Scuola Internazionale Superiore di Studi Avanzati - Trieste

Phases of $\mathcal{N} = 1$ theories in $d = 2 + 1$ and non-supersymmetric conformal manifolds,

or

Is there life beyond holomorphy?

Candidate: Vladimir F. Bashmakov

Supervisor: Professor Matteo Bertolini

SISSA - Via Bonomea 265 - 34136 TRIESTE - ITALY

Abstract

In this thesis we mainly review two works, one regarding three-dimensional $\mathcal{N}=1$ field theories and the other about an interesting structure which may exist in conformal field theories, known as conformal manifolds.

After an Introduction in which we put things into context, we discuss in great detail in chapter 2 the dynamics of a special class of three-dimensional theories, that is $\mathcal{N}=1$ QCD with non-vanishing Chern-Simons level coupled to one adjoint matter multiplet. The important feature of $\mathcal{N}=1$ supersymmetric theories in $2+1$ dimensions is that the Witten index can jump on co-dimension one walls in parameter space, where new vacua come from infinity of field space. We demonstrate that this physics is captured by the two-loop effective potential. Together with the decoupling limit at large masses for matter fields, it allows to formulate a robust conjecture regarding the phase diagram of the theory. Another interesting result is the appearance of metastable supersymmetry breaking vacua for sufficiently small values of Chern-Simons level.

The third chapter focuses on the constraints that a conformal field theory should enjoy to admit exactly marginal deformations, *i.e.* to be part of a conformal manifold. While in two spacetime dimensions conformal manifolds are rather common, their existence in $d>2$ is absolutely non-trivial. In fact, in absence of supersymmetry, no single example of a conformal manifold is known in $d>2$ dimensions. Using tools from conformal perturbation theory, we derive a sum rule from which one can extract restrictions on the spectrum of low spin and low dimension operators. We then focus on conformal field theories admitting a gravity dual description, and as such a large-N expansion. We discuss the relation between conformal perturbation theory and loop expansion in the bulk, and show how such connection could help in the search for conformal manifolds beyond the planar limit. Our results do not rely on supersymmetry, here, and therefore apply also outside the realm of superconformal field theories.

Both chapters end with conclusion and outlook sections.

List of publications

The present PhD thesis is based on the following publications listed in chronological order:

1. V. Bashmakov, M. Bertolini and H. Raj, *On non-supersymmetric conformal manifolds: field theory and holography*, **JHEP 1711 (2017) 167** [arXiv:1709.01749];
2. V. Bashmakov, J. Gomis, Z. Komargodski and A. Sharon, *Phases of $\mathcal{N} = 1$ theories in 2+1 dimensions*, **JHEP 1807 (2018) 123** [arXiv:1802.10130].

The research performed by the author during his PhD studies has also led to the following articles:

3. V. Bashmakov, M. Bertolini, L. Di Pietro and H. Raj, *Scalar multiplet recombination at large N and holography*, **JHEP 1605** (2016) 183 [arXiv:1603.00387];
4. V. Bashmakov, M. Bertolini and H. Raj, *Broken current anomalous dimensions, conformal manifolds, and renormalization group flows*, **Phys. Rev. D 95 (2017) no. 6, 066011** arXiv:1609.09820.

Contents

Chapter 1

Introduction

In the same way as the theory of differential equations was the language of theoretical physics in the nineteenth and the first half of twentieth centuries, quantum field theory (QFT) has been the main language of theoretical physics in the last sixty years. Being initially born to describe the dynamics of elementary particles, it then became an essential tool in such brunches of modern physics as statistical physics, condensed matter physics, cosmology, and impacted some developments in pure mathematics.

Unfortunately, together with its importance and ubiquity, QFT is also notorious for being quite difficult to study. One can attempt to develop the perturbation theory, formulated in terms of Feynman diagrams, which gives a great deal of information about the behavior of quantum fields and can grasp a lot of interesting physics. Still, perturbation theory can be reliably applied only in the case of a weak coupling constant. In a sense it describes quantum corrections to the classical picture, and cannot shed light on intrinsically quantum phenomena. The latter, though, are very important in our understanding of nature, with the canonical example of Quantum Chromodynamics (QCD) and its confinement and chiral symmetry breaking.

Having said this, a natural question to ask is whether there are some models where also non-perturbative dynamics can be analyzed, at least partially. The answer is "yes", and one possible property that makes the theory more tractable is supersymmetry (SUSY). In the following we will assume that a QFT under consideration is Lorentz invariant [1] . The supersymmetry algebra is an extension of the Poincaré algebra by introducing fermionic generators (also called supercharges) $\mathcal{Q}$ [1].[2] Roughly, supersymmetry is a symmetry between bosonic and fermionic states of the theory, or said in a more fancy parlance a symmetry between matter and forces.

[1] Sometimes instead of Lorentz invariance we will assume Euclidean symmetry.

[2] In fact, supersymmetry is a very specific way to introduce fermionic generators, in fact the only possible extension of the Poincaré algebra, if one works under the assumptions of Haag-Łopuszański-Sohnius theorem (*e.g.* assuming mass gap).

While supersymmetric models are proved to be very useful in understanding non-perturbative physics, there is a question that always remains unanswered: how much our supersymmetric results are applicable as soon as we depart away from the realm of SUSY? How much are we able to understand about the dynamics of models lacking supersymmetry? In this thesis we will try to touch these issues, even though we will briefly discuss, before, some instances where supersymmetry makes it possible to gain a much deeper understanding of certain aspects of QFT dynamics, than it would be without it.

1.1 Vacua and Infrared Phases

One of the important questions about QFTs is the vacuum structure of the theory. One would like to know what is the space of vacua and what are the expectation values of gauge invariant operators in these vacua. In a generic QFT this boils down to the analysis of minima of the scalar potential V, which is the non-derivative function of the scalar fields present in the theory. Unfortunately, V can receive quantum corrections [2] and so strictly speaking is an unknown function.

In supersymmetric contexts it makes sense to distinguish between supersymmetry breaking vacua and supersymmetry preserving ones. The former are of phenomenological interest, since if supersymmetry has something to do with the real world, it must be spontaneously broken; these vacua are generically hard to study, even though we will say something more about them later. The latter are under much more control. In fact, in the $4d$ case already the minimal supersymmetry, containing four real supercharges and denoted by $\mathcal{N}=1$, tremendously facilitates the analysis. In this case supersymmetric vacua are given by the critical points of the *superpotential* $\mathcal{W}$ — a function of *chiral superfields*, encoding interactions of matter fields. One can think that the situation is not much better also in this case, since in principle also $\mathcal{W}$ can receive quantum corrections. In fact, this is true only partially. A crucial statement about the superpotential is that it does not receive any corrections in the perturbation theory, and can be modified only non-perturbatively,

$$\mathcal{W}_{\text{eff}} = \mathcal{W}_{\text{tree}} + \mathcal{W}_{\text{non-pert}}, \tag{1.1}$$

where $\mathcal{W}_{\text{eff}}$ is the effective superpotential in the Wilsonian sense, $\mathcal{W}_{\text{tree}}$ is the tree-level value and $\mathcal{W}_{\text{non-pert}}$ is given by non-perturbative contributions. This statement was first discovered within perturbation theory, performed in the supersymmetric case in terms of the so-called *supergraphs*, a supersymmetric generalization of familiar Feynman diagrams [3]. Later it was understood that this non-renormalization theorem can be attributed to property of superpotential called *holomorphicity* [4]. The tree-level superpotential is a holomorphic function of the chiral superfields and of the couplings (which can actually also be treated as background chiral superfields).

The idea put forward by Seiberg states that not only the tree-level superpotential, but the full Wilsonian effective superpotential must be a holomorphic function of chiral superfields and coupling constants. It implies the absence of perturbative corrections, and in some cases it can even fix the possible non-perturbative coontribution up to a constant. Because of the holomorphicity property, supersymmetry in $4d$ is sometimes said to be "complex". This complex structure is also extremely instrumental in determining VEVs of chiral operators, which form the *chiral ring*, but this idea will not be discussed further in this thesis [5].

An instance where holomorphicity was extremely useful in understanding the IR properties of $SU(N)$ SQCD (super QCD) with $0 < N_f < N$ (N_f is the number of flavors)[3], Seiberg duality further gave the description for SQCD with $N+1 \leq N_F \leq 3N$ flavors [6]. However, prior to the discussion of gauge theories with matter one should address a more basic issue. The question concerning all supersymmetric gauge theories is about the dynamics of the pure gauge sector. In $4d$ $\mathcal{N}=1$ the theory of vector multiplets only is known as Super Yang-Mills (SYM). There are classical results about this theory (again, we concentrate on the case of $SU(N)$), like the exact effective superpotential, which is just a constant in this case, given by

$$\mathcal{W}_{\text{SYM}} = N\Lambda^3, \tag{1.2}$$

where $\Lambda = \mu e^{2\pi i\tau/(3N)}$ is the strong coupling scale, μ is the sliding scale and τ is the complexified gauge coupling. This result follows from the one-loop exactness of the Wilsonian gauge coupling's β function, which is again a consequence of holomorphicity, (with respect to τ this time). Another important result about SYM is that gaugini condense [7], with the VEV for their bilinear being

$$\langle\lambda\lambda\rangle = -32\pi^2\Lambda^3. \tag{1.3}$$

The theory possesses a discrete $\mathbb{Z}_{2N}$ symmetry, under which gaugini have charge one. This symmetry is spontaneously broken by the condensate down to $\mathbb{Z}_2$, and the action of the group sweeps N isolated supersymmetric vacua of the theory.

A similar question, namely what is the IR dynamics of pure glue, can be asked for $4d$ theories with higher amount of supersymmetry. For eight supercharges this is known as Seiberg-Witten theory [8, 9], and for sixteen supercharges this is the subject of gigantic amount of literature devoted to $\mathcal{N}=4$ SYM.

Instead of discussing these fascinating topics, we will turn to the $3d$ case. $\mathcal{N}=2$ SUSY in three dimensions has the same number of supercharges as $\mathcal{N}=1$ in four dimensions, and in fact one can get multiplets of the former by dimensional reduction of the latter. Still, three dimensions bring in new ingredients in the story. First, Abelian gauge theories are strongly coupled in the IR. Second, there are real

[3] Cases of different gauge groups were also discussed in the literature

parameters such as real masses and Fayet-Iliopoulos terms. They reside in *real background superfields* and so are not controlled by holomorphy [10]. Third, there are Chern-Simons (CS) parameters, which are quantized and cannot be continuously varied. Finally, there are Coulomb branches, associated with expectation values of scalars in the vector multiplets.

The dynamics of non-Abelian $\mathcal{N}=2$ vector multiplets with vanishing CS coupling was settled long time ago [11]. We will review it later, but the rough picture is the following. Classically there is a Coulomb brunch, which is lifted non-perturbatively by the *monopole superpotential*. There are no SUSY vacua, and the theory exhibits runaway behavior. If we turn on non-zero CS couplings, one ought to expect some different picture: indeed, in the presence of CS coupling monopoles are not gauge invariant anymore, and so one cannot use their VEVs to characterize the vacua. In [10] Witten index is computed for arbitrary values of the gauge group rank N, CS level k (and also number of flavors N_f). In particular, it is noted that for $k<N$ supersymmetry is dynamicaly broken. Still, their picture is not complete. For instance, the IR dynamics of SUSY breaking vacuum, which cannot be represented by the Goldstino only, is not discussed: it does not match the 1-form symmetry anomaly, present in the UV.

In this thesis we address the question of the IR phase of $SU(N)_k$ $\mathcal{N}=2$ with $k\neq 0$. We will approach the problem deforming the theory by breaking $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$, which is the minimal supersymmetry in three dimensions. This can be done by decomposing the $\mathcal{N}=2$ vector multiplet into $\mathcal{N}=1$ vector and matter multiplets, and turning on a mass for the matter multiplet. Our strategy will then be to study IR phases of the deformed theory as a function of the mass parameter. One point of the phase diagram will correspond to the original $\mathcal{N}=2$ theory. Along the way we will discover a very rich phase space of the $\mathcal{N}=1^*$ theory.

It may look strange to attempt to study a supersymmetric theory by means of a less supersymmetric one: at the very end, we are loosing some of the power of supersymmetry by this. But actually, the same idea was recently used with much success to understand the phases of $\mathcal{N}=1$ $SU(N)_k$ theories, where $\mathcal{N}=1$ supersymmetry was broken to nothing by the mass term for gaugini. The results for $\mathcal{N}=1$ vector multiplet obtained in this way will be crucial for our discussion.

Minimal supersymmetry in $3d$ is real, as opposed to the complex minimal supersymmetry in $4d$. Consequently, all the powerful statements stemming from holomorphicity, including non-renormalization theorems, do not hold anymore. At first glance, $3d$ $\mathcal{N}=1$ supersymmetry is not much better than no supersymmetry at all. Still, there are some consequences of supersymmetry that are true also here. For example, the scalar potential is bounded from below $V\geq 0$, and vanishes only at the SUSY preserving vacua. Another property that is shared by any supersymmetric theory is the possibility to define the Witten index, originally introduced as a criterion for the possibility of SUSY breaking [12]. The idea is the following. While the supersymmetry algebra requires that all states with non-zero energy must appear in pairs - one bosonic and one fermionic, this is not necessary the case for the states with zero energy. The index is defined as the difference between the number of bosonic zero-energy states and the number of fermionic zero-energy states, and if it does not vanish supersymmetry cannot be spontaneously broken. This definition can be expressed as

$$I_{\mathrm{W}} = \mathrm{Tr}(-1)^F, \tag{1.4}$$

where I_{W} is the Witten index, F is the fermion number operator, and the trace is over the Hilbert space of the system. Clearly, this last expression is well-defined only when a QFT is put on a compact space manifold, for example on a torus, since in this case the spectrum of Hamiltonian is discrete. The index is a topological object, in the sense that it is invariant under a broad class of deformations of the action. In particular, it is invariant under those deformations of the superpotential which do not change its asymptotic behaviour at infinity. Sometimes this property makes it possible to compute the index by going to the weak coupling regime.

In Chapter 2 we use this simple but powerful idea to formulate a rather robust conjecture about the phase diagram of $3d$ $\mathcal{N}=1$ gauge theory with $SU(N)$ gauge group, arbitrary CS coupling and matter in the adjoint representation. We start by reviewing some basic facts about $3d$ gauge theories and $3d$ supersymmetry, while the rest of the Chapter is dedicated to a detailed analysis of the phase diagram as a function of the mass of matter multiplets. Briefly, the picture which emerges is the following. For large negative mass $m \to -\infty$ we have one vacuum. Depending on the relation between the rank of the group N and the CS level k, this vacuum is either supersymmetric or supersymmetry breaking. In both cases it also contains certain topological quantum field theory (TQFT). When we increase the mass and cross the zero value $m = 0$, new vacua come from infinity in field space, and then merge in a sequence of second order phase transitions, such that after we cross some critical value m_{cr} only one supersymmetric vacuum remains. As a byproduct of this analysis, we will be able to answer the original question and give more precise description of the IR phase of $SU(N)_k$ $\mathcal{N}=2$ CS theory.

1.2 Conformal Manifolds

An important class of quantum field theories is conformal field theories (CFT). Conformal field theories are theories invariant under an extension of the Poincaré

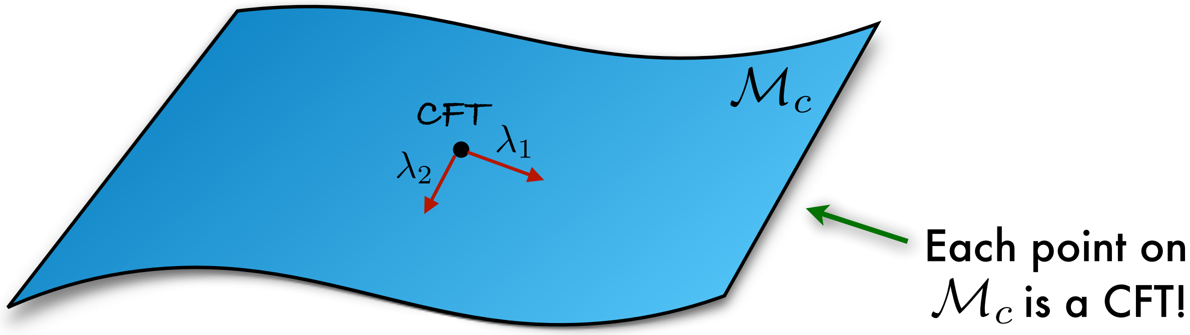


Figure 1.1. A conformal manifold is a surface in the space of couplings with every point corresponding to a CFT.

group known as the conformal group[4]. This implies, in particular, scale invariance of the theory. Conformal field theories arise as UV or IR fixed points of renormalization group (RG) flows. Having a CFT, one can deform it by adding some operators to the action[5]:

$$\delta S = \int d^d x \sum_i \lambda_i \mathcal{O}_i \ . \tag{1.5}$$

Generally, the resulting theory is not conformal anymore: new couplings introduce scale dependence into the theory, and so the deformation triggers an RG flow, with a new CFT in the IR. Conformal invariance is not always preserved even if operators $\mathcal{O}_i$ are *marginal*, that is if their dimension is equal to the dimensionality of spacetime, $\Delta_{\mathcal{O}_i} = d$. In this case coupling constants are dimensionless at tree level, but scale dependence enters quantum mechanically through the β function for the couplings. It is possible, though, that the β function vanishes, due to some mechanism. If this is the case, we say that an operator $\mathcal{O}$ is *exactly marginal*. By deforming a CFT with exactly marginal operators, by definition we get another CFT. So, we are actually dealing with a continuous family of CFTs, parametrized by exactly marginal couplings. These couplings form a surface in the space of couplings, and this surface is known as *conformal manifold* (fig. (1.1)).

