computations, the latter is more subtle. This is because, although an algorithm exists to treat this case, it happens to be not directly applicable in some instances of interest. This have lead us to develop a *desingularization algorithm* [2] aimed to reduce the singular cases to the regular ones, essentially engineering some perturbations of the poles of the integrand such that the final result does not depend on the perturbation parameters. This simplifies concrete computations very much.

In chap. 4, we introduce the $\mathcal{N} = (2,2)$ partition function on S^2 [40, 41] and its Higgs branch representation showing its factorization property: it can be written as a sum over the (discrete) Higgs vacua of a certain product of a "vortex" and an "anti-vortex" partition function weighted by semiclassical factors. This is a common feature even in higher dimension [63–65]: an intriguing aspect of these constructions is that the structure of the Riemannian manifold in patches reflects in factorization properties of the partition function in partial "blocks". In our case of interest is devoted to the "elliptic" vortices on $\mathbb{C} \times T^2$: this is done by considering the 2d quiver gauge theory describing their moduli spaces. The elliptic genus of these moduli space is the elliptic version of vortex partition function 4d theory. We focus on two examples: the first is a $\mathcal{N} = 1$, $\mathrm{U}(N)$ gauge theory with fundamental and anti-fundamental matter (its moduli space is described a $\mathcal{N} = (0,2)$ 2d model); the second is a $\mathcal{N} = 2$, $\mathrm{U}(N)$ gauge theory with matter in the fundamental representations (its moduli space is described a $\mathcal{N} = (0,2)$ 2d model). The results are instances of 4d "holomorphic blocks" generalizing or reproducing (from a first-principle computation) result in literature [40, 41, 66–71]. The computation is done following the recipe given in the previous chapter: we classify the poles of the matrix models (which turn out to be all regular), and we compute the residues. To do that we need a straightforward but lengthy classification of the charge matrix, which is relegated to the app. C. It is found that each contribution to the k-vortex PI is in one to one correspondence with a collection of N positive integers which sum to k. For the first example it is also possible to write a "grand-canonical" vortex partition function, summing the contributions for each k against a vorticity parameter v. The result is written in terms of elliptic hypergeometric functions, defined in app. B.

In chap. 5 we analyze the a system with N D7 branes and k D1 branes in which the extended dimensions of the D1's are compactified. As already mentioned, the effective dynamics of the D1's is captured by a $\mathcal{N} = (2,2)$ GLSM whose classical vacua describe the moduli space of rank-N sheaves on $\mathbb{C}^3$. In the first part we perform the computation of the elliptic genus of the moduli space of this GLSM for $N = 1$: first of all, we classify the poles (some of which turn out to be singular) and we prove, in app. C.3.2, that they arrange in the 3d generalization of Young diagram, called *plane partitions* (which are introduced in app C.3.1). Dimensional reductions to D0/D6 (*trigonometric*) and to D(−1)/D5 (*rational*) are also discussed: they correspond to the refined Witten index and the equivariant volume of the same moduli space. The last two cases were extensively studied (for $N = 1$), in view of their relation with black-hole entropy, microstate counting [72] and the DT invariants. The latter are in turn mapped to Gromov–Witten invariants by MNOP relation [38, 39]. These cases are of interest since it is possible to compare the results we get with our method with the ones appearing in literature. Less is known in the higher-rank case, except for the D(−1)/D5 system whose partition function was conjectured to factorize as the N^{th} power of the Abelian one [73, 74]. The second part of this chapter is devoted to the computation of the elliptic genus with general N. We also provide evidence for such a factorization conjecture, also in accordance the results of [75, 76]. On the other hand, we find that the elliptic genus and the generalized Witten index do not factorize and give new interesting results. In Proposition 5.1 of [77], a relation between the higher-rank equivariant K-theoretic DT invariants on a three-fold X and the M2-brane contribution to the M-theory index on a A_{N-1} surface fibration over X was established. A conjectural plethystic exponential[9] form for the equivariant K-theoretic DT invariants in higher rank was proposed in [73] for the case $X = \mathbb{C}^3$: we confirm this proposal. For rank one, the D0/D6 system on a circle is known to compute the eleven-dimensional supergravity index, which can indeed be expressed in an elegant plethystic exponential form [74]. We show that the same is true in the higher-rank case. In fact, extending the construction of [78], the M-theory lift of the D0/D6 system in the presence of an Omega background is given by a $\text{TN}_N \times \mathbb{C}^3$ fibration over a circle [74], where TN_N is a multi-center Taub-NUT space and whose charge N equals the number of D6-branes. The fibration is such that the fiber space is rotated by a $U(1)^3$ action as we go around the circle. The multi-center Taub-NUT space looks asymptotically as a lens space $S^3/\mathbb{Z}_N \times \mathbb{R}^+$, precisely as the asymptotic behavior of the A_{N-1} surface singularity $\mathbb{C}^2/\mathbb{Z}_N$. This implies the appearance in the higher-rank index of twisted sectors carrying irreducible representations of the cyclic group, which spoils the factorization property. In the elliptic case, describing the D1/D7 system, a novelty appears: because of anomalies in the path integral measure, there are non-trivial constraints on the fugacities of the corresponding symmetries. Once these constraints are taken into account, the higher-rank elliptic index takes a particularly simple form, which can be traced back to a suitable geometric lift to F-theory [79]. In the final part of this chapter we propose a realization of the elliptic genus as a chiral correlator of free fields on the torus generalizing the construction of [80], with the aim of exploring the underlying integrable structure in the spirit of BPS/CFT correspondence [81].

[9] We define this operation in app. C.2.

2. Supersymmetry in two dimensions

In this chapter we will introduce two dimensional supersymmetric theories on Euclidean flat space. In particular we will review how to build the action of $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (0, 2)$ super Yang–Mills (SYM) with matter which be one of the main ingredients in the following. We will begin our exposition with $\mathcal{N} = (2, 2)$ supersymmetry (SUSY) and then we will switch to $\mathcal{N} = (0, 2)$.

2.1 $\mathcal{N} = (2, 2)$ Supersymmetry

There are essentially two ways in which this kind of supersymmetry can be introduced: either one can exploit the fact that two dimensional $\mathcal{N} = (2, 2)$ is the dimensional reduction of four dimensional $\mathcal{N} = 1$ (see for instance [**82**]) or it is possible to build it from scratch (as in [**83**]). We will use this second approach since we want to be as self-contained as possible.

2.1.1 Superspace and Superfields

Let us take $\mathbb{C}$ as our two dimensional space having coordinates z and $\bar{z}$. We introduce four fermionic (anticommuting) coordinates organized in two spinors (see app. A for the conventions)

$$\theta = \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix} , \qquad \bar{\theta} = \begin{pmatrix} \bar{\theta}^+ \\ \bar{\theta}^- \end{pmatrix} . \tag{2.1}$$

We will use $\tilde{\theta}$ to refer collectively to θ and $\bar{\theta}$. Under a translation of parameters $(\zeta, \bar{\zeta})$ we have that

$$z \mapsto z + \zeta , \qquad \bar{z} \mapsto \bar{z} + \bar{\zeta} , \qquad \theta \mapsto \theta , \qquad \bar{\theta} \mapsto \bar{\theta} , \tag{2.2}$$

while under a $\mathrm{U}(1)$ rotation of parameter φ

$$z \mapsto e^{\mathrm{i}\varphi} z , \qquad \bar{z} \mapsto e^{-\mathrm{i}\varphi} \bar{z} , \qquad \theta \mapsto e^{-\frac{\mathrm{i}}{2}\gamma_3} \theta , \qquad \bar{\theta} \mapsto e^{-\frac{\mathrm{i}}{2}\gamma_3} \bar{\theta} . \tag{2.3}$$

We introduce two differential operators acting on superspace:

$$\mathsf{Q}_a = \boldsymbol{\partial}_a - \tfrac{\mathrm{i}}{2}(\gamma\bar{\theta})_a \cdot \partial , \qquad \bar{\mathsf{Q}}_a = \bar{\boldsymbol{\partial}}_a - \tfrac{\mathrm{i}}{2}(\gamma\theta)_a \cdot \partial , \tag{2.4}$$

where we denoted $\boldsymbol{\partial}_a = \frac{\partial}{\partial\theta^a}$ and $\bar{\boldsymbol{\partial}} = \frac{\partial}{\partial\bar{\theta}^a}$. They close the algebra

$$\{\mathsf{Q}_a, \mathsf{Q}_b\} = 0 , \qquad \{\mathsf{Q}_a, \bar{\mathsf{Q}}_b\} = -\mathrm{i}(\gamma)_{ab} \cdot \partial , \qquad \{\bar{\mathsf{Q}}_a, \bar{\mathsf{Q}}_b\} = 0 , \tag{2.5}$$

which is a typical SUSY algebra. A function of the superspace is called *superfield*: since the fermionic variable are nilpotent, the Taylor expansion of a generic superfield in monomials of θ and $\bar{\theta}$ is finite. The most general scalar function one can write is the following:

$$\boldsymbol{\mathcal{B}}(\tilde{\theta}, z, \bar{z}) = a + \theta\chi + \bar{\theta}\xi + \theta\theta M + \bar{\theta}\bar{\theta}N + \theta\gamma\bar{\theta}\cdot v - \mathrm{i}\theta \mathrm{P}_{-}\bar{\theta}\sigma - \mathrm{i}\theta\mathrm{P}_{+}\bar{\theta}\eta + \\ + \tfrac{\mathrm{i}}{2}\theta\theta\bar{\theta}(\psi + \tfrac{1}{2}\gamma\cdot\partial\chi) + \tfrac{\mathrm{i}}{2}\bar{\theta}\bar{\theta}\theta(\lambda + \tfrac{1}{2}\gamma\cdot\partial\xi) - \tfrac{\mathrm{i}}{4}\theta\theta\bar{\theta}\bar{\theta}(D - \mathrm{i}\partial\bar{\partial}\varphi) \ , \quad (2.6)$$

where every component fields depends on z and $\bar{z}$. We see that we have 8 complex bosonic d.o.f. (a, M, N, v, $\bar{v}$ and σ, η and D) as well as 8 complex fermionic d.o.f. (χ, ξ, ψ and λ). It is also possible to write a superfield carrying some spin: in general we say that a superfield is bosonic if $[\tilde{\theta}, \boldsymbol{\mathcal{F}}] = 0$ (i.e. if its bottom component is a boson) and fermionic if $\{\tilde{\theta}, \boldsymbol{\mathcal{F}}\} = 0$ (i.e. if its bottom component is a fermion). An important tool concerning SUSY algebra is the group of its outer automorphism: in the present case we have $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ (resp. left- and right-moving R-charges). We assign the following charges

$$\begin{aligned} &\mathrm{R}_R[\theta^+] = +1 \ , && \mathrm{R}_R[\theta^-] = 0 \ , && \mathrm{R}_R[\bar{\theta}^+] = -1 \ , && \mathrm{R}_R[\bar{\theta}^-] = 0 \ , \\ &\mathrm{R}_L[\theta^+] = 0 \ , && \mathrm{R}_L[\theta^-] = +1 \ , && \mathrm{R}_L[\bar{\theta}^+] = 0 \ , && \mathrm{R}_L[\bar{\theta}^-] = -1 \ . \end{aligned} \quad (2.7)$$

Once fixed $\mathrm{R}_R[\boldsymbol{\mathcal{F}}]$ and $\mathrm{R}_L[\boldsymbol{\mathcal{F}}]$ the respective charge of the components fields follows. Denoting the generators of the R-symmetry by R_R and R_L, from eq. (2.5) follows

$$\begin{aligned} &[\mathsf{R}_R, \mathsf{Q}_+] = -\mathsf{Q}_+ \ , && [\mathsf{R}_R, \mathsf{Q}_-] = 0 \ , && [\mathsf{R}_R, \bar{\mathsf{Q}}_+] = -\bar{\mathsf{Q}}_+ \ , && [\mathsf{R}_R, \bar{\mathsf{Q}}_-] = 0 \ , \\ &[\mathsf{R}_L, \mathsf{Q}_+] = 0 \ , && [\mathsf{R}_L, \mathsf{Q}_-] = +\mathsf{Q}_- \ , && [\mathsf{R}_L, \bar{\mathsf{Q}}_+] = 0 \ , && [\mathsf{R}_L, \bar{\mathsf{Q}}_-] = +\bar{\mathsf{Q}}_- \ . \end{aligned} \quad (2.8)$$

A extensive treatment of superspace formalism in two dimensions is carried out in [84]. Now we will discuss some class of multiplets which will be the building block of our computations in the next chapters. Of course the list is not exhaustive: the interested reader can see [85].

2.1.2 *Chiral Superfield*

Let us introduce the following derivative operators:

$$\mathsf{D}_a = \boldsymbol{\partial}_a + \tfrac{\mathrm{i}}{2}(\gamma\bar{\theta})_a \cdot \partial \ , \qquad \bar{\mathsf{D}}_a = \bar{\boldsymbol{\partial}} + \tfrac{\mathrm{i}}{2}(\gamma\theta)_a \cdot \partial \ . \quad (2.9)$$

They anticommute with the supercharges Q and $\bar{\mathsf{Q}}$ and satisfy the algebra

$$\{\mathsf{D}_a, \mathsf{D}_b\} = 0 \ , \qquad \{\mathsf{D}_a, \bar{\mathsf{D}}_b\} = \mathrm{i}(\gamma)_{ab}\cdot\partial \ , \qquad \{\bar{\mathsf{D}}_a, \bar{\mathsf{D}}_b\} = 0 \ . \quad (2.10)$$

We define a *chiral superfield* with

$$\bar{\mathsf{D}}_a\boldsymbol{\Phi} = 0 \ , \quad (2.11)$$

plugging the general expression for a superfield eq. (2.6) and solving for the constraint, we have the following expansion

$$\boldsymbol{\Phi} = \varphi + \theta\psi + \tfrac{1}{2}\theta\theta F + \tfrac{\mathrm{i}}{2}\theta\gamma\bar{\theta}\cdot\partial\varphi + \tfrac{\mathrm{i}}{4}\theta\theta\bar{\theta}\gamma\cdot\partial\psi - \tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial\bar{\partial}\varphi \ , \quad (2.12)$$

where numerical factor has been set for future convenience. Notice that we have 2 complex bosonic d.o.f. (φ and F) as well as 2 fermionic d.o.f. (ψ). To get the SUSY variation of the component fields we have just to act with $\boldsymbol{\delta} = \epsilon\mathsf{Q} + \bar{\epsilon}\bar{\mathsf{Q}}$ on $\boldsymbol{\Phi}$. Since[1] $\{\tilde{\mathsf{D}}, \tilde{\mathsf{Q}}\} = 0$, $\boldsymbol{\delta\Phi}$ will be again a chiral superfield: reading its components we find

$$\delta\varphi = \epsilon\psi \ ,$$

[1] Again we use tilde sign to denote barred and unbarred operators collectively.

$$\begin{aligned}\delta\psi^a &= \epsilon^a F + \mathrm{i}(\gamma\bar{\epsilon})^a \cdot \partial\varphi \, , \\ \delta F &= \mathrm{i}\bar{\epsilon}\gamma \cdot \partial\psi \, . \end{aligned} \tag{2.13}$$

The anti-chiral superfield $\bar{\mathbf{\Phi}}$ (defined by $\mathsf{D}_a\bar{\Phi} = 0$) and its variation are obtained simply swapping barred with unbarred terms and vice-versa. An action for the chiral superfield is

$$S_{\mathbf{\Phi}} = \int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\bar{\mathbf{\Phi}}\mathbf{\Phi} = \int \mathrm{d}z\mathrm{d}\bar{z}\left(4\bar{\partial}\bar{\varphi}\partial\varphi - \mathrm{i}\bar{\psi}\gamma\cdot\partial\psi + \bar{F}F + \text{total derivatives}\right) \, . \tag{2.14}$$

This action is manifestly SUSY invariant: one can verify using variations (2.13), however, since Berezin integration is equal to a derivative, a function integrated in all $\tilde{\theta}$ variables is automatically SUSY invariant. That kind of term is called *D-term*. From this action we see that F does not have a dynamics: for this reason it is called *auxiliary field*. As a general remark we want to stress that any D-term must be uncharged under left and right moving R-symmetries since we integrate it against both θ and $\bar{\theta}$. If we set $\mathrm{R}_+[\mathbf{\Phi}] = r_R$ and $\mathrm{R}_L[\mathbf{\Phi}] = r_L$, from eqs. (2.7) and (2.12) we have the following assignment

Field	R_R	R_L
φ	r_R	r_L
$\bar{\varphi}$	$-r_R$	$-r_L$
ψ^+	r_R	$r_L - 1$
ψ^-	$r_R - 1$	r_L
$\bar{\psi}^+$	$-r_R$	$1 - r_L$
$\bar{\psi}^-$	$1 - r_R$	$-r_L$
F	$r_R - 1$	$r_L - 1$
$\bar{F}$	$1 - r_R$	$1 - r_L$

Table 2.1: R_R and R_L charge of component field of chiral superfield.

2.1.3 Vector Superfield

Let G be a gauge group with Lie algebra $\mathfrak{g}$. We can consider a $\mathfrak{g}$-valued superfield $\mathbf{V}$ obeying the condition

$$\mathbf{V}^\dagger = \mathbf{V} \, , \tag{2.15}$$

where "†" is a certain antilinear operator. If we were in the Minkowski spacetime it would be the usual complex conjugation, in Euclidean signature we can take a "Wick rotated version" of the former: it will simply exchange barred symbols with unbarred one. We can expand such a superfield as

$$\begin{aligned}\mathbf{V} = a &+ \theta\chi + \bar{\theta}\bar{\chi} + \theta\theta M + \bar{\theta}\bar{\theta}\bar{M} + \theta\gamma\bar{\theta}\cdot A - \mathrm{i}\theta\mathrm{P}_{-}\bar{\theta}\sigma - \mathrm{i}\theta\mathrm{P}_{+}\bar{\theta}\bar{\sigma} + \\ &+ \tfrac{\mathrm{i}}{2}\theta\theta\bar{\theta}(\bar{\lambda} + \tfrac{1}{2}\gamma\cdot\partial\chi) + \tfrac{\mathrm{i}}{2}\bar{\theta}\bar{\theta}\theta(\lambda + \tfrac{1}{2}\gamma\cdot\partial\bar{\chi}) - \tfrac{\mathrm{i}}{4}\theta\theta\bar{\theta}\bar{\theta}(D - \mathrm{i}\partial\bar{\partial}a) \, . \end{aligned} \tag{2.16}$$

We impose the super-gauge transformation of the vector superfield as

$$e^{\mathbf{V}} \mapsto e^{\mathrm{i}\bar{\mathbf{\Omega}}} e^{\mathbf{V}} e^{-\mathrm{i}\mathbf{\Omega}} \, , \tag{2.17}$$

where $\mathbf{\Omega}$ is a chiral superfield of components

$$\mathbf{\Omega} = \omega + \theta\xi + \tfrac{1}{2}\theta\theta w + \tfrac{\mathrm{i}}{2}\theta\gamma\bar{\theta}\cdot\partial\omega + \tfrac{\mathrm{i}}{4}\theta\theta\bar{\theta}\gamma\cdot\partial\xi - \tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial\bar{\partial}\omega\ . \tag{2.18}$$

We see that eq. (2.17) is highly non-linear when G is non-abelian, in fact using BCH formula we find that

$$\mathbf{V} \mapsto \mathbf{V} - \mathrm{i}(\mathbf{\Omega}-\bar{\mathbf{\Omega}}) - \tfrac{\mathrm{i}}{2}[\mathbf{V},\mathbf{\Omega}+\bar{\mathbf{\Omega}}] - \tfrac{\mathrm{i}}{12}[\mathbf{V},[\mathbf{V},\mathbf{\Omega}-\bar{\mathbf{\Omega}}]] + O(\mathbf{\Omega}^2), +O(\bar{\mathbf{\Omega}}^2) + O(\mathbf{V}^3)\ . \tag{2.19}$$

However we see that at lowest order (which is the only one present if G is abelian), we can set $a = \chi = \bar{\chi} = M = \bar{M} = 0$. This is called Wess-Zumino (WZ) gauge[2]:

$$\mathbf{V}_{\mathrm{WZ}} = \theta\gamma\bar{\theta}\cdot A - \mathrm{i}\theta\mathrm{P}_{-}\bar{\theta}\sigma - \mathrm{i}\theta\mathrm{P}_{+}\bar{\theta}\bar{\sigma} + \tfrac{\mathrm{i}}{2}\theta\theta\bar{\theta}\bar{\lambda} + \tfrac{\mathrm{i}}{2}\bar{\theta}\bar{\theta}\theta\lambda - \tfrac{\mathrm{i}}{4}\theta\theta\bar{\theta}\bar{\theta}D\ , \tag{2.20}$$

this gauge has the remarkable property that $\mathbf{V}^n = 0$ for $n > 2$, this simplify computations very much; for instance in eq. (2.19) we can remove $O(\mathbf{V}^3)$. For this reason we will always use WZ gauge. A comparison with the four dimensional multiplet (in WZ gauge as well) shows us that here σ and $\bar{\sigma}$ appear: they come from the dimensional reduction of the four vector to the two vector. It is easy to see that a transformation governed by

$$\mathbf{\Omega}_{\mathrm{WZ}} = \omega + \tfrac{\mathrm{i}}{2}\theta\gamma\bar{\theta}\cdot\partial\omega - \tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial\bar{\partial}\omega\ , \qquad \omega\in\mathbb{R}\ , \tag{2.21}$$

parametrizes the residual gauge: i.e. if we $\mathbf{V}$ is in WZ, after such a transformation remains in WZ gauge. It acts infinitesimally on component field as

$$\begin{aligned} &A \mapsto A + \partial\omega - \mathrm{i}[A,\omega]\ , &&\lambda \mapsto \lambda - \mathrm{i}[\lambda,\omega]\ , &&\sigma \mapsto \sigma - \mathrm{i}[\sigma,\omega]\ , \\ &\bar{A} \mapsto \bar{A} + \bar{\partial}\omega - \mathrm{i}[\bar{A},\omega]\ , &&\bar{\lambda} \mapsto \bar{\lambda} - \mathrm{i}[\bar{\lambda},\omega]\ , &&\bar{\sigma} \mapsto \bar{\sigma} - \mathrm{i}[\bar{\sigma},\omega]\ . \end{aligned} \tag{2.22}$$

We see that transformations (2.22) are the transformation that we expect for the gauge field $\tilde{A}$ and for the bosonic and fermionic matter in the adjoint representation. Unfortunately, a SUSY transformation breaks WZ gauge:

$$\begin{aligned} \delta a &= 0\ , \\ \delta\chi^a &= (\gamma\bar{\epsilon})^a\cdot A - \mathrm{i}(\mathrm{P}_{-}\bar{\epsilon})^a\sigma - \mathrm{i}(\mathrm{P}_{+}\bar{\epsilon})^a\bar{\sigma}\ , \\ \delta\bar{\chi}^a &= -(\gamma\epsilon)^a\cdot A - \mathrm{i}(\mathrm{P}_{+}\epsilon)^a\sigma - \mathrm{i}(\mathrm{P}_{-}\epsilon)^a\bar{\sigma}\ , \\ \delta M &= \tfrac{\mathrm{i}}{2}\bar{\epsilon}\bar{\lambda}\ , \\ \delta\bar{M} &= \tfrac{\mathrm{i}}{2}\epsilon\lambda\ , \\ \delta A &= \tfrac{\mathrm{i}}{2}(\epsilon\gamma\bar{\lambda} - \bar{\epsilon}\gamma\lambda)\ , \\ \delta\bar{A} &= \tfrac{\mathrm{i}}{2}(\epsilon\bar{\gamma}\bar{\lambda} - \bar{\epsilon}\bar{\gamma}\lambda)\ , \\ \delta\sigma &= (\epsilon\mathrm{P}_{+}\bar{\lambda}) + (\bar{\epsilon}\mathrm{P}_{-}\lambda)\ , \\ \delta\bar{\sigma} &= (\epsilon\mathrm{P}_{-}\bar{\lambda}) + (\bar{\epsilon}P_{+}\lambda)\ , \\ \delta\lambda^a &= -\tfrac{1}{2}\epsilon^a\partial\cdot A + (\gamma_3\epsilon)^a F_{z\bar{z}} - \epsilon^a D + \tfrac{\mathrm{i}}{2}(\gamma\mathrm{P}_{+}\epsilon)^a\cdot\partial\sigma + \tfrac{\mathrm{i}}{2}(\gamma\mathrm{P}_{-}\epsilon)^a\cdot\partial\bar{\sigma}\ , \\ \delta\bar{\lambda}^a &= \tfrac{1}{2}\bar{\epsilon}^a\partial\cdot A - (\gamma_3\bar{\epsilon})^a F_{z\bar{z}} - \bar{\epsilon}^a D + \tfrac{\mathrm{i}}{2}(\gamma\mathrm{P}_{-}\bar{\epsilon})^a\cdot\partial\sigma + \tfrac{\mathrm{i}}{2}(\gamma\mathrm{P}_{+}\bar{\epsilon})^a\cdot\partial\bar{\sigma}\ , \\ \delta D &= -\tfrac{\mathrm{i}}{2}\epsilon\gamma\cdot\partial\bar{\lambda} - \tfrac{\mathrm{i}}{2}\bar{\epsilon}\gamma\cdot\partial\lambda\ , \end{aligned} \tag{2.23}$$

[2]It is possible to show that, at least perturbatively, one can find $\mathbf{\Omega}$ and $\bar{\mathbf{\Omega}}$ such that we can transform $\mathbf{V}$ to WZ gauge.

where we used $F_{z\bar{z}} = \partial\bar{A} - \bar{\partial}A$. In order to restore WZ gauge we can perform a compensating transformation, usually called de Witt–Freedman (dWF), parametrized by

$$-\,\mathrm{i}\boldsymbol{\Omega}_{\mathrm{dWF}} = -\theta(\gamma\bar{\epsilon}\cdot A - \mathrm{i}\mathrm{P}_{-}\bar{\epsilon}\sigma - \mathrm{i}\mathrm{P}_{+}\bar{\epsilon}\bar{\sigma}) - \tfrac{\mathrm{i}}{2}\theta\theta\bar{\epsilon}\bar{\lambda} - \\ - \tfrac{\mathrm{i}}{4}\theta\theta\bar{\theta}(\bar{\epsilon}\partial\cdot A + 2(\gamma_3\bar{\epsilon})F_{z\bar{z}} - \mathrm{i}(\gamma\mathrm{P}_{-}\bar{\epsilon})\cdot\partial\sigma - \mathrm{i}(\gamma\mathrm{P}_{+}\bar{\epsilon})\cdot\partial\bar{\sigma})\ . \quad (2.24)$$

Usually one calls the the total transformation (SUSY + dWF) *dressed SUSY* (which we denote by $\tilde{\delta}$): in this case, plugging (2.24) into (2.19) we get

$$\begin{aligned}
\tilde{\delta}A &= \tfrac{\mathrm{i}}{2}(\epsilon\gamma\bar{\lambda} - \bar{\epsilon}\gamma\lambda)\ ,\\
\tilde{\delta}\bar{A} &= \tfrac{\mathrm{i}}{2}(\epsilon\bar{\gamma}\bar{\lambda} - \bar{\epsilon}\bar{\gamma}\lambda)\ ,\\
\tilde{\delta}\sigma &= (\epsilon\mathrm{P}_{+}\bar{\lambda}) + (\bar{\epsilon}\mathrm{P}_{-}\lambda)\ ,\\
\tilde{\delta}\bar{\sigma} &= (\epsilon\mathrm{P}_{-}\bar{\lambda}) + (\bar{\epsilon}P_{+}\lambda)\ ,\\
\tilde{\delta}\lambda^a &= 2(\gamma_3\epsilon)^a\mathcal{F}_{z\bar{z}} - \epsilon^a D + \tfrac{\mathrm{i}}{2}(\mathrm{P}_{+} - \mathrm{P}_{-})\epsilon[\sigma,\bar{\sigma}] + \mathrm{i}(\gamma\mathrm{P}_{+}\epsilon)\cdot\boldsymbol{\mathcal{D}}\sigma + \mathrm{i}(\gamma\mathrm{P}_{-}\epsilon)^a\cdot\boldsymbol{\mathcal{D}}\bar{\sigma}\ ,\\
\tilde{\delta}\bar{\lambda}^a &= -2(\gamma_3\bar{\epsilon})^a\mathcal{F}_{z\bar{z}} - \bar{\epsilon}^a D + \tfrac{\mathrm{i}}{2}(\mathrm{P}_{+} - \mathrm{P}_{-})\bar{\epsilon}[\sigma,\bar{\sigma}] + \mathrm{i}(\gamma\mathrm{P}_{+}\bar{\epsilon})\cdot\boldsymbol{\mathcal{D}}\sigma + \mathrm{i}(\gamma\mathrm{P}_{-}\bar{\epsilon})\cdot\boldsymbol{\mathcal{D}}\bar{\sigma}\ ,\\
\tilde{\delta}D &= -\tfrac{\mathrm{i}}{2}(\epsilon\gamma\cdot\boldsymbol{\mathcal{D}}\bar{\lambda} + \bar{\epsilon}\gamma\cdot\boldsymbol{\mathcal{D}}\lambda) + \\
&\quad + \mathrm{i}([\lambda,\bar{\sigma}]\mathrm{P}_{+}\bar{\epsilon} + [\lambda,\sigma]\mathrm{P}_{-}\bar{\epsilon} - [\bar{\lambda},\bar{\sigma}]\mathrm{P}_{-}\epsilon - [\bar{\lambda},\sigma]\mathrm{P}_{+}\epsilon)\ ,
\end{aligned} \quad (2.25)$$

where we used $\tilde{\boldsymbol{\mathcal{D}}} = \tilde{\partial} + [\tilde{A},\cdot]$ and $\mathcal{F}_{z\bar{z}} = \partial\bar{A} - \bar{\partial}A - \mathrm{i}[A,\bar{A}]$. To conclude this subsection, from eqs. (2.7) and (2.20) we write

Field	R_R	R_L
$\tilde{A}$	0	0
λ^+	+1	0
λ^-	0	+1
$\bar{\lambda}^+$	−1	0
$\bar{\lambda}^-$	0	−1
σ	−1	+1
$\bar{\sigma}$	+1	−1
D	0	0

Table 2.2: R_R and R_L assignment for vector superfield.

2.1.4 Twisted Chiral Superfield and Field Strength

A novelty that we have in two dimension is the twisted chiral superfield, defined by

$$\bar{\mathsf{D}}_{+}\boldsymbol{\Sigma} = \mathsf{D}_{-}\boldsymbol{\Sigma} = 0\ , \quad (2.26)$$

here we will not describe this superfield in detail (the interested reader can consult [**82**, **85**] or [**83**] for its implications in mirror symmetry), however we want to stress that it is possible to build a gauge invariant[3] field strength for the Abelian vector multiplet in the form of a twisted

[3]From eq. (2.10) we have that $\bar{\mathsf{D}}_{+}\mathsf{D}_{-}(\mathbf{V} - i(\boldsymbol{\Omega} - \bar{\boldsymbol{\Omega}})) = \bar{\mathsf{D}}_{+}\mathsf{D}_{-}\mathbf{V}$, since in eq. (2.19) commutators drop in the Abelian case.

chiral

$$
\begin{aligned}
\mathbf{\Sigma} &= \bar{\mathsf{D}}_+ \mathsf{D}_- \mathbf{V} \\
&= \bar{\sigma} + \mathrm{i}\bar{\theta}^- \lambda^+ - \mathrm{i}\theta^+ \bar{\lambda}^- + \mathrm{i}\theta^+ \bar{\theta}^- (D - 2F_{z\bar{z}}) + \theta^- \bar{\theta}^- \partial \bar{\sigma} + \theta^+ \bar{\theta}^+ \bar{\partial} \bar{\sigma} - \\
&\quad - \mathrm{i}\theta^+ \theta^- \bar{\theta}^- \partial \bar{\lambda}^- + \mathrm{i}\bar{\theta}^+ \bar{\theta}^- \theta^+ \bar{\partial} \lambda^+ - \theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- \partial \bar{\partial} \bar{\sigma} \; .
\end{aligned}
\tag{2.27}
$$

Similarly the twisted anti-chiral superfield is $\bar{\mathbf{\Sigma}} = \mathsf{D}_+ \bar{\mathsf{D}}_- \mathbf{V}$. The action for this twisted chiral superfield amounts to be the action for an abelian vector superfield

$$
S_{\mathbf{V},\mathrm{ab.}} = \int \mathrm{d}z \mathrm{d}\bar{z} \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \bar{\mathbf{\Sigma}} \mathbf{\Sigma} = \int \mathrm{d}z \mathrm{d}^2\bar{z} (-4F_{z\bar{z}}^2 + 4\bar{\partial}\bar{\sigma}\partial\sigma - \mathrm{i}\bar{\lambda}\gamma \cdot \partial\lambda + D^2 + \text{total derivative}) \; ;
\tag{2.28}
$$

for this reason $\mathbf{\Sigma}$ is called *field strength multiplet*. It is possible to generalize field strength in case of non-Abelian gauge group. Using notations of [**86**] we define

$$
\mathbf{\Sigma} = \bar{D}_+ \left(e^{-V} D_- e^{V} \right) \; , \qquad \bar{\mathbf{\Sigma}} = D_+ \left(e^{V} \bar{D}_- e^{-V} \right) \; ,
\tag{2.29}
$$

which reduce to (2.27) in the case of Abelian gauge group. Under a gauge transformation (2.17) they transform as

$$
\mathbf{\Sigma} \mapsto e^{\mathrm{i}\mathbf{\Omega}} \mathbf{\Sigma} e^{-\mathrm{i}\mathbf{\Omega}} \; , \qquad \bar{\mathbf{\Sigma}} \mapsto e^{\mathrm{i}\bar{\mathbf{\Omega}}} \bar{\mathbf{\Sigma}} e^{-\mathrm{i}\bar{\mathbf{\Omega}}} \; ,
\tag{2.30}
$$

so that

$$
\mathrm{tr} \left(e^{-V} \bar{\mathbf{\Sigma}} e^{V} \mathbf{\Sigma} \right)
\tag{2.31}
$$

is gauge invariant. With this we can write down the action

$$
\begin{aligned}
S_{\mathbf{V}} &= \int \mathrm{d}z \mathrm{d}\bar{z} \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \mathrm{tr} \left(e^{-V} \bar{\mathbf{\Sigma}} e^{V} \mathbf{\Sigma} \right) \\
&= \int \mathrm{d}z \mathrm{d}\bar{z}\, \mathrm{tr}(-4\mathcal{F}_{z\bar{z}}^2 + 4\bar{\mathcal{D}}\bar{\sigma}\mathcal{D}\sigma - \mathrm{i}\bar{\lambda}\gamma \cdot \partial\lambda + D^2 + \mathrm{i}D[\sigma, \bar{\sigma}] - \mathrm{i}\bar{\lambda}\mathrm{P}_+[\sigma, \lambda] - \mathrm{i}\bar{\lambda}\mathrm{P}_-[\bar{\sigma}, \lambda]) \; ,
\end{aligned}
\tag{2.32}
$$

which is invariant under (2.25). We notice again that the presence of the scalars σ and $\bar{\sigma}$ is arguable if we think $S_{\mathbf{V}}$ as a dimensional reduction of SYM in four dimensions[4]. Another remarkable fact is that

$$
(\bar{\epsilon}\bar{\mathsf{Q}})(\epsilon\mathsf{Q}) \int \mathrm{d}z \mathrm{d}\bar{z} \left(\tfrac{1}{2}\bar{\lambda}\lambda - D(\sigma + \bar{\sigma}) \right) = \epsilon\bar{\epsilon} S_{\mathbf{V}} \; ,
\tag{2.33}
$$

this will turn out to be very important in the following.

