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DIFFERENT FORMULATIONS OF GRAVITY AND THEIR EQUIVALENCE AT CLASSICAL AND QUANTUM LEVEL

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A mis abuelos.

Abstract

General Relativity is usually formulated as a theory with gauge invariance under the diffeomorphism group, but there is a "dilaton" formulation where it is in addition invariant under Weyl transformations, and a "unimodular" formulation where it is only invariant under the smaller group of special diffeomorphisms. Other formulations with the same number of gauge generators, but a different gauge algebra, also exist. These different formulations provide examples of what we call "inessential gauge invariance", "symmetry trading" and "linking theories"; they are locally equivalent, but may differ when global properties of the solutions are considered. We discuss these notions in the Lagrangian and Hamiltonian formalism. The discussion is then extended to the quantum level. By making suitable choices of parametrization and gauge we show that the alternative formulations are equivalent to quantum EG, in the sense that the effective actions are the same. In particular, in the dilaton formulation Weyl invariance can be maintained also in the quantum theory.

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Contents

Chapter 1

Introduction

"I carried it around with me for days and days.
Playing little games like not looking at it for a whole day.
And then... looking at it. To see if I still liked it. I did."
King Crimson, Indiscipline.

At low energies with respect to the Plank scale, General relativity or Einstein Gravity (EG) is the well tested description of spacetime. It is based on the (vacuum) Einstein's equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 0 \tag{1.0.1}$$

where $g_{\mu\nu}$, the metric tensor, encodes the gravitational dynamics and $\mathcal{R}_{\mu\nu}$, $\mathcal{R}$ are the Ricci tensor and scalar respectively. In this framework gravity is just a manifestation of the (pseudo-Riemannian) geometry of spacetime. The equation can be derived from the action defined in d-spacetime dimensions by

$$S[g_{\mu\nu}] = Z_N \int \mathrm{d}^d x \sqrt{-g}\,\mathcal{R}(g) \tag{1.0.2}$$

Dimensional analysis tell us that the energy dimensions of Z_N must be $d-2$ and for $d = 4$ is given by

$$Z_N = \frac{c^4}{16\pi G_N} \tag{1.0.3}$$

where G_N is the Newton's constant and c the speed of light. The action 1.0.2 is diffeomorphism invariant (reparametrization invariance or general covariance) and local. As it is shown in [1], $\mathcal{R}$ is the only invariant that contains derivatives with respect to the metric up to second order and second order only linearly. The equation is obtained from Hamilton's principle and in order to have a consistent variational principle a boundary term should be added to the action [2], [3] or assume that the manifold is closed (for a review see [4]). Boundary terms are a subtle issue since they encode physical information [2], [5], [6] and they are fundamental in the context of the gauge/gravity duality (see [7] for a short introduction).

The Lagrangian formulation of EG allow us to describe the same physical phenomena as a spin-2 field $h_{\mu\nu}$, called the graviton, its action is obtain by substituting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ corresponds to the metric of flat spacetime. In [8], [9] is discussed that the action 1.0.2 can be derived from Lorentz and gauge invariance (local reparametrization invariance). This formulation highlights the fact that the classical field theory of the graviton enjoys a geometrical interpretation.

The Hamiltonian formulation of EG was started by Dirac [10] and culminated by Arnowitt, Deser and Misner (ADM) [11]. In this approach general covariance is broken since spacetime is viewed as being foliated by space-like hypersurfaces of codimension one. It can be restored considering all possible foliations and as shown by Dirac [12], [13], [14], due to reparametrization invariance, the Hamiltonian is constrained to (weakly) vanish. The Hamiltonian of EG is given by

$$H = H_{\perp}[N] + H_{||}[\mathbf{N}] + C_1[\zeta] + C_2[\vec{\zeta}] \tag{1.0.4}$$

with

$$\begin{aligned} C_1[\zeta] &= \int \mathrm{d}^{d-1}x \, \zeta P_N^0 && (1.0.5) \\ C_2[\vec{\zeta}] &= \int \mathrm{d}^{d-1}x \, \delta^{ij}\zeta_i (P_N)_j && (1.0.6) \\ H_{\perp}[N] &= \int \mathrm{d}^{d-1}x \, N \left[\frac{1}{Z_N\sqrt{q}} \left(P_{qij}P_q^{ij} - \frac{P_q^2}{(d-2)} \right) - Z_N\sqrt{q}R(q) \right] && (1.0.7) \\ H_{||}[\mathbf{N}] &= \int \mathrm{d}^{d-1}x \, P_q^{ij} \pounds_{\mathbf{N}} q_{ij} && (1.0.8) \end{aligned}$$

C_1 and C_2 are called primary (smeared) constraints. Demanding that they should be preserved during evolution (consistency condition) one obtain the secondary (smeared) constraints $H_{\perp}[N], H_{||}[\mathbf{N}]$. They are called Hamiltonian and momentum constraint respectively. The momentum constraint corresponds simply to the generator of spatial diffeomorphisms and the Hamiltonian constraint to the generator of dynamics. The Hamiltonian and momentum constraint satisfy the following algebra

$$\begin{aligned} \{H_{\perp}[N], H_{\perp}[M]\} &= H_{||}[q^{ij}\,(N\partial_j M - M\partial_j N)\} && (1.0.9) \\ \{H_{||}[\mathbf{N}], H_{\perp}[M]\} &= H_{\perp}[\pounds_{\mathbf{N}} M] && (1.0.10) \\ \{H_{||}[\mathbf{N}], H_{||}[\mathbf{M}]\} &= H_{||}[\pounds_{\mathbf{N}}\mathbf{M}] && (1.0.11) \end{aligned}$$

where $\{,\}$ is the Poisson bracket. The constraints form a first class system, i.e. the Poisson bracket of each constraint with every other constraint (weakly) vanishes. Following [15], the above algebra can also be obtained from the so called path independence principle. It provides the embeddability conditions in order to ensure that the evolution of a hypersurface can be thought of as the deformation of a hypersurface cut in spacetime (ambient space). This alternative description emphasizes the geometrical nature of the algebra.

Quantization

The method of quantization is a procedure of promoting a classical theory to a quantum theory. It is common to use two schemes: Path Integral Quantization and Canonical Quantization. Independently of the scheme, the procedure relies on the structure of the classical theory. Canonical quantization is based entirely on the Hamiltonian formalism. Following Dirac, the quantization for a non-relativistic particle in a line is achieved by

$$\{f(x,p), g(x,p)\} \to \frac{1}{i\hbar}[\hat{f}(\hat{x},\hat{p}), \hat{g}(\hat{x},\hat{p})] \tag{1.0.12}$$

where f and g are functions of the phase space coordinates x, p. The coordinates are promoted to hermitian operators $\hat{x}, \hat{p} = \frac{\hbar}{i}\frac{\mathrm{d}}{\mathrm{d}x}$. We think of quantization as a map Q, called quantization map, such that

$$\begin{aligned} f &\mapsto \hat{f} = Q(f) && (1.0.13)\\ Q(\{f,g\}) &= \frac{1}{i\hbar}[Q(f), Q(g)] && (1.0.14)\end{aligned}$$

The construction of this map is limited by Groenewold's theorem [16]; which deals with ordering ambiguities of the operators. For example: $xp - px = 0$ is valid classically but not quantum mechanically[1]. In the Schrödinger picture the wave function of the system $\psi(t,x)$ (defined in the configuration space) must satisfy the Schrödinger equation

$$\hat{H}(\hat{x},\hat{p})\psi(t,x) = i\hbar\frac{\partial\psi(t,x)}{\partial t} \tag{1.0.15}$$

with $\hat{H}(\hat{x},\hat{p})$ as the Hamiltonian operator. For a field theory the procedure is the same, we promote fields to operators. The main difficulty is that we must consider gauge invariant field theories which will correspond to the quantization of constraint systems (see [14], [19]). Following [20] and [21] for the quantization of EG in the Schrödinger picture, the configuration space is defined by $S(\Sigma) = \mathrm{Riem}\Sigma/\mathit{Diff}\,\Sigma$, known as the superspace. $\mathrm{Riem}\Sigma$ corresponds to the set of all possible metrics on the hypersurface Σ and $\mathit{Diff}\,\Sigma$ the group of space diffeomorphisms. In the canonical formalism each

[1] Weyl quantization tackle this issue since for a general monomial, it averages all the possible orderings of the position and momentum operators. See [17] and [18].

classical constraint is promoted to a restriction to the wave function(al). Therefore the physical wave functional $\Psi[q_{ij}]$ must satisfy

$$\begin{aligned}
\hat{C}_1[\zeta]\Psi[q_{ij}] &= 0 && (1.0.16)\\
\hat{C}_1[\vec{\zeta}]\Psi[q_{ij}] &= 0 && (1.0.17)\\
\hat{H}_\perp[N]\Psi[q_{ij}] &= 0 && (1.0.18)\\
\hat{H}_\parallel[\mathbf{N}]\Psi[q_{ij}] &= 0 && (1.0.19)
\end{aligned}$$

with

$$\begin{aligned}
\hat{P}^0_N &= \frac{\hbar}{i}\frac{\delta}{\delta N} && (1.0.20)\\
\hat{P}^i_N &= \frac{\hbar}{i}\frac{\delta}{\delta N_i} && (1.0.21)\\
\hat{P}^{ij}_q &= \frac{\hbar}{i}\frac{\delta}{\delta q_{ij}} && (1.0.22)
\end{aligned}$$

Equations 1.0.16 and 1.0.17 indicates that the functional only depends on the metric. Following [22], 1.0.19 is just the statement that Ψ is invariant under spatial diffeomorphism. Hence it only depends on the geometry of the hypersurface not on specific form of the metric. Finally the equation 1.0.18 is known as the Wheeler-De Witt equation (WDW). It describes the dynamical evolution of the wave functional in superspace. Following [21] a metric, called De Witt metric, is defined in superspace as

$$\begin{aligned}
G^{ijkl} &= \frac{1}{2}\sqrt{\det q}\left[q^{ik}q^{jl} + q^{il}q^{jk} - 2q^{ij}q^{kl}\right] && (1.0.23)\\
G_{ijkl} &= \frac{1}{2}\frac{1}{\sqrt{\det q}}\left[q_{ik}q_{jl} + q_{il}q_{jk} - \frac{2}{d-2}q_{ij}q_{kl}\right] && (1.0.24)\\
G^{ijmn}G_{mnkl} &= \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right) && (1.0.25)
\end{aligned}$$

In $d = 4$ it has a signature $(- + + + + +)$ (which is independent of the signature of the spacetime) and the equation becomes

$$\int \mathrm{d}^3x \left[\frac{1}{Z_N}G_{ijkl}\hat{P}^{ij}_q\hat{P}^{kl}_q - Z_N\sqrt{q}R(q)\right]\Psi[q_{ij}] = 0 \qquad (1.0.26)$$

Since the signature is indefinite, WDW has the structure of a second order hyperbolic equation. There are many issues regarding WDW equation such as operator ordering (a particular choice has being made in 1.0.26), problem of time, probabilistic interpretation, semiclassical limit and boundary conditions (see [23], [24] for a review). Nevertheless some solutions are found for finite dimensional degrees of freedom in superspace, this subspace is called minisuperspace [25]. They corresponds to toy models for quantum cosmology [26] and regardless of their simplicity they are useful to study the conceptual issues previously mentioned.

On the other hand path integral quantization can be done from the Lagrangian and Hamiltonian description; for specific forms of the Hamiltonian it can be developed in terms of a Lagrangian only and the difficulties encounter from gauge invariant field theories are dealt by the Fadeev-Popov procedure. This imply that we can perform the quantization in an explicit covariant way and there is not necessity to work with operators. Path integral quantization has some formal issues regarding the measure, summing over the topologies and convergence, nevertheless in the physical framework it provides a connection between quantum field theory and statistical mechanics (for EG see [27]) and also it allows us to compute non-perturbative effects (such as gravitational instantons [28] [29]).

In the perturbative regime, the one-loop correction to EG path integral has been computed for pure gravity by [30]. Their work is based on the background field method [31] [32], the metric splits as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ corresponds to a general classical background and we integrate out the quantum fluctuation $h_{\mu\nu}$.

$$Z[\bar{g}] = \int \mathcal{D}h_{\mu\nu} \mathrm{e}^{-S_E[\bar{g}_{\mu\nu}+h_{\mu\nu}]} \tag{1.0.27}$$

As required by the Fadeev-Popov procedure we add the gauge fixing and ghost actions, the divergent part of the effective action (without a cosmological constant), defined by $\Gamma = -\log Z$, result

$$\Gamma_{div}[\bar{g}_{\mu\nu}] = -\frac{1}{(4\pi)^2\epsilon}\int \mathrm{d}^4x\sqrt{\bar{g}}\left[\frac{7}{10}\bar{\mathcal{R}}_{\mu\nu}\bar{\mathcal{R}}^{\mu\nu} + \frac{1}{60}\bar{\mathcal{R}}^2 + \frac{53}{45}\bar{E} - \frac{19}{15}\bar{\nabla}^2\bar{\mathcal{R}}\right] \tag{1.0.28}$$

On-shell this term vanishes, therefore pure gravity is renormalizable at one-loop. The divergent part of the effective action with a cosmological constant, computed in [33], is given by

$$\begin{aligned}\Gamma_{div}[\bar{g}_{\mu\nu}] \quad = \quad & -\frac{1}{(4\pi)^2\epsilon}\int \mathrm{d}^4x\sqrt{\bar{g}}\left[\frac{7}{20}\bar{W}_{\mu\nu\rho\sigma}\bar{W}^{\mu\nu\rho\sigma} + \frac{1}{4}\bar{\mathcal{R}}^2 + \frac{149}{180}\bar{E}\right.\\ & \left. -\frac{26}{3}\bar{\mathcal{R}}\Lambda + 20\Lambda^2\right]\end{aligned} \tag{1.0.29}$$

where $W_{\mu\nu\rho\sigma}$ corresponds to the Weyl tensor and E the Euler characteristic. If we rewrite this expression in terms of Ricci tensor by means of

$$\bar{W}_{\mu\nu\rho\sigma}\bar{W}^{\mu\nu\rho\sigma} = \bar{E} + 2\bar{\mathcal{R}}_{\mu\nu}\bar{\mathcal{R}}^{\mu\nu} - \frac{2}{3}\bar{\mathcal{R}}^2 \tag{1.0.30}$$

and set $\Lambda = 0$, we recover the result given in 1.0.28.We can also rewrite 1.0.29 in terms of Riemann tensor by using

$$\bar{W}_{\mu\nu\rho\sigma}\bar{W}^{\mu\nu\rho\sigma} = \bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma} - 2\bar{\mathcal{R}}_{\mu\nu}\bar{\mathcal{R}}^{\mu\nu} + \frac{1}{3}\bar{\mathcal{R}}^2 \qquad (1.0.31)$$

$$\bar{E} = \bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma} - 4\bar{\mathcal{R}}_{\mu\nu}\bar{\mathcal{R}}^{\mu\nu} + \bar{\mathcal{R}}^2 \qquad (1.0.32)$$

and the equations $\bar{\mathcal{R}}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$, $\bar{\mathcal{R}} = 4\Lambda$:

$$\Gamma_{div}[\bar{g}_{\mu\nu}] = -\frac{1}{(4\pi)^2\epsilon}\int \mathrm{d}^4x\sqrt{\bar{g}}\left[\frac{53}{45}\bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma} - \frac{58}{5}\Lambda^2\right] \qquad (1.0.33)$$

This result agrees with [34]. At two-loop the divergence of pure gravity was calculated in [35]. However beyond two loops and for one-loop with matter [30], EG is found to be perturbatively non-renormalizable. Therefore, if we decide to think of gravity as a field theory only, it is natural to adopt an effective field description of quantum gravity [36], [37] or require UV completness and demand non-perturbative renormalizability such as in asymptotic safety program (see [38] for a review).

Purpose and outline

Independently on the classical formalism and quantization approach of the theory, there are procedures to deal with gauge invariant field theories. We learned that gauge invariance corresponds to the art of introducing new (not physical) degrees of freedom in order to work with a manifest symmetry. The action for electrodynamics is local and explicitly Lorentz invariant if written in a gauge invariant way. As discussed above the same situation occur for EG. Once quantum mechanics is added locality and gauge invariance ensure that the theory is manifestly unitary.

This thesis is devoted to the study of theories that have gauge groups that are either larger or smaller then the diffeomorphism group, *Diff* in short, the gauge group of EG.

As an example of a theory with a larger gauge group we analyze a theory that is diffeomorphism and Weyl invariant. This theory was studied by Dirac [39] and it is based on Weyl's geometrical idea that units of length should also co-variate locally [40]. Here we study a general Weyl invariant scalar-tensor gravity (WSTG) and show that Weyl invariance is "fake" or "inessential", in the sense that the scalar field corresponds to a gauge artefact. When the sign of the kinetic term in the Lagrangian is the right one to describe gravity, the scalar is a ghost and we refer to this theory as dilaton or Dirac gravity (DG). This theory amounts to applying the so called "Stuckelberg trick" [41], [42], which was originally developed to achieve a gauge invariant description of a massive gauge theory, to EG. From the field theory side its physical appeal resides in the conjecture that field theories in the UV should be scaleless. In [43] and more recently [44], [45] it is shown that in such theories, where Weyl invariance is achieved by introducing a non-physical scalar field, Weyl invariance can be maintained in the quantum theory.

As an example of a theory with smaller gauge group we studied *SDiff* (special diffeomorphisms) which leaves the determinant of the metric invariant. This theory has the well known property that the cosmological constant, at the level of the equation of motion, corresponds to an integration constant rather than a coupling in the action. The theory with this property is often referred as Unimodular gravity (UG) [46], [47], [48], [49], [50]. Therefore in this work, UG is defined as a theory of gravity invariant under *SDiff*'s.

Since EG is a physical theory that enjoys many equivalent classical descriptions here we also discuss the equivalence at the quantum level. By classical equivalence we mean that the physical degrees of freedom of EG , $d(d-3)/2$, are the same (at least locally) for theories with larger or smaller gauge group and dynamically they obey the same equations of motion. At the quantum level, equivalence corresponds to the statement that the one-loop effective action are the same. Weyl invariance is generally known to be anomalous at the quantum level [51] [52]. However, as shown in [43] [44], [45] the theory can be quantized in a Weyl invariant way. The anomaly is absent or present if Weyl invariance is achieved by introducing a dilaton or not, this is summarized in a footnote in [53] and discuss it with more details in this work. On the other hand, in [54], [55] it is showed that UG and EG are also equivalent at one loop.

Let us introduce the following terminology: When two theories A and B can be obtained from a theory C by fixing the gauge in different ways, we say that C is a "linking theory" for A and B. In particular, this is a way of proving that A and B are physically equivalent. The theories A and B may have different residual gauge groups. In this case we say that there is a "symmmetry trading" between A and B.

EG can be recovered from DG by choosing the Weyl gauge in such a way that the scalar field is set to a constant. On the other hand, there are other gauge choices for DG that yield different formulations of GR. DG thus acts a linking theory for these different formulations of GR, this is sketched in figure 1.1. They are characterized by different gauge groups, which in some cases still contains Weyl transformations.

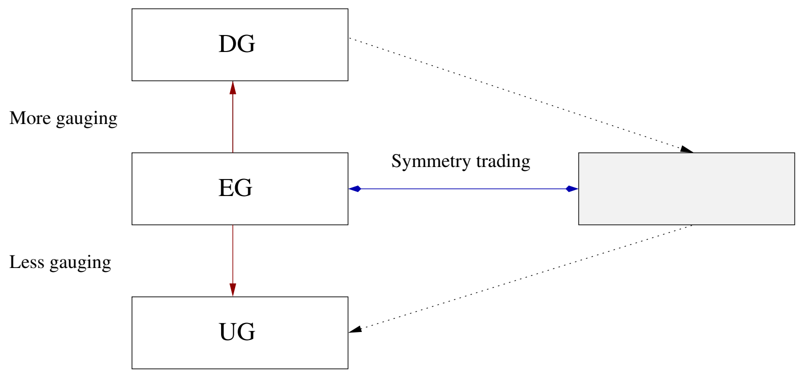

Figure 1.1: DG as a linking theory.

The structure of the thesis is the following: in Chapter 1 and 2 we analyse the classical (Lagrangian and Hamiltonian formalism) and quantum (canonical and path integral quantization) aspects of DG respectively, in Chapter 3 the same aspects for UG. This separation has been made for clarity since the mathematical techniques are different for each theory. Finally in Chapter 4 we study DG as a linking theory and its consequences; the natural question of which description is preferable is discussed. The signature convention of the metric is mostly plus and from now on we set $G_N = \hbar = c = 1$.

This work is based on:

- R. de León Ardón, N. Ohta and R. Percacci, "Path integral of unimodular gravity," Phys. Rev. D **97**, no. 2, 026007 (2018) doi:10.1103/PhysRevD.97.026007 [arXiv:1710.02457 [gr-qc]].

- Gielen et al 2018 Class. Quantum Grav. https://doi.org/10.1088/1361-6382/aadbd1, arXiv:1805.11626 [gr-qc].

Chapter 2

Classical aspects of DG

Consider the following general action of a real scalar field ϕ non-minimally coupled to gravity:

$$S[\phi, g_{\mu\nu}] = \int \mathrm{d}^d x \sqrt{-g} \left[f(\phi)\mathcal{R} + \frac{\lambda(\phi)}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + U(\phi) \right] \tag{2.0.1}$$

The fields are taken to be dimensionless and therefore the energy dimensions of f, λ and U are $d-2, d-2, d$ respectively. The variation of the action gives

$$\delta S = \int \mathrm{d}^d x \sqrt{-g} \left[E^{(g)}_{\mu\nu} \delta g^{\mu\nu} + E^{(\phi)} \delta\phi \right] + \int \mathrm{d}^d x \, \partial_\mu \mathcal{K}^\mu \tag{2.0.2}$$

where

$$\begin{aligned} E^{(g)}_{\mu\nu} &= f(\phi)\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left[f(\phi)\mathcal{R} + \frac{\lambda(\phi)}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + U(\phi) \right] \\ &\quad + \frac{\lambda(\phi)}{2} \nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu} \nabla^2 f(\phi) - \nabla_\mu \nabla_\nu f(\phi) && (2.0.3) \\ E^{(\phi)} &= f'(\phi)\mathcal{R} - \frac{\lambda'(\phi)}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + U'(\phi) - \lambda(\phi) \nabla^2 \phi && (2.0.4) \\ \mathcal{K}^\mu &= \sqrt{-g} \left[f(\phi) g_{\rho\sigma} \nabla^\mu \delta g^{\rho\sigma} - f(\phi) \nabla_\rho \delta g^{\mu\rho} - g_{\rho\sigma} \delta g^{\rho\sigma} \nabla^\mu f(\phi) \right. \\ &\quad \left. + \delta g^{\mu\rho} \nabla^\rho f(\phi) + \lambda(\phi) (\nabla^\mu \phi) \delta\phi \right] && (2.0.5) \end{aligned}$$

For convenience let us drop the boundary term; Hamilton's principle give us the equations of motion: $E^{(g)}_{\mu\nu} = 0$ and $E^{(\phi)}_{\mu\nu} = 0$. Notice that under Weyl transformation defined by

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \quad \phi \to \frac{\phi}{\Omega} \tag{2.0.6}$$

the action 2.0.1 transforms as

$$S' = S - 2(d-1) \int \mathrm{d}^d x \sqrt{-g} \nabla_\mu \left[f(\phi) \Omega^{-1} \nabla^\mu \Omega \right] \tag{2.0.7}$$

if and only if the following conditions are satisfied:

$$\Omega^{d-2} f\left(\frac{\phi}{\Omega}\right) = f(\phi) \tag{2.0.8}$$

$$\Omega^{d-4} \lambda\left(\frac{\phi}{\Omega}\right) = \lambda(\phi) \tag{2.0.9}$$

$$\Omega^{d} U\left(\frac{\phi}{\Omega}\right) = U(\phi) \tag{2.0.10}$$

$$(d-1)(d-2) f(\phi) = \frac{\phi^2 \lambda(\phi)}{2} \tag{2.0.11}$$

$$2(d-1) f'(\phi) = \phi \lambda(\phi) \tag{2.0.12}$$

Combining the last two equations one finds $f(\phi)$ and the final form of the functions are

$$f(\phi) = \pm Z_N \phi^{d-2} \tag{2.0.13}$$

$$\lambda(\phi) = \pm 2 Z_N (d-1)(d-2) \phi^{d-4} \tag{2.0.14}$$

$$U(\phi) = Z_N \alpha \phi^{d} \tag{2.0.15}$$

By dimensional analysis the energy dimensions α is 2, 1.0.2 is recovered for positive solutions of 2.0.13 with $\phi = 1$ and $\alpha = 0$. The theory with an action given by 2.0.1 and the functions restricted to 2.0.13, 2.0.14, 2.0.15 is denoted WSTG. Some remarks are in order:

- For closed manifolds the action is exactly invariant under 2.0.6 otherwise is invariant up to a boundary term.
- For WSTG: $g^{\mu\nu} E^{(g)}_{\mu\nu} = \frac{\phi}{2} E^{(\phi)}$.
- One can see that for DG, i.e. positive solution of 2.0.13, the kinetic term for the field has the wrong sign. Therefore ϕ corresponds to a classical ghost field [56]. In the literature it is also referred as "dilaton", "spurion", "Stueckelberg field" or "Weyl compensator". The Hamiltonian analysis of the theory indicates that this does not lead to any instability since it corresponds to a gauge degree of freedom. For 2.0.13 negative we have physical scalar field conformally coupled to gravity.
- The coupling constant can be absorbed by a redefinition of the field

$$\psi^2 = 8\left(\frac{d-1}{d-2}\right) Z_N \phi^{d-2} \tag{2.0.16}$$

This field has dimension $(d-2)/2$. The action becomes

$$S[\psi, g_{\mu\nu}] = \int \mathrm{d}^d x \sqrt{-g} \left[\pm \frac{1}{8}\left(\frac{d-2}{d-1}\right) \psi^2 \mathcal{R} \pm \frac{1}{2} g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi + \tilde{\alpha} \psi^{\frac{2d}{d-2}}\right] \tag{2.0.17}$$

with $\tilde{\alpha} = \left[\frac{1}{8}\left(\frac{d-2}{d-1}\right)\right]^{\frac{d}{d-2}} Z_N^{-\frac{2}{d-2}}\alpha$ as a dimensionless parameter.