The only known mechanism for the situation described above to happen is again provided by supersymmetry[6] [14]. Let's put ourselves again in the realm of $4d$ $\mathcal{N} = 1$ supersymmetry (the situation is identical in the case of other dimensions, but with the same number of supercharges.). Given the superpotential $\mathcal{W} = \lambda\Phi_1 ... \Phi_n$

[4]In footnote 2 we noticed that supersymmetry is the only possible extension of Poincaré group. This is true if the assumption about mass gap is accepted. Instead, if we relax this assumption, then the Poincaré group can be extended also to conformal and superconformal groups.

[5]Evidently, we consider only scalar operators $\mathcal{O}_i$ in order to preserve Lorentz invariance.

[6]Non-supersymmetric examples are known in two dimensions (see *e. g.* [66] and references therein.), which is very special in different respects.

($\Phi_i,\ i = 1, ..., k$ are chiral superfields), its non-renormalization together with the property of the composite chiral operators not to receive additional renormalization (both can be traced back to holomorphicity) implies the following beta function for the coupling constants λ:

$$\beta_\lambda \equiv \frac{\partial h(\mu)}{\partial \ln \mu} = \lambda(\mu) \left(-d_W + \sum_k \left[d(\Phi_k) + \tfrac{1}{2}\gamma(\Phi_k) \right] \right). \tag{1.6}$$

Above d_W is the canonical dimension of the superpotential, $d(\Phi_k)$ is the canonical dimension of the superfield Φ_k and $\gamma(\Phi_k)$ is its anomalous mass dimension. Similarly, for the complexified gauge coupling one has

$$\beta_g \equiv \frac{\partial g(\mu)}{\partial \ln \mu} = -f(g(\mu)) \left(\left[3C_2(G) - \sum_k T(R_k) \right] + \sum_k T(R_k)\gamma(\Phi_k) \right), \tag{1.7}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation, $T(R_k)$ is the quadratic Casimir of the representation in which Φ_k transforms, and $f(g)$ is a function of the gauge coupling.

For n couplings we have n constraints, coming from β function equations. So the generic possibility is to have some discrete set of solutions for the coupling constants. But it can happen that some constraints are not independent. Then, one can get a space of solutions of dimension $n - p$, where p is the number of independent constraints. In fact, this is perfectly possible due to a peculiarity of the formulas (1.6), (1.7): they are proportional to some linear combinations of anomalous dimensions, and sometimes the matrix built out of the coefficients in front of the anomalous dimensions can be degenerate. This is indeed the mechanism which allows to construct a lot of examples of (supersymmetric) conformal manifolds.

Though this criterion is useful practically and conceptually transparent, it is not completely satisfactory. First, it is formulated in terms of gauge non-invariant objects - β functions and γ functions. Second, the same objects are not scheme-independent beyond one loop. Finally, in practice one usually use some operators, related by global symmetry transformations, which provides the linear dependence of β functions, and this formulation is not covariant with respect to the global group of the theory. The hint how to give a more invariant description of supersymmetric conformal manifolds is provided by the AdS/CFT correspondence. In the context of the AdS/CFT correspondence conformal manifolds are dual to moduli space of AdS vacua [15–17]. In the bulk global symmetries of the boundary CFT are gauged, and one of the obstructions for the existence of the moduli space are D-term equations. It was proposed to use a kind of D-term constraint also on the CFT side [16, 18, 19], which for the case of superpotential deformations take the form

$$D^a = 2\pi^2\, \lambda^i\, T^a_{i\bar{j}}\, \lambda^{\bar{j}}. \tag{1.8}$$

Above λ^i are superpotential couplings and $T^a_{i\bar{j}}$ is a generator of the global symmetry group G in the representation in which the deformation operator transforms. Having then $\{\lambda_i\}$ to be the set of all marginal superpotential couplings, the conformal manifold is given by the quotient

$$\mathcal{M}_c = \{\lambda^i | D^a = 0\}/G = \{\lambda^i\}/G^{\mathbb{C}}. \tag{1.9}$$

Similar constructions appear when also gauge coupling deformations are taken into account.

We cannot retain from mentioning some geometrical structures associated with conformal manifolds. It has been pointed out by Zamolodchikov that conformal manifolds possess a natural Riemannian structure with the metric given by the two-point functions of exactly marginal operators [20]:

$$g_{ij}(\lambda) = \langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \; |x|^{2d}. \tag{1.10}$$

It was also noticed [21] that contact terms in the operator product expansion (OPE) can be treated as the connection on the manifold,

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \frac{g_{ij}(\lambda)}{|x|^{2d}} + \Gamma^k_{ij}(\lambda)\; \delta(x)\mathcal{O}_k(0) + \ldots \,. \tag{1.11}$$

The Riemann tensor is given by

$$R_{ijkl}(\lambda) = \frac{1}{2}\left(A_{kilj} - A_{kjli} + A_{ljki} - A_{likj}\right), \tag{1.12}$$

$$\text{where } A_{klij} = \int d^d x_1 d^d x_2 \; \langle \mathcal{O}_k(x_1)\mathcal{O}_l(x_2)\mathcal{O}_i(0)\mathcal{O}_j(1)\rangle. \tag{1.13}$$

It is obvious that conformal manifolds of $4d$ $\mathcal{N}=1$ SCFTs are complex manifolds. It was further proven that, in accordance with the expectation from holography, these conformal manifolds are also Kähler manifolds [22]. In the same work it was suggested that the Kähler potential must be related to the free energy of the theory, and this statement was made precise for $\mathcal{N}=2$ conformal manifolds in [23].

As we have already mentioned, all known examples of conformal manifolds in dimensions $d > 2$ are supersymmetric. It is clear from the discussion above that supersymmetry provides a mechanism for the existence of exactly marginal operators, while in the non-supersymmetric setup it is not clear what can provide vanishing β functions. Nevertheless, a no-go theorem for non-supersymmetric conformal manifolds does not exist. In fact, the existence of conformal manifolds without supersymmetry *is* compatible at least with unitarity and crossing symmetry. Indeed, as we will review later, the requirement of vanishing β functions imposes stringent constraints on the CFT data, but only regarding operators with integer spins. One can then take any of the known SCFTs belonging to a conformal manifold and truncate the spectrum of operators, excluding all operators with half-integer spins, while leaving CFT data of integer-spin operators unmodified. This is consistent, because half-integer spin operators cannot appear in the OPE of two integer spin operators. The operator algebra one ends up with is crossing-symmetric because initially it was, and also the truncated Hilbert space does not contain any negative-norm states, because the original one did not, consistently with unitarity. CFT data still obey the β function constraints, because the original theory had a conformal manifold by assumption. And, finally, the resulting operator algebra does not form a representation of the supersymmetry algebra, because it contains only integer spin operators. This might suggest it to be simple, eventually, to construct non-supersymmetric CFTs living on a conformal manifold. In fact, unitarity and crossing symmetry are necessary but not sufficient conditions to get a consistent theory. For instance, there are further conditions coming from modular invariance in two dimensions or, more generally, by requiring the consistency of the CFT at finite temperature in any number of dimensions, see, *e.g.*, [24]. This is why the truncation described above does not allow for getting non-supersymmetric conformal manifolds for free. The truncated operator algebra might not form a consistent CFT, eventually. To summarize, it remains an open question to establish the (non-)existence of conformal manifolds without supersymmetry.

In Chapter 3 we start by reminding some basic ingredients of conformal field theories, in particular the conformal perturbation theory computation of β functions up to two-loop order. We will then apply these results to the problem of constraining CFT data of theories being part of a conformal manifold and discuss, finally, some applications in a holographic context. The important point for us will be that this analysis does not rely on supersymmetry, and so can be useful for studying non-supersymmetric conformal manifolds, if they exist, or it can be a starting point for the formulation of a no-go theorem. Our results can also be useful for supersymmetric conformal manifolds, in fact, since they can give some information about the behavior of non-protected operators.

Chapter 2

Phases of $\mathcal{N} = 1$ vector multiplet with adjoint matter

This chapter is devoted to the IR dynamics of $\mathcal{N} = 1$ vector multiplets coupled to an adjoint matter multiplet in three spacetime dimensions. We will start highlighting some special properties of physics in $3d$, paying attention to fermions, CS terms and three-dimensional supersymmetry. We then review some known facts about $\mathcal{N} = 2$ vector multiplets with zero CS coupling as well as $\mathcal{N} = 1$ vector multiplets with arbitrary CS coupling. Finally, we will analyze the infrared phases of the $SU(N)$ $\mathcal{N} = 1$ coupled to an adjoint matter as a function of the mass of the adjoint. As a result of this analysis, we will come up with a proposal for the phase diagram, which in particular will let us understand the IR dynamics of $\mathcal{N} = 2$ vector multiplets with arbitrary CS coupling.

2.1 Aspects of QFT in $d = 3$

2.1.1 Fermions

Dirac algebra in three dimensions is represented by three two by two matrices γ^μ, $\mu = 0, 1, 2$, given in a Majorana basis by

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_3, \tag{2.1}$$

with the commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad g^{\mu\nu} = (1, -1, -1), \tag{2.2}$$

$$[\gamma^\mu, \gamma^\nu] = -2i\epsilon^{\mu\nu\rho}\gamma_\rho. \tag{2.3}$$

The smallest spinorial representation is given by the two-component Majorana (real) fermion χ^α. It is convenient to define the conjugation operation $\bar{\chi}=\chi^T\gamma_0$. The Lagrangian for a massive Majorana fermion is then given by

$$\mathcal{L}_M = \tfrac{1}{2}\, i\, \bar{\chi}\gamma^\nu \partial_\mu \chi + \tfrac{1}{2}\, m\, \bar{\chi}\chi. \tag{2.4}$$

One can define parity transformations

$$P: \ (x^0, x^1, x^2) \to (x^0, -x^1, x^2), \quad \chi \to \pm i\gamma^1 \chi, \tag{2.5}$$

and time reversal transformations

$$T: \ (x^0, x^1, x^2) \to (-x^0, x^1, x^2), \quad \chi \to \pm i\gamma^0 \chi. \tag{2.6}$$

While the kinetic term is invariant under both P and T, the mass term transforms as $\bar{\chi}\chi \to -\bar{\chi}\chi$, and so breaks both symmetries. One can restore the symmetries having two Majorana fermions with opposite signs for the mass terms:

$$\mathcal{L}_{2M} = \tfrac{1}{2}\, i\, \bar{\chi}_1\gamma^\nu \partial_\mu \chi_1 + \tfrac{1}{2}\, i\, \bar{\chi}_2\gamma^\nu \partial_\mu \chi_2 + \tfrac{1}{2}\, m\, (\bar{\chi}_1\chi_1 - \bar{\chi}_2\chi_2). \tag{2.7}$$

Defining P and T in a different way, as

$$P: \ \chi_1 \to i\gamma_1\chi_2, \quad \chi_2 \to i\gamma_1\chi_1, \tag{2.8}$$

$$T: \ \chi_1 \to i\gamma_0\chi_2, \quad \chi_2 \to i\gamma_0\chi_1, \tag{2.9}$$

we see that the Lagrangian (2.4) is invariant under these transformations.

Since there is no complex representations of $so(1,2)$ algebra, there is no chiral anomaly in $3d$. Still, there are anomalies involving discrete symmetries, P and T (we will loosely call these two transformations parity transformation). In particular, there is no way to couple a single Dirac fermion of charge 1 to a $U(1)$ gauge field in $P-$ or $T-$invariant way [25]: this is known as parity anomaly. This mixed gauge-time reversal anomaly disappears if one has two Dirac fermions (or, more generally, an even number of them), and the theory is parity invariant even with the gauge field turned on. One can ask then if it is possible to gauge the parity symmetries. This would imply to put the theory on a non-orientable manifold. The answer is yes if the number of Dirac fermions y is multiple of four, $y = 0 \bmod 4$ [26]. Another anomaly one can consider is the mixed gravitational-parity anomaly (see again [26]). This 't Hooft anomaly represents an obstruction to put a theory of Majorana fermions coupled to gravity on unorientable manifolds. It is conventionally denoted by ν and defined mod sixteen. Every Majorana fermion contributes $\nu = 1$.

2.1.2 Gauge fields and Chern-Simons term

An important peculiarity of $3d$ is the possibility to add CS terms to the gauge fields Lagrangian. The pure gauge part of the Lagrangian then takes the form

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2g^2}\text{Tr}F^2 + \frac{k}{4\pi}\left(A \wedge dA - \frac{2i}{3}A \wedge A \wedge A\right). \tag{2.10}$$

The coefficient k is called CS level. The last term is topological, in the sense that it does not contain metric in its definition. One can compare this CSs term with the theta term in four dimensions, but the difference is that the latter does not affect equation of motion, while the former does. In fact, Yang-Mills equation is now modified to be

$$D_\mu F^{\mu\nu} + \frac{kg^2}{2\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} = 0 \ . \tag{2.11}$$

This equation implies that there are propagating modes with mass $m_t = \frac{kg^2}{2\pi}$ (called also topological mass) and zero modes, corresponding to flat connections $F = 0$.

The CS term contains the gauge connection explicitly, and so it is not manifestly gauge invariant. A convenient way to think about the CS action is to imagine the three-manifold under consideration M to be the boundary of some four-manifold X. Then one can continue gauge fields on X and write

$$S_{\text{CS}}[A] = \frac{k}{4\pi}\int_{\partial X}\left(A \wedge dA - \frac{2i}{4}A \wedge A \wedge A\right) = \frac{k}{4\pi}\int_X F \wedge F = S_X[A] \ . \tag{2.12}$$

The expression $S_X[A]$ is manifestly gauge invariant, but it can depend on the choice of the four-manifold and on the way we continue the fields. Choosing some other manifold X', we consider the difference between the two definitions of the CS term:

$$S_X[A] - S_{X'}[A] = \frac{k}{4\pi}\int_{X \cup \bar{X}'} F \wedge F \ . \tag{2.13}$$

Here the bar means orientation inversion. But then

$$\frac{1}{8\pi^2}\int_{X \cup \bar{X}'} F \wedge F \in \mathbb{Z} \tag{2.14}$$

is the second Chern class of the bundle over the compact manifold $X \cup \bar{X}'$. Taking into account that not the action itself, but the partition function must be gauge invariant, we have

$$e^{iS_X[A] - iS_{X'}[A]} = e^{2\pi i n k} \ . \tag{2.15}$$

We conclude that gauge invariance requires the level k to be integer.

As it is obvious from the form of the CS term, time reversion changes the sign of the CS level:

$$T:\ k \to -k, \tag{2.16}$$

so CS term breaks time reversal symmetry (or parity) explicitly.

When gauge fields are coupled to fermions, the transformation law for the CS level under the action of T is more complicated:

$$T:\ k_0 \to -k_0 + \sum_f T(R_f)\ , \tag{2.17}$$

where k_0 is the bare UV CS level, the sum is over all Majorana fermions, and $T(R_f)$ is the index of the representation. Therefore, it is convenient to label the theory by the shifted (or effective) CS level $k = k_0 - \frac{1}{2}\sum_f T(R_f)$, on which time reversal acts as in (2.16).

When massive fermions are integrated out in the Wilsonian sense, the CS level gets renormalized:

$$k_{IR} = k + \frac{1}{2}\,\mathrm{sign}(m)\sum_f T(R_f). \tag{2.18}$$

According to the Coleman-Hill theorem [28], this renormalization is one-loop exact. Note that if k_0 is integer, then also is k_{IR}, while k can be either integer of half-integer.

In the IR only light degrees of freedom matter. Since propagating gauge field degrees of freedom are massive, we are left only with topological ones, described by the pure CS Lagrangian (2.12). It is a topological quantum field theory (TQFT), and due to the equation of motion $F=0$ it does not have any local observables. Still, there are non-trivial non-local observables, given by the Wilson lines,

$$W_R[\mathcal{C}] = \mathrm{Tr}_R\ e^{i\int_{\mathcal{C}} A}, \tag{2.19}$$

where R stand for representation. While there are infinitely many Wilson loops in Yang-Mills theory - one for every representation of the gauge group - in CS theories there are just finitely many of them. For example, in the case of Abelian $U(1)_k$ CS theory, independent lines are given by

$$W_n = e^{in\int A} \tag{2.20}$$

with

$$n = 0,\ \pm 1,\ ...,\ \tfrac{k-2}{2},\ \tfrac{k}{2} \tag{2.21}$$

for even k and

$$n = 0,\ \pm 1,\ ...,\ k-1,\ k \tag{2.22}$$

for odd k [27].

In order to count the number of line operators in the non-Abelian theory, one can exploit the correspondence between CS theories and $2d$ rational conformal field theories (RCFT) [29]. Then the lines are in one-to-one correspondence with the *integral* representations of the corresponding affine Kac-Moody algebra $\hat{g}_k$. These representations are labeled by the usual Dynkin indices, but with the sum of them less or equal than k

$$(\lambda_1, ..., \ \lambda_N), \qquad \sum_{i=1}^{N} \lambda_i \leq k. \tag{2.23}$$

In the case of $SU(N)_k$ it amounts to count the number of Young tableaux, fitting in the rectangle with horizontal size k and vertical size $N-1$. The solution of this combinatorial problem is

$$\frac{(k+N-1)!}{k!(N-1)!}. \tag{2.24}$$

In the following, we will consider also $U(N)_{k_1,k_2}$ CS theory, defined by

$$U(N)_{k_1,k_2} = \frac{SU(N)_{k_1} \times U(1)_{Nk_2}}{\mathbb{Z}_N}, \tag{2.25}$$

where consistency requires $k_1 = k_2 \mod N$.

An interesting and important statement is that certain CS theories, defined through different gauge groups and levels, give physically equivalent theories. This statement is known under the name of *Level-Rank duality*, and the prominent examples are (see [30] for the recent discussion)

$$SU(N)_{\pm K} \leftrightarrow U(K)_{\mp N, \mp N}, \tag{2.26}$$

$$U(N)_{K, N \pm K} \leftrightarrow U(K)_{-N, -N \mp K}. \tag{2.27}$$

2.1.3 Review of 3d $\mathcal{N} = 1$ SUSY

In this section we will sketch some basic elements of the superspace construction and multiplet structure of minimal supersymmetry in three dimensions. A detailed treatment can be found in Chapter 2 of [31].