2.1.5 Gauge–Matter Coupling

Now we want to couple matter to SYM. Under a gauge transformation we have that matter change as

$$
\mathbf{\Phi} \mapsto e^{-\mathrm{i}\mathbf{\Omega}} \mathbf{\Phi} \; , \qquad \bar{\mathbf{\Phi}} \mapsto \bar{\mathbf{\Phi}} e^{\mathrm{i}\bar{\mathbf{\Omega}}} \; ,
\tag{2.34}
$$

[4] In many references the action is written with the shift $D \mapsto D - \frac{\mathrm{i}}{2}[\sigma, \bar{\sigma}]$.

therefore we see that the term

$$\bar{\mathbf{\Phi}}e^{-\mathbf{V}}\mathbf{\Phi} \ , \tag{2.35}$$

is gauge invariant and reproduces the integrand of (2.14) when the gauge coupling is set to zero. The action therefore will be

$$S_{\mathbf{\Phi V}} = \int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\bar{\mathbf{\Phi}}e^{-\mathbf{V}}\mathbf{\Phi} = \int \mathrm{d}z\mathrm{d}\bar{z}(4\bar{\mathcal{D}}\bar{\varphi}\mathcal{D}\varphi - \mathrm{i}\bar{\psi}\gamma\cdot\mathcal{D}\psi + \bar{F}F + \bar{\varphi}(\bar{\sigma}\sigma + \mathrm{i}D)\varphi + \\ + \mathrm{i}\bar{\psi}(\mathrm{P}_{-}\bar{\sigma} + \mathrm{P}_{+}\sigma)\psi + \mathrm{i}\bar{\psi}\bar{\lambda}\varphi + \mathrm{i}\psi\lambda\bar{\varphi} + \text{total derivative}) \tag{2.36}$$

Having coupled matter to the vector superfield, which we want to be in WZ gauge, we have to remember to perform the dWF transformation after a a SUSY variation, therefore eqs. (2.13) get modified in

$$\begin{aligned}\tilde{\delta}\varphi &= \epsilon\psi \ , \\ \tilde{\delta}\psi^a &= \epsilon^a F + \mathrm{i}(\gamma\bar{\epsilon})^a\cdot\mathcal{D}\varphi + \mathrm{i}(\mathrm{P}_{-}\bar{\epsilon})^a\sigma\varphi + \mathrm{i}(\mathrm{P}_{+}\bar{\epsilon})\bar{\sigma}\varphi \ , \\ \tilde{\delta}F &= \mathrm{i}\bar{\epsilon}\gamma\cdot\mathcal{D}\psi - \mathrm{i}\bar{\epsilon}\bar{\lambda}\varphi - \mathrm{i}\bar{\epsilon}\mathrm{P}_{+}\psi\sigma - \mathrm{i}\bar{\epsilon}\mathrm{P}_{-}\psi\bar{\sigma} \ .\end{aligned} \tag{2.37}$$

One can easily verify that $S_{\mathbf{\Phi V}}$ (2.36) is invariant under variations (2.37). Another remarkable fact is that

$$(\bar{\epsilon}\bar{\mathsf{Q}})(\epsilon\mathsf{Q})\int \mathrm{d}z\mathrm{d}\bar{z}(\bar{\psi}\psi - \mathrm{i}\bar{\varphi}(\sigma+\bar{\sigma})\varphi) = \epsilon\bar{\epsilon}S_{\mathbf{\Phi V}} \ , \tag{2.38}$$

this will turn out to be very important in the following.

2.1.6 Interactions

Since now we have written D-term densities, which are to be integrated over θ and $\bar{\theta}$ to yield the corresponding Lagrangian density. However if we take a holomorphic function of some chiral superfields $W(\mathbf{\Phi}_i)$, it turns out that

$$S_W = -\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}^2\theta W(\mathbf{\Phi})|_{\bar{\theta}=0} = -\int \mathrm{d}z\mathrm{d}\bar{z}\left(F_i\partial_i W(\varphi) + \tfrac{1}{2}\partial_i\partial_j W(\varphi)F\psi_i\psi_j\right) \ , \tag{2.39}$$

is SUSY invariant as well as its hermitian conjugate, which we must add to the action as well to keep it hermitian. Usually W is called *superpotential*, and it is an instance of *F-term*, i.e. a term which is integrated just in half of Grassmann variables. A fundamental difference with D-terms is that, to keep action invariant under R-symmetries, we must have $\mathrm{R}_R[W] = \mathrm{R}_L[W] = 1$. We notice that since W is a holomorphic function of chiral superfields it satisfies itself $\bar{\mathsf{D}}_a W = 0$. In its expansion as a chiral superfield there is the term $\frac{1}{2}\theta\theta F_W$, which is the one selected by Berezin integration: explicitly we have $F^{(W)} = F_i\partial_i W + \frac{1}{2}\partial_i\partial_j W F\psi_i\psi_j$. Since W is uncharged under G, using eq. (2.13) to see that

$$(\epsilon\mathsf{Q})\int \mathrm{d}z\mathrm{d}\bar{z}\bar{\epsilon}\psi^{(W)} = \bar{\epsilon}\epsilon S_W \ , \tag{2.40}$$

where $\psi_{(W)}$ and $\bar{\psi}_{(\bar{W})}$ are the fermion present in the superpotential. This, again, will be important in the following. An illustrative example of superpotential is the addition of $W(\mathbf{\Phi}) = \frac{1}{2}m\Phi^2$ to the matter action

$$S_{\mathbf{\Phi}} + S_W + S_{\bar{W}} = \int \mathrm{d}z\mathrm{d}\bar{z}\left(4\bar{\partial}\bar{\varphi}\partial\varphi - \mathrm{i}\bar{\psi}\gamma\cdot\partial\psi + \bar{F}F - mF\varphi + m\bar{F}\bar{\varphi} + \tfrac{1}{2}m\psi\psi + \tfrac{1}{2}m\bar{\psi}\bar{\psi}\right) \ , \tag{2.41}$$

in which we can integrate out $F = m\bar{\varphi}$ and $\bar{F} = m\varphi$. We see that we recover a matter action in which φ, ψ and $\bar{\psi}$ acquired mass m[5]. As far as R-charges are concerned we see that the correct assignment is $\mathrm{R}_R[\mathbf{\Phi}] = \mathrm{R}_L[\mathbf{\Phi}] = \frac{1}{2}$. Notice that this constraint arise form the presence of W: had not we had W, we were free to choose any R-charge for $\mathbf{\Phi}$. In the case of $\mathrm{G} = \mathrm{G}' \times \mathrm{U}(1)^r$ we can consider also some other terms. If we introduce a holomorphic function $\widetilde{W}(\mathbf{\Sigma}_A)$, with $A = 1, \dots r$, called *twisted superpotential* we can add to the action the following term

$$S_{\widetilde{W}} = \sum_{A=1}^{r} \int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+\mathrm{d}\bar{\theta}^- \mathbf{\Sigma}_A|_{\theta^-=\bar{\theta}^+=0} =$$

$$= \sum_{A=1}^{r} \int \mathrm{d}z\mathrm{d}\bar{z}(2\mathrm{i}(D_A - 2F_{A;z\bar{z}})\partial_A \widetilde{W}(\sigma_A) + 2\lambda_A^+ \bar{\lambda}_A^- \partial_A^2 \widetilde{W}(\sigma_A)) \ , \quad (2.42)$$

together with its hermitian conjugate. For $\widetilde{W} = \sum_{A=1}^{r} \frac{1}{4}(\mathrm{i}\xi_A + \frac{\vartheta_A}{2\pi})\mathbf{\Sigma}_A$ one recovers the usual Fayet–Iliopulos and theta term[6]

$$S_{\widetilde{W}} + S_{\bar{\widetilde{W}}} = \sum_{A=1}^{r} \int \mathrm{d}z\mathrm{d}\bar{z}\left(-\xi_A D_{A;z\bar{z}} - 2\mathrm{i}F_{A;z\bar{z}}\frac{\vartheta_A}{2\pi}\right) \ . \quad (2.43)$$

As a concluding remark of this section we want to stress that we do not review *all* possible term that can be written in a $\mathcal{N} = (2, 2)$ SUSY action, but only those that will be useful for our purposes. Even if we tried to organize every term in such a way it can be written in term of superspace, there exist some interactions that do not fit this formalism [87] but are not relevant for our purposes.

2.2 $\mathcal{N} = (0, 2)$ Supersymmetry

In two dimension it is also possible to build actions which are less supersymmetric than the those described above, since they preserve just two supercharges. In this sense $\mathcal{N} = (2, 2)$ supersymmetry is a special case of the one we are briefly going to discuss. These actions will be actually the building blocks for the computations of the Elliptic Genera. Interested reader can consult [82, 88–91].

2.2.1 Superspace

The rough idea to mimic what we did in previous section is to remove half of theta variables and to remain just with θ^+ and $\bar{\theta}^+$:

$$\mathsf{Q}_+ = \boldsymbol{\partial}_+ - \bar{\theta}^+\bar{\partial} \ , \qquad \bar{\mathsf{Q}}_+ = \bar{\boldsymbol{\partial}}_+ - \theta^+\bar{\partial} \ , \qquad \mathsf{D}_+ = \boldsymbol{\partial}_+ + \bar{\theta}^+\bar{\partial} \ , \qquad \bar{\mathsf{D}}_+ = \bar{\boldsymbol{\partial}}_+ + \theta^+\bar{\partial} \ , \quad (2.44)$$

having non-trivial anticommutation relation

$$\{\mathsf{Q}_+, \bar{\mathsf{Q}}_+\} = -2\bar{\partial} \ , \qquad\qquad \{\mathsf{D}_+, \bar{\mathsf{D}}_+\} = 2\bar{\partial} \ . \quad (2.45)$$

The most general bosonic superfield is simply

$$\boldsymbol{\mathcal{B}}^{(0,2)} = \varphi + \theta^+\psi_+ + \bar{\theta}^+\bar{\psi}_+ + \theta^+\bar{\theta}^+ s \ , \quad (2.46)$$

[5] Recall that massive Dirac equation is equivalent to two coupled Weyl equations.

[6] Recall that $-2\mathrm{i}F_{z\bar{z}} \in \mathbb{R}$.

while the most general fermionic superfield is

$$\boldsymbol{\mathcal{F}}^{(0,2)} = \lambda_+ + \theta^+ G + \bar{\theta}^+ H + \theta^+\bar{\theta}^+\chi_+ \ . \tag{2.47}$$

More information about $\mathcal{N} = (0,2)$ superspace (also for the SUGRA point of view) can be found in [84].

2.2.2 Chiral and Fermi Superfield

Imposing the constraint

$$\bar{\mathsf{D}}_+\mathbf{\Phi}^{(0,2)} = 0 \ , \tag{2.48}$$

the general solution, called *chiral superfield*, is

$$\mathbf{\Phi}^{(0,2)} = \varphi - \mathrm{i}\theta^+\psi_+ + \theta^+\bar{\theta}^+\bar{\partial}\varphi \ . \tag{2.49}$$

A novelty is that is possible to introduce also a *Fermi superfield*, defined by

$$\bar{\mathsf{D}}_+\mathbf{\Lambda}^{(0,2)} = \mathcal{E}(\mathbf{\Phi}_i) \ , \tag{2.50}$$

where $\mathcal{E}(\mathbf{\Phi}_i) = \mathcal{E}(\varphi_i) + \theta^+\partial_i\mathcal{E}(\varphi_i)\psi_{+;i} + \theta^+\bar{\theta}^+\bar{\partial}\mathcal{E}(\varphi_i) = E - \mathrm{i}\theta^+\psi_+^{(E)} + \theta^+\bar{\theta}^+\bar{\partial}E$ is an holomorphic function of some chiral superfield; therefore we have

$$\mathbf{\Lambda}^{(0,2)} = \mathrm{i}\psi_- + \mathrm{i}\theta^+ G - \bar{\theta}^+ E + \mathrm{i}\theta^+\bar{\theta}^+(\bar{\partial}\psi_- - \psi_+^{(E)}) \ . \tag{2.51}$$

SUSY variations are easily obtained by the action of the operator $\boldsymbol{\delta} = \epsilon^+\mathsf{Q}_+ + \bar{\epsilon}^+\bar{\mathsf{Q}}_+$ to the $\mathbf{\Phi}$ and $\mathbf{\Lambda}_-$, so that we get

$$\begin{aligned} \delta\varphi &= -\mathrm{i}\epsilon^+\psi_+ \ , & \delta\psi_- &= \epsilon^- G + \mathrm{i}\bar{\epsilon}^+ E \ , \\ \delta\psi_+ &= 2\mathrm{i}\bar{\epsilon}^+\bar{\partial}\varphi \ , & \delta G &= 2\bar{\epsilon}^+\bar{\partial}\psi_- - \bar{\epsilon}^+\psi_+^{(E)} \ . \end{aligned} \tag{2.52}$$

It is possible to see that

$$\begin{aligned} \mathbf{\Phi}^{(0,2)} &= \mathbf{\Phi}^{(2,2)}|_{\theta^-=\bar{\theta}^-=0} \ , \\ \mathbf{\Lambda}^{(0,2)} &= \mathsf{D}_-\mathbf{\Phi}^{(2,2)}|_{\theta^-=\bar{\theta}^-=0} \qquad \text{with } \mathcal{E} = 0 \ , \end{aligned} \tag{2.53}$$

with suitable definitions of component fields. The actions for the chiral and Fermi superfields are respectively

$$\begin{aligned} S_{\mathbf{\Phi}}^{(0,2)} &= -\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\bar{\mathbf{\Phi}}\partial\mathbf{\Phi} = \int \mathrm{d}z\mathrm{d}\bar{z}(-2\bar{\varphi}\partial\bar{\partial}\varphi + \bar{\psi}^-\partial\psi^-) \ , \\ S_{\mathbf{\Lambda}}^{(0,2)} &= -\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\bar{\mathbf{\Lambda}}\mathbf{\Lambda} \\ &= \int \mathrm{d}z\mathrm{d}\bar{z}(-\bar{\psi}^+\bar{\partial}\psi^+ + \tfrac{1}{2}\bar{E}E + \tfrac{1}{2}\bar{G}G + \tfrac{1}{2}\bar{\psi}^+\psi_{(E)}^- - \tfrac{1}{2}\bar{\psi}_{(E)}^-\psi^+) \ . \end{aligned} \tag{2.54}$$

2.2.3 Vector Multiplet

For the vector multiplet the story is a bit more delicate. We will discuss what happen in the Abelian case and then we will generalize our result to the non-Abelian case. In fact vector multiplet consist of a pair of superfields $\boldsymbol{\mathcal{A}}$ and $\mathbf{V}^{(0,2)}$. A super-gauge transformation acts in the following way:

$$\mathbf{V}^{(0,2)} \mapsto \mathbf{V}^{(0,2)} - \tfrac{i}{2}(\boldsymbol{\Omega} - \bar{\boldsymbol{\Omega}}) \,, \qquad \boldsymbol{\mathcal{A}} \mapsto \boldsymbol{\mathcal{A}} - \tfrac{i}{2}\partial(\boldsymbol{\Omega} + \bar{\boldsymbol{\Omega}}) \,. \tag{2.55}$$

Using these transformations it is possible to write fields in WZ gauge as

$$\mathbf{V}^{(0,2)} = \tfrac{1}{2}\theta^+\bar{\theta}^+\bar{A} \,, \qquad \boldsymbol{\mathcal{A}} = \tfrac{1}{2}(A + \theta^+\lambda_- - \bar{\theta}^+\bar{\lambda}_- + \theta^+\bar{\theta}^+ D) \,. \tag{2.56}$$

A residual gauge is parameterized by

$$\boldsymbol{\Omega}_{\mathrm{WZ}} = \omega + \theta^+\bar{\theta}^+\bar{\partial}\omega \,, \qquad \omega \in \mathbb{R} \,, \tag{2.57}$$

under which we have that

$$A \mapsto A + \partial\omega \,, \qquad \bar{A} \mapsto \bar{A} + \bar{\partial}\omega \,. \tag{2.58}$$

SUSY variations have to be accompanied with a compensating dWF gauge transformation (with $\boldsymbol{\Omega} = -\frac{1}{2}\theta^+\bar{\epsilon}^+\bar{A}$) in order to remain in the WZ gague: the result is

$$\begin{aligned}
\tilde{\delta} A &= \tfrac{1}{2}(\epsilon^+\lambda^+ - \bar{\epsilon}^+\bar{\lambda}^+) \,, \\
\tilde{\delta}\bar{A} &= 0 \,, \\
\tilde{\delta}\lambda^+ &= \bar{\epsilon}^+(-D + 2F_{z\bar{z}}) \,, \\
\tilde{\delta}\bar{\lambda}^+ &= \epsilon^+(-D - 2F_{z\bar{z}}) \,, \\
\tilde{\delta} D &= -(\epsilon^+\bar{\partial}\lambda^+ + \bar{\epsilon}^+\partial\bar{\lambda}^+) \,.
\end{aligned} \tag{2.59}$$

The gauge-invariant field strength (see [82]) is expressed with the following Fermi superfields.

$$\begin{aligned}
\boldsymbol{\Upsilon} &= 4[\partial + \boldsymbol{\mathcal{A}}, e^{\mathbf{V}^{(0,2)}}\bar{\mathsf{D}}_+ e^{-\mathbf{V}^{(0,2)}}] = -\lambda^+ + \bar{\theta}^+(D + 2F_{z\bar{z}}) + \theta^+\bar{\theta}^+\bar{\partial}\lambda^+ \,. \\
\bar{\boldsymbol{\Upsilon}} &= -4[\partial + \boldsymbol{\mathcal{A}}, e^{-\mathbf{V}^{(0,2)}}\mathsf{D}_+ e^{\mathbf{V}^{(0,2)}}] = \bar{\lambda}^+ + \theta^+(D - 2F_{z\bar{z}}) + \theta^+\bar{\theta}^+\bar{\partial}\bar{\lambda}^+ \,.
\end{aligned} \tag{2.60}$$

We can build the action as

$$S^{(0,2)}_{\mathbf{V},\mathrm{ab.}} = \int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\bar{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon} = \int \mathrm{d}z\mathrm{d}\bar{z}(-4F^2_{z\bar{z}} + D^2 - 2\bar{\lambda}^+\bar{\partial}\lambda^+) \,. \tag{2.61}$$

It is easy to guess the generalization of eqs. (2.59) and (2.61) for a non-Abelian gauge group since the field content is very limited: we have

$$\begin{aligned}
\tilde{\delta} A &= \tfrac{1}{2}(\epsilon^+\lambda^+ - \bar{\epsilon}^+\bar{\lambda}^+) \,, \\
\tilde{\delta}\bar{A} &= 0 \,, \\
\tilde{\delta}\lambda^+ &= \bar{\epsilon}^+(-D + 2\mathcal{F}_{z\bar{z}}) \,, \\
\tilde{\delta}\bar{\lambda}^+ &= \epsilon^+(-D - 2\mathcal{F}_{z\bar{z}}) \,, \\
\tilde{\delta} D &= -(\epsilon^+\bar{\mathcal{D}}\lambda^+ + \bar{\epsilon}^+\mathcal{D}\bar{\lambda}^+) \,,
\end{aligned} \tag{2.62}$$

and

$$S_{\mathbf{V}}^{(0,2)}=\int \mathrm{d}z\mathrm{d}\bar{z}(-4\mathcal{F}_{z\bar{z}}^2+D^2-2\bar{\lambda}^+\bar{\mathcal{D}}\lambda^+)\ . \tag{2.63}$$

An $\mathcal{N}=(2,2)$ vector superfield $\mathbf{V}^{(2,2)}$ as in eq. (2.20) splits in a $N=(0,2)$ $\mathbf{V}^{(0,2)}$ vector with associated $\boldsymbol{\Upsilon}$ as in eq. (2.60) (with $\mathcal{E}=0$)[7], and in a chiral multiplet in the adjoint $\bar{\boldsymbol{\chi}}=\boldsymbol{\Sigma}|_{\theta^-=\bar{\theta}^-=0}$ containing component fields $\bar{\sigma}$ and $\bar{\lambda}^-$. A remarkable fact concerning the action for vector multiplet is that

$$\mathsf{Q}_+\int \mathrm{d}z\mathrm{d}\bar{z}(\lambda^+(D+2\mathcal{F}_{z\bar{z}}))=S_{\mathbf{V}}\ , \tag{2.64}$$

we will use this property later. To conclude this subsection we want to observe that, upon redefinition of component fields, one has

$$S_{\mathbf{V}}^{(2,2)}=S_{\mathbf{V}}^{(0,2)}+S_{\mathbf{\Phi}}^{(0,2)}\ , \tag{2.65}$$

where $\mathbf{\Phi}$ is in the adjoint representation.

2.2.4 Gauge–Matter Coupling

Under gauge transformation matter change as

$$\mathbf{\Phi}\mapsto e^{-\mathrm{i}\mathbf{\Omega}}\mathbf{\Phi}\ ,\qquad \bar{\mathbf{\Phi}}\mapsto\bar{\mathbf{\Phi}}e^{\mathrm{i}\bar{\mathbf{\Omega}}}\ ,\qquad \mathbf{\Lambda}\mapsto e^{-\mathrm{i}\mathbf{\Omega}}\mathbf{\Lambda}\ ,\qquad \bar{\mathbf{\Lambda}}\mapsto\bar{\mathbf{\Phi}}e^{\mathrm{i}\bar{\mathbf{\Omega}}}\ . \tag{2.66}$$

The actions for that kind of matter interacting with SYM are

$$\begin{aligned}
S_{\mathbf{\Phi V}}^{(0,2)}&=-\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\bar{\mathbf{\Phi}}e^{-\frac{1}{2}\mathbf{V}}(\partial+\boldsymbol{\mathcal{A}})e^{-\frac{1}{2}\mathbf{V}}\mathbf{\Phi}\\
&=\int \mathrm{d}z\mathrm{d}\bar{z}(-2\bar{\varphi}\mathcal{D}\bar{\mathcal{D}}\varphi-\tfrac{i}{2}\bar{\varphi}(2\mathcal{F}_{z\bar{z}}-D)\varphi+\bar{\psi}^-\mathcal{D}\psi^--\tfrac{1}{2}\bar{\psi}^-\lambda^+\varphi+\tfrac{1}{2}\bar{\varphi}\bar{\lambda}^+\psi^-)\ ,\\
S_{\mathbf{\Lambda V}}^{(0,2)}&=-\int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\bar{\mathbf{\Lambda}}e^{-\mathbf{V}}\mathbf{\Lambda}\\
&=\int \mathrm{d}z\mathrm{d}\bar{z}(-\bar{\psi}^+\bar{\mathcal{D}}\psi^++\tfrac{1}{2}\bar{E}E+\tfrac{1}{2}\bar{G}G+\tfrac{1}{2}\bar{\psi}^+\psi^-_{(E)}-\tfrac{1}{2}\bar{\psi}^-_{(E)}\psi^+)\ .
\end{aligned} \tag{2.67}$$

Due to dWF gauge transformation, eqs. (2.52) gets modified:

$$\begin{aligned}
\delta\varphi&=-\mathrm{i}\epsilon^+\psi_+\ , & \delta\psi_-&=\epsilon^-G+\mathrm{i}\bar{\epsilon}^+E\ ,\\
\delta\psi_+&=2\mathrm{i}\bar{\epsilon}^+\bar{\mathcal{D}}\varphi\ , & \delta G&=2\bar{\epsilon}^+\bar{\mathcal{D}}\psi_--\bar{\epsilon}^+\psi_+^{(E)}\ .
\end{aligned} \tag{2.68}$$

Also these two actions (2.67) have the property that

$$\begin{aligned}
(\mathsf{Q}_++\bar{\mathsf{Q}}_+)\int \mathrm{d}z\mathrm{d}\bar{z}(\mathrm{i}\bar{\varphi}\mathcal{D}\psi^--\tfrac{\mathrm{i}}{2}\bar{\varphi}\lambda^+\varphi)&=S_{\mathbf{\Phi}}\ ,\\
(\mathsf{Q}_++\bar{\mathsf{Q}}_+)\int \mathrm{d}z\mathrm{d}\bar{z}(\tfrac{1}{2}\bar{\psi}^+G-\tfrac{\mathrm{i}}{2}\bar{E}\psi^+)&=S_{\mathbf{\Lambda}}\ ;
\end{aligned} \tag{2.69}$$

note that, this time, we specialized $\epsilon^+=\bar{\epsilon}^+=1$. To conclude this subsection we want to observe that

$$S_{\mathbf{\Phi V}}^{(2,2)}=S_{\mathbf{\Phi V}}^{(0,2)}+S_{\mathbf{\Lambda V}}^{(0,2)}\ , \tag{2.70}$$

with $\mathcal{E}_{\mathbf{\Lambda}}=\boldsymbol{\chi}\mathbf{\Phi}$.[8]

[7] In the non-Abelian case with the replacements $F_{z\bar{z}}\mapsto\mathcal{F}_{z\bar{z}}$ and $\bar{\partial}\mapsto\bar{\mathcal{D}}$.

[8] We have that $\mathcal{E}\neq 0$ because of the coupling with vector superfield, see [**82**] for details.

2.2.5 Interactions

Now we want to describe more generic interactions in $\mathcal{N} = (0,2)$ theories. They can be expressed in terms of a holomorphic function of the chiral multiplets $\mathcal{J}(\mathbf{\Phi}_a) = J - \mathrm{i}\theta^+\psi_+^{(J)} + \theta^+\bar{\theta}^+\bar{\partial}J$, together with its "barred" version $\bar{\mathcal{J}}(\bar{\mathbf{\Phi}}_a)$. We can consider the term

$$S_{\mathcal{J}} = -\mathrm{i}\sum_a \int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}\theta^+ \mathbf{\Lambda}_a \mathcal{J}^a|_{\bar{\theta}=0} = \sum_a \int \mathrm{d}z\mathrm{d}\bar{z}(G_a J^a + \mathrm{i}\psi_a^+ \psi_{(J)}^{-a}) \ , \tag{2.71}$$

which are SUSY invariant (up to a total derivative) if

$$\sum_a E_a J^a = 0 \ , \tag{2.72}$$

where E_a are the bottom component of $\mathcal{E}_a$ which are used in the definition of Fermi superfield. Another property we have is that

$$(\mathsf{Q}_+ + \bar{\mathsf{Q}}_+) \int \mathrm{d}z\mathrm{d}\bar{z} \sum_a \psi_a^+ J^a = S_{\mathcal{J}} \ . \tag{2.73}$$

In the case of a theory having $\mathcal{N} = (2,2)$ SUSY with respectively a superpotential and a twisted superpotential we have that

$$J^a(\varphi) = \frac{\partial W}{\partial \varphi_a} \ , \qquad\qquad J^a(\sigma) = \frac{\partial \widetilde{W}}{\partial \sigma_a} \ . \tag{2.74}$$

3. Elliptic Genus from Gauge Theories

In this chapter we introduce one of the main ingredient of this thesis: the *Elliptic Genus*. We will give the mathematical definition and then we will see how to compute it in the context of $\mathcal{N} = (0,2)$ SUSY gauge theories using the technique of *supersymmetric localization* mainly following [**60**, **61**].

3.1 Witten Index and Elliptic Genus

As already mentioned in the introduction, topology and SUSY gauge theories have a deep relationship. If one consider a SUSY QM having a Riemaniann manifold M as target space, one can introduce an important quantity, the *Witten Index* (see [**92**])

$$\mathcal{I}_{\mathrm{W}} = \mathrm{tr}[(-1)^F e^{\beta H}] \,, \tag{3.1}$$

which turns out to compute the Euler characteristic $\chi(M)$ of the manifold under consideration (see also [**83**]). The Euler Characteristic is computed as

$$\chi(M) = \int_M \prod_i^{d_{\mathbb{R}}} x_i \,, \tag{3.2}$$

where x_i are the Chern roots and $d_{\mathbb{R}}$ is the real dimension of M[1]. It is possible to generalize genus (3.2) for a complex manifold M to (see [**60**, **61**, **93**, **94**])

$$\mathcal{I}_{\mathrm{EG}}(q,y) = \int \prod_i^{d_{\mathbb{R}}} x_i \frac{\theta_1(\tau|\frac{x_i}{2\pi\mathrm{i}})}{\theta_1(\tau|\frac{x_i}{2\pi\mathrm{i}} - z)} \,, \tag{3.4}$$

where the θ_1 is defined in App. B, $q = e^{2\pi\mathrm{i}\tau}$ and $y = e^{2\pi\mathrm{i}z}$. In terms of bundles, eq. (3.4) means that we are computing the holomorphic Euler characteristic of the formal power series of vector bundles (see [**95**])

$$E_{q,y} = \bigotimes_{n=1}^{\infty} \left[\bigwedge\nolimits^{\bullet}_{-yq^n} T_M \otimes \bigwedge\nolimits^{\bullet}_{-y^{-1}q^{n-1}} T_M^* \otimes \mathrm{Sym}^{\bullet}_{q^n}(T_M \oplus T_M^*) \right] \,, \tag{3.5}$$

where T_X is the holomorphic tangent bundle, and we defined the formal power series

$$\bigwedge\nolimits^{\bullet}_t V := \sum_{i=0}^{\infty} t^i \bigwedge\nolimits^i V \,, \qquad \mathrm{Sym}^{\bullet}_t V := \sum_{i=0}^{\infty} t^i \,\mathrm{Sym}^i V \,. \tag{3.6}$$

[1]By Hirzebruch–Riemann–Roch theorem Chern classes of M splits as

$$c(M) = \prod_i^{d_{\mathbb{R}}} (1 + x_i) \,, \tag{3.3}$$

which implicitly defines x_i's.

This means that

$$\mathcal{I}_{\text{EG}}(y,q) = y^{\frac{d_{\mathbb{C}}}{2}} \int_M \text{ch}(E_{q,y})\,\text{td}(M)\ . \tag{3.7}$$

Although (3.4) seems very complicate, it has some important properties. First of all it captures many other invariants that appear in literature: in the limit of $q \to 0$ it reduces to the so called χ_y *genus*, while for $y = 1$ we simply recover the Euler characteristic, then for $y = -1$ we have the *signature*, for $y = -\sqrt{q}$ the $\hat{A}$ *genus*. Moreover, if M is Calabi–Yau (CY), the elliptic genus is a Jacobi form of weight 0 and index $\frac{d_{\mathbb{C}}}{2}$, that is, under modular transformation we have

$$\mathcal{I}_{\text{EG}}\left(\frac{a\tau+b}{c\tau+d}\middle|\frac{z}{c\tau+d}\right) = e^{\mathrm{i}\pi\frac{c}{c\tau+d}d_{\mathbb{C}}z^2}\mathcal{I}_{\text{EG}}(\tau|z)\ . \tag{3.8}$$

In Physics, elliptic genus appeared in late 80's, for instance [**96**–**101**] where different computations of elliptic genera were carried out. In the following years, elliptic genus for a Landau–Ginzburg model was computed [**102**] using localization; then the result was extended to Gepner models using the orbifold Landau–Ginzburg description [**103**–**106**]. The elliptic genus of a subspace of a Kähler quotient was studied in [**107**, **108**]. In [**60**] there is a path-integral derivation of elliptic genus for $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ SUSY gauge theory in two dimension with rank-one gauge group; in [**61**] the result is extended to arbitrary rank gauge groups. If the theory has a smooth geometric phase the formulae found in these papers reproduce the mathematical results of [**94**, **109**]. The definition that is given for $\mathcal{N} = (2,2)$ theories having flavor symmetry group K (with Cartan generators K_a) and left-moving R-symmetry R_L (which is discrete if the theory is not conformal)

$$\mathcal{I}_{\text{EG}} = \text{tr}_{\text{RR}}(-1)^F q^{H_L}\bar{q}^{H_R}y^{R_L}\prod\nolimits_a \zeta_a^{K_a}\ , \tag{3.9}$$

where the trace is taken over Ramond-Ramond sector (periodic boundary condition for fermion) and $H_L = \frac{1}{2}(H + \mathrm{i}P)$ and $H_R = \frac{1}{2}(H - \mathrm{i}P)$ are respectively the left- and right-moving Hamiltonians in Euclidean signature. In the case of an $\mathcal{N} = (0,2)$ theory we have

$$\mathcal{I}_{\text{EG}} = \text{tr}_{\text{RR}}(-1)^F q^{H_L}\bar{q}^{H_R}\prod\nolimits_a \zeta_a^{K_a}\ . \tag{3.10}$$

Notice that in the latter case we do not have any left-moving U(1) R-symmetry, but only flavor symmetries. It is obvious that any $\mathcal{N} = (2,2)$ theory can be regarded as an $\mathcal{N} = (0,2)$ theory, and therefore, once one finds an expression for eq. (3.10) one has also one for eq. (3.9). Of course, left-moving R-symmetry that we have in the $\mathcal{N} = (2,2)$ case is regarded simply as a flavor symmetry in the $\mathcal{N} = (0,2)$ case. In the following sections we will focus on this latter case, which is, at least conceptually, more general. As a final comment, notice that the elliptic genus seems to depend both on τ and $\bar{\tau}$; we will however discover that the dependence on $\bar{\tau}$ disappears. This is because states with $H_R \neq 0$ come in pairs with all equal charges but opposite fermionic number, thus only states with $H_R = 0$ will contribute. Moreover, since the spin $H_R - H_L$ does not renormalize, the elliptic genus is a quantity that does not change under small deformations of the theory provided that there are not states coming in and escaping to infinity in field space (this happens whenever there are flat directions).

3.2 Derivation via Localization

Supersymmetric localization (since [15]) is a powerful technique that allows us to compute exactly some kind of PI including partition functions and vacuum expectation values of certain operators. It is beyond the purposes of this thesis to explain in detail this technique: the interested reader can see the review article [27] and reference therein (for shorter introduction see also [25, 26]); we will limit ourselves to sketch the main idea and to apply it to the case at hand.

3.2.1 Localization Argument

Suppose we have an action S of a given theory and that we want to compute the VEV of a $\mathcal{Q}$-closed operator $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \int [\mathrm{d}\Phi] e^{-S[\Phi]} \mathcal{O}[\Phi] \ , \tag{3.11}$$

where Φ denotes all the field collectively. Suppose that the theory possess a fermionic symmetry charge (called *localizing charge*) $\mathcal{Q}$, that is $\mathcal{Q}S = 0$, that squares to a bosonic symmetry of the action K. Choosing a functional $V[\Phi]$ such that $KV = 0$, we can consider the following *localizing action*, $S_{\text{loc}} = \mathcal{Q}V[\Phi]$, and deform the PI[2] (3.11) in the following way

$$\langle \mathcal{O} \rangle(t) = \int [\mathrm{d}\Phi] e^{-S[\Phi] - tS_{\text{loc}}[\Phi]} \mathcal{O}[\Phi] \ , \tag{3.12}$$

which have a (fake) dependence on the parameter t, indeed since

$$\frac{\partial Z}{\partial t} = -\int [\mathrm{d}\Phi] \mathcal{Q}V[\Phi] e^{-S[\Phi] - tS_{\text{loc}}[\Phi]} \mathcal{O}[\Phi] = -\int [\mathrm{d}\Phi] \mathcal{Q}\left(V[\Phi] e^{-S[\Phi] - tS_{\text{loc}}[\Phi]} \mathcal{O}[\Phi] \right) \ , \tag{3.13}$$

if the measure $[\mathrm{d}\Phi]$ is $\mathcal{Q}$-invariant, i.e. the fermionic symmetry is non-anomalous, we have[3]

$$\frac{\partial \langle \mathcal{O} \rangle}{\partial t} = 0 \qquad \Rightarrow \qquad \langle \mathcal{O} \rangle(0) = \lim_{t \to \infty} \langle \mathcal{O} \rangle(t) \ . \tag{3.14}$$

It is important to notice the VEV of the operator is unchanged if we add to it a $\mathcal{Q}$-exact term: this means that what actually counts is the cohomology class of $\mathcal{Q}$. Similarly, this VEV is independent also of the coupling constants of $\mathcal{Q}$-exact term in the action. We can now evaluate $\langle \mathcal{O} \rangle$ using eq. (3.14). Suppose that[4] $S_{\text{loc}}|_{\text{bos}} \geq 0$, then in the limit $t \to \infty$ all the fields configuration with $S_{\text{loc}}|_{\text{bos}} > 0$ are infinitely suppressed and we remain just with field configurations Φ_0 such that $S_{\text{loc}}[\Phi_0]|_{\text{bos.}} = 0$. This is called *localization locus*[5] $\mathcal{M}_{\text{loc}}$. Expanding the fields as[6]

$$\Phi = \Phi_0 + t^{-1/2} \tilde{\Phi} \ , \tag{3.15}$$

[2] We remind that *all* PI in the present thesis are *Euclidean PI*. Consequently a filed that in Minkowski realm we would call conjugate of another, here becomes independent of that.