2.1 Noether current for Weyl transformations

Before discussing the result for WSTG we review the Noether procedure: consider a Lagrangian for a scalar field φ of the form $\mathcal{L}(\varphi, \partial\varphi)$ and the infinitesimal transformation $\varphi \to \varphi + \delta\varphi$. This transformation is called a *symmetry* if and only if the corresponding variation of the Lagrangian is given by

$$\delta\mathcal{L} = \partial_\mu \mathcal{F}^\mu, \quad \mathcal{F}^\mu = \mathcal{F}^\mu(\varphi) \tag{2.1.1}$$

On the other hand for any transformation of a Lagrangian we have

$$\delta\mathcal{L} = E^{(\varphi)}\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right) \tag{2.1.2}$$

where $E^{(\varphi)} = 0$ gives the equation of motion. Equating previous variations one finds

$$E^{(\varphi)}\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi - \mathcal{F}^\mu\right) = 0 \tag{2.1.3}$$

The Noether current is defined by

$$\mathcal{J}^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi - \mathcal{F}^\mu \tag{2.1.4}$$

and we can see that it is conserved on-shell. For a Lagrangian of the type $\mathcal{L}(\phi, \partial\phi, g_{\mu\nu}, \partial g_{\mu\nu}, \partial\partial g_{\mu\nu})$, the Noether current has the form

$$\mathcal{J}^\mu = \mathcal{K}^\mu - \mathcal{F}^\mu \tag{2.1.5}$$

with

$$\mathcal{K}^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu g_{\rho\sigma}}\delta g_{\rho\sigma} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu g_{\rho\sigma}}\partial_\nu\delta g_{\rho\sigma} - \left[\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu g_{\rho\sigma}}\right]\delta g_{\rho\sigma} \tag{2.1.6}$$

For the transformation given by 2.0.6 we already compute $\mathcal{F}$ and $\mathcal{K}$. The former is extracted from equation 2.0.7 and considering $\Omega = 1 + \omega$

$$\mathcal{F}^\mu = -2(d-1)\sqrt{-g}f(\phi)\nabla^\mu\omega \tag{2.1.7}$$

For the latter we substitute

$$\delta g_{\mu\nu} = 2\omega g_{\mu\nu}, \quad \delta\phi = -\omega\phi, \quad \delta g^{\mu\nu} = -2\omega g^{\mu\nu} \tag{2.1.8}$$

into equation 2.0.5 in order to obtain

$$\mathcal{K}^\mu = -2(d-1)\sqrt{-g}f(\phi)\nabla^\mu\omega \tag{2.1.9}$$

Hence, the Noether current for Weyl transformations vanishes (notice that this result is independent of the sign of $f(\phi)$). This is in agreement with [57] and for its physical interpretation we also follow [58].

By way of contrast let us consider the case of a scalar electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \left(D_\mu \varphi\right)^* D^\mu \varphi \qquad (2.1.10)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \varphi = \partial_\mu \varphi - iqA_\mu \varphi$. It is invariant under $U(1)$ gauge symmetry

$$\begin{aligned} \varphi(x) &\rightarrow \mathrm{e}^{-iq\omega(x)} \varphi(x) && (2.1.11) \\ A_\mu(x) &\rightarrow A_\mu(x) + \partial_\mu \omega(x) && (2.1.12) \end{aligned}$$

where ω stands for the gauge parameter. The equation of motion for the gauge field is

$$\partial_\mu F^{\mu\nu} = qJ^\nu, \qquad J^\nu = -i\varphi^* D_\nu \varphi + c.c. \qquad (2.1.13)$$

Using the equations of motion the Noether current is

$$\begin{aligned} \mathcal{J}^\mu &= -F^{\mu\nu} \delta A_\nu - \left(\delta\varphi^* D^\mu \varphi + c.c.\right) \\ &= -F^{\mu\nu} \partial_\nu \omega + q\omega J^\mu \\ &= \partial_\nu \left(F^{\nu\mu} \omega\right) - \omega \left(\partial_\nu F^{\nu\mu} - qJ^\mu\right) \\ &= \partial_\nu \left(F^{\nu\mu} \omega\right) && (2.1.14) \end{aligned}$$

Notice that the current is conserved due to the asymmetry of the electromagnetic tensor. The Noether charge is given by the boundary integral

$$Q = \int \mathrm{d}\Sigma_i \, F^{i0} \omega \qquad (2.1.15)$$

For finiteness and time independence of the charge a suitable boundary condition must be given, the authors take as an example

$$F^{i0} \sim O\left(\frac{1}{r^2}\right), \quad \partial_0 F^{i0} \sim O\left(\frac{1}{r^3}\right), \quad \omega \sim \text{const.} \quad \text{as} \quad r \rightarrow \infty \qquad (2.1.16)$$

If we extend the asymptotic condition for ω through all space , from equations 2.1.13 and 2.1.14 we find $\mathcal{J}^\mu = q\omega_0 J^\mu$, thus in the global limit the Noether charge is proportional to the physical charge.

Returning to Weyl transformations we see that there is no global limit since there current vanishes exactly and therefore within this framework the authors of [57] conclude that Weyl invariance does not have physical relevance.

This can be discussed in a different manner. We will say that a gauge invariance is "inessential" if there is a way of fixing it which results in a description that is still in terms of local fields. This is the case when there is a field that transforms by a shift under infinitesimal gauge transformations. The standard example is a nonlinear sigma model with values in G/H coupled to gauge fields for the group G. One can partly gauge fix G by fixing the nonlinear scalar, leaving a massive Yang-Mills field with values in the Lie algebra of G, but only an H gauge invariance. By contrast, a simple example of an "essential" gauge invariance is the one of QED. Indeed, any attempt to fix the $U(1)$ gauge will result in the new variables being related to the original ones by a non-local transformation. For example, if we fix the Lorentz gauge, the remaining, physical, degree of freedom is a transverse vector, which is obtained from the original gauge potential acting with a projector that involves an inverse d'Alembertian.

Consider the re-scaled metric

$$\bar{g}_{\mu\nu} = \phi^2 g_{\mu\nu} \tag{2.1.17}$$

After integrating by parts the scalar kinetic term in the WSTG action and taking $\alpha = 0$, the resulting action is equal upto a sign to the action of EG given by 1.0.2 for the metric $\bar{g}_{\mu\nu}$. For a strictly positive result the physical scalar field should be analytical continued. We see that the transformations 2.0.6 are trivially realized in the right hand side of 2.1.17. Therefore we can reconcile this result with the physical irrelevance of the Noether current associated just by stating that the invariance under 2.0.6 is just simply "inessential".

Moreover, the WSTG action after integrate by parts the scalar kinetic term is exactly invariant under 2.0.6 ($\mathcal{F}^\mu = 0$) and the Noehter current is given by 2.1.9. For this case in the global limit we still do not recover any physical current since 2.0.6 is obviously inessential. Hence, we have shown that Weyl is inessential for ϕ as a physical or unphysical field conformally coupled to the metric.

Let us study the Noether current for a generic action for matter and gravity as stated in [44]. Let $S[g_{\mu\nu}, \chi_a; g_i]$ be the generic action with χ_a and g_i as the matter fields and couplings respectively. One can express the couplings as $g_i = \phi^{d_i}\hat{g}_i$, d_i is the mass dimension, ϕ the dilaton and $\hat{g}_i$ is dimensionless. Replacing all covariant derivatives ∇ by Weyl covariant derivatives $\mathscr{D}$ and all curvatures $\mathcal{R}$ by the Weyl-covariant curvatures $\mathscr{R}$ we obtain, by construction, a Weyl invariant action $\hat{S}[g_{\mu\nu}, \phi, \chi_a, \hat{g}_i]$ [1]. As an

[1]The technical details of this procedure will be explain in detail in the next chapter.

example consider

$$S[g_{\mu\nu},\chi]=\int \mathrm{d}^4x\sqrt{-g}\left[Z_N\mathcal{R}-2\Lambda-\frac{1}{2}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi-\frac{m^2}{2}\chi^2\right] \tag{2.1.18}$$

After following the procedure one obtains

$$\begin{aligned}\hat{S}[g_{\mu\nu},\phi,\chi] &= \int \mathrm{d}^4x\sqrt{-g}\left[\phi^2\hat{Z}_N\mathscr{R}-2\hat{\Lambda}\phi^4-\frac{1}{2}g^{\mu\nu}\mathscr{D}_\mu\chi\mathscr{D}_\nu\chi-\frac{\hat{m}^2\phi^2}{2}\chi^2\right]\\ &= \int \mathrm{d}^4x\sqrt{-g}\left[\phi^2\hat{Z}_N\mathcal{R}+\hat{Z}_N 6g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-2\hat{\Lambda}\phi^4\right]\\ &\quad+\int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\phi^2\nabla_\mu\left(\frac{\chi}{\phi}\right)\nabla_\nu\left(\frac{\chi}{\phi}\right)-\frac{\hat{m}^2}{2}\phi^2\chi^2\right]\end{aligned} \tag{2.1.19}$$

Clearly this action is Weyl invariant and it can be written as $\hat{S}[g_{\mu\nu},\phi,\chi]=\hat{S}_{DG}[g_{\mu\nu},\phi]+\hat{S}_m[g_{\mu\nu},\phi,\chi]$, direct computation shows that the Noether current for Weyl transformation is

$$\mathcal{J}^\mu=\sqrt{-g}2\omega\frac{\chi^2}{\phi}\nabla^\mu\phi \tag{2.1.20}$$

which arise from matter. Therefore, for this matter choice Weyl current is not zero. As another example let us consider a Weyl invariant matter

$$S[g_{\mu\nu},\chi]=\int \mathrm{d}^4x\sqrt{-g}\left[Z_N\mathcal{R}-2\Lambda-\frac{1}{2}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi-\frac{1}{12}\chi^2\mathcal{R}\right] \tag{2.1.21}$$

Then

$$\begin{aligned}\hat{S}[g_{\mu\nu},\phi,\chi] &= \int \mathrm{d}^4x\sqrt{-g}\left[\phi^2\hat{Z}_N\mathcal{R}+\hat{Z}_N 6g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-2\hat{\Lambda}\phi^4\right]\\ &\quad+\int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi-\frac{1}{12}\mathcal{R}\chi^2\right]\\ &\quad+\int \mathrm{d}^4x\sqrt{-g}\,\frac{1}{2}\nabla_\mu\left(\frac{\chi^2}{\phi}\nabla^\mu\phi\right)\end{aligned} \tag{2.1.22}$$

If the boundary term is dropped, the action corresponds to a WSTG for a physical χ and unphysical ϕ scalar fields as expected; the Noether current for Weyl transformation vanishes. Taking into account the boundary term one finds

$$\mathcal{J}^\mu=\sqrt{-g}2\omega\frac{\chi^2}{\phi}\nabla^\mu\phi-\chi 2\omega\nabla^\mu\omega \tag{2.1.23}$$

With these simple examples we have learn that the way matter fields coupled affects the Noether current.

2.2 Hamiltonian Analysis of DG

In the ADM variables the general scalar-tensor action 2.0.1 becomes[2]

$$
\begin{aligned}
S[\phi, h_{ij}, \dot{h}_{ij}, N, N_i] \quad = \quad & \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}Nf(\phi)\left(R + K_{ij}K^{ij} - K^2\right) \\
& + \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}N\left[2Kf'(\phi)\nabla_n\phi - \frac{1}{2}\lambda(\phi)(\nabla_n\phi)^2\right] \\
& + \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}N\left[\frac{1}{2}\lambda(\phi)h^{ij}D_i\phi D_j\phi\right] \\
& - \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}N\left[2D_i\left(f'(\phi)D^i\phi\right) - U(\phi)\right]
\end{aligned}
\tag{2.2.1}
$$

The boundary term dropped in the action is

$$
\begin{aligned}
S_B = & -2\int \mathrm{d}^dx\,\partial_\mu\left(\sqrt{-g}f(\phi)Kn^\mu\right) \\
& + 2\int \mathrm{d}t\int \mathrm{d}^{d-1}x\,\partial_i\left[\sqrt{h}\left(Nf'(\phi)D^i\phi - f(\phi)D^iN\right)\right]
\end{aligned}
\tag{2.2.2}
$$

Here D corresponds to the covariant derivative defined with respect the induced metric h_{ij} and $\nabla_n\phi = n^\mu\partial_\mu\phi$. If $\lambda(\phi) \neq 0$ we can define the function

$$
\mathcal{W}(\phi) \equiv \left(\frac{d-2}{d-1}\right)f(\phi) - \frac{2}{\lambda(\phi)}[f'(\phi)]^2 \tag{2.2.3}
$$

It has the property that for WSTG it vanishes, this can be seen by substituting 2.0.11 and 2.0.12 into the definition. The action written in terms of $\mathcal{W}$ result

$$
\begin{aligned}
S[\phi, h_{ij}, \dot{h}_{ij}, N, N_i] = & \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}Nf(\phi)\left(R + \hat{K}_{ij}\hat{K}^{ij}\right) \\
& - \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}N\left[\mathcal{W}(\phi)K^2 + \frac{1}{2}\lambda(\phi)\left(\nabla_n\phi - \frac{2K}{\lambda(\phi)}f'(\phi)\right)^2\right] \\
& + \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}N\left[\frac{1}{2}\lambda(\phi)h^{ij}D_i\phi D_j\phi\right] \\
& - \int \mathrm{d}t\mathrm{d}^{d-1}x\sqrt{h}N\left[2D_i\left(f'(\phi)D^i\phi\right) - U(\phi)\right]
\end{aligned}
\tag{2.2.4}
$$

with

$$
\hat{K}_{ij} = K_{ij} - \frac{K}{d-1}h_{ij} \tag{2.2.5}
$$

[2]For the conventions consult section A of the Appendix.

The momenta are

$$\begin{aligned}
\Pi_N^0 &= 0 && (2.2.6)\\
\Pi_N^i &= 0 && (2.2.7)\\
\Pi_\phi &= -\sqrt{h}\lambda(\phi)\left(\nabla_n\phi - \frac{2K}{\lambda(\phi)}f'(\phi)\right) && (2.2.8)\\
\Pi_h^{ij} &= \sqrt{h}\left[-f(\phi)\hat{K}^{ij}\right.\\
&\quad \left.+\left(\mathcal{W}(\phi)K - f'(\phi)\left(\nabla_n\phi - \frac{2K}{\lambda(\phi)}f'(\phi)\right)\right)h^{ij}\right] && (2.2.9)
\end{aligned}$$

Let us define another function by

$$\mathcal{C} = \frac{1}{d-1}\Pi_h - \frac{f'(\phi)}{\lambda(\phi)}\Pi_\phi \tag{2.2.10}$$

where $\Pi_h = h_{ij}\Pi_h^{ij}$. Taking the trace to 2.2.9 and comparing with 2.2.10 we obtain an important relation

$$\mathcal{Q} = \mathcal{C} - \sqrt{h}\mathcal{W}(\phi)K = 0 \tag{2.2.11}$$

Notice that for

1. Weyl invariance $\mathcal{W}(\phi) = 0$ and for all $K \neq 0$.
2. $K = 0$ and for all $\mathcal{W}(\phi) \neq 0$

we can interpret $\mathcal{Q}$ as a constraint (i.e. the function $\mathcal{C}$ is promoted to a constraint)[3]. Hence for DG the primary constraints are $\Pi_N^0, \Pi_N^i, \mathcal{C}$ and the primary Hamiltonian result

$$\begin{aligned}
H_p = \int \mathrm{d}^{d-1}x \Bigg\{ N\left[\frac{\hat{\Pi}_{hij}\hat{\Pi}_h^{ij}}{f(\phi)\sqrt{h}} - \frac{\Pi_\phi^2}{2\lambda(\phi)\sqrt{h}} - L\right]\\
+ \Pi_h^{ij}\pounds_{\mathbf{N}}h_{ij} + \Pi_\phi\pounds_{\mathbf{N}}\phi + \zeta_0\Pi_N^0 + \delta^{ij}\zeta_i(\Pi_N)_j + \theta\mathcal{C}\Bigg\}
\end{aligned} \tag{2.2.12}$$

with ζ_0, ζ_i, θ as Lagrange multipliers, $\pounds_{\mathbf{N}}$ the Lie derivative along the shift $\mathbf{N}$ and

$$L = \sqrt{h}\left[f(\phi)R - \frac{1}{2}\left(\frac{d-5}{d-1}\right)\lambda(\phi)h^{ij}D_i\phi D_j\phi - 2f'(\phi)D_iD^i\phi + U(\phi)\right] \tag{2.2.13}$$

Consider the canonical transformation (of type 2)

$$\left(\phi, h_{ij}, \Pi_\phi, \Pi_h^{ij}\right) \to \left(\Phi, q_{ij}, P_\Phi, P_q^{ij}\right) \tag{2.2.14}$$

[3] There is a particular sub-case of interest: Weyl invariance $\mathcal{W}(\phi) = 0$ and K as a spatial constant but time dependent.

generated by [4]

$$\mathcal{G}_2 = \int \mathrm{d}^{d-1}x \left(\mathrm{e}^{2\sigma(\phi)} h_{ij} P_q^{ij} + \phi P_\Phi \right) \tag{2.2.15}$$

where σ is a scalar function of ϕ. Then

$$\begin{aligned} q_{ij} &= \mathrm{e}^{2\sigma(\phi)} h_{ij} && (2.2.16) \\ \Phi &= \phi && (2.2.17) \\ \Pi_h^{ij} &= \mathrm{e}^{2\sigma(\phi)} P_q^{ij} && (2.2.18) \\ \Pi_\phi &= 2\sigma'(\phi) P_q + P_\Phi && (2.2.19) \end{aligned}$$

Notice that the scalar field and the trace of the momentum associated to the metric remain invariant under the transformation. We take

$$\sigma(\Phi) = \left(\frac{d-2}{d-1} \right) \ln \Phi \tag{2.2.20}$$

in order to eliminate the dependence on $f(\phi)$ in $\frac{\hat{\Pi}_{hij}\hat{\Pi}_h^{ij}}{f(\phi)\sqrt{h}}$. The new Hamiltonian is given by

$$\begin{aligned} H_p = \int \mathrm{d}^{d-1}x \Bigg\{ N \left[\frac{1}{Z_N \sqrt{q}} \left(\hat{P}_{qij} \hat{P}_q^{ij} - \frac{P_q^2}{(d-1)(d-2)} \right) - Z_N \sqrt{h} R(q) - \tilde{L} \right] \\ + P_q^{ij} \pounds_{\mathbf{N}} q_{ij} + \frac{2}{d-1} P_q \pounds_{\mathbf{N}} \ln \Phi + \zeta_0 \Pi_N^0 + \delta^{ij} \zeta_i (\Pi_N)_j + \tilde{\theta} \tilde{\mathcal{C}} \Bigg\} \end{aligned} \tag{2.2.21}$$

where $\tilde{D}$ correspond to the covariant derivative with respect to q,

$$\begin{aligned} \tilde{L} &= Z_N \sqrt{q} \Big[\gamma_1(\Phi) R(q) + \gamma_2(\Phi) q^{ij} \tilde{D}_i \sigma(\Phi) \tilde{D}_j \sigma(\Phi) \\ &\quad + \gamma_3(\Phi) q^{ij} \tilde{D}_i \tilde{D}_j \sigma(\Phi) + \alpha \mathrm{e}^{\frac{d(d-1)}{d-2}\sigma(\Phi)} \Big] && (2.2.22) \\ \gamma_1(\Phi) &= \mathrm{e}^{2\sigma(\Phi)} - 1 && (2.2.23) \\ \gamma_2(\Phi) &= \left[d(d-3) - \frac{1}{2} \frac{(d-1)^3}{d-2} \right] \mathrm{e}^{2\sigma(\Phi)} && (2.2.24) \\ \gamma_3(\Phi) &= -\mathrm{e}^{2\sigma(\Phi)} && (2.2.25) \\ \tilde{\mathcal{C}} &= \frac{P_q}{d-1} - \frac{1}{2} \Phi P_\Phi && (2.2.26) \end{aligned}$$

and $\tilde{\theta}$ absorb many terms proportional to $\tilde{\mathcal{C}}$. Notice that the dependence on P_Φ in 2.2.21 is only via a constraint. Let us introduce the following

[4] See section C.1 of the appendix for a short review.

functionals

$$
\begin{aligned}
H_\perp^{Total}[N] &= H_\perp[N] + \Delta H_\perp[N] &(2.2.27)\\
H_\|^{Total}[\mathbf{N}] &= H_\|[\mathbf{N}] + \Delta H_\|[\mathbf{N}] &(2.2.28)\\
\Delta H_\perp[N] &= -\int \mathrm{d}^{d-1}x\, N\tilde{L} &(2.2.29)\\
\Delta H_\|[\mathbf{N}] &= \frac{2}{d-1}\int \mathrm{d}^{d-1}x\, P_q \pounds_{\mathbf{N}} \ln\Phi &(2.2.30)\\
C_1[\zeta] &= \int \mathrm{d}^{d-1}x\, \zeta_0 \Pi_N^0 &(2.2.31)\\
C_2[\vec{\zeta}] &= \int \mathrm{d}^{d-1}x\, \delta^{ij}\zeta_i (\Pi_N)_j &(2.2.32)\\
C_3[\tilde{\theta}] &= \int \mathrm{d}^{d-1}x\, \tilde{\theta}\tilde{\mathcal{C}} &(2.2.33)
\end{aligned}
$$

Then

$$
H_p = H_\perp^{Total}[N] + H_\|^{Total}[\mathbf{N}] + C_1[\zeta] + C_2[\vec{\zeta}] + C_3[\tilde{\theta}] \tag{2.2.34}
$$

We see that $H_\perp[N]$ and $H_\|[\mathbf{N}]$ are exactly equal to the generators in Einstein gravity, equations 1.0.7 and 1.0.8. Therefore we coin the term "deformations" to $\Delta H_\perp[N]$ and $\Delta H_\|[\mathbf{N}]$. Following the result from [15] we interpret $\Delta H_\perp$ as matter; thus we are dealing with a "non-derivative coupling" theory and this generator satisfy independently the same Poisson bracket as 1.0.9. Then

$$
\begin{aligned}
\{H_\perp^{Total}[N], H_\perp^{Total}[M]\} &= \{H_\perp[N], H_\perp[M]\} + \{\Delta H_\perp[N], \Delta H_\perp[M]\}\\
&= H_\|[q^{ij}(N\partial_j M - M\partial_j N)\} &(2.2.35)
\end{aligned}
$$

since $\Delta H_\perp[N]$ does not depend on any momenta. We can learn about the nature of $\Delta H_\|[\mathbf{N}]$ by comparing the following Poisson brackets:

$$
\begin{aligned}
\{q_{ij}, \Delta H_\|[\mathbf{N}]\} &= \frac{2}{d-1}(\pounds_{\mathbf{N}}\ln\Phi)\, q_{ij} &(2.2.36)\\
\{P_q^{ij}, \Delta H_\|[\mathbf{N}]\} &= -\frac{2}{d-1}(\pounds_{\mathbf{N}}\ln\Phi)\, P_q^{ij} &(2.2.37)\\
\{\Phi, \Delta H_\|[\mathbf{N}]\} &= 0 &(2.2.38)\\
\{P_\Phi, \Delta H_\|[\mathbf{N}]\} &= \frac{2}{d-1}\frac{P_\Phi}{\Phi}(\pounds_{\mathbf{N}}\ln\Phi) + \frac{2}{d-1}\tilde{D}_i\left[\frac{N^i P_q}{\Phi}\right] &(2.2.39)
\end{aligned}
$$

with

$$\begin{aligned}
\{q_{ij}, C_3[\tilde{\theta}]\} &= \frac{\tilde{\theta}}{d-1} q_{ij} && (2.2.40)\\
\{P_q^{ij}, C_3[\tilde{\theta}]\} &= -\frac{\tilde{\theta}}{d-1} P_q^{ij} && (2.2.41)\\
\{\Phi, C_3[\tilde{\theta}]\} &= -\frac{1}{2}\tilde{\theta}\Phi && (2.2.42)\\
\{P_\Phi, C_3[\tilde{\theta}]\} &= \frac{1}{2}\tilde{\theta} P_\Phi && (2.2.43)
\end{aligned}$$

We see that for the metric and its conjugate momentum, $\Delta H_{\parallel}[\mathbf{N}]$ and $C_3[\tilde{\theta}]$ generates infinitesimal Weyl transformation which are indistinguishable for a particular choice of foliation and Φ or a particular Lagrange multiplier $\tilde{\theta}$. From

$$\{q_{ij}, H_{\parallel}^{Total}[\mathbf{N}]\} = \pounds_{\mathbf{N}} q_{ij} + \frac{2}{d-1}\left(\pounds_{\mathbf{N}} \ln \Phi\right) q_{ij} \tag{2.2.44}$$

we see that $H_{\parallel}^{Total}[\mathbf{N}]$ not only generates displacements on the hypersurface but also rescalings. From

$$\begin{aligned}
\{H_{\parallel}^{Total}[\mathbf{N}], H_{\parallel}^{Total}[\mathbf{M}]\} &= H_{\parallel}^{Total}[\pounds_{\mathbf{N}}\mathbf{M}] + \Delta H_{\parallel}[\pounds_{\mathbf{N}}\mathbf{M}] && (2.2.45)\\
\{C_3[\tilde{\theta}], H_{\parallel}^{Total}[\mathbf{N}]\} &= C_3[\pounds_{\mathbf{N}}\tilde{\theta}] + \Delta H_{\parallel}[\tilde{\theta}\mathbf{N}] && (2.2.46)\\
\{C_3[\tilde{\theta}_1], C_3[\tilde{\theta}_2]\} &= 0 && (2.2.47)
\end{aligned}$$

We conclude that the generators of reparamentrizations on the hypersurface and Weyl rescaling form a first class system. Moreover, the number of degrees of freedom (d.o.f) is

$$2 \times \left(\underbrace{1}_{N} + \underbrace{d-1}_{\mathbf{N}} + \underbrace{1}_{\Phi} + \underbrace{\frac{d(d-1)}{2}}_{q_{ij}} \right) = d^2 + d + 2 \tag{2.2.48}$$

and the current number of constraints is

$$\underbrace{1}_{\frac{\delta C_1}{\delta N}} + \underbrace{d-1}_{\frac{\delta C_2}{\delta \mathbf{N}}} + \underbrace{1}_{\frac{\delta C_3}{\delta \tilde{\theta}}} + \underbrace{1}_{\frac{\delta H_{\perp}^{Total}}{\delta N}} + \underbrace{d-1}_{\frac{\delta H_{\parallel}^{Total}}{\delta \mathbf{N}}} = 2d+1 \tag{2.2.49}$$

Assuming that they form a first class system, the number of d.o.f is $d(d-3)$. Therefore if we suppose that there exist a further constraints it would not only change the d.o.f but also contradict the results of the generators according to the deformation algebra. The remaining bracket of interest is $\{C_3[\tilde{\theta}], H_{\perp}^{Total}[N]\}$. From 2.2.40,2.2.41, 2.2.42,2.2.43, we notice that for the functional

$$F[T] = \int \mathrm{d}^{d-1}x\, T^{i\ldots j}{}_{k\ldots l} f^{k\ldots l}{}_{i\ldots j}(q, P_q, \Phi, P_\phi) \tag{2.2.50}$$

with T a general tensor that cannot be a canonical variable or momenta, f a tensor of these variables; the variation with respect Weyl transformation result

$$\begin{aligned} \delta_W F[T] &= -\{C_3[\tilde{\theta}], F[T]\} \\ &= \int \mathrm{d}^{d-1}x\, \tilde{\theta} T^{i\ldots j}{}_{k\ldots l} \delta_W f^{k\ldots l}{}_{i\ldots j}(q, P_q, \Phi, P_\phi) \end{aligned} \quad (2.2.51)$$

If f transform as a shift, i.e. $\delta_W f = \beta f$ with β as a constant, then

$$\{C_3[\tilde{\theta}], F[T]\} = -\beta F[\tilde{\theta} T] \quad (2.2.52)$$

For $F = H_\perp^{Total}[N]$ the counting of degrees of freedom imply that it must transform as a shift[5].