In $3d$ the Lorentz group is $SL(2, \mathbb{R})$, and the fundamental representation acts on a two-component Majorana spinor $\psi^\alpha = (\psi^+, \psi^-)$. One can use spinor indeces to denote all other representations, with *e.g.* the vector being given by a symmetric second-rank spinor $V_{\alpha,\beta} = (V^{++}, V^{+-}, V^{--})$. Spinor indices are raised and lowered by the second-rank antisymmetric tensor

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}., \tag{2.28}$$

$$\psi_\alpha = \psi^\beta \epsilon_{\beta\alpha}, \quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta. \tag{2.29}$$

As usually, the product of two spinors is defined by $\psi\chi = \frac{1}{2}\psi^\alpha\chi_\alpha$.

Superspace is labled by three spacetime coordinates and two real Grassmann variables, denoted collectively by $z^M = (x^{\mu\nu}, \theta^\mu)$. The standard rules for differentiation and integration are summarized in the following formulas:

$$\partial_\mu \theta^\nu \equiv \{\partial_\mu, \theta^\nu\} \equiv \delta_\mu^\nu, \tag{2.30}$$

$$\partial_{\mu\nu} x^{\rho\sigma} \equiv [\partial_{\mu\nu}, x^{\rho\sigma}] \equiv \frac{1}{2}\delta^\rho_{(\mu}\delta^\sigma_{\nu)}, \tag{2.31}$$

$$\int d\theta_\alpha = 0, \tag{2.32}$$

$$\int d\theta_\alpha\, \theta^\beta = \delta^\beta_\alpha. \tag{2.33}$$

We will define *superfields* $\Phi_{\dots}(x,\theta)$ as functions on superspace (the dotes stand for Lorentz indices). Poincaré algebra is extended by the spinorial generators Q_α, such as

$$\{Q_\alpha, Q_\beta\} = 2P_{\alpha\beta}, \tag{2.34}$$

$$[Q_\alpha, P_{\beta\gamma}] = 0, \tag{2.35}$$

together with the usual commutators with the Lorentz generators $M_{\mu\nu}$. Supercharges are represented on superfields as

$$Q_\mu = i(\partial_\mu - \theta^\nu i \partial_{\nu\mu}). \tag{2.36}$$

Another commonly defined object is the supercovariant derivative

$$D_M = (D_{\mu\nu}, D_\rho) = (\partial_{\mu\nu}, \partial_\mu + \theta^\nu i \partial_{\mu\nu}). \tag{2.37}$$

Every integral over superspace of an arbibreary function of superfields and their supercovariant derivatives will provide an invariant under the whole supersymmetry algebra

$$S = \int d^3x d^2\theta\; f(\Phi,\; D_\alpha\Phi,\; \dots\;). \tag{2.38}$$

Scalar multiplet. Matter fields reside in the *scalar multiplet*

$$\Phi(x,\theta) = \phi + \theta\psi - \theta\theta F. \tag{2.39}$$

Canonical kinetic terms arise from

$$S_{\text{kin}} = -\frac{1}{2}\int d^3x d^2\theta \; (D_\alpha \Phi)^2 \, , \tag{2.40}$$

or in components

$$S_{\text{kin}} = \frac{1}{2}\int d^3x \left(-\phi\Box\phi + i\psi^\alpha \partial_\alpha^\beta \psi_\beta + F^2\right). \tag{2.41}$$

Interactions can be added by the superpotential term

$$S_{\text{int}} = \int d^3x d^2\theta \; \mathcal{W}(\Phi). \tag{2.42}$$

In components it takes the form

$$S_{\text{int}} = \int d^3x \left(\mathcal{W}''(\phi)\psi\psi + \mathcal{W}'(\phi)F\right). \tag{2.43}$$

Vector multiplet. Consider a complex scalar multiplet $\Phi(x,\theta)$. The kinetic term $|D_\alpha\Phi|^2$ is invariant under the transformation

$$\Phi \to e^{iK}\Phi, \tag{2.44}$$

$$\bar{\Phi} \to e^{-iK}\bar{\Phi} \tag{2.45}$$

with a constant K. We can promote this symmetry into a local one, considering now the superspace function $K(x,\theta)$ and covariantizing superderivative by introducing the connection superfield Γ_α:

$$D_\alpha \to D_\alpha \mp i\Gamma_\alpha. \tag{2.46}$$

where $\mp$ is for the action on Φ, $\bar{\Phi}$, respectively. Corresponding transformation rules are

$$\delta\Gamma_\alpha = D_\alpha K, \tag{2.47}$$

$$\nabla'_\alpha = e^{iK}\nabla_\alpha e^{-iK}. \tag{2.48}$$

Define the components of Γ by

$$\chi_\alpha = \Gamma_\alpha|, \qquad B = \frac{1}{2}D^\alpha\Gamma_\alpha|,$$

$$V_{\alpha\beta} = -\frac{i}{2}D_{(\alpha}\Gamma_{\beta)}|, \qquad \lambda_\alpha = \frac{1}{2}D^\beta D_\alpha \Gamma_\beta|. \tag{2.49}$$

Above χ_α and B can be gauged away (with the choice of Wess-Zumino gauge), while $V_{\alpha\beta}$ are physical and correspond to gauge field and gaugini, respectively. The vertical line stands for the bottom component of a multiplet.

In order to construct kinetic terms for gauge fields, we define a field strength superfield $W_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma_\beta$, satisfying the condition $D^\alpha W_\alpha = 0$ due to the Bianchi identity. Kinetic term then takes the form, similar to the expression in four dimensions:

$$S_{\text{kin}} = \frac{1}{g^2} \int d^3x d^2\theta \, W^2 = \frac{1}{g^2} \int d^3x \left(-\frac{1}{2} f^{\alpha\beta} f_{\alpha\beta} + \lambda^\alpha i \partial_\alpha^\beta \lambda_\beta \right). \tag{2.50}$$

CS term in the $\mathcal{N}=1$ superspace takes the form

$$S_{\text{CS}} = \frac{k}{8\pi} \int d^3x d^2\theta \, \Gamma^\alpha W_\alpha = \frac{k}{4\pi} \int d^3x \, V^{\alpha\beta} f_{\alpha\beta}. \tag{2.51}$$

All this can be straightforwardly generalized to the non-Abelian case. The only non-trivial point perhaps is the generalization of the CS term, which is now given by

$$S_{\text{CS}} = \frac{k}{8\pi} \int d^3x d^2\theta \left(\Gamma^\alpha W_\alpha + \frac{i}{6} \{\Gamma^\alpha, \Gamma^\beta\} D_\beta \Gamma_\alpha + \frac{1}{12} \{\Gamma^\alpha, \Gamma^\beta\} \{\Gamma_\alpha, \Gamma_\beta\} \right). \tag{2.52}$$

Gauge-matter coupling. To couple gauge fields and matter, we just replace superderivatives by covariantized derivatives, described above. The result is

$$\begin{aligned} S &= -\frac{1}{2} \int d^3x d^2\theta \, (\nabla^\alpha \bar{\Phi})(\nabla_\alpha \Phi) = \\ &= \int d^3x \, \left(\bar{F} F + \bar{\psi}^\alpha (i \partial_\alpha^\beta + V_\alpha^\beta) \psi_\beta + (-\bar{\psi} \lambda_\alpha + h.c.) - \bar{\phi} (\partial_{\alpha\beta} - i V_{\alpha\beta})^2 \, \phi \right). \end{aligned} \tag{2.53}$$

2.1.4 Review of 3d $\mathcal{N}=2$ SUSY

Superspace formulation and multiplet structure of $\mathcal{N}=2$ SUSY in three dimensions is very similar to that of $\mathcal{N}=1$ SUSY in four dimensions. In fact, they are related by dimensional reduction and have the same number of supercharges. The chiral multiplet consists of a complex scalar, a Dirac fermion and a complex auxiliary field:

$$\Phi = \phi + \theta\psi - \theta\theta \, F + \ \dots \ . \tag{2.54}$$

It can be decomposed into two $\mathcal{N} = 1$ scalar multiplets

$$\text{Chiral}_{\mathcal{N}=2} = \text{Scalar}_{\mathcal{N}=1} \oplus \text{Scalar}_{\mathcal{N}=1}. \tag{2.55}$$

A $U(1)$ gauge fields sits in the vector multiplet, given in the Wess-Zumino gauge by [32] (see also [10], Appendix A)

$$V = -i\theta\bar{\theta}\,\sigma - \theta\gamma^{\mu}\bar{\theta}\,A_{\mu} + i\theta\theta\,\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\,\theta\lambda + \tfrac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta}\,D. \tag{2.56}$$

In the $\mathcal{N} = 1$ it can be written as

$$\text{Vector}_{\mathcal{N}=2} = \text{Vector}_{\mathcal{N}=1} \oplus \text{Scalar}_{\mathcal{N}=1}. \tag{2.57}$$

The gauge field strength is contained in the so called linear multiplet:

$$\Sigma \equiv -\frac{i}{2}\bar{D}D\;V \equiv 2\pi\mathcal{J}_J = \sigma + \theta\bar{\lambda} + \bar{\theta}\lambda + \tfrac{1}{2}\theta\gamma^{\mu}\bar{\theta}\,F^{\nu\rho}\,\epsilon_{\mu\nu\rho} + i\bar{\theta}\theta\;D + \;\ldots\;. \tag{2.58}$$

This same superfield can be also considered as the current superfield for the $U(1)_J$ topological symmetry. The gauge kinetic term, CS term and Fayet-Iliopoulos term are contained in

$$S_{\text{gauge}} = \int d^3x d^4\theta \left(-\frac{1}{e^2}\Sigma^2 - \frac{k}{4\pi}\Sigma V - \frac{\zeta}{2\pi} \right). \tag{2.59}$$

In the non-Abelian case it is straightforward to write down the Yang-Mills term, using $4d$ technology. Defining the (anti)chiral field strength superfield

$$W_{\alpha} = -\frac{1}{4}\bar{D}\bar{D}\,e^{-V}D_{\alpha}\,e^{V}, \qquad \bar{W}_{\alpha} = -\frac{1}{4}DD\,e^{-V}\bar{D}_{\alpha}\,e^{V}\;, \tag{2.60}$$

kinetic terms are given by

$$S_{\text{kin}} = \frac{1}{g^2}\int d^3x d^2\theta\;\text{Tr}\;W_{\alpha}^2 + h.c. \tag{2.61}$$

It is trickier to come up with the superspace version of the CS term, which takes the form [33–36]

$$S_{\text{CS}} = \frac{k}{4\pi}\int d^3x d^4\theta \int_0^1 dt\;\text{Tr}\left[V\bar{D}^{\alpha}\left(e^{-tV}D_{\alpha}\,e^{tV}\right)\right]. \tag{2.62}$$

In the following, it will be useful to have the picture of the IR dynamics of a pure $SU(N)$ vector multiplet. We start with the $SU(2)$ case. Classically this theory has a Coulomb branch on the moduli space, where at an arbitrary point the gauge group is partially broken, with an unbroken $U(1)$ factor. This Coulomb branch is parametrized by the VEV of a scalar σ in the vector multiplet, but also by the *dual photon* a. The latter is a real compact scalar ($a \sim a+2\pi$), on which the topological symmetry acts by the shift transformation:

$$U(1)_J: \ a \to \ a+\alpha. \tag{2.63}$$

In fact, it is possible to dualize not just the gauge field, but the whole Abelian vector multiplet, describing the infrared physics. The effective Lagrangian is given by a real function f:

$$\mathcal{L}_{\text{eff}} = -\int d^4\theta \ f(\Sigma). \tag{2.64}$$

One can change the description by dualizing the linear multiplet Σ. Locally, this is achieved by the addition of a Lagrange multiplier chiral superfield U to the action. This chiral superfield is at the moment non-dynamical. The resulting Lagrangian is

$$\int d^4\theta \ \left(-f(\Sigma) + (U+\bar{U})\frac{\Sigma}{2\pi}\right). \tag{2.65}$$

In order to get a description in terms of the chiral superfield U, we should integrate out Σ. This can be done semiclassically, since the effective theory under consideration is IR free, and this integration out is equivalent to a Legendre transform. The equation of motion for Σ is

$$U+\bar{U} = 2\pi\, \partial_\Sigma\, f(\Sigma). \tag{2.66}$$

Solving this equation for Σ and denoting the solution by $\Sigma(U)$, we get the Lagrangian in terms of new variables

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \int d^4\theta\, K(U,\bar{U}), \\ K(U+\bar{U}) &= -f\left(\Sigma(U)\right) + (U+\bar{U})\,\frac{\Sigma(U)}{2\pi}. \end{aligned} \tag{2.67}$$

In particular, we get the Kähler potential for U in terms of the function f.

As a result, the moduli space can be equally well described by the VEV of the complex scalar, the bottom component of U. The new and the old moduli are related by $\phi = \sigma + iga$. It is convenient to define the chiral superfield $Y = e^{U/g}$, on which the topological symmetry acts linearly. In fact, Y has charge one under the topological symmetry.

$\mathcal{N}=2$ $3d$ supersymmetry possesses $SO(2)_R$ R-symmetry, and it is useful to determine the charge of Y under this R-symmetry. For this purpose we couple the theory to a background $SO(2)_R$ field B. Integrating out heavy (off-diagonal) fermions, we generate a mixed CS term for the background field B and the gauge $U(1)$ field A:

$$\frac{2}{2\pi} B \wedge dA. \tag{2.68}$$

Indeed, there are two off-diagonal complex fields with opposite sign of mass, and with the R-symmetry and gauge charges $(1, 2)$ and $(1, -2)$. Each of the two fermions contributes $q_B q_A/(2\pi)$, with the result of (2.68). This CS coupling can be viewed as the coupling of the field B to the topological current, under which Y is charged. So, from (2.68) we can read off that the R-charge of Y is -2.

We can now come back to the analysis of the moduli space. The superpotential of $\mathcal{N} = 2$ SUSY does not receive any perturbative corrections, but there are possible non-perturbative ones. In fact,

$$\mathcal{W} = \frac{1}{Y} \tag{2.69}$$

has exactly the suitable quantum numbers, in particular the correct R-charge 2 (the overall scale needed in the expression above for dimensional reasons, is omitted). In fact, the instanton calculation of [11] exactly demonstrates the presence of this term. It follows that the classical moduli space is lifted at quantum level, and the theory exhibits runaway behavior.

Similarly, for $SU(N)$ gauge group one defines monopole operators

$$Y_i = e^{(U_i - U_{i+1})/g}, \quad i = 1, \ \ldots, N-1 \tag{2.70}$$

amd there is a non-perturbatively generated Toda-like superpotential

$$\mathcal{W} = \sum_i \frac{1}{Y_i}. \tag{2.71}$$

2.2 Dynamics of $\mathcal{N} = 1$ Vector Multiplet

An important aspect to be understood is the dynamics of pure $\mathcal{N} = 1$ vector multiplets, in particular the structure of IR phases. This section is dedicated to the review of this issue. The vector multiplet consists of a gauge field A and a Majorana fermion λ in the adjoint representation of the gauge group. The most general renormalizable Lagrangian has rather restricted form and is given by

$$\mathcal{L} = -\frac{1}{4g^2}\mathrm{Tr}F^2 + i\mathrm{Tr}\lambda D\lambda + \frac{k}{4\pi}\mathrm{Tr}\left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A\right) + m\,\mathrm{Tr}\lambda\lambda. \tag{2.72}$$

In accordance with our previous discussion, the level k is integral for N even, and half-integral for N odd.

The theory becomes $\mathcal{N}=1$ supersymmetric for $m=-\frac{kg^2}{2\pi}$, but it is sometimes convenient to think about it as a special point in the family of theories parametrized by the real parameter m. We will call this theory $\mathcal{N}=1$ $SU(N)_k$ vector multiplet below.

Consider the theory with supersymmetric value of the mass parameter in the large k limit, when the theory is weakly coupled. We can integrate out fermions (whose mass has negative sign), which renormalize the CS level at one loop, and then get a $SU(N)_{k-N/2}$ TQFT in the IR. We can then conclude that for large k Witten index is given by the number of states of $SU(N)_{k-N/2}$ on a torus, namely

$$I=\frac{(k+\frac{N}{2}-1)!}{(N-1)!(k-\frac{N}{2})!}. \tag{2.73}$$

Witten showed [37] that this result in fact holds for all values of k. One can easily see that $I\neq 0$ for $k\geq N/2$, which implies that supersymmetry is unbroken. It is therefore natural to suppose that the theory still reduces to $SU(N)_{k-N/2}$ TQFT in the IR.

Instead, for $0\leq k<N/2$ one gets $I=0$. The standard interpretation is that supersymmetry is spontaneously broken in this case[7], and so there is a massless Majorana Goldstino in the IR. However, this can not be the whole story because a single massless Majorana fermion can not match various discrete 't Hooft anomaly of the theory. The simplest one is time reversal anomaly, which is there for $k=0$, when the theory is time reversal invariant. There are also non-trivial 1-form symmetry anomalies for generic values of k. The conjecture of [38] is that the IR theory consists of a Majorana Goldstino G_α together with a TQFT.

$$U\left(\tfrac{N}{2}-k\right)_{\frac{N}{2}+k,N}\overset{\text{LR}}{\simeq}U\left(\tfrac{N}{2}+k\right)_{-\frac{N}{2}+k,-N}. \tag{2.74}$$

It is worth to note that these IR gauge fields are not directly related to the UV gauge degrees of freedom, but emerge from the strong coupling dynamics. This proposal for the IR theory matches all discrete anomalies, and passes some other very non-trivial checks.

[7]Vanishing Witten index does not necessarily implies supersymmetry breaking. Though, some arguments in favor of it are given in [37], and the consistency of picture advocated in [38] leads to the same conclusion. We will assume thereof that this is indeed the case.