[3] We have also to assume that there are no boundary terms at the infinity of the filed configuration space. This may be done with a suitable choice of V.

[4] Actually, it is important that this holds on the “contour” of integration. This originates from the fact that, in Euclidean, we doubled the field content, and therefore, have to impose some “reality conditions”. These specify the “contour”. Along this contour the PI have to be convergent.

[5] It is know also as BPS locus since it turns out that one localizes on some BPS configurations.

[6] The power of t has been chosen such that the kinetic terms are canonically normalized.

for large values of t we have

$$S[\Phi] + t\mathcal{Q}V[\Phi] = S[\Phi_0] + \iint \left(\tilde{\Phi} \frac{\delta^2 S_{\text{loc}}[\Phi]}{\delta \Phi^2}\bigg|_{\Phi=\Phi_0} \tilde{\Phi} \right) + O(t^{-1/2}) \,, \tag{3.16}$$

where in the second term on RHS the integral are on the worldsheet coordinates, that we omitted. We see that this is a standard Gaussian integral, therefore

$$\langle \mathcal{O} \rangle(0) = \lim_{t\to\infty} \langle \mathcal{O} \rangle(t) = \int_{\mathcal{M}_{\text{loc}}} [\mathrm{d}\Phi_0] e^{S[\Phi_0]} \left[\text{sdet} \left(\frac{\delta^2 S_{\text{loc}}[\Phi]}{\delta \Phi^2}\bigg|_{\Phi=\Phi_0} \right) \right]^{-1} \mathcal{O}[\Phi_0] \,, \tag{3.17}$$

where sdet is the super-determinant. Some comments are in order: first, when $\mathcal{M}_{\text{loc}}$ degenerates to constant field configurations, we end up with a finite dimensional integral: we reduced the original PI to a so-called *matrix model*. Second, the computation of sdet, that is the fermions one-loop functional determinants over that ones for bosons, can be made easy, or feasible, thanks to SUSY: several cancellations occur and sometimes is possible to compute it even if the boson and fermion one-loop determinants unknown [**25**, **110**]. Third, there is a caveat in computing sdet: there can appear zero-modes both for bosons and fermions. In this case, the sdet has to be computed without these zero-modes, and the integration over these has to be carried out separately. Finally we remark that formula (3.17) is *exact*.

3.2.2 Application to the Elliptic Genus

Let us now apply this technique to the computation of elliptic genera. Consider a $\mathcal{N} = (0,2)$ theory with gauge group G with some matter in certain representation $\mathfrak{R}$. The effect of the insertion of $\zeta_a^{K_a}$ in the path integral amounts to switch on a flat background connection for the flavor symmetry:

$$\xi_a = \oint_{a\text{-cycle}} A_a^{\text{flav.}} - \tau \oint_{b\text{-cycle}} A_a^{\text{flav.}} \,, \tag{3.18}$$

this is equivalent to specifying non-trivial boundary condition twisted by flavor symmetry. The effect of this flat connection is to covariantize the derivatives w.r.t. the flavor group K. Then the elliptic genus reads

$$\mathcal{I}_{\text{EG}} = \int_{\text{PBC}} [\mathrm{d}\Phi] e^{-S[\Phi,\xi]} \,, \tag{3.19}$$

where "PBC" means periodic boundary conditions[7], with $S = S_{\mathbf{V}} + S_{\mathbf{\Phi V}} + S_{\mathbf{\Lambda\Phi}} + S_{\mathcal{J}}$ and $[\mathrm{d}\Phi]$ we denote the functional measure of all fields collectively. At this point we have to choose a localizing charge: we set $\mathcal{Q} = \mathsf{Q}_+ + \bar{\mathsf{Q}}_+$. Clearly S is $\mathcal{Q}$-closed, that is $\mathcal{Q}S = 0$, but, thanks to eqs. (2.64), (2.69) and (2.73), we discover that S is also $\mathcal{Q}$-exact, that is $S = \mathcal{Q}V$, for a certain V. In particular, we can take $e^{-2}S_{\mathbf{V}} + g^{-2}(S_{\mathbf{\Phi V}} + S_{\mathbf{\Lambda V}})$ as localizing actions: this means that the quantity

$$\int_{\text{PBC}} [\mathrm{d}A][\mathrm{d}\bar{A}][\mathrm{d}\lambda^+][\mathrm{d}\bar{\lambda}^+][\mathrm{d}D][\mathrm{d}\varphi][\mathrm{d}\bar{\varphi}][\mathrm{d}\psi^-][\mathrm{d}\bar{\psi}^-][\mathrm{d}G][\mathrm{d}\bar{G}] e^{-e^{-2}S_{\mathbf{V}} - g^{-2}(S_{\mathbf{\Phi V}} + S_{\mathbf{\Lambda V}}) - S_{\mathcal{J}}} \,, \tag{3.20}$$

[7] In particular, the "RR" subscript regards spatial periodicity (for both right- and left-movers), while the insertion of $(-1)^F$ specify that also time boundary conditions should be periodic (for both right- and left-movers).

does not depend on e and g, since its derivative w.r.t. these parameters is $\mathcal{Q}$-exact. We say that the path integral localizes on zeroes of $S_{\mathbf{V}}$, $S_{\mathbf{\Phi V}}$ and $S_{\mathbf{\Lambda V}}$ in the limit $e, g \to 0$. In order to determine the localization locus $\mathcal{M}_{\text{loc}}$ we have to impose some reality conditions. We choose

$$A^\dagger = \bar{A}\,, \qquad (\lambda^+)^\dagger = \bar{\lambda}^+\,, \qquad D^\dagger = D\,, \qquad \varphi^\dagger = \bar{\varphi}\,, \qquad (\psi^-)^\dagger = \bar{\psi}^- \qquad G^\dagger = \bar{G}\,. \tag{3.21}$$

We see from eq. (2.63) that the localization locus for the vector superfield consists of flat connections of $\mathcal{F}_{z\bar{z}} = 0$ (modulo gauge transformation) which, if G has connected and simply-connected non-Abelian part, turns out to be

$$\mathcal{M}_{\text{loc}} = \mathfrak{M}/W\,, \qquad\qquad \mathfrak{M} = \mathfrak{h}/(\Gamma_{\mathfrak{h}} + \tau\Gamma_{\mathfrak{h}}) \simeq T^{2r}\,, \tag{3.22}$$

where r is the rank of G, $\mathfrak{h}$ is the Cartan subalgebra, $\Gamma_{\mathfrak{h}}$ is the coroot lattice and W is the Weyl group. We parameterize this flat connections along Cartan generators with u_a. As far as the gaugino is concerned, every component along the Cartan has a fermionic zero-mode since is not charged under anything. Then, to attain the zero of $S_{\mathbf{V}}$ we must have also $D = 0$. The matter actions eqs. (2.67) attain zero value when all the fields are vanishing. In order to go on with the computation let us analyze the rank-one case which is simpler and then we will state the result for arbitrary rank.

3.2.3 Rank one case

A generic field ϕ with PBC on T^2 can be decomposed into its Fourier modes [**111**]

$$\phi(z,\bar{z}) = \phi_0 + \sum_{\{n,m\}\neq\{0,0\}} c_{n,m}\phi_{n,m}(z,\bar{z})\,, \quad \phi_{n,m}(z,\bar{z}) = \exp\left\{\frac{\pi}{\tau_2}\left(n(z-\bar{z}) + m(\tau\bar{z} - \bar{\tau}z)\right)\right\}\,, \tag{3.23}$$

where ϕ_0 is the constant zero-mode. Then it is convenient to split the field in the following way

$$A = \bar{u} + ea\,, \qquad \bar{A} = u + e\bar{a}\,, \qquad \tilde{\lambda}^+ = \tilde{\lambda}_0^+ + e\tilde{\tilde{\lambda}}^+\,, \qquad D = D_0 + e\hat{D}\,. \tag{3.24}$$

Then the elliptic genus amounts to

$$\mathcal{I}_{\text{EG}} = \frac{1}{|W|}\int_{\mathfrak{M}} \mathrm{d}u\mathrm{d}\bar{u}\mathrm{d}\lambda_0^+\mathrm{d}\bar{\lambda}_0^+ \int_{\mathbb{R}} \mathrm{d}D_0 \mathcal{Z}(u,\bar{u},\lambda_0^+,\bar{\lambda}_0^+,D_0)\,, \tag{3.25}$$

where $\mathcal{Z}$ is

$$\mathcal{Z}(u,\bar{u},\lambda_0^+,\bar{\lambda}_0^+,D_0) = \int_{\text{PBC}} [\mathrm{d}a][\mathrm{d}\bar{a}][\mathrm{d}\hat{\lambda}^+][\mathrm{d}\hat{\bar{\lambda}}^+][\mathrm{d}\hat{D}][\mathrm{d}\varphi][\mathrm{d}\bar{\varphi}][\mathrm{d}\psi^-][\mathrm{d}\bar{\psi}^-][\mathrm{d}G][\mathrm{d}\bar{G}]e^{-e^{-2}S_{\mathbf{V}} - g^{-2}(S_{\mathbf{\Phi V}} + S_{\mathbf{\Lambda V}}) - S_J}\,. \tag{3.26}$$

Notice that switching off the interactions, that is performing the limit $e, g \to 0$, the expression $\mathcal{Z}(u,\bar{u},0,0,D_0)$ reduces to a products of one-loop determinants[8] with zero-mode removed and possibly regulated by D_0:

$$\lim_{e,g\to 0} \mathcal{Z}(u,\bar{u},0,0,D_0) = Z_{1-\text{loop}}^{\text{vec}}(u) \prod_{\rho\in\mathfrak{R}^{\text{c}}} Z_{1-\text{loop}}^{\text{chiral}}(\rho;u,\bar{u},D_0) \prod_{\rho\in\mathfrak{R}^{\text{f}}} Z_{1-\text{loop}}^{\text{Fermi}}(\rho;u)\,, \tag{3.27}$$

[8] The contribution from the classical action is absent since all the action is $\mathcal{Q}$-exact.

where ρ are the weights of representations $\mathfrak{R}^{\mathrm{c}}$ and $\mathfrak{R}^{\mathrm{f}}$ of the group $\mathrm{G}\times\mathrm{K}$. For the latter we have introduced the fugacities along the Cartan $\zeta_a = e^{2\pi\mathrm{i}\xi_a}$, as in eq. (3.18). Using expansion (3.23) it is easy to compute that functional determinants. Notice that, as far as the vector multiplet is concerned, we have to perform a gauge fixing: we take[9]

$$(\partial\bar{a})^2 + (\bar{\partial}a)^2 = 0 \ . \tag{3.29}$$

We have also to include the ghost action[10]

$$S_{\mathrm{ghost}} = 4\int \mathrm{d}z\mathrm{d}\bar{z}\,(\bar{c}\boldsymbol{D}\bar{\boldsymbol{D}}c) \ . \tag{3.30}$$

The contribution from the vector field cancels against that from ghost and we remain with

$$Z^{\mathrm{vec}}_{1-\mathrm{loop}}(u) = \prod_{n,m}{}' (n-\tau m) \times \prod_{\alpha\in\mathcal{R}_{\mathrm{G}}}\prod_{n,m}(n-\tau m+\alpha(u)) \tag{3.31}$$

$$= -2\pi\mathrm{i}\eta^2(\tau)\prod_{\alpha\in\mathcal{R}}\mathrm{i}\frac{\theta_1(\tau|\alpha(u))}{\eta(\tau)} \ , \tag{3.32}$$

where the prime means that we have excluded $\{n,m\}=\{0,0\}$ from the product and we denoted the roots of G as $\mathcal{R}_{\mathrm{G}}$. Using the ζ-regularization (see [**111**, **112**]), and Dedekind η, defined in B, the contribution form the chiral multiplet reads

$$Z^{\mathrm{chiral}}_{1-\mathrm{loop}}(\rho;u,\bar{u},D_0) = \prod_{m,n}\frac{n+\bar{\tau}m+\rho(\bar{u},\bar{\xi})}{|n+\tau m+\rho(u,\xi)|^2+\mathrm{i}\rho(D_0)} \ , \tag{3.33}$$

which, for $D_0 = 0$ simplifies to

$$Z^{\mathrm{chiral}}_{1-\mathrm{loop}}(\rho;u,\bar{u},0) = \mathrm{i}\frac{\eta(\tau)}{\theta_1(\tau|\rho(u,\xi))} \ ; \tag{3.34}$$

notice that the dependence on $\bar{u}$ disappears. The one-loop determinant for a Fermi multiplet is

$$Z^{\mathrm{Fermi}}_{1-\mathrm{loop}}(\rho;u) = \prod_{n,m}(n-\tau m+\rho(u,\xi)) \tag{3.35}$$

$$= \mathrm{i}\frac{\theta_1(\tau|\rho(u,\xi))}{\eta(\tau)} \ , \tag{3.36}$$

We used the shorthand $\rho(u,\xi) = \rho_{\mathrm{G}}(u) + \rho_{\mathrm{K}}(z)$, where $\rho_{\mathrm{G}}(u)$ is the weight of gauge representation while $\rho_{\mathrm{K}}(z)$ is the weight of flavor one. We observe that the "roots part" of determinant for vector multiplet (3.31) equals that of a Fermi multiplet (3.35); this is because in two dimensions the gauge field is not dynamical and thus vector a Fermi multiplet have the same degrees of freedom. We observe that the one-loop determinant for the chiral multiplet, when $D_0 = 0$, (3.34), develops some poles on the hyperplane

$$H_i = \{Q_i u + \rho_{\mathrm{K}}(\xi) = 0 \mod \mathbb{Z}+\tau\mathbb{Z}\} \ , \tag{3.37}$$

[9]The reader is certainly familiar with this condition enforced by the gauge fixing Lagrangian

$$\frac{1}{2\kappa}(\boldsymbol{D}^{\mu}a_{\mu}) \ , \tag{3.28}$$

written with Lorentz indices.

[10]Even if in the Abelian case ghosts do not interact with other field, we have to take in account of them when computing functional determinants.

where $Q_i = \rho_{\mathrm{G}}(\cdot)$. Let us define in $\mathfrak{M}$ the set

$$\mathfrak{M}_{\text{sing.}} = \bigcup_i H_i \ . \tag{3.38}$$

We are now ready to compute the elliptic genus using eq. (3.25): we have already observed that it is independent of e and g and moreover we have explicitly computed one-loop determinants in the limit $e, g \to 0$, so we will take this limit to evaluate $\mathcal{I}_{\mathrm{EG}}$. The limit $g \to 0$ can be brought inside the integral while we saw that for $u^* \in \mathfrak{M}_{\text{sing.}}$ we encounter some poles which appears in the limit $e \to 0$. A careful analysis (see [**60**, **113**]) shows that, for finite values of e, the integrand is convergent and it is bounded by a function of e; so we can remove a small ε-neighborhood of u^* which we call Δ_ε and take a scaling limit $\varepsilon, e \to 0$ in which ε goes to zero fast enough[11], since the integral inside the region Δ_ε gives a vanishing contribution. Thus we have from eq. (3.25)

$$\mathcal{I}_{\mathrm{EG}} = \frac{1}{|W|} \lim_{e,\varepsilon\to 0} \int_{\mathfrak{M}\setminus\Delta_\varepsilon} \mathrm{d}u\mathrm{d}\bar{u} \int_{\mathbb{R}+\mathrm{i}\eta} \mathrm{d}D_0 \int \mathrm{d}\lambda_0^+ \mathrm{d}\bar{\lambda}_0^+ \mathcal{Z}(u, \bar{u}, \lambda_0^+, \bar{\lambda}_0^+)\big|_{g=0} \ . \tag{3.39}$$

Notice that we shifted the contour of integration on D_0 by $\mathrm{i}\eta$ (with $\eta \in \mathbb{R}$) since for $u \notin \Delta_\varepsilon$ the function $\mathcal{Z}$ is holomorphic around the origin. To proceed we notice that zero-modes can be arranged[12] into off-shell supermultiplet:

$$\mathsf{Q}_+ u = 0 \ , \quad \mathsf{Q}_+ \bar{u} = \tfrac{1}{2}\bar{\lambda}_0^+ \ , \quad \mathsf{Q}_+ \lambda_0^+ = 0 \ , \quad \mathsf{Q}_+ \bar{\lambda}_0^+ = -D_0 \ , \quad \mathsf{Q}_+ D_0 = 0 \ , \tag{3.40}$$

$$\bar{\mathsf{Q}}_+ u = 0 \quad \bar{\mathsf{Q}}_+ \bar{u} = -\tfrac{1}{2}\lambda_0^+ \ , \quad \bar{\mathsf{Q}}_+ \lambda_0^+ = -D_0 \ , \quad \bar{\mathsf{Q}}_+ \bar{\lambda}_0^+ = 0 \ , \quad \bar{\mathsf{Q}}_+ D_0 = 0 \ , \tag{3.41}$$

therefore, since $\bar{\mathsf{Q}}_+ \mathcal{Z} = 0$ we have

$$\frac{\partial^2 \mathcal{Z}}{\partial\lambda_0^+ \partial\bar{\lambda}_0^+} = \frac{1}{2D_0} \frac{\partial Z}{\partial \bar{u}}\bigg|_{\lambda_0^+ = \bar{\lambda}_0^+ = 0} \ . \tag{3.42}$$

We can plug this relation into eq. (3.39) and using Stokes theorem we have

$$\mathcal{I}_{\mathrm{EG}} = \frac{1}{|W|} \lim_{e,\varepsilon\to 0} \int_{\partial\Delta_\varepsilon} \mathrm{d}u \int_{\mathbb{R}+\mathrm{i}\eta} \frac{\mathrm{d}D_0}{D_0} \mathcal{Z}(u, \bar{u}, 0, 0, D_0)\big|_{g=0} \ . \tag{3.43}$$

From the chiral one-loop determinant (3.33) we see that for every ρ and the corresponding Q_i the contribution giving singularities for $\varepsilon \to 0$ is of the form

$$\frac{1}{\varepsilon^2 + \mathrm{i}Q_i D_0} \ , \tag{3.44}$$

since we have excised the region Δ_ε and therefore $|\rho(u, z)| \sim \varepsilon$ on $\partial\Delta_\varepsilon$. Therefore, in the integrand of eq. (3.43) we will have product of functions such as (3.44). Under the technical assumption that all the charges Q_i have the same sign[13] we see that the poles of that integrand occur at

$$D_0 = 0 \ , \qquad\qquad D_0 = \mathrm{i}\frac{\varepsilon^2}{Q_i} \ . \tag{3.45}$$

[11] We must have $\varepsilon^2 e^{-M} \to 0$ being M the number of quasi-zero modes, that is the zero-modes that we have for finite e.

[12] This was first noted in [**50**] in the context of SUSY 3d theories, see [**113**] for the 2d case.

[13] We see that in the case of just one charge, this assumption is automatically met. More generally, as will be explained, the case in which the number of singular hyperplanes that met at certain u^* equals the rank r is called *regular*, otherwise *singular*. We will deal with singular case, for general r in the following sections.

If $\eta > 0$ and $Q_i < 0$ the poles lies all above the contour of integration (they collapse towards $D_0 = 0$ as $\varepsilon \to 0$) therefore we can close the contour in the upper half plane and see that the integral vanishes. If $\eta > 0$ and $Q_i > 0$ instead we can close the contour from below[14] and we get for each u^*

$$\frac{1}{|W|}\lim_{e,\varepsilon\to 0}\int_{\partial\Delta_\varepsilon} \mathrm{d}u\mathcal{Z}(u,\bar{u},0,0,0) = \frac{1}{|W|}\operatorname*{Res}_{u=u^*} Z_{1-\mathrm{loop}}(u)\ , \tag{3.46}$$

where, from eq. (3.27) we set

$$Z_{1-\mathrm{loop}}(u) = Z^{\mathrm{vec}}_{1-\mathrm{loop}}(u,\bar{u})\prod_{\rho\in\mathfrak{R}^{\mathrm{c}}} Z^{\mathrm{chiral}}_{1-\mathrm{loop}}(\rho;u,\bar{u},0)\prod_{\rho\in\mathfrak{R}^{\mathrm{f}}} Z^{\mathrm{Fermi}}_{1-\mathrm{loop}}(\rho;u)\ . \tag{3.47}$$

Notice that $Z_{1-\mathrm{loop}}$ is a meromorphic function of u on T^2. Then, summing over all the poles with $Q_i > 0$ we have

$$\mathcal{I}_{\mathrm{EG}} = \frac{1}{|W|}\sum_{u^*\in\mathfrak{M}^+_{\mathrm{sing.}}}\operatorname*{Res}_{u=u^*} Z_{1-\mathrm{loop}}(u)\ . \tag{3.48}$$

We called $\mathfrak{M}^+_{\mathrm{sing.}}$ the subset of $\mathfrak{M}_{\mathrm{sing.}}$ for which all $Q_i > 0$. Of course we could have done the same reasoning starting with $\eta < 0$ then, keeping in account the orientation of integration we would have get

$$\mathcal{I}_{\mathrm{EG}} = -\frac{1}{|W|}\sum_{u^*\in\mathfrak{M}^-_{\mathrm{sing.}}}\operatorname*{Res}_{u=u^*} Z_{1-\mathrm{loop}}(u)\ , \tag{3.49}$$

with obvious definition $\mathfrak{M}^-_{\mathrm{sing.}}$ the subset of $\mathfrak{M}_{\mathrm{sing.}}$ for which all $Q_i < 0$. Notice that eqs. (3.48) and (3.49) are consistent since $\mathfrak{M}_{\mathrm{sing.}} = \mathfrak{M}^+_{\mathrm{sing.}} \sqcup \mathfrak{M}^-_{\mathrm{sing.}}$ and the sum of poles of a meromorphic function on T^2 vanishes. We see therefore that the final result does not depend on the choice of η. We conclude this subsection by mentioning [**114**] in which the authors compute the 2d superconformal index of $\mathcal{N} = (2,2)$ gauge theory. They get the same $Z_{1-\mathrm{loop}}$ however the origin of the JK prescription is not straightforward.

3.2.4 Higher Rank: Jeffrey–Kirwan residue

The result presented in the previous subsection can be generalized for arbitrary r: while physical ideas are the same, the derivation become technically more involved because of the richer topology of $\mathfrak{M}$ and its singular subset $\mathfrak{M}_{\mathrm{sing.}}$. The original generalization was carried out in [**61**] while the strategy of derivation was a bit simplified in [**50**] in the context of three dimensional theories. As before we have to compute one-loop determinants: the only modification with respect to the rank-one case is the vector determinant that becomes

$$Z^{\mathrm{vec}}_{1-\mathrm{loop}}(u) = \left[{\prod_{n,m}}'(n-\tau m)\right]^r \times \prod_{\alpha\in R_G}\prod_{n,m}(n-\tau m+\alpha(u)) \tag{3.50}$$

$$= [-2\pi\mathrm{i}\eta^2(\tau)]^r\prod_{\alpha\in\mathrm{roots}}\mathrm{i}\frac{\theta_1(\tau|\alpha(u))}{\eta(\tau)}\ , \tag{3.51}$$

[14] Remember that the ε limit is outside the integral, therefore we have to perform the integral with a fixed (finite) value of ε.

the exponent r is due to the fact that the Cartan subalgebra is now r-dimensional. As before we have to integrate over the moduli space $\mathfrak{M} \sim T^{2r}$, roughly

$$\mathcal{I}_{\text{EG}} = \frac{1}{|W|} \int_{\mathfrak{M}} \mathrm{d}^r u \mathrm{d}^r \bar{u} \mathrm{d}^r \lambda_0^+ \mathrm{d}^r \bar{\lambda}_0^+ \int_{\mathbb{R}^r} \mathrm{d}^r D_0 \mathcal{Z}(u, \bar{u}, \lambda_0^+, \bar{\lambda}_0^+, D_0) \ , \tag{3.52}$$

where again $\mathcal{Z}$ is the result of path-integration over all massive modes. The difference is that this time our integrals are multi-dimensional. As before we define singular hyperplanes[15] $H_i \subset \mathfrak{M}$

$$H_i = \{Q_i(u) + \rho_{\text{K}}(\xi) = 0 \mod \mathbb{Z} + \tau\mathbb{Z}\} \ , \tag{3.53}$$

this time $Q_i \in \mathfrak{h}^*$. Then in $\mathfrak{M}_{\text{sing.}} = \bigcup_i H_i$ we define

$$\mathfrak{M}^*_{\text{sing.}} = \{u^* \in \mathfrak{M}_{\text{sing.}} |\ \text{at least } r \text{ linearly independent } H_i\text{'s meet at } u^*\} \ . \tag{3.54}$$

We will denote by Q the set of all charge covectors and by $Q(u^*) = \{Q_i^\intercal\}_{i=1}^l$ the set of charge of the hyperplanes meeting at u^*. We will call it the *charge matrix* at u^*. The dimensions of that matrix are $r \times l$ where l is the number of hyperplanes meeting at u^*. The case in which $l = r$ is called *regular*[16] while, the case in which $l > r$ is called *singular*.[17]. We will explain in the next section that, it is possible, in some cases, to reduce the singular case to regular ones; therefore we will be mainly interested in the latter. A technical hypothesis that is needed to proceed [**115**] is that for any $u^* \in \mathfrak{M}^*_{\text{sing.}}$ the set $Q(u^*)$ is contained in a half-space of $\mathfrak{h}^*$; such arrangement of plane is called *projective*. This hypothesis is automatically fulfilled in the regular cases. Furthermore let us introduce $\text{Cone}_{\text{sing.}}(Q) \subset \mathfrak{h}^*$ as the union of all cones generated by all subset of Q with $r-1$ elements. Each connected component of $\mathfrak{h}^* \setminus \text{Cone}_{\text{sing.}}(Q)$ is called a *chamber*. Then we have to choose a $\eta \in \mathfrak{h}^*$ which is generic, i.e. $\eta \notin \text{Cone}_{\text{sing.}}(Q)$: such η identifies a chamber in $\mathfrak{h}^*$. This parameter is actually a generalization of the η we encountered before, since its rôle is to specify the integration contour, i.e. to specify which are the poles we have to take the residue at. Under the genericity assumption the elliptic genus is computed by

$$\mathcal{I}_{\text{EG}} = \frac{1}{|W|} \sum_{u^* \in \mathfrak{M}^*_{\text{sing.}}} \underset{u=u^*}{\text{JK-Res}}(Q(u^*), \eta) Z_{1-\text{loop}}(u) \prod_{i=1}^r \mathrm{d}u_i \ , \tag{3.55}$$

where JK-Res is the so called *Jeffrey–Kirwan residue* [**62**] (see also [**116**] for the conjecture from which it originates) which will be explained in the following. Differently from the usual (iterated) residue, this operation depends on the charge matrix $Q(u^*)$, which is an external data[18], and also on η. Actually it is a locally constant function of η: it can jump as η corsses form one chamber to another; however the sum of all the contributions is independent of η. The Jeffrey-Kirwan residue operation is a linear functional on the space of meromorphic r-forms (this is the reason why in eq. (3.55) we included the differentials) that are holomorphic in the complement of the singular hyperplanes arrangement $\mathfrak{M}_{\text{sing.}}$. In the regular case, for U a

[15] We want to stress that while H_i are defined in the complex space $\mathbb{C}^r$, the coefficients determining them $Q_i \in \mathbb{R}$.

[16] Intuitively the regular case is when the "order of singularity", that is l equals the number of integrals r we are performing.

[17] The case in which $l < r$ is not interesting since the result trivially vanishes.

[18] This means that it is not contained in $Z_{1-\text{loop}}$.

neighborhood of a pole u^*, the homology group $H_r(U \setminus \mathfrak{M}_{\text{sing.}}, \mathbb{Z}) = \mathbb{Z}$, so that, its generator is defined up to a sign. This means that we can define the residue at u^* by its integral over $\prod_{i=1}^r \mathcal{C}_i$, where $\mathcal{C}_i$ is a small circle around H_i. In a singular case instead, the homology group has more than one generator and therefore one must specify unambiguously the contour; however, as already said, we will not need to do that, since we will subsume the singular case to the regular one. For a projective arrangement the JK residue is defined by[19]

$$\underset{u=u^*}{\text{JK-Res}}(Q(u^*), \eta) \frac{\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_r}{Q_{j_1}(u-u^*) \ldots Q_{j_r}(u-u^*)} = \begin{cases} |\det(Q_{j_1}, \ldots, Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}, \ldots, Q_{j_r}) \\ 0 & \text{otherwise} \end{cases} . \tag{3.56}$$

A constructive definition of JK residue as a sum of iterated residues has been given in [**115**] and reviewed in [**61**]. Let us now specialize $r = 1$ in eq. (3.56): we have a single charge Q so that

$$\underset{u=0}{\text{JK-Res}}(Q, \eta) \frac{\mathrm{d}u}{Qu} = \begin{cases} |Q|^{-1} & \text{if } Q\eta > 0 \\ 0 & \text{if } Q\eta < 0 \end{cases} , \tag{3.57}$$

which implies that

$$\underset{u=0}{\text{JK-Res}}(Q, \eta) \frac{\mathrm{d}u}{u} = \begin{cases} \operatorname{sign} Q & \text{if } Q\eta > 0 \\ 0 & \text{if } Q\eta < 0 \end{cases} . \tag{3.58}$$

From this expression we see that first of all eq. (3.55) for rank one gauge group coincides with eqs. (3.48) (setting $\eta = 1$) and (3.49) (setting $\eta = -1$). Moreover, as anticipated, we see that the operation of taking JK residue does depend also on data which are not inside the function we are taking residue of.

The reader interested in explicit examples of application of the technique of JK residue can consult [**61**]. Another renowned example in which JK residues are used, albeit surreptitiously, is the computation of [**11**], as has been explained in [**118**].

3.2.5 *Explicit formulae for $\mathcal{N} = (0,2)$ and $\mathcal{N} = (2,2)$ theories*

Here we recap formulae that we derived or explained below and we will also extend them to $\mathcal{N} = (2,2)$ theories. It is convenient to absorb the differentials of eq. (3.55) inside the vector one-loop determinant. Therefore we have

$$Z^{(0,2)}_{\mathbf{V},\mathrm{G}}(\tau|u) = (-2\pi \mathrm{i} \eta^2(\tau))^r \prod_{\alpha \in \mathcal{R}_\mathrm{G}} \mathrm{i} \frac{\theta_1(\tau|\alpha(u))}{\eta(\tau)} \prod_{i=1}^r \mathrm{d}u_i , \tag{3.59}$$

$$Z^{(0,2)}_{\mathbf{\Phi},\mathfrak{R}}(\tau|u,\xi) = \prod_{\rho \in \mathfrak{R}} \mathrm{i} \frac{\eta(\tau)}{\theta_1(\tau|\rho(u,\xi))} \tag{3.60}$$

$$Z^{(0,2)}_{\mathbf{\Lambda},\mathfrak{R}}(\tau|u,\xi) = \prod_{\rho \in \mathfrak{R}} \mathrm{i} \frac{\theta_1(\tau|\rho(u,\xi))}{\eta(\tau)} . \tag{3.61}$$

[19] See [**117**] for details regarding consistency of the definition.

Then we collect all these contributions

$$Z^{(0,2)}_{1-\text{loop}}(\tau|u,\xi) = Z_{\mathbf{V},\text{G}}(\tau|u)\prod_c Z_{\mathbf{\Phi}_c,\mathfrak{R}_c}(\tau|u,\xi)\prod_f Z_{\mathbf{\Lambda}_f,\mathfrak{R}_f}(\tau|u,\xi)\ ; \tag{3.62}$$

therefore

$$\mathcal{I}^{(0,2)}_{\text{EG}}(\xi) = \frac{1}{|W|}\sum_{u^*\in\mathfrak{M}^*_{\text{sing.}}} \underset{u=u^*}{\text{JK-Res}}(Q(u^*),\eta) Z^{(0,2)}_{1-\text{loop}}(\tau|u,\xi)\ , \tag{3.63}$$

From the point of view of a $\mathcal{N}=(0,2)$ theory, the left-moving $\mathcal{R}$-charge of a $\mathcal{N}=(2,2)$ one, is simply a flavor symmetry. Therefore we will consider the "effective" flavor group $\text{U}(1)_L\times\text{K}$ to subsume eq. (3.9) to eq. (3.10). In particular, from our analysis in chap. 2, we have the following decomposition

$$\text{R}_L[\mathbf{V}^{(2,2)}]=0 \qquad \Rightarrow \qquad \text{R}_L[\mathbf{V}^{(0,2)}]=0\ , \qquad \text{R}_L[\boldsymbol{\chi}^{(0,2)}]=-1\ , \tag{3.64}$$

$$\text{R}_L[\mathbf{\Phi}^{(2,2)}]=r_L \qquad \Rightarrow \qquad \text{R}_L[\mathbf{\Phi}^{(0,2)}]=r_L\ , \qquad \text{R}_L[\mathbf{\Lambda}^{(0,2)}]=r_L-1\ . \tag{3.65}$$

Setting $y=e^{2\pi i\epsilon}$ it follows that

$$Z^{(2,2)}_{\mathbf{V},\text{G}}(\tau|u,\epsilon) = \left[\frac{2\pi\eta^3(\tau)}{\theta_1(\tau|-\epsilon)}\right]^r \prod_{\alpha\in\mathcal{R}_\text{G}}\left[-\frac{\theta_1(\tau|\alpha(u))}{\theta_1(\tau|\alpha(u)-\epsilon)}\right]\prod_{i=1}^r \text{d}u_i\ , \tag{3.66}$$

$$Z^{(2,2)}_{\mathbf{\Phi},\mathfrak{R},r_L}(\tau|u,\xi,\epsilon) = \prod_{\rho\in\mathfrak{R}}\frac{\theta_1(\tau|\rho(u,\xi)+(r_L-1)\epsilon)}{\theta_1(\tau|\alpha(u,\xi)+r_L\epsilon)}\ . \tag{3.67}$$

Then

$$Z^{(2,2)}_{1-\text{loop}}(\tau|u,\xi,\epsilon) = Z^{(2,2)}_{\mathbf{V},\text{G}}(\tau|u,\epsilon)\prod_c Z^{(2,2)}_{\mathbf{\Phi}_c,\mathfrak{R}_c,r_{-,c}}(\tau|u,\xi,\epsilon)\ . \tag{3.68}$$

We observe that the q-expansion of eq. (3.62) can start with some non-trivial powers q^E, and we can interpret E as the Casimir energy of the multiplet; the expansion of eq. (3.68), on the other hand, has always $E=0$. We also notice that in the standard literature [**61**, **113**], $Z^{(2,2)}_{\mathbf{V},\text{G}}$ has an irrelevant minus sing in the product over the roots of the group. We will also drop this sign in the following computations, not to deviate from common usage. Finally the elliptic genus is obtained by

$$\mathcal{I}^{(2,2)}_{\text{EG}}(\xi,\epsilon) = \frac{1}{|W|}\sum_{u^*\in\mathfrak{M}^*_{\text{sing.}}} \underset{u=u^*}{\text{JK-Res}}(Q(u^*),\eta) Z^{(2,2)}_{1-\text{loop}}(\tau|u,\xi,\epsilon)\ , \tag{3.69}$$

where we restored the dependence on ξ and ϵ.