2.3 Construction of the spacetime

Let us review (see [59]) the construction procedure of a vacuum solution in EG:

1. Give an initial data set (q_{ij}, P_q^{ij}) which satisfy the constraints

$$\frac{\delta H_\perp}{\delta N} \approx 0 \quad (2.3.1)$$
$$\frac{\delta H_\parallel}{\delta N^i} \approx 0 \quad (2.3.2)$$
$$(2.3.3)$$

2. Choose freely the lapse N and the shift $\mathbf{N}$.

3. Solve the Hamilton equations for the canonical variables.

4. Reconstruct the spacetime metric and verify that solves Einstein's equation.

The first step is by no means trivial since satisfying the constraints restricts the form of the hypersurface which may not yield to a relevant physical solution. For analytical results the conformal decomposition of the constraints is a standard tool which has been developed by Lichnerowicz [60] , Choquet-Bruhat [61] [62] and York and Ó Murchadha [63] [64] [65] [66]. Let us write the constraints 2.3.5 and 2.3.6 in terms of the extrinsic curvature by substituting

$$P_q^{ij} = Z_N \sqrt{q} \left[K_q q^{ij} - K_q^{ij} \right] \quad (2.3.4)$$

[5]Similar result is made for $H_\parallel[\mathbf{N}]$ in Appendix D

into 1.0.7 and 1.0.8. The constraints become

$$K_q^{ij}K_{ij}^q - K_q^2 - R(q) = 0 \tag{2.3.5}$$

$$\tilde{D}_j\left[K_q q^{ij} - K_q^{ij}\right] = 0 \tag{2.3.6}$$

Consider an auxiliary metric λ_{ij}; the traceless, transverse and symmetric tensor σ^{ij} and a positive scalar field χ with the ansatz

$$q_{ij} = \chi^{\gamma_1}\lambda_{ij}, \quad \hat{K}_q^{ij} = \chi^{\gamma_2}\sigma^{ij} \tag{2.3.7}$$

with γ_1 and γ_2 as a powers to be determined. From

$$\tilde{D}_j\hat{K}_q^{ij} = \chi^{-\frac{(d+1)}{2}\gamma_1}\bar{D}_j\left[\chi^{\frac{(d+1)}{2}\gamma_1}\hat{K}_q^{ij}\right] = \chi^{-\frac{(d+1)}{2}\gamma_1}\bar{D}_j\left[\chi^{\frac{(d+1)}{2}\gamma_1+\gamma_2}\sigma^{ij}\right] \tag{2.3.8}$$

Then the constraint 2.3.6 becomes

$$\tilde{D}^i K_q - \chi^{-\frac{(d+1)}{2}\gamma_1}\bar{D}_j\left[\chi^{\frac{(d+1)}{2}\gamma_1+\gamma_2}\sigma^{ij}\right] = 0 \tag{2.3.9}$$

We see that it is reasonable to choose $\gamma_2 = -\frac{(d+1)}{2}\gamma_1$ and $K_g = 0$ (Lichnerowiczs ansatz) or $K_g = K_g(t)$ (York ansatz), since the constraints become

$$\chi^{-(d-1)\gamma_1}\sigma^{ij}\sigma_{ij} - \left(\frac{d-2}{d-1}\right)K_q^2 - \chi^{-\gamma_1}R(\lambda) - (d-2) \tag{2.3.10}$$

$$\gamma_1\left[\frac{d-3}{4}\gamma_1 - 1\right]\chi^{-\gamma_1-2}\bar{D}_i\chi\bar{D}^i\chi + (d-2)\gamma_1\chi^{-\gamma_1-1}\bar{D}_i\bar{D}^i\chi = 0 \tag{2.3.11}$$

$$\bar{D}_j\sigma^{ij} = 0 \tag{2.3.12}$$

In order to simplify the first expression we choose $\gamma_1 = \frac{4}{d-3}$, then

$$\chi^{-4\left(\frac{d-1}{d-3}\right)}\sigma^{ij}\sigma_{ij} - \left(\frac{d-2}{d-1}\right)K_q^2 - \chi^{-\frac{4}{d-3}}R(\lambda) + 4\left(\frac{d-2}{d-3}\right)\chi^{-\left(\frac{4}{d-3}\right)-1}\bar{D}_i\bar{D}^i\chi = 0 \tag{2.3.13}$$

$$\bar{D}_j\sigma^{ij} = 0 \tag{2.3.14}$$

The elliptic equation 2.3.13 with $K_g = 0$ is known as the "Lichnerowicz equation" and for $K_g = K_g(t)$ is known as the "Lichnerowicz-York equation". The problem of satisfying the constraints become that of finding the scalar χ. Following [59] the solvability for $d = 4$ is resumed in Table 2.1. The Yamabe classes $\mathcal{Y}^+, \mathcal{Y}^0, \mathcal{Y}^-$ are 3-dimensional Riemannian closed manifolds on which every metric can be conformally transformed to a metric of constant scalar curvature. A solution of 2.3.13 exist for $\lambda_{ij}, \sigma^{ij}, K_q$ exist if and only if there exist a solution of the conformally transformed data $\nu^4\lambda_{ij}, \nu^{-2}\sigma^{ij}, K_q$ with $R(\nu^4\lambda) = +1, 0, -1$ $\nu > 0$, depending upon the Yamabe class.

For DG we follow the same procedure. Let us start before the canonical transformation, we give an initial data set $(\phi, h_{ij}, \Pi_\phi, \Pi_h^{ij})$ and solve the

	$\sigma^2=0, K_g=0$	$\sigma^2\neq 0,\ K_g=0$	$\sigma^2=0,\ K_g\neq 0$	$\sigma^2\neq 0,\ K_g\neq 0$
$\mathcal{Y}^+$	N	Y	N	Y
$\mathcal{Y}^0$	Y	N	N	Y
$\mathcal{Y}^-$	N	N	Y	Y

Table 2.1: Solutions of "Lichnerowicz-York equation" for conformally transformed data in a certain Yamabe class, σ^2 stands for $\sigma^{ij}\sigma_{ij}$.

constraints calculated from the primary Hamiltonian given by 2.2.12:

$$\begin{aligned} \frac{\hat{\Pi}_{hij}\hat{\Pi}_h^{ij}}{f(\phi)\sqrt{h}} - \frac{\Pi_\phi^2}{2\lambda(\phi)\sqrt{h}} - L &\approx 0 && (2.3.15)\\ -2D_i\Pi^i_{hj} + \Pi_\phi D_j\phi &\approx 0 && (2.3.16)\\ \Pi_h + \frac{1}{2}\phi\Pi_\phi &\approx 0 && (2.3.17)\end{aligned}$$

We use the last expression to eliminate the dependence on Π_h:

$$\begin{aligned} \frac{\hat{\Pi}_{hij}\hat{\Pi}_h^{ij}}{f(\phi)\sqrt{h}} - \frac{\Pi_\phi^2}{2\lambda(\phi)\sqrt{h}} - L &\approx 0 && (2.3.18)\\ D_i\hat{\Pi}^i_{hj} + \frac{d}{d-1}\Pi_\phi D_j\phi + \phi D_j\Pi_\phi &\approx 0 && (2.3.19)\end{aligned}$$

In order to disentangle the constraints it is natural to gauge fix the theory: $\Pi_\phi \approx 0$. This gauge is called "Lichnerowicz gauge" and imply that $\hat{\Pi}_{hij}$ is also transverse, $\Pi_h \approx 0$ and ϕ must satisfy

$$\begin{aligned} & f(\phi)\hat{K}_{ij}\hat{K}^{ij} - f(\phi)R - \frac{1}{2}\left(\frac{d-5}{d-1}\right)\lambda(\phi)h^{ij}D_i\phi D_j\phi \\ & -2f'(\phi)D_iD^i\phi + U(\phi) \approx 0 \qquad (2.3.20)\end{aligned}$$

For the field redefinition $\phi = \chi^{\frac{2}{d-3}}$ we obtain again an elliptic equation. The constraint must be preserved in time, therefore $\dot{\Pi}_\phi \approx 0$. This will give in turn a lapse-gauge fixing equation. After choosing $\mathbf{N}$ and θ one must solve Hamilton's equation for the remaining canonical variables and finally check if the satisfy Einstein's equation.

Notice that for the "Einstein gauge" $\phi \approx 1$ $(\alpha = 0)$2.3.15, 2.3.16 and 2.3.17 reduced to 2.3.5 and 2.3.6. The consistency condition $\dot{\phi} \approx 0$ gives a fixing equation for θ. From table 2.2 we see that the previous gauges

Lichnerowicz gauge	Einstein gauge
$\Pi_\phi \approx 0$	$\phi \approx 1$
$\dot{\Pi}_\phi \approx 0 \Rightarrow N$ is fixed	$\dot{\phi} \approx 0 \Rightarrow \theta$ is fixed
$\mathbf{N}, \theta$ are free	$N, \mathbf{N}$ are free

Table 2.2: Symmetry trading.

give different theories. Effectively there has been a trade of generators: $\frac{\delta \mathcal{H}}{\delta N} \leftrightarrow \mathcal{C}$. Nevertheless the canonical variables are the same in both theories (a spatial metric and its conjugate momentum); the same number of first class constraints implies the same number of (local) degrees of freedom.

Chapter 3

Quantum aspects of DG

3.1 Path integral of DG

Let us study the path integral in which we integrate out the scalar field, i.e. we treat $g_{\mu\nu}$ as a classical field and the scalar field as pure quantum fluctuation. After the analytic continuation to the Euclidean time, the action 2.0.17 [1] can be written as

$$S[\psi, g_{\mu\nu}] = -\frac{\varepsilon}{2}\int \mathrm{d}^d x\sqrt{g}\,\psi\mathcal{O}_{\tilde{\alpha}}\psi \tag{3.1.1}$$

with $\varepsilon = \pm 1$ and

$$\mathcal{O}_{\tilde{\alpha}} \;=\; -\nabla^2 + \frac{1}{4}\left(\frac{d-2}{d-1}\right)\mathcal{R} + \frac{2}{\varepsilon}\tilde{\alpha}\psi^{\frac{4}{d-2}} \tag{3.1.2}$$

The path integral becomes

$$Z[g_{\mu\nu}] = \int \mathcal{D}\psi\, \mathrm{e}^{\frac{\varepsilon}{2}\int \mathrm{d}^d x\sqrt{g}\,\psi\mathcal{O}_{\tilde{\alpha}}\psi} \tag{3.1.3}$$

with

$$\mathcal{D}\psi = \prod_x \left(\frac{\mathrm{d}\psi}{\mu}\right) \tag{3.1.4}$$

where μ corresponds to an arbitrary mass. For $\tilde{\alpha} = 0$ the path integral is Gaussian if ψ is a physical field ($\varepsilon = -1$) and as a dilaton one must analytic continued it to a pure imaginary field (similar to [67]). The integral result $Z[g_{\mu\nu}] = \mathcal{N}\mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\log\frac{\mathcal{O}_0}{\mu^2}}$ where $\mathcal{N}$ corresponds to a (infinite) normalization constant. Notice that given choice of measure Weyl invariance is broken due to presence of μ, i.e. it is anomalous. The effective action can be computed

[1] It amounts to an overall change of sign of the action and the manifold is assumed to be closed. Any reference to Euclidean metric is not shown explicitly.

by means of the heat kernel method. The heat kernel trace and the effective action are related by the formal expression [68] [33]:

$$\Gamma[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\mathrm{Tr}K_{\mathcal{O}_0}(s) \tag{3.1.5}$$

where small s encodes the UV behavior, large s the IR and $\mathrm{Tr}K_{\mathcal{O}_0}(s) = \mathrm{Tr}\mathrm{e}^{-s\mathcal{O}_0}$. The local expansion of the heat kernel trace is on the form

$$\mathrm{Tr}K_{\mathcal{O}_0}(s) = \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty} s^{n-\frac{d}{2}} B_{2n}(\mathcal{O}_0) \tag{3.1.6}$$

where the $B_{2n}(\mathcal{O}_0)$ are the integrated heat kernel coefficients (see [33] and [69] for details). Before substituting 3.1.6 into 3.1.5 we add a mass term e^{-sm^2} in the integrand in order to ensure convergence. We will obtain a general result on the form $\Gamma = \Gamma_{finite} + \Gamma_{divergent}$. The divergent part arises from the lower extrema of the integral, therefore some regularization method must be employ. If we choose dimensional regularization the effective action is

$$\Gamma[g_{\mu\nu}] = -\frac{1}{2(4\pi)^{d/2}}\sum_{n=0}^{\infty} m^{d-2n}\Gamma\left(n-\frac{d}{2}\right) B_{2n}(\mathcal{O}_0) \tag{3.1.7}$$

where $\Gamma\left(n-\frac{d}{2}\right)$ corresponds to the Gamma function. From its analytical continuation we see that for $n-\frac{d}{2} \leq 0$ the effective action has UV divergences (also finite terms) and the terms starting from $n = \frac{d}{2}+1$ are finite. Consider the dimensionless integrated heat kernel coefficients

$$\hat{B}_{2n}(\mathcal{O}_0) = \Lambda^{d-2n} B_{2n}(\mathcal{O}_0) \tag{3.1.8}$$

then for $d = 4-\epsilon$ the divergent part is

$$\begin{aligned}
\Gamma_{div}[g_{\mu\nu}] \quad = \quad & -\frac{1}{2(4\pi)^2}\left[\frac{m^4}{\Lambda^4}\left(\frac{1}{\epsilon}-\frac{\gamma}{2}+\frac{3}{4}+\ln 4\pi - \ln\frac{m^2}{\Lambda^2}\right)\hat{B}_0(\mathcal{O}_0)\right. \\
& +\frac{m^2}{\Lambda^2}\left(-\frac{2}{\epsilon}+\gamma-1-\ln 4\pi+\ln\frac{m^2}{\Lambda^2}\right)\hat{B}_2(\mathcal{O}_0) \\
& \left. +\left(\frac{2}{\epsilon}-\gamma+\ln 4\pi-\ln\frac{m^2}{\Lambda^2}\right)\hat{B}_4(\mathcal{O}_0)\right]
\end{aligned} \tag{3.1.9}$$

Notice that in the limit $m \to 0$ the third term is IR divergent. Ignoring this limit we obtain a well known result

$$\Gamma_{div}[g_{\mu\nu}] = -\frac{1}{(4\pi)^2\epsilon}\int \mathrm{d}^4x\sqrt{g}\left[\frac{1}{120}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - \frac{1}{360}E - \frac{1}{30}\nabla^2\mathcal{R}\right] \tag{3.1.10}$$

Recall that $W^{\mu}{}_{\nu\rho\sigma}$ is invariant under metric rescalings. In $d=4$, $\sqrt{g}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ is invariant under metric rescalings and E is a total derivative [70]. Since we are dealing with closed manifolds we can conclude that the divergent part of the effective action is Weyl invariant. Therefore the anomaly must arise from the finite part.

In order to take into account the potential in the path integral we are obliged to split the scalar field

$$\psi = \bar{\psi} + \chi \tag{3.1.11}$$

where $\bar{\psi}$ corresponds to the background field (not yet a dimensionful constant) and χ the quantum fluctuation. Therefore the (Euclidean) action 2.0.17 have the following expansion

$$S[\bar{\psi} + \chi, g_{\mu\nu}] = S[\bar{\psi}, g_{\mu\nu}] + S^{(1)}[\chi, g_{\mu\nu}] + S^{(2)}[\chi, g_{\mu\nu}] + \dots \tag{3.1.12}$$

with

$$\begin{aligned} S^{(1)}[\chi, g_{\mu\nu}] &= -\int \mathrm{d}^d x \sqrt{-g} E^{\bar{\psi}} \chi & (3.1.13) \\ S^{(2)}[\chi, g_{\mu\nu}] &= -\frac{\varepsilon}{2} \int \mathrm{d}^d x \sqrt{-g} \chi \mathcal{O}_{\tilde{\alpha}} \chi & (3.1.14) \\ \mathcal{O}_{\tilde{\alpha}} &= -\nabla^2 + \frac{1}{4} \left(\frac{d-2}{d-1} \right) \mathcal{R} \\ &\quad + \frac{1}{\varepsilon} \left(\frac{2d}{d-2} \right) \left(\frac{2d}{d-2} - 1 \right) \tilde{\alpha} \bar{\psi}^{\frac{4}{d-2}} & (3.1.15) \end{aligned}$$

As before χ must be pure imaginary if the scalar is a dilaton (as in [67]). The divergent part of the (Euclidean) effective action result

$$\begin{aligned} \Gamma_{div}[\bar{\psi}, g_{\mu\nu}] &= -\frac{1}{(4\pi)^2 \epsilon} \int \mathrm{d}^4 x \sqrt{g} \left[72 \tilde{\alpha}^2 \bar{\psi}^4 + \frac{1}{120} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} \right. \\ &\quad \left. - \frac{1}{360} E - \frac{1}{180} \nabla^2 \mathcal{R} + 2 \tilde{\alpha} \nabla^2 \bar{\psi}^2 \right] \end{aligned} \tag{3.1.16}$$

where $\bar{\psi}$ satisfy $E^{\bar{\psi}} = 0$. Again we find that the divergent part of the effective action is Weyl invariant. Moreover the anomaly must arise from the finite quantum correction.

Let us study the opposite case, the scalar now is completely classical and the metric is of the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{3.1.17}$$

where $\bar{g}_{\mu\nu}$ corresponds to the background metric and $h_{\mu\nu}$ the fluctuation. We use the convention that indices are raised and lowered with the background metric, e.g. $h = \bar{g}^{\mu\nu} h_{\mu\nu}$. After the splitting, the (Euclidean) action 2.0.17 have the following expansion

$$S[\psi, \bar{g}_{\mu\nu} + h_{\mu\nu}] = S[\psi, \bar{g}_{\mu\nu}] + S^{(1)}[\psi, h_{\mu\nu}] + S^{(2)}[\psi, h_{\mu\nu}] + \dots \tag{3.1.18}$$

with

$$\begin{aligned}
S^{(1)}[\psi, h_{\mu\nu}] &= \int \mathrm{d}^4x\sqrt{-\bar{g}}E^{(\bar{g})}_{\mu\nu}h^{\mu\nu} \qquad (3.1.19)\\
S^{(2)}[\psi, h_{\mu\nu}] &= -\int \mathrm{d}^4x\sqrt{-\bar{g}}\left[\frac{\psi^2}{24}\left(\frac{1}{2}h_{\mu\nu}\bar{\nabla}^2h^{\mu\nu} - \frac{1}{2}h\bar{\nabla}^2h - h_{\mu\nu}\bar{\nabla}^\mu\bar{\nabla}_\rho h^{\rho\nu}\right.\right.\\
&\left. +h\bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} + \bar{\mathcal{R}}_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} + h^\alpha_{\ \beta}\bar{\mathcal{R}}_{\gamma\alpha}h^{\gamma\beta} - h\bar{\mathcal{R}}_{\mu\nu}h^{\mu\nu}\right)\\
&+\left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)\left(\frac{\psi^2}{12}\bar{\mathcal{R}} + \tilde{\alpha}\psi^4\right)\\
&-\frac{1}{8}h_{\mu\nu}\psi\bar{\nabla}_\alpha\psi\bar{\nabla}^\alpha h^{\mu\nu} + \frac{1}{6}h^\rho_{\ \mu}\psi\bar{\nabla}_\rho\psi\bar{\nabla}_\nu h^{\nu\mu}\\
&-\frac{1}{6}h^\rho_{\ \mu}\psi\bar{\nabla}_\rho\psi\bar{\nabla}^\mu h + \frac{1}{12}h_{\nu\alpha}\psi\bar{\nabla}_\mu\psi\bar{\nabla}^\alpha h^{\mu\nu} + \frac{1}{24}h\psi\bar{\nabla}_\mu\psi\bar{\nabla}^\mu h\\
&+\frac{1}{2}\left(\frac{1}{8}h^2 - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta}\right)\bar{g}^{\mu\nu}\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi - \frac{1}{4}hh^{\mu\nu}\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi\\
&\left.+\frac{1}{2}h^{\mu\alpha}h_\alpha^{\ \nu}\bar{\nabla}_\mu\psi\bar{\nabla}_\nu\psi\right]
\end{aligned}$$

$$(3.1.20)$$

Following [44], [45] and [71] we introduce the Weyl connection

$$\Upsilon^\rho_{\ \mu\nu} = \bar{\Gamma}^\rho_{\ \mu\nu} - \delta^\rho_\mu\upsilon_\nu - \delta^\rho_\nu\upsilon_\mu + g_{\mu\nu}\upsilon^\rho \qquad (3.1.21)$$

where υ_ν is a pure-gauge Abelian gauge field and $\bar{\Gamma}^\rho_{\ \mu\nu}$ is the Levi-Civita connection with respect to $\bar{g}_{\mu\nu}$. Under $\bar{g}_{\mu\nu} \to \Omega^2\bar{g}_{\mu\nu}$ the gauge fields transform as $\upsilon_\mu \to \upsilon_\mu + \Omega^{-1}\partial_\mu\Omega$, thus leaving the Weyl connection invariant. We define ∇^W as the covariant derivative associated with the Weyl connection. Moreover we define the Weyl and diffeomorphic invariant covariant derivative

$$\mathscr{D}_\sigma T^{\mu_1\cdots}_{\ \ \nu_1\cdots} = \nabla^W_\sigma T^{\mu_1\cdots}_{\ \ \nu_1\cdots} - w\upsilon_\sigma T^{\mu_1\cdots}_{\ \ \nu_1\cdots} \qquad (3.1.22)$$

where w is defined from the transformation $T^{\mu_1\cdots}_{\ \ \nu_1\cdots} \to \Omega^w T^{\mu_1\cdots}_{\ \ \nu_1\cdots}$. Notice that $\mathscr{D}_\sigma\bar{g}_{\mu\nu} = 0$. If υ_μ is constructed from the unphysical scalar field $\upsilon_\mu = -\psi^{-1}\partial_\mu\psi$, we also have $\mathscr{D}_\sigma\psi = 0$ (recall that in $d = 4$, ψ transforms as $\psi \to \Omega^{-1}\psi$). The curvature tensor with respecto to $\mathscr{D}$ is defined as

$$[\mathscr{D}_\rho, \mathscr{D}_\sigma]V^\alpha = \mathscr{R}_{\rho\sigma\ \ \beta}^{\ \ \ \alpha}V^\beta \qquad (3.1.23)$$

The tensor $\mathscr{R}_{\rho\sigma\ \ \beta}^{\ \ \ \alpha}$ is Weyl invariant and can be written in terms of the Riemmann tensor as

$$\begin{aligned}
\mathscr{R}_{\mu\nu\rho\sigma} &= \bar{\mathcal{R}}_{\mu\nu\rho\sigma} + \left[\bar{g}_{\mu\rho}\left(\bar{\nabla}_\nu\upsilon_\sigma + \upsilon_\nu\upsilon_\sigma\right) - \rho \leftrightarrow \sigma\right]\\
&- \left[\bar{g}_{\nu\rho}\left(\bar{\nabla}_\mu\upsilon_\sigma + \upsilon_\mu\upsilon_\sigma\right) - \rho \leftrightarrow \sigma\right]\\
&-(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\upsilon_\lambda\upsilon^\lambda \qquad (3.1.24)
\end{aligned}$$

Notice that 3.1.20 can be rewritten in terms of Weyl-covariant derivatives $\mathscr{D}$ and curvatures $\mathscr{R}$. Then

$$\begin{aligned} S^{(2)}[\psi, h_{\mu\nu}] &= \int \mathrm{d}^4x\sqrt{-\bar{g}}\left[\frac{\psi^2}{24}\left(-\frac{1}{2}h_{\mu\nu}\mathscr{D}^2h^{\mu\nu} + \frac{1}{2}h\mathscr{D}^2h + h_{\mu\nu}\mathscr{D}^\mu\mathscr{D}_\rho h^{\rho\nu}\right.\right. \\ &\quad \left. -h\mathscr{D}_\mu\mathscr{D}_\nu h^{\mu\nu} - \mathscr{R}_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} - h^\alpha{}_\beta\mathscr{R}_{\gamma\alpha}h^{\gamma\beta} + h\mathscr{R}_{\mu\nu}h^{\mu\nu}\right) \\ &\quad \left. -\left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)\left(\frac{\psi^2}{12}\mathscr{R} + \tilde{\alpha}\psi^4\right)\right] \end{aligned} \tag{3.1.25}$$