2.3 $\mathcal{N} = 1$ Vector Multiplet with an Adjoint Matter Multiplet

IIn this section we investigate the main model of interest of the present chapter — $\mathcal{N}=1$ vector multiplets coupled to one adjoint matter multiplet. The field content of the theory is given by a gauge field A, two Majorana fermions λ and ψ and a real scalar field X, all in the adjoint representation.

The Lagrangian consists of several parts,

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{Yuk}} + \mathcal{L}_{\mathcal{W}}, \tag{2.75}$$

with the kinetic terms

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4g^2}\,\text{Tr}F^2 + i\text{Tr}\lambda D\lambda + i\text{Tr}\psi D\psi + \text{Tr}(DX)^2, \tag{2.76}$$

the CS terms

$$\mathcal{L}_{\text{CS}} = \frac{k}{4\pi}\,\text{Tr}\left(A\wedge dA - \tfrac{2i}{3}A\wedge A\wedge A\right) - \frac{kg^2}{2\pi}\,\text{Tr}\lambda\lambda, \tag{2.77}$$

the Yukawa coupling

$$\mathcal{L}_{\text{Yuk}} = \sqrt{2}i\,g\,\text{Tr}[\lambda, X]\psi, \tag{2.78}$$

and the superpotential term

$$\mathcal{L}_{\mathcal{W}} = \text{Tr}(m^2X^2 + m\psi\psi), \tag{2.79}$$

corresponding to the superpotential $\mathcal{W} = m\text{Tr}X^2$.

For the value of the mass $m = -\frac{kg^2}{2\pi}$ the theory has $\mathcal{N}=2$ supersymmetry and the Lagrangian is that of pure $\mathcal{N}=2$ vector multiplet with $\mathcal{N}=2$ CS term. In the following we will analyze the infrared phases of this theory as a function of k and m.

2.3.1 Large Mass Asymptotic Phases

Generically one expects strong coupling dynamics in the infrared, but things simplify in the large mass limit $m \to \pm\infty$, where real multiplet can be integrated out semiclassicaly. In this regime X is getting zero VEV, providing a single vacuum, and integrating out the Majorana fermion the CS level is renormalized at one loop. As a result, we get $SU(N)_{k+N/2}$ $\mathcal{N}=1$ for large positive mass and $SU(N)_{k-N/2}$ $\mathcal{N}=1$ for large negative mass.

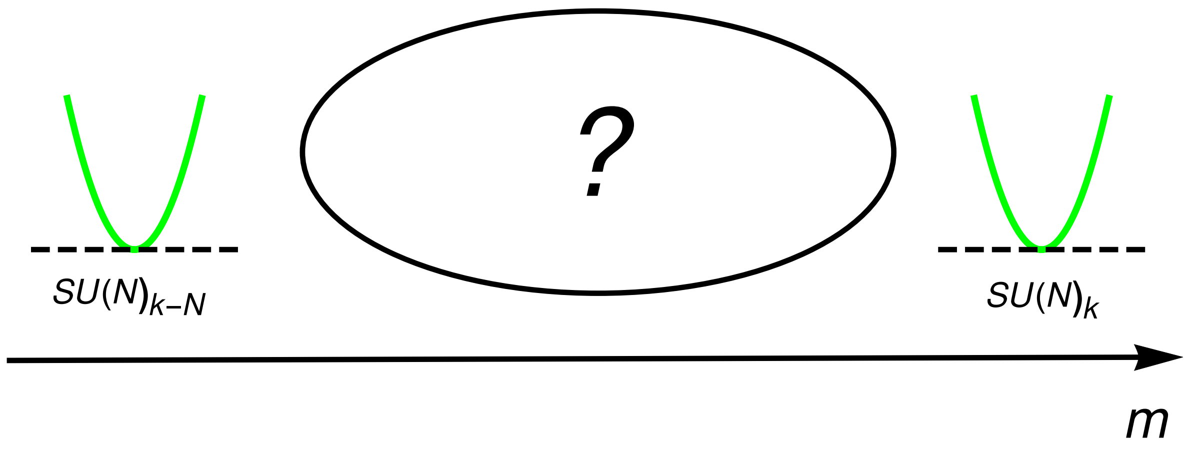


Figure 2.1. Asymptotic phases for $k \geq N$, from which it is obvious that the Witten index jumps.

At this point the dynamics reduces to that of a pure $\mathcal{N}=1$ vector multiplet, discussed in the previous section. There the relation between the rank of the gauge group and the CS level was important for the qualitative picture, and so the discussion of asymptotic phases splits into three distinct cases:

1. $k \geq N$. In this case $k \pm N/2 \geq N/2$, and so supersymmetry is unbroken in both limits. The IR theory is represented by a $SU(N)_k$ for the large positive mass limit and by $SU(N)_{k-N}$ for the large negative mass limit. Clearly, Witten index jumps as a function of m. This is depicted on figure (2.1).
2. $0 < k < N$. Now $|k - N/2| < N/2$, supersymmetry is still unbroken for large positive mass, but gets broken in the opposite limit. Correspondingly, one gets in the infrared $SU(N)_k$ TQFT for $m \to \infty$ and a Majorana Goldstino G_α together with a $U(N-k)_{k,N}$ TQFT for $m \to -\infty$. Witten index jumps also in this case: it vanishes for large negative mass and is nonzero for large positive mass. This is depicted on figure (2.2).
3. $k = 0$. Since $|k - N/2| = N/2$, supersymmetry is preserved by the vacuum in both limits. Moreover, infrared phases in both theories are gapped and trivial. Correspondingly, $I = 1$ in both cases, and this case possesses the simplest dynamics.

Above we have determined asymptotic phases for large mass limits. It was found that for the large negative mass limit one can have both supersymmetry breaking vacua and vacua with unbroken supersymmetry, depending on the value of k. Also, apart from the $|k| = 0$ case, Witten index jumps as a function of m. It is natural to suspect that the transition point must be $m = 0$, since this is when

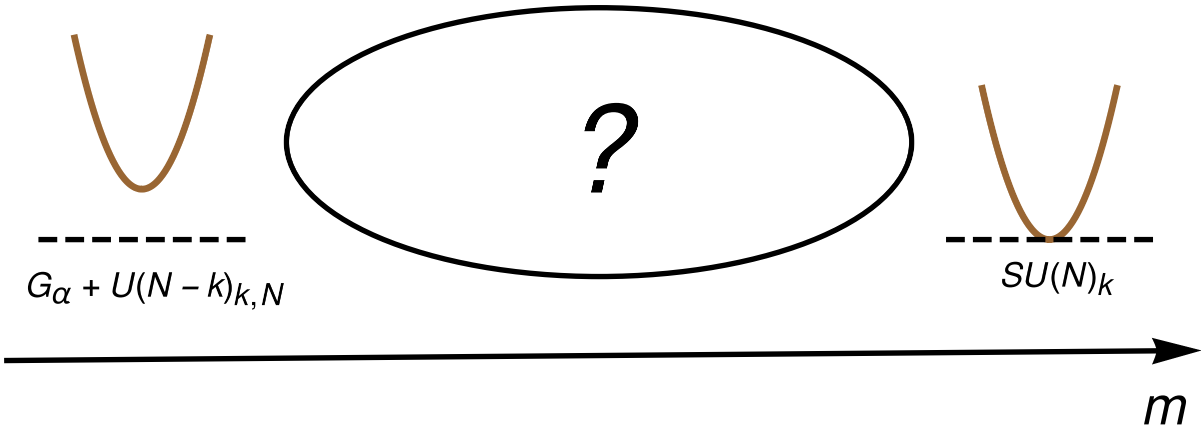


Figure 2.2. Asymptotic phases for $0 < k < N$, from which it is obvious that the Witten index jumps.

the asymptotic behavior of superpotential changes [12], and so we turn now our analysis to this point of the phase diagram.

2.3.2 Classical Moduli Space of Vacua at m=0

For the value $m = 0$ the tree level superpotential vanishes, and so classically the theory has a moduli space of vacua, parametrized by the VEV of the scalar in the real multiplet. More precisely, gauge nonequivalent configurations are parametrized by the eigenvalues of X,

$$X = \begin{pmatrix} X_1 & 0 & 0 & \dots & 0 \\ 0 & X_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & X_{N-1} & 0 \\ 0 & 0 & 0 & \dots & X_N. \end{pmatrix}, \tag{2.80}$$

where X_i are real and obey tracelessness condition $\sum_{i=1}^{N} X_i = 0$. The leftover gauge symmetry, Weyl group of $SU(N)$, permutes eigenvalues of X. As a result, the classical moduli space at $m = 0$ is given by

$$\mathbb{R}^{N-1}/S_N. \tag{2.81}$$

At the generic point of moduli space gauge symmetry is broken down to $U(1)^{N-1}$. However, there are singular loci, where some eigenvalues coincide and unbroken gauge symmetry contains non-Abelian factors. We will now discuss these distinct cases in order.

Classical Abelian Vacua

We first discuss low energy effective theory around an Abelian vacuum. The off-diagonal W-bosons together with their superpartners acquire mass from CS coupling as well as from Higgs mechanism. Their masses are given by [39]

$$m_{\pm} = \frac{kg^2}{2}\left(\sqrt{1+\frac{4m_h^2}{k^2g^4}} \pm 1\right). \tag{2.82}$$

where $\pm$ refers to positive helicity and negative helicity states, and for the off-diagonal gauge bosons W_{ij} the Higgs mechanism contribution is given by $m_h^2 = g^2 X_{ij}^2$, where $X_{ij} \equiv X_I - X_j$. Then there are $U(1)^{N-1}$ gauge fields and corresponding fermions, which are also massive, but due to the CS coupling only. And, finally, there are also $N-1$ massless $\mathcal{N}=1$ moduli multiplets (X_i, ψ_i). In the IR limit what is left are these moduli multiplets together with the TQFT, associated with the unbroken gauge symmetry $U(1)^{N-1}$.

The massless $\mathcal{N}=1$ moduli multiplets (X_i, ψ_i) are free in the infrared. The IR TQFT is just the CS theory for the unbroken gauge fields, induced from the UV CS term. To determine its precise form, one should substitute the matrix

$$\begin{pmatrix} A_1 & 0 & 0 & ... & 0 \\ 0 & A_2 - A_3 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & A_{N-2} - A_{N-1} & 0 \\ 0 & 0 & 0 & ... & A_{N-1} \end{pmatrix}. \tag{2.83}$$

into the CS action for the $SU(N)_k$ with the result

$$\frac{K^{ij}}{4\pi}\int A_i \wedge dA_j, \tag{2.84}$$

where the level matrix K^{ij} is given by

$$K = k\begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & -1 & 2 \end{pmatrix}. \tag{2.85}$$

Note that this form of the matrix K is not unique, but subject to a $SL(N-1,\mathbb{Z})$ ambiguity with the transformation

$$A \to LA, \quad K \to LKL^T, \quad L \in SL(N-1,\mathbb{Z})\ . \tag{2.86}$$

The determinant $\det K = k^{N-1}N$ has a physical meaning as the number of states of this $U(1)^{N-1}$ CS theory on the torus. From the expression for the determinant it is also easy to see that K/k cannot be put in a diagonal form. Indeed, since the original CS theory does not depend on the spin structure, also after the diagonalization it must not depend on it. But then all entries must be even, and so the determinant must be a multiple of 2^{N-1}. But this is impossible for $N>2$ because $2^{N-1}>N$ for all $N>2$.

An important result for us is that this Abelian TQFT for $k=1$ is dual to $U(1)_{-N}$ CS theory [40]

$$\frac{K^{ij}}{4\pi}\int A_i\wedge dA_j \;\leftrightarrow\; -\frac{N}{4\pi}\int A\wedge dA\ , \tag{2.87}$$

which can be thought of as a sort of Level-Rank duality.

Summarizing the classical picture just described, everywhere on the moduli space, except for the singular loci where some of X_{ij} vanish, the theory flows to a theory of $N-1$ free massless real multiplets, accompanied by the CS theory (2.84). It is important to remember though, that quantum correction to $\mathcal{N}=1$ superpotential can in principle lift some (or all) of the just described vacua.

Classical non-Abelian vacua

We now turn to the classical analysis of the non-generic vacua, where some $X_{ij}=0$, and so there is leftover non-Abelian dynamics over there. Such vacua will be important for charting the phase diagram of the theory.

X now can be represented as a diagonal matrix with L blocks labeled by $i=1,...,L$ of the size $S_i\times S_i$ and with eigenvalues X_i, such that

$$\sum_{i=1}^{L} S_i = N,\quad \sum_{i=1}^{L} S_iX_i = 0\ . \tag{2.88}$$

It is worth to note that these are not *partitions* of N, defined as the number of ways N can be written as a sum of positive integers, with the sums different just by the order of summands treated to be the same. Indeed, we have already used the Weyl group to order eigenvalues, so their order does metter and the sums different just by the order of summands must be now treated as different. Those are called *compositions* of N and there are 2^{N-1} of them.

By assumption all X_is are distinct (otherwise they would be joined into the same block). Then, all gauge fields away from the blocks (as well as their superpartners) are Higgsed, and the unbroken gauge symmetry in such a vacuum is given by

$$S[U(S_1)\times ... \times U(S_L)]. \tag{2.89}$$

One can think about the effective low energy $\mathcal{N}=1$ gauge theory in the following way. First, we extend the original gauge group from $SU(N)$ to $U(N)$. The inclusion of the extra $U(1)$ is irrelevant dynamically, since all matter is in the adjoint representation. But then it is easy to read off the structure of low energy CS theory, which is given by the product theory

$$U(S_1)_{k,k} \times U(S_2)_{k,k} \times \ \dots \ \times U(S_l)_{k,k}. \tag{2.90}$$

In order to get rid of the extra $U(1)$ gauge field, we introduce an auxiliary gauge field B with the coupling

$$\frac{1}{2\pi} B \wedge \sum_{i=1}^{L} S_i \, \mathrm{Tr} A_i, \tag{2.91}$$

which sets the overall trace to zero.

At energies below the gauge symmetry breaking scale, we are left with an $\mathcal{N}=1$ vector multiplet with gauge group and CS levels of (2.90) together with the real matter multiplets (X, ψ) in the adjoint representation of the unbroken non-Abelian gauge group. Differently from the Abelian vacua case, we are still left with a theory strongly coupled in the infrared. In order to understand the ultimate fate of these vacua, quantum corrections will be crucial.

2.3.3 Semiclassical Moduli Space of Vacua

It was already mentioned above that quantum corrections are important for understanding the vacuum structure at $m=0$, so we turn now to the semiclassical analysis.

In three-dimensional theories with minimal supersymmetry there is no obstructions for the perturbative generation of a superpotential. So, in the Wilsonian low energy effective action one should expect a superpotentail, depending on the $N-1$ coordinates, parametrizing the moduli space (2.81):

$$\mathcal{W}(X_i), \qquad \sum_{i=1}^{N} X_i = 0. \tag{2.92}$$

Semiclassical regime can be trasted in the region far away from the origin, that is when X_i are large compared with g. Also, for the semiclassical analysis to be valid we should be far away from the singular loci, supporting the residual non-Abelian dynamics, in other words, also X_{ij} must be large. So, we start our discussion, assuming the above mentioned conditions, and postponding the discussion of quantum behaviour of the vacua with non-Abelian gauge symmetry until later.

It is useful first to gain some intuition about the perturbative expansion of radiatively induced superpotential just from dimensional analysis. We will develop the perturbation theory in $1/X_{ij}$, assuming that they all scale in the same way: $X_{ij} \sim X$. We will also assume canonical normalization for the fields. Gauge will be chosen such that cubic CS term vanishes. We then have cubic interactions weighted by g, and quartic interactions weighted by g^2. CS level k then appears only in quadratic terms that are proportional to kg^2. Masses of the heavy particles (those which get mass also from the Higgs mechanism) that run in the loops are of order $M \sim gX$. Since there is also the CS contribution, this estimate is correct as far as $X \gg gk$, which is indeed the case far away on the moduli space.

Consider a vacuum L-loop diagram contribution to the quantum effective potential (i.e. the Coleman-Weinberg potential [2]). Such a diagram is weighted by a factor of g^{2L-2}. For $k = 0$ (and $m = 0$) the theory under consideration is $\mathcal{N} = 2$, where the superpotential is perturbatively non-renormalizable. This means that our L-loop must be proportional to k. Furthermore, parity implies that only even powers of k appear. We therefore have at L loops a perturbative series in $1/X$ of the form

$$V^{(L)}(X) = g^{2L-2} \sum_{n>0} d_{n,L} \frac{(kg^2)^{2n}}{(gX)^{L+2n+4}} \tag{2.93}$$

with some coefficients $d_{n,L}$. The full scalar effective potential is given by the sum over all loops:

$$V(X) = \sum_L V^{(L)}(X). \tag{2.94}$$

It turns out that the one-loop contribution vanishes,

$$V^{(1)} = 0. \tag{2.95}$$

This is expected since on general ground one expects the perturbative expansion for the generated superpotential to start with a one-loop contribution $W^{(1)}(X)$ or higher, and in the absence of a tree-level superpotential this results into the expansion of the scalar potential, starting from 2-loops $V^{(1)} = (\partial_X W^{(1)}(X))^2$.

As was reviewed above, integrating out massive fermions can induce a 1-loop correction to the CS levels. A closer look at the mass matrix for the off-diagonal Majorana fermions in $\mathcal{N} = 1$ vector and matter multiplets around the generic vacuum out of the singular loci shows that for each massive charged fermion there is a massive charged fermion with a mass of opposite sign. This can easily be illustrated e.g. by the example of $SU(2)$ gauge group broken to $U(1)$. The scalar

VEV is given in this case by

$$X = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} . \tag{2.96}$$

The off-diagonal components of two Majorana fermions at disposal, λ and ψ, can be expressed in terms of two Dirac fermions λ^+ and ψ^+ of charge two under the unbroken $U(1)$, together with their complex conjugate. The mass matrix comes from the CS coupling and the Yukawa term and takes the form

$$\begin{pmatrix} \bar{\lambda}^+ & \bar{\psi}^+ \end{pmatrix} \begin{pmatrix} -\frac{kg^2}{2\pi} & 2\sqrt{2}i\, gx \\ -2\sqrt{2}i\, gx & 0 \end{pmatrix} \begin{pmatrix} \lambda^+ \\ \psi^+ \end{pmatrix} . \tag{2.97}$$

Clearly, it has one positive and one negative eigenvalue. Furthermore, the Coleman-Hill theorem [28] guarantees that the CS term does not receive any corrections from the higher orders in perturbation theory. One can also suspect that CS term can receive renormalization from the integrated out heavy W-bosons. This issue is discussed in the Appendix A, with the conclusion that this extra renormalization actually does not take place.