In both $\mathcal{N}=(0,2)$ and $\mathcal{N}=(2,2)$ cases, to compute the elliptic genus the following procedure has to be followed:

1. Write down the matter content of the theory specifying representations of every multiplet (in the $\mathcal{N}=(2,2)$ fix also the left-moving R-symmetry);
2. With this information write $Z_{1-\text{loop}}$;
3. Classify the poles of $Z_{1-\text{loop}}$ that arise from the intersection of *at least* r singular hyperplanes; in particular for every pole u^* specify $Q(u^*)$;

4. Choose a suitable η; this choice will select some poles and discard some others;

5. Compute the JK residues at the selected poles.

While point 1, 2 and 4 are straightforward and can be carried out painlessly, point 3 and 5 require much attention. Classify the poles of the one-loop determinant can be much lengthy in a generic situation. First of all one should list all the singular hyperplane (whose number we call A) then one should check how many all the $\binom{A}{r}$ intersections of these hyperlanes give rise to a singularity point $u^* \in \mathfrak{M}^*_{\text{sing.}}$. In fact one has to check $\binom{A}{r}$ system of r equation that can or cannot have solution. We will see that in our explicit computations it is possible to find an argument which allows us to classify all the poles at once. Another delicate point in the procedure is that, for each u^* one has to check on how many singular hyperplanes it sits. Of course this number, l, will be $l \geq r$ since we explicitly solved system of linear equation requiring u^* to be at least on r hyperplanes. As stated above this number l discriminates between regular and singular cases. For the regular case the computation of the residue is straightforward, while for the singular case extra work must be done. In the following section we will describe a useful method to reduce singular cases to regular one [**2**]. This method will be applied in our computations of elliptic genus.

3.3 Desingularization Procedure

Let us now consider a theory with some gauge group G and some flavor group K. For the sake of clarity let us restrict ourselves to the case of matter in fundamental and adjoint representation only[20]. Keeping just the dependence on gauge fugacities, the one-loop determinant can be written as[21]

$$Z(u) = \prod_{i=1}^{k} Z_i(u_1, \dots, u_i) \ , \quad Z_i(u_1, \dots, u_i) = \frac{\prod_{c_i=1}^{C_i(u^*)} \theta_1\left(\tau \middle| u_i - u_{\gamma_{i,c_i}} + s_{i,c_i}\right)}{\prod_{a_i=1}^{A_i(u^*)} \theta_1\left(\tau \middle| u_i - u_{\alpha_{i,a_i}} + r_{i,c_i}\right)} f_i(u_1, \dots, u_i) \ , \tag{3.70}$$

where f contains all the factors which are both regular and non-zero for[22] $\{u_i \to u_i^*\}_{i=1}^r$, while in the fraction we put all the others. Thus for $\{u_i \to u_i^*\}_{i=1}^r$ there will be $A(u^*) := \sum_{i=1}^r A_i(u^*)$ singular hyperplanes and $C(u^*) := \sum_{i=1}^r C_i(u^*)$ hyperplanes in on which our one-loop determinant vanishes: we will call these *zero hyperplanes*. As stated before, the interesting case is when $A \geq r$, since in tho other cases, the residue is trivially vanishing. Then $\alpha_\bullet$ and $\gamma_\bullet$ are sequences such that $0 \leq \alpha_{i,a_i} \leq i$ and $0 \leq \gamma_{i,c_i} \leq i$. In this way every Z_i depends only on u_j with $j \leq i$. We allowed also to have $u_0 := 0$ in order to subsume the contribution coming from the fundamental representation. Coefficients r_{i,a_i} and s_{i,c_i} are combination of flavor fugacities, which we do not need to specify. If $A = r$ we are in the regular case of JK procedure and we can compute recursively

$$\lim_{\{u_i \to u_i^*\}_{i=1}^r} \frac{Z(u)}{\mathcal{M}^r \prod_{i=1}^r (u_i - u_i^*)} = \mathcal{M}^{-r} \prod_{i=1}^r \lim_{u_i \to u_i^*} \frac{Z_i(u_1, \dots, u_{i-1}, u_i)}{(u_i - u_i^*)} \ , \tag{3.71}$$

[20]Every contribution coming from the fundamental representation will contain a single u_i while the ones coming from the adjoint will depend on the differences $u_i - u_j$, as the weights of these two representations suggest us.

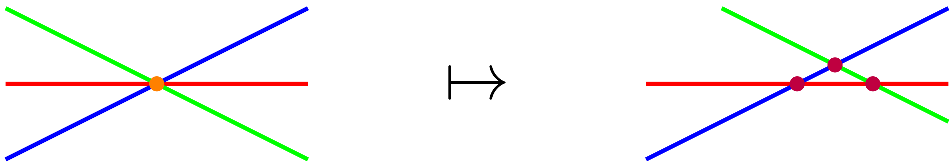
[21]This is nothing but an ordering of the factor of the one-loop determinant.

[22]Here we will explicitly write the components of the r-vector $u = (u_1, \dots, u_r)$.

where the normalization is $\mathcal{M} = -2\pi i\eta^2(\tau)$ for $\mathcal{N} = (0,2)$ theories and $\mathcal{M} = 2\pi\eta^3(\tau)$ for $\mathcal{N} = (2,2)$ ones. When $A > k$ we perturb singularities appearing in eq. (3.70) introducing some parameters β

$$Z_i(u_1, \dots, u_i) \mapsto \tilde{Z}_i(u_1, \dots, u_i) :=$$
$$:= \frac{1}{\theta_1\left(\tau \middle| u_i - u_{\alpha_i,1} + r_{i,1}\right)} \times \frac{\prod_{c_i=1}^{A_i(u^*)-1} \theta_1\left(\tau \middle| u_i - u_{\gamma_{i,c_i}} + s_{i,c_i} + \beta_{i,c_i}\right)}{\prod_{a_i=2}^{A_i(u^*)} \theta_1\left(\tau \middle| u_i - u_{\alpha_{i,a_i}} + r_{i,a_i} + \beta_{i,a_i-1}\right)} \times$$
$$\times \prod_{c_i=A_i}^{C_i(u^*)} \theta_1\left(\tau \middle| u_i - u_{\gamma_{i,c_i}} + s_{i,c_i}\right) \times f_i(u_1, \dots, u_i) \ . \quad (3.72)$$

We observe that the second factor has neither poles nor zeroes since numerator and denominator vanish simultaneously, by construction. This kind of desingularization amounts to "explode" our pole into $\binom{A(u^*)}{r}$ non-singular[23] poles, as we see in fig. 3.1.



(a) A degenerate pole. (b) Three non-degenerate poles.

Figure 3.1: An example with $k = 2$ and $A = 3$.

We can number all these poles with a r-ple $(t,p) := ((t_1,p_1), \dots, (t_r,p_r))$, where $t_i = 1, \dots, r$, $p_i = 1, \dots, A_i(u^*)$ and no duplicates (t_i,p_i) are possible. The new poles occur at[24] $u = u^*_{(t,p)}$ where $u^*_{(t,p),i}$ solves the linear system of equations

$$\left\{ u^*_{(t,p),t_i} - u^*_{(t,p),\alpha_{t_i,p_i}} + r_{t_i,p_i} + \beta_{t_i,p_i} = 0 \right\}_{i=1}^{r} , \quad (3.73)$$

whose solution, when it exists is of the form

$$u^*_{(t,p),i} = u^*_i + \sum_{i=1}^{r} \ell_{(t,p),i} \beta_{t_i,p_i} , \quad (3.74)$$

for certain coefficients $\ell_{(t,p),i}$. Now it is easy to compute residue

$$\underset{\{u_i = u^*_{(t,p),i}\}_{i=1}^{r}}{\text{Res}} \tilde{Z}(u) = \mathcal{M}^{-r} \lim_{u_i \to u^*_{(t,p),i}} \prod_{i=1}^{r} \frac{\tilde{Z}_i(u_1, \dots, u_i)}{(u_i - u^*_{(t,p),i})} , \quad (3.75)$$

in the following cases (which will be cases of interest in the following of this thesis):

- if $(t_i,p_i) = (i,1)$ for[25] $i = 1, \dots, r$ and $A_i(u^*) = C_i(u^*) + 1$ for all $i = 1, \dots, r$ we have:

$$\underset{\{u_i = u^*_{(t,p),i}\}_{i=1}^{t}}{\text{Res}} \tilde{Z}(u) = \mathcal{M}^{-r} \prod_{i=1}^{r} f_i(u^*_1, \dots, u^*_i) ; \quad (3.76)$$

[23] The new poles are non-singular for generic values of β's.
[24] The (t_i,p_i) means that we are using the p_i^{th} singular hyperplane of Z_{t_i}, to determine the intersection point.
[25] This is actually the "unshifted pole" at $u = u^*$.

- if $(t_i, p_i) \neq (i, 1)$ and $A_i(u^*) = C_i(u^*) + 1$ for at least one $i = 1, \dots, r$ we have

$$\operatorname*{Res}_{\{u_i = u^*_{(t,p),i}\}_{i=1}^r} \tilde{Z}(u) = 0 \; ; \tag{3.77}$$

- if $A_i(u^*) < C_i(u^*) + 1$ for at least one $i = 1, \dots, r$, for every pole we have

$$\operatorname*{Res}_{\{u_i = u^*_{(t,p),i}\}_{i=1}^r} \tilde{Z}(u) = 0 \; . \tag{3.78}$$

This is because in eq. (3.72) the numerator and the denominator in the second factor take the same value for $u_i = u^*_{(t,p),i}$, by construction, and because if $C_i(u^*) > A_i(u^*) - 1$ for some i, the last factor sets the whole expression to zero. The condition $A_i(u^*) = C_i(u^*) + 1$ for every i means that the order of singularity of Z is 1 for every u_i. If this condition is satisfied, we saw that, after desingularization procedure, only the "unshifted pole" (i.e. $u_i = u_i^*$) gives non-zero contribution and this contribution is independent of the desingularization parameters β's. This means that once the pole is selected by JK condition, no matter if it lies in the regular or singular case, after the (possibly required) desingularization procedure, it yields one and just one contribution. Moreover eq. (3.76) suggests also a very simple way to evaluate residues provided $A_i(u^*) = C_i(u^*) + 1$ for all $i = 1, \dots, r$: it implies that we have to evaluate $\mathcal{M}^{-r} Z(u)$ at $u = u^*$ simply dropping form it all factors (in the numerator as well as in the denominator) that vanish at this point. We call this way of evaluating the residue *regular representation* of the product. In this way the result is both finite and non-zero.

4. Holomorphic Blocks from Elliptic Genus

In this chapter we will compute in detail the elliptic genera of two moduli spaces: we will see all the procedure described in the previous chapter at work. We will learn how it is possible to classify poles of the one-loop determinant when this has a certain form. While the problem that we solve here is interesting per se, the analysis we are going to carry out will be paradigmatic for more involved computations, as the one in the next chapter. We will follow [**1**].

4.1 Elliptic Vortices and Holomorphic Blocks

In the previous chapter we see how it has been possible to apply localization to a SUSY GLSM on T^2, that is a *flat* manifold. However, as already mentioned, several exact computations have been done in curved space as well. Of course, the first step in this direction is to implement rigid SUSY in curved space; this can be done systematically [**24**]. Roughly[1], one has to couple the theory with off-shell SUGRA: this amounts to introduce a pairing between the $\mathcal{S}$-multiplet of the SUSY theory and the gravity multiplet. Then one decouples the gravitational dynamics in a given SUSY vacuum: this amounts to take the rigid limit $G_D \to 0$ keeping fixed the background for the metric and for the auxiliary fields; successively, one has to get rid of fields in gravity multiplet without imposing equations of motions nor integrating out auxiliary fields. The vanishing of gravitino variation does not contain matter fields and leads to the generalized Killing spinor equation (KSE). The solutions of this equation[2] tell us how to generalize the constant ϵ that we had in the flat case.

Exploiting this construction, the partition function of $\mathcal{N} = (2, 2)$ gauge theories on S^2 was found [**40**, **41**]. A very interesting point is that one can get the same result using localization in two different ways: that is choosing two different S_{loc}. The first closely follows what we did in flat space[3] and it known as *Coulomb branch localization*

$$Z_{S^2} = \frac{1}{|W|} \sum_{\mathfrak{m} \in \mathbb{Z}^r} \int_{\mathbb{R}^r} \left(\prod_{i=1}^{r} \mathrm{d}\sigma_i \right) e^{-S_{\text{FI}}(\sigma,\mathfrak{m})} Z^{\text{vec}}_{1-\text{loop}}(\sigma, \mathfrak{m}) \prod_{\rho \in \mathfrak{R}} Z_{1-\text{loop}^{\text{chiral}}}(\rho; \sigma, \mathfrak{m}) \,, \tag{4.1}$$

where integers over which we sum, $\mathfrak{m} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_r)$, are numbered by the flux $\mathfrak{m} = -2\pi^{-1} \int_{S^2} \mathcal{F}_{z\bar{z}}$. It is possible to find another representation of Z_{S^2} choosing another localization action depending on a size regulator χ: this is dubbed *Higgs branch localization.* In this case one integrates out the auxiliary field D and then looks for localization locus. It can be divided into three categories: the Higgs branch, the deformed Coulomb branch (which gives a vanishing contribution

[1] We suggest [**26**, **119**].

[2] The interested reader can consult [**85**].

[3] We use σ instead of u as the integration variable since, in this case $\mathcal{M}_{\text{loc}}$ can be parametrized by the scalars in the vector multiplet.

in the limit $\chi \to \pm\infty$) and the point-like *vortices/anti-vortices*, defined in the neighborhood of the poles. Therefore, to get the partition function, we have to sum over the Higgs branches $p \in \tilde{\mathcal{M}}_{\text{Higgs}}$ and in each Higgs branch we have to integrate over the moduli space of vortices/anti-vortices. The total moduli space is therefore

$$\mathcal{M}_{\text{Higgs}} = \bigsqcup_{p \in \tilde{\mathcal{M}}_{\text{Higgs}}} \left[\left(\bigcup_{k=0}^{\infty} \mathcal{M}_{p,k}^{\text{vortex}} \right) \oplus \left(\bigcup_{k=0}^{\infty} \mathcal{M}_{p,k}^{\text{anti-vortex}} \right) \right] , \quad k = -2\pi^{-1} \int_{S^2} \operatorname{tr} \mathcal{F} . \tag{4.2}$$

The partition function turns out to be

$$Z_{S^2} = \sum_{p \in \tilde{\mathcal{M}}} e^{-S_{\text{FI}}(p)} Z'_{1-\text{loop}}(p) Z_{\text{vortex}}(v,p) Z_{\text{anti-vortex}}(\bar{v},p) , \tag{4.3}$$

where $Z'_{1-\text{loop}}$ is the one-loop determinant of all fields with vanishing VEV on the Higgs branch. One has to be careful to evaluate these determinants on the vortex/anti-vortex background; this can be done by using a cohomolocal argument [40, 41, 110]. The vortex/anti-vortex contributions are the equivariant volumes of the respective moduli spaces

$$Z_{\text{vortex}}(v,p) = \int_{\mathcal{M}_{p,k}^{\text{vortex}}} e^{\omega} , \qquad Z_{\text{anti-vortex}}(\bar{v},p) = \int_{\mathcal{M}_{p,k}^{\text{anti-vortex}}} e^{\omega} , \tag{4.4}$$

here ω is U(1)-equivariantly closed[4]. The upshot is that we found a representation of the partition function which is the sum over a finite number of points; for each point the PI gets contribution from point-like vortices at the North pole and anti-vortices at the South pole [66–68, 120]. This kind of phenomenon has been observed also in higher dimensions [63–65] where the factorization of supersymmetric partition function on squashed three-spheres and on their products with circles has been noticed to happen in terms of 3d and 4d "holomorphic blocks". Such factorization has also been derived using Higgs branch localization in [121, 122]. Further examples of this phenomenon in 4d can be found in [123, 124]: the former were identified with vortex particles blocks on $\mathbb{C} \times S^1$ and the latter await a first principle computation as elliptic vortices on $\mathbb{C} \times T^2$; their description in terms of elliptic hypergeometric functions [125] is given by holomorphic factorization. In what follows we provide such a first principle evaluation, when specified to the appropriate case. This is done computing the SUSY gauge theory partition function on $\mathbb{C} \times T^2$ by resumming its expansion in rotational modes on the complex plane (elliptic vortices), each term being the elliptic genus of the corresponding moduli space: this is a generalization of eqs. (4.4) to the elliptic genus. This reproduces the results of [64]. Another motivation for this computation is to understand the algebraic structure of BPS vacua of such theories. Indeed, it is by now well known that Virasoro algebra and its generalization of W-algebrae acts on the moduli space of instantons of four dimensions [126–129]. Vortices are the two-dimensional analogue of instantons and indeed their moduli space can be obtained as special Lagrangian submanifolds of the instanton moduli space [66, 130]. It is thus interesting to investigate and unveil the algebrae acting on their equivariant cohomology spaces. A specialization of the vortex partition function we analyze in this paper can be also obtained from a six-dimensional gauge theory on $\mathbb{R}^4 \times T^2$ in presence of a codimension two defect along $\mathbb{R}^2 \times T^2$ [71, 130]. Indeed, this gives rise to a coupled 6d/4d system which reduces to the elliptic genus computation in the decoupling limit of the 6d dynamics.

[4]This is nothing but the computation on $\mathbb{R}^2$ but with Ω-background [11, 12, 67].

4.2 $\mathcal{N}=(0,2)$ theories

In this section we study the elliptic genus of moduli space of k vortices in a four-dimensional $\mathcal{N}=1$ theory with $\mathrm{U}(N)$ gauge group, N_F fundamental chirals and $\tilde{N}_\mathrm{F}$ anti-fundamental chirals, [**56**]. This moduli space is described by 2d $\mathcal{N}=(0,2)$ quiver gauge theory depicted in fig. 4.1.

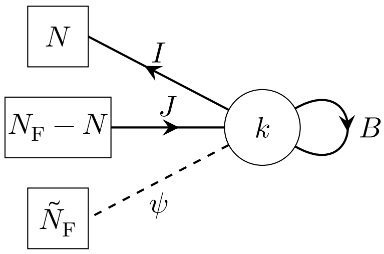


Figure 4.1: $\mathcal{N}=(0,2)$ quiver gauge theory: it has[5] $\mathrm{U}(k)$ vector multiplet, I, J and B chiral multiplets and ψ Fermi multiplet. It describes the moduli space of vortices of 4d, $\mathcal{N}=1$, $\mathrm{U}(N)$ gauge theory with N_F fundamental chirals and $\tilde{N}_\mathrm{F}$ anti-fundamental chirals.

In order to compute the elliptic genus we consider the flavor group $\mathrm{S}(\mathrm{U}(N)\times\mathrm{U}(N_\mathrm{F}-N)\times\mathrm{U}(\tilde{N}_\mathrm{F}))$. Notice that we have excluded the global $\mathrm{U}(1)$ phase from the flavor group since it is gauged. We also allow a $\mathrm{U}(1)$ holonomy for the chiral multiplet in the adjoint B.

Group	I	J	ψ	B	Chemical Potential
$\mathrm{U}(k)$	$\square$	$\overline{\square}$	$\square$	**Adj**	u_i
$\mathrm{U}(N)$	$\overline{\square}$	$\bullet$	$\bullet$	$\bullet$	z_α
$\mathrm{U}(N_\mathrm{F}-N)$	$\bullet$	$\square$	$\bullet$	$\bullet$	μ_A
$\mathrm{U}(\tilde{N}_\mathrm{F})$	$\bullet$	$\bullet$	$\overline{\square}$	$\bullet$	ν_I
$\mathrm{U}(1)_B$	0	0	0	1	$-\ell$

Table 4.1: Groups representations and related chemical potentials. $\square$ is the fundamental representation $\overline{\square}$ is the anti-fundamental, $\bullet$ is the trivial and **Adj** is the adjoint.

The chemical potentials, listed in tab. 4.1, are subject to the constraint

$$\sum_{\alpha=1}^{N} z_\alpha + \sum_{A=1}^{N_\mathrm{F}-N} \mu_A - \sum_{I=1}^{\tilde{N}_\mathrm{F}} \nu_I = 0\,. \tag{4.5}$$

The one-loop determinants are easily found

$$Z_{\mathbf{V}}^{(0,2)}(\tau|u) = \frac{1}{k!}(-2\pi\mathrm{i}\eta^2(\tau))^k \prod_{\substack{i,j=1\\ i\neq j}}^{k} \mathrm{i}\frac{\theta_1(\tau|u_{ij})}{\eta(\tau)} \prod_{i=1}^{k} \mathrm{d}u_i\,, \tag{4.6}$$

$$Z_I^{(0,2)}(\tau|u,z) = \prod_{i=1}^{k}\prod_{\alpha=1}^{N} \mathrm{i}\frac{\eta(\tau)}{\theta_1(\tau|u_i-z_\alpha)}\,, \tag{4.7}$$

[5] All the multiplets here are $\mathcal{N}=(0,2)$ multiplets.

$$Z_J^{(0,2)}(\tau|u,\mu) = \prod_{i=1}^{k}\prod_{A=1}^{N_{\mathrm{F}}-N} \mathrm{i}\frac{\eta(\tau)}{\theta_1(\tau|-u_i+\mu_A)} \ , \tag{4.8}$$

$$Z_\psi^{(0,2)}(\tau|u,\nu) = \prod_{i=1}^{k}\prod_{I=1}^{\tilde{N}_{\mathrm{F}}} \mathrm{i}\frac{\theta_1(\tau|u_i-\nu_I)}{\eta(\tau)} \ , \tag{4.9}$$

$$Z_B^{(0,2)}(\tau|u,\ell) = \prod_{i,j=1}^{k} \mathrm{i}\frac{\eta(\tau)}{\theta_1(\tau|u_{ij}-\ell)} \ , \tag{4.10}$$

where we used the shorthand $u_{ij} := u_i - u_j$. Therefore

$$\boldsymbol{\mathcal{I}}^{(0,2)}_{k,N,N_{\mathrm{F}},\tilde{N}_{\mathrm{F}}}(\tau|z,\mu,\nu,\ell) = \int_{\mathrm{JK}} Z_{\mathbf{V}}^{(0,2)}(\tau|u) Z_I^{(0,2)}(\tau|u,z) Z_J^{(0,2)}(\tau|u,\mu) Z_\psi^{(0,2)}(\tau|u,\nu) Z_B^{(0,2)}(\tau|u,\ell) \ , \tag{4.11}$$

where "JK" subscript of the integral means that we are going to take JK resiude, as explained in the preceding chapter.

4.2.1 Anomaly Cancellation

Since our 2d theory is manifestly chiral, we have to find out under which conditions gauge anomalies cancel. A very instructive way to find such conditions is to impose the double periodicity of the integrand of the partition function eq. (4.11). Such property is *not* trivially enjoyed since eq. (B.7) shows us that θ_1 is just *quasiperiodic* under a shift which is an integer times the modulus of the torus. Let us study the behaviour of the integrand in eq. (4.11) under the shift: $u_i \mapsto u_i + a + b\tau$ $(a, b \in \mathbb{Z})$:

- $Z_{\mathbf{V}}^{(0,2)}$ and $Z_B^{(0,2)}$ are left unchanged;
- $Z_I^{(0,2)} \mapsto (-1)^{kN(a+b)} e^{i\pi k N b^2 \tau} e^{2\pi \mathrm{i} N b \sum_{j=1}^{k} u_j} e^{-2\pi \mathrm{i} k b \sum_{\alpha=1}^{N} z_\alpha} Z_I^{(0,2)}$;
- $Z_J^{(0,2)} \mapsto (-1)^{k(N_{\mathrm{F}}-N)(a+b)} e^{\mathrm{i}\pi k(N_{\mathrm{F}}-N) b^2 \tau} e^{2\pi \mathrm{i}(N_{\mathrm{F}}-N) b \sum_{j=1}^{k} u_j} e^{-2\pi \mathrm{i} k b \sum_{A=1}^{N_{\mathrm{F}}-N} \mu_A} Z_J^{(0,2)}$;
- $Z_\psi^{(0,2)} \mapsto (-1)^{-k\tilde{N}_{\mathrm{F}}(a+b)} e^{-\mathrm{i}\pi k \tilde{N}_{\mathrm{F}} b^2 \tau} e^{-2\pi \mathrm{i} \tilde{N}_{\mathrm{F}} b \sum_{j=1}^{k} u_i} e^{2\pi \mathrm{i} k b \sum_{I=1}^{\tilde{N}_{\mathrm{F}}} \nu_I} Z_\psi^{(0,2)}$.

Combining all these contributions together and imposing shift invariance we get:

$$(-1)^{k(N_{\mathrm{F}}-\tilde{N}_{\mathrm{F}})(a+b)} e^{\mathrm{i}\pi k (N_{\mathrm{F}}-\tilde{N}_{\mathrm{F}}) b^2 \tau} e^{2\pi \mathrm{i}(N_{\mathrm{F}}-\tilde{N}_{\mathrm{F}}) b \sum_{j=1}^{k} u_j} \times \\ \times e^{-2\pi \mathrm{i} k b \left(\sum_{\alpha=1}^{N} z_\alpha + \sum_{A=1}^{N_{\mathrm{F}}-N} \mu_A - \sum_{I=1}^{\tilde{N}_{\mathrm{F}}} \nu_I\right)} = 1 \ . \tag{4.12}$$

The last exponential is one thanks to eq. (4.5), thus the anomaly cancels iff:

$$N_{\mathrm{F}} = \tilde{N}_{\mathrm{F}} \ , \tag{4.13}$$

This condition also ensure the anomaly cancellation in the "parent" 4d theory.

4.2.2 Polology

There are three sources of poles: the chiral multiplets. The singular hyperplanes are

$$H_{B;i,j} = \{u_{ij} - \ell = 0\} , \qquad H_{I;i,\alpha} = \{u_i = z_\alpha\} , \qquad H_{J;i,A} = \{u_i = \mu_A\} , \tag{4.14}$$

whose (gauge) charges are

$$\vec{h}_{B;i,j} = (0, \dots , \underbrace{1}_{i}, \dots , \underbrace{-1}_{j}, \dots , 0) , \quad \vec{h}_{I;i,\alpha} = (0, \dots , \underbrace{1}_{i}, \dots , 0) , \quad \vec{h}_{J;i,A} = (0, \dots , \underbrace{-1}_{i}, \dots , 0) . \tag{4.15}$$

We now have to find the intersection points u^* of k hyperplanes described by eqs. (4.14) that is we have to select all the possible groups of k equations among (4.14) that have a unique solution. This corresponds to look for a solution of the following system of linear equations

$$Q^{\mathsf{T}}(u^*) \begin{pmatrix} u_1^* \\ \vdots \\ u_k^* \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix} ; \tag{4.16}$$

here the components of $Q(u^*) = (Q_1, \dots , Q_k)$ are chosen among charges (4.15) corresponding to a given sets of hyperplanes: $Q_i = \vec{h}$ for certain[6] $\vec{h}$. Then $d_i = \ell$ if the corresponding hyperplane is of the type H_B; $d_i = z_\alpha$ for some α, if the corresponding hyperplane is of the type H_I; and $d_i = -\mu_A$, for some A, if the corresponding hyperplane is of the type H_J. However, as explained in the previous chapter, not all these u^*'s will contribute to the computation of JK residue. When exactly k hyperplanes meet at u^* and they are linearly independent the contribution is selected by JK procedure if $\eta \in \mathrm{Cone}(Q_1, \dots Q_k)$; namely if

$$Q(u^*) \begin{pmatrix} \varsigma_1 \\ \vdots \\ \varsigma_k \end{pmatrix} = \eta^{\mathsf{T}} \qquad \text{for all } \varsigma_i > 0. \tag{4.17}$$

The problem of finding the general form for such a matrix $Q(u^*)$ is solved in app. C.1: first of eq. (C.14) tells us that hyperplanes of type $H_{J;i}$ and $H_{B;i,j}$ with $i < j$ are excluded since their charge covectors are related to those appearing in eq. (C.14) by a sign flip. This would lead to a flip in the corresponding ς which would become negative. Therefore we remain with hyperplanes of type $H_{I;i}$ and of type $H_{B:i,j}$ with $i > j$. The first hyperplane of each block $Q_q(u^*)$ is of type H_I, the next ones in the same block are of type H_B. The first coordinate of a pole can therefore be taken $u_1^* = z_\alpha$ and each coordinates differ by one of the previous ones by ℓ. Therefore the poles will be labeled by a the chemical potential index α and another index r_α:

$$u^*_{\alpha,r_\alpha} = z_\alpha + (r_\alpha - 1)\ell , \tag{4.18}$$

with $r_\alpha = 1, \dots , k_\alpha$ and $\sum_\alpha k_\alpha = k$. These poles can be represented as a collection of "colored" stripes of boxes: each color represent a different α, as in fig. 4.2.

Then we see that for every pole we are in the regular case of JK procedure and we do not need desingularization procedure 3.3. We remark that all the u^*'s have to be different, otherwise the residue would vanish because of the presence of the $\theta_1(\tau|u_{ij})$ in the numerator. Of course, given a set of k poles that contribute to the elliptic genus, any of its $k!$ permutation will contribute as well; this multiplicity factor cancels the order of the Weyl group in eq. (4.6).

[6] We omit the vector sign over Q_i, as already done.

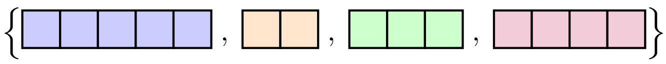


Figure 4.2: An example of "colored strip". In this case we have $k = 14$ divided in 4 colors $k_1 = 5$, $k_2 = 2$, $k_3 = 4$, $k_4 = 4$

4.2.3 Computation

Now we have to compute residues at the poles found in the preceding paragraph: this is pretty simple since we are dealing with simple poles. We have just to use eq. (B.8) with $a = b = 1$ for the θ_1 giving the pole, and to evaluate the other non singular factors at that point. Since we labelled poles with two indices (eq. (4.18)) it will be convenient to rearrange products as $\prod_{i=1}^{k} = \prod_{\alpha=1}^{N}\prod_{r_\alpha=1}^{k_\alpha}$. In this way it's easy to write the contribution of every multiplet:

$$Z_{\mathbf{V}}^{(0,2)} = \left(-2\pi \mathrm{i}\eta^2(q)\right)^k \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\prod_{r_\alpha=1}^{k_\alpha}\prod_{s_\beta=1}^{k_\beta} \mathrm{i}\frac{\theta_1(\tau|z_{\alpha\beta}+(r_\alpha-s_\beta)\ell)}{\eta(q)}\times$$
$$\times\prod_{\alpha=1}^{N}\prod_{\substack{r_\alpha,s_\alpha=1\\ r_\alpha\neq s_\alpha}}^{k_\alpha}\mathrm{i}\frac{\theta_1(\tau|(r_\alpha-s_\alpha)\ell)}{\eta(q)}\,, \tag{4.19}$$

$$Z_{I}^{(0,2)} = \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\prod_{r_\alpha=1}^{k_\alpha}\mathrm{i}\frac{\eta(q)}{\theta_1(\tau|z_{\alpha\beta}+(r_\alpha-1)\ell)}\times\prod_{\alpha=1}^{N}\left(\frac{\mathrm{i}\eta(q)}{2\pi\eta^3(q)}\right)\prod_{r_\alpha=2}^{k_\alpha}\mathrm{i}\frac{\eta(q)}{\theta_1(\tau|(r_\alpha-1)\ell)}\,, \tag{4.20}$$

$$Z_{J}^{(0,2)} = \prod_{\alpha=1}^{N}\prod_{A=1}^{N_{\mathrm{F}}-N}\prod_{r_\alpha=1}^{k_\alpha}\mathrm{i}\frac{\eta(q)}{\theta_1(\tau|\mu_A-z_\alpha-(r_\alpha-1)\ell)}\,, \tag{4.21}$$

$$Z_{\psi}^{(0,2)} = \prod_{\alpha=1}^{N}\prod_{I=1}^{N_{\mathrm{F}}}\prod_{r_\alpha=1}^{k_\alpha}\mathrm{i}\frac{\theta_1(\tau|z_\alpha-\nu_I+(r_\alpha-1)\ell)}{\eta(q)}\,, \tag{4.22}$$

$$Z_{B}^{(0,2)} = \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\prod_{r_\alpha=1}^{k_\alpha}\prod_{s_\beta=1}^{k_\beta}\mathrm{i}\frac{\eta(q)}{\theta_1(\tau|z_{\alpha\beta}+(r_\alpha-s_\beta-1)\ell)}\times$$
$$\times\prod_{\alpha=1}^{N}\left(\frac{\mathrm{i}\eta(q)}{2\pi\eta^3(q)}\right)^{k_\alpha-1}\prod_{\substack{r_\alpha,s_\alpha=1\\ r_\alpha\neq s_\alpha+1}}^{k_\alpha} i\frac{\eta(q)}{\theta_1(\tau|(r_\alpha-s_\alpha-1)\ell)}\,, \tag{4.23}$$

$$\tag{4.24}$$

where $z_{\alpha\beta} = z_\alpha - z_\beta$. Now it is simply a matter of collecting and rearranging factors. According to:

$$\frac{\prod_{\substack{i_\alpha,s_\alpha=1\\ r_\alpha\neq s_\alpha}}^{k_\alpha}\theta_1(\tau|(r_\alpha-s_\alpha)\ell)}{\prod_{\substack{r_\alpha,s_\alpha=1\\ r_\alpha\neq s_\alpha+1}}^{k_\alpha}\theta_1(\tau|(r_\alpha-s_\alpha-1)\ell)\times\prod_{r_\alpha=2}^{k_\alpha}\theta_1(\tau|(r_\alpha-1)\ell)} = \frac{1}{\prod_{r_\alpha=1}^{k_\alpha}\theta_1(\tau|-r_\alpha\ell)}\,, \tag{4.25}$$

and:

$$\frac{\prod_{r_\alpha=1}^{k_\alpha}\prod_{s_\beta=1}^{k_\beta}\theta_1(\tau|z_{\alpha\beta}+(r_\alpha-s_\beta)\ell)}{\prod_{r_\alpha=1}^{k_\alpha}\prod_{s_\beta=1}^{k_\beta}\theta_1(\tau|z_{\alpha\beta}+(r_\alpha-s_\beta-1)\ell)\times\prod_{r_\alpha=1}^{k_\alpha}\theta_1(\tau|z_{\alpha\beta}+(s_\alpha-1)\ell)}= \\ =\frac{1}{\prod_{r_\alpha=1}^{k_\alpha}\theta_1(\tau|z_{\alpha\beta}+(r_\alpha-k_\beta-1)\ell)}, \quad (4.26)$$

we can write:

$$\mathcal{I}^{(0,2)}_{\vec{k},N,N_{\rm F}}(\tau|z,\mu,\nu,\ell)= \\ (-1)^{k(N+1)}\prod_{\alpha=1}^{N}\frac{\prod_{I=1}^{N_{\rm F}}\Theta_1(\tau,\ell|z_\alpha-\nu_I)_{k_\alpha}}{\Theta_1(\tau,\ell|\ell)_{k_\alpha}\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\Theta_1(\tau,\ell|z_{\alpha\beta}-k_\beta\ell)_{k_\alpha}\times\prod_{A=1}^{N_{\rm F}-N}\Theta_1(\tau,\ell|z_\alpha-\mu_A)_{k_\alpha}}, \quad (4.27)$$

where the Θ_1 function is a generalization of Pochhammer symbols and it is defined and studied in app. B.2. This result is an elliptic generalization of k-vortex partition functions found in [**40**, **41**, **66**–**70**]. If we now define de grand-canonical partition function summing over all N-colored partitions of k, as:

$$\mathcal{I}^{(0,2)}_{N,N_{\rm F}}(\tau|z,\mu,\nu,\ell)=\sum_{k_1=1}^{\infty}\dots\sum_{k_N=1}^{\infty}\mathcal{I}^{(0,2)}_{\vec{k},N,N_{\rm F}}(\tau|z,\mu,\nu,\ell)\left[-\left(-e^{\mathrm{i}\pi\ell}\right)^N z\right]^{|\vec{k}|}, \quad (4.28)$$

where $|\vec{k}|=\sum_\alpha k_\alpha$. Applying eq. (B.13) to eq. (4.27), and then rewriting the prefactor as a differential operator, making use of eqs. (B.14) and (4.5), we can write:

$$\mathcal{I}^{(0,2)}_{N,N_{\rm F}}(\tau|z,\mu,\nu,\ell)=\mathcal{D}^{(0,2)}_N\prod_{\alpha=1}^{N}{}_{N_{\rm F}}E_{N_{\rm F}-1}\left(\begin{matrix}\vec{A}_\alpha\\ \vec{B}_\alpha,\vec{C}_\alpha\end{matrix}\middle|\tau,\ell\middle|z_\alpha\right) \quad (4.29)$$

in which we set:

$$\vec{A}_\alpha=\left(z_\alpha-\nu_1,\dots,z_\alpha-\nu_{N_{\rm F}}\right); \quad (4.30)$$

$$\vec{B}_\alpha=(z_\alpha-z_1,\dots,\widehat{z_\alpha-z_\alpha},\dots,z_\alpha-z_N); \quad (4.31)$$

$$\vec{C}_\alpha=\left(z_\alpha-\mu_A,\dots,z_\alpha-\mu_{N_{\rm F}-N}\right); \quad (4.32)$$

$$\mathcal{D}^{(0,2)}_N=\prod_{1\le\alpha<\beta\le N}e^{-\mathrm{i}\pi\ell(z_\alpha\partial_{z_\alpha}-z_\beta\partial_{z_\beta})}\frac{\theta(\tau|z_{\alpha\beta}+\ell(z_\alpha\partial_{z_\alpha}-z_\beta\partial_{z_\beta}))}{\theta(\tau|z_{\alpha\beta})}, \quad (4.33)$$

where the wide hat means omission. We notice also that, using eq. (B.16) is possible to write the grand-partition function (4.29) as a *finite* combination of elliptic hypergeometric functions. Notice that in the case of $N=1$, the differential operator is not there and we recover the result of [**64**].