This result is the same as in EG since ψ behaves as a constant with respect to $\mathscr{D}$ (see [33] for comparison with EG). For $E^{(\bar{g})}_{\mu\nu} = 0$ (this amount to fix ψ due to Weyl invariance), we ought to compute the path integral on the form

$$Z[\psi, \bar{g}_{\mu\nu}] = e^{-S[\psi,\bar{g}_{\mu\nu}]}\int \mathcal{D}h_{\mu\nu}\,\mathrm{e}^{-\frac{1}{2}\int \mathrm{d}^4x\sqrt{\bar{g}}\,h_{\alpha\beta}\mathcal{O}^{\alpha\beta\gamma\delta}(\psi)h_{\gamma\delta}} \tag{3.1.26}$$

Due to diffeomorphism invariance we follow Faddeev-Popov's procedure, then

$$\begin{aligned} Z[\psi, \bar{g}_{\mu\nu}] &= e^{-S[\psi,\bar{g}_{\mu\nu}]}\int \mathcal{D}h_{\mu\nu}\mathcal{D}C\mathcal{D}\bar{C} \\ &\quad \times\mathrm{e}^{-\frac{1}{2}\int \mathrm{d}^4x\sqrt{\bar{g}}\,h_{\alpha\beta}\mathcal{O}^{\alpha\beta\gamma\delta}(\psi)h_{\gamma\delta} - S_{G.F.}[\psi,\bar{g}_{\mu\nu},h_{\mu\nu}] - S_{ghost}[\psi,\bar{g}_{\mu\nu},C,\bar{C}]} \end{aligned} \tag{3.1.27}$$

The replacement ($\bar{\nabla} \to \mathscr{D}$ and $\bar{\mathcal{R}}'s \to \mathscr{R}'s$) allow us to work as if we where dealing with EG, thus the gauge fixing term is given by

$$S_{G.F.}[\psi, \bar{g}_{\mu\nu}, h_{\mu\nu}] = \frac{1}{\alpha}\int \mathrm{d}^4x\sqrt{\bar{g}}\,\frac{\psi^2}{24}F_\mu\bar{g}^{\mu\nu}F_\nu \tag{3.1.28}$$

where

$$F_\mu = \mathscr{D}_\sigma h^\sigma{}_\mu - \frac{\beta+1}{4}\mathscr{D}_\mu h \tag{3.1.29}$$

and α, β as gauge parameters. The corresponding ghost action for the gauge condition $F_\mu = 0$ is

$$S_{ghost}[\psi, \bar{g}_{\mu\nu}, C, \bar{C}] = -\int \mathrm{d}^4x\sqrt{\bar{g}}\,\psi^2\bar{C}_\mu\left[-\bar{g}^{\mu\nu}\mathscr{D}^2 - \mathscr{R}^{\mu\nu}\right]C_\nu \tag{3.1.30}$$

where $C_\mu, \bar{C}_\mu$ are anticommuting dimensionless vector fields. For the de Donder-Feynmann gauge $\beta = 1$, $\alpha = 1$, the final (contracted) Hessian is on the form

$$\begin{aligned} \int \mathrm{d}^4x\sqrt{\bar{g}}\,h_{\alpha\beta}\mathcal{H}^{\alpha\beta\gamma\delta}(\psi)h_{\gamma\delta} &= \int \mathrm{d}^4x\sqrt{\bar{g}}\left[\frac{\psi^2}{12}\left(-\frac{1}{2}h_{\mu\nu}\mathscr{D}^2h^{\mu\nu}\right.\right. \\ &\quad \left. -\mathscr{R}_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} - h^\alpha{}_\beta\mathscr{R}_{\gamma\alpha}h^{\gamma\beta} + h\mathscr{R}_{\mu\nu}h^{\mu\nu}\right) \\ &\quad \left. -\left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)\left(\frac{\psi^2}{12}\mathscr{R} + \tilde{\alpha}\psi^4\right)\right] \end{aligned} \tag{3.1.31}$$

We see that the Hessian corresponds to a bilinear form in the space of symmetric tensors, it maps covariant tensors to contravariant tensors $h_{\gamma\delta} \to \mathcal{H}^{\alpha\beta\gamma\delta}(\psi) h_{\gamma\delta}$. We wish to think of it as a differential operator, i.e. an operator that maps covariant tensors to covariant tensors, since its trace and determinant are base independent. In order to acomplish this we need to define a metric in the space of metrics (analogous to the De Witt's metric)

$$\mathcal{G}_2(h,h) = \int \mathrm{d}^4x \sqrt{\bar{g}}\, \psi^4 h_{\alpha\beta} \bar{g}^{\alpha\gamma} \bar{g}^{\beta\delta} h_{\gamma\delta} \tag{3.1.32}$$

which is invariant under the background diffeomorphisms defined by

$$\delta_\epsilon \bar{g}_{\mu\nu} = \pounds_\epsilon \bar{g}_{\mu\nu} = \bar{\nabla}_\mu \epsilon_\nu + \bar{\nabla}_\nu \epsilon_\mu, \quad \delta_\epsilon h_{\mu\nu} = \pounds_\epsilon h_{\mu\nu}, \quad \delta_\epsilon \psi = \epsilon^\mu \partial_\mu \psi \tag{3.1.33}$$

and background Weyl transformations ($\Omega = 1 + \omega$):

$$\delta_\omega \bar{g}_{\mu\nu} = 2\omega \bar{g}_{\mu\nu}, \quad \delta_\omega h_{\mu\nu} = 2\omega h_{\mu\nu}, \quad \delta_\omega \psi = -2\omega\psi \tag{3.1.34}$$

Then we can write

$$\int \mathrm{d}^4x \sqrt{\bar{g}}\, h_{\alpha\beta} \mathcal{H}^{\alpha\beta\gamma\delta}(\psi) h_{\gamma\delta} = \mathcal{G}_2(h, \mathcal{O}_2 h) \tag{3.1.35}$$

with

$$\mathcal{O}_{2\mu\nu}{}^{\gamma\delta} = \psi^{-4} \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} \mathcal{H}^{\alpha\beta\gamma\delta}(\psi) \tag{3.1.36}$$

Notice that $\mathcal{O}_2$ is dimensionless. From the invariance of $\mathcal{G}_2$ one can deduce the covariance property

$$\mathcal{O}_{2\alpha\beta}{}^{\gamma\delta}\Big|_{\Omega^2 g_{\mu\nu}, \Omega^{-1}\psi} \left(\Omega^2 h_{\gamma\delta}\right) = \Omega^2\, \mathcal{O}_{2\alpha\beta}{}^{\gamma\delta}\Big|_{g_{\mu\nu},\psi} (h_{\gamma\delta}) \tag{3.1.37}$$

and from it we deduce that the spectrum of $\mathcal{O}_2$ is Weyl invariant. Finally a Weyl invariant measure for $h_{\mu\nu}$ can be achieved by writing it as

$$\mathcal{D}h_{\mu\nu} = \prod_x \left(\psi^2 \mathrm{d}h_{\mu\nu}\right) \tag{3.1.38}$$

For the anticommuting vector fields we define invariant metric

$$\mathcal{G}_1(\bar{C}, C) = \int \mathrm{d}^4x \sqrt{\bar{g}} \psi^2 \bar{C}_\mu \bar{g}^{\mu\nu} C_\nu \tag{3.1.39}$$

Then

$$S_{ghost}[\psi, \bar{g}_{\mu\nu}, C, \bar{C}] = -\mathcal{G}_1(\bar{C}, \mathcal{O}_1 C) \tag{3.1.40}$$

with

$$\mathcal{O}_{1\mu}{}^\nu = \psi^{-2}\left(-\delta^\nu_\mu \mathscr{D}^2 - \mathscr{R}_\mu{}^\nu\right) \tag{3.1.41}$$

Hence the effective action result

$$\Gamma[\psi, \bar{g}_{\mu\nu}] = S[\psi, \bar{g}_{\mu\nu}] + \frac{1}{2} \mathrm{tr}\log \mathcal{O}_2 - \mathrm{tr}\log \mathcal{O}_1 \tag{3.1.42}$$

If we now consider also the splitting $\psi = \bar{\psi} + \chi$ we need to expand the quadratic action further:

$$\begin{aligned} S^{(2)} &= (3.1.20) + \int \mathrm{d}^4x\sqrt{\bar{g}}\left[\frac{1}{12}\bar{\psi}\chi h\bar{\mathcal{R}} + \frac{1}{12}\chi^2\bar{\mathcal{R}}\right. \\ &\quad + \frac{1}{6}\bar{\psi}\chi\left(\bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - \bar{\nabla}^2 h - \bar{\mathcal{R}}_{\mu\nu}h^{\mu\nu}\right) \\ &\quad \left. + \frac{1}{2}\bar{g}^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi - h^{\mu\nu}\nabla_\mu\bar{\psi}\nabla_\mu\chi + 6\tilde{\alpha}\bar{\psi}^2\chi^2\right] \end{aligned} \tag{3.1.43}$$

We choose the gauge in which $\chi = 0$, this will not give any ghost since χ transforms as a shift under Weyl. After choosing this particular gauge we return to the quadratic action 3.1.20 and arrive to an important result:

$$\left.\begin{array}{l} S_{DG}[\bar{\psi}, \bar{g}_{\mu\nu}] \\ \Gamma_{DG}[\bar{\psi}, \bar{g}_{\mu\nu}] \end{array}\right\} \xRightarrow{\bar{\psi}=\sqrt{12 Z_N}} \left\{\begin{array}{l} S_{EG}[\bar{g}_{\mu\nu}] \\ \Gamma_{EG}[\bar{g}_{\mu\nu}] \end{array}\right. \tag{3.1.44}$$

In particular the divergent part of the effective action of DG is given again by equation 1.0.28.

3.1.1 Remark on anomalies

The dilaton and the metric field have been quantized employing the background field method and the resulting effective action is invariant under background diffeomorphisms and Weyl, i.e. is not anomalous. If we add matter fields, such as in 2.1.19 and 2.1.22, we choose the measure such that

$$\begin{aligned} 0 = \delta_\omega\Gamma &= \int \mathrm{d}^4x\,\left[\frac{\delta\Gamma}{\delta\phi}\delta_\omega\phi + \frac{\delta\Gamma}{\delta\chi}\delta_\omega\chi + \frac{\delta\Gamma}{\delta g_{\mu\nu}}\delta_\omega g_{\mu\nu}\right] \\ &= \int \mathrm{d}^4x\,\sqrt{g}\,\left[\omega\phi E^\phi + \omega g^{\mu\nu}\langle T_{\mu\nu}\rangle\right] \end{aligned} \tag{3.1.45}$$

The second line is on-shell. E^ϕ is just the expression given by 2.0.4 since the actual equation of motion must take into account contributions from the matter action. Is clear that due to the presence of the dilaton Weyl is not anomalous.

3.2 Minisuperspace DG

The metric of a universe that is homogeneous and isotropic in space is given by:

$$\mathrm{d}s^2 = \sigma^2\left\{-N^2(t)\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right]\right\} \tag{3.2.1}$$

where σ normalization factor with energy dimension -1, therefore the coordinates and κ are dimensionless. The lapse and scale factor transform under Weyl as

$$
\begin{aligned}
N &\rightarrow \Omega N & (3.2.2)\\
a &\rightarrow \Omega a & (3.2.3)
\end{aligned}
$$

Recall that $N = 1$ imply that t corresponds to the proper time in the each point on hypersurface. The geometrical objects of interest are

$$
\begin{aligned}
\sqrt{-\det g} &= \sigma^4 N a^3 \frac{r^2}{\sqrt{1-\kappa r^2}} \sin\theta & (3.2.4)\\
\mathcal{R} &= \frac{6}{\sigma^2}\left[\frac{1}{N^2}\frac{\ddot{a}}{a} + \frac{1}{N^2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} - \frac{\dot{a}}{a}\frac{\dot{N}}{N^3}\right] & (3.2.5)
\end{aligned}
$$

In $d = 4$ the action for DG with $\phi = \phi(t)$ [2], after some temporal integration by parts [3], becomes

$$
S[\phi, a, N] = 6 Z_N \mathcal{V} \sigma^2 \int \mathrm{d}t N\left[-\frac{\dot{a}^2 a \phi^2}{N^2} - \frac{2\phi\dot{\phi}\dot{a}a^2}{N^2} + \kappa a \phi^2 - \frac{1}{N^2} a^3 \dot{\phi}^2 + \frac{\sigma^2 \alpha}{6} a^3 \phi^4\right] \tag{3.2.6}
$$

with

$$
\mathcal{V} = 2\pi\left[\frac{\arcsin\left(\sqrt{\kappa}L\right)}{\kappa^{\frac{3}{2}}} - \frac{L}{\kappa}\sqrt{1-\kappa L^2}\right] \tag{3.2.7}
$$

For the choice $\sigma^2 = \frac{1}{Z_N \mathcal{V}}$ we define the dimensionless parameter $\tilde{\alpha} = \frac{1}{144}\sigma^2\alpha$ and also we rescale the scalar field $\phi \rightarrow \sqrt{12}\phi$, then the action in minisuperspace is given by

$$
S[\phi, a, N] = \int \mathrm{d}t N\left[-\frac{\dot{a}^2 a \phi^2}{2N^2} - \frac{\phi\dot{\phi}\dot{a}a^2}{N^2} + \kappa a \phi^2 - \frac{1}{2N^2} a^3 \dot{\phi}^2 + \tilde{\alpha} a^3 \phi^4\right] \tag{3.2.8}
$$

Notice that it can be rewritten as

$$
S[\phi, a, N] = \int \mathrm{d}t\left[-\frac{a^3\phi^2}{2N}\left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{\phi}\right)^2 - N V(a, \phi)\right] \tag{3.2.9}
$$

with $V(a, \phi) = -\kappa a \phi^2 - \tilde{\alpha} a^3 \phi^4$. Consider the following field redefinitions

$$
\begin{aligned}
a &= \mathrm{e}^{\tau} & (3.2.10)\\
\phi &= \mathrm{e}^{\chi} & (3.2.11)
\end{aligned}
$$

[2] The field must be homogeneous in order to respect the symmetries

[3] We stress that the boundary terms are not just simply dropped, they are not present due to the cancellation with the Gibbons-Hawking-York action.

Then

$$S[\tau,\chi,N]=\int \mathrm{d}t\left[-\frac{\mathrm{e}^{3\tau+2\chi}}{2N}\,(\dot{\tau}+\dot{\chi})^2-NV(\tau,\chi)\right] \tag{3.2.12}$$

We again redefine these fields as

$$\begin{aligned}\vartheta &= 3\tau+2\chi &(3.2.13)\\ \varphi &= \tau+\chi &(3.2.14)\end{aligned}$$

The action finally becomes

$$S[\varphi,\vartheta,N]=\int \mathrm{d}t\left[-\frac{\mathrm{e}^{\vartheta}}{2N}\dot{\varphi}^2-N\mathrm{e}^{-\vartheta}V(\varphi)\right],\quad V(\varphi)=-\kappa\mathrm{e}^{-4\varphi}-\tilde{\alpha}\mathrm{e}^{6\varphi} \tag{3.2.15}$$

Notice that under Weyl transformations parametrized by $\Omega=\mathrm{e}^{\sigma}$, φ remains invariant and $\vartheta\to\vartheta+\sigma$, i.e. it transform as a shift. Since the combinations $N\mathrm{e}^{-\vartheta}$, e^{ϑ}/N are Weyl invariant, so is the action. The canonical momenta are

$$\begin{aligned}P_\varphi &= -\frac{\mathrm{e}^{\vartheta}}{N}\dot{\varphi} &(3.2.16)\\ P_\vartheta &= 0 &(3.2.17)\\ P_N &= 0 &(3.2.18)\end{aligned}$$

we find two primary constraints P_ϑ, P_N. We identify P_ϑ as the generator of Weyl transformations. The Hamiltonian becomes

$$H=N\mathrm{e}^{-\vartheta}\left(-\frac{1}{2}P_\varphi^2+V(\varphi)\right)+\zeta_N P_N+\zeta_\vartheta P_\vartheta \tag{3.2.19}$$

Consistency condition for P_N and P_ϑ gives the secondary constraint

$$H_0=\mathrm{e}^{-\vartheta}\left(-\frac{1}{2}P_\varphi^2+V(\varphi)\right) \tag{3.2.20}$$

Before continuing to the quantization procedure let us study the EG gauge defined by:

$$C=3\varphi-\vartheta \tag{3.2.21}$$

Then

$$H_0^{EG}=\mathrm{e}^{-3\varphi}\left(-\frac{1}{2}P_\varphi^2+V(\varphi)\right) \tag{3.2.22}$$

The quantization follows by promoting the canonical variables to hermitian operators:

$$\begin{aligned}\varphi &\to \hat{\varphi} &(3.2.23)\\ P_\varphi &\to \frac{1}{i}\frac{\partial}{\partial\varphi} &(3.2.24)\\ \vartheta &\to \hat{\vartheta} &(3.2.25)\end{aligned}$$

Then immediately we can see that the ordering issues in EG 3.2.22 are avoided in DG 3.2.20 since $[\hat{\varphi}, \hat{\vartheta}] = 0$. For DG the Wheeler-De Witt equation is

$$\mathrm{e}^{-\vartheta}\left(\frac{1}{2}\frac{\partial^2}{\partial\varphi^2} + V(\varphi)\right)\psi(\varphi,\vartheta) = 0 \tag{3.2.26}$$

For the ansatz $\psi(\varphi,\vartheta) = \psi(\vartheta)\psi(\varphi)$ the equation is separable and we see that Wheeler-De Witt equation only restricts the the physical wave function $\psi(\varphi)$ of the only physical degree of freedom.

Chapter 4

Classical and quantum aspects of UG

4.1 Special diffeomorphism *SDiff*

Let us consider an infinitesimal diffeomorphism generated by the vector field $\epsilon^\mu(x)$: $x^\mu \to x^\mu - \epsilon^\mu(x)$. Then

$$\delta_\epsilon \sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}g^{\mu\nu}\delta g_{\mu\nu} = \sqrt{-\det g}\nabla_\mu \epsilon^\mu \tag{4.1.1}$$

SDiff's are generated by divergenceless vector fields, i.e.

$$\nabla_\mu \epsilon^\mu = 0, \quad (g^{\mu\nu}\delta g_{\mu\nu} = 0) \tag{4.1.2}$$

Hence the determinant of the metric remains invariant. This can be though as an analogous case of the groups $O(n)$ and $SO(n)$. *SDiff*'s have the following properties:

- There is not distinction between tensors and tensor densities. As an example let $a(x)$ be a scalar density field of weight 2, under a diffeomorphism it transforms as

$$a'(x') = \left|\frac{\partial x^\mu}{\partial x'^\nu}\right|^2 a(x) \tag{4.1.3}$$

  For an infinitesimal diffeomorphism one finds that [1]

$$\pounds_\epsilon a = \epsilon^\mu \partial_\mu a + 2a\partial_\mu \epsilon^\mu = \epsilon^\mu \nabla_\mu a + 2a\nabla_\mu \epsilon^\mu = \epsilon^\mu \nabla_\mu a \tag{4.1.4}$$

- From the volume form

$$\mathrm{Vol}_d = \underbrace{\sqrt{-\det g}}_{\substack{\text{Covariant under } \textit{Diff}'\text{s.}\\ \text{Invariant under } \textit{SDiff}'\text{s}}} \quad dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{d-1}$$

[1]Recall that $\nabla_\mu a = \partial_\mu a - 2\left(\partial_\mu \ln\sqrt{|g|}\right) a$

We consider the exponential splitting of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\rho} \left(e^h\right)^\rho_{\ \nu} \tag{4.1.5}$$

The form result

$$\mathrm{Vol}_d = \sqrt{-\det \bar{g}} e^{\mathrm{tr}h} dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{d-1} \tag{4.1.6}$$

Therefore for *SDiff* we set $\mathrm{tr}h = 0$.

4.2 UG as a *SDiff* invariant theory

Consider 1.0.2 with matter

$$S[g_{\mu\nu}] = Z_N \int \mathrm{d}^d x \sqrt{-g}\, \mathcal{R} + S_m \tag{4.2.1}$$

where S_m corresponds to the matter action. From the invariance of the action under general *Diff*'s we find

$$\nabla^\mu G_{\mu\nu} = \frac{1}{2Z_N} \nabla^\mu \theta_{\mu\nu} \tag{4.2.2}$$

where

$$\theta_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{4.2.3}$$

Since the Einstein tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu}$ is covariantly conserved due to the contracted Bianchi identity: $\nabla^\mu G_{\mu\nu} = 0$, this imply that also the energy-momentum tensor $\theta_{\mu\nu}$ is conserved. If now we demand invariance of the action under *SDiff*'s we obtain

$$\nabla^\mu \left(Z_N \mathcal{R}_{\mu\nu} - \frac{1}{2}\theta_{\mu\nu} \right) = 0 \tag{4.2.4}$$

We notice that the energy-momentum tensor is not immediately conserved. The equations of motion for a general variation that leaves invariant the determinant of the metric is on the form

$$\mathcal{R}_{\mu\nu} - \frac{1}{d} g_{\mu\nu} \mathcal{R} = \frac{1}{2Z_N} \left(\theta_{\mu\nu} - \frac{1}{d} g_{\mu\nu} \theta \right) \tag{4.2.5}$$

Notice that the right hand side is invariant under

$$\theta_{\mu\nu} \to \theta_{\mu\nu} + \vartheta g_{\mu\nu} \tag{4.2.6}$$

We choose ϑ such that the improved energy momentum tensor $T_{\mu\nu}$ be covariantly conserved:

$$T_{\mu\nu} = \theta_{\mu\nu} + \vartheta g_{\mu\nu}; \quad \nabla^\mu T_{\mu\nu} = 0 \tag{4.2.7}$$

As an example consider the energy-momentum tensor of a free massless scalar field:

$$\theta_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi \tag{4.2.8}$$

we can improve it by taking $\vartheta = -\frac{1}{2}\nabla_\alpha \phi \nabla^\alpha \phi$. This example should make it clear that the improvement consists of adding always a term where ϑ is the classical matter Lagrangian. This is precisely the term that would come in the definition of $T_{\mu\nu}$ from the variation of $\sqrt{-g}$ in the *Diff*-invariant formulation of the theory, but is not there in UG, because $\sqrt{-g}$ is not varied. By simply adding such a term to the energy-momentum tensor one is guaranteed to obtain an energy-momentum tensor that is symmetric and conserved.

Taking the covariant derivative of (4.2.5) (with the improved energy momentum tensor) and using the twice contracted Bianchi identity one recover the Einstein field equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + \Lambda g_{\mu\nu} = \frac{1}{2Z_N} T_{\mu\nu} \tag{4.2.9}$$

where the cosmological constant Λ corresponds to an integration constant instead of a coupling in the Lagrangian. Therefore the action of UG is defined as

$$S[g_{\mu\nu}] = Z_N \int \mathrm{d}^d x \, \omega \, \mathcal{R} \tag{4.2.10}$$

with ω as a scalar density of weight one. There exist local coordinate systems where ω is constant and can be set to unity, in this case the metric would be unimodular in the proper sense of the word.

Following [55] the generalization to an arbitrary Lagrangian $\mathcal{L} = \mathcal{L}(g)$ is straightforward. First consider the action

$$S[g_{\mu\nu}] = \int \mathrm{d}^d x \, \sqrt{-g} \, \mathcal{L}(g) \tag{4.2.11}$$

The variation under *Diff*'s result

$$\delta_\epsilon S[g_{\mu\nu}] = \int \mathrm{d}^d x \, \sqrt{-g} \left[\nabla^\mu \left(\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{L} \right) \right] \epsilon^\nu \tag{4.2.12}$$

From the invariance of the action we obtain the "generalized" Bianchi identity

$$\nabla^\mu \left(\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{L} \right) = 0 \tag{4.2.13}$$

Now let us consider

$$S[g_{\mu\nu}] = \int \mathrm{d}^d x \, \omega \, \mathcal{L}(g) + S_m \tag{4.2.14}$$

where S_m corresponds to the matter action. Varying this expression, with the determinant of the metric fixed, one obtains the equation of motion for the metric

$$\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - \frac{1}{d} g_{\mu\nu} g^{\alpha\beta} \frac{\delta \mathcal{L}}{\delta g^{\alpha\beta}} = \frac{1}{2}\left(\theta_{\mu\nu} - \frac{1}{d} g_{\mu\nu}\theta\right) \tag{4.2.15}$$

By the same argument as in equations 4.2.6,4.2.7, the energy-momentum tensor $\theta_{\mu\nu}$ can be replaced by the improved energy-momentum tensor $T_{\mu\nu}$ in the right hand side. Taking the divergence of equation 4.2.15 and using 4.2.13 one finds

$$\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu}\mathcal{L} + \Lambda g_{\mu\nu} = \frac{1}{2} T_{\mu\nu} \tag{4.2.16}$$

We thus see that the equation of motion derived from the unimodular theory is the same as the one coming from the full theory, up to an arbitraty term proportional to the metric, which must be determined from the boundary conditions.