To summarize, the moduli space parametrized by $N-1$ scalars from the $\mathcal{N}=1$ real multiplet is not lifted at one loop. Also, to all orders in pertutbation theory the infrared CS theory is $U(1)^{N-1}$ with the level matrix (2.85). In order to uncover the first non-trivial quantum corrections to the moduli space, one has to go to higher-loop order.

The scalar potential (2.94) can be recasted in terms of a superpotential

$$\mathcal{W}(X) = kg^3 \sum_{L>1} g^L \sum_{n>0} c_{n,L} \frac{(kg^2)^{2n-2}}{g^{2n} X^{L+2n-5}}, \tag{2.98}$$

where $c_{n,L}$ are some coefficients, and the fact that one-loop contribution vanishes was taken into account. A very nice fact about the expansion (2.98) is that any given term in $1/X$ expansion receives contributions from finitely many loop orders in perturbation theory. The leading term corresponds to $L=2$, $n=1$, and it scales linearly: $W(X) \sim X$. This term receives contributions from 2-loop diagrams only, and it is crucial for the understanding of the low-energy dynamics of this $\mathcal{N}=1$ theories to check whether it is present or not.

In fact, the computation of a two-loop superpotential was already performed in [41, 42] in some other context, with the result

$$\mathcal{W} = -\sum_{ij} g^3\, k\, \sqrt{g^2k^2 + X_{ij}^2}, \tag{2.99}$$

expressed in terms of the eigenvalue differences $X_{ij}=X_i-X_j$, defined before. Note that it vanishes for $k=0$, in accordance with the fact that quantum corrections to the superpotential vanish when supersymmetry is enhanced to $\mathcal{N}=2$. To be precise, [41, 42] computed the scalar potential. The superpotential we have quoted can be reconstructed from the scalar potential, but up to some ambiguities. The first one is related to the additive constant, but it is anyway unphysical. The second one is the overall sign. It can be determined by studying diagrams with external fermions. Here we have just chosen it such that the whole picture is consistent.

Recall that we were interested in the two-loop superpotential in order to extract the leading term in the $1/X$ expansion. Basically, this is the only information coming from the superpotential (2.99) that we can trust as far as the "far zone" $X \gg gk$ is concerned. Indeed, subleading terms in the $1/X$ expansion such as X^0 (i.e. it could be $\log X$) or $1/X$ receive contribution also from three loops and four loops, respectively. So the only reliable information, coming from (2.99) is

$$X \gg gk : \qquad \mathcal{W}=-g^3\, k \sum_{ij} |X_{ij}| + \mathcal{O}\left((1/X)^0\right). \tag{2.100}$$

It follows from (2.100) that at two loops the classical moduli space is lifted, and there is a flat non-supersymmetric direction in the limit $X\to\infty$. We are now turning to the consequences of the two-loop superpotential for the phases of the theory around $m=0$.

2.3.4 Semiclassical Abelian Vacuum near $m=0$

Having understood the IR dynamics of the theory for $m=0$, we now can address a small deformation around this point,

$$\delta\mathcal{W}=m\,\mathrm{Tr}\,X^2+\lambda\,\mathrm{Tr}\,X, \tag{2.101}$$

where we have also added a Lagrange multiplier λ to enforce that X is a traceless matrix. Taking into account the two loop effective superpotential, and working at leading order in m (such that the effective potential is computed for $m=0$), we find that the superpotential on the moduli space takes the form

$$\mathcal{W}=-\sum_{ij} g^3\, k\,\sqrt{g^2k^2+X_{ij}^2}+m\sum_i X_i^2+\lambda\sum_i X_i, \tag{2.102}$$

with i,j ranging over $1,\ldots,N$. We will be mostly interested in the "far zone" regime $X_{ij}\gg gk$, where the superpotential can be consistently approximated by

$$\mathcal{W}=-g^3\, k\sum_{ij}|X_{ij}|+m\sum_i X_i^2+\lambda\sum_i X_i. \tag{2.103}$$

In order to find supersymmetric vacua, we should solve standard F-term equations

$$\frac{\partial \mathcal{W}}{\partial X_i} = 0, \tag{2.104}$$

$$\frac{\partial \mathcal{W}}{\partial \lambda} = 0. \tag{2.105}$$

For the case under consideration it takes the form

$$-g^3\, k \sum_j \operatorname{sign}(X_{ij}) + mX_i + \frac{1}{2}\lambda = 0, \tag{2.106}$$

$$\sum_i X_i = 0. \tag{2.107}$$

Summing over i in the first equation (and using the second), we get $\lambda = 0$. As a result, we get

$$-g^3\, k \sum_j \operatorname{sign}(X_{ij}) + mX_i = 0, \tag{2.108}$$

$$\sum_i X_i = 0. \tag{2.109}$$

The last equation implies that at least one of the X_i must be positive. Using the residual S_N gauge group, we can order all eigenvalues by their values, from the greatest (positive) X_1 to the lowest (negative) X_N. Having done this and choosing $i=1$, we see that the first term in (2.108) is negative, while the second is positive for $m>0$ and negative for $m<0$. It then follows that there is no any supersymmetric Abelian vacua in the "far zone" for $m<0$.

For small positive m one can find a solution of the equations above, which is (up to the action of Weyl group)

$$X_i = \frac{g^3 k}{m}\,(N+1-2i). \tag{2.110}$$

The eigenvalue differences X_{ij} are indeed parametrically large for $m \ll g^2$, and so this solution is consistent with our approximations, and not just an artifact of two-loop perturbation theory. It implies that there exists one $\mathcal{N}=1$ supersymmetric vacuum, supporting the $U(1)^{N-1}$ CS theory (2.84) in the infrared.

As an intermediate conclusion, here we don't observe any supersymmetric vacua in the "far zone" for the small negative mass, but a new Abelian vacuum appears when the mass is small but positive. One can keep in mind a picture with a potential, that grows everywhere at large X for $m<0$, develops asymptotically flat

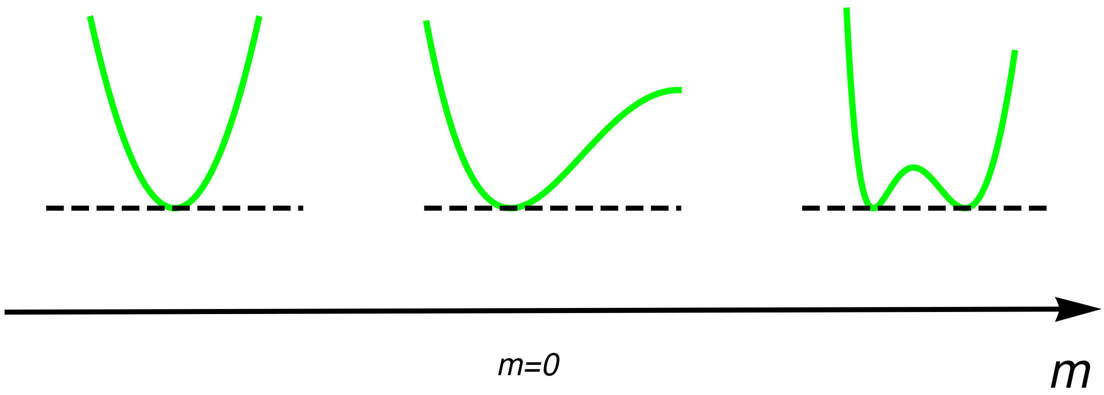


Figure 2.3. Intuitive picture for the behavior of the effective potential near the $m = 0$ point: the potential grows in all directions for $m < 0$, developes a flat direction for $m = 0$, and a new vacuum comes for $m > 0$.

direction for $m = 0$, and gets a new minimum, corresponding to the supersymmetric vacuum, when m is slightly positive, see fig. (2.3). This new vacuum comes from infinity in the field space and carry non-vanishing Witten index.

2.3.5 Phases of the Theory with $k \geq N$

Analysis of the previous subsection is valid for any values of $k > 0$. Strictly speaking, we were working under the assumption of being in the "far zone", but as will be clear later all the conclusions regarding the existence of the supersymmetric Abelian vacuum can be continued to arbitrary values of X. It was crucial though that all the eigenvalues of X are distinct, and so the unbroken group is Abelian, with a free theory in the infrared. We now turn to the consideration of singular loci, where some eigenvalues coincide, and the unbroken gauge group is non-Abelian. To make the analysis tractable, we will assume the large k limit, which makes the dynamics "semiclassical". Practically, it will mean that $k \geq N$.

Critical Points of the Superpotential

As it was mentioned, we can assume that k is large in order to gain an understanding of degenerate vacua. Deforming the two-loop superpotential (2.98) by mass terms, but not assuming large X this time, we get the following equations for critical points of the superpotential:

$$-g^3\,k\sum_j \frac{X_{ij}}{\sqrt{g^2k^2+X_{ij}^2}}+\frac{1}{2}\lambda+mX_i=0, \tag{2.111}$$

$$\sum_i X_i=0. \tag{2.112}$$

Summing over i in the first equation we find $\lambda=0$. Therefore the first equations simplifies to

$$mX_i=g^3\,k\sum_j \frac{X_{ij}}{\sqrt{g^2k^2+X_{ij}^2}}. \tag{2.113}$$

It is more convenient to work with the rescaled quantities $gk\tilde{X}=X$ such that the equation takes the form

$$\frac{m}{g^2}\,\tilde{X}=\sum_j \frac{\tilde{X}_{ij}}{\sqrt{1+\tilde{X}_{ij}^2}}. \tag{2.114}$$

Clearly, we have $X_i=0$ as a solution.

Let us now search for solutions where not all X_is vanish. This is possible only if $\frac{m}{g^2}\in(0,N)$. To prove this, assume that indeed $\tilde{X}_1>0$, and it is the greatest eigenvalue. Then, $\tilde{X}_i\geq 0$ for all j and we have

$$\sum_j \frac{\tilde{X}_{ij}}{\sqrt{1+\tilde{X}_{ij}^2}}\leq\sum_j\tilde{X}_{ij}=\sum_j\tilde{X}_1-\sum_j\tilde{X}_j=N\tilde{X}_1, \tag{2.115}$$

so that

$$\tfrac{m}{g^2}\,\tilde{X}_1\leq N\tilde{X}_1. \tag{2.116}$$

From this it follows that the system has non-trivial solutions only for $\frac{m}{g^2}\in(0,N)$. The argument above is based on an estimate, and so the precise critical value for the mass to forbid the non-trivial solutions can be modified (*e.g.*, by higher loop corrections or by a more accurate estimate).

One can write down the solutions explicitly, assuming again that the differences between eigenvalues in different blocks are large. In this case the superpotential

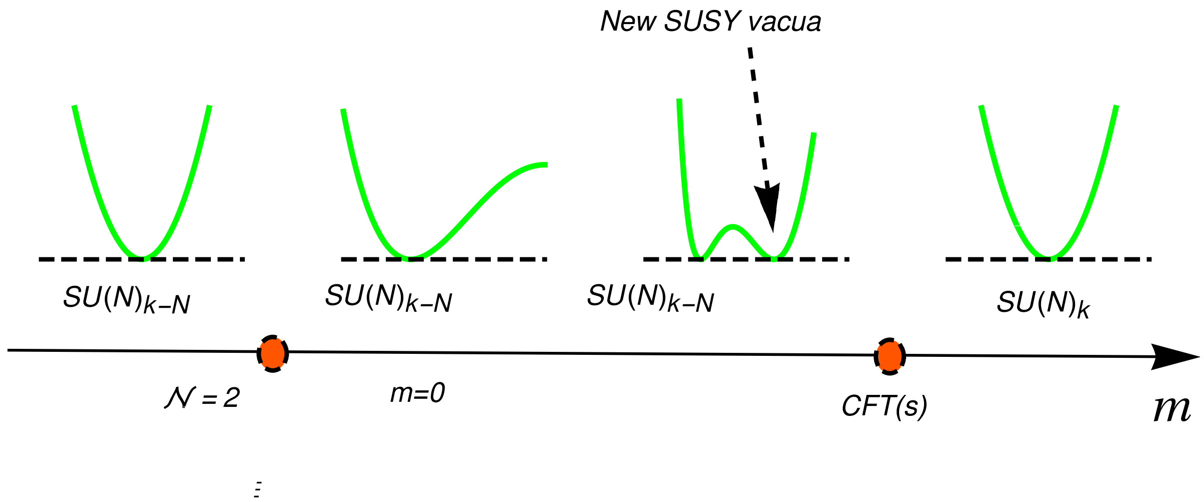


Figure 2.4. Phase diagram for the case $k \geq N$.

can again be approximated by (2.100). Choosing the block of size $S_I \times S_I$ (recall that there are L of them) and ordering the eigenvalues $X_I, \quad I = 1, ..., L$ as before,

$$X_1 > X_2 > ... > X_L, \tag{2.117}$$

we get the solution

$$X_I = \frac{g^3 k}{m} \left[(S_I + ... + S_L) - (S_1 + ... + S_{I-1}) \right]. \tag{2.118}$$

We conclude that for every composition of N there is a corresponding vacuum. In total there are 2^{N-1} vacua, including the one at the origin and the Abelian vacuum discussed in the previous subsection. The only supersymmetric vacuum that exists at small negative mass is the one at the origin.

On the basis of our findings we can establish the following picture (summarized in fig. (2.4)). For negative mass there is just one supersymmetric gapped vacuum with $SU(N)_{k-N}$ TQFT in the infrared. As soon as we cross the $m = 0$ value, new vacua come from infinity, with the total number of 2^{N-1} and with various TQFTs each. While we increase the value of m, all these vacua merge in a sequence of second order phase transitions, such that for (roughly) $m > g^2 N$ there is just one vacuum left, with the $SU(N)_k$ TQFT in the infrared, and persisting for arbitrarily large values of the mass. The merging of the intermediate vacua must indeed be of second order, because all of them carry non-trivial Witten index, and we don't expect the total index to jump at other values of the mass but $m = 0$. The precise merging pattern and the sequence of CFTs arising from that are rather complicated.

It is important to determine the precise TQFTs, appearing in the 2^{N-1} vacua. In particular, it will allow to check that these vacua indeed account for the required jump of Witten index. For the sake of simplicity this analysis will be performed for the gauge group $U(N)$ instead of the original $SU(N)$. It simplifies the counting problem but doesn't change physics much, since the $U(1)$ factor just decouples. We start once again with the superpotential

$$\mathcal{W} = -\sum_{ij} g^3\, k\,\sqrt{g^2k^2 + X_{ij}^2} + m\sum_i X_i^2, \tag{2.119}$$

this time without the Lagrange multiplier term. Our final interest lies in getting the fermions mass matrix, so we expand the superpotential around the critical point X_i^0, with the result

$$\begin{aligned}\mathcal{W} \;&= -\tfrac{1}{2}\textstyle\sum_{ij} g^3\, k\,\sqrt{g^2k^2 + (X_{ij}^0)^2}\left(\tfrac{2X_{ij}^0\delta X_{ij}+\delta X_{ij}^2}{g^2k^2+(X_{ij}^0)^2} - \left(\tfrac{X_{ij}^0\delta X_{ij}^2}{g^2k^2+(X_{ij}^0)^2}\right)^2\right)\\ &\quad +2m\textstyle\sum_i X_i^0\delta X_i + m\sum_i \delta X_i^2.\end{aligned} \tag{2.120}$$

The piece linear in δX_i drops out, since the expansion is performed around the critical point of the superoptential. We then find after some simplification

$$\mathcal{W} = -\frac{1}{2}\sum_{ij}\frac{g^5k^3\delta X_{ij}^2}{(g^2k^2+(X_{ij}^0)^2)^{3/2}} + m\sum_i \delta X_i^2. \tag{2.121}$$

We will work in the small m/g^2 limit, where X_i^0 can be approximated by (2.118). Now, in analyzing (2.121) it is convenient to distinguish two cases. When i and j lie in the same block, then X_{ij}^0 vanishes, and the first term scales like g^2, which is much greater than m. If on the contrary i and j belong to two different blocks, then the coefficient in front of δX_{ij} scales like m^3/g^4, which is much smaller then m. The conclusion is that for $i,\ j$ in the same block we should take into account the first term in (2.121), while for $i,\ j$ in different blocks we can neglect it. The desired mass matrix for the fermions ψ_i, the superpartners of X_i, then takes the form

$$\mathcal{M} = m\mathbb{I}_{S_I\times S_I} + g^2\begin{pmatrix} -S_I+1 & 1 & 1 & ... & 1\\ 1 & -S_I+1 & 1 & ... & 1\\ 1 & 1 & -S_I+1 & ... & 1\\ . & . & . & ... & .\\ 1 & 1 & 1 & ... & -S_I+1\end{pmatrix} \tag{2.122}$$

in each $S_I\times S_I$ block, and it vanishes otherwise. Even though $m \ll g^2$, we didn't neglect the first term with respect to the second one in the expression for the mass matrix: it serves to lift the zero mode and so to make all eigenvalues of the mass matrix non-vanishing. The latter are given by

$$(-g^2S_I+m,\ -g^2S_I+m,\ ...,\ -g^2S_I+m, m). \tag{2.123}$$

With the mass matrix at hands, we can finally compute quantum corrections to the classical result (2.90) and compute the TQFTs. For sufficiently small m all eigenvalues but one are negative. Thus, under the unbroken gauge group

$$U(S_1) \times U(S_2) \times U(S_L) \tag{2.124}$$

for any factor there are $S_I - 1$ charged matter fermions and gaugini, all with negative masses (recall that in the $\mathcal{N} = 1$ vector multiplet gaugini have negative mass for positive CS level). We were in the "large k" phase with respect to the original gauge group, which is *a fortiori* true for the unbroken components. Integrating out all fermions at one loop, we get a supersymmetric vacuum with

$$U(S_1)_{k-S_1,k} \times U(S_2)_{k-S_2,k} \times ... \times U(S_L)_{k-S_L,k} \tag{2.125}$$

TQFT. The Witten index carried by this vacuum is given by

$$\prod_I \frac{k!}{S_I!(k-S_I)!}. \tag{2.126}$$

Computing the total index, we should sum over all vacua. But here it comes a subtle issue: there is no canonical way to determine if the particular vacuum is bosonic or fermionic. It is not a problem if we have just one vacuum, since we can just assign the corresponding zero modes to be bosonic; at the very end, often we are just interested on wether the index vanishes or not. But if there are several vacua, they can come with relative signs, and it is crucial to know the relative signs in order to compute the total index. Similar issue arises when one considers mass-deformed supersymmetric sigma-model, studied in [12]. The prescription proposed there is to count the number of negative mass fermions n_- in each vacuum. The relative sign is then given by $(-1)^{n_-}$.