4.3 $\mathcal{N}=(2,2)$ theories

In this section we discuss the elliptic genus of moduli space of k vortices in a four-dimensional $\mathcal{N}=2$ theory with U(N) gauge group with $N_{\rm F}$ hypermultiplets in the fundamental [**56**]. This moduli space is described by 2d $\mathcal{N}=(2,2)$ quiver gauge theory depicted in fig. 4.3.

[7] All the multiplets here are $\mathcal{N}=(2,2)$ multiplets.

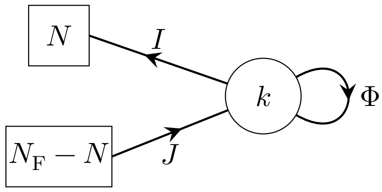


Figure 4.3: $\mathcal{N} = (2, 2)$ quiver gauge theory: it has[7] $U(k)$ vector multiplet, I, J and Φ chiral multiplets and vanishing superpotential. It describes the moduli space of vortices of 4d, $\mathcal{N} = 2$, $U(N)$ gauge theory with N_F fundamental chirals; it becomes superconformal if $N_F = 2N$.

In order to compute the elliptic genus we consider the flavor group $S(U(N) \times U(N_F - N))$ together with a $U(1)$ holonomy for Φ; in this case we have also an extra holonomy for the left-moving R-charge. Since the theory does not have superpotential, we are free to choose the left-moving R-charge of each multiplet: we will take all them as zero[8].

Group	I	J	Φ	Chemical Potential
$U(k)$	$\square$	$\overline{\square}$	**Adj**	u_i
$U(N)$	$\overline{\square}$	$\bullet$	$\bullet$	z_α
$U(N_F - N)$	$\bullet$	$\square$	$\bullet$	μ_A
$U(1)_\Phi$	0	0	1	$-\ell$
$U(1)_L$	0	0	0	ϵ

Table 4.2: Groups representations and related chemical potentials. $\square$ is the fundamental representation $\overline{\square}$ is the anti-fundamental, $\bullet$ is the trivial and **Adj** is the adjoint.

As before, these chemical potentials, listed in tab. 4.2, are not independent:

$$\sum_{\alpha=1}^{N} z_\alpha + \sum_{A=1}^{N_F - N} \mu_A = 0 \, . \tag{4.34}$$

The one-loop determinants read

$$Z_{\mathbf{V}}^{(2,2)}(\tau|u, \epsilon) = \frac{1}{k!} \left(-\frac{2\pi\eta(\tau)}{\theta_1(\tau|\ell)} \right)^k \prod_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{\theta_1(\tau|u_{ij})}{\theta_1(\tau|u_{ij} - \epsilon)} \prod_{i=1}^{k} \mathrm{d}u_i \, , \tag{4.35}$$

$$Z_I^{(2,2)}(\tau|u, z, \epsilon) = \prod_{\alpha=1}^{N} \prod_{i=1}^{k} \frac{\theta_1(\tau|u_i - \epsilon - z_\alpha)}{\theta_1(\tau|u_i - z_\alpha)} \, , \tag{4.36}$$

$$Z_J^{(2,2)}(\tau|u, \mu, \epsilon) = \prod_{A=1}^{N_F - N} \prod_{i=1}^{k} \frac{\theta_1(\tau|-u_i - \epsilon + \mu_A)}{\theta_1(\tau|-u_i + \mu_A)} \, , \tag{4.37}$$

$$Z_\Phi^{(2,2)}(\tau|u, \ell, \epsilon) = \prod_{i,j=1}^{k} \frac{\theta_1(\tau|u_{ij} - \ell - \epsilon)}{\theta_1(\tau|u_{ij} - \ell)} \, . \tag{4.38}$$

Notice that a different left-moving R-charge assignment would lead to the same partition function provided we reabsorb various ϵ factors in a subgroup of the flavor group. As before we

[8]In general, when a theory is superconformal, the superconformal R-charges can be computed with c-extremization [131, 132].

see that the theory is anomaly-free at the superconformal point $N_{\text{F}} = 2N$. The elliptic genus is therefore

$$\mathcal{I}^{(2,2)}_{k,N}(\tau|z,\mu,\ell,\epsilon) = \int_{\text{JK}} Z^{(2,2)}_{\mathbf{V}}(\tau|u,\ell) Z^{(2,2)}_{I}(\tau|u,z,\epsilon) Z^{(2,2)}_{J}(\tau|u,\mu,\epsilon) Z^{(2,2)}_{\Phi}(\tau|u,\ell,\epsilon) \; . \tag{4.39}$$

4.3.1 Polology

The pole classification in the present case is very similar to the preceding one. The singular hyperplane are

$$H_{\mathbf{V};i,j} = \{u_{ij} - \epsilon = 0\} \; , \qquad H_{\Phi;i,j} = \{u_{ij} - \ell = 0\} \; , \tag{4.40}$$

$$H_{I;i,\alpha} = \{u_r = z_\alpha\} \; , \qquad H_{J;i,A} = \{u_r = \mu_A\} \; , \tag{4.41}$$

the respective charge

$$\vec{h}_{\mathbf{V};i,j} = \vec{h}_{\Phi;i,j} = (0, \dots, \underbrace{1}_{i}, \dots, \underbrace{-1}_{j}, \dots, 0) \; , \tag{4.42}$$

$$\vec{h}_{I;i,\alpha} = (0, \dots, \underbrace{1}_{i}, \dots, 0) \; , \qquad \vec{h}_{J;i,A} = (0, \dots, \underbrace{-1}_{i}, \dots, 0) \; . \tag{4.43}$$

As before poles coming from H_J are discarded by JK procedure. We will organize the contributions as before: fixing a color we have that the first pole occurs at $u^*_{\alpha,1} = z_\alpha$. To compute the second pole of that color, we can use either $H_{\mathbf{V}}$ or H_Φ; in the first case we have $u^*_{\alpha,2} = z_\alpha + \epsilon$. This value of u indeed, set the numerator of eq. (4.36) to zero and so we drop this case; the remaining choice is to take $u^*_{\alpha,2} = z_\alpha + \ell$. Then we proceed to the third pole of the same color: as before, we can have either $u^*_{\alpha,3} = z_\alpha + \ell + \epsilon$ or $z_{\alpha,3} = z_\alpha + 2\ell$. Again, in the first case we encounter a zero in the numerator of eq. (4.38). The iteration of this argument shows that the poles are of the form:

$$u^*_{\alpha,r_\alpha} = z_\alpha + (r_\alpha - 1)\ell \; , \tag{4.44}$$

with $r_\alpha = 1, \dots, k_\alpha$ and $\sum_\alpha k_\alpha = k$. We conclude that the structure of the poles is the same as in the preceding case.

4.3.2 Computation

As in the previous case, we have just to plug the poles (4.44) into the expression of one loop determinants of the multiplets and to use eq. (B.8):

$$\begin{aligned} Z^{(2,2)}_{\mathbf{V}} = \left(\frac{2\pi\eta^3(q)}{\theta_1(\tau|-\epsilon)}\right)^k \prod_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \prod_{r_\alpha=1}^{k_\alpha} \prod_{s_\beta=1}^{k_\beta} \frac{\theta_1(\tau|z_{\alpha\beta} + (r_\alpha - s_\beta)\ell)}{\theta_1(\tau|z_{\alpha\beta} + (r_\alpha - s_\beta)\ell - \epsilon)} \times \\ \times \prod_{\alpha=1}^{N} \prod_{\substack{r_\alpha,s_\alpha=1 \\ r_\alpha\neq s_\alpha}}^{k_\alpha} \frac{\theta_1(\tau|(r_\alpha - s_\beta)\ell)}{\theta_1(\tau|(r_\alpha - s_\beta)\ell - \epsilon)} \; , \end{aligned} \tag{4.45}$$

$$Z^{(2,2)}_{I} = \prod_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \prod_{r_\alpha=1}^{k_\alpha} \frac{\theta_1(\tau|z_{\alpha\beta} + (r_\alpha - 1)\ell - \epsilon)}{\theta_1(\tau|z_{\alpha\beta} + (r_\alpha - 1)\ell)} \times \prod_{\alpha=1}^{N} \left(\frac{\theta_1(\tau|-\epsilon)}{2\pi\eta^3(q)}\right) \prod_{r_\alpha=2}^{k_\alpha} \frac{\theta_1(\tau|(r_\alpha - 1)\ell - \epsilon)}{\theta_1(\tau|(r_\alpha - 1)\ell)} \; , \tag{4.46}$$

$$Z_J^{(2,2)} = \prod_{\alpha,\beta=1}^{N} \prod_{r_\alpha=1}^{k_\alpha} \frac{\theta_1(\tau|\mu_\alpha - z_\beta - (r_\alpha - 1)\ell - \epsilon)}{\theta_1(\tau|\mu_\alpha - z_\beta - (r_\alpha - 1)\ell)}\,, \tag{4.47}$$

$$Z_\Phi^{(2,2)} = \prod_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \prod_{r_\alpha=1}^{k_\alpha} \prod_{s_\beta=1}^{k_\beta} \frac{\theta_1(\tau|z_{\alpha\beta} + (r_\alpha - s_\beta - 1)\ell - \epsilon)}{\theta_1(\tau|z_{\alpha\beta} + (r_\alpha - s_\beta - 1)\ell)} \times$$
$$\times \prod_{\alpha=1}^{N} \left(\frac{\theta_1(\tau|-\ell)}{2\pi\eta^3(q)} \right)^{k_\alpha - 1} \prod_{\substack{r_\alpha, s_\alpha=1\\ r_\alpha\neq s_\alpha+1}}^{k_\alpha} \frac{\theta_1(\tau|(r_\alpha - s_\beta - 1)\ell - \epsilon)}{\theta_1(\tau|(r_\alpha - s_\beta - 1)\ell)}\,. \tag{4.48}$$

Using again eqs. (4.25) and (4.26) both for numerators and denominators, and collecting all the factors, it is possible to write the result as:

$$\mathcal{I}_{\vec{k},N}^{(2,2)}(\tau|z,\mu,\ell,\epsilon) = \prod_{\alpha,\beta=1}^{N} \frac{\Theta_1(\tau,\ell|z_{\alpha\beta} - k_\beta\ell - \epsilon)_{k_\alpha}}{\Theta_1(\tau,\ell|z_{\alpha\beta} - k_\beta\ell)_{k_\alpha}} \frac{\Theta_1(\tau,\ell|z_\beta - \mu_\alpha + \ell)_{k_\alpha}}{\Theta_1(\tau,\ell|z_\beta - \mu_\alpha)_{k_\alpha}}. \tag{4.49}$$

which, unfortunately, has not a simple resummation in terms of elliptic hypergeometric equation, due to the presence of the shift $-\epsilon$ in the numerator of the first factor. This result reproduces holomorphic 4d blocks and free-field correlators of elliptic Virasoro algebrae [**71**]. Eq. (4.49) can be used to generalize the result for the topological vertex presented in [**66**] to its elliptic version [**133**], in which rotational modes are resummed. Following [**66**] we define the "elliptic-lift" of the amplitude of the open topological string as a double copy of our original system with opposite ϵ:

$$\mathcal{A}_N^{\text{ell.}}(\tau|z,\mu,\ell,\epsilon) = \sum_{k_1=1}^{\infty} \dots \sum_{k_N=1}^{\infty} \mathcal{I}_{\vec{k}}^{(2,2),N}(\tau|z,\mu,\ell,\epsilon)\mathcal{I}_{\vec{k}}^{(2,2),N}(\tau|z,\mu,\ell,-\epsilon) z^{|\vec{k}|}. \tag{4.50}$$

Now we can apply, as before, eq. (B.13) in the denominator as well as in the numerator of (4.50). Using our bag of tricks as before the result reads:

$$\mathcal{A}_N^{\text{ell.}}(\tau|z,\mu,\ell,\epsilon) = \mathcal{D}_N^{(2,2)} \prod_{\alpha=1}^{N} {}_{4N}E_{4N-1} \left(\begin{matrix} \vec{A}_\alpha^+, \vec{A}_\alpha^-, \vec{B}_\alpha^+, \vec{B}_\alpha^- \\ \ell, \vec{C}_\alpha, \vec{C}_\alpha, \vec{D}_\alpha, \vec{D}_\alpha \end{matrix} \middle| \tau, \ell \middle| z_\alpha \right), \tag{4.51}$$

where we set:

$$\vec{A}_\alpha^\pm = (z_\alpha - z_1 \pm \ell + \ell, \dots, z_\alpha - z_N \pm \epsilon + \ell)\,; \tag{4.52}$$
$$\vec{B}_\alpha^\pm = (z_1 - \mu_\alpha \pm \epsilon, \dots, z_N - \mu_\alpha \pm \epsilon)\,; \tag{4.53}$$
$$\vec{C}_\alpha = (z_\alpha - z_1, \dots, \widehat{z_\alpha - z_\alpha}, \dots, z_\alpha - z_N)\,; \tag{4.54}$$
$$\vec{D}_\alpha = (z_1 - \mu_\alpha, \dots, z_N - \mu_\alpha)\,; \tag{4.55}$$
$$\mathcal{D}_N^{(2,2)} = \prod_{1\le\alpha<\beta\le N} e^{\mathrm{i}\pi\ell(z_\alpha\partial_{z_\alpha} - z_\beta\partial_{z_\beta})} \frac{\theta(\tau|z_{\alpha\beta} - \epsilon)\theta(\tau|z_{\alpha\beta} + \epsilon)}{\theta^2(\tau|z_{\alpha\beta})} \times$$
$$\times \frac{\theta^2(\tau|z_{\alpha\beta} + \ell(z_\alpha\partial_{z_\alpha} - z_\beta\partial_{z_\beta}))}{\theta(\tau|z_{\alpha\beta} + \epsilon + \ell(z_\alpha\partial_{z_\alpha} - z_\beta\partial_{z_\beta}))\theta(\tau|z_{\alpha\beta} - \epsilon + \ell(z_\alpha\partial_{z_\alpha} - z_\beta\partial_{z_\beta}))}. \tag{4.56}$$

This result is the elliptic analogue of the $\mathcal{N} = 2$ vortex partition presented in [**66**].

5. Elliptic Genus of D1/D7 brane system

In this chapter we will compute in detail the elliptic genus of D1/D7 brane system, using the same technique employed in the preceding chapter. This time, however, the computation is more involved and we will use the desingularization procedure described in sec. 3.3. We will follow [2].

5.1 Elliptic non-Abelian Donaldson–Thomas Invariants of $\mathbb{C}^3$

To study (equivariant) Donaldson–Thomas invariants [37] of a three-fold X, one can employ a string theory brane construction [38, 39]. In particular, in order to study the so called "Hilbert scheme of points" on[1] X, we place a single Euclidean D5 brane on the threefold, and some number k of D(-1) branes on its worldvolume. Then, SUSY is preserved if we set a B-field along the D5 brane [54]. The resulting SUSY field theory from the point of view of the D(-1) branes, which in this case is 0d, that is, a matrix model, contains information about the sought-after invariants. Much information can be extracted with supersymmetric field theory techniques. Actually, we can similarly study K-theoretic generalization of the DT invariants by adding one direction to the brane setup. Specifically, we can study D6 brane wrapped on X and k D0 branes on its worldvolume: the quantum mechanics on the D0 branes captures the K-theoretic DT invariants of X [135]. Furtherly we can add one more direction and study a D7 brane wrapped on X and k D1 branes on its worldvolume: the two dimensional theory on the D1 branes allows us to define the *elliptic DT invariants* of X. Mathematically speaking we define them as the elliptic genera of the Hilbert scheme of k points on X. Physically they are the elliptic genera of the theories living on those D1 brane. One can generalize the construction under discussion using N D5, D6 or D7 branes instead of a single one. We call these higher rank DT invariants *elliptic non-Abelian DT invariants*. Here we are interested in the simplest three-fold, that is $X = \mathbb{C}^3$.

Let us consider a type IIB superstring theory with both D7 and D1 brane arranged as in tab. 5.1.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
D1	–	–	•	•	•	•	•	•	•	•
D7	–	–	–	–	–	–	–	–	•	•
	T^2		$\mathbb{C}^3$							

Table 5.1: Arrangement of the D1/D7 brane system.

[1]The Hilbert Scheme of n points on X is usually denoted as $X^{[n]}$. It is an algebraic desingularization of the space $\mathrm{Sym}^n X$. We recommend [134].

In this situation we see that the D7 brane are wrapped on $\mathbb{C}^3$. The 2d theory living on k D1-branes probing N D7-branes is a $\mathcal{N}=(2,2)$ GLSM and it is described by the quiver depicted in 5.1.

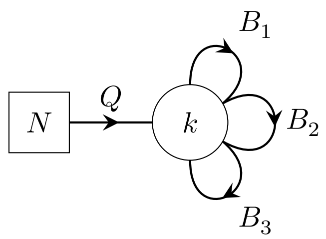


Figure 5.1: $\mathcal{N}=(2,2)$ quiver gauge theory: it has[2] $\mathrm{U}(k)$ vector multiplet; Q is a chiral multiplet in the fundamental representation of the gauge group and in the anti-fundamental representation of the $\mathrm{SU}(N)$ flavor group. B_1, B_2 and B_3 are chiral multiplets in the adjoint representation of the gauge group. The superpotential $W=\mathrm{tr}(B_1[B_2,B_3])$.

It has a superpotential

$$W=\mathrm{tr}(B_1[B_2,B_3])\ . \tag{5.1}$$

The field content has the following interpretation from the string theory point of view: there are three (complex) chiral multiplet B_1, B_2 and B_3 parametrizing the fluctuation of the D1 brane along the D7 (namely in the directions $3,...,8$), the two scalars in the vector multiplet σ and $\bar{\sigma}$ parametrize the fluctuation in the directions 9 and 10, there is a chiral multiplet Q originating from the open string stretched between the D1 and D7.

On general grounds, from chap. 2, such GLSM has a scalar potential

$$V\sim\frac{1}{2g^2}D^2+\sum_i|F_i|^2+2\sigma_A\bar{\sigma}_B\bar{\varphi}T_AT_B\varphi\ , \tag{5.2}$$

where T's belongs to the Lie algebra of the gauge group and the D- and F-terms read

$$F_i=\frac{\partial W}{\partial\varphi_i}\ , \qquad D_A=-\tfrac{\mathrm{i}}{2}\left([\sigma,\bar{\sigma}]_A+g\bar{\varphi}T_A\varphi+g\xi_A\right)\ , \tag{5.3}$$

In the Higgs branch, $\sigma=\bar{\sigma}=0$, SUSY solutions $V=0$ for our matter content give us ADHM-like equations[3]

$$[B_a,B_b]=0\ , \qquad \sum_{i=1}^{3}[B_a,B_b^\dagger]+QQ^\dagger=\xi\mathbf{I}\ , \tag{5.4}$$

which describe the moduli space. Dimensional reduction to D0/D6 and D(-1)/D5, in the string theoretic setup can be obtained by means of a T-duality along direction 1 and 2. In these cases the number of direction transverse to both branes increase, correspondingly the same happens with the number of scalars in the vector multiplet (which have to be set to zero to parametrize the Higgs branch).

[2] All the multiplets are $\mathcal{N}=(2,2)$ multiplets.

[3] We switch from "bar" to "dagger" sign to conform ourselves to the literature. Undaggered fields have the index of the fundamental representation of the gauge group, daggered field have the anti-fundamental representation of the gauge group

Besides the gauge symmetry, the theory has SU(N) flavor symmetry acting on the chiral multiplet Q and a $\mathrm{U}(1)^2$ flavor symmetry acting on $B_{a=1,2,3}$. There is also a $\mathrm{U}(1)_L$ of the left-moving R-symmetry[4].

Group	Q	B_1	B_2	B_3	Chemical Potential
$\mathrm{U}(k)$	$\square$	**Adj**	**Adj**	**Adj**	u_i
$\mathrm{SU}(N)$	$\overline{\square}$	$\bullet$	$\bullet$	$\bullet$	$-z_\alpha$
$\mathrm{U}(1)_1$	0	1	-1	0	ζ_1
$\mathrm{U}(1)_2$	0	0	1	-1	ζ_2
$\mathrm{U}(1)_L$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	ϵ

Table 5.2: Groups representations and related chemical potentials.

The chemical potential, listed in tab. 5.2, are constrained

$$\sum_{\alpha=1}^{N} z_\alpha = 0 \, . \tag{5.5}$$

For future convenience we define the following variables

$$\epsilon_1 = \tfrac{1}{3}\epsilon + \zeta_1 \, , \qquad \epsilon_2 = \tfrac{1}{3}\epsilon + \zeta_2 - \zeta_1 \, , \qquad \epsilon_3 = \tfrac{1}{3}\epsilon - \zeta_2 \tag{5.6}$$

that satisfy the relation

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon \, . \tag{5.7}$$

In the following section we will compute the elliptic genus for $N = 1$ and then we will generalize the result for generic N.

5.2 The Abelian Case

In this section we focus on the Abelian case $N = 1$ which contains more subtleties with respect to the above computations. Putting together the one-loop determinant one has

$$\begin{aligned} \mathcal{I}_{k,1}(\tau|\vec{\epsilon}) = \frac{1}{k!} \left[\frac{2\pi\eta(\tau)^3 \, \theta_1(\tau|\epsilon_{12}) \, \theta_1(\tau|\epsilon_{13}) \, \theta_1(\tau|\epsilon_{23})}{\theta_1(\tau|\epsilon_1) \, \theta_1(\tau|\epsilon_2) \, \theta_1(\tau|\epsilon_3) \, \theta_1(\tau|\epsilon)} \right]^k \int_{\mathrm{JK}} \prod_{i=1}^{k} \mathrm{d}u_i \prod_{i=1}^{k} \frac{\theta_1(\tau|u_i - \epsilon)}{\theta_1(\tau|u_i)} \times \\ \times \prod_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{\theta_1(\tau|u_{ij}) \, \theta_1(\tau|u_{ij} - \epsilon_{12}) \, \theta_1(\tau|u_{ij} - \epsilon_{13}) \, \theta_1(\tau|u_{ij} - \epsilon_{23})}{\theta_1(\tau|u_{ij} + \epsilon_1) \, \theta_1(\tau|u_{ij} + \epsilon_2) \, \theta_1(\tau|u_{ij} + \epsilon_3) \, \theta_1(\tau|u_{ij} - \epsilon)} \; ; \end{aligned} \tag{5.8}$$

where we used the shorthand $\epsilon_{ab} := \epsilon_a + \epsilon_b$. Two comments are in order. First, we have to check for anomaly cancellation: this is done as before, requiring that the integrand of (5.8) is invariant under $u_i \mapsto u_i + a + b\tau$ (with $a, b \in \mathbb{Z}$). The net effect of this shift comes from the one-loop determinant for Q (as was expected since all other contribution originates from

[4] Actually, at the classical level there is the full R-symmetry group $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$, however in the quantum theory the anomaly breaks the anti-diagonal axial part to a discrete group ($\mathbb{Z}_N$ in the case at hand). This is relate to the fact that the theory is not conformal, rather it is gapped with a dynamically generated scale.

matter in the adjoint representation): it pick up a phase $e^{2\pi i b\epsilon}$, as a consequence R-symmetry parameter is quantized $\epsilon \in \mathbb{Z}$. This is how anomaly manifests itself in the localized path-integral formulation. There is also a 't Hooft anomaly for the left moving R-symmetry: under the shift $\epsilon \mapsto \epsilon+1$, $\zeta_1 \mapsto \zeta_1 - \frac{1}{3}$ and $\zeta_2 \mapsto \zeta_2 - \frac{2}{3}$ we have

$$\mathcal{I}_{k,1}(\tau|(\epsilon_1,\epsilon_2,\epsilon_3+1)) = (-1)^k \mathcal{I}_{k,1}(\tau|(\epsilon_1,\epsilon_2,\epsilon_3)) \; ; \tag{5.9}$$

Exactly the same sign is picked up if we shift one of the other ϵ_a's. Second, the prefactor outside the integral in (5.8) is ill-defined for $\epsilon \in \mathbb{Z}$ because $\theta_1(\tau|\epsilon) = 0$. To solve this conflict, we proceed as in [**60**, **61**]. We introduce an extra chiral multiplet P in the $\det^{-1}$ representation[5] of U(k). In the new theory, the continuous R-symmetry is non-anomalous and we can take generic values of ϵ. In particular, the limit $\epsilon \to 0$ is well-defined and finite. Of course, the theory with P is different from the one we are interested in. However, at $\epsilon = 0$ we can introduce a real mass for P and remove it from the low-energy spectrum[6]. Therefore the elliptic genus of the theory without P at $\epsilon = 0$ is equal to the $\epsilon \to 0$ limit of the elliptic genus of the theory with P. Notice that the one-loop determinant of P satisfies $\lim_{\epsilon\to 0} Z_P(u_i) = 1$. With a suitable choice of the parameter η in the JK residue, i.e. with a suitable choice of contour, the poles of Z_P at $\epsilon \neq 0$ do not contribute to the integral. Thus, with this particular choice, the multiplet P can be completely ignored: one computes the integral (5.8) for generic ϵ and then takes the $\epsilon \to 0$ limit. More details and examples can be found in [**60**, **61**].

5.2.1 Evaluation

In order to evaluate the Jeffrey-Kirwan residue integral in (5.8) we follow similar what we have done before. We first identify the hyperplanes where the integrand has pole singularities:

$$H_{Q;i} = \{u_i = 0\} \; , \quad H_{\mathbf{V};i,j} = \{u_{ij} - \epsilon = 0\} \; , \quad H^a_{B;i,j} = \{u_{ij} + \epsilon_a = 0\} \quad a = 1,2,3 \; . \tag{5.12}$$

The singular hyperplanes H_Q are due to the one-loop determinant of the chiral multiplet Q, the hyperplanes H_B are due to B_a while the hyperplanes $H_{\mathbf{V}}$ are due to vector multiplets associated to the roots of U(k). The associated charge vectors, which are the charge vectors of the chiral or vector multiplets responsible for the singularities, are:

$$\vec{h}_{F;i} = (0,\dots,\underbrace{1}_{i},\dots,0) \; , \qquad \vec{h}_{\mathbf{V};i,j} = \vec{h}_{B;ij} = (0,\dots,\underbrace{1}_{i},\dots,\underbrace{-1}_{j},\dots,0) \; . \tag{5.13}$$

First, we have to find poles u^*:

$$Q^{\mathsf{T}}(u^*) \begin{pmatrix} u_1^* \\ \vdots \\ u_k^* \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix} \; , \tag{5.14}$$

---

[5] Its one-loop determinant is

$$Z_P(\tau|u,\epsilon) = \frac{\theta_1\left(\tau\middle| -\sum_{i=1}^k u_i - \epsilon\right)}{\theta_1\left(\tau\middle| -\sum_{i=1}^k u_i\right)} \; , \tag{5.10}$$

that compensates the anomaly. Its charge vector is

$$\vec{h}_P = (-1,\dots,-1) \; . \tag{5.11}$$

[6] A real mass has R-charge 2, therefore it is compatible with the elliptic genus computation only at $\epsilon = 0$.

where the components of $Q(u^*) = (Q_1, \dots, Q_k)$ are chosen among charges (5.13) corresponding to a given sets of hyperplanes: $Q_i = \vec{h}$ for certain $\vec{h}$. Then $d_i = 0$ if the corresponding hyperplane is of type H_Q, $d_i = \epsilon$ if the corresponding hyperplane is of type $H_{\mathbf{V}}$, and $d_i = -\epsilon_a$ if the corresponding hyperplane is of type H_B^a. Moreover, according JK procedure, we choose η, as before, and a certain u^* will give contribution iff

$$Q(u^*) \begin{pmatrix} \varsigma_1 \\ \vdots \\ \varsigma_k \end{pmatrix} = \eta^{\mathsf{T}} \qquad \text{for all } \varsigma_i > 0. \tag{5.15}$$

We remark that, in general, the sum of JK residue on the u-torus T^{2k} does not depend on the choice of η; in our case, this would be true if we kept the multiplet P throughout the computation. If, instead, we want to neglect P, we should make a special choice of η such that the would-be poles from P would not be picked up. From eq. (5.11) one argues that $\eta = (1, \dots, 1)$ is such a good choice. The general form of $Q(u^*)$ is again the one we have in eq. (C.14). From this we can read off that the firs hyperplane in each block $Q_q(u^*)$ is of type H_Q, while the other ones in the same block are either of type $H_{\mathbf{V};i,j}$ or of type $H_{B;i,j}$ with $i > j$. It follows that a poles u^* will have $u_1^* = 0$, and then each coordinate differs from one of the previous one by either ϵ or $-\epsilon_a$. Thus we argue that the coordinates of a singular point u^* take values on a 3d lattice

$$u^*_{(l,m,n)} = (1-l)\epsilon_1 + (1-m)\epsilon_2 + (1-n)\epsilon_3 \ . \tag{5.16}$$

The main difference with respect to subsec. 4.2.2 is that there there the difference of two coordinates could be just ϵ, here we have four different choices ϵ and $-\epsilon_a$. This complicates the analysis of the location of poles. There is a one-to-one correspondence between a singular (modulo Weyl permutations) point and some arrangement of boxes

$$u^* = \{u^*_{(l,m,n)}\} \qquad\qquad \leftrightarrow \qquad\qquad \mathsf{U}_k = \{U_{(l,m,n)}\} \tag{5.17}$$

where $U_{(l,m,n)}$ is the box that sit at position (l, m, n) in the lattice. We will use $\hat{u}^* = \mathsf{U}_k$ to implement this correspondence even between coordinates of u^* and boxes of U_k, i.e. $\hat{u}^*_{l,m,n} = U_{(l,m,n)}$. It is possible to prove, as we do in subsec. C.3.2, that the only poles that give contributions are in those represented by certain U_k that are *plane partitions*, defined in C.3.1. We denote the set of all plane partition with k box as $\mathscr{P}_k$.

The elliptic genus (5.8) reduces to a sum o residues at those singular points that are picked up the JK contour prescription

$$\boldsymbol{\mathcal{I}}_{k,1} = \sum_{\hat{u}^* \in \mathscr{P}_k} \boldsymbol{\mathcal{I}}_{u^*,1} \ , \tag{5.18}$$

where the sum is over u^* corresponding to plane partitions with k boxes. As we have already said, each plane partition encodes the position of a pole. For fixed plane partition, each box at position $\vec{l} = (l, m, n)$ specifies the values of the coordinates $u_i^* = u^*_{(l,m,n)}$ according to (5.16), and the order of coordinates is not important because of the residual Weyl permutation symmetry. The summands in (5.18) are

$$\boldsymbol{\mathcal{I}}_{u^*,1} = \theta_1(\tau|\epsilon) \left[-\frac{\theta_1(\tau|\epsilon_{12})\,\theta_1(\tau|\epsilon_{13})\,\theta_1(\tau|\epsilon_{23})}{\theta_1(\tau|\epsilon_1)\,\theta_1(\tau|\epsilon_2)\,\theta_1(\tau|\epsilon_3)\,\theta_1(\tau|\epsilon)} \right]^k \prod_{u^*_{\vec{l}} \in u^* \setminus u^*_{(1,1,1)}} \frac{\theta_1\big(\tau\big|u^*_{\vec{l}} - \epsilon\big)}{\theta_1\big(\tau\big|u^*_{\vec{l}}\big)} \times$$

$$\times \prod_{\substack{u^*_{\vec{l}},\, u^*_{\vec{l}'} \in u^* \\ u^*_{\vec{l}} \neq u^*_{\vec{l}'}}}{}^{\prime} \frac{\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}\right)\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}-\epsilon_{12}\right)\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}-\epsilon_{13}\right)\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}-\epsilon_{23}\right)}{\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}+\epsilon_1\right)\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}+\epsilon_2\right)\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}+\epsilon_3\right)\theta_1\left(\tau\middle|u^*_{\vec{l},\vec{l}'}-\epsilon\right)} \ . \tag{5.19}$$

where $u^*_{\vec{l},\vec{l}'} := u^*_{\vec{l}} - u^*_{\vec{l}'}$. The first product is over all coordinates of u^* corresponding to boxes of the plane partition $\hat{u}^*$, but the one located at the origin $(1,1,1)$. The second product is over all ordered pairs of coordinates of u^* corresponding to boxes in the plane partition; prime means that vanishing factors, both in the numerator and denominator, are excluded from the product (as explained in sec. 3.3, it is an instance of regular product representation). Many cancellations occur and the product can be recast in the form

$$\boldsymbol{\mathcal{I}}_{u^*,1} = (-1)^k \, \frac{N_{u^*,1}}{D_{u^*,1}} \ , \tag{5.20}$$

where

$$\begin{aligned} N_{u^*,1} = \prod_{u^*_{(l,m,n)} \in u^*} \Bigg\{ & \theta_1\left(\tau\middle|l\epsilon_1 + m\epsilon_2 + (n - h^{xy}_{1,1})\epsilon_3\right) \times \\ & \times \prod_{n'=1}^{h^{xy}_{1,1}} \Big[\theta_1\left(\tau\middle|(l - h^{yz}_{m,n'})\epsilon_1 + (1 + h^{xz}_{l,n} - s)\epsilon_2 + (1 + n - n')\epsilon_3\right) \times \\ & \times \theta_1\left(\tau\middle|(1 + h^{yz}_{m,n'} - l)\epsilon_1 + (m - h^{xz}_{l,n})\epsilon_2 + (1 + n' - n)\epsilon_3\right) \Big] \Bigg\} \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} D_{u^*,1} = \prod_{u^*_{(l,m,n)} \in u^*} \Bigg\{ & \theta_1\left(\tau\middle|(1-l)\epsilon_1 + (1-m)\epsilon_2 + (1 + h^{xy}_{1,1} - n)\epsilon_3\right) \times \\ & \times \prod_{n'=1}^{h^{xy}_{1,1}} \Big[\theta_1\left(\tau\middle|(l - h^{yz}_{m,n'})\epsilon_1 + (1 + h^{xz}_{l,n} - m)\epsilon_2 + (n - n')\epsilon_3\right) \times \\ & \times \theta_1\left(\tau\middle|(1 + h^{yz}_{m,n'} - l)\epsilon_1 + (m - h^{xz}_{l,n})\epsilon_2 + (n' - n)\epsilon_3\right) \Big] \Bigg\} \ . \end{aligned} \tag{5.22}$$

Each product is (with a slight abuse of notation[7]) over the $\vec{l}$ parametrizing the coordinates of u^* corresponding to boxes of the plane partition $\hat{u}^*$. Then $h^{xy}_{l,m}$ is the depth of the pile of boxes laying at $(l,m,*)$; $h^{xz}_{l,n}$ is the height of the column of boxes at $(l,*,n)$; and $h^{yz}_{m,n}$ is the length of the row of boxes laying at $(*,m,n)$. In fact, (5.20)–(5.22) are the elliptic Abelian version of similar equations in Section 4.1 of [**76**].