4.3 Hamiltonian analysis of UG

In the ADM parametrization we have

$$\sqrt{-g} = N\sqrt{q} \tag{4.3.1}$$

Following [48] UG is obtained from EG by fixing the lapse

$$N = \frac{\omega}{\sqrt{q}} \tag{4.3.2}$$

The primary Hamiltonian becomes

$$H = H_{\perp}\left[\frac{\omega}{\sqrt{q}}\right] + H_{\parallel}[\mathbf{N}] + C_2[\vec{\zeta}] \tag{4.3.3}$$

with

$$C_2[\vec{\zeta}] = \int \mathrm{d}^{d-1}x\, \delta^{ij}\zeta_i (P_N)_j \tag{4.3.4}$$

$$H_{\perp}\left[\frac{\omega}{\sqrt{q}}\right] = \int \mathrm{d}^{d-1}x\, \frac{\omega}{\sqrt{q}}\left[\frac{1}{Z_N\sqrt{q}}\left(P_{qij}P_q^{ij} - \frac{P_q^2}{(d-2)}\right)\right.$$
$$\left. - Z_N\sqrt{q}R(q)\right] \tag{4.3.5}$$

$$H_{\parallel}[\mathbf{N}] = \int \mathrm{d}^{d-1}x\, P_q^{ij}\,\pounds_{\mathbf{N}} q_{ij} \tag{4.3.6}$$

where C_2 is the only primary (smear) constraint and applying the consistency condition one obtain the secondary (smear) constraint $H_{\parallel}[\mathbf{N}]$. Consistency condition for the secondary constraint imply the tertiary constraint

$$C_3 = \int \mathrm{d}^{d-1}x\, N^k D_k\left[\frac{1}{Z_N\sqrt{q}}\left(P_{qij}P_q^{ij} - \frac{P_q^2}{(d-2)}\right) - Z_N\sqrt{q}R(q)\right] \tag{4.3.7}$$

The term in the bracket must be proportional to the metric. This is equivalent to the constraint

$$\begin{aligned} H_\perp^\Lambda\left[\frac{\omega}{\sqrt{q}}\right] &= \int \mathrm{d}^{d-1}x\,\frac{\omega}{\sqrt{q}}\left[\frac{1}{Z_N\sqrt{q}}\left(P_{qij}P_q^{ij}-\frac{P_q^2}{(d-2)}\right)\right. \\ &\quad \left. -\left(Z_N\sqrt{q}R(q)-2\Lambda\right)\right] \end{aligned} \tag{4.3.8}$$

with Λ as an integration constant. We can write the above constraint as

$$H_\perp^{Total}\left[N\right] = H_\perp\left[N\right] + \Delta H_\perp\left[N\right], \quad N=\frac{\omega}{\sqrt{q}} \tag{4.3.9}$$

with

$$\Delta H_\perp\left[N\right] = -\int \mathrm{d}^{d-1}x\, N2\Lambda \tag{4.3.10}$$

Then by the same argument used in the Hamiltonian analysis for DG this deformation does not involve the momenta conjugate to the metric, therefore the constraints satisfy the deformation algebra given by 1.0.9,1.0.10, 1.0.11 and no further constraint exist.

4.4 Path integral of UG

We need to expand 4.2.10 up to second order in the fluctuation and since we want that the background be *SDiff* invariant we choose the exponential parametrization of the metric, given in 4.1.5 with $\mathrm{tr}h=0$. The (Euclidean) quadratic action is obtain by substituting 4.1.5 into 1.0.2 and setting $\mathrm{tr}h=0$.

$$S^{(2)}[h_{\mu\nu}] = -\frac{Z_N}{2}\int \mathrm{d}^dx\,\omega\left[\frac{1}{2}h_{T\mu\nu}\bar{\nabla}^2h_T^{\mu\nu} - h_{\mu\nu}^T\bar{\nabla}^\mu\bar{\nabla}_\rho h_T^{\rho\nu} + \bar{\mathcal{R}}_{\alpha\beta\gamma\delta}h_T^{\alpha\gamma}h_T^{\beta\delta}\right] \tag{4.4.1}$$

where T indicates that the fluctuation is traceless. We can rewrite the quadratic action by introducing the Lichnerowicz Laplacians. They are defined for scalar field ϕ, vector field A_μ and symmetric tensor $h_{\mu\nu}$ as

$$\Delta_{L0}\phi = -\bar{\nabla}^2\phi \tag{4.4.2}$$
$$\Delta_{L1}A_\mu = -\bar{\nabla}^2A_\mu + \bar{\mathcal{R}}_\mu{}^\rho A_\rho \tag{4.4.3}$$
$$\Delta_{L2}h_{\mu\nu} = -\bar{\nabla}^2h_{\mu\nu} + \bar{\mathcal{R}}_\mu{}^\rho h_{\rho\nu} + \bar{\mathcal{R}}_\nu{}^\rho h_{\mu\rho} - \bar{\mathcal{R}}_{\mu\rho\nu\sigma}h^{\rho\sigma} - \bar{\mathcal{R}}_{\mu\rho\nu\sigma}h^{\sigma\rho} \tag{4.4.4}$$

Using the equation of motion of the background $\bar{\mathcal{R}}_{\mu\nu}=\frac{\bar{\mathcal{R}}}{d}\bar{g}_{\mu\nu}$, the Lichnerowicz Laplacians becomes

$$\Delta_{L_2}h_{\mu\nu} = -\bar{\nabla}^2h_{\mu\nu} + \frac{2}{d}\bar{\mathcal{R}}h_{\mu\nu} - 2\bar{\mathcal{R}}_{\mu\rho\nu\sigma}h^{\rho\sigma} \tag{4.4.5}$$

$$\Delta_{L_1}A_\mu = \left(-\bar{\nabla}^2 + \frac{1}{d}\bar{\mathcal{R}}\right)A_\mu \tag{4.4.6}$$

and we obtain the following properties[2]

$$\begin{aligned}
\Delta_{L_1}\bar{\nabla}_\mu\phi &= \bar{\nabla}_\mu\Delta_{L_0}\phi && (4.4.7)\\
\bar{\nabla}_\mu\Delta_{L_1}A^\mu &= \Delta_{L_0}\bar{\nabla}_\mu A^\mu && (4.4.8)\\
\Delta_{L_2}(\bar{\nabla}_\mu\bar{\nabla}_\nu)\phi &= \bar{\nabla}_\mu\bar{\nabla}_\nu\Delta_{L_0}\phi && (4.4.9)\\
\Delta_{L_2}(\bar{\nabla}_\mu A_\nu + \bar{\nabla}_\nu A_\mu) &= \bar{\nabla}_\mu\Delta_{L_1}A_\nu + \bar{\nabla}_\nu\Delta_{L_1}A_\mu && (4.4.10)\\
\Delta_{L_2}\bar{g}_{\mu\nu}\phi &= \Delta_{L_0}\bar{g}_{\mu\nu}\phi && (4.4.11)
\end{aligned}$$

The quadratic action becomes

$$S^{(2)}[\bar{g}_{\mu\nu}] = \frac{Z_N}{2}\int \mathrm{d}^dx\,\omega\left[\frac{1}{2}h^T_{\mu\nu}\left(\Delta_{L_2} - \frac{2}{d}\mathcal{R}\right)h_T^{\mu\nu} + h^T_{\mu\nu}\bar{\nabla}^\mu\bar{\nabla}_\rho h_T^{\rho\nu}\right] \quad (4.4.12)$$

We introduce the York decomposition of the fluctuation

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu + \bar{\nabla}_\mu\bar{\nabla}_\nu\sigma - \frac{1}{d}\bar{g}_{\mu\nu}\bar{\nabla}^2\sigma + \frac{\bar{g}_{\mu\nu}}{d}h \quad (4.4.13)$$

where TT stands for traceless and transverse ($\bar{\nabla}^\mu h^{TT}_{\mu\nu} = 0$), ξ_μ is transverse vector field ($\bar{\nabla}_\mu\xi^\mu = 0$) and σ is a scalar field. Using the properties 4.4.7-4.4.11 one obtain

$$\begin{aligned}
\bar{\nabla}^\rho h^T_{\rho\nu} &= -\left(\Delta_{L_1} - \frac{2}{d}\bar{\mathcal{R}}\right)\xi_\nu - \left(\frac{d-1}{d}\right)\bar{\nabla}_\nu\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma && (4.4.14)\\
h_T^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}^\rho h^T_{\rho\nu} &\approx -(\bar{\nabla}_\mu h_T^{\mu\nu})(\bar{\nabla}^\rho h^T_{\rho\nu})\\
&\approx -\xi^\mu\left(\Delta_{L_1} - \frac{2}{d}\bar{\mathcal{R}}\right)^2\xi_\mu\\
&\quad -\frac{(d-1)^2}{d^2}\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma\Delta_{L_0}\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma && (4.4.15)\\
h_T^{\mu\nu}\Delta_{L_2}h^T_{\mu\nu} &\approx h^{\mu\nu}_{TT}\Delta_{L_2}h^{TT}_{\mu\nu} + 2\xi^\mu\Delta_{L_1}\left(\Delta_{L_1} - \frac{2}{d}\bar{\mathcal{R}}\right)\xi_\mu\\
&\quad +\frac{d-1}{d}\sigma\Delta^2_{L_0}\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma && (4.4.16)\\
h_T^{\mu\nu}h^T_{\mu\nu} &\approx h^{\mu\nu}_{TT}h^{TT}_{\mu\nu} + 2\xi^\mu\left(\Delta_{L_1} - \frac{2}{d}\bar{\mathcal{R}}\right)\xi_\mu\\
&\quad +\frac{d-1}{d}\sigma\Delta_{L_0}\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma && (4.4.17)
\end{aligned}$$

where $\approx$ indicates that some partial integration has been done. The quadratic action becomes

$$\begin{aligned}
S^{(2)}[\bar{g}_{\mu\nu}] &= \frac{Z_N}{2}\int \mathrm{d}^dx\,\omega\left[\frac{1}{2}h^{TT}_{\mu\nu}\left(\Delta_{L_2} - \frac{2}{d}\bar{\mathcal{R}}\right)h_{TT}^{\mu\nu}\right.\\
&\quad\left. -\frac{(d-1)(d-2)}{2d^2}\sigma\Delta^2_{L_0}\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma\right] && (4.4.18)
\end{aligned}$$

[2]The Lichnerowicz Laplacians have these useful properties in general for Einstein spaces.

Notice that there is not a dependence on the vector field. The measure is obtained from

$$\int \mathcal{D}h^{TT}_{\mu\nu}\mathcal{D}\xi_\mu\mathcal{D}\sigma\, J\, \mathrm{e}^{-\int \mathrm{d}^d x\sqrt{\bar{g}}\, h^{\mu\nu}_T h^T_{\mu\nu}} = 1 \tag{4.4.19}$$

using 4.4.17 one finds the Jacobian

$$J = \sqrt{\det\left(\Delta_{L_1} - \frac{2}{d}\bar{\mathcal{R}}\right)}\sqrt{\det\left(\Delta_{L_0}\right)}\sqrt{\det\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)} \tag{4.4.20}$$

4.4.1 Gauge fixing condition

In order to impose a suitable gauge condition first recall that for EG one usually consider

$$S_{GF} = \frac{Z_N}{2\alpha}\int \mathrm{d}^d x\sqrt{\bar{g}}\, F_\mu \bar{g}^{\mu\nu} F_\nu \tag{4.4.21}$$

where

$$F_\mu = \bar{\nabla}_\rho h^\rho{}_\mu - \frac{\beta+1}{d}\bar{\nabla}_\mu h \tag{4.4.22}$$

and α, β as gauge parameters. Using York's decomposition and assuming a maximally symmetric background one finds

$$F_\mu = -\left(\Delta_{L_1} - \frac{2}{d}\bar{\mathcal{R}}\right)\xi_\mu - \left(\frac{d-1}{d}\right)\bar{\nabla}_\mu\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma - \frac{\beta}{d}\bar{\nabla}_\mu h \tag{4.4.23}$$

It can be writing as

$$F_\mu = F^{\mathrm{T}}_\mu + \bar{\nabla}_\mu F^{\mathrm{L}} \tag{4.4.24}$$

where

$$F^{\mathrm{T}}_\mu = -\left(\Delta_{L_1} - \frac{2}{d}\bar{\mathcal{R}}\right)\xi_\mu \tag{4.4.25}$$

$$F^{\mathrm{L}} = -\left(\frac{d-1}{d}\right)\left(\Delta_{L_0} - \frac{1}{d-1}\bar{\mathcal{R}}\right)\sigma - \frac{\beta}{d}h \tag{4.4.26}$$

Since we can decompose the vector field ϵ^μ into its longitudinal and transverse parts with respect to the background metric

$$\epsilon^\mu = \epsilon^{\mathrm{T}\mu} + \bar{\nabla}^\mu\epsilon^{\mathrm{L}}; \qquad \bar{\nabla}_\mu\epsilon^{\mathrm{T}\mu} = 0 \tag{4.4.27}$$

therefore

$$\delta\xi_\mu = \epsilon^{\mathrm{T}}_\mu \tag{4.4.28}$$

$$\delta\sigma = 2\epsilon^{\mathrm{L}} \tag{4.4.29}$$

$$\delta h = -2\Delta_{L_0}\epsilon^{\mathrm{L}} \tag{4.4.30}$$

We can see that F^{T}_μ only transform under the transverse modes and F^{L} under the longitudinal modes. It is convenient to define

$$\chi = \frac{((d-1)\Delta_{L0} - \bar{\mathcal{R}})\sigma + \beta h}{(d-1-\beta)\Delta_{L0} - \bar{\mathcal{R}}} \ , \tag{4.4.31}$$

Thus the gauge fixing condition reads

$$F_\mu = -\left(\Delta_{L1} - \frac{2}{d}\bar{\mathcal{R}}\right)\xi_\mu - \frac{d-1-\beta}{d}\nabla_\mu\left(\Delta_{L0} - \frac{1}{d-1-\beta}\bar{\mathcal{R}}\right)\chi \ , \tag{4.4.32}$$

The gauge fixing action is then equal to

$$\begin{aligned} S_{GF} &= \frac{Z_N}{2\alpha}\int d^dx\sqrt{\bar{g}}\left[\xi_\mu\left(\Delta_{L1} - \frac{2}{d}\bar{\mathcal{R}}\right)^2\xi^\mu\right. \\ &\left. + \frac{(d-1-\beta)^2}{d^2}\chi\Delta_{L0}\left(\Delta_{L0} - \frac{1}{d-1-\beta}\bar{\mathcal{R}}\right)^2\chi\right] \end{aligned} \tag{4.4.33}$$

Under the transformations 4.4.29,4.4.30 the variable χ transforms in the same way as σ. Therefore ξ and χ can be viewed as the gauge degrees of freedom. Decomposing the ghost into transverse and longitudinal parts

$$C_\nu = C^T_\nu + \nabla_\nu\frac{1}{\sqrt{-\bar{\nabla}^2}}C^L \tag{4.4.34}$$

and likewise for $\bar{C}$, the ghost action splits in two terms

$$\begin{aligned} S_{gh} &= \int d^dx\sqrt{\bar{g}}\left[\bar{C}^{T\mu}\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)C^T_\mu\right. \\ &\left. + 2\frac{d-1-\beta}{d}\bar{C}^L\left(\Delta_{L0} - \frac{\bar{R}}{d-1-\beta}\right)C^L\right] . \end{aligned} \tag{4.4.35}$$

We note that the change of variables (4.4.34) has unit Jacobian.

For *SDiff*'s we have $\Delta_{L_0}\epsilon^{\mathrm{L}} = 0$ ($h = 0$) and in order to define a suitable gauge-fixing for these transformations, let

$$L^\mu{}_\nu = \bar{\nabla}^\mu\frac{1}{\bar{\nabla}^2}\bar{\nabla}_\nu \ ; \qquad T^\mu{}_\nu = \delta^\mu_\nu - L^\mu{}_\nu \tag{4.4.36}$$

be the longitudinal and transverse projectors defined relative to the background metric. We choose our gauge-fixing function as [3]

$$F_\mu = T_{\mu\nu}\bar{\nabla}_\rho h^{\rho\nu}_T = -\left(\Delta_{L1} - \frac{2}{d}\bar{\mathcal{R}}\right)\xi_\mu \ . \tag{4.4.37}$$

[3] It may be better to have a local gauge-fixing condition. This can be achieved by inserting a power of Δ_{L1} in the gauge fixing term (4.4.38) below, see [72]. Ultimately the additional determinant is canceled by a Nielsen-Kallosh ghost term, so that the final result is the same. In order to minimize the number of determinants we stick to a non-local gauge fixing term.

The gauge-fixing term is given by

$$S_{G.F.} = \frac{Z_N}{2\alpha}\int d^dx\,\omega F_\mu T^{\mu\nu}F_\nu = \frac{Z_N}{2\alpha}\int d^dx\,\omega\,\xi_\mu\left(\Delta_{L1}-\frac{2}{d}\bar{\mathcal{R}}\right)^2\xi^\mu\ , \tag{4.4.38}$$

and the ghost action

$$S_{ghost} = \int d^dx\,\omega\,\bar{C}^T_\mu\left(\Delta_{L1}-\frac{2}{d}\bar{\mathcal{R}}\right)C^{\mu T}\ , \tag{4.4.39}$$

where ghost and antighost fields are transverse vectors.

4.4.2 Effective action

The path integral result

$$\begin{aligned} Z &= \int \mathcal{D}h^{TT}_{\mu\nu}\mathcal{D}\xi_\mu\mathcal{D}\sigma\, J\,\mathrm{e}^{-\left(S[\bar g_{\mu\nu}]+S^{(2)}[\bar g_{\mu\nu}]+S_{G.F.}+S_{ghost}\right)} \\ &= e^{-S[\bar g]}\int\mathcal{D}\xi_\mu\frac{1}{\sqrt{\det\Delta_{L0}}^{1/2}}\frac{\sqrt{\det\left(\Delta_{L1}-\frac{2}{d}\bar{\mathcal{R}}\right)_{\mathrm{T}}}}{\sqrt{\det\left(\Delta_{L2}-\frac{2}{d}\bar{\mathcal{R}}\right)_{\mathrm{TT}}}} \end{aligned} \tag{4.4.40}$$

From 4.4.28 4.4.29 we can identify

$$\xi_\mu \leftrightarrow \epsilon^{\mathrm{T}}_\mu \tag{4.4.41}$$

$$\sigma \leftrightarrow \epsilon^{\mathrm{L}} \tag{4.4.42}$$

Then

$$Z = e^{-S[\bar g_{\mu\nu}]}\int\mathcal{D}\epsilon^{\mathrm{T}}_\mu\frac{1}{\sqrt{\det\Delta_{L0}}^{1/2}}\frac{\sqrt{\det\left(\Delta_{L1}-\frac{2}{d}\bar{\mathcal{R}}\right)_{\mathrm{T}}}}{\sqrt{\det\left(\Delta_{L2}-\frac{2}{d}\bar{\mathcal{R}}\right)_{\mathrm{TT}}}} \tag{4.4.43}$$

and we notice that scalar determinant arises from σ. Let us define the volume of *Diff*'s as $V(\mathit{Diff}) = \int\mathcal{D}\epsilon_\mu$. The measure is defined from

$$\int\mathcal{D}\epsilon_\mu\mathrm{e}^{-\int\mathrm{d}^dx\sqrt{\bar g}\epsilon_\mu\epsilon^\mu} = 1 \tag{4.4.44}$$

After the decomposition results

$$\int\mathcal{D}\epsilon^{\mathrm{T}}_\mu\mathcal{D}\epsilon^{\mathrm{L}}\,J\mathrm{e}^{-\int\mathrm{d}^dx\sqrt{\bar g}\left[\epsilon_{\mathrm{T}\mu}\epsilon^{\mathrm{T}\mu}+\epsilon^{\mathrm{L}}\left(-\bar\nabla^2\right)\epsilon^{\mathrm{L}}\right]} = 1 \tag{4.4.45}$$

where J corresponds to the Jacobian and is given by $J = \sqrt{\det\left(-\bar\nabla^2\right)}$. The volume becomes

$$V(\mathit{Diff}) = \int\mathcal{D}\epsilon^{\mathrm{T}}_\mu\mathcal{D}\epsilon^{\mathrm{L}}\sqrt{\det\left(-\bar\nabla^2\right)} \tag{4.4.46}$$

For *SDiff*'s the divergenceless condition is implemented via a Dirac delta:

$$\begin{aligned}
V(SDiff) &= \int \mathcal{D}\epsilon_\mu \delta(-\nabla_\mu \epsilon^\mu) \\
&= \int \mathcal{D}\epsilon_\mu^{\mathrm{T}} \mathcal{D}\epsilon^{\mathrm{L}} \sqrt{\det\left(-\bar{\nabla}^2\right)}\delta(-\nabla^2 \epsilon^L) \\
&= \int \mathcal{D}\epsilon_\mu^{\mathrm{T}} \mathcal{D}\epsilon^{\mathrm{L}} \frac{\delta(\epsilon^L)}{\sqrt{\det\left(-\bar{\nabla}^2\right)}} \\
&= \int \mathcal{D}\epsilon_\mu^{\mathrm{T}} \frac{1}{\sqrt{\det\left(-\bar{\nabla}^2\right)}}
\end{aligned} \tag{4.4.47}$$

One can see that

$$\frac{V(Diff)}{V(SDiff)} = \int \mathcal{D}\epsilon^{\mathrm{L}} \det\left(-\bar{\nabla}^2\right) \tag{4.4.48}$$

If we consider quotient space $Q = \frac{Diff}{SDiff}$, the volume of this space must be $V(Q) = \int \mathcal{D}\epsilon^{\mathrm{L}} \det\left(-\bar{\nabla}^2\right)$. We arrive to an important result

$$V(Diff) = V(SDiff)V(Q) \tag{4.4.49}$$

The quotient space can be identified with the space of volume-forms. We demand that the measure on the quotient space agrees with the measure on the volume forms. An infinitesimal change of volume form is a trace deformation of the metric. We can see this by considering the infinitesimal variation of trace of $h \equiv \bar{g}^{\mu\nu}h_{\mu\nu}$ under diffeomorphism

$$\delta h = 2\bar{\nabla}^2 \epsilon^{\mathrm{L}} \tag{4.4.50}$$

One can identify

$$h \leftrightarrow -\bar{\nabla}^2 \epsilon^{\mathrm{L}} \tag{4.4.51}$$

and find

$$V(Q) = \int \mathcal{D}\epsilon^{\mathrm{L}} \det\left(-\bar{\nabla}^2\right) = \int \mathcal{D}h \tag{4.4.52}$$

Therefore by the definition of the volume group for *SDiff*'s given by 4.4.47, we obtain

$$Z = e^{-S[\bar{g}_{\mu\nu}]}\, V(SDiff) \frac{\sqrt{\det\left(\Delta_{L1} - \frac{2}{d}\bar{\mathcal{R}}\right)_{\mathrm{T}}}}{\sqrt{\det\left(\Delta_{L2} - \frac{2}{d}\bar{\mathcal{R}}\right)_{\mathrm{TT}}}} \tag{4.4.53}$$

After dropping this volume, the effective action result

$$\Gamma[\bar{g}_{\mu\nu}] = S[\bar{g}_{\mu\nu}] + \frac{1}{2}\mathrm{tr}\log\left(\frac{\Delta_{L2} - \frac{2}{d}\bar{\mathcal{R}}}{\mu^2}\right)_{\mathrm{TT}} - \mathrm{tr}\log\left(\frac{\Delta_{L1} - \frac{2}{d}\bar{\mathcal{R}}}{\mu^2}\right)_{\mathrm{T}} \tag{4.4.54}$$

Employing the heat kernel technique we need to compute

$$B_{2n}(\mathcal{O}) = \int \mathrm{d}^d x\, \omega\, b_{2n}(\mathcal{O}) \tag{4.4.55}$$

Hence we use the following formulas

$$
\begin{aligned}
b_0(\Delta_{L_0}) &= 1 && (4.4.56)\\
b_2(\Delta_{L_0}) &= \frac{1}{6}\bar{\mathcal{R}} && (4.4.57)\\
b_4(\Delta_{L_0}) &= \frac{1}{180}\bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma} + \frac{5d-2}{360d}\bar{\mathcal{R}}^2 && (4.4.58)\\
b_0\left(\Delta_{L_1}\right)_T &= d-1 && (4.4.59)\\
b_2\left(\Delta_{L_1}\right)_T &= \frac{d-7}{6}\bar{\mathcal{R}} && (4.4.60)\\
b_4\left(\Delta_{L_1}\right)_T &= \frac{d-16}{180}\bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma} + \frac{5d^2-67d+182}{360d}\bar{\mathcal{R}}^2 && (4.4.61)\\
b_0\left(\Delta_{L_2}\right)_{TT} &= \frac{(d+1)(d-2)}{2} && (4.4.62)\\
b_2\left(\Delta_{L_2}\right)_{TT} &= \frac{d^2-13d-14}{12}\bar{\mathcal{R}} && (4.4.63)\\
b_4\left(\Delta_{L_2}\right)_{TT} &= \frac{d^2-31d+508}{360}\bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma}\\
&\quad + \frac{5d^3-127d^2+592d+1804}{720d}\bar{\mathcal{R}}^2 && (4.4.64)
\end{aligned}
$$

and for a Lichnerowicz Laplacian on the form $\Delta_L + a\bar{\mathcal{R}}$

$$
\begin{aligned}
b_0(\Delta_L + a\bar{\mathcal{R}}) &= b_0(\Delta_L) && (4.4.65)\\
b_2(\Delta_L + a\bar{\mathcal{R}}) &= b_2(\Delta_L) - a\bar{\mathcal{R}}b_0(\Delta_L) && (4.4.66)\\
b_4(\Delta_L + a\bar{\mathcal{R}}) &= b_4(\Delta_L) - a\bar{\mathcal{R}}b_2(\Delta_L) + \frac{1}{2}a^2\bar{\mathcal{R}}^2 b_0(\Delta_L) && (4.4.67)
\end{aligned}
$$

In $d = 4$ we compute the divergent part of the effective action by dimensional regularization and as we learn in Chapter 2 only the B_4 coefficients are needed:

$$
\begin{aligned}
\Gamma_{div.} &= -\frac{1}{(4\pi)^2\epsilon}\left[B_4\left(\Delta_{L2} - \frac{1}{2}\bar{\mathcal{R}}\right)_{\mathrm{TT}} - B_4\left(\Delta_{L1} - \frac{1}{2}\bar{\mathcal{R}}\right)_{\mathrm{T}}\right] && (4.4.68)\\
&= -\frac{1}{(4\pi)^2\epsilon}\int \mathrm{d}^4x\,\omega\left[\frac{53}{45}\bar{\mathcal{R}}_{\mu\nu\rho\sigma}\bar{\mathcal{R}}^{\mu\nu\rho\sigma} - \frac{29}{40}\bar{\mathcal{R}}^2\right] && (4.4.69)
\end{aligned}
$$

The final result is the same as in equation 1.0.33 for $\Lambda = \frac{\bar{\mathcal{R}}}{4}$. Hence we conclude that

$$\Gamma_{UG}[\bar{g}_{\mu\nu}] = \Gamma_{EG}[\bar{g}_{\mu\nu}]\big|_{\sqrt{\bar{g}}=\omega} \tag{4.4.70}$$

In Appendix E the equivalence of the path integral of EG and UG is also discussed at the linearized order using Hamiltonian formalism.