Applying this prescription to our problem and using again (2.123), we get the relative signs

$$(-1)^{\sum_I (S_I - 1)} = (-1)^{N-L}, \tag{2.127}$$

where L is the length of the partition. The totoal index is given by the sum over the vacua, or by the sum over different compositions of the rank, $N = \sum_{I=1}^{L(P)} S_I$ ($L(P)$ is the length of a partition P). The Witten index is then given by

$$I = \sum_P (-1)^{N-L} \prod_{I=1}^{L} \frac{k!}{S_I!(k-S_I)!}. \tag{2.128}$$

It turns out that the sum above can be dramatically simplified. As it is proven in Appendix B, summation gives the following result:

$$\sum_P (-1)^{N-L} \prod_{I=1}^{L} \frac{k!}{S_I!(k-S_I)!} = \frac{(N+k-1)!}{N!(k-1)!}. \tag{2.129}$$

The right-hand side is just the number of states of the TQFT $U(N)_{k,k}$ on a torus, the theory expected to live in the single supersymmetric vacuum, remaining in the $m \to \infty$ limit. This is a very nice confirmation of the whole picture, demonstrating that $2^{N-1}-1$ vacua coming from infinity exactly reproduce the jump of the Witten index.

2.3.6 Phases of the Theory with $0 < k < N$

Having understood the dynamics of the theory for $k \geq N$, we now turn to the case $0 < k < N$. In certain sense this is the small k regime, as opposed to the large k regime considered above, and so non-perturbative effects dominate. Still, we will see some similarity with the picture developed in the previous section.

Let us first recollect what happens in the large $|m|$ limit. When mass is large and negative, integrating out matter we get a pure $\mathcal{N}=1$ vector multiplet with gauge group $SU(N)$ and CS level $k - N/2$. According to the discussion of section (2.2), supersymmetry in this case is dynamically broken and in the IR one has one Majorana Goldstino and a TQFT

$$U(N-k)_{k,N} \leftrightarrow U(k)_{-N+k,k}. \tag{2.130}$$

There is no reason to expect any phase transitions for negative values of the mass, therefore also at the $\mathcal{N}=2$ point we are going to have (2.130) together with a Dirac goldstino. Soft mass deformation makes one Majorana component massive, while the other is left massless.

In the opposite regime of large positive mass we again integrate out matter and get a pure vector multiplet with gauge group $SU(N)$ and CS level $k + N/2$. This theory does not break supersymmetry and flows to a TQFT

$$SU(N)_k \leftrightarrow U(k)_{-N,-N}. \tag{2.131}$$

Evidently, the Witten index jumps also in this case, being zero at large negative mass and non-zero at large positive mass, and one would like to trace the mechanism of this jump. In the previous section we were studying the vicinity of the point $m = 0$. The large k limit allowed us to build a semiclassical picture with 2^{N-1} vacua. In each vacuum there were living one or several vector multiplets, each with coupled adjoint matter. Then we were able to study these IR theories semiclassicaly, integrate out matter and the flow to TQFTs in the IR.

In the present case we cannot use the large k limit, essentially because appropriate for the system 't Hooft coupling N/k is always greater than one. We will see that, as a support for this intuition, the IR phases are not the ones one would be able to obtain semiclassically.

Keeping this in mind, we propose the following way to think about this regime. We suppose that there are the same 2^{N-1} vacua, analogous to (2.118), corresponding to compositions of N. All these vacua preserve supersymmetry at perturbative level, as before, and support a bunch of vector multiplets with CS terms

$$S[U(S_1)_{k,k} \times ... \times U(S_L)_{k,k}] \tag{2.132}$$

and coupled adjoint matter with negative mass eigenvalues. The difference is that some of these theories break supersymmetry non-perturbatively, and so some of these vacua are lifted due to the non-perturbative dynamics.

It is easy to describe the lifted vacua. A vacuum breaks supersymmetry, if, for some I, $S_I > k$: this is possible since $k < N$. The vacua that remain correspond to compositions of N, satisfying the condition:

$$N = \sum_I S_I, \quad S_I \le k. \tag{2.133}$$

In order to perform the counting of vacua and indices, we again switch to $U(N)$ gauge group. IR theories at the vacua obeying condition (2.133) effectively behave semiclassically, and so flow to the TQFT

$$U(S_1)_{k-S_1,k} \times U(S_2)_{k-S_2,k} \times ... \times U(S_L)_{k-S_L,k}. \tag{2.134}$$

Now we would like to match the Witten index, carried by these vacua, with the one required for the jump, as it was done in the previous section. At first sight it seems that we have to perform a new computation, since summation now is going to be over the different set of vacua. But in fact we can use the old expression, since contributions from those vacua, which were supersymmetric for $k \ge N$ and break SUSY for $k < N$ vanish (in accordance with the fact that the Witten index for those vacua vanish). As the result, we again observe that vacua coming from infinity are just right to match the jump of the index.

We can now summarize the picture (see fig. (2.5)). For $m \le 0$ there are no supersymmetric ground states, and the system has at least one supersymmetry breaking vacuum, hosting a TQFT. At $m = 0$ an asymptotically flat direction with non-zero energy density opens, and for $m > 0$ new supersymmetric vacua appear. They coexist with one or more non-supersymmetric metastable vacua. While m is increased, supersymmetric vacua coalesce, such that for large enough m there is just one supersymmetric ground state, supporting a $SU(N)_k$ TQFT.

An interesting special case is $SU(N)_1$. The only supersymmetric vacuum corresponds to the composition given by

$$N = 1 + 1 + ... + 1\ . \tag{2.135}$$

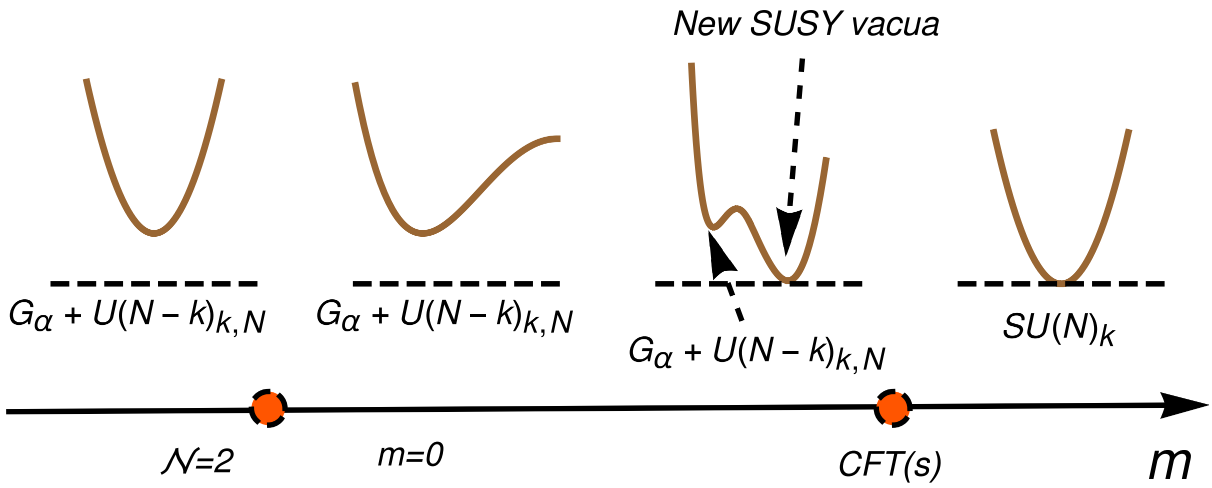


Figure 2.5. Phase diagram for the case $0 < k < N$.

The TQFT in that vacuum is Abelian $U(1)^{N-1}$ with CS level matrix (2.85) and $k = 1$. We have already noticed above that this theory is dual to the TQFT $U(1)_{-N}$, which in turn is level-rank dual to $SU(N)_1$. Therefore, for small positive mass a single supersymmetric vacuum supports $SU(N)_1$ theory, coinciding with the one expected in the large positive mass limit. This is a nice consistency check of the whole proposal.

2.3.7 A simple example: $SU(2)_k$

In this section we explore in details the simplest example to illustrate the general discussion above – the $SU(2)_k$ theory.

In this case, the most general adjoint matrix can be brought to the form

$$X = gk \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} . \tag{2.136}$$

Plugging this into (2.114), we obtain

$$\frac{m}{g^2} = \frac{2x}{\sqrt{1+4x^2}} . \tag{2.137}$$

For $m \leq 0$ the only solution is $x = 0$. For $k \geq 2$ it is a supersymmetric vacuum with $SU(N)_{k-2}$ TQFT. For $k = 1$ supersymmetry is broken, and the IR theory contains a Majorana Goldstino plus the $U(1)_2$ TQFT.

For $m > 0$ there is again the $x = 0$ solution, corresponding to the partition $2 = 2$. with the vacuum described in the same way as for $m \leq 0$. There is also a new solution, $x=\frac{1}{2}\sqrt{\frac{4g^4}{m^2}-1}$, which corresponds to the partition $2=1+1$ (note that solutions must be counted up to the action of Weyl group). This vacuum is supersymmetric and hosts a $U(1)_{2k}$ TQFT.

If $k\geq 2$, we have two supersymmetric vacua with $SU(2)_{k-2}$ and $U(1)_{2k}$ TQFTs, respectively. At a certain value of the mass they coalesce in a second order phase transition and create a single supersymmetric vacuum with the $SU(2)_k$ TQFT. The Witten index is conserved along the transition,

$$I_{SU(2)_{k-2}}+I_{U(1)_{2k}}=-(k-1)+2k=k+1=I_{SU(2)_k}, \tag{2.138}$$

where it was taken into account that the vacuum in the origin is fermionic, while the Abelian vacuum is bosonic.

If $k=1$, the new supersymmetric vacuum, appearing for small positive mass, has a $U(1)_2$ TQFT. It is dual to its time reversal, which is level-rank dual to $SU(2)_1$:

$$U(1)_2\leftrightarrow U(1)_{-2}\leftrightarrow SU(2)_1. \tag{2.139}$$

The last TQFT is in agreement with the $m\to\infty$ vacuum.

2.3.8 Phases of the Theory with $k=0$

Till this moment we were retaining to discuss the case of vanishing CS level, and now we will fill in this gap.

The $\mathcal{N}=1$ theory with an adjoint matter multiplet at $k=m=n$ has $\mathcal{N}=2$ supersymmetry. This implies that the superpotential on the classical moduli space of vacua (2.81) does not receive any perturbative corrections, and in particular, the two-loop potential (2.99) vanishes. As a consequence, the phase diagram of this theory is much simpler than for the theory with $k\neq 0$.

In section (2.3.1) we showed that the theory with large positive and large negative mass flows to a trivial, gapped supersymmetric vacuum (*i.e.* with no TQFT). Our goal is now to describe what happens between these asymptotic phases. Let us start at the $\mathcal{N}=2$ supersymmetric point $m=0$ and consider first the $SU(2)$ case. As it was reviewed in section (2.1.4), a non-perturbative superpotential

$$\mathcal{W}=\frac{1}{Y} \tag{2.140}$$

is generated, where Y is the monopole chiral superfield $Y=e^{U/g}$, and U is the chiral superfield dual to the linear superfield Σ. It is convenient to use U for the IR descriprion.

The Kähler potential for U comes from the dualization and far out in the moduli space is approximately given by

$$K \sim (U+\bar{U})^2 = (\log Y + \log \bar{Y})^2. \tag{2.141}$$

Therefore, there is a runaway potential, scaling like $V \sim \frac{1}{|Y|^2} \sim e^{-2\sigma}$, where σ is the real scalar from the vector multiplet.

Let us now turn on a small mass deformation for the $\mathcal{N}=1$ matter multiplet and determine where the theory flows to. This can be done by writing the deformation in the ultraviolet using an $\mathcal{N}=2$ spurion superfield M. In terms of this , the mass deformation preserving $\mathcal{N}=1$ takes the form

$$\delta\mathcal{L} = \frac{1}{2}\int d^4\theta\ M\ \mathrm{Tr}(\Sigma) \tag{2.142}$$

with

$$M = m(\theta-\bar{\theta})^2. \tag{2.143}$$

This choice of M preserves $\mathcal{N}=1$ supersymmetry. This choice is not of course unique: we could have used use R-symmetry to relate any two such choices of M.

In the presence of M the standard transformation from Σ to the chiral superfield U is modified as

$$\frac{1}{2}\int d^4\theta\ (-1+M)\Sigma^2 + \Sigma\,(U+\bar{U}). \tag{2.144}$$

Above we have rescaled the field U such that a factor of 2π disappears. Integrating out Σ leads to the effective action in terms of U

$$\frac{1}{2}\int d^4\theta\ (1-M)^{-1}(U+\bar{U})^2 = \int d^4\theta \left(U\bar{U} + \frac{1}{2}M(U+\bar{U})^2 + ...\right), \tag{2.145}$$

where on the right hand side we have only kept terms to linear order in M. Expanding this action in components, and including the non-perturbative superpotential, we find (ignoring terms with derivatives)

$$e^{-u/g}F_u + \text{c.c.} - (e^{-u/g}\psi_u\psi_u + \text{c.c.}) + |F_u|^2 - m(u+\bar{u})(F_u+\bar{F}_u) - \frac{1}{2}m(\psi_u+\bar{\psi}_u)^2. \tag{2.146}$$

Integrating out auxiliary fields we find the potential

$$|e^{-u/g} - (u+\bar{u})m|^2. \tag{2.147}$$

This can be viewed as arising from the $\mathcal{N}=1$ superpotential

$$\mathcal{W}_{\mathcal{N}=1} = e^{-\operatorname{Re}U/g}\cos(\operatorname{Im}U/g) + m(\operatorname{Re}U)^2. \tag{2.148}$$

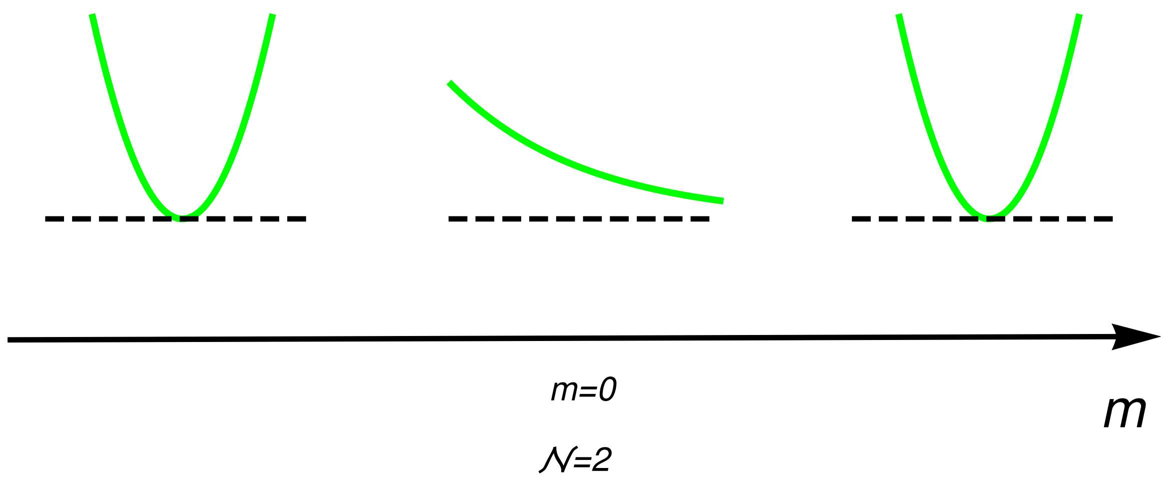


Figure 2.6. Phase diagram for the case $k=0$.

For small positive m, the minimum is at $u \sim -g \log 2m$ with u real, while for small negative m it is at $u \sim -g \log 2m + i\pi g$. The scalar fields are massive in these vacua, and since the vacua are $\mathcal{N}=1$ supersymmetric, so are the fermions. Therefore, we have shown that the theory flows to a trivial phase for both positive and negative small m, leading to a very simple phase diagram, with a trivial massive vacuum everywhere except at $m=0$, where there is no stable vacuum.