Surprisingly, we observe that for $\epsilon \in \mathbb{Z}$ the expression $\boldsymbol{\mathcal{I}}_{u^*,1}$ in (5.20) simplifies: as a matter of fact we find

$$\boldsymbol{\mathcal{I}}_{u^*,1} = (-1)^{k\epsilon} \ . \tag{5.23}$$

The dependence on ϵ is dictated by the 't Hooft anomaly (5.9). There is no other dependence on ϵ_a nor on τ. This implies that, up to a sign, $\boldsymbol{\mathcal{I}}_{k,1}$ equals the integer number of plane partitions

[7] One in principle should multiply over (l,m,n) such that $u^*_{(l,m,n)} \in u^*$.

with k boxes. It is then convenient to define a "grand canonical" elliptic genus, function of a new fugacity v, by resumming all contributions from the sectors at fixed k:

$$\mathcal{I}_1 := 1 + \sum_{k=1}^{\infty} \mathcal{I}_{k,1} v^k \,. \tag{5.24}$$

Up to a sign, this is the generating function of the number of plane partitions, namely the MacMahon function:

$$\mathcal{I}_1(v) = \Phi((-1)^{\epsilon}\, v)\,, \tag{5.25}$$

where

$$\Phi(v) := \prod_{k=1}^{\infty} \frac{1}{(1-v^k)^k} = \mathtt{PE}_v\left[\frac{v}{(1-v)^2}\right] \tag{5.26}$$

is the MacMahon function and PE is the plethystic exponential operator (see sec. C.2).

5.2.2 Dimensional Reductions

We can consider dimensional reductions of the system. Reducing on a circle, we obtain the Witten index of an $\mathcal{N}=4$ SUSY quantum mechanics. This case, known as *trigonometric* or *motivic*, has been studied in [**74**]. It can be obtained from the elliptic case in the limit $q \to 0$, where $q = e^{2\pi i \tau}$. By a further reduction on a second circle, we obtain a SUSY matrix integral with 4 supercharges. This case, known as *rational*, has been studied in [**76**]. It can be obtained from the trigonometric case in the limit $\alpha \to 0$, where α is the radius of the circle used to compute the Witten index in the path integral formulation.

It is important to notice that in the trigonometric and rational cases, corresponding to field theories in 1d and 0d respectively, there is no anomaly constraint and one can take generic real values for the parameter descending from ϵ. This means that, in order to have access to all values of the parameters, we should apply the two limits to the integrand in (5.8) and then recompute the contour integral.

Given a quantity X in the elliptic case, we use the notation $\widetilde{X}$ for the corresponding quantity in the trigonometric case and $\overline{X}$ in the rational case. We also use $\dot{X}$ to refer to the three cases at the same time.

Trigonometric Limit. To obtain the trigonometric limit, we use that $\theta_1(\tau|z) \to 2q^{1/8}\sin(\pi z)$ as $q \to 0$. We express the result in terms of new variables

$$p_a = e^{2\pi i \epsilon_a}\,, \qquad p = e^{2\pi i \epsilon}\,, \qquad x_i = e^{2\pi i u_i}\,, \tag{5.27}$$

with $q_1 q_2 q_3 = q$. We find the integral expression for the Witten index of the $\mathcal{N}=4$ SUSY quantum mechanics corresponding to the quiver in fig 5.1:

$$\begin{aligned}\widetilde{\mathcal{I}}_{k,1}(q,\vec{p}) = \frac{1}{k!}\left[-p^{\frac{1}{2}}\frac{(1-p_1p_2)(1-p_1p_3)(1-p_2p_3)}{(1-p_1)(1-p_2)(1-p_3)(1-p)}\right]^k \int_{\mathrm{JK}} \prod_{i=1}^{k}\frac{\mathrm{d}x_i}{x_i}\prod_{i=1}^{k}\frac{1-p^{-1}x_i}{1-x_i}\times \\ \times \prod_{\substack{i,j=1\\ i\neq j}}^{k} p\,\frac{(1-x_{ij})(1-p_1^{-1}p_2^{-1}x_{ij})(1-p_1^{-1}p_3^{-1}x_{ij})(1-p_2^{-1}p_3^{-1}x_{ij})}{(1-p_1x_{ij})(1-p_2x_{ij})(1-p_3x_{ij})(1-p^{-1}x_{ij})}\,.\end{aligned} \tag{5.28}$$

Since there are no anomalies this time, the value of ϵ is unconstrained. The Witten index of SUSY quantum mechanics can jump when flat directions open up at infinity in field space. From the point of view of the 7D theory on the D6-brane, or DT invariants of $\mathbb{C}^3$, this is the wall crossing phenomenon. In the quantum mechanics, the parameter we vary is the Fayet-Iliopoulos (FI) term and it corresponds to the stability parameter in DT theory. The integral in (5.28) is a contour integral in $(\mathbb{C}^*)^k$, and in general it includes boundary components. However, choosing the auxiliary parameter $\vec{\eta}$ parallel to the FI parameter guarantees that the JK contour has no boundary components [**118**, **136**, **137**] (see also [**50**, **138**]). The chamber with non-trivial DT invariants corresponds to $\vec{\eta} = (1, \dots, 1)$.

The result can be expressed as before:

$$\tilde{\mathcal{I}}_{k,1} = \sum_{\hat{u}^* \in \mathscr{P}_k} \tilde{\mathcal{I}}_{u^*,1} \,, \qquad \tilde{\mathcal{I}}_{u^*,1} = (-1)^k \frac{\widetilde{N}_{u^*,1}}{\widetilde{D}_{u^*,1}} \,, \tag{5.29}$$

where

$$\begin{aligned} \widetilde{N}_{u^*,1} = \prod_{u^*_{(l,m,n)} \in u^*} \Bigg\{ & \hat{a}\left(p_1^l \, p_2^m \, p_3^{n-h^{xy}_{1,1}}\right) \times \\ & \times \prod_{n'=1}^{h^{xy}_{1,1}} \left[\hat{a}\left(p_1^{r-h^{yz}_{l,n'}} p_2^{1+h^{xz}_{l,n}-m} p_3^{1+n-n'}\right) \hat{a}\left(p_1^{1+h^{yz}_{m,n'}-l} p_2^{m-h^{xz}_{l,n}} p_3^{1+n'-n}\right) \right] \Bigg\} \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} \widetilde{D}_{u^*,1} = \prod_{u^*_{(l,m,n)} \in u^*} \Bigg\{ & \hat{a}\left(p_1^{1-l} \, p_2^{(1-m)} \, p_3^{(1+h^{xy}_{1,1}-n)}\right) \times \\ & \times \prod_{n'=1}^{h^{xy}_{1,1}} \left[\hat{a}\left(p_1^{l-h^{yz}_{m,n'}} p_2^{1+h^{xz}_{l,n}-m} p_3^{n-n'}\right) \hat{a}\left(p_1^{1+h^{yz}_{m,n'}-l} p_2^{m-h^{xz}_{l,n}} p_3^{n'-n}\right) \right] \Bigg\} \,. \end{aligned} \tag{5.31}$$

The notation is the same as in (5.21) and (5.22). We defined the function

$$\hat{a}\,(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}} \,, \tag{5.32}$$

in other words $\hat{a}\,(e^{2\pi i z}) = 2i \sin(\pi z)$. Notice that (5.30) and (5.31) are simply obtained from (5.21) and (5.22) by substituting $\theta_1(\tau|z) \mapsto \sin(\pi z)$, because the extra powers of q cancel out.

Rational Limit. To obtain the rational limit, we place the SUSY quantum mechanics on a circle of radius α and shrink it. This can be done, starting from (5.27) and (5.28), by substituting $\epsilon_a \mapsto \alpha\epsilon_a$ and $u_i \mapsto \alpha u_i$, then taking a $\alpha \to 0$ limit. The result is

$$\begin{aligned} \overline{\mathcal{I}}_{k,1}(\vec{\epsilon}) = \frac{1}{k!} \left[\frac{\epsilon_{12}\epsilon_{13}\epsilon_{23}}{\epsilon_1\epsilon_2\epsilon_3\epsilon} \right]^k \int_{\mathrm{JK}} \prod_{i=1}^k \mathrm{d}u_i \prod_{i=1}^k \frac{u_i - \epsilon}{u_i} \times \\ \times \prod_{\substack{i,j=1 \\ i \neq j}} \frac{u_{ij}(u_{ij}-\epsilon_{12})(u_{ij}-\epsilon_{13})(u_{ij}-\epsilon_{23})}{(u_{ij}+\epsilon_1)(u_{ij}+\epsilon_2)(u_{ij}+\epsilon_3)(u_{ij}-\epsilon)} \,. \end{aligned} \tag{5.33}$$

This expression can be cast in the same form as in previous cases:

$$\overline{\mathcal{I}}_{k,1} = \sum_{\hat{u}^* \in \mathscr{P}_k} \overline{\mathcal{I}}_{u^*,1} \,, \qquad\qquad \overline{\mathcal{I}}_{u^*,1} = (-1)^k \frac{\overline{N}_{u^*,1}}{\overline{D}_{u^*,1}} \,, \tag{5.34}$$

with

$$\begin{aligned} \overline{N}_{u^*,1} = \prod_{u_{(l,m,n)} \in u^*} \Bigg\{ & \Big(l\epsilon_1 + m\epsilon_2 + (n - h^{xy}_{1,1})\epsilon_3 \Big) \times \\ & \times \prod_{n'=1}^{h^{xy}_{1,1}} \Bigg[\Big(\Big(l - h^{yz}_{m,n'}\Big)\,\epsilon_1 + \Big(1 + h^{xz}_{l,n} - m\Big)\,\epsilon_2 + (1 + n - n')\epsilon_3 \Big) \times \\ & \qquad \times \Big(\Big(1 + h^{yz}_{m,n'} - l\Big)\,\epsilon_1 + \Big(m - h^{xz}_{l,n}\Big)\,\epsilon_2 + (1 + n' - n)\epsilon_3 \Big) \Bigg] \Bigg\} \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} \overline{D}_{u^*,1} = \prod_{u^*_{(l,m,n)} \in u^*} \Bigg\{ & \Big((1-l)\epsilon_1 + (1-m)\epsilon_2 + \Big(1 + h^{xy}_{1,1} - n\Big)\,\epsilon_3 \Big) \times \\ & \times \prod_{n'=1}^{h^{xy}_{1,1}} \Bigg[\Big(\Big(l - h^{yz}_{m,n'}\Big)\,\epsilon_1 + \Big(1 + h^{xz}_{l,n} - m\Big)\,\epsilon_2 + (n - n')\epsilon_3 \Big) \times \\ & \qquad \times \Big(\Big(1 + h^{yz}_{m,n'} - l\Big)\,\epsilon_1 + \Big(m - h^{xz}_{l,n}\Big)\,\epsilon_2 + (n' - n)\epsilon_3 \Big) \Bigg] \Bigg\} \,. \end{aligned} \tag{5.36}$$

Once again, (5.35) and (5.36) are obtained from (5.21) and (5.22) by substituting $\theta_1(\tau|z) \to z$.

5.2.3 The Plethystic Ansätze

As we observed in (5.23)–(5.25), the elliptic Abelian DT invariants are very simple and count the number of plane partitions. This is because the dependence of the elliptic genera on $\epsilon \in \mathbb{Z}$ is fixed by the anomaly, and there is no dependence on τ. The latter is a general property of gapped systems (see e.g. [**61**] for other examples) due to the fact that the elliptic genus of a gapped vacuum does not depend on τ.

By dimensional reduction, this implies that also the trigonometric and rational DT invariants, evaluated at $\epsilon = 0$, are captured by MacMahon's function. Defining a grand canonical partition function

$$\dot{\mathcal{I}}_1(v) \,:=\, 1 + \sum_{k=1}^{\infty} \dot{\mathcal{I}}_{k,1}\, v^k \tag{5.37}$$

both in the elliptic, trigonometric and rational case, we find that they are all equal to the MacMahon function:

$$\mathcal{I}_1(v)\big|_{\epsilon=0} = \tilde{\mathcal{I}}_1(v)\big|_{\epsilon=0} = \overline{\mathcal{I}}_1(v)\big|_{\epsilon=0} = \Phi(v) \,. \tag{5.38}$$

In the trigonometric and rational case, it is natural to ask whether a similar plethystic expression holds also when $\epsilon \neq 0$ (since there is no constraint on ϵ). It is clear that such an expression cannot be derived from the elliptic case.

It has been proved in [38, 39] that in the rational case the grand canonical partition function is simply

$$\overline{\mathcal{I}}_1 = \Phi(v)^{-\frac{\epsilon_{12}\epsilon_{13}\epsilon_{23}}{\epsilon_1\epsilon_2\epsilon_3}} = \mathrm{PE}_v\left[-\frac{\epsilon_{12}\epsilon_{13}\epsilon_{23}}{\epsilon_1\epsilon_2\epsilon_3}\,\frac{v}{(1-v)^2}\right] . \tag{5.39}$$

Notice that in this formula the plethystic variable is just v (not ϵ_a). In the trigonometric case, the following plethystic expression was conjectured by Nekrasov [74]:

$$\widetilde{Z}_1 = \mathrm{PE}_{v;\vec{p}}\left[-\frac{(1-p_1p_2)(1-p_1p_3)(1-p_2p_3)}{(1-p_1)(1-p_2)(1-p_3)}\,\frac{v}{q^{\frac{1}{2}}(1-vp^{-\frac{1}{2}})(1-vp^{\frac{1}{2}})}\right] . \tag{5.40}$$

We have verified that this expression reproduces (5.29) up to $k=12$.

5.3 The non-Abelian Case

In this section we generalize the result of the preceding section for a generic N: eq. (5.8) becomes

$$\begin{aligned}\mathcal{I}_{k,N}(\tau|\vec{\epsilon},z) = \frac{1}{k!}\left[\frac{2\pi\eta^3(\tau)\,\theta_1(\tau|\epsilon_{12})\,\theta_1(\tau|\epsilon_{13})\,\theta_1(\tau|\epsilon_{23})}{\theta_1(\tau|\epsilon_1)\,\theta_1(\tau|\epsilon_2)\,\theta_1(\tau|\epsilon_3)\,\theta_1(\tau|\epsilon)}\right]^k \int_{\mathrm{JK}}\prod_{i=1}^{k}\mathrm{d}u_i\,\times \\ \times\prod_{i=1}^{k}\prod_{\alpha=1}^{N}\frac{\theta_1(\tau|u_i+z_\alpha-\epsilon)}{\theta_1(\tau|u_i+z_\alpha)}\prod_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\theta_1(\tau|u_{ij})\,\theta_1(\tau|u_{ij}-\epsilon_{12})\,\theta_1(\tau|u_{ij}-\epsilon_{13})\,\theta_1(\tau|u_{ij}-\epsilon_{23})}{\theta_1(\tau|u_{ij}+\epsilon_1)\,\theta_1(\tau|u_{ij}+\epsilon_2)\,\theta_1(\tau|u_{ij}+\epsilon_3)\,\theta_1(\tau|u_{ij}-\epsilon)} ,\end{aligned} \tag{5.41}$$

we see that, respect to the $N=1$ case, the flavor fugacities along the Cartan on $\mathrm{SU}(N)$ appears, with the constraint (5.5). Again, in order to find the condition under which the R-symmetry anomaly cancels we have to impose that the integrand is doubly periodic: under a shift $u_i \mapsto u_i + a + b\tau$ (with $a,b\in\mathbb{Z}$) it pick up a phase $e^{2\pi\mathrm{i}Nb\epsilon}$ that tells us that $\epsilon\in\mathbb{Z}/N$. Besides, the R-symmetry 't Hooft anomaly dictates

$$\mathcal{I}_{k,N}(\tau|(\epsilon_1,\epsilon_2,\epsilon_3+1)) = (-1)^{kN}\mathcal{I}_{k,N}(\tau|(\epsilon_1,\epsilon_2,\epsilon_3)) , \tag{5.42}$$

and similarly for other ϵ_a's. The evaluation of the contour integral proceeds in the same way we did in the preceding section, but keeping into account the fugacities for the flavor group. When $N>1$ the charge matrix $Q(u^*)$ is block diagonal (C.14): we have one block for each flavor. The poles live in the union of N different lattices, eq. (5.16) becomes

$$u^*_{\alpha,(l,m,n)} = -z_\alpha + (1-l)\epsilon_1 + (1-m)\epsilon_2 + (1-n)\epsilon_3 . \tag{5.43}$$

Representing poles by arrangement of boxes on collection of lattices, it turns out that the poles contributing to the JK residue are those represented by N distinct plane partition labelled by α. This is because for each block of the charge matrix we can apply what we discover in the previous section. Such type of arrangement is known as *colored* plane partition (see sec. C.3.1). We denote a pole corresponding to ha colored plane partition as $\vec{u}^* = (u_1^*, \ldots u_N^*)$ and by $\mathscr{P}_{k,N}$ the set of N-colored plane partitions with k boxes. The partition function is the sum of residues

$$\mathcal{I}_{k,N} = \sum_{\hat{\vec{u}}^*\in\mathscr{P}_{k,N}} \mathcal{I}_{\vec{u}^*,N} . \tag{5.44}$$

In order to compute the residue at the pole represented by a colored plane partition, we observe that there are no factor in the denominator involving more the one z_α. It follows that the residue can be written as

$$\mathcal{I}_{\vec{u}^*,N} = \prod_{u^* \in \vec{u}^*} \mathcal{I}_{u^*,1} \times \prod_{\substack{u^*_\alpha, u^*_\beta \in \vec{u}^* \\ \alpha \neq \beta}} \left[\prod_{u^*_{\vec{l}} \in u^*_\alpha} \frac{\theta_1\left(\tau \middle| u^*_{\vec{l}} - z_{\alpha\beta} - \epsilon\right)}{\theta_1\left(\tau \middle| u^*_{\vec{l}} - z_{\alpha\beta}\right)} \times \right.$$
$$\left. \times \prod_{\substack{u^*_{\vec{l}} \in u^*_\alpha \\ u^*_{\vec{l}'} \in u^*_\beta}} \frac{\theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta}\right) \theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta} - \epsilon_{12}\right) \theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta} - \epsilon_{13}\right) \theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta} - \epsilon_{23}\right)}{\theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta} + \epsilon_1\right) \theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta} + \epsilon_2\right) \theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta} + \epsilon_3\right) \theta_1\left(\tau \middle| u^*_{\vec{l},\vec{l}'} - z_{\alpha\beta} - \epsilon\right)} \right], \tag{5.45}$$

we remark that here u^*_α is a component of the pole $\vec{u}^*$, and $u^*_{\vec{l}} \in u^*_\alpha$ indicates a coordinate of such component: $u^*_{\vec{l}} = (1-l_1)\epsilon_1 + (1-l_2)\epsilon_2 + (1-l_3)\epsilon_3$. In (5.45), $\mathcal{I}_{u^*,1}$ is the expression (5.19) from the Abelian case, while $z_{\alpha\beta} := z_\alpha - z_\beta$ and again $u^*_{\vec{l},\vec{l}'} = u^*_{\vec{l}} - u^*_{\vec{l}'}$. We stress that $u^*_{\vec{l}}$ does not depend on z_α, as this is different from $u^*_{\alpha,\vec{l}}$. Also in this case, several cancellations occur in evaluating (5.45) and it is possible to recast the result in a form similar to (5.20)–(5.22). We find:

$$\mathcal{I}_{\vec{u}^*,N} = (-1)^{kN} \prod_{\alpha,\beta=1}^{N} \frac{N_{\vec{u}^*,N;\alpha\beta}(z_{\alpha\beta})}{D_{\vec{u}^*,N:\alpha\beta}(z_{\alpha\beta})}, \tag{5.46}$$

with:

$$N_{\vec{u}^*,N;\alpha\beta}(z) = \prod_{u^*_{(l,m,n)} \in u^*_\alpha} \left\{ \theta_1\left(\tau \middle| z + l\epsilon_1 + m\epsilon_2 + (n - h^{xy;\beta}_{1,1})\epsilon_3\right) \times \right.$$
$$\times \prod_{n'=1}^{h^{xy;\beta}_{1,1}} \left[\theta_1\left(\tau \middle| z + (l - h^{yz;\beta}_{m,n'})\epsilon_1 + (1 + h^{xz;\alpha}_{l,n} - m)\epsilon_2 + (1 + n - n')\epsilon_3\right) \times \right.$$
$$\left.\left. \times \theta_1\left(\tau \middle| -z + (1 + h^{yz;\beta}_{m,n'} - l)\epsilon_1 + (m - h^{xz;\alpha}_{l,n})\epsilon_2 + (1 + n' - n)\epsilon_3\right) \right] \right\}, \tag{5.47}$$

and

$$D_{\vec{u}^*,N;\alpha\beta}(z) = \prod_{u^*_{(l,m,n)} \in u^*_\alpha} \left\{ \theta_1\left(\tau \middle| -z + (1 - l)\epsilon_1 + (1 - m)\epsilon_2 + (1 + h^{xy;\beta}_{1,1} - n)\epsilon_3\right) \times \right.$$
$$\times \prod_{n'=1}^{h^{xy;\beta}_{1,1}} \left[\theta_1\left(\tau \middle| z + (l - h^{yz;\beta}_{m,n'})\epsilon_1 + (1 + h^{xz;\alpha}_{l,n} - m)\epsilon_2 + (n - n')\epsilon_3\right) \times \right.$$
$$\left.\left. \times \theta_1\left(\tau \middle| -z + (1 + h^{yz;\beta}_{l,n'} - m)\epsilon_1 + (m - h^{xz;\alpha}_{l,n})\epsilon_2 + (n' - n)\epsilon_3\right) \right] \right\}. \tag{5.48}$$

Notice that now the function h has an index α that clarifies which plane partition in the colored set it refers to. These expressions are the elliptic version of similar equations in [**76**], where the rational case was analyzed.

The dimensional reduction of these formulae to the trigonometric case is the following:

$$\tilde{\mathcal{I}}_{\vec{u}^*,N} = (-1)^{kN} \prod_{\alpha,\beta=1}^{N} \frac{\widetilde{N}_{\vec{u}^*,N;\alpha\beta}(a_{\alpha\beta})}{\widetilde{D}_{\vec{u}^*,N;\alpha\beta}(a_{\alpha\beta})}, \tag{5.49}$$

where we set $a_\alpha = e^{2\pi i z_\alpha}$, $a_{\alpha\beta} = a_\alpha / a_\beta$ and

$$\widetilde{N}_{\vec{u}^*,N;\alpha\beta}(a) = \prod_{u^*_{(l,m,n)} \in u^*_\alpha} \Bigg\{ \hat{a}\left(a\, p_1^l\, p_2^m\, p_3^{n-h^{xy;\beta}_{1,1}} \right) \times \\ \times \prod_{n'=1}^{h^{xy;\beta}_{1,1}} \left[\hat{a}\left(a\, p_1^{l-h^{yz;\beta}_{m,n'}}\, p_2^{1+h^{xz;\alpha}_{l,n}-m}\, p_3^{1+n-n'} \right) \hat{a}\left(a^{-1}\, p_1^{1+h^{yz;\beta}_{m,n'}-l}\, p_2^{m-h^{xz;\alpha}_{l,n}}\, p_3^{1+n'-n} \right) \right] \Bigg\} , \quad (5.50)$$

$$\widetilde{D}_{\vec{u}^*,N;\alpha\beta}(a) = \prod_{u^*_{(l,m,n)} \in u^*_\alpha} \Bigg\{ \hat{a}\left(a^{-1}\, p_1^{1-l}\, p_2^{(1-m)}\, p_3^{(1+h^{xy;\beta}_{1,1}-n} \right) \times \\ \times \prod_{n'=1}^{h^{xy;\beta}_{1,1}} \left[\hat{a}\left(a\, p_1^{l-h^{yz;\beta}_{m,n'}}\, p_2^{1+h^{xz;\alpha}_{l,n}-m}\, p_3^{n-n'} \right) \hat{a}\left(a^{-1}\, p_1^{1+h^{yz;\beta}_{m,n'}-l}\, p_2^{m-h^{xz;\alpha}_{l,n}}\, q_3^{n'-n} \right) \right] \Bigg\} . \quad (5.51)$$

The reduction to the rational case gives the following:

$$\overline{\mathcal{I}}_{\vec{u}^*,N} = (-1)^{kN} \prod_{\alpha,\beta=1}^{N} \frac{\overline{N}_{\vec{u}^*,N;\alpha\beta}(z_{\alpha\beta})}{\overline{D}_{\vec{u}^*,N;\alpha\beta}(z_{\alpha\beta})} , \quad (5.52)$$

with

$$\overline{N}_{\vec{u}^*,N;\alpha\beta}(z) = \prod_{u^*_{(l,m,n)} \in u^*_\alpha} \Bigg\{ \left(z + l\epsilon_1 + m\epsilon_2 + \left(n - h^{xy;\beta}_{1,1} \right) \epsilon_3 \right) \times \\ \times \prod_{n'=1}^{h^{xy;\beta}_{1,1}} \Bigg[\left(z + \left(l - h^{yz;\beta}_{m,n'} \right) \epsilon_1 + \left(1 + h^{xz;\alpha}_{l,n} - s \right) \epsilon_2 + (1+n-n')\epsilon_3 \right) \times \\ \times \left(-z + \left(1 + h^{yz;\beta}_{m,n'} - l \right) \epsilon_1 + \left(m - h^{xz;\alpha}_{l,n} \right) \epsilon_2 + (1+n'-n)\epsilon_3 \right) \Bigg] \Bigg\} , \quad (5.53)$$

and

$$\overline{D}_{\vec{u}^*,N;\alpha\beta}(z) = \prod_{u^*_{(l,m,n)} \in u^*_\alpha} \Bigg\{ \left(-z + (1-l)\epsilon_1 + (1-m)\epsilon_2 + \left(1 + h^{xy;\beta}_{1,1} - n \right) \epsilon_3 \right) \times \\ \times \prod_{n'=1}^{h^{xy;\beta}_{1,1}} \Bigg[\left(z + \left(l - h^{yz;\beta}_{m,n'} \right) \epsilon_1 + \left(1 + h^{xz;\alpha}_{l,n} - m \right) \epsilon_2 + (n-n')\epsilon_3 \right) \times \\ \times \left(-z + \left(1 + h^{yz;\beta}_{m,n'} - l \right) \epsilon_1 + \left(m - h^{xz;\alpha}_{l,n} \right) \epsilon_2 + (n'-n)\epsilon_3 \right) \Bigg] \Bigg\} . \quad (5.54)$$

This reproduces the expressions in Section 4 of [76].

5.3.1 Resummation Conjectures and Factorization

We are interested in the generating functions of non-Abelian Donaldson-Thomas invariants, namely in the "grand canonical" partition functions

$$\dot{\mathcal{I}}_N(v) = 1 + \sum_{k=1}^{\infty} \dot{\mathcal{I}}^{k,N}\, v^k , \quad (5.55)$$

in the three cases: elliptic, trigonometric and rational.

As in the Abelian case, we observe that (5.46), (5.49) and (5.52) drastically simplify when we set $\epsilon = 0$:

$$\mathcal{I}_{\vec{u}^*,N}\big|_{\epsilon=0} = \tilde{\mathcal{I}}_{\vec{u}^*,N}\big|_{\epsilon=0} = \overline{\mathcal{I}}_{\vec{u}^*,N}\big|_{\epsilon=0} = 1 \,. \tag{5.56}$$

This implies that the grand canonical partition function reduces to the N^{th} power of MacMahon's function,

$$\mathcal{I}_N\big|_{\epsilon=0} = \tilde{\mathcal{I}}_N\big|_{\epsilon=0} = \overline{\mathcal{I}}_N\big|_{\epsilon=0} = \Phi(v)^N \,, \tag{5.57}$$

with no dependence on the flavor fugacities, nor on τ in the elliptic case.

Next, we observe that in all cases the dependence on the flavor fugacities cancels out in $\dot{\mathcal{I}}^{k,N}$, after summing the various contributions from colored plane partitions. We have verified this claim up to a certain order in k. Assuming that the cancellation persists to all orders, our task of identifying the grand canonical partition functions simplifies.

Let us start with the elliptic DT invariants. As opposed to the Abelian case, for $N > 1$ (5.56) and the anomalous quasi-periodicity (5.42) are not enough to fix the partition function, since now $\epsilon = n/N$ with $n \in \mathbb{Z}$. Nevertheless, inspecting the result for various values of N and k, we were able to propose the following formula:

$$\mathcal{I}_{k,N}\Big|_{\epsilon=\frac{n}{N}} = \begin{cases} (-1)^{nk}\, \Phi^{(\gcd(n,N))}_{\frac{k}{N}\gcd(n,N)} & \text{if } \frac{N}{\gcd(n,N)}|k \,, \\ 0 & \text{otherwise} \,. \end{cases} \tag{5.58}$$

Here the coefficients $\Phi_k^{(N)}$, defined in subsec. C.3.1, are those of the series expansion of $\Phi(v)^N$. Moreover recall that $\gcd(0, N) = N$. The proposal (5.58) satisfies the anomalous quasi-periodicity (5.42). It is then easy to resum the series:

$$\mathcal{I}_N\big|_{\epsilon=\frac{n}{N}}(v) = \Phi\left((-1)^{nN}\, v^{\frac{N}{\gcd(n,N)}}\right)^{\gcd(n,N)} \,. \tag{5.59}$$

We provide a string theory derivation of this formula in subsec. 5.3.2. As in the Abelian case, we should expect no dependence on τ because the two-dimensional theory is gapped. The lack of dependence on the flavor fugacities is also observed in other gapped models, for instance the Grassmannians (see e.g. [**61**]).

In the trigonometric case, the following expression was proposed in [**73**]:[8]

$$\tilde{\mathcal{I}}_N = \texttt{PE}_{v,\vec{p}}\left[-\frac{(1-p_1p_2)(1-p_1p_3)(1-p_2p_3)}{(1-p_1)(1-p_2)(1-p_3)} q^{-\frac{N}{2}} \frac{1-p^N}{1-p} \frac{v}{(1-vp^{-\frac{N}{2}})(1-vp^{\frac{N}{2}})}\right] \,. \tag{5.60}$$

This reproduces Nekrasov's ansatz (5.40) for $N = 1$. We provide an M-theory derivation of this formula in subsec 5.3.3. It is possible to show that

$$\tilde{\mathcal{I}}_N\big|_{\epsilon=\frac{n}{N}} = \mathcal{I}_N\big|_{\epsilon=\frac{n}{N}} \,. \tag{5.61}$$

In order to evaluate the left-hand-side some care is needed: if we set $p = e^{2\pi i \frac{n}{N}}$ we find a vanishing argument in the plethystic exponential. Applying the definition (C.19), though, we

[8] We have verified it up to $k = 5$ and $N = 5$.

see that the terms that survive in the expansion are those for which $\frac{kn}{N} \in \mathbb{Z}$, namely such that $\frac{N}{\gcd(n,N)}|k$. We can compute those terms by substituting $n \mapsto \mu n$ and the taking the limit $\mu \to 1$.

Finally, for the rational case a conjecture was already put forward in [74–76]:

$$\overline{\mathcal{I}}_N(v) = \left(\overline{Z}_1(v)\right)^N = \Phi(v)^{-N\frac{\epsilon_{12}\epsilon_{13}\epsilon_{23}}{\epsilon_1\epsilon_2\epsilon_3}} . \tag{5.62}$$

We have verified this conjecture up to $k = 8$ and $N = 8$. As a check, the trigonometric expression (5.60) reduces to (5.62) in the rational limit. It is particularly simple to see that the trigonometric expression has a well-defined $q \to 1$ limit yielding $\Phi(v)^N$.

5.3.2 F-theoretic Interpretation of Elliptic DT Counting

We can give an interpretation of the elliptic non-Abelian DT invariants (5.58) from their realization in type IIB string theory, or F-theory, in terms of the D1/D7 brane system.

The setup consists of N D7-branes wrapping $T^2 \times \mathbb{C}^3$, as well as k D1-branes on the worldvolume of the D7's and wrapping T^2. There is a further complex plane $\mathbb{C}$ orthogonal to all branes. We can introduce a complex coordinate w on T^2, complex coordinates $x_{1,2,3}$ on $\mathbb{C}^3$ and u on $\mathbb{C}$. The Ω-background is geometrically implemented by fibering $\mathbb{C}^3 \times \mathbb{C}$ on T^2 in a non-trivial way, controlled by four complex parameters $\epsilon_{1,2,3,4}$. The fibering of complex structure that corresponds to the scheme we chose in field theory is such that each of the complex factors in the fiber is rotated by a complexified phase $e^{2\pi i \epsilon_a}$ for $a = 1, 2, 3, 4$, respectively, when we go around the b-cycle of T^2, while they are not rotated when we go around the a-cycle. Supersymmetry requires to impose a Calabi-Yau condition to the total geometry, $\sum_{a=1}^4 \epsilon_a = 0$. This means that we can identify $\epsilon_4 = -\epsilon = -\sum_{a=1}^3 \epsilon_a$.

The D7-branes source a non-trivial holomorphic profile for the axio-dilaton τ_{IIB} along the $\mathbb{C}$ fiber:

$$\tau_{\text{IIB}}(z) = \frac{1}{2\pi i} \sum_{\alpha=1}^N \log(u - u_\alpha) , \tag{5.63}$$

where u_α are the positions of the D7-branes on $\mathbb{C}$. Such parameters are controlled by real masses associated to the $\text{SU}(N)$ flavor symmetry in field theory. Going around the b-cycle, the fiber is rotated as $u \to e^{-2\pi i \epsilon} u$. Considering the case $u_\alpha = 0$, the condition that the axio-dilaton be periodic up to $\text{SL}(2, \mathbb{Z})$ transformations imposes the constraint

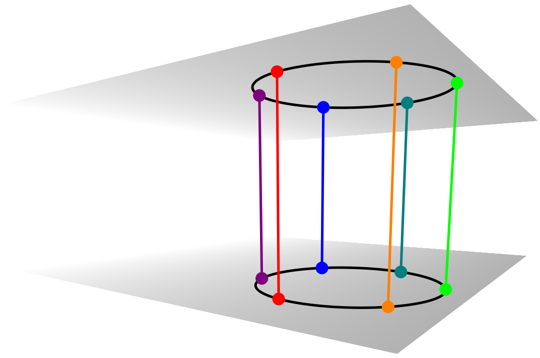
$$N\epsilon \in \mathbb{Z} . \tag{5.64}$$

This reproduces the anomaly constraint $\epsilon \in \mathbb{Z}/N$ in field theory, and forces us to set $\epsilon = n/N$ with $n \in \mathbb{Z}$.

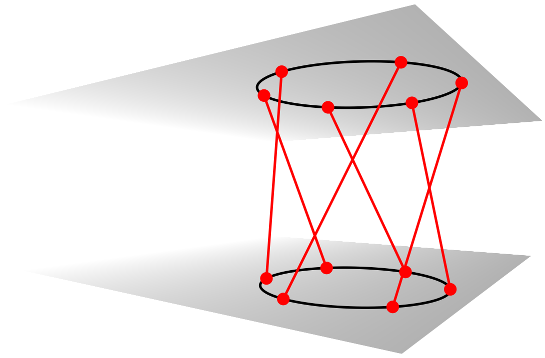
Next, we turn on the mass parameters u_α in a way compatible with the twisted geometry. For $\epsilon \neq 0 \mod 1$, periodicity around the b-cycle of T^2 imposes constraints on u_α. The simplest allowed choice is

$$u_\alpha = e^{2\pi i \alpha/N} u_{(0)} \qquad\qquad \text{for } \alpha = 1, \dots, N \tag{5.65}$$

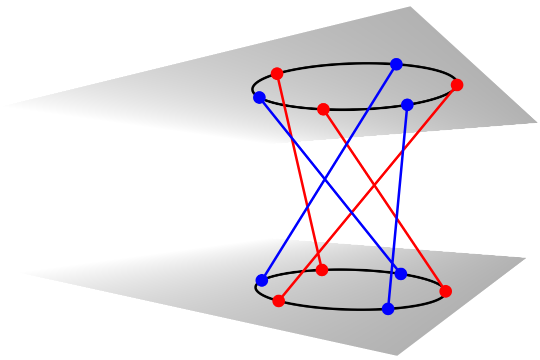
and generic $u_{(0)} \in \mathbb{C}$. This is a configuration where the branes homogeneously distribute on a circle around the origin. See fig. 5.2 for a pictorial representation of the various cases when



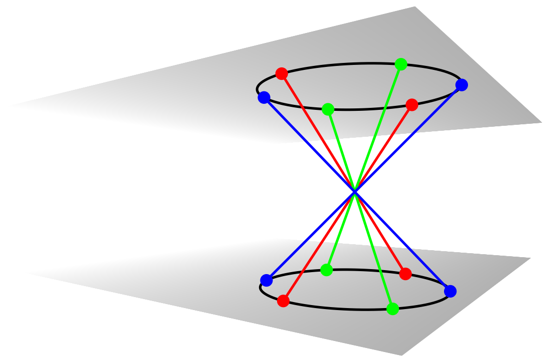
(a) Case $n = 0$: $\gcd(n, N) = 6$ different branes.