4.5 Minisuperspace UG

Starting from the metric given by 3.2.1 with $\kappa = 0$ we have

$$\omega = \sigma^4 N a^3 \tag{4.5.1}$$

and we choose $N = \frac{1}{a^3}$, then the action for UG becomes

$$S[a] = \int \mathrm{d}t - \frac{1}{2} a^4 \dot{a}^2 \tag{4.5.2}$$

for the choice $\sigma^2 = \frac{1}{24 Z_N \mathcal{V}}$. We consider the field redefinition $a = \mathrm{e}^{\varphi}$, the action becomes

$$S[\varphi] = \int \mathrm{d}t - \frac{1}{2} \mathrm{e}^{6\varphi} \dot{\varphi}^2 \tag{4.5.3}$$

The canonical momenta is

$$P_\varphi = -\mathrm{e}^{6\varphi} \dot{\varphi} \tag{4.5.4}$$

The Hamiltonain becomes

$$H^{UG} = -\frac{\mathrm{e}^{-6\varphi}}{2} P_\varphi^2 \tag{4.5.5}$$

Comparing with 3.2.22 we see that the above hamiltonian is the same as in EG with the special choice of the shift and without the cosmological constant. Using the equations of motion of UG

$$\dot{\varphi} = \frac{2}{P_\varphi} H^{UG} \tag{4.5.6}$$

$$\dot{P}_\varphi = 6 H^{UG} \tag{4.5.7}$$

It is easy so see that $\frac{\mathrm{d}H^{UG}}{\mathrm{d}t} = 0$, then we find the constraint

$$H_\Lambda = -\Lambda \tag{4.5.8}$$

with the cosmological constant as an integration constant. We can write

$$H_\Lambda^{UG} = -\frac{\mathrm{e}^{-6\varphi}}{2} P_\varphi^2 + H_\Lambda \tag{4.5.9}$$

The Wheeler-De Witt equation with a specific ordering becomes

$$-\frac{\mathrm{e}^{-6\varphi}}{2} \hat{P}_\varphi^2 \psi + \hat{H}_\Lambda \psi = 0 \tag{4.5.10}$$

Consider the wave function ansatz: $\psi = \Psi(\varphi)\psi_\Lambda$ with $H_\Lambda \psi_\Lambda = \Lambda \psi_\Lambda$ we obtain

$$-\frac{\mathrm{e}^{-6\varphi}}{2} \hat{P}_\varphi^2 \Psi(\varphi) = \Lambda \Psi(\varphi) \tag{4.5.11}$$

which has the form of a Schrödinger type equation. Therefore, for UG the time issue of WDW is absent.

Chapter 5

A circle of theories

"... I played with this tools as well as I could
just because it was beautiful.
In the same way a musician plays the violin,
not expecting to change the world
but just because he loves the instrument."
Freeman Dyson. Quanta Magazine.

In the previous chapters we have studied in great detail many aspects DG and UG. The counting of degrees of freedom of these theories is explicit in the Hamiltonian formalism and it is summarised in table 5.1. For $d \geq 3$ in EG there are $d(d+1)$ canonical variables and $2d$ first class constraints. Since each first class constraint has to be accompanied by a gauge condition, the total number of canonical degrees of freedom is $d(d+1) - 2 \times 2d = d(d-3)$. In $d = 4$ this agrees with the two polarization states of the graviton.

	DG	**EG**	**UG**
fields	q_{ij},$\mathbf{N}$,N,ϕ	q_{ij},$\mathbf{N}$,N	q_{ij},$\mathbf{N}$
momenta	P_q^{ij},P_N^i,P_N,P_ϕ	P_q^{ij},P_N^i,P_N	P_q^{ij},P_N^i
# of canonical variables	$d(d+1)+2$	$d(d+1)$	$d(d+1)-2$
primary constraints	$\mathcal{C}_1[\zeta]$, $\mathcal{C}_2[\vec{\zeta}]$, $\mathcal{C}_3[\theta]$	$\mathcal{C}_1[\zeta]$, $\mathcal{C}_2[\vec{\zeta}]$	$\mathcal{C}_2[\vec{\zeta}]$
secondary constraints	$H_\perp^{Total}[N]$,$H_\parallel^{Total}[\mathbf{N}]$	$H_\perp[N]$,$H_\parallel[\mathbf{N}]$	$H_\perp^\Lambda[N]$,$H_\parallel[\mathbf{N}]$
# of first class constraints	$2d+1$	$2d$	$2d-1$
# of canonical d.o.f.	$d(d-3)$	$d(d-3)$	$d(d-3)$

Table 5.1: Summary of the constraint analysis of Dirac, Einstein and unimodular gravity.

In DG one has two more canonical variables (the scalar field and its momentum). There is also one more primary constraint and the same number of secondary constraints, so the number of constraints is one higher. This, and the associated gauge condition, removes the additional variables. In

UG there are two less canonical variables, due to the condition of unimodularity of the spacetime metric, which we use to eliminate the lapse and the associated momentum. There is then one less primary constraint than in EG, because the momentum conjugate to the lapse is not a canonical variable. There is also one less secondary constraint, but then there is a tertiary constraint. The fact that there is one less constraint is related to the fact that the gauge group *SDiff* has one less free parameter. Altogether, the constraints and their gauge condition remove two variables less than in EG, so the final number of degrees of freedom is the same.

We wish as well to have a similar discussion at the Lagrangian level with the symmetry groups. For that reason we focus on DG since it has a larger symmetry group: *Diff* $\ltimes$ Weyl. The metric and the unphysical field transform under *Diff*'s as

$$\delta_\epsilon g_{\mu\nu} = \pounds_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu, \qquad \delta_\epsilon \psi = \pounds_\epsilon \psi = \epsilon^\mu \partial_\mu \psi \tag{5.0.1}$$

and under Weyl as

$$\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}, \qquad \delta_\sigma \psi = -\frac{(d-2)}{2}\sigma\psi \tag{5.0.2}$$

where ϵ and σ are the generators and from the properties of the Lie derivative it follows that

$$[\delta_\xi, \delta_\epsilon] = \delta_{[\xi,\epsilon]} \tag{5.0.3}$$

and the Lie algebra of the diffeomorphism group is given by the Lie algebra of vector fields. We can choose σ in such a way that $\psi = \psi_0$, i.e. a constant, and reduce the symmetry group to *Diff*, a.k.a. the EG gauge. Since every gauge theory has infinitely many gauge fixings, we think of DG as a linking theory.

Consider the following "Weyl-compensated" diffeomorphism

$$\delta_\epsilon g_{\mu\nu} = \pounds_\epsilon g_{\mu\nu} + \beta_\epsilon g_{\mu\nu}, \qquad \delta_\epsilon \psi = \pounds_\epsilon \psi = \epsilon^\mu \partial_\mu \psi - \frac{(d-2)}{4}\beta_\epsilon \psi \tag{5.0.4}$$

where β_ϵ is an unknown function. We see that

$$g^{\mu\nu}\delta_\epsilon g_{\mu\nu} = 2\nabla_\mu \epsilon^\mu + d\beta_\epsilon \tag{5.0.5}$$

Then for the choice $\beta_\epsilon = -\frac{2}{d}\nabla_\mu \epsilon^\mu$ the the determinant remains invariant. Moreover it is now easy to check explicitly that

$$[\delta_\xi, \delta_\epsilon] g_{\mu\nu} = \pounds_{[\xi,\epsilon]} g_{\mu\nu} - \frac{2}{d} g_{\mu\nu} \nabla_\rho [\xi, \epsilon]^\rho\,, \quad [\delta_\xi, \delta_\epsilon]\psi = [\xi, \epsilon]^\mu \partial_\mu \psi + \frac{(d-2)}{2d}\psi \nabla_\rho [\xi, \epsilon]^\rho\,. \tag{5.0.6}$$

Hence we have, again, $[\delta_\xi, \delta_\epsilon] = \delta_{[\xi,\epsilon]}$, the Lie algebra of *Diff*. In order to clarify that this group acts differently on fields than usual diffeomorphisms, we call the group generated by *Diff**. This theory can be realized by simply fixing the determinant $\sqrt{-\det g} = \omega$ of DG, it is referred to as unimodular DG, UD in short. At this point we can close the circle of theories by noting that the formulation of UG can be obtained from UD by fixing the scalar $\psi = \psi_0$. This fixes the Weyl invariance leaving just the group *SDiff*.

On the other hand if we restrict ourselves to $\nabla_\mu \epsilon^\mu = 0$ the group becomes *SDiff* $\ltimes$ Weyl. This gauge can be realized by the particular choice of the scalar

$$\psi = \psi_0 \left(\frac{-\det g}{\omega} \right)^{\frac{2-d}{4d}} , \qquad \psi_0 = \left[8 \left(\frac{d-1}{d-2} \right) Z_N \right]^{\frac{1}{4}} \tag{5.0.7}$$

This leads to the following action [73], [74], [75], [76]

$$S[g] = Z_N \int \mathrm{d}^d x \, (-g)^{\frac{1}{d}} \, \omega^{\frac{d-2}{d}} \left[\mathcal{R} + \frac{(d-1)(d-2)}{4d^2} \left((-g)^{-1} \nabla(-g) - 2\omega^{-1} \nabla \omega \right)^2 \right] . \tag{5.0.8}$$

This action is invariant under *SDiff* $\ltimes$ Weyl, hence this theory is referred as *WTDiff* (*TDiff* being synonymous to *SDiff*). The invariance under Weyl can be read easly if one consider the re-scaled metric:

$$\tilde{g}_{\mu\nu} = \left(\frac{\psi}{\psi_0} \right)^{\frac{4}{d-2}} g_{\mu\nu} \tag{5.0.9}$$

$$= \left(\frac{-\det g}{\omega} \right)^{-\frac{1}{d}} g_{\mu\nu} \tag{5.0.10}$$

From the right hand side of 5.0.9 we see that the metric $\tilde{g}_{\mu\nu}$ is Weyl invariant and from 5.0.10 that is unimodular. The action 5.0.8 can be obtained from the EG action by setting $S_{WTDiff}[g] = S_{EG}[\tilde{g}]$. This is also can be viewed as a form of unimodular gravity: instead of removing the determinant by a constraint on the metric, it is removed by making the action independent of it.

The above discussion of circle of theories is nicely summarized in figure 5.1. We refer to the transitions between EG, UD and *WTDiff* gravity as "symmetry trading". In order to clarify this point, let us compare EG and *WTDiff* gravity: they are both formulations of general relativity in terms of a Lorentzian metric, but with different action and different symmetries. Understanding the relation of the two can be done either via UG [77] or, as we have done here, via DG. As we have seen, EG and *WTDiff* are obtained by different choices for the scalar ψ in DG. Thus, DG provides a "linking theory" from which EG and *WTDiff* can be obtained rather straightforwardly. A linking theory, when it exists, also clarifies the global differences

$$
\begin{tikzcd}
& \begin{array}{c}\mathbf{DG}\\ (g,\psi);\ \mathit{Diff} \ltimes \mathit{Weyl}\end{array} \arrow[dl, "\psi=\psi_0"'] \arrow[d, "\psi=\psi_0\left(\frac{|g|}{\omega^2}\right)^{\frac{2-d}{4d}}"'] \arrow[dr, "\sqrt{|g|}=\omega"] & \\
\begin{array}{c}\mathbf{EG}\\ (g);\ \mathit{Diff}\end{array} \arrow[dr, "\sqrt{|g|}=\omega"'] & \begin{array}{c}\mathit{WTDiff}\\ (g);\ \mathit{SDiff} \ltimes \mathit{Weyl}\end{array} \arrow[d, "\sqrt{|g|}=\omega"'] & \begin{array}{c}\mathbf{UD}\\ (g_U,\psi);\ \mathit{Diff}^*\end{array} \arrow[dl, "\psi=\psi_0"] \\
& \begin{array}{c}\mathbf{UG}\\ (g_U);\ \mathit{SDiff}\end{array} &
\end{tikzcd}
$$

Figure 5.1: Dynamical fields and symmetry groups for different formulations of vacuum EG; g_U stands for a metric constrained to be unimodular.

between different formulations, which correspond to the failure of the respective gauge-fixing conditions for certain solutions. Symmetry trading then becomes more directly understandable in terms of the linking theory.

In order to complete the circle given in figure 5.1 at the Hamiltonian level we first study UD in this formalism. It proceeds very similarly to the case of minimal UG, with one small twist. As for minimal UG and EG, the only difference between UD and DG when deriving the Hamiltonian is that the lapse is fixed by the unimodularity condition:

$$H_p = H_\perp^{Total}\left[\frac{\omega}{\sqrt{q}}\right] + H_\parallel^{Total}[\mathbf{N}] + C_1[\zeta] + C_2[\vec{\zeta}] + C_3[\tilde{\theta}] \tag{5.0.11}$$

As expected there is no primary constraint associated to the lapse. Since

$$\left\{C_3[\tilde{\theta}], H_\perp^{Total}\left[\frac{\omega}{\sqrt{q}}\right]\right\} \sim H_\perp^{Total}\left[\frac{\tilde{\theta}\omega}{\sqrt{q}}\right] \tag{5.0.12}$$

we find a tertiary constraint. In UD, unlike in UG, there is no additional global degree of freedom corresponding to the cosmological constant, as the Weyl symmetry forbids the appearance of a new dimensionful parameter.

At the quantum level, figure 5.1 also holds. This should be expected since in the classical limit the degree's of freedom must be preserved. We have shown this explicitly for QDG, QEG and QUG. The main conclusion is there is no anomaly associated with Weyl for DG. This is in agreement with [78], since they also found no Weyl anomaly in *WTDiff*.

Which description is preferable?

Let us answer this question for minisuperspace models. From the Hamiltonians of DG, EG and UG, we learned that more symmetries imply more constraints. At the quantum level this imply that the physical wave function must satisfy this (quantum) constraints, i.e. the Hilbert space is projected to a physical Hilbert space. Since we are dealing with highly simplified models, in EG the only remaining constraint corresponds to the Hamiltonian constraint. Quantizing this constraint imply that we have to choose a specific order of the operators. In DG we have an extra constraint, the momentum of the dilaton. The quantum Hamiltonian for this case does not have the issue of ordering and since the WDW equation is separable, only the part of the wave function that depends on the physical degree of freedom is projected. On the other hand, the Hamiltonian of UG corresponds to the Hamiltonian of EG with a specific foliation, the cosmological constant as a integration constant appears as a constraint. The resulting WDW equation has the form of a Schrödinger like equation and therefore the issue of time is absent. Hence more symmetries allow us to deal with quantization issues.

Consider the case of a cosmological Friedmann-Lemaître-Robertson-Walker universe, which can be written in terms of conformal time as

$$\mathrm{d}s^2 = a^2(\eta)(-\mathrm{d}\eta^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j) \equiv a^2(\eta)\, g^0_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{5.0.13}$$

From 5.0.9, it is now evident that such EG solutions can be "lifted" to DG solutions for which

$$g_{\mu\nu} = g^0_{\mu\nu}, \quad \psi \equiv a(\eta)\psi_0 \tag{5.0.14}$$

i.e. solutions with static spacetime metric and time-dependent ψ field; $a(\eta)$ solves the Friedmann equations of usual cosmology. From a mathematical point of view, the singularity has been shifted from the metric $g_{\mu\nu}$ to a zero of the dilaton. (Notice that $\psi \to 0$ means a divergence in the effective Newton's constant $\sim \psi^{-2}$.) Whether this is to be regarded as a physical singularity of the geometry depends on whether free falling test particles are assumed to follow the geodesics of the metric $g_{\mu\nu}$ (in which case the physical singularity has been removed) or of the "original" metric $\tilde{g}_{\mu\nu}$ of 5.0.9 (in which case it is still present). In the former case, such cosmological solutions of DG allow an extension of the spacetime manifold through what would normally be the Big Bang/Big Crunch singularity at $a(\eta) = 0$, the point where Einstein gauge breaks down.

This mathematical result can be interpreted in analogous way to the singularity for the origin in a plane using polar coordinates. But it should be taken with a grain of salt since in the physical point of view we just translate the singularity to a bad gauge choice of Weyl; which does not have

Noether charge, contrary to diffeomorphisms as shown in [84]. If we interpret singularities as a window for new fundamental physics, their presence should be removed by means of another dynamical physical theory; is in this sense that the "resolution" proposed by DG is not "illuminating" in a physical context.

Nevertheless the issue of resolving the Big Bang/Big Crunch singularity of EG either classically or quantum mechanically in this line is discussed furthermore in [79], [80], [81], [82] and [83]. Finally if we add matter to DG, we see that is most natural to choose matter fields that are themselves conformal (like 2.1.22 without the boundary term), and do not "see" any singularities in the conformal factor of the metric, nor in the dilaton ψ. For non-conformal matter there is, as usual, a choice of whether it couples only to the metric $g_{\mu\nu}$ or also to ψ, and coupling to ψ would in general still lead to a singularity even if $g_{\mu\nu}$ is made to be non-singular.

Just as the larger gauge group of DG means that certain singular fields in EG are not singular in DG, the smaller gauge group of UG means that certain regular fields in EG would be singular in UG. This can be seen from the Schwarzschild metric in isotropic coordinates

$$\mathrm{d}s^2 = -\left(\frac{r - M/2}{r + M/2}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{M}{2r}\right)^4 (\mathrm{d}x_1^1 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2) \tag{5.0.15}$$

If we base UG on the fixed volume element $\omega = \left(1 - \frac{M}{2r}\right)\left(1 + \frac{M}{2r}\right)^5$, there is a (mild) singularity at $r = M/2$ that cannot be removed by gauge transformations.

On the other hand UG translates the issue of the cosmological constant to a different domain. The classical cosmological constant "problem" (see [85], [86], [87]) deals with the fine tuning of the bare cosmological constant and the vacuum fluctuations of matter fields (coupled to gravity via $\sqrt{-g}$). In UG the cosmological constant is an integration constant and its value corresponds to boundary conditions, since vacuum fluctuations do not coupled to gravity the issue, if any, resides in why it has that value. The quantum cosmological constant problem arises from radiative instability, the tuning of the effective cosmological constant must be re-done for each loop correction, i.e. it is sensitive to UV physics. In UG this is mathematically solved but physically does not give a dynamical resolution to the tuning of the cosmological constant.

Appendix

A ADM Formalism and equations of motion

The spacetime dimension is d and the convention are the ones in [88]. The metric is parametrized as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t) \tag{A.1}$$

where N and N^i corresponds to the lapse and shift respectively and q_{ij} is the induced metric on the spacelike hypersurface Σ. The second fundamental form is defined by

$$K_{\alpha\beta} = -\nabla_\beta n_\alpha - a_\alpha n_\beta = -\frac{1}{2}\pounds_n q_{\alpha\beta} \tag{A.2}$$

∇ is the covariant derivative associated with g, $n_\alpha = (-N, \mathbf{0})$ is a unit time-like vector normal to the hypersurfaces and a_α, the acceleration, is given by $a_\alpha = n^\beta \nabla_\beta n_\alpha$. In terms of the induced metric and the lapse and the shift, the extrinsic curvature is given by

$$K_{ij} = \frac{1}{2N}\left(q_{ik}D_jN^k + q_{ik}D_jN^k - \dot{q}_{ij}\right) \tag{A.3}$$

The dot represents partial differentiation with respect to t and D the covariant derivative associated with h. The trace $K = q^{ij}K_{ij}$ can be regarded as a the expansion of a geodesic congruence orthogonal to Σ. Secondly the time-like vector n_α also enjoys the interpretation of a d-velocity of some observer (called Eulerian observers), therefore a_α is corresponds to d-acceleration which is tangent to Σ. In terms of the shift, the acceleration of the Eulerian observer is given by

$$a_\alpha = \frac{1}{N}D_\alpha N \tag{A.4}$$

Let $\mathcal{R}$ be the Ricci scalar with respect to g , then

$$\mathcal{R} = R + K_{ij}K^{ij} - K^2 - 2\nabla_\mu(Kn^\mu) - \frac{2}{N}D_iD^iN \tag{A.5}$$

$$\sqrt{-\det g} = \sqrt{\det q}N \tag{A.6}$$

where R is the Ricci scalar with respect to q. The EG action has the form

$$\begin{aligned} S[h_{ij}, N, N_i] &= Z_N \int_{t_i}^{t_f} \mathrm{d}t \int_{\Sigma_t} \mathrm{d}^{d-1}x \sqrt{q} N \left(R - 2\Lambda + K_{ij}K^{ij} - K^2 \right. \\ &\left. -2\nabla_\mu (K n^\mu) - \frac{2}{N} D^i D_i N \right) \end{aligned} \tag{A.7}$$

The canonical momenta is

$$P_N^0 = 0 \tag{A.8}$$

$$P_N^i = 0 \tag{A.9}$$

$$P_q^{ij} = Z_N \sqrt{q} \left[K q^{ij} - K^{ij} \right] \tag{A.10}$$

we can invert this relation

$$K_q^{ij} = \frac{1}{Z_N \sqrt{q}} \left[\frac{1}{d-2} P_q q^{ij} - P_q^{ij} \right] \tag{A.11}$$

where $P_q = q_{ab} P_q^{ab}$. From A.3 we find the equation of motion for the induced metric

$$\dot{q}_{ij} = -\frac{2N}{Z_N \sqrt{q}} \left[\frac{1}{d-2} P_q q_{ij} - P_{qij} \right] + q_{ik} D_j N^k + q_{ik} D_j N^k \tag{A.12}$$

The Hamiltonian of EG is given by

$$H = H_\perp[N] + H_{||}[\mathbf{N}] + C_1[\zeta] + C_2[\vec{\zeta}] \tag{A.13}$$

with

$$C_1[\zeta] = \int \mathrm{d}^{d-1}x \, \zeta P_N^0 \tag{A.14}$$

$$C_2[\vec{\zeta}] = \int \mathrm{d}^{d-1}x \, \delta^{ij} \zeta_i (P_N)_j \tag{A.15}$$

$$H_\perp[N] = \int \mathrm{d}^{d-1}x \, N \left[\frac{1}{Z_N \sqrt{q}} \left(P_{qij} P_q^{ij} - \frac{P_q^2}{(d-2)} \right) - Z_N \sqrt{q} R(q) \right] \tag{A.16}$$

$$H_{||}[\mathbf{N}] = \int \mathrm{d}^{d-1}x \, P_q^{ij} \pounds_{\mathbf{N}} q_{ij} \tag{A.17}$$

C_1 and C_2 are called primary (smear) constraints. Demanding that they should be preserved during evolution (consistency condition) one obtain the secondary (smear) constraints $H_\perp[N], H_{||}[\mathbf{N}]$. The equations of motion for the momentum is[1]

$$\dot{P}_q^{ij} = -\frac{\delta H_\perp[N]}{\delta q_{ij}} - \frac{\delta H_{||}[\mathbf{N}]}{\delta q_{ij}} \tag{A.18}$$

[1]For computational details the reader can check [89].

with

$$
\begin{aligned}
\frac{\delta H_\perp[N]}{\delta q_{ij}} &= \frac{2N}{Z_N\sqrt{q}}\left(P_q^{ai}P_{qa}{}^{j} - \frac{P_q P_q^{ij}}{(d-2)}\right) - \frac{N}{2Z_N\sqrt{q}}q^{ij}\left(P_{qab}P_q^{ab} - \frac{P_q^2}{(d-2)}\right) \\
&\quad + Z_N N\sqrt{q}\left(R^{ij} - \frac{R}{2}q^{ij}\right) + Z_N\sqrt{q}\left(q^{ij}D^2N - D^{(i}D^{j)}N\right) \quad \text{(A.19)} \\
\frac{\delta H_\parallel[\mathbf{N}]}{\delta q_{ij}} &= 2P_q^{a(i}D_a N^{j)} - \sqrt{q}D_a\left(\frac{q^{ij}N^a}{\sqrt{q}}\right) \quad \text{(A.20)}
\end{aligned}
$$

B Dirac's analysis of constraint systems

This section is a LaTeX summary of some chapters of [90].

B.1 Regular finite systems

Consider a system with a finite number of degrees of freedom (N) whose dynamics can be derived from the action

$$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) \tag{B.1}$$

where L is the Lagrangian which is a function of the generalized coordinates $\{q^k\}$, generalized velocities $\{\dot{q^k}\}$ with $\dot{q} = \frac{dq^k}{dt}$ and time t ($k = 1 \ldots N$). The Lagrangian is called **regular** if the determinant of the Hessian

$$[W]_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \tag{B.2}$$

is not identically zero as a function of q^k, $\dot{q^k}$ and t. Regularity is a property of a Lagrangian and not of the system describe by it. From the variation of the action (Hamilton principle) we obtain the Euler-Lagrange equations of motion

$$\frac{\partial L}{\partial q^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} = 0 \tag{B.3}$$

They can be written in matrix notation as

$$W\ddot{\mathbf{q}} = \mathbf{v} \tag{B.4}$$

with

$$v_k = \frac{\partial L}{\partial q^k} - \frac{\partial^2 L}{\partial q^l \partial \dot{q}^k}\dot{q}^l - \frac{\partial^2 L}{\partial t \partial \dot{q}^k} \tag{B.5}$$

In the regular case where $\det W \neq 0$ all equations are of second order and functionally independent. Under rather weak conditions there exist solutions in the interval $t_1 \leq t \leq t_2$. They are unique after $2N$ integration constants are fixed, say by the initial data $q(t_1)^k$ and $\dot{q}(t_1)^k$. Except for some pathological cases these $2N$ constants can be expressed by the values $q(t_1)^k$ and $q(t_2)^k$ which are kept fixed in the variation of the action.