For the case of $SU(N)$, $N>2$ the story is rather similar. We have

$$\mathcal{W}(U) = \sum_{i=1}^{N-1} e^{-(U_i - U_{i+1})/g}, \tag{2.149}$$

where the Coulomb branch is parametrized such that $\sum_i \phi_i = 0$, $\phi_1 > \phi_2 > ... > \phi_N$, and

$$\mathcal{W}_{\mathcal{N}=1} = 2\sum_i e^{-(\operatorname{Re} U_i - \operatorname{Re} U_{i+1})/g} \cos\left(\operatorname{Im}(U_i - \operatorname{Im} U_{i+1})/g\right) + 4m \sum_i \operatorname{Re} U_i^2. \tag{2.150}$$

From the $F_{\operatorname{Im} U}=0$ equations it follows that in the SUSY vacuum we have $\operatorname{Im} U_i = 0$. The equations $F_{\operatorname{Re} U_i}=0$ then take the form

$$e^{(\operatorname{Re} u_{i-1} - \operatorname{Re} u_i)/g} - e^{(\operatorname{Re} u_i - \operatorname{Re} u_{i+1})/g} + 4m \operatorname{Re} u_i = 0. \tag{2.151}$$

It is not difficult to prove that the system above can have only one solution. Moreover, low-N examples suggest that there is indeed a solution with $\mathrm{Re}\, U_i = -\mathrm{Re}\, U_{N-i}$, and so there is a single supersymmetric vacuum.

The resulting phase diagram is summarized in fig. (2.6)

2.3.9 Comparison with other results

Our proposal involves certain claims about the $\mathcal{N}=2$ point, and so can be confronted with the $\mathcal{N}=2$ literature.

First, let us mention [10], where the Witten index for $SU(N)_k$ gauge theory was computed[8]. The result is that for $k \geq N$ the index coincides with the number of primary operators of G_{k-N} Wess-Zumino-Witten (WZW) theory. It follows from the correspondence between CS theories and rational conformal field theories (RCFT) [29] that this is the same as the number of states of the $SU(N)_{k-N}$ TQFT on a torus, in agreement with our findings. This check is in some sense trivial, since the infrared TQFT can be seen semiclassically by integrating out matter and gaugini. It is also stated in [10] that for $0 \leq k < N$ the index vanishes and supersymmetry is spontaneously broken.

Other evidences in favour of our proposal can be extracted from some $\mathcal{N}=2$ dualities that have appeared in the literature. In [43], based on the equality of Z-functions and superconformal indices, a duality between the $\mathcal{N}=2$ $SU(2)_1$, coupled to an adjoint chiral multiplet Φ, and a single free chiral multiplet X plus a topological sector was suggested:

$$SU(2)_1 + \text{adjoint } \Phi \leftrightarrow \text{ free chiral multiplet } X \;+\; U(1)_{-2} \text{ TQFT} \,, \tag{2.152}$$

with the dual of X being $\mathrm{Tr}\Phi^2$. It was then generalized in [44] as

$$SU(N)_1 + \text{adjoint } \Phi \leftrightarrow \text{ free chiral multiplets } X_1,\ \ldots,\ X_N \;+\; U(1)_{-N} \text{ TQFT} \,. \tag{2.153}$$

Let us start with the duality (2.152) and add a supersymmetric mass to the adjoint Φ. The theory then flows to the $\mathcal{N}=2$ $SU(2)_1$ vector multiplet. On the right hand side, the deformation amounts to add a linear superpotential $\mathcal{W} \propto X$. This breaks supersymmetry spontaneously (*a la* Polonyi model), leading to a Dirac Goldstino. Thus the right hand side flows to $U(1)_2$ TQFT + Dirac Goldstino, consistently with our proposal.

Analogously, we can add a mass for the afjoint on the left hand side of (2.153), getting $\mathcal{N}=2$ $SU(N)_1$ in the IR. This deformation corresponds to adding $\mathcal{W} \propto X_1$

[8] In fact, in [10] the more general case of arbitrary gauge group G, together with some matter fields, was considered.

superpotential on the right hand side. As before, supersymmetry is broken and we get $U(1)_{-N}$ TQFT + Dirac Goldstino in the IR. Applying level-rank duality,

$$U(1)_{-N} \leftrightarrow U(N-1)_{1,N} \ , \tag{2.154}$$

we again observe matching.

2.4 Conclusion and Outlook

In this chapter we have described in details phase diagrams of three-dimensional $\mathcal{N}=1$ $SU(N)_k$ gauge theories coupled to an adjoint matter multiplet. Let us summarize the key phenomena and the physical picture which emerges.

- There are walls (in our case zero-dimensional) in parameter space, where the Witten index jumps. In our case this is the point $m=0$.
- The jump is provided by a bunch of new vacua, appearing from infinity in field space and carrying non-zero index.
- There is an intermediate regime, where the theory possesses 2^{N-1} vacua. They undergo a chain of second-order phase transitions, such that for large enough positive mass only one vacuum remains. It implies that there is at least one non-trivial fixed point present on the phase diagram.
- For small enough values of CS level, namely for $0<k<N$, supersymmetry is broken for large negative mass.

It is an important question to understand the fate of this supersymmetry breaking vacuum, when we cross the point $m=0$. It always stays in the vicinity of the origin in field space, as can be seen from the fact that it carries a TQFT, which cannot be seen semiclassically. But we don't have much control on this region, since there we cannot rely on finite orders of perturbation theory anymore, when computing quantum corrections to the superpotential. Moreover, there are non-pertubative effects, related to the dynamics of the vector multiplet and which basically lead to supersymmetry breaking. Still, it seem plausible that in the intermediate region of the phase space this supersymmetry breaking vacuum coexist with newly appeared supersymmetric vacua, and so becomes metastable.

Most of the elements of the proposed picture seem to be generic for minimally supersymmetric gauge theories with CS coupling, and are not peculiar to the specific model under consideration. Indeed, in our paper [45] also $\mathcal{N}=1$ $SU(N)_k$ with one fundamental flavor was studied. In particular, the same question of how IR phases depend on matter mass was addressed. It turns out that this theory enjoys phenomena very similar to those we have just seen. There are two asymptotic phases for large negative and positive mass, carrying certain topological sector.

Again, it is clear that the Witten index must jump somewhere in-between. Analysis of two-loop potential indeed reveals that a new vacuum comes in from infinity, when the mass becomes positive. Two vacua - the one existing for large negative mass and the one coming from infinity merge at a certain value of the mass, and produce the single vacuum visible in the large positive mass limit. This second order phase transition between two topological vacua on one side and one topological vacuum on other side, can be given a dual description, which leads to a SU/U duality

$$U(N)_{k+N/2+1/2,k+1/2}+\Phi \leftrightarrow SU(k+1)_{-N-k/2}+\Psi\ . \tag{2.155}$$

The duality map for deformations is $\Phi\Phi^\dagger \leftrightarrow -\Psi\Psi^\dagger$. Recently this analysis was generalized to arbitrary number of flavors [46]. An interesting special case of this setup to consider could be $\mathcal{N}=1$ $SU(3)_k$ theory with $N_f\geq 3$. In this case baryon operators provide another interesting relevant deformation, which in the dual description would correspond to a deformation by monopole operators [47].

$\mathcal{N}=1$ gauge theories with fundamentals were also discussed in [48]. There the authors were looking for a supersymmetric analog of non-Abelian bosonization. In the canonical non-supersymmetric examples bosonization maps *critical scalar* (that is, the one with quartic interaction) to *regular fermion*. A natural expectation is that critical scalars are paired with *critical fermions* (*i.e.* with interactions of Gross-Neveu-Yukawa type), and *regular scalars* are paired with regular fermions. In order to construct critical matter in the supersymmetric setup, the authors used gauge singlet superfields (otherwise it is impossible to get ϕ^4 interaction from gauge invariant superpotential). Finally, they come up with the following dualities

$$\begin{array}{ccc} U(k)_{N+\frac{k}{2}-\frac{1}{2},N-\frac{1}{2}} & & SU(N)_{-k-\frac{N}{2}+\frac{1}{2}} \text{ with 1 flavor } P \\ \text{with 1 flavor } Q & \longleftrightarrow & \text{and a gauge-singlet } H \\ \mathcal{W}=-\frac{1}{4}\big(\sum_{i=1}^k Q_iQ_i^\dagger\big)^2 & & \mathcal{W}=H\,\sum_{i=1}^N P_iP_i^\dagger-\frac{1}{3}H^3\ . \end{array} \tag{2.156}$$

and a similar proposal with critical U-theory and regular SU-theory.

Another interesting development is related to $\mathcal{N}=1$ theories invariant under time reversal symmetry [49]. The superpotential transforms as a pseudoscalar under the time reversal, and it turns out that this fact severely constraints possible quantum corrections it can receive. In particular, sometimes it leads to exact moduli spaces, with the simplest example being a theory of three real superfields A, B, C with superpotential $\mathcal{W}=ABC$. Another illustration of this phenomenon is a $U(1)$ gauge field coupled to a charge 2 superfield. This theory has a dual description consisting of a pure $U(1)_2$ TQFT tensored with a charge 1 superfield coupled to a $U(1)_{\frac{3}{2}}$ gauge field. Even though in the second description time reversal invariance is emergent in the IR, the theory still has a space of $\mathcal{N}=1$ supersymmetric ground states.

The authors also discuss an instance of global symmetry enhancement and supersymmetry enhancment in the infrared. They propose a duality between an $\mathcal{N}=2$ SQED and a Wess-Zumino model

$$\mathcal{N}=2 \;\; U(1)+2 \text{ charge } 1 \;\leftrightarrow\; \mathcal{N}=1 \;\; \mathcal{W}=\mathrm{Tr}\Phi^3 \,, \tag{2.157}$$

where Φ stands for eight real superfields in the adjoint representation of $SU(3)$. The left hand side of the duality has manifest $U(2)$ global symmetry, which is enhanced to $SU(3)$ in the IR. The right hand side has just $\mathcal{N}=1$ supersymmetry, which is enhanced to $\mathcal{N}=2$ in the IR.

Finally, also the non-Abelian example is discussed, namely an $\mathcal{N}=1$ $SU(N)$ minimally coupled to N_f fundamental multiplet. For $N_f<N$ there is a moduli space of vacua, which is not corrected at any order in perturbation theory, but is lifted non-perturbatively.

The issue of symmetry enhancement in $\mathcal{N}=1$ was also raised in [50]. In particular, the results about $U(2)\to SU(3)$ symmetry enhancement was confirmed, as well as other examples with *e.g.* $U(2)\to O(4)$ were suggested.

There is certainly much more to learn about three-dimensional $\mathcal{N}=1$ theories. Some of them can be realized on domain walls and interfaces (as it was discussed in [51] for non-supersymmetric QCD), and in these cases the understanding of world-volume theories can teach us something about BPS spectra of $4d$ theories. It can also be interesting to consider the suggested dual pairs upon compactification on a circle [52]: this may open the door for interesting interplay between $\mathcal{N}=1$ $3d$ dualities and $\mathcal{N}=(1,1)$ $2d$ dualities.

Chapter 3

Non-supersymmetric conformal manifolds and holography

The aim of this chapter is to study conformal manifolds, without relying on supersymmetric tools. Upon deforming a CFT as

$$S_{CFT} \to g \int d^d x \ \mathcal{O} \ , \tag{3.1}$$

where $\mathcal{O}$ ia a scalar *primary* of the CFT with scaling dimension $\Delta_{\mathcal{O}} = d$, a β function for the coupling g is induced, at the quantum level. The necessary and sufficient condition for a conformal manifold to exist is the vanishing of the β function:

$$\beta(g) = 0 \ . \tag{3.2}$$

In fact, the deformation triggered by the coupling g can also generate new couplings at quantum level, and the corresponding β functions must also be set to zero, if we were to preserve conformal invariance.

Using conformal perturbation theory, one can express β function coefficients in terms of CFT *data* of the original CFT, namely in terms of dimensions of primary operators and operator product expansion (OPE) coefficients $\{\Delta_i,\ C_{ijk}\}$. Then, demanding the β function coefficients to vanish gives in principle infinitely many conditions on CFT data. Discussing these conditions, up to two-loop order, will be the subject of this chapter.

The vanishing of tree-level β function is equivalent to the fact that the operator $\mathcal{O}$ is marginal in the undeformed CFT. Note that already this condition can imply that the CFT we start with is either supersymmetric or free, since otherwise there are no simple reasons to expect operator dimensions to be integers. One can still adopt the point of view that we start with a SCFT, such that there are operators of integer dimensions at our disposal, but the deformation does not preserve supersymmetry.

We start by reviewing the conformal perturbation theory (CPT) computation of a β function, showing how β-function coefficients depend on CFT data. Then, using these results, we will then try to extract some predictions about the structure and properties a CFTs belonging to a conformal manifold should satisfy. In particular, we will extract a new sum rule and point out some restrictions on the content of low spin and low dimension operators in the spectrum of the CFT. In the second part of this chapter we focus on CFTs admitting a gravity dual description. First, we discuss the relation between conformal perturbation theory and the $1/N$ expansion, and the role that Witten diagrams play in this matter. Then, focusing on a toy-model, we discuss, from a holographic point of view, the conditions for the existence of conformal manifolds at planar and non-planar levels. More specifically, we investigate under which conditions a conformal manifold existing at leading order in $1/N$, can survive at higher orders, and show that, even in absence of supersymmetry, this is a non-empty set.

3.1 Conformal field theories: basic notions

In this section we review some basic facts about CFTs in $d > 2$ [9]. A more detailed account can be found in [53–55]

The conformal algebra is an extension of Poincaré algebra by the *dilatation operator* D, together with *special conformal transformations* K_μ. The resulting algebra is defined by the following commutation relations:

$$\begin{aligned}
[M_{\mu\nu},\, P_\rho] &= \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu, &(3.3)\\
[M_{\mu\nu},\, K_\rho] &= \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu, &(3.4)\\
[M_{\mu\nu},\, M_{\rho\sigma}] &= \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\mu\rho} - \delta_{\mu\sigma} M_{\nu\rho}, &(3.5)\\
[D,\, P_\mu] &= P_\mu, &(3.6)\\
[D,\, K_\mu] &= -K_\mu, &(3.7)\\
[K_\mu, P_\nu] &= 2\delta_{\mu\nu} D - 2M_{\mu\nu}, &(3.8)
\end{aligned}$$

with all the other commutators vanishing. The first two relations imply that P_μ and K_μ transform as vectors under $so(d)$ rotations. One can show that the resulting algebra is isomorophic to $so(d+1,1)$. States of the theory are classified according to their representations under $so(d)$ (spins) and according to the eigenvalues of D, the so-called *scaling dimensions*. The last three commutation relations show that P_μ and K_μ can be considered as raising and lowering operators for D. One can then construct the lowest weight representation of the conformal algebra in the following way. First we define *primary* operators as operators satisfying

$$[K_\mu, \mathcal{O}(0)] = 0, \tag{3.9}$$

[9] We will concentrate on Euclidean case here.

which correspond to the lowest-weight state of a conformal representation, or *conformal family.* All other states, called descendants, can be obtained by successively applying P_μ to $\mathcal{O}$, which just acts as a derivative:

$$\text{Conformal family} = \{\mathcal{O},\ \partial_\mu\mathcal{O},\ \partial_\mu\partial_\nu\mathcal{O},\ ...\}. \tag{3.10}$$

The corresponding scaling dimensions are given by

$$\{\Delta_\mathcal{O},\ \Delta_\mathcal{O}+1,\ \Delta_\mathcal{O}+2,\ ...\}. \tag{3.11}$$

Evidently, it is enough to study correlation functions of primary operators, since those of descendants can be obtained by differentiation. In fact, conformal invariance imposes severe restrictions on the possible correlation functions of primary operators. Let us consider some simple examples, involving scalar operators only (for operators with spin additional tensor structures appear). The two-point function is fixed up to a constant to be

$$\langle\mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{\delta_{ij}}{|x|^{2\Delta_i}}. \tag{3.12}$$

Note that two-point functions of operators with different dimensions vanish. In the above equation the overall constant has been absorbed into the operator normalization, and a diagonal basis of operators with the same dimensions was chosen. So, in order to compute the two-point function it is enough to know the dimension of the operator.

Three-point functions are fixed to be

$$\langle\mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle = \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|x-z|^{\Delta_i+\Delta_k-\Delta_j}|y-z|^{\Delta_j+\Delta_k-\Delta_i}}. \tag{3.13}$$

Hence, in order to compute three-point functions it is enough to know dimensions of the operators as well as the coefficient C_{ijk}.

The four-point function is constrained by conformal kinematics only up to an unknown function, because one can construct two conformal invariants out of four coordinates:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{3.14}$$

For instance, the four-point function of four identical scalars takes the form

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{f(u,v)}{|x_{12}|^{2\Delta_\mathcal{O}}|x_{34}|^{2\Delta_\mathcal{O}}}\ , \tag{3.15}$$

where $f(u,v)$ is an unknown function.

At this point operator product expansion (OPE) appears extremely helpful. The statement of OPE is that the product of two local operators with nearby insertion points can be substituted by the infinite sum over all local operators in the theory:

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k d_{ij}^k(x,\partial)\mathcal{B}_k \ . \tag{3.16}$$

For a general QFT $\mathcal{B}$ are all operators, both primaries and descendants, $d_{ij}^k(x,\partial)$ are some differential operators, and the series has zero radius of convergence. In the case of a CFT the radius of convergence is finite (and equals the distance to the other closest insertions). Moreover, contributions from the descendants to the OPE are related to that of the primaries. So, the series can be simplified to

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ij}^k\, \mathcal{F}_{\mathcal{O}}(x,\partial)\, \mathcal{O}_k = \sum_k \frac{C_{ij}^k\, \mathcal{O}_k}{|x|^{\Delta_i+\Delta_j-\Delta_k}} + \text{descendants}. \tag{3.17}$$

Above C_{ij}^k are the same constants appearing in the tree-point function and are known as OPE coefficients, and $\mathcal{F}_{\mathcal{O}}(x,\partial)$ are known differential operators, depending on kinematics only, *i.e.* on scaling dimensions and spins of the primaries $\mathcal{O}_i$, $\mathcal{O}_j$, $\mathcal{O}_k$.

The power of OPEs stems from the observation that one can use them to reduce n-point functions to $n-1$-point functions. Indeed, applying OPE inside the correlator of n operators, one can reduce it to an infinite sum of correlators of $n-1$ operators, and so on until the problem is reduced to the computation of two- and three-point functions, which are known. It is clear from this procedure that all possible correlators can be expressed in terms of operator dimensions and OPE coefficients $\{\Delta_i,\, C_{ijk}\}$, known together as *CFT data*.