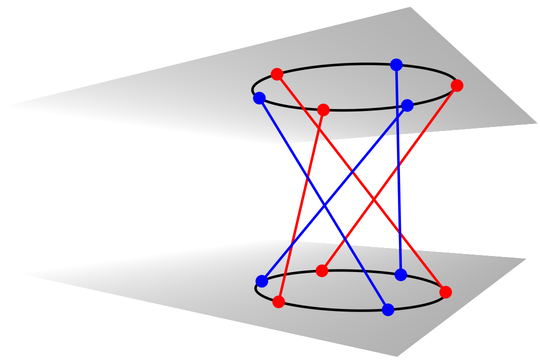
(b) Case $n = 1$: just $\gcd(n, N) = 1$ brane.



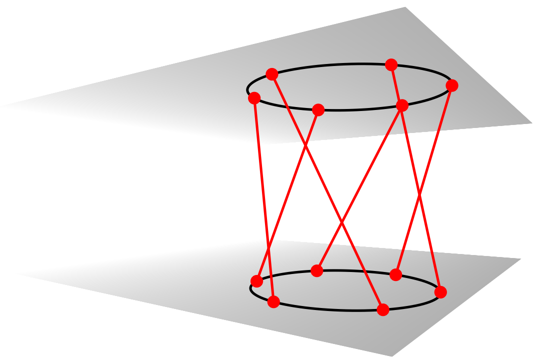
(c) Case $n = 2$: $\gcd(n, N) = 2$ different branes.



(d) Case $n = 3$: $\gcd(n, N) = 3$ different branes.



(e) Case $n = 4$: $\gcd(n, N) = 2$ different branes.



(f) Case $n = 5$: just $\gcd(n, N) = 1$ brane.

Figure 5.2: The case with $N = 6$.

$N = 6$. From the field theory point of view, twisted masses are in general not compatible with the SUSY background that gives rise to the elliptic genus, because they are charged under the (left-moving) R-symmetry for which we turn on a background flat connection. However the special choice (5.65) is invariant under a combination of R-symmetry rotation and Weyl transformation within SU(N).

The elliptic genus does not depend on the twisted masses, therefore we can safely evaluate it for u_α as in (5.65). Because of the twist, the N segments of D7-branes organize themselves into $\gcd(n, N)$ disconnected branes, each made of $N/\gcd(n, N)$ segments (see fig. 5.2). Notice that these numbers are correct even in the case of no twist, $n = 0$, in which the N D7's are simply taken apart. The twisted geometry has a $\mathbb{Z}_{N/\gcd(n,N)}$ symmetry, therefore if the number k of D1-branes is not a multiple of that, they cannot be moved from the origin to the worldvolumes of the D7's. This reproduces the condition in (5.58).

Finally, taking into account that each D7-brane is made of $N/\gcd(n, N)$ segments and so its worldvolume should be rescaled, we are left with a system of $\gcd(n, N)$ decoupled D7-branes, with a total of $k \gcd(n, N)/N$ D1-branes per segment to be distributed among the D7's. This is precisely the content of (5.58), or its generating function 5.59, up to the sign which is fixed by the R-symmetry anomaly. The extreme cases $n = 0$ and $n = 1$ are easier to understand.

5.3.3 M-theory Graviton Index: an Exercise on "Membranes and Sheaves"

We can give a geometric interpretation to the expression (5.60) in the realm of M-theory. This can be done as an exercise on [**77**].

Let us study our D-brane system from the viewpoint of M-theory. A bound state of N D6-branes and k D0-branes on S^1 can be lifted to an 11-dimensional bound state of k gravitons on $S^1 \times \mathbb{C}^3 \times \mathrm{TN}_N$, where TN_N is the N-center Taub-NUT space [**139**, **140**]. The Ω-deformation of this lift is a twisted equivariant fibration, which has been considered in [**77**]. Essentially, the toric space $\mathbb{C}^3 \times \mathrm{TN}_N$ is rotated by an action of $\mathrm{U}(1)^5$ as we circle around S^1, with a BPS constraint that the diagonal element does not act.

In the special case $N = 1$ [**74**], the 11-dimensional lift contains a single-center Taub-NUT space whose topology is the same as $\mathbb{C}^2$. Upon Ω-deformation, the BPS graviton states localize towards the center of TN_1 and become insensitive to the fact that its metric is different from that of $\mathbb{C}^2$. Therefore, one can compute the BPS index of gravitons on the Ω-deformed space by looking at the near-core geometry $\mathbb{C}^3 \times \mathbb{C}^2 \cong \mathbb{C}^5$. The index of BPS single-particle graviton states (plus anti-BPS states) turns out to be [**74**, **77**]

$$F_1^{(11)}(p_1, p_2, p_3, p_4, p_5) = \frac{\sum_{i=1}^5 p_i}{\prod_{i=1}^5 (1 - p_i)} + \frac{\sum_{i=1}^5 p_i^{-1}}{\prod_{i=1}^5 (1 - p_i^{-1})} . \tag{5.66}$$

For $\prod_{i=1}^5 p_i = 1$, it can be decomposed as

$$F_1^{(11)}(p_1, p_2, p_3, p_4, p_5) = F^{(6)}(p_1, p_2, p_3) + F^{(6)}(p_1^{-1}, p_2^{-1}, p_3^{-1}) + \mathcal{F}_1(p_1, p_2, p_3; v) , \tag{5.67}$$

where

$$F^{(6)}(p_1, p_2, p_3) = \frac{p}{\prod_{i=1}^3 (1 - p_i)} , \tag{5.68}$$

$$\mathcal{F}_1(p_1, p_2, p_3; v) = \frac{\prod_{i=1}^3 (1 - p/p_i)}{\prod_{i=1}^3 (1 - p_i)} \times \frac{1}{(1 - p^{1/2} v)(1 - p^{1/2} v^{-1})} , \tag{5.69}$$

we set $p = p_1 p_2 p_3$ and solved $p_4 = v p^{-1/2}$ and $p_5 = v^{-1} p^{-1/2}$. One can interpret $F^{(6)}$ as the perturbative contribution to the free energy of the 7-dimensional theory on the D6-brane on $S^1 \times \mathbb{C}^3$, and $\mathcal{F}_1$ as the instanton part. In fact, $\mathcal{F}_1$ is precisely the single-particle seed of the plethystic exponential in 5.40.

We can extend the computation of the BPS single-particle graviton index to the case $N > 1$. As we said, the 11-dimensional lift of the D0/D6 system is a bound state of gravitons on $S^1 \times \mathbb{C}^3 \times \mathrm{TN}_N$, and after Ω-deformation this becomes a fibration of $\mathbb{C}^3 \times \mathrm{TN}_N$ on S^1. Because the Ω-deformation localizes the graviton states around the origin of TN_N, we can safely substitute TN_N by its near-core geometry, the orbifold space $\mathbb{C}^2/\mathbb{Z}_N$.

The index of BPS single-particle graviton states (plus anti-BPS states) on $\mathbb{C}^3 \times [\mathbb{C}^2/\mathbb{Z}_N]$ is easily obtained by projecting to the $\mathbb{Z}_N$-invariant sector:

$$F_N^{(11)}(p_1,p_2,p_3,p_4,p_5) = \frac{1}{N}\sum_{a=1}^{N} F_1^{(11)}\left(p_1,p_2,p_3,p_4^{(a)},p_5^{(a)}\right) , \tag{5.70}$$

where the fugacities along the orbifold directions are

$$p_4^{(a)} = \omega^{(a)} v^{1/N} p^{-1/2} , \qquad p_5^{(a)} = \omega^{(-a)} v^{-1/N} p^{-1/2} , \tag{5.71}$$

and $\omega^{(a)} = e^{2\pi i a/N}$. To isolate the instanton counting factor, we subtract from the free energy the 7-dimensional perturbative contribution, and notice that $F^{(6)}$ is invariant under the $\mathbb{Z}_N$ action. Setting

$$F_N^{(11)}(p_1,p_2,p_3,p_4,p_5) = F^{(6)}(p_1,p_2,p_3) + F^{(6)}\left(p_1^{-1},p_2^{-1},p_3^{-1}\right) + \mathcal{F}_N(p_1,p_2,p_3;v) , \tag{5.72}$$

we obtain

$$\mathcal{F}_N(q_1,q_2,q_3;v) = \frac{\prod_{i=1}^3(1-p/p_i)}{\prod_{i=1}^3(1-p_i)} \times \frac{1}{N}\sum_{a=1}^{N} \frac{1}{(1-\omega^{(a)}p^{1/2}v^{1/N})(1-\omega^{(-a)}p^{1/2}v^{-1/N})} . \tag{5.73}$$

After resumming the last factor,[9] we obtain

$$\mathcal{F}_N(q_1,q_2,q_3;v) = \frac{\prod_{i=1}^3(1-p/p_i)}{\prod_{i=1}^3(1-p_i)} \times \frac{p^N-1}{p-1} \times \frac{1}{(1-p^{N/2}v)(1-p^{N/2}v^{-1})} . \tag{5.75}$$

This is precisely the single-particle seed of the plethystic exponential in (5.60).

5.4 Free Field Representation of Matrix Integral

In this section we give a representation of the elliptic genus partition function in terms of chiral free bosons on the torus. The very existence of such a representation indicates that the elliptic vertex algebra, i.e. the algebra of chiral vertex operators on the torus, might act on the cohomology of the moduli spaces that we have been studying so far and offer the language to detect a link to integrable systems in the spirit of the BPS/CFT correspondence [**81**].

The rational case in dimension 0 has a well-known free field representation in terms of chiral free bosons on the plane [**80**, **141**]. In the following we will represent the grand canonical partition function for the elliptic genera as a combination of two factors: the torus (chiral)

[9] A convenient way to perform the sum is the following. Consider the function

$$f(z) = \frac{1}{z^N - v}\cdot\frac{1}{z}\cdot\frac{1}{(1-p^{1/2}z)(1-p^{1/2}z^{-1})} , \tag{5.74}$$

which has $N+2$ poles: at $z = v^{1/N}\omega^{(a)}$, $z = p^{1/2}$ and $z = p^{-1/2}$. Computing the residues and using that their sum is zero, one obtains the desired formula.

correlator of an exponentiated integrated vertex (whose power expansion reproduces the contributions from multiplets in the adjoint representation), and a linear source (that reproduces the contributions from multiplets in the fundamental representation).

It is well known that an off-shell formulation of the chiral boson is difficult, therefore we will define it on-shell in the following way. Consider the usual free massless scalar boson two-point function

$$\left\langle \phi(u,\bar{u})\,\phi(w,\bar{w})\right\rangle_{T^2} = \log G(u,\bar{u};w,\bar{w}) \tag{5.76}$$

where

$$G(u,\bar{u};w,\bar{w}) = e^{-\frac{2\pi}{\tau_2}(\Im(u-w))^2}\left|\frac{\theta_1(\tau|u-w)}{2\pi\eta(\tau)^3}\right|^2 . \tag{5.77}$$

Here $\tau_2 = \Im\tau$. Using this propagator, one computes the elliptic vertex algebra and the correlation functions of vertex fields of the usual type $:e^{\lambda\phi}:$. A generic higher-point correlation function is the product of three factors: a holomorphic (in u and w) contribution proportional to a product of functions θ_1, an anti-holomorphic contribution proportional to $\bar{\theta}_1$'s, and a mixed contribution proportional to a product of exponentials. If the last term cancels out, then we can define, up to a pure c-number phase, the chiral projection of the correlation function by picking the holomorphic contribution.

Let us consider the following vertex operator:

$$\mathcal{V}_{\vec{\epsilon}}(u) = \prod_{i=1}^{7} :e^{\lambda_i\phi_i(u_{+i})}::e^{-\lambda_i\phi_i(u_{-i})}: , \tag{5.78}$$

where $\vec{\lambda} = (\mathrm{i},\mathrm{i},\mathrm{i},\mathrm{i},1,1,1)$ and

$$\begin{aligned} u_{\pm i} &= u \pm \frac{\tilde{\epsilon}_i}{2} , & \tilde{\epsilon}_1 &= \epsilon_1 , & \tilde{\epsilon}_2 &= \epsilon_2 , & \tilde{\epsilon}_3 &= \epsilon_3 , \\ \tilde{\epsilon}_4 &= \epsilon , & \tilde{\epsilon}_5 &= \epsilon_{12} , & \tilde{\epsilon}_6 &= \epsilon_{13} , & \tilde{\epsilon}_7 &= \epsilon_{23} , \end{aligned} \tag{5.79}$$

are the vertices of two cubes with sides $\pm\epsilon_i/2$ as shown in fig. 5.3.

At each vertex we placed one of 7 non-interacting scalar fields on the torus with normalized two-point function

$$\left\langle \phi_i(u,\bar{u})\,\phi_j(w,\bar{w})\right\rangle_{T^2} = \delta_{ij}\log G(u,\bar{u};w,\bar{w}) . \tag{5.80}$$

Using Wick's theorem it is straightforward to find

$$\mathcal{V}_{\vec{\epsilon}}(u) = \prod_{i=1}^{7}\Big[G(u_{+i},\bar{u}_{+i};u_{-i},\bar{u}_{-i})\Big]^{\lambda_i^2} :\mathcal{V}_{\vec{\epsilon}}(u): \tag{5.81}$$

$$= \left|\frac{2\pi\eta^3(\tau)\,\theta_1(\tau|\epsilon_{12})\,\theta_1(\tau|\epsilon_{13})\,\theta_1(\tau|\epsilon_{23})}{\theta_1(\tau|\epsilon_1)\,\theta_1(\tau|\epsilon_2)\,\theta_1(\tau|\epsilon_3)\,\theta_1(\tau|\epsilon)}\right|^2 :\mathcal{V}_{\vec{\epsilon}}(u): , \tag{5.82}$$

where, in the second line, the exponent of the imaginary parts squared cancels since

$$\sum\nolimits_{i=1}^{7}\lambda_i^2\left(\Im(\tilde{\epsilon}_i)\right)^2 = 0 . \tag{5.83}$$

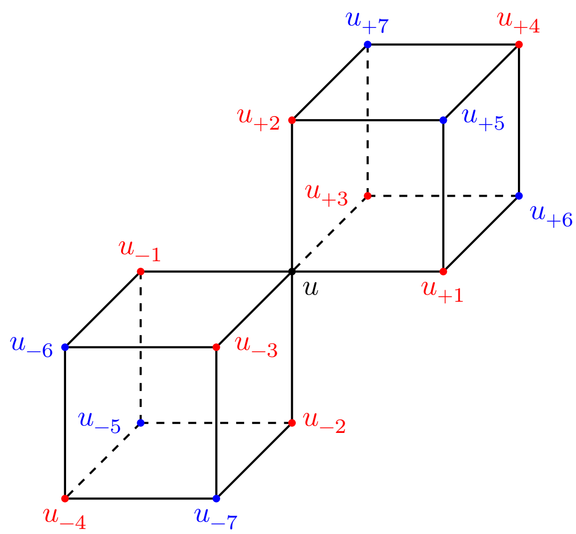


Figure 5.3: Representation of the multi-local vertex operator $\mathcal{V}_{\vec{\epsilon}}$. Red vertices are those for which $\lambda = \mathrm{i}$ while blue ones are those for with $\lambda = 1$.

Again, using Wick's theorem, we find:

$$:e^{\lambda_i\phi_i(u_{+i})}e^{-\lambda_i\phi_i(u_{-i})}::e^{\lambda_j\phi_j(u_{+j})}e^{-\lambda_j\phi_j(u_{-j})}:=$$
$$= \left[\frac{G\left(u_{+i},\bar{u}_{+i};u_{-j},\bar{u}_{-j}\right)G\left(u_{-i},\bar{u}_{-i};u_{+j},\bar{u}_{+j}\right)}{G\left(u_{+i},\bar{u}_{+i};u_{+j},\bar{u}_{+j}\right)G\left(u_{-i},\bar{u}_{-i};u_{-j},\bar{u}_{-j}\right)}\right]^{\delta_{ij}\lambda_i\lambda_j}\times$$
$$\times :e^{\lambda_i\phi_i(u_{+i})}e^{-\lambda_i\phi_i(u_{-i})}e^{\lambda_j\phi_j(u_{+j})}e^{-\lambda_j\phi_j(u_{-j})}: \,. \quad (5.84)$$

The factor in square brackets, when, $i = j$ is

$$\left|\frac{\theta_1(\tau|u-v+\tilde{\epsilon}_i)\,\theta_1(\tau|u-v-\tilde{\epsilon}_i)}{\theta_1^2(\tau|u-v)}\right|^{\lambda_i^2} e^{-\frac{4\pi}{\tau_2}\lambda_i^2(\Im(\tilde{\epsilon}_i))^2}\,. \quad (5.85)$$

by which it follows that

$$\langle:\mathcal{V}_{\vec{\epsilon}}(u)::\mathcal{V}_{\vec{\epsilon}}(w):\rangle = \left|\frac{\theta_1^2(\tau|u-w)\theta_1(\tau|z-w+\epsilon_{12})\,\theta_1(\tau|u-w+\epsilon_{13})\,\theta_1(\tau|u-w+\epsilon_{23})}{\theta_1(\tau|u-w+\epsilon_1)\,\theta_1(\tau|u-w+\epsilon_2)\,\theta_1(\tau|u-w+\epsilon_3)\,\theta_1(\tau|u-w+\epsilon)}\times\right.$$
$$\left.\times\frac{\theta_1(\tau|u-w-\epsilon_{12})\,\theta_1(\tau|u-w-\epsilon_{13})\,\theta_1(\tau|u-w-\epsilon_{23})}{\theta_1(\tau|u-w+\epsilon_1)\,\theta_1(\tau|u-w+\epsilon_2)\,\theta_1(\tau|u-w+\epsilon_3)\,\theta_1(\tau|u-w+\epsilon)}\right|^2. \quad (5.86)$$

Notice that again because of eq. (5.83) the exponent of the imaginary part squared cancels in eq. (5.86) and we can define its holomorphic projection as:

$$\langle:\mathcal{V}_{\vec{\epsilon}}(u)::\mathcal{V}_{\vec{\epsilon}}(w):\rangle_{\text{hol.}} = \frac{\theta_1^2(\tau|u-w)\theta_1(\tau|z-w+\epsilon_{12})\,\theta_1(\tau|u-w+\epsilon_{13})\,\theta_1(\tau|u-w+\epsilon_{23})}{\theta_1(\tau|u-w+\epsilon_1)\,\theta_1(\tau|u-w+\epsilon_2)\,\theta_1(\tau|u-w+\epsilon_3)\,\theta_1(\tau|u-w+\epsilon)}\times$$
$$\times\frac{\theta_1(\tau|u-w-\epsilon_{12})\,\theta_1(\tau|u-w-\epsilon_{13})\,\theta_1(\tau|u-w-\epsilon_{23})}{\theta_1(\tau|u-w+\epsilon_1)\,\theta_1(\tau|u-w+\epsilon_2)\,\theta_1(\tau|u-w+\epsilon_3)\,\theta_1(\tau|u-w+\epsilon)}. \quad (5.87)$$

which is the contribution of single modes in the adjoint.

The other term that we need to give a free-boson representation of our matrix model is the following source operator:

$$H = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \partial\phi_4(w)\omega(w)\mathrm{d}w, \tag{5.88}$$

where ω is a locally analytic function in the inner region bounded by the contour Γ. The contour Γ is chosen to be a closed path around $w = 0$ encircling all $u_{\pm i}$ for $i = 1, \dots, 7$ where $u = 0$. Then we can compute[10]:

$$e^{H} :e^{\lambda_j\phi_j(u_{+j})}e^{-\lambda_j\phi_j(u_{-j})}: = e^{W} :e^{H}e^{\lambda_j\phi_j(u_{+j})}e^{-\lambda_j\phi_j(u_{-j})}: , \tag{5.94}$$

where

$$W = \delta_{4j}\lambda_j \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \mathrm{d}w\omega(w) \left[\partial_w\langle\phi_4(w)\phi_j(u_{+j})\rangle - \partial_w\langle\phi_4(w)\phi_j(u_{-j})\rangle\right] \tag{5.95}$$

$$= \delta_{j4}\lambda_j \left[\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \mathrm{d}w\omega(w)[\zeta_{\mathrm{W}}(w - u_{+j}) - \zeta_{\mathrm{W}}(w - u_{-j})] - \frac{2i}{\tau_2}\oint_{\Gamma} \mathrm{d}w\omega(w)\Im(\tilde{\epsilon}_j)\right] , \tag{5.96}$$

where we introduced the Weierstrass-ζ function: $\zeta_{\mathrm{W}}(u) = \partial \log\theta_1(\tau|u)$ which has a simple pole around the origin:

$$\zeta_{\mathrm{W}}(u) = \frac{1}{u} + \text{holomorphic in } u. \tag{5.97}$$

The second term in the last line of eq. (5.95) is zero since ω is holomorphic inside Γ. It follows that:

$$\langle e^{H} :\mathcal{V}_{\vec{\epsilon}}(u):\rangle = e^{\frac{1}{2\pi}\oint_{\Gamma}[(w-u_{+4})^{-1}-(w-u_{-4})^{-1}]\omega(w)\mathrm{d}w} = e^{\mathrm{i}\omega(u+\epsilon/2)-\mathrm{i}\omega(u-\epsilon/2)} , \tag{5.98}$$

Choosing (up to an irrelevant additive constant)

$$\omega(u) = \mathrm{i}\sum_{\alpha=1}^{N}\log\theta_1\left(\tau\middle|u + z_\alpha - \frac{\epsilon}{2}\right) , \tag{5.99}$$

which is holomorphic inside Γ for generic values[11] of the Cartan parameters $\{z_\alpha\}$, eq. (5.98) reads

$$\langle e^{H} :\mathcal{V}_{\vec{\epsilon}}(u):\rangle_{\text{hol.}} = \prod_{\alpha=1}^{N}\frac{\theta_1(\tau|u + z_\alpha - \epsilon)}{\theta_1(\tau|u + z_\alpha)}. \tag{5.100}$$

[10] In the following formula we can trade e^H as $:e^H:$ since ω is holomorphic inside Γ. Indeed we have that

$$:e^{H}:= e^{\mathfrak{N}}e^{H} , \tag{5.89}$$

where the normal ordering operator is defined

$$\mathfrak{N} = \int \mathrm{d}^2z\mathrm{d}^2w\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle\frac{\delta}{\delta\phi(z,\bar{z})}\frac{\delta}{\delta\phi(w,\bar{w})} . \tag{5.90}$$

We consider now:

$$\mathfrak{N}e^{H} = \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \oint_{\Gamma}\mathrm{d}u\omega(u)\oint_{\Gamma}\mathrm{d}u'\omega(u')\partial_u\partial_{u'}\langle\phi(u,\bar{u})\phi(u',\bar{u}')\rangle e^{H} \tag{5.91}$$

$$= -\oint_{\Gamma}\mathrm{d}u\omega(u)\partial\omega(u)e^{H} \tag{5.92}$$

$$= 0 . \tag{5.93}$$

This implies our claim.

[11] The branch-cuts of the logarithms generically extend outside the contour.

Moreover notice that, since only the chiral part of the scalar boson enters eq. (5.88), eq. (5.100) is already holomorphic, so we add the subscript "hol." without further ado. Now using eqs. (5.87) and (5.100) we can expand

$$\langle e^H e^{v \oint_{\mathcal{C}} \mathcal{V}_{\vec{\epsilon}}(u)\mathrm{d}u} \rangle_{\text{hol.}} = \sum_{k=0}^{\infty} \frac{v^k}{k!} \left[\frac{2\pi\eta^3(\tau)\theta_1(\tau|\epsilon_{12})\,\theta_1(\tau|\epsilon_{13})\,\theta_1(\tau|\epsilon_{23})}{\theta_1(\tau|\epsilon_1)\,\theta_1(\tau|\epsilon_2)\,\theta_1(\tau|\epsilon_3)\,\theta_1(\tau|\epsilon)} \right]^k \times$$
$$\times \oint_{\mathcal{C}} \mathrm{d}u_1 \dots \oint_{\mathcal{C}} \mathrm{d}u_k \prod_{i=1}^{k} \prod_{\alpha=1}^{N} \frac{\theta_1(\tau|u_i + z_\alpha - \epsilon)}{\theta_1(\tau|u_i + z_\alpha)} \times$$
$$\times \prod_{\substack{i,j=1 \\ i\neq j}}^{k} \frac{\theta_1(\tau|u_{ij})\,\theta_1(\tau|u_{ij}+\epsilon_{12})\,\theta_1(\tau|u_{ij}+\epsilon_{13})\,\theta_1(\tau|u_{ij}+\epsilon_{23})}{\theta_1(\tau|u_{ij}+\epsilon_1)\,\theta_1(\tau|u_{ij}+\epsilon_2)\,\theta_1(\tau|u_{ij}+\epsilon_3)\,\theta_1(\tau|u_{ij}+\epsilon)} \,. \quad (5.101)$$

Notice that the prefactor in the first line arises from the fact that in the l.h.s. $\mathcal{V}_{\vec{\epsilon}}$ is present *without* normal ordering (see the holomorphic part of eq. (5.82)). Comparing eqs. (5.24) and (5.101) we realize that

$$\mathcal{I}_{(N)}(v) = \langle e^H e^{v \oint_{\mathcal{C}} \mathcal{V}_{\vec{\epsilon}}(u)\mathrm{d}u} \rangle_{\text{hol.}} \quad (5.102)$$

provided the contour $\mathcal{C}$ to be the one specified by JK prescription. Let's remark that the function defined trough H can be lifted to T^2 in the cases for which the R-symmetry is not anomalous, that is $\epsilon \in \mathbb{Z}/N$.

6. Conclusions and Outlook

In the present thesis we reviewed some features of elliptic genus for 2d GLSM, and we explored some new aspects about its computation and its application to brane systems. The main results are can be summarized as follows:

- we gave a prescription to treat degenerate cases of JK procedure and to reconduct them as the non-degenerate ones;
- we explored the moduli space of D1/D3 system and we gave a first principle computation of the elliptic vortex partition function;
- we explored the moduli space of D1/D7 and discussed its factorization properties as well as those of its dimensional reductions D0/D6 and D(−1)/D5. Moreover we gave a representation of the elliptic genus of that moduli space in terms of free bosons on a torus.

There are certain directions in which our analysis can be extended. As far a the desingularization algorithm is concerned we expect that it can be generalized also for some situations in which $A_i(u^*) > C_i(u^*) + 1$, providing a regular product representation also for these cases that can be of some interest in most generic situations.

In would be interesting to change target manifold in our computations, for instance one can study the D1/D7 system on more general toric geometries, such as the conifold, where a wall crossing phenomenon among different geometric phases of the moduli space is expected to arise, see [**142**] for a review. On such geometries, bound states including D2-branes become important, and a description of D2/D6 systems in terms of 3d Chern–Simons–matter theories [**143–145**] might turn useful. In our approach, the different phases should be related to different choices of the integration contour. Moreover, it would be interesting to investigate whether the factorization property of the matrix model limit is spoiled on more general geometries. It would be also interesting to investigate along these lines the supersymmetric partition function on compact toric three-folds, as for example $\mathbb{P}^3$ or $\mathbb{P}^1 \times \mathbb{P}^2$, in order to compute topological invariants of higher-rank stable sheaves on them. Analogous computations in two complex dimensions have been performed in [**146–148**], while some results for three-folds already appeared in the mathematical literature [**149**]. The free field representation of the elliptic genus seems to signal the existence of an elliptic vertex algebra acting on the associated moduli space of sheaves, see [**150**] for recent progress in this direction. We also expect this result to prompt a constructive connection with integrable hierarchies, which would be very interesting to investigate.

To conclude, it would be interesting to apply our technique to the case of D0/D8 brane system as recently suggested in [**151**].

A. Conventions

Here we discuss the conventions used in the present thesis.

A.1 Coordinates

We consider a two dimensional Euclidean spacetime. We use small Greek indices (μ, ν and so on) to label coordinates. We will also use complex coordinates

$$\begin{aligned} z &= x^1 + \mathrm{i}x^2 \ , \\ \bar{w} &= x^1 - \mathrm{i}x^2 \ , \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} \partial &= \tfrac{1}{2}(\partial_1 - \mathrm{i}\partial_2) \ , \\ \bar{\partial} &= \tfrac{1}{2}(\partial_1 + \mathrm{i}\partial_2) \ . \end{aligned} \tag{A.2}$$

Also for general one-form

$$\begin{aligned} A &= \tfrac{1}{2}(A_1 - \mathrm{i}A_2) \ , \\ \bar{A} &= \tfrac{1}{2}(A_1 + \mathrm{i}A_2) \ . \end{aligned} \tag{A.3}$$

A.2 Spinors

We introduce Dirac spinors

$$\psi^a = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \ . \tag{A.4}$$

The Clifford algebra generated by Gamma matrices

$$(\gamma_1)^a{}_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad (\gamma_2)^a{}_b = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \ , \tag{A.5}$$

The chirality matrix is

$$(\gamma_3)^a{}_b = \tfrac{1}{2\mathrm{i}}[\gamma_1, \gamma_2]^a{}_b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{A.6}$$

We have that

$$(\gamma_\mu \gamma_\nu)^a{}_b = \delta_{\mu\nu}\delta^a{}_b + \mathrm{i}\epsilon_{\mu\nu}(\gamma_3)^a{}_b \ . \tag{A.7}$$

The charge conjugation matrix

$$\mathcal{C}_{ab} = (\gamma_2)^a{}_b \ , \tag{A.8}$$

defines the invariant Majorana product

$$\psi\chi = \psi^{\rm t}\mathcal{C}\chi \ , \tag{A.9}$$

which can be expressed in index notation as

$$\psi\chi = \psi^a\chi_a = \psi^a\mathcal{C}_{ab}\chi^b \ . \tag{A.10}$$

$$\psi_+ = -{\rm i}\psi^- \ , \qquad \psi_- = {\rm i}\psi^+ \ . \tag{A.11}$$

and

$$\psi\chi = \psi^+\chi_+ + \psi^-\chi_- = -{\rm i}\psi^+\chi^- + {\rm i}\psi^-\chi^+ \ . \tag{A.12}$$

One can, as usually, introduce the n-form[1]

$$\psi\gamma_{\mu_1\cdots\mu_n}\chi = \psi^{\rm t}\mathcal{C}\gamma_{\mu_1\cdots\mu_n}\chi = \psi^a\mathcal{C}_{ab}(\gamma_{\mu_1\cdots\mu_n})^b{}_c\chi^c \ , \tag{A.13}$$

where

$$\gamma_{\mu_1\cdots\mu_n} = \gamma_{[\mu_1}\cdots\gamma_{\mu_n]} \ . \tag{A.14}$$

One can also introduce gamma matrices with lowered indices

$$(\gamma_{\mu_1\cdots\mu_n})_{ac} = \mathcal{C}_{ab}(\gamma_{\mu_1\cdots\mu_n})^b{}_c \tag{A.15}$$

so that

$$\psi\gamma_{\mu_1\cdots\mu_n}\chi = \psi^a(\gamma_{\mu_1\cdots\mu_n})_{ab}\chi^b \ . \tag{A.16}$$

In particular,

$$(\gamma_1)_{ab} = \begin{pmatrix} -{\rm i} & 0 \\ 0 & {\rm i} \end{pmatrix} \ , \qquad (\gamma_2)_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \qquad (\gamma_3)^a{}_b = \begin{pmatrix} 0 & {\rm i} \\ {\rm i} & 0 \end{pmatrix} \ . \tag{A.17}$$

Now, since

$$\gamma_2\gamma_\mu^{\rm t}\gamma_2 = -\gamma_\mu \ , \tag{A.18}$$

it follows that, for anticommuting spinors

$$\psi\gamma_{\mu_1}\cdots\gamma_{\mu_n}\chi = (-1)^n\chi\gamma_{\mu_n}\cdots\gamma_{\mu_1}\psi \ , \tag{A.19}$$

while for commuting ones

$$\psi\gamma_{\mu_1}\cdots\gamma_{\mu_n}\chi = (-1)^{n+1}\chi\gamma_{\mu_n}\cdots\gamma_{\mu_1}\psi \ , \tag{A.20}$$

[1] include the pseudo-tensors built with the chirality matrix

where the μ's run from 1 to 3, and the result holds for the boundary case of $n = 0$ as well. We also define chiral projectors

$$\mathrm{P}_{\pm} = \tfrac{1}{2}(\mathbf{I} \pm \gamma^3) \ , \tag{A.21}$$

explicitly

$$(\mathrm{P}_+)_{ab} = \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 0 \end{pmatrix} \ , \qquad (\mathrm{P}_-)_{ab} = \begin{pmatrix} 0 & -\mathrm{i} \\ 0 & 0 \end{pmatrix} \ . \tag{A.22}$$

We will also use Gamma matrices in complex basis

$$\gamma = \gamma_1 + \mathrm{i}\gamma_2 \ , \qquad \bar{\gamma} = \gamma_1 - \mathrm{i}\gamma_2 \ . \tag{A.23}$$

It holds

$$(\gamma^\mu \partial_\mu)_{ab} = (\gamma\partial)_{ab} + (\bar{\gamma}\bar{\partial})_{ab} = \begin{pmatrix} -2\mathrm{i}\partial_w & 0 \\ 0 & 2\mathrm{i}\partial_{\bar{w}} \end{pmatrix} \ . \tag{A.24}$$

It is also useful to denote

$$\gamma \cdot \partial = \gamma^\mu \partial_\mu = \gamma\partial + \bar{\gamma}\bar{\partial} \ . \tag{A.25}$$

A.2.1 Useful Identities

Starting from Fierz master formula[2]

$$\psi^a \chi^b = -\tfrac{1}{2}\psi\chi \mathbf{I}^{ab} + \tfrac{1}{2}\psi\gamma^\mu\chi(\gamma_\mu)^{ab} + \tfrac{1}{2}\psi\gamma_3\chi(\gamma_3)^{ab} \ , \tag{A.26}$$

we get

$$\begin{aligned} \theta\psi\,\theta\chi &= -\tfrac{1}{2}\theta\theta\,\psi\chi \ , \\ \theta\gamma^\mu\psi\,\theta\gamma^\nu\chi &= \tfrac{1}{2}\theta\theta(\delta^{\mu\nu}\psi\chi + \mathrm{i}\epsilon^{\mu\nu}\psi\gamma^3\chi) \ , \\ \theta\gamma^\mu\bar{\theta}\,\theta\gamma^\nu\bar{\theta} &= \tfrac{1}{2}\delta^{\mu\nu}\theta\theta\,\bar{\theta}\bar{\theta} \ , \\ \lambda\eta\,\chi\psi &= \tfrac{1}{2}(\lambda\gamma^\mu\chi)(\eta\gamma_\mu\psi) - (\lambda P_-\chi)(\eta P_+\psi) - (\lambda P_+\chi)(\eta P_-\psi) \ , \\ \theta\mathrm{P}_-\bar{\theta}\,\theta\mathrm{P}_+\bar{\theta} &= -\tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta} \ . \end{aligned} \tag{A.27}$$

[2] Recall that indices are raised by $\mathcal{C}$

B. Special Functions

Here we list the special function that have been used in our discussion with some of their useful properties.