The generalized momenta is define as

$$p_k = \frac{\partial L}{\partial \dot{q}^k} \tag{B.6}$$

and the Hamiltonian, a function of q^k, p_k and t, as

$$H(q,p,t) = p_k \dot{q}^k - L \tag{B.7}$$

However only in the regular case the equation defining the generalized momenta can be solved uniquely for the $\dot{q}^k$. This comes about since

$$\frac{\partial p_i}{\partial \dot{q}^j} = [W]_{ij} \tag{B.8}$$

and if $\det W \neq 0$ the inverse function theorem guarantees that there is a unique solution $\dot{q}^k = \dot{q}^k(q,p,t)$. Only in this case can one switch from one set of variables $(q, \dot{q}, t)$ to the other one (q,p,t) in a one-to-one manner.

From the comparison of dL and the definition of the Hamiltonian with dH one finds the following equations

$$\begin{aligned} \dot{q}^k &= \frac{\partial H}{\partial p_k} && \text{(B.9)} \\ \dot{p}_k &= -\frac{\partial H}{\partial q^k} && \text{(B.10)} \\ \frac{\partial L}{\partial t} &= -\frac{\partial H}{\partial t} && \text{(B.11)} \end{aligned}$$

For the second equation we used the Euler-Lagrange equations and the first two equations are called Hamilton equations of motion. They are a set of $2N$ coupled differential equations of first order which have a unique solution after N coordinates and N momenta for $t = t_1$ have been supplied.

B.2 Poisson bracket and canonical transformations

Consider the functions $A(q,p,t)$ and $B(q,p,t)$ the Poisson bracket is defined as

$$\{A, B\} = \frac{\partial A}{\partial q^k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q^k} \tag{B.12}$$

Then the time evolution of any function is given by

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\} \tag{B.13}$$

From a geometrical point of view the bracket is the most central object. It has the following properties:

- $\{A, B\} = -\{B, A\}$, antisymmetry.
- $\{c_1 A + c_2 B, C\} = c_1\{A, C\} + c_2\{A, C\}$, linearity ($c_1, c_2$ are constants).
- $\{c_1, A\} = 0$, null elements.
- $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$, Jacobi identity.
- $\{AB, C\} = A\{B, C\} + \{A, C\}B$, product rule.
- $\{q^i, q^j\} = 0 = \{p_i, p_j\}$, $\{q^i, p_j\} = \delta^i_j$, fundamental brackets.

Consider the following transformations

$$\tilde{q}^k = \tilde{q}^k(q, p) \tag{B.14}$$

$$\tilde{p}_k = \tilde{p}_k(q, p) \tag{B.15}$$

the transformation is *canonical* if and only if

$$\{\tilde{q}^i, \tilde{q}^j\} = 0 = \{\tilde{p}_i, \tilde{p}_j\}, \quad \{\tilde{q}^i, \tilde{p}_j\} = \delta^i_j \tag{B.16}$$

In general a canonical transformation does not necessarily imply a symmetry (i.e. a transformation that leaves invariant the action). In phase space, a space in which local coordinates are (q^k, p_k), we can collect the variables into one vector $\{x_\alpha\} = \{q^1, \ldots, q^N, p_1, \ldots, p_N\}$, then the Poisson bracket becomes

$$\{A, B\} = \Gamma_{\alpha\beta} \frac{\partial A}{\partial x_\alpha} \frac{\partial B}{\partial x_\beta} \tag{B.17}$$

where Γ is a $2N \times 2N$ matrix

$$\Gamma = \begin{pmatrix} 0_N & 1_N \\ -1_N & 0_N \end{pmatrix} \tag{B.18}$$

The matrix Γ has the properties

$$\Gamma^T = -\Gamma, \quad \Gamma^2 = -\Gamma \tag{B.19}$$

The fundamental brackets become

$$\{x_\alpha, x_\beta\} = \Gamma_{\alpha\beta} \tag{B.20}$$

and a transformation $\tilde{x}_\alpha = \tilde{x}_\alpha(x)$ is canonical if

$$\{\tilde{x}_\alpha, \tilde{x}_\beta\} = \Gamma_{\alpha\beta} \tag{B.21}$$

Let

$$X_{\alpha\beta} = \frac{\partial \tilde{x}_\alpha}{\partial x_\beta} \tag{B.22}$$

then we can specify a canonical transformations in matrix notation

$$X\Gamma X^T = \Gamma \tag{B.23}$$

From this it is easy to see that canonical transformations constitute a group. Since $2N \times 2N$ real matrices M with property $M\Gamma M^T = \Gamma$ form the symplectic group $Sp(2N, \mathbb{R})$ we can call a transformation $x_\alpha \to \tilde{x}_\alpha$ canonical if the matrix X formed out of the derivatives belongs to $Sp(2N, \mathbb{R})$ for any x.

If x and $\tilde{x}$ are correlated by a canonical transformation it does not matter whether we calculate a Poisson bracket with respect to the new or old set of phase-space variables, that is

$$\{A(x), B(x)\}_x = \{A(x(\tilde{x}))), B(x(\tilde{x}))\}_{\tilde{x}} \tag{B.24}$$

B.3 Dirac-Bergmann algorithm

We start again with the definition

$$p_i = \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}), \quad i = 1, \ldots, N \tag{B.25}$$

If the rank R of W is maximal ($R = N$) this relation can (at least locally) be solved for all velocities to give $\dot{q}^k = \dot{q}^k(q, p)$. However if $R < N$ there is only a nondegenerate $R \times R$ matrix $[W]_{\alpha a}$, with $\alpha, a = 1, \ldots, R$. Then (at least locally) it will be possible to solve equations (B.25) for the $\dot{q}^a$ and express these as functions of the position, the momenta p_α and the remaining velocities $\dot{q}^\rho$ ($\rho = R+1, \ldots, N$):

$$\dot{q}^a = f^a(q, p_\alpha, \dot{q}^\rho) \tag{B.26}$$

Now we substitute formally this expression in (B.25)

$$p_i = \tilde{g}_i(q, \dot{q}^a, \dot{q}^\rho) = \tilde{g}_i(q, f^a(q, p_\alpha, \dot{q}^\rho), \dot{q}^\rho) = g_i(q, p_\alpha, \dot{q}^\rho) \tag{B.27}$$

Then for $i = 1, \ldots, R$ we simply have that $g_\alpha = p_\alpha$. For $i = R+1, \ldots, N$ the $N - R$ functions cannot depend on $\dot{q}^\rho$ any longer since otherwise one could solve for more of velocities, thus

$$p_r = g_r(q, p_\alpha), \quad r = R+1, \ldots, N \tag{B.28}$$

These $N-R$ relations between coordinates and momenta are called **primary constrains**.

For this non regular case we will take the N generalized position q^i, the R momenta p_α and the $N-R$ velocities $\dot{q}^\rho$ as coordinates. The function H_c is defined as

$$\begin{aligned} H_c(q, p_\alpha, \dot{q}^\rho) &= p_i \dot{q}^i - L(q, \dot{q}) \\ &= p_a \dot{q}^a + p_r \dot{q}^r - L(q, \dot{q}) \end{aligned} \tag{B.29}$$

where we must substitute the expressions (B.26) and (B.28). Then one finds

$$\frac{\partial H_c}{\partial \dot{q}^\rho} = 0 \tag{B.30}$$

$$\frac{\partial H_c}{\partial q^i} = \frac{\partial g_r}{\partial q^i}\dot{q}^r - \frac{\partial L}{\partial q^i} \tag{B.31}$$

$$\frac{\partial H_c}{\partial p_\alpha} = \dot{q}^\alpha + \frac{\partial g_r}{\partial p_\alpha}\dot{q}^r \tag{B.32}$$

Using the Euler-Lagrange equations one obtain

$$\dot{q}^a = \frac{\partial H_c}{\partial p_a} - \frac{\partial g_r}{\partial p_\alpha}\dot{q}^r \tag{B.33}$$

$$\dot{p}_i = -\frac{\partial H_c}{\partial q^i} + \frac{\partial g_r}{\partial q^i}\dot{q}^r \tag{B.34}$$

These are reminiscent to the Hamilton equations for regular systems however there are extra terms on the right-hand side depending on $N-R$ functions $\dot{q}^r$, and related to this, in the singular case there are only $N+R$ equations as opposed to $2N$ equations on the regular case.

Due to the constrains (B.28) the motion is restricted to a subspace Γ_p of the full phase space Γ and H_c is only defined on Γ_p. We would like to extend the equations (B.33) and (B.34) into Γ.

Weak and strong equations

Let $F(q,p)$ be a function defined in a finite neighborhood of Γ_p. The restriction of F to Γ_p is achieved by replacing the p_r by $g_r(q,p_a)$

$$F(q,p)|_{\Gamma_p} = F(q,p_a,g_r(q,p_a)) \tag{B.35}$$

If F is identically zero after this replacement it is called **weakly zero**; denote it by $F \approx 0$. If the gradient of F is zero on Γ_p too, F is called **strongly zero**; denoted it by $F \simeq 0$:

$$\left.\begin{array}{c} F|_{\Gamma_p} = 0 \\ \left(\dfrac{\partial F}{\partial q^i}, \dfrac{\partial F}{\partial p_i}\right)\Bigg|_{\Gamma_p} = 0 \end{array}\right\} F \simeq 0 \tag{B.36}$$

This definition is specially useful since the equations of motion we are dealing with contain gradients of functions on Γ_p. The hypersurface Γ_p can itself be defined by weak equations, We have

$$G_r(q,p) = p_r - g_t(q,p) \approx 0 \tag{B.37}$$

but $G_r \not\approx 0$ since $\dfrac{\partial G_r}{\partial p_s} = \delta_r^s$. In order to establish a relation between weak and strong equality consider the variation of F

$$\delta F = \frac{\partial F}{\partial q^i}\delta q^i + \frac{\partial F}{\partial p_a}\delta p_a + \frac{\partial F}{\partial p_r}\delta p_r \tag{B.38}$$

then using (B.28) for $\delta F|_{\Gamma_p} = 0$ $(\delta F \approx 0)$ we obtain

$$\begin{aligned} \left.\left(\frac{\partial F}{\partial q^i} + \frac{\partial F}{\partial p_r}\frac{\partial g_r}{\partial q^i}\right)\right|_{\Gamma_p} &= 0 \\ \left.\left(\frac{\partial F}{\partial p_a} + \frac{\partial F}{\partial p_r}\frac{\partial g_r}{\partial p_a}\right)\right|_{\Gamma_p} &= 0 \end{aligned} \tag{B.39}$$

Then for the first equation

$$\begin{aligned} \tfrac{\partial F}{\partial q^i} + \tfrac{\partial F}{\partial p_r}\tfrac{\partial g_r}{\partial q^i} &= \tfrac{\partial F}{\partial q^i} - \tfrac{\partial F}{\partial p_r}\tfrac{\partial G_r}{\partial q^i} \\ &= \tfrac{\partial F}{\partial q^i} - \tfrac{\partial}{\partial q^i}\left(\tfrac{\partial F}{\partial p_r}G_r\right) + \tfrac{\partial^2 F}{\partial q^i \partial p_r}G_r \\ &= \tfrac{\partial}{\partial q^i}\left(F - \tfrac{\partial F}{\partial p_r}G_r\right) + \tfrac{\partial^2 F}{\partial q^i \partial p_r}G_r \end{aligned} \tag{B.40}$$

if we neglect terms which are proportional to G_r, they are weakly zero anyhow, one obtain for (B.39) a set of weak equations:

$$\begin{aligned} \frac{\partial}{\partial q^i}\left(F - \frac{\partial F}{\partial p_r}G_r\right) &\approx 0 \\ \frac{\partial}{\partial p_i}\left(F - \frac{\partial F}{\partial p_r}G_r\right) &\approx 0 \end{aligned} \tag{B.41}$$

In the second set of equations the index a has been replaced by i. This extension is possible because of $\frac{\partial G_r}{\partial p_s} = \delta_r^s$ so the left hand-side vanishes identically. In conclusion

$$F \approx 0 \leftrightarrow F \simeq G_r \frac{\partial F}{\partial p_r} \tag{B.42}$$

that is: *a weakly vanishing function is a linear combination of the weakly vanishing functions defining the hypersurface* Γ_p.

The extension with Poisson brackets

H_c may be the restriction to the hypersurface Γ_p of a function H' defined all over phase space (or at least in the neighborhood of Γ_p), that is

$$H_c - H' \approx 0 \tag{B.43}$$

Let $F = H_c - H'$ and $F \approx 0$ then (B.41) results

$$\begin{aligned} \frac{\partial H_c}{\partial q^i} - \frac{\partial}{\partial q^i}\left(H' - \frac{\partial H'}{\partial p_r}G_r\right) &\approx 0 \\ \frac{\partial H_c}{\partial p_i} - \frac{\partial}{\partial p_i}\left(H' - \frac{\partial H'}{\partial p_r}G_r\right) &\approx 0 \end{aligned} \tag{B.44}$$

Comparing these expressions with (B.31) and (B.32), again using G_r instead of g_r, one obatins

$$\begin{array}{rcl} \dot{q}^a - \dfrac{\partial G_r}{\partial p_a}\dot{q}^r & \approx & \dfrac{\partial}{\partial p_a}\left(H' - \dfrac{\partial H'}{\partial p_r}G_r\right) \\ \dfrac{\partial G_r}{\partial q^i}\dot{q}^r + \dfrac{\partial L}{\partial q^i} & \approx & -\dfrac{\partial}{\partial q^i}\left(H' - \dfrac{\partial H'}{\partial p_r}G_r\right) \end{array} \tag{B.45}$$

In the first set of these relations the range of the index a may now be extended to include all i since again the equations for the superfluos indices are trivially satisfied. With

$$H = H' - \frac{\partial H'}{\partial p_r}G_r \tag{B.46}$$

one can thus write the previous equations as

$$\begin{array}{rcccl} \dot{q}^i & \approx & \dfrac{\partial H}{\partial p_i} + \dfrac{\partial G_r}{\partial p_i}\dot{q}^r & \approx & \{q^i, H + G_r\dot{q}^r\} \\ \dfrac{\partial L}{\partial q^i} & \approx & -\dfrac{\partial H}{\partial q^i} - \dfrac{\partial G_r}{\partial q^i}\dot{q}^r & \approx & \{p_i, H + G_r\dot{q}^r\} \end{array} \tag{B.47}$$

Here the Poisson brackets notation is used and it is understood that the brackets are calculated as if the $q's$ and $p's$ were independent. Only after that are allowed to impose the constrains. Although H is not completely defined, the restriction $H \approx H_c$ allows to use H_c instead of H in these equations. And for a solution of the Euler-Lagrange equations we finally get

$$\begin{array}{rcl} \dot{q}^i & \approx & \{q^i, H_c + G_r\dot{q}^r\} \\ \dot{p}_i & \approx & \{p_i, H_c + G_r\dot{q}^r\} \end{array} \tag{B.48}$$

Besides there are still the equations defining the primary constraint space Γ_p:

$$G_r(q,p) \approx 0 \tag{B.49}$$

Remarks:

- Although G_r vanishes it can influence the dynamics, the clue being weakly vanishing of G_r, that is vanishing on Γ_p.
- Equations (B.48) reveal the price we have to pay for using the Poisson brackets defined in the full phase space, they are weak conditions only.
- They appear however in a nice symmetric form and bear more similarity to the Hamilton equations of motion we are used to from unconstrained systems.
- Nevertheless equations (B.48) still contain arbitrary functions, the $\dot{q}^r$.

Implicit constrains

Suppose that the constrains are written in a manifest *covariant* manner one would prefer not to destroy this. So let us write (B.48) in terms of implicit constrains

$$\phi_r(q,p) \approx 0, \quad r = R+1,\dots,N \tag{B.50}$$

One may consider the previous $G_r = p_r - g_r(q,p_a)$ to be an explicit solution of $\phi_r \approx 0$. There is some ambiguity in the functional form of (B.50) since together with $\phi_r \approx 0$ also $\phi_r^2 \approx 0$, the latter being even strongly zero.

One has to impose a condition of minimality on the form of (B.50) in the sense that a weakly vanishing function should strongly be equal to a linear combination of the constraints defining the hypersurface Γ_p, see (B.42). This is guaranteed provided that the $2N \times (N-R)$ matrix:

$$\begin{pmatrix} \frac{\partial\phi_{R+1}}{\partial q^1} & \dots & \frac{\partial\phi_N}{\partial q^1} \\ \vdots & \dots & \vdots \\ \frac{\partial\phi_{R+1}}{\partial q^N} & \dots & \frac{\partial\phi_N}{\partial q^N} \\ \frac{\partial\phi_{R+1}}{\partial p_1} & \dots & \frac{\partial\phi_N}{\partial p_1} \\ \vdots & \dots & \vdots \\ \frac{\partial\phi_{R+1}}{\partial p_N} & \dots & \frac{\partial\phi_N}{\partial p_N} \end{pmatrix} \tag{B.51}$$

(where the derivatives have to be evaluated before using the constrains) has finite matrix elements and is of rank $N-R$.

Differentiate (B.50) with respect to q^i and p_a

$$\begin{aligned} \frac{\partial\phi_r}{\partial q^i} + \frac{\partial\phi_r}{\partial p_s}\frac{\partial g_s}{\partial q^i} &\approx 0 \\ \frac{\partial\phi_r}{\partial p_a} + \frac{\partial\phi_r}{\partial p_s}\frac{\partial g_s}{\partial p_a} &\approx 0 \end{aligned} \tag{B.52}$$

The matrix

$$[V]_{rs} = \frac{\partial\phi_r}{\partial p_s} \tag{B.53}$$

must be nondegenerate since otherwise one would be able to eliminate some $p's$ and would obtain constraints involving the $q's$ only, which is impossible since the primary constraints originate from the definition of the momenta. Therefore (B.52) can be inverted leading to

$$\begin{aligned} \frac{\partial g_s}{\partial q^i} &\approx -\frac{\partial G_s}{\partial q^i} \approx -[V^{-1}]_{sr}\frac{\partial\phi_r}{\partial q^i} \\ \frac{\partial g_s}{\partial p_a} &\approx -\frac{\partial G_s}{\partial p_a} \approx -[V^{-1}]_{sr}\frac{\partial\phi_r}{\partial p_a} \end{aligned} \tag{B.54}$$

Insert this into (B.47)

$$\begin{aligned} \dot{q}^i &\approx \{q^i, H_c\} + \dot{q}^s [V^{-1}]_{sr} \frac{\partial \phi_r}{\partial p_i} \\ \dot{p}_i &\approx \{p_i, H_c\} + \dot{q}^s [V^{-1}]_{sr} \frac{\partial \phi_r}{\partial q^i} \end{aligned} \tag{B.55}$$

Then with

$$\mu_r = \dot{q}^s [V^{-1}]_{sr} \tag{B.56}$$

and the **primary Hamiltonian**

$$H_p = H_c + \mu^r \phi_r \tag{B.57}$$

the equation of motion for any phase-space function $A(q,p)$ becomes

$$\dot{A} = \frac{dA}{dt} \approx \{A, H_p\} \approx \{A, H_c\} + \mu^r \{A, \phi_r\} \tag{B.58}$$

and the μ^r take over the role as multipliers.

From the Lagrangian treatment of constrained dynamics we know that some of the originally arbitrary functions eventually become determined by consistency arguments. Consistency meant that time derivative of constraints are bound to vanish (modulo the constraints themselves). Time development is characterized here by (B.58). The constraints $\phi_r \approx 0$ ought to be preserved, therefore one has to have

$$0 \overset{!}{\approx} \dot{\phi}_r \approx \{\phi_r, H_p\} \approx \{\phi_r, H_c\} + \mu^s \{\phi_r, \phi_s\} \tag{B.59}$$

In order to discuss the implications of this we define

$$h_r = \{\phi_r, H_c\}, \quad [P]_{rs} = \{\phi_r, \phi_s\} \tag{B.60}$$

and distinguish four cases:

Case IA

$h \not\approx 0$ (not all $h_r \approx 0$), $\det P \not\approx 0$. Then (B.59) constitute an inhomogeneous system of linear equations for the $\mu's$ with solutions

$$\mu_s \approx -[P^{-1}]_{sr} h_r \tag{B.61}$$

The $\mu's$ (weakly) fixed, the equation of motion for any phase-space function $A(p,q)$ becomes

$$\dot{A} \approx \{A, H_c\} - \{A, \phi_s\}[P^{-1}]_{sr}\{\phi_r, H_c\} \tag{B.62}$$

After specifying initial values for coordinates and momenta subject to the restrictions $\phi_{r(q,p)=0}$ can these equations be solved without ambiguity.

Case IB

$h \not\approx 0$, $\det P \approx 0$. In order to (B.59) to posses a solution certain relations among the components of h must be fulfilled: Let the rank of P be M. Since P is a $(N-R) \times (N-R)$ matrix this implies the existence of $(N-R)-M$ linearly independent null-eigenvectors $e^{(\alpha)}$, i.e.

$$e_s^{(\alpha)}(q,p)[P]_{sr} \approx 0, \quad \alpha = 1, \ldots, (N-R)-M \tag{B.63}$$

Multiplication of (B.59) with these eigenvectors yields the conditions

$$0 \overset{!}{\approx} e_r^{(\alpha)} h_r \tag{B.64}$$

Now either these equations are fulfilled or they lead to a certain number of L' of new constrains independent from each other and independent of the previous constraints $\phi_r \approx 0$, these are called **secondary constraints**. These restrict the motion in phase-space to a hypersurface Γ' of lower dimension than Γ_p.

Case IIA

$h \approx 0$, $\det P \not\approx 0$. There is only the trivial solution $\mu_r \approx 0$, i.e. $H_p = H_c$. If $h \approx 0$ originates from $H_c \approx 0$ this would be difficult to interpret since a vanishing Hamiltonian does not allow for any dynamics at all. To avoid this situation one should impose as a secondary constraint $\det P \overset{!}{\approx} 0$.

Case IIB

$h \approx 0$, $\det P \approx 0$. In this case of a homogeneous system of equations for the $\mu's$ non-trivial solutions exist. If M is the rank of P, $(N-R)-M$ multipliers are (weakly) fixed.

Remark: we see that there are situations in which new constraints emerge: IB, IIA.

Secondary constraints in case IB

The hypersurface Γ' is defined by the $(N-R)+L'$ weak equations

$$\left.\begin{array}{ccc} \phi_r & \approx & 0 \\ \chi_{\rho'} & \approx & 0 \end{array}\right|_{\Gamma'} \quad r = R+1, \ldots, N, \quad \rho' = 1, \ldots, L' \tag{B.65}$$

where the $\chi's$ are just the independent functional relations implied by (B.64) and, as indicated, weak equality refers now to Γ'.

(B.59) after being evaluated on Γ, has now for consistency to be checked on Γ'. By this the rank if P may decrease (or stay the same) and the number of relations (B.64) may increase (or stay the same). So one might gain more (tertiary) constrains, independent among themselves and of the previous primary and secondary constraints. Then all of the constraints define a hypersurface until the following situation is reached: The motion is restricted to a hypersurface Γ'' defined by the $N-R$ primary and L'' constraints

$$\left.\begin{array}{rcl}\phi_r & \approx & 0 \\ \chi_{\rho''} & \approx & 0\end{array}\right|_{\Gamma''} \qquad r = R+1,\dots,N, \quad \rho'' = 1,\dots,L'', \quad L'' \geq L' \tag{B.66}$$

The matrix with elements $\{\phi_r, \phi_s\}$ has the rank P ($P \leq (N-R)-L''$), and for every null-eigenvector $e^{(p)}$, i.e.

$$e_r^{(p)}\{\phi_r, \phi_s\} \overset{\Gamma''}{\approx} 0 \tag{B.67}$$

is obeyed. The remaining constraints are simply called secondary constraints.

For consistency one also has to requiere that the secondary constraints are preserved in time. This implies the set of conditions

$$\left.\begin{array}{rcl}\{\phi_r, H_c\} + \mu^s\{\phi_r, \phi_s\} & \approx & 0 \\ \{\chi_{\rho''}, H_c\} + \mu^s\{\chi_{\rho''}, \phi_s\} & \approx & 0\end{array}\right|_{\Gamma''} \tag{B.68}$$

The solution of this linear system of equations for the $\mu's$ is ruled by the rectangular matrix

$$\left(\ \{\phi_r, \phi_s\}\{\phi_{\rho''}, \phi_s\}\ \right) \tag{B.69}$$

which has $N-R$ columns and $(N-R)+L''$ rows. Every left null-eigenvector $(e_r^{(i)}, e_{\rho''}^{(i)})$ of this matrix, i.e.

$$e_r^{(i)}\{\phi_r, \phi_s\} + e_{\rho''}^{(i)}\{\phi_{\rho''}, \phi_s\} \approx 0\Big|_{\Gamma''} \tag{B.70}$$

the superscripts i numbering the eigenvector, implies a condition

$$0 \overset{!}{\approx} e_r^{(i)}\{\phi_r, H_c\} + e_{\rho''}^{(i)}\{\phi_{\rho''}, H_c\} \tag{B.71}$$

These relations again are either fulfilled or lead to a new set of independent constraints, which together with the old ones define still another hypersurface. In the latter case one has to start the iteration scheme on this hypersurface.

This process ends after a finite number of steps at the following point: There is a hypersurface Γ_c in $2N$ dimensional phase-space defined by

$$\left.\begin{array}{rcl}\{\phi_r, H_c\} + \mu^s\{\phi_r, \phi_s\} & \approx & 0 \\ \{\chi_\rho, H_c\} + \mu^s\{\chi_\rho, \phi_s\} & \approx & 0\end{array}\right|_{\Gamma_c} \qquad r = R{+}1,\dots,N, \quad \rho = 1,\dots,L, \quad L \geq L'' \tag{B.72}$$

For every left null-eigenvector $e^{(i)}$ of the matrix

$$D = \left(\; \{\phi_r, \phi_s\} \{\phi_\rho, \phi_s\} \; \right) \tag{B.73}$$

the conditions

$$e_r^{(i)} \{\phi_r, H_c\} + e_\rho^{(i)} \{\phi_\rho, H_c\} \approx 0 \Big|_{\Gamma_c} \tag{B.74}$$

are fulfilled. For the multipliers functions μ^s there are finally the equations

$$\left. \begin{array}{rcl} \{\phi_r, H_c\} + \mu^s \{\phi_r, \phi_s\} & \approx & 0 \\ \{\chi_\rho, H_c\} + \mu^s \{\chi_\rho, \phi_s\} & \approx & 0 \end{array} \right|_{\Gamma_c} \tag{B.75}$$

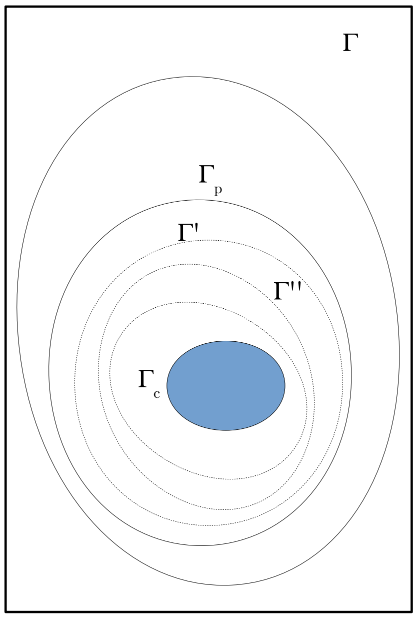

Figure 5.2: Representation of the algorithm. The blue region represents the part of the phase-space in which the equations of motion are restricted.

First and second class constraints

One can ask: how many of the multipliers $\mu' s$ in (B.75) may eventually determined? The answer depends on the rank of the matrix D, defined in (B.73). If the rank of D is $N - R$, all $\mu' s$ are fixed. This is just case IA. On the other hand if the rank is $K < N - R$, only K independent combinations of the multipliers are fixed, leaving $(N-R)-K$ linear combinations complete free. The rank of D being K means that there are $(N-R)-K$ relations

$$\begin{array}{rcl} \{\phi_r, \phi_s\}e_s^{(J)} & \approx & 0 \\ \{\chi_\rho, \phi_s\}e_s^{(J)} & \approx & 0 \end{array} \qquad J = 1, \ldots, N-R-K \tag{B.76}$$

Any linear combination of the constraints ϕ_r is again a constraint. Defining especially

$$\Phi^J = e_r^{(J)}\phi_r \tag{B.77}$$

then (B.76) tells that all the Φ^j have weakly vanishing Poisson brackets with all (primary and secondary) constraints:

$$\begin{array}{rcl} \{\phi_r, \Phi^J\} & \approx & 0 \\ \{\chi_\rho, \Phi^J\} & \approx & 0 \end{array} \tag{B.78}$$

A quantity is called **first class constraints** if its Poisson brackets with all constraints vanish (at least weakly). Let us denote with the symbol Φ^I $(I = N-R-K+1, \ldots, N-R)$ those of the primary constraints which are not exhausted in (B.77). Then Φ^J being first class constraints, the ϕ^i cannot be, they are called **second class constraints**.

In a next step separate the secondary constraints χ_ρ into these classes too. This can be achieved with a suitable nonsingular matrix A and further rectangular matrices B and C:

$$\chi'_\rho = [A]_{\rho\sigma} + [B]_{\rho J}\Phi^J + [C]_{\rho I}\phi^i \tag{B.79}$$

Assume that you have chosen these matrices such that as many as possible of the $\chi's$ are first class, call them $\mathcal{X}^A$ an call the rest χ^a. We write the sum $\phi_s\mu^s$ appearing in (B.75) in terms of first and second class quantities

$$\phi_s\mu^s = \Phi^J v_J + \phi^i u_i \tag{B.80}$$

Equations (B.75) become

$$\begin{array}{rcl} \{\Phi^J, H_c\} & \approx & 0 \\ \{\mathcal{X}^A, H_c\} & \approx & 0 \\ \{\phi^i, H_c\} + \{\phi^i, \phi^j\}u_j & \approx & 0 \\ \{\chi^a, H_c\} + \{\chi^a, \phi^j\}u_j & \approx & 0 \end{array} \tag{B.81}$$

One can demonstrate that al multipliers u_J are determined by the least two equations, and because all v_J disappeared, we draw the conclusion that: *there are as many undetermined combinations of multipliers functions as there are primary first class constraints.*

To prove that the coefficients u_J are completely fixed we start with the fact (see the book for the proof) that the matrix of all second class constraints

$$\Delta = \begin{pmatrix} \{\phi^i, \phi^j\} & \{\phi^i, \chi^b\} \\ \{\chi^a, \phi^j\} & \{\chi^a, \chi^b\} \end{pmatrix} \tag{B.82}$$

is non-singular. For a convenient notation let us arrange all second class constraints into one set

$$\xi_\lambda = (\phi^i, \chi^a) \tag{B.83}$$

In terms of Δ and ξ_μ, the last two equations of (B.81) simply becomes

$$\{\xi_\mu, H_c\} + [\Delta]_{\mu J} u_J \approx 0 \tag{B.84}$$

and if we multiply these equations with the inverse of Δ we obtain

$$u_J \approx -[\Delta^{-1}]_{j\mu}\{\xi_\mu, H_c\} \tag{B.85}$$

and

$$[\Delta^{-1}]_{a\mu}\{\xi_\mu, H_c\} \approx 0 \tag{B.86}$$

Since the u_J are fixed by (B.85), the equation of motion for a phase-space function A,

$$\dot{A} \approx \{A, H_c\} + v_J\{A, \Phi^J\} + u_j\{A, \phi^j\} \tag{B.87}$$

becomes

$$\dot{A} \approx \{A, H_c\} + v_J\{A, \Phi^J\} - \{A, \phi^j\}[\Delta^{-1}]_{j\mu}\{\xi_\mu, H_c\} \tag{B.88}$$

With the help of (B.86) this may be brought onto a form in which all second class constraints appear in a symmetric manner:

$$\dot{A} \approx \{A, H_c\} + v_J\{A, \Phi^J\} - \{A, \xi_\nu\}[\Delta^{-1}]_{\nu\mu}\{\xi_\mu, H_c\} \tag{B.89}$$

Consistency guarantees that the time derivative of any constraint vanishes on the constraint hypersurface Γ_c defined by the first class constraints $\varphi^\alpha = (\Phi^J, \mathcal{X}^A)$, and the second class constraints ξ_μ.

C Canonical transformations

The transformations between the two sets of canonical variables (q, p) and (Q, P) satisfy

$$p\dot{q} - H(q, p) = P\dot{Q} - K(Q, P) + \dot{F} \tag{C.1}$$

with F as the generating function. A type-2 generating function is on the form

$$F = F_2(q, P, t) - QP \tag{C.2}$$

Taking de total derivative and substituting in C.1 one obtains

$$p = \frac{\partial F_2}{\partial q} \tag{C.3}$$

$$Q = \frac{\partial F_2}{\partial P} \tag{C.4}$$

$$K = H + \frac{\partial F_2}{\partial t} \tag{C.5}$$

For fields we consider the transformation of (ϕ, Π_ϕ) to (Φ, P_Φ). From

$$\int \mathrm{d}t\mathrm{d}x \left(\Pi_\phi \dot{\phi} - \mathcal{H}\right) = \int \mathrm{d}t\mathrm{d}x \left(P_\Phi \dot{\Phi} - \mathcal{K}\right) + \int \mathrm{d}t \frac{\partial \mathcal{F}}{\partial t} \tag{C.6}$$

Let

$$\mathcal{F} = \mathcal{F}_2 - \int \mathrm{d}x P_\Phi \Phi = \int \mathrm{d}x \left(f(\phi, P_\Phi) - P_\Phi \Phi\right) \tag{C.7}$$

Then

$$\Pi_\phi = \frac{\delta \mathcal{F}_2}{\delta \phi} \tag{C.8}$$

$$\Phi = \frac{\delta \mathcal{F}_2}{\delta P_\Phi} \tag{C.9}$$

$$\mathcal{K} = \mathcal{H} \tag{C.10}$$

D Poisson Brackets

Some useful Poisson Brackets for the metric q_{ij} and conjugate momenta P_q^{ij} and the connection D associated with q.

$$\{q_{ij}(x), P_q^{lm}(y)\} = \delta^l_{(i}\delta^m_{j)}\delta(x-y) \tag{D.1}$$

$$\{q_{ij}(x), q_{i'j'}(y)\} = 0 \tag{D.2}$$

$$\{P_q^{lm}(x), P_q^{l'm'}(y)\} = 0 \tag{D.3}$$

$$\{q^{ij}(x), P_q^{lm}(y)\} = -q^{ip}(x)q^{p'j}(x)\delta^l_{(p}\delta^m_{p')}\delta(x-y) \tag{D.4}$$

$$\{\sqrt{q(x)}, P_q^{lm}(y)\} = \frac{1}{2}\sqrt{q(x)}q^{lm}(x)\delta(x-y) \tag{D.5}$$

$$\{\pounds_{\mathbf{N}} q_{ij}(x), P_q^{lm}(y)\} = \delta^l_{(i}\delta^m_{j)}\pounds^x_{\mathbf{N}}\delta(x-y) - 2N^{(l}(x)\delta^{m)}_{(i} D^x_{j)}\delta(x-y) \tag{D.6}$$

$$\begin{aligned}\{R(q(x)), P_q^{lm}(y)\} &= -R^{lm}(q(x))\delta(x-y) - q^{lm}(x)D^2_x\delta(x-y) \\ &\quad + D^{(l}_x D^{m)}_x \delta(x-y)\end{aligned} \tag{D.7}$$

$$\{q_{ij}, H_{\|}[\mathbf{N}]\} = \pounds_{\mathbf{N}} q_{ij} \tag{D.8}$$

$$\{P_q^{ij}, H_{\|}[\mathbf{N}]\} = \pounds_{\mathbf{N}} P_q^{ij} \tag{D.9}$$

$$\tag{D.10}$$

For the functional

$$F[T] = \int \mathrm{d}^{d-1}x\, T^{i\ldots j}{}_{k\ldots l} f^{k\ldots l}{}_{i\ldots j}(q, P_q) \tag{D.11}$$

with T a general tensor that cannot be q or its momenta, f a tensor density of these variables of weight $+1$; we find

$$\{F[T], H_{\|}[\mathbf{N}]\} = -F[\pounds_{\mathbf{N}} T] \tag{D.12}$$

E Linearized QUG

In this section we prove, based on constrained Hamiltonian formalism, that the path integrals of GR and UG contain the same determinants. Insofar as the Lagrangian path integral is derived from the Hamiltonian one, this proof is somewhat more fundamental. For our purpose it is going to be enough to consider the case of a flat background. Then GR reduces to the Fierz-Pauli theory with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \partial_\alpha h_\mu{}^\alpha \partial_\beta h^{\mu\beta} - \partial_\alpha h_\mu{}^\alpha \partial^\mu h + \frac{1}{2}\partial_\alpha h \partial^\alpha h \ . \tag{E.1}$$

The canonical analysis of the Fierz-Pauli theory has been discussed earlier in [91]. Here we shall redo it in a different set of variables.

In view of the canonical analysis, we begin by decomposing all tensors into time (0) and space $(i, j \ldots)$ components. We rename the variables as follows: $h_{00} = -2\phi$, $h_{0i} = v_i = v_i^T + \partial_i v$, where $\partial_i v_i^T = 0$, and for the space metric we use the York decomposition

$$h_{ij} = h_{ij}^{\mathrm{TT}} + \partial_i \zeta_j + \partial_j \zeta_i + \left(\partial_i \partial_j - \frac{1}{d-1}\delta_{ij}\partial^2\right)\tau + \frac{1}{d-1}\delta_{ij} t\,, \tag{E.2}$$

where $\partial_i \zeta_i = 0$, $\partial_i h_{ij}^{TT} = 0$, $h_{ii}^{TT} = 0$ (summed over i). These are the variables that are often used in the analysis of cosmological perturbations.

Under an infinitesimal gauge transformation $\epsilon_\mu = \{\epsilon_0, \epsilon_i\}$

$$\delta\phi = \dot{\epsilon}_0 \tag{E.3}$$

$$\delta v_i = \partial_i \epsilon_0 + \dot{\epsilon}_i \tag{E.4}$$

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i \tag{E.5}$$

The transformation parameter can be decomposed in transverse and longitudinal parts

$$\epsilon_i = \epsilon_i^T + \partial_i \epsilon \tag{E.6}$$

Then we get

$$\delta\phi = \dot{\epsilon}_0 \tag{E.7}$$

$$\delta v = \epsilon_0 + \dot{\epsilon} \tag{E.8}$$

$$\delta v_i^T = \dot{\epsilon}_0^T \tag{E.9}$$

$$\delta t = 2\partial^2 \epsilon \tag{E.10}$$

$$\delta\tau = 2\epsilon \tag{E.11}$$

$$\delta\zeta_i = \epsilon_i^T \tag{E.12}$$

$$\delta h_{ij}^{TT} = 0 \tag{E.13}$$

¿From here we see that the scalar combinations:

$$\Phi = -\frac{1}{2(d-1)}(t - \partial^2\tau) \ ; \qquad 2\phi - 2\dot{v} + \ddot{\tau} \tag{E.14}$$

(Φ is the Bardeen potential) and the vector combination $v_i^T - \dot{\zeta}_i$, as well as h_{ij}^{TT} are gauge invariant. Also, the combination $\dot{t} - 2\partial^2 v$ is invariant under spacial diffeomorphisms ($\epsilon_0 = 0$).

We insert the new variables in (E.1) and calculate the Lagrangian $L = \int d^{d-1}x\mathcal{L}$. We allow integration by parts of spacial derivatives. Also, we remove all time derivatives of v_i^T and v by adding suitable total time derivative terms. In this way we arrive at:

$$\begin{aligned} L &= \int d^{d-1}x \Bigg[\frac{1}{2}\left(\dot{h}_{ij}^{TT}\right)^2 - \dot{\zeta}_i\partial^2\dot{\zeta}_i + 2(d-1)(d-2)\dot{\Phi}^2 + 2(d-2)\dot{\Phi}(\dot{t} - 2\partial^2 v) \\ &\quad + 2\dot{\zeta}_i\partial^2 v_i^T + \frac{1}{2}h_{ij}^{TT}\partial^2 h_{ij}^{TT} - v_i^T\partial^2 v_i^T - 2(d-2)(d-3)\Phi\partial^2\Phi + 4(d-2)\phi\partial^2\Phi \Bigg] \end{aligned} \tag{E.15}$$

We perform the Dirac constraint analysis on this Lagrangian. From (E.15) we derive the conjugate momenta

$$\Pi_{ij}^{TT} = \dot{h}_{ij}^{TT} \tag{E.16}$$
$$\Pi_i^{vT} = 0 \tag{E.17}$$
$$\Pi_i^{\zeta} = -2\partial^2(\dot{\zeta}_i - v_i^T) \tag{E.18}$$
$$\Pi^{\phi} = 0 \tag{E.19}$$
$$\Pi^t = 2(d-2)\dot{\Phi} \tag{E.20}$$
$$\Pi^v = 0 \tag{E.21}$$
$$\Pi^{\Phi} = 4(d-1)(d-2)\dot{\Phi} + 2(d-2)(\dot{t} - 2\partial^2 v) \tag{E.22}$$

Equations (E.16), (E.18), (E.20) and (E.22) can be solved for the velocities $\dot{h}_{ij}^{TT}$, $\dot{\zeta}_i$, $\dot{\Phi}$, $\dot{t}$, whereas (E.17), (E.19), (E.21) give d primary constraints.

Their preservation leads to d secondary constraints

$$\Pi_i^{\zeta}\ , \qquad \partial^2\Phi\ , \qquad \partial^2\Pi^t\ . \tag{E.23}$$

There are no further constraints, and all constraints are first class. We then have to gauge fix the system. We can take the gauge fixing conditions

$$v_i^T = 0\ , \qquad \phi = 0\ , \qquad v = 0\ , \qquad \zeta_i = 0\ , \qquad \Pi^{\Phi} = 0\ , \qquad t = 0. \tag{E.24}$$

The situation is very simple, because each gauge condition is conjugate to one of the constraints. If we order all the constraints as $\phi_a = (\Pi_i^{vT}, \Pi^{\phi}, \Pi^v, \Pi_i^{\zeta}, \partial^2\Phi, \partial^2\Pi^t)$ and all gauge conditions as $\chi_a = (v_i^T, \phi, v, \zeta_i, \Pi^{\Phi}, t)$. Then the matrix of Poisson brackets $M_{ab} = \{\phi_a, \chi_b\}$ is diagonal and has determinant

$$\det M = (\det(-\partial^2))^2\ . \tag{E.25}$$

The gauge conditions determine the Lagrange multipliers in the extended Hamiltonian, which in the chosen gauge becomes

$$\begin{aligned} H_{GF} &= \int d^{d-1}x \left[\frac{1}{2}\left(\Pi_{ij}^{TT}\right)^2 + \frac{1}{4}\Pi_i^{\zeta}\frac{1}{\partial^2}\Pi_i^{\zeta} + \frac{d-1}{2(d-2)}\left(\Pi^t\right)^2 \right. \\ &\quad \left. - \frac{1}{2}h_{ij}^{TT}\partial^2 h_{ij}^{TT} - 2(d-2)(d-3)\Phi\partial^2\Phi \right] \end{aligned} \tag{E.26}$$

In particular on the constrained surface it is

$$H_C = \int d^{d-1}x \left[\frac{1}{2} \left(\Pi_{ij}^{TT} \right)^2 + \frac{1}{2} \left(\partial_k h_{ij}^{TT} \right)^2 \right] . \tag{E.27}$$

Let us now come to the path integral. The measure is

$$d\mu_{GR} = \mathcal{D}h_{ij}^{TT} \, \mathcal{D}\Pi_{ij}^{TT} \, \mathcal{D}v_i^T \, \mathcal{D}\Pi_i^{vT} \, \mathcal{D}\zeta_i \, \mathcal{D}\Pi_i^{\zeta} \, \mathcal{D}\phi \, \mathcal{D}\Pi^{\phi} \, \mathcal{D}v \, \mathcal{D}\Pi^{v} \, \mathcal{D}t \, \mathcal{D}\Pi^{t} \, \mathcal{D}\Phi \, \mathcal{D}\Pi^{\Phi}$$

The Hamiltonian path integral is

$$\begin{aligned} Z_{GR} = & \int d\mu_{FP} \, \Pi_a \delta(\phi_a) \, \Pi_b \delta(\chi_b) \det M \\ & \times \exp \left\{ i \int dt \left[\int d^{d-1}x \left[\dot{h}_{ij}^{TT} \Pi_{ij}^{TT} + \dot{v}_i^T \Pi_{v_i^T} + \dot{\zeta}_i \Pi_{\zeta_i} + \dot{\phi} \Pi^{\phi} + \dot{v} \Pi^{v} \right. \right. \right. \\ & \left. \left. \left. + \dot{t} \Pi^{t} + \dot{\Phi} \Pi^{\Phi} \right] - H \right] \right\} . \end{aligned} \tag{E.28}$$

Two of the secondary constraints contain ∂^2. Using $\delta(ax) = (1/a)\delta(x)$, they give

$$(\det(-\partial^2))^{-2} \delta(\Phi) \delta(\Pi^t) \ .$$

This power of the determinant exactly cancels $\det M$ in the path integral. All the variables are now integrated against a delta function, except for the transverse traceless tensor and its momentum, so the path integral reduces to

$$Z_{GR} = \int \mathcal{D}h_{ij}^{TT} \, \mathcal{D}\Pi_{ij}^{TT} \, \exp \left\{ i \int dt \int d^{d-1}x \, \left(\dot{h}_{ij}^{TT} \Pi_{ij}^{TT} - H_C \right) \right\} \tag{E.29}$$

where H_C is given by (E.27). Finally integrating out the momentum

$$Z_{GR} = \int \mathcal{D}h_{ij}^{TT} \, \exp \left\{ \int dt L_C \right\} . \tag{E.30}$$

where

$$L_C = \int d^{d-1}x \left[\frac{1}{2} \left(\dot{h}_{ij}^{TT} \right)^2 - \frac{1}{2} \left(\partial_k h_{ij}^{TT} \right)^2 \right] \ .$$

This is just the Lgrangian of the free fields h_{ij}^{TT}. It can be written in the "semi-covariant" way

$$L_C = \int d^{d-1}x \left[-\frac{1}{2} \partial_\mu h_{ij}^{TT} \partial^\mu h_{ij}^{TT} \right] \ ,$$

so that the Gaussian integral gives

$$Z_{GR} = \left(\det \Box_{TT} \right)^{-1/2} \ , \tag{E.31}$$

where $\Box_{TT}$ is the d'Alembertian acting on h_{ij}^{TT}. The number of independent components of a transverse traceless tensor in $d-1$ space dimensions is $d(d-3)/2$, so

$$Z_{GR} = \left(\det \Box \right)^{-d(d-3)/4} \ . \tag{E.32}$$

Linearized minimal UG

UG in minimal formulation corresponds to just setting $h = 0$ in (E.1), which effectively just removes the last two terms. In terms of the variables introduced in the preceding section, the constraint $h = 0$ implies $2\phi + t = 0$. Using this in (E.15), one finds

$$
\begin{aligned}
L_{UG} \;=\; & \int d^{d-1}x \Bigg[\frac{1}{2}\left(\dot{h}_{ij}^{TT}\right)^2 - \dot{\zeta}_i \partial^2 \dot{\zeta}_i + 2(d-1)(d-2)\dot{\Phi}^2 + 2(d-2)\dot{\Phi}\dot{t} \\
& -4(d-2)\dot{\Phi}\partial^2 v + 2\dot{\zeta}_i \partial^2 v_i^T + \frac{1}{2} h_{ij}^{TT} \partial^2 h_{ij}^{TT} - v_i^T \partial^2 v_i^T \\
& -2(d-2)(d-3)\Phi\partial^2\Phi - 2(d-2)\Phi\partial^2 t \Bigg] . \qquad \text{(E.33)}
\end{aligned}
$$

Compared to GR, we have one less scalar. From (E.33) we derive the conjugate momenta

$$
\begin{aligned}
\Pi_{ij}^{TT} &= \dot{h}_{ij}^{TT} && \text{(E.34)} \\
\Pi_i^{vT} &= 0 && \text{(E.35)} \\
\Pi_i^{\zeta} &= -2\partial^2(\dot{\zeta}_i - v_i^T) && \text{(E.36)} \\
\Pi^t &= 2(d-2)\dot{\Phi} && \text{(E.37)} \\
\Pi^v &= 0 && \text{(E.38)} \\
\Pi^\Phi &= 4(d-1)(d-2)\dot{\Phi} + 2(d-2)\dot{t} - 4(d-2)\partial^2 v && \text{(E.39)}
\end{aligned}
$$

Equations (E.34), (E.36), (E.37) and (E.39) can be solved for the velocities $\dot{h}_{ij}^{TT}$, $\dot{\zeta}_i$, $\dot{\Phi}$ and $\dot{t}$, whereas (E.35) and (E.38) give $d-1$ primary constraints

$$
C_i^{vT} = \Pi_i^{vT} \ , \qquad C^v = \Pi^v \ . \qquad \text{(E.40)}
$$

Taking Poisson brackets with the Hamiltonian one obtains $d-1$ secondary constraints

$$
S_i^{vT} = \Pi_i^{\zeta} \ , \qquad S^v = \partial^2 \Pi^t \ . \qquad \text{(E.41)}
$$

The first of these commutes with the Hamiltonian and is therefore automatically conserved. The second, however, generates a "tertiary" constraint

$$
T^v = \partial^2\partial^2\Phi \ , \qquad \text{(E.42)}
$$

which is ∂^2 times the second constraint in (E.23). Its Poisson bracket with the Hamiltonian is weakly zero, so there are no further constraints. All the constraints commute with each others and form a first class system. Their total number is one less than in Fierz-Pauli theory.

We can impose the following gauge fixing conditions:

$$
v_i^T = 0 \ , \qquad v = 0 \ , \qquad \zeta_i = 0 \ , \qquad t = 0 \ , \qquad \Pi^\Phi = 0 \ . \qquad \text{(E.43)}
$$

Let us order all the constraints as $\phi_a = (C_i^{vT}, C^v, S_i^{vT}, S^v, T^v)$ and all gauge conditions as $\chi_a = (v_i^T, v, \zeta_i, t, \Pi^\Phi)$. Then the matrix of Poisson brackets is the same as in GR except that the rows for ϕ and Π^ϕ are missing, and furthermore the term T^v-Π^Φ has an extra factor ∂^2. Therefore

$$\det M_{UG} = (\det(-\partial^2))^3 \; . \tag{E.44}$$

The time conservation of the gauge constraints fixes the Lagrange multipliers in the Hamiltonian. When this is done, one finds that the gauge-fixed Hamiltonian is again equal to (E.26).

We finally come to the unimodular path integral. The measure in the unimodular case is the same as for FP except that ϕ and Π^ϕ are missing:

$$d\mu_{UG} = \mathcal{D}h_{ij}^{TT}\,\mathcal{D}\Pi_{ij}^{TT}\,\mathcal{D}v_i^T\,\mathcal{D}\Pi_i^{vT}\,\mathcal{D}\zeta_i\,\mathcal{D}\Pi_i^\zeta\,\mathcal{D}v\,\mathcal{D}\Pi^v\,\mathcal{D}t\,\mathcal{D}\Pi^t\,\mathcal{D}\Phi\,\mathcal{D}\Pi^\Phi$$

The Hamiltonian path integral is

$$\begin{aligned} Z_{UG} \;&=\; \int d\mu_{UG}\,\Pi_a\delta(\phi_a)\,\Pi_b\delta(\chi_b)\det M_{UG} \\ &\quad \times\exp\left\{i\int dt\left[\int d^{d-1}x\left[\dot h_{ij}^{TT}\Pi_{ij}^{TT} + \dot v_i^T\Pi_{v_i^T} + \dot\zeta_i\Pi_{\zeta_i} + \dot v\Pi^v\right.\right.\right. \\ &\quad \left.\left.\left. +\dot t\Pi^t + \dot\Phi\Pi^\Phi\right] - H\right]\right\} . \end{aligned}$$

The constraints S^v and T^v contain ∂^2 and $\partial^2\partial^2$. Using $\delta(ax) = (1/a)\delta(x)$, they become

$$(\det(-\partial^2))^{-3}\delta(\Phi)\delta(\Pi^t) \; .$$

Again, the power of the determinant exactly cancels $\det M_{UG}$ in the path integral. All the variables are now integrated against a delta function, except for the transverse traceless tensor and its momentum. Thus,

$$Z_{UG} = Z_{GR} \; . \tag{E.45}$$

In these "unitary" gauges, the two path integrals are the same.

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