B.1 q-Pochhammer, η and θ

First of all we define the modular parameter to be $q = e^{2\pi i\tau}$, with $\Im(\tau) > 0$. The q-Pochhammer is defined:

$$(y, q)_\infty := \prod_{k=0}^{\infty}(1 - yq^k) \ . \tag{B.1}$$

In term of this function is possible to define the Dedekind eta:

$$\eta(q) := q^{\frac{1}{24}}(q; q)_\infty \ . \tag{B.2}$$

and the "core" of Jacobi theta functions:

$$\theta(\tau|z) := (y; q)_\infty (qy^{-1}; q)_\infty \ , \tag{B.3}$$

where we set for convenience $y = e^{2\pi iz}$. The most ubiquitous function in this paper is the Jacobi theta of first kind:

$$\theta_1(\tau|z) := iq^{\frac{1}{8}}y^{-\frac{1}{2}}(q; q)_\infty \theta(\tau|z) \tag{B.4}$$

$$= -iq^{\frac{1}{8}}y^{\frac{1}{2}}(q; q)_\infty \theta(\tau| - z) \ . \tag{B.5}$$

From its definition is possible to see that Jacobi function is odd:

$$\theta_1(\tau| - z) = -\theta_1(\tau|z) \ . \tag{B.6}$$

Moreover, under shift of the argument $z \mapsto z + a + b\tau$ $(a, b \in \mathbb{Z})$, the function transforms as:

$$\theta_1(\tau|z + a + b\tau) = (-1)^{a+b}e^{-2\pi ibz}e^{-i\pi b^2\tau}\theta_1(\tau|z) \ . \tag{B.7}$$

So we see that it is 1-periodic and τ-quasiperiodic. Then we can reduce to study its behaviour in a "fundamental domain" of the lattice $\mathbb{Z}+\tau\mathbb{Z}$. The function $\theta_1(\tau|z)$ has no poles and simple zeroes occur at $z = \mathbb{Z} + \tau\mathbb{Z}$. The residues of its inverse are

$$\frac{1}{2\pi i}\oint_{z=a+b\tau}\frac{\mathrm{d}z}{\theta_1(\tau|z)} = \frac{(-1)^{a+b}e^{i\pi b^2\tau}}{2\pi\eta^3(q)} \ ; \tag{B.8}$$

and for small values of q and z we have:

$$\theta_1(\tau|z) \xrightarrow{q\to 0} 2q^{\frac{1}{8}}\sin(\pi z) \xrightarrow{z\to 0} 2\pi q^{\frac{1}{8}}z \ . \tag{B.9}$$

B.2 Θ and Elliptic Hypergeometric Functions

We give the following definitions which is suitable for our results:

$$\Theta_\bullet(\tau,\sigma|a)_n := \begin{cases} \prod_{k=0}^{n-1} \theta_\bullet(\tau|a+k\sigma) & n\in\mathbb{Z}_\geq\ , \\ \left[\prod_{k=0}^{|n|-1} \theta_\bullet(\tau|a-(k+1)\sigma)\right]^{-1} & n\in\mathbb{Z}_<\ ; \end{cases} \tag{B.10}$$

where $\theta_\bullet$ can be just θ or θ_1, and the same for $\Theta_\bullet$. This function enjoys the key property of Pochhammer symbol:

$$\Theta_\bullet(\tau,\sigma|a)_m = \Theta_\bullet(\tau,\sigma|a)_n\Theta_\bullet(\tau,\sigma|a+n\sigma)_{m-n}. \tag{B.11}$$

Using eq. (B.9), indeed, it is possible to show that for small values of its arguments:

$$\Theta(\tau,\sigma|a) \xrightarrow[a+n\sigma\to0]{q\to0} [-2\pi i\sigma]^n\left(\frac{a}{\sigma}\right)_n\ ; \qquad \Theta_1(\tau,\sigma|a) \xrightarrow[a+n\sigma\to0]{q\to0} \left[2\pi q^{\frac{1}{8}}\sigma\right]^n\left(\frac{a}{\sigma}\right)_n\ . \tag{B.12}$$

Another identity which will be useful is the following which holds[1] just for Θ_1:

$$\Theta_1(\tau,\sigma|a-l\sigma)\Theta_1(\tau,\sigma|-a-m\sigma) = \frac{\theta_1(\tau|a)}{\theta_1(\tau|a+(m-l)\sigma)}\Theta_1(\tau,\sigma|a+\sigma)_m\Theta_1(\tau,\sigma|-a+\sigma)_l\ . \tag{B.13}$$

We shall use Θ to write products of Θ_1. In the present cases we have the following situation:

$$\frac{\prod_{i=1}^M\Theta_1(\tau,\epsilon|z_i)_n}{\prod_{i=1}^M\Theta_1(\tau,\epsilon|a_i)_n} = e^{i\pi n\left(\sum_{i=1}^M a_i-\sum_i^M z_i\right)}\frac{\prod_{i=1}^M\Theta(\tau,\sigma|z_i)_n}{\prod_{i=1}^M\Theta(\tau,\sigma|a_i)_n}\ . \tag{B.14}$$

Elliptic hypergeometric series [**125**] are defined as follows:

$${}_rE_s\left(\begin{matrix}t_0,\dots,t_{r-1}\\ w_1,\dots,w_s\end{matrix}\middle|\tau,\sigma\middle|z\right) := \sum_{k=0}^{\infty}\frac{\prod_{i=0}^{r-1}\Theta(\tau,\sigma|t_i)_k}{\Theta(\tau,\sigma|\sigma)_k\prod_{j=1}^{s}\Theta(\tau,\sigma|w_j)_k}z^k\ . \tag{B.15}$$

A remarkable property is the following[2]:

$$\theta(\tau|b+z\partial_z)\,{}_rE_s\left(\begin{matrix}t_0,\dots,t_{r-1}\\ w_1,\dots,w_s\end{matrix}\middle|\tau,\sigma\middle|z\right) = \theta(\tau|b)\,{}_{r+1}E_{s+1}\left(\begin{matrix}t_0,\dots,t_{r-1},b+\sigma\\ w_1,\dots,w_s,b\end{matrix}\middle|\tau,\sigma\middle|z\right)\ . \tag{B.16}$$

In the case of $b=0$, eq. (B.16) reduces to:

$$\theta(\tau|z\partial_z)\,{}_rE_s\left(\begin{matrix}t_0,\dots,t_{r-1}\\ w_1,\dots,w_s\end{matrix}\middle|\tau,\sigma\middle|z\right) = \frac{\prod_{i=0}^{r-1}\theta(\tau|t_i)}{\prod_{j=1}^{s}\theta(\sigma|w_i)}\,{}_rE_s\left(\begin{matrix}t_0+\sigma,\dots,t_{r-1}+\sigma\\ w_1+\sigma,\dots,w_s+\sigma\end{matrix}\middle|\tau,\sigma\middle|z\right)\ ; \tag{B.17}$$

which resembles the usual relation between ${}_rF_s$ and its derivatives.

[1]This is actually a property that θ_1 and its products share with all odd functions.
[2]Similar operators appear in [**152**].

C. Technical Details

In this appendix we collect some technical details that we omitted in the main text for the sake of readability.

C.1 Canonical Form of Charge Matrix

In this section we want to determine what is the most general form of a $r \times r$ charge matrix

$$Q(u^*) = (Q_1^\mathsf{T}, \dots, Q_r^\mathsf{T}) \ , \tag{C.1}$$

whose charge covector can be either of "fundamental" type that is

$$Q_{\mathrm{F};i} = (0, \dots, \underbrace{1}_{i}, \dots, 0) \ , \tag{C.2}$$

whose number will be denoted by f, or of "adjoint" type, namely

$$Q_{\mathrm{A};i,j} = (0, \dots, \underbrace{1}_{i}, \dots, \underbrace{-1}_{j}, \dots, 0) \ . \tag{C.3}$$

Moreover we want

$$Q(u^*)^\mathsf{T} \begin{pmatrix} u_1^* \\ \vdots \\ u_r^* \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} \ , \tag{C.4}$$

to have isolated solution for u^* for generic values of d_i's and that

$$Q(u^*) \begin{pmatrix} \varsigma_1 \\ \vdots \\ \varsigma_r \end{pmatrix} = \eta^\mathsf{T} \qquad \text{for all } \varsigma_i > 0. \tag{C.5}$$

with $\eta = (1, \dots, 1)$. The quest for isolated solution is possible if $f > 0$, otherwise $\det Q(u^*) = 0$. In order to find a canonical form of $Q(u^*)$ we will use two moves:

- swap columns: this is equivalent to relabeling the ς's;
- swap rows: this is equivalent to a Weyl transformation, i.e. to a permutation of u's.

The algorithm to reach the canonical form goes as follows:

`Step 1:` Choose v_1, a vector of type Q_F among Q_i^T with $i = 1, \dots, r$. Shuffle rows so that the only non-vanishing entry of v_1 sits at the first row. Shuffle the columns so that v_1 is Q_1^T.

`Step 2:` Choose v_2 among $Q_i^\intercal$ with $i = 2, \dots, r$ such that its first entry is non-vanishing. If there is no such a vector, go to `Intermezzo`. The vector v_2 will have another non-zero entry to maintain $\det Q(u^*) \neq 0$: shuffle the rows after the first so that the first two entries of v_2 are non-zero while the other vanish. Shuffle the columns after the first so that v_2 is $Q_2^\intercal$.

`Step` p`:` Choose v_p among $Q_i^\intercal$ with $i = p, \dots, r$ such that its first p entry are not all vanishing. If there is no such a vector go to `Intermezzo`. The vector v_p will have another non-vanishing component after the $(p-1)^{\text{th}}$ entry, otherwise $Q_1, \dots, Q_p$ would be linear dependent and $\det Q(u^*) = 0$. Shuffle the rows after the $(p-1)^{\text{th}}$ so that this non-vanishing value sits in the p^{th} entry. Shuffle the columns after the $(p-1)^{\text{th}}$ so that v_p is $Q_p^\intercal$.

`Intermezzo:` After having chosen r_1 vectors $v_1, \dots, v_{r_1}$ (since they are in finite number) we are in the situation in which there are no more vectors $Q_i^\intercal$ with $i = r_1 + 1, \dots, r$ having the first r_1 entries not all vanishing. At this step the charge matrix looks like

$$Q(u^*) = \begin{pmatrix} \overbrace{\begin{matrix} 1 & \tilde{*} & \tilde{*} & \dots & \tilde{*} \\ 0 & \pm 1 & \tilde{*} & \dots & \tilde{*} \\ 0 & 0 & \pm 1 & \dots & \tilde{*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \pm 1 \end{matrix}}^{r_1} & \overbrace{\begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}}^{r - r_1} \\ \begin{matrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} \tilde{*} & \dots & \tilde{*} \\ \vdots & \ddots & \vdots \\ \tilde{*} & \dots & \tilde{*} \end{matrix} \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\} r_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} r - r_1 \end{matrix} \quad . \tag{C.6}$$

Every $\tilde{*}$ represent a value that can be either 0 or ± 1 so that every column is a charge vector like Q_{F} or Q_{A}.

`Steps from` $r_1 + 1$ `to` r_2`:` Repeat `Steps` above on the right-bottom block with $f - 1$ vectors $Q_i^\intercal$ of type Q_{F}.

`Steps from` $r_2 + 1$ `to` r_f`:` Repeat `Steps` above until there are no more vectors in the right-bottom block:

$$\sum_{q=1}^{f} r_q = r \ . \tag{C.7}$$

`Coda:` At the end of this procedure the charge matrix is block diagonal

$$Q(u^*) = \text{diag}(Q_1(u^*), \dots, Q_f(u^*)) \ , \qquad \text{with } Q_q(u^*) = \begin{pmatrix} 1 & \tilde{*} & \tilde{*} & \dots & \tilde{*} \\ 0 & \pm 1 & \tilde{*} & \dots & \tilde{*} \\ 0 & 0 & \pm 1 & \dots & \tilde{*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \pm 1 \end{pmatrix} \ , \tag{C.8}$$

for $q = 1, \dots, f$. Until here we did not use the condition $\varsigma_j > 0$ as in (C.5). Since we have proven that $Q(u^*)$ is block diagonal we can impose block by block the condition of positivity of ς's:

$$Q_q(u^*)\varsigma_q = \eta_q \ , \qquad q = 1, \dots, f \ , \qquad \text{with } \varsigma_q = \begin{pmatrix} \varsigma_{q,1} \\ \vdots \\ \varsigma_{q,r_q} \end{pmatrix} \ , \tag{C.9}$$

where ς_q is the part of ς corresponding to the q^{th} block. The same is for η_q. Comparing eq. (C.9) with eq. (C.8) we see that the solution for positive ς_{q,k_q} is

$$\varsigma_{q,k_q} = 1 \ , \qquad Q_q(u^*) = \begin{pmatrix} 1 & \tilde{*} & \tilde{*} & \dots & * \\ 0 & \pm 1 & \tilde{*} & \dots & * \\ 0 & 0 & \pm 1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & +1 \end{pmatrix} \ , \tag{C.10}$$

that is, we have restricted the values of the last columns of $Q(u^*)$: the values of $*$ can be just either 0 or -1. We can go ahead with this procedure: in order to do so we introduce the following notation: $Q_q^{(i)}(u^*)$ indicates the matrix $Q_q(u^*)$ with the last i rows and i columns removed; while $v^{(i)}$ denotes the vector v with the last i entries removed. From eq. (C.9) follows

$$Q_q^{(1)}(u^*)\varsigma_q^{(1)} = \eta_q^{(1)} - \varsigma_{q,k_q}\mathsf{q}_{q,k_q}^{(1)} \ , \tag{C.11}$$

where we introduced $\mathsf{q}_{q,i}$ as the i^{th} column vector of $Q_q(u^*)$. We see that on the r.h.s. we have a vector which is made of all 1 except an entry, which is 2. From this fact, we can infer as above that

$$\varsigma_{q,r_q-1} \geq 1 \ , \qquad Q_q^{(i)}(u^*) = \begin{pmatrix} 1 & \tilde{*} & \tilde{*} & \dots & * \\ 0 & \pm 1 & \tilde{*} & \dots & * \\ 0 & 0 & \pm 1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & +1 \end{pmatrix} \ . \tag{C.12}$$

The argument above can be easily iterated:

$$Q_q^{(i)}(u^*)\varsigma_q^{(i)} = \eta_q^{(i)} - \sum_{j=0}^{i-1} \varsigma_{q,r_q-j}\mathsf{q}_{q,r_q-j}^{(i)} \ , \tag{C.13}$$

at every step we discover that $\varsigma_{q,r_q-j} \geq \varsigma_{q,r_q-j+1}$. Therefore we have that

$$Q(u^*) = \mathrm{diag}(Q_1(u^*), \dots, Q_f(u^*)) \ , \qquad \text{with } Q_q(u^*) = \begin{pmatrix} 1 & -1 & * & \dots & * \\ 0 & +1 & * & \dots & * \\ 0 & 0 & +1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & +1 \end{pmatrix} \ , \tag{C.14}$$

for $q = 1, \dots, f$, and

$$r_q = \varsigma_{q,r_q} \leq \varsigma_{q,r_q-1} \leq \dots \leq \varsigma_{q,2} \leq \varsigma_{q,1} = 1 \ . \tag{C.15}$$

The fact that $\varsigma_{q,k_q} = k_q$ can be argued summing all the rows in eq. (C.9) and plugging the result (C.14).

With this new information we can write eq. (C.4) block by block

$$Q_q^{\mathsf{T}}(u^*)\vec{u}_q = \vec{d}_q \,, \qquad \text{with}^1 \vec{u}_q = \begin{pmatrix} u_{q,1} \\ \vdots \\ u_{q,k_q} \end{pmatrix} \,, \text{ and } \vec{d}_q = \begin{pmatrix} d_{q,1} \\ \vdots \\ d_{q,k_q} \end{pmatrix} \,. \tag{C.16}$$

We conclude this section with a comment on the disungularization procedure 3.3 in presence of a charge matrix satisfying (C.4) and (C.5). Of all the $\binom{A(u^*)}{r}$ regular poles into which the singular pole have been "exploded", only $\prod_{i=1}^r A_i(u^*)$ respect the JK condition. They are the ones $u^*_{(t,p)}$ with all different t_i's. In fact if $t_i = t_j$ the charge matrix will look like

$$\begin{pmatrix} & * & \cdots & * & \\ & \vdots & \ddots & \vdots & \\ & * & \cdots & * & \\ \cdots & 1 & \cdots & 1 & \cdots \\ & 0 & \cdots & 0 & \\ & \vdots & \ddots & \vdots & \\ & \underbrace{0}_{i} & \cdots & \underbrace{0}_{j} & \end{pmatrix} \,. \tag{C.18}$$

It is clear that such a matrix cannot be but in the form (C.14): none of our to moves can separate the 1's that are in the same row.

C.2 Plethystic Exponential

Let us define the plethystic exponential, following [153, 154]. Given a function $f(x_1, \dots, x_n)$ of n variables, such that it vanishes at the origin, $f(0, \dots, 0) = 0$, we set

$$\mathtt{PE}_{x_1,\dots,x_n}\left[f(x_1,\dots,x_n)\right] \equiv \exp\left\{\sum_{r=1}^{\infty} \frac{f(x_1^r,\dots,x_n^r)}{r}\right\} \,. \tag{C.19}$$

If f is C^ω with expansion

$$f(x_1,\dots,x_n) = \sum_{m_1,\dots,m_n=1}^{\infty} f_{m_1,\dots,m_n}\, x_1^{m_1} \cdots x_n^{m_n} \,, \tag{C.20}$$

then (C.19) can be rewritten as

$$\mathtt{PE}_{x_1,\dots,x_n}\left[f(x_1,\dots,x_n)\right] = \prod_{m_1,\dots,m_n}^{\infty} \left(1 - x_1^{m_1} \cdots x_n^{m_n}\right)^{-f_{m_1,\dots,m_n}} \,. \tag{C.21}$$

[1]We are relabeling components of u and d:

$$u_{q,i} = u_{i+\sum_{s=1}^{q-1} r_s} \,, \qquad d_{q,i} = d_{i+\sum_{s=1}^{q-1} r_s} \,. \tag{C.17}$$

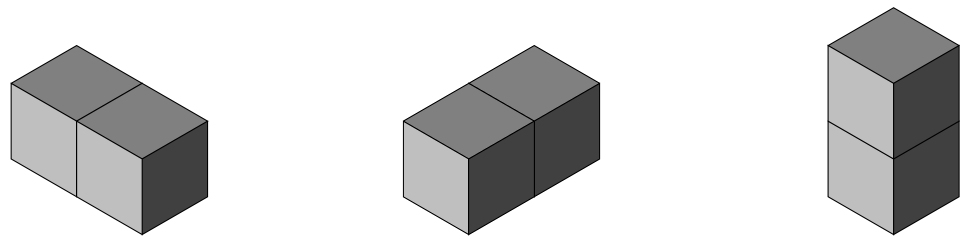
C.3 Plane Partition

C.3.1 Definition

A set of boxes arranged in a 3d lattice labeled by (l, m, n) is called plane partition iff:

- each box sits at a different lattice point;
- only points with $l, m, n \geq 1$ can be occupied;
- the point (l, m, n) can be occupied only if all point (l', m, n) with $1 \leq l' < l$, all point (l, m', n) with $1 \leq m' < m$ and all points (l, m, n') with $1 \leq n' < n$ are also occupied.

In fact, these are 3d version of Young diagram, and can be built in terms of these, as we do below.



(a) (b) (c)

Figure C.1: Plane partitions of the case $k = 2$.

A list of integers $\pi^{(1)} = \{a_1, \dots, a_k\}$ such that $a_i \geq a_{i+1}$ and whose sum is a given integer k, is called a *partition* of k. We define $|\pi^{(1)}| = k$. Partitions of k are in one-to-one correspondence with Young diagrams with k boxes. We call ϕ_k the number of partitions of k, and their generating function is

$$\phi(v) \equiv \sum_{k=0}^{\infty} \phi_k \, v^k = \prod_{k=1}^{\infty} \frac{1}{1-v^k} = \mathtt{PE}_v \left[\frac{v}{1-v} \right] . \tag{C.22}$$

We can introduce a partial order relation $\succeq$ among partitions: we say that $\pi_1^{(1)} \succeq \pi_2^{(1)}$ if the Young diagram representing $\pi_1^{(1)}$ "covers" the one representing $\pi_2^{(1)}$. We can then iterate the process. We define a *plane partition* of k as a collection of Young diagrams

$$\pi^{(2)} = \left\{ \pi_1^{(1)}, \dots, \pi_\ell^{(1)} \right\} \qquad \text{such that } \pi_i^{(1)} \succeq \pi_{i+1}^{(1)} \qquad \text{and } |\pi^{(2)}| \equiv \sum_{r=1}^{\ell} |\pi_r^{(1)}| = k \, . \tag{C.23}$$

We can imagine $\pi^{(2)}$ as a pile of ℓ Young diagrams placed one on top of the other. We call Φ_k the number of plane partitions of k. Their generating function Φ was found by MacMahon to be

$$\Phi(v) \equiv \sum_{k=0}^{\infty} \Phi_k \, v^k = \prod_{k=1}^{\infty} \frac{1}{(1-v^k)^k} = \mathtt{PE}_v \left[\frac{v}{(1-v)^2} \right] . \tag{C.24}$$

In this paper we denote a plane partition simply by π without any superscript.

A *colored plane partition* is a collection of N plane partitions. The generating function of the numbers $\Phi_k^{(N)}$ of colored plane partitions of k is simply the N-th power of the generating function of uncolored plane partitions:

$$\sum_{k=0}^{\infty} \Phi_k^{(N)} v^k = \Phi(v)^N . \tag{C.25}$$

For instance:

$$\Phi_0^{(N)} = 1 , \quad \Phi_1^{(1)} = N , \quad \Phi_2^{(N)} = 3N + \binom{N}{2} , \quad \Phi_3^{(N)} = 6N + 6\binom{N}{2} + \binom{N}{3} . \tag{C.26}$$

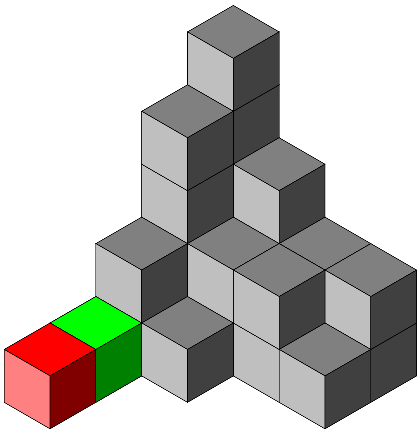
C.3.2 Construction

In this subsection we prove that the only arrangements of boxes U_k corresponding to poles (through eq. (5.17)) which give a non-vanishing residue for (5.8), are plane partitions. We start assigning to every factor in the integrand (5.8) an *order of singularity* according tab. C.1.

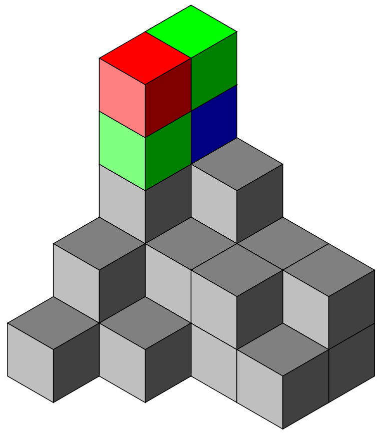
Factor		Hyperplane	Order of singularity
$\theta_1(\tau\|u_i)$	H_Q:	$u_i = 0$	$+1$
$\theta_1(\tau\|u_{ij}+\epsilon_a)$	$H_B^{(a)}$:	$u_i = u_j - \epsilon_a$	$+1$
$\theta_1(\tau\|u_{ij}-\epsilon)$	$H_{\mathbf{V}}$:	$u_i = u_j + \epsilon$	$+1$
$\theta_1(\tau\|u_i-\epsilon)$	Z_Q:	$u_i = \epsilon$	-1
$\theta_1(\tau\|u_{ij})$	$Z_{\mathbf{V}}$:	$u_i = u_j$	-1
$\theta_1(\tau\|u_{ij}-\epsilon_{ab})$	$Z_B^{(ab)}$:	$u_i = u_j + \epsilon_{ab}$	-1

Table C.1: Contributions to the order of singularity from the integrand in (5.8).

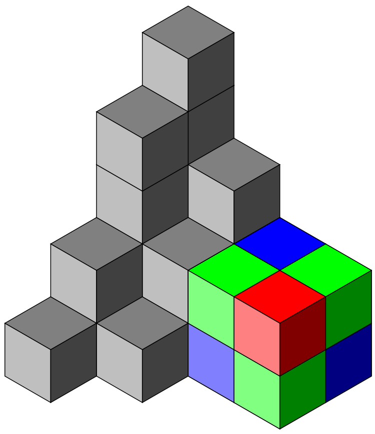
Each singular hyperplane through the point contributes $+1$ to the singularity order, while each vanishing hyperplane through the point, coming from a zero of a function θ_1 in the numerator, contributes -1. Fixed k, the total order of singularity at u^* is denoted by $\mathcal{S}_k(u^*)$. A necessary condition such that a singular u^* has a non-vanishing JK residue is that $\mathcal{S}_k(u^*) \geq k$. This is trivial if we are in the regular case, since the JK residue is simply the iterated residue, otherwise it follows from the desingularization procedure 3.3. Defining, in a general situation $\mathcal{S}_k(u^*) = C(u^*) - A(u^*)$, here we will prove that the non-vanishing contributions to the JK residue come from pole u^* satisfying $\mathcal{S}_k(u^*) = k$, and that the U_k corresponding to these u^* are plane partitions. Notice that in this case we can compute residue easily thanks to eq. (3.76). We proceed in the proof by induction on k. The case $k = 1$ is trivial: the only pole we have is at $u^* = 0$ and the only box representing it is $U_{(1,1,1)}$; clearly, it is a plane partition and, according to the definition, it is the only plane partition we can form with just one box; in addition we have $C(u^*) = 1$ and $A(u^*) = 0$. Then we suppose that we have already built a plane partition of order k, $\mathsf{U}_k \equiv \{U_{(l,m,n)}\}$ and see what happens when we "add a box", $U_{(l',m',n')}$ so that we have the new arrangement $\mathsf{U}'_{k+1} = \mathsf{U}_k \cup U_{(l',m',n')}$. "Adding a box" means, at the level of integral (5.8), that we are spotting the poles of the integrand of $\mathcal{I}_{k+1,1}$ once we have already classified the poles of the integrand of $\mathcal{I}_{k,1}$. Our claim is that $\mathcal{S}_{k+1}(u'^*) = \mathcal{S}_k(u^*) + 1$ if U'_{k+1} is again a plane partition while, if the new arrangement is not a plane partition, its residue is trivially zero. Once this claim is proved we have the correspondence stated above by induction on k.



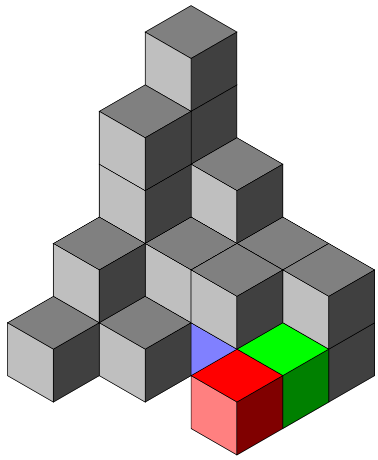
(a) Adding a box along an *edge.*



(b) Adding a box to a *face.*



(c) Adding a box to the *bulk.*



(d) Adding a box such that the new arrangement is not a plane partition.

Figure C.2: Several ways to add the $(k+1)^{\text{th}}$ box (the red one) given an arrangement of k boxes. At the same time we add an integral over u_{k+1}. We colored in *green* those boxes whose position differs, from that of the red one, by ϵ_a; in *blue* those boxes whose position differs by ϵ_{ab}. From Table C.1 we see that a green box increases the singularity order of the integrand by 1, while a blue box decreases it by 1. In case (a) we increase the order by 1, therefore the pole contributes. In case (b) we increase the order by $2-1=1$, therefore the pole contributes. In case (c) we increase the order by $3+1-3=1$, therefore the pole contributes. In case (d) there is no change in the order of the singularity, therefore the pole does not contribute.

Let us prove the claim. We distinguish two main cases to organize the proof. Consider the case in which $U_{(l',m',n')} \notin \mathsf{U}_k$, which in terms of boxes means that $U_{(l',m',n')}$, the new box, does not coincide with another box in U_k. In order to increase the singularity, we see from Tab. C.1 there are four possibilities: either $(l',m',n') = (a+1,b,c)$ or $(l',m',n') = (a,b+1,c)$, or $(l',m',n') = (a,b,c+1)$ or $(l',m',n') = (a-1,b-1,c-1)$, where $U_{(a,b,c)} \in \mathsf{U}_k$. We treat the first three possibilities together as a first case and the last possibility as a second case. Let us now introduce some useful terminology and notation: for practical reason it is convenient to denote $l'_1 \equiv l'$, $l'_2 \equiv m'$ and $l'_3 \equiv n'$, moreover we define[2] $\vec{e}_i$ $(i = 1, 2, 3)$ directions, as the direction along which the plane partition increases, corresponding to ϵ_i. We will call the "direction (and orientation) of a face" of the boxes, the direction (and orientation) of the unit vector normal to this face, pointing outward the box. Thus, every box in the plane partition has three external faces (EFs), which are the ones whose orientation is aligned[3] with one of the $\vec{e}_i$, and three internal faces (IFs), which are the ones whose orientation is anti-aligned[3] with one of the $\vec{e}_i$. We will say that a face is free if it is not in common with any other boxes (there is no boxes attached there). Let's start the proof in the first case. The box $U_{(l',m',n')}$ can have either 0, 1, 2 o 3 free IFs:

- If there are 3 free IFs this mean that the box sits in the origin and we have already considered that case $k = 1$;
- If there are 2 free IFs, let us suppose[4] that they have direction $-\vec{e}_1$ and $-\vec{e}_2$ while the face which is not free have direction $-\vec{e}_3$. Since, by inductive hypothesis, we have the box $U_{(l_1,l_2,l_3-1)}$ in the plane partition, there is one poles arising from a singular hyperplane of type[5] $H_B^{(3)}$. Then we can make the following distinction:
  - if $l'_1 = l'_2 = 1$ the new arrangement is by definition a plane partition. There are neither source of zeroes nor other sources of poles. So $\Delta\mathcal{S} := \mathcal{S}_{k+1}(u'^*) - \mathcal{S}_k(u^*) = 1$;
  - if $l'_1 = 1$ but $l'_2 \neq 1$ we do not have a plane partition. In this case there is a zero from $Z_B^{(23)}$ since the box $U_{(l',m'-1,n'-1)}$ is present. There are no other source of poles. We have therefore $\Delta\mathcal{S} \leq 0$;
  - if $l'_1 \neq 1$ and $l'_2 \neq 1$ the new arrangement is not a plane partition. In this cases the following boxes are present: $U_{(l',m'-1,n'-1)}$, $U_{(l'-1,m',n'-1)}$ from which we get two zeroes ($Z_B^{(23)}$ and $Z_B^{(13)}$) and $U_{(l'-1,m'-1,n'-1)}$ from which we get a pole thanks to H_V. There are not any other source of poles. So we have $\Delta\mathcal{S} \leq 0$.
- If there is 1 free IF, let us suppose that it has direction $-\vec{e}_1$ and that the direction of non-free IF are $-\vec{e}_2$ and $-\vec{e}_3$. Then we have the following boxes: $U_{(l',m'-1,n')}$ and $U_{(l',m',n'-1)}$, which give us two poles (from $H_B^{(2)}$ and $H_B^{(3)}$) and $U_{(l',m'-1,n'-1)}$ which gives a zero (from $Z_B^{(23)}$). Then we can distinguish the following subcases:
  - if $l'_1 = 1$ the new arrangement is a plane partition. There are neither sources of poles nor sources of zeroes; then $\Delta\mathcal{S} = 1$;
  - if $l'_1 \neq 1$ we have several boxes to consider: from $U_{(l'-1,m'-1,n'-1)}$ we have a pole (from $H_{\mathbf{V}}$), while from $U_{(l'-1,m'-1,n-)}$, $U_{(l'-1,m',n'-1)}$ and $U_{(l',m'-1,n'-1)}$ we have zeroes (from $H_B^{(12)}$, $H_B^{(13)}$ and $H_B^{(23)}$). There are no more source of poles. Then $\Delta\mathcal{S} \leq 0$.
- If there are not free IFs, this means that we have several boxes: $U_{(l'-1,m',n')}$, $U_{(l',m'-1,n')}$, $U_{(l',m',n'-1)}$ from which we get three poles (from $H_B^{(1)}$, $H_B^{(2)}$ and $H_B^{(3)}$), another pole from

[2] Explicitly $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$ and $\vec{e}_3 = (0, 0, 1)$.

[3] For aligned we mean same direction and same orientation while for antialigned we mean same direction but different orientation.

[4] The other cases are easily obtained by permuting 1, 2 and 3.

[5] We recall that the name of singular and zero hyperplane are listed in Tab. C.1.

$U_{(l'-1,m'-1,n'-1)}$ (from $H_{\mathbf{V}}$), while from $U_{(l'-1,m'-1,n)}$, $U_{(l'-1,m',n'-1)}$ and $U_{(l',m'-1,n'-1)}$ we have zeroes (from $H_B^{(12)}$, $H_B^{(13)}$ and $H_B^{(23)}$). Then $\Delta\mathscr{S} = 1$.

We have now to consider the second case in which $(l', m', n') = (a-1, b-1, c-1)$ for some $U_{(a,b,c)} \in \mathsf{U}_k$. Since we want $U_{(l',m',n')} \notin \mathsf{U}_k$, at least one among a or b or c must be equal to 1. The hyperplane $H_{\mathbf{V}}$ provide us a pole, then:

- if $l'_1 = l'_2 = l'_3 = 1$, there is a zero from Z_Q, so $\Delta\mathscr{S} = 0$;
- if, suppose, $l_1 \neq 1$ then we have the box $U_{(l'-1,m',n')}$ that gives a zero by $Z_B^{(23)}$. So $\Delta\mathscr{S} = 0$.

This exhausts the way one can add $U_{(l',m',n')} \notin \mathsf{U}_k$ to U_k. Until now we proved that if U_{k+1} is a plane partition $\Delta\mathscr{S} = 1$ and so the residue computed in this case is not zero. In fig. C.2 we depict some of the situations mentioned in the text.

We have finally to examine what happens if we add a box $U'_{(l',m',n')}$ which coincides with another box $U_{(l',m',n')}$ of U_k.

Using the notation of the section 3.3 [6], if one takes some ${u'^*}_{i'} = {u'^*}_i$, the ordering (3.70) will be of the form

$$I(\vec{u}) = I_1(u_1)\cdot \ldots \cdot I_i(u_1, \ldots, u_i)\times \\ \times I_{i'}(u_1, \ldots, u_i, u_{i'}) I_{i+1}(u_1, \ldots, u_i, u_{i'}, u_{i+1})\cdot \ldots \cdot I_k(u_1, \ldots, u_k)\ . \quad \text{(C.27)}$$

Now we can desingularize $I(\vec{u})$ and get $\tilde{I}(\vec{u})$. Now let us examine the following product[7]

$$\tilde{I}_1(u_1)\cdot \ldots \cdot \tilde{I}_i(u_1, \ldots, u_i)\tilde{I}_{i'}(u_1, \ldots, u_i, u_{i'})\ , \quad \text{(C.28)}$$

we will have that $A_j(u^*) = C_j(u^*) + 1$ for $j = 1, \ldots, i$ and also for $j = i'$. Then, from the integrand (5.8) we have that $\tilde{I}_{i'}$ contains a term which is $\theta_1^2(\tau|u_i - u_{i'})$, and therefore vanishes when one take the residue w.r.t. the "unshifted pole" $u_i = u_{i'} = u_i^* = u_{i'}^*$. From this we conclude that an arrangement of boxes in which two of them occupy the same place do not give contribution.

This proves that, for $N = 1$, the number of the fundamentals charge vector in $Q(u^*)$ can just be $f = 1$ and so there is only one block.

[6] It is always possible to order boxes $U_{(l',m',n')}$ first by l' then by m' and lastly by n'. In this way one finds the corresponding u_i^*.

[7] We just need to do this because of the last comment of sec. C.1.

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