A natural question to ask is if all possible sets of CFT data define a consistent theory, with the answer being negative. In particular, there are elementary constraints coming from the requirement of unitarity. For instance, dimensions of scalar primary operators must satisfy

$$\Delta_{\text{scalar}} \geq \frac{d-2}{2} \ , \tag{3.18}$$

where the equality holds if and only if the scalar is a free field. For operators with spin l we have

$$\Delta_l \geq d + l - 2 \ , \tag{3.19}$$

with the equality saturated if and only if the operator is a conserved current. There are also more intricate constraints: the point is that sometimes OPE can be performed in several different ways, by choosing the order in which we pair the operators inside the correlator, and all these different ways must eventually give the same result. This condition is called *OPE associativity*, and it imposes nontrivial constraints on the CFT data.

The simplest example comes considering the four-point function

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \ . \tag{3.20}$$

One can evaluate it either applying OPE to the operators $\mathcal{O}_1\mathcal{O}_2$ and simultaneously to $\mathcal{O}_3\mathcal{O}_4$, or otherwise to $\mathcal{O}_1\mathcal{O}_3$ and $\mathcal{O}_2\mathcal{O}_4$. The first way, called s-channel, gives

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_k C_{12}^k\, C_{34}^k\, \mathcal{F}_k(x_1-x_2, \partial_{x_2})\, \mathcal{F}_k(x_3-x_4, \partial_{x_4})\, \frac{1}{|x_2-x_4|^{2\Delta_k}}. \tag{3.21}$$

One can now introduce the functions

$$g_k^{12;34}(x_1,x_2,x_3,x_4) \equiv \mathcal{F}_k(x_1-x_2, \partial_{x_2})\, \mathcal{F}_k(x_3-x_4, \partial_{x_4})\, \frac{1}{|x_2-x_4|^{2\Delta_k}}. \tag{3.22}$$

The functions $g_k^{12;34}$ are known as *conformal blocks*, and they are some known functions of dimensions, spins and insertion points (see *e.g.* [56, 57] and also below in this thesis for explicit expressions in $d=4$).

Apart from the s-channel $(12) \to (34)$, one can also use the t-channel $(14) \to (23)$ which leads to a similar expression, with the roles of the points 2 and 4 interchanged. The requirement that the two results must coincide leads to the condition

$$\sum_k C_{12}^k\, C_{34}^k\, g_k^{12,34}(x_1,x_2,x_3,x_4) = \sum_k C_{14}^k\, C_{23}^k\, g_k^{14,23}(x_1,x_2,x_3,x_4)\ , \tag{3.23}$$

which is called *crossing symmetry equations*, or *bootstrap conditions*. Any consistent CFT data must satisfy these condition. It turns out that one does not need to check higher correlation functions, since they do not give any new constraints. It is also worth noting, though, that crossing symmetry is not the only constraint one need to satisfy, see [24] for a review.

3.1.1 Holographic CFTs

One way to attempt to solve the crossing symmetry constraints is to develop the large central charge expansion, starting from *generalized free fields* (GFF) as the zero order approximation. Let as define a scalar generalized free field $\mathcal{O}$ of dimension $\Delta > \frac{d-2}{2}$ as an operator whose correlation functions factorize:

$$\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \ldots \langle \mathcal{O}(x_{n-1})\mathcal{O}(x_n)\rangle + \text{permutations}. \tag{3.24}$$

In particular, n-point functions vanish for odd n. This kind of Wick theorem is what makes GFF similar to free fields in the usual sense.

One can ask how does the OPE of two GFFs may look like. First of all, it is evident that the GFF itself can not appear there, since it would violate the factorization property. In fact, the detailed consideration of the OPE inside the four-point function reveals [56] that the $\mathcal{O}\mathcal{O}$ OPE must contain an infinite tower of the conformal primary "double-trace operators" of the form

$$\mathcal{O}^{(2)}_{n,l} =: \mathcal{O}\partial_{[\mu_1}...\partial_{[\mu_l]}\Box^n\mathcal{O} :, \tag{3.25}$$

where the brackets denote the symmetric traceless part and are projecting out descendants, and dimensions of such operators are given by $\Delta^{(2)} = 2\Delta + 2n + l$. The resulting expansion takes the form:

$$\mathcal{O}(x)\mathcal{O}(0) = \frac{1}{|x|^{2\Delta}} + \sum_{n,l} C_{n,l}\left(|x|^{2n+l}\,\mathcal{O}^{(2)}_{n,l} + \text{descendants}\right) \tag{3.26}$$

with

$$\begin{aligned} C_{n,l} &= (1+(-1)^l)\,\tfrac{2(l+1)(2\Delta+2n+l-2)}{(\Delta-1)^2}\,A_n\,A_{n+l+1}, \\ A_n &= \tfrac{\Gamma^2(\Delta+n-1)\Gamma(\Delta+n-2)}{n!\,\Gamma^2(\Delta-1)\Gamma(\Delta+2n-2)} \end{aligned} \tag{3.27}$$

for $d = 4$. Correlation functions of operators $\mathcal{O}^{(2)}_{n,l}$ do not factorize, but they follow completely from the correlation functions of $\mathcal{O}$ and from $\mathcal{O}\mathcal{O}$ OPE. This CFT data solve crossing equations by construction.

Considering further OPEs of $\mathcal{O}\mathcal{O}^{(2)}_{n,l}$ and $\mathcal{O}^{(2)}_{n,l}\mathcal{O}^{(2)}_{n,l}$, it is possible to infer the existence of higher-trace operators of the form $:\mathcal{O}\mathcal{O}\mathcal{O}:$, $:\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}:$, and so on. This multi-trace operators have an interpretation of multi-particle states, and so we see how the structure of freely-generated Fock space emerges in this setup.

While generalized free fields cannot be described by a local Lagrangian in d dimensions, they can be given a local description in $d+1$ dimensions [58–62]. Let us denote by $\mathbf{x}$ coordinates in d-dimensional space, and by $(\mathbf{y}, z)$ coordinates in $d+1$-dimensional space. Define the function

$$\phi(\mathbf{y}, z) = \int d^d\mathbf{x}\; T(\mathbf{x};\, \mathbf{y}, z)\, \mathcal{O}(\mathbf{x}), \tag{3.28}$$

where the kernel $T(\mathbf{x}; \mathbf{y}, z)$ is called *transfer function* and satisfies the equation

$$\left(\Box_{d+1} - m^2\right) T(\mathbf{x};\, \mathbf{y}, z) = 0. \tag{3.29}$$

Here $\Box_{d+1}$ is the Laplacian on the space with the AdS$_{d+1}$ metric

$$ds^2 = \frac{dz^2 + d\mathbf{y}^2}{z^2}. \tag{3.30}$$

The mass m and conformal dimension Δ are related by the standard formula of AdS/CFT dictionary:

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}. \tag{3.31}$$

One can still think about $\phi(\mathbf{y}, z)$ as a non-local operator acting on the CFT Hilbert space, but more natural is to consider it as a field, living in the (by now) auxiliary $d+1$-dimensional space. In order to support this idea, let us note that $\phi(\mathbf{y}, z)$ satisfies a linear wave equation in AdS_{d+1}:

$$\left(\Box_{d+1} - m^2\right) \phi = 0. \tag{3.32}$$

Correlation functions of ϕ are determined in terms of correlation functions of $\mathcal{O}$, and in particular coincide with those of a free scalar in AdS_{d+1}, as far as the factorization property for $\mathcal{O}$ is satisfied. In particular, we have in the Lorentzian signature that

$$[\phi(\mathbf{y}, z), \phi(\mathbf{y}', z')] = 0 \tag{3.33}$$

whenever points are space-like separated. All these facts show that ϕ behaves as a local free field in AdS space.

Similar consideration can be performed for operators with spin, which give rise to fields with spin in AdS. In particular, conserved currents correspond to gauge fields, stress-energy tensor to the graviton, and so on.

So far we have been discussing the limit of GFFs (corresponding to $c \to \infty$, or $N \to \infty$ for CFTs with a gauge theory origin) and we have come to an alternative description in terms of free fields in AdS. One then would like to depart from the infinite central charge limit, considering $1/c$ corrections and giving up the factorization property of single-trace operators. One way to do this is to keep the form of OPEs unmodified, but correct the CFT data. More concretely, we can still assume that there are single trace operators, fusing which we get double-trace operators, but the double-trace operator dimensions and OPE coefficients are slightly corrected:

$$\Delta^{(2)} = 2\Delta + 2n + l + \delta, \tag{3.34}$$

$$C_{n,l} = C^0_{n,l} + \zeta, \tag{3.35}$$

where the superscript 0 stands for the GFF value.

Possible deformations of this kind are restricted by bootstrap conditions, and one can develop $1/N$ expansion in solving for δ and ζ. It was shown in [63] that such solutions of bootstrap equations at leading order in $1/N$ are in one-to-one correspondence with quartic (generically derivative) interactions in the bulk. The four-point function $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$ then acquires a connected part, described by the contact Witten diagramm (3.8). One can also modify the form of $\mathcal{O}\mathcal{O}$ OPE by introducing a new single-trace operator on the r.h.s.; this would correspond to a new cubic interaction in the bulk [24]. The single-trace operator four-point function gets, again, a connected contribution, described by the exchange Witten diagram (3.7). Higher-order corrections to the CFT data in $1/N$ were discussed in [64] and correspond to loops in AdS.

It is worth noting that GFFs, even with $1/N$ corrections, cannot form a consistent CFT by themselves. Instead, one should think about CFTs with a holographic description as theories with a gap in the spectrum of operator dimensions, with the low-dimension spectrum approximately described by GFFs [63].

3.2 Constraints from conformal perturbation theory

Given a CFT and a deformation as that in eq. (3.1), one expects that a β function for the coupling g is generated and that conformal invariance is lost. The β function reads

$$\beta(g) = \beta_1\, g^2 + \beta_2\, g^3 + \dots \; . \tag{3.36}$$

Loop coefficients are expected to depend on the data of undeformed CFT. In order to find such dependence a perturbative analysis can be conveniently done in the context of conformal perturbation theory [66].

One can extract the β function by considering cleverly chosen physical observables and demand them to be UV-cutoff independent. Following [67] (see also [68]), we consider the overlap

$$\langle \mathcal{O}(\infty)|0\rangle_{g,V} \tag{3.37}$$

where $\mathcal{O}(\infty) = \lim_{x\to\infty} x^{2d}\mathcal{O}(x)$, while $|0\rangle_{g,V} = e^{g\int_V d^dx\,\mathcal{O}(x)}|0\rangle$ is the state obtained by deforming the theory by (3.1) in a finite region around the origin. The choice of a finite volume V allows one to get rid of IR divergences, while not affecting the UV behavior we are interested in. Expanding (3.37) in g one gets a perturbative expansion in terms of integrals of n-point functions of $\mathcal{O}$. These are generically plagued by logarithmic divergences, which can be absorbed by demanding that the coupling g runs with scale μ in a way that the final result is μ-independent. This, in turns, lets one extract the β function.

Proceeding this way one gets for the β function at two loops (which to this order is universal, hence independent of the renormalization scheme) the following expressions

$$\beta_1 \;=\; -\frac{1}{2}\, S_{d-1} C_{\mathcal{O}\mathcal{O}\mathcal{O}} \tag{3.38}$$

$$\begin{aligned}\beta_2 \;=\; & -\frac{1}{6}\,S_{d-1}\int d^dx\Big[\langle\mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(e)\mathcal{O}(\infty)\rangle_c - \\ & -\sum_{\Phi}\frac{1}{2}C^2_{\mathcal{O}\mathcal{O}\Phi}\left(\frac{1}{x^d(x-e)^d}+\frac{1}{x^d}+\frac{1}{(x-e)^d}\right) \\ & -\sum_{\Psi}C^2_{\mathcal{O}\mathcal{O}\Psi}\left(\frac{1}{x^{2d-\Delta_\Psi}}+\frac{1}{(x-e)^{2d-\Delta_\Psi}}+x^{-\Delta_\Psi}\right)\Big]\,, \end{aligned} \tag{3.39}$$

where S_{d-1} is the volume of the $(d-1)$-dimensional unit sphere, e is a unit vector in some fixed direction and the subscript c in the four-point function refers to the connected contribution. Sums are over marginal operators Φ and relevant operators Ψ appearing in the $\mathcal{O}\mathcal{O}$ OPE. In principle, one can go to higher orders in g. In particular, marginality of $\mathcal{O}$ at order $O(g^{n-1})$ would require the vanishing of logarithmic divergences of an integral in $d^dx_1\cdots d^dx_{n-3}$ of the n-point function $\langle\mathcal{O}\ldots\mathcal{O}\rangle$.

The deformation (3.1) does not cause the running of g, only. In general, any coupling g_Φ dual to a marginal operator Φ appearing in the OPE of $\mathcal{O}(x)\mathcal{O}(0)$ will start running, due to quantum effects.[10] Following the same procedure described above, one gets the following contribution at order g^2 to $\beta(g_\Phi)$

$$\beta(g_\Phi) \supset -\frac{1}{2}\,S_{d-1}C_{\mathcal{O}\mathcal{O}\Phi}\,g^2\,. \tag{3.40}$$

Therefore, at one loop in CPT, the persistence of a conformal manifold under the deformation (3.1) implies the following constraints on the OPE coefficients of the CFT

$$C_{\mathcal{O}\mathcal{O}\Phi}=0\quad,\quad \forall\,\Phi\ \text{ such that }\ \Delta_\Phi=d\,. \tag{3.41}$$

Taking into account the above constraint, eq. (3.39) simplifies and we get the following condition at two-loops, eventually

$$\int d^dx\left[\langle\mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(e)\mathcal{O}(\infty)\rangle_c-\sum_\Psi C^2_{\mathcal{O}\mathcal{O}\Psi}\left(\frac{1}{x^{2d-\Delta_\Psi}}+\frac{1}{(x-e)^{2d-\Delta_\Psi}}+x^{-\Delta_\Psi}\right)\right]=0\,. \tag{3.42}$$

Eqs. (3.41) and (3.42) are the two constraints the existence of a conformal manifold under the deformation (3.1) imposes on the CFT at two-loop order in CPT.[11]

[10]Runnings are also induced for relevant operators appearing in the OPE. However, these effects are associated to power-law divergences and can be reabsorbed by local counter-terms. This is equivalent to be at a fixed point, to $\mathcal{O}(g^2)$ order, of the corresponding β functions $\beta(g_\Psi)$ [66].

[11]One can obtain similar expressions for two-loop β function of other marginal operators, if there are any, and get additional constraints.

3.2.1 Two-loop constraint and integrated conformal blocks

One can try to translate the constraint (3.42) into a sum rule in terms of conformal blocks, which can provide, in turn, constraints on the CFT data.

Let us first rewrite (3.42) as an integral of the full four-point function, that is

$$\int d^d x \quad \Big(\langle \mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(e)\mathcal{O}(\infty)\rangle - \tfrac{1}{x^{2d}} - \tfrac{1}{(x-e)^{2d}} - 1 - \\ - \textstyle\sum_\Psi C^2_{\mathcal{O}\mathcal{O}\Psi} \left(\tfrac{1}{x^{2d-\Delta_\Psi}} + \tfrac{1}{(x-e)^{2d-\Delta_\Psi}} + x^{-\Delta_\Psi} \right) \Big) = 0 \, . \tag{3.43}$$

The integrand above is axial-symmetric, hence the integration can be seen as an integration over a two-plane $(z, \bar{z})$ containing the unit vector e, followed by integration over a $(d-2)$-dimensional sphere, whose coordinates the integrand does not depend on. So, for the integration measure, we get

$$d^d x \to \frac{\pi^{\frac{d-1}{2}}}{2\Gamma\left(\frac{d-1}{2}\right)} d^2 z \left(\frac{z - \bar{z}}{2i} \right)^{d-2} \, . \tag{3.44}$$

Notice that the integrand together with the measure is inversion-invariant. Therefore, instead of integrating over the whole R^d, one can integrate over a unit disk, $B_{r=1}(0) = \{z \in C \, , |z| \leq 1\}$, where the coordinate z is chosen such that $x = e$ corresponds to $z = 1$.

The integrand in eq. (3.43) is expected to be a singularity-free function, but among the terms coming with a minus sign, there are some which have manifest singularities. Hence, they must be compensated by the corresponding singularities of the four-point function. Due to divergences both at $z = 0$ and $z = 1$, one cannot use just one OPE channel. However, it turns out that one can reduce the integration domain to a fundamental one [70], for which a single channel suffices. The integral (3.43) is invariant under transformations generated by $z \to 1/z$ and $z \to 1 - z$ and complex conjugation. Hence, choosing one of the following domains

$$\begin{aligned} D_1 &= \{z \in C| \; |1-z|^2 < 1, \;\; \mathrm{Re}(z) < 1/2, \;\; \mathrm{Im}(z) > 0\} \\ D_2 &= \{z \in C| \; |1-z|^2 < 1, \;\; \mathrm{Re}(z) < 1/2, \;\; \mathrm{Im}(z) < 0\} \\ D_3 &= \{z \in C| \; |1-z|^2 > 1, \;\; |z|^2 < 1, \;\; \mathrm{Im}(z) > 0\} \\ D_4 &= \{z \in C| \; |1-z|^2 > 1, \;\; |z|^2 < 1, \;\; \mathrm{Im}(z) < 0\} \, , \end{aligned} \tag{3.45}$$

one can use s-channel OPE only. For the sake of computational convenience we will not do the minimal choice, but use the union of all four domains, $D = D_1 \cup D_2 \cup D_3 \cup D_4$. Using s-channel OPE, we get

$$\langle \mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(e)\mathcal{O}(\infty)\rangle = \frac{\sum_{\mathcal{O}'} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'} g_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}}}{x^{2d}} \, , \tag{3.46}$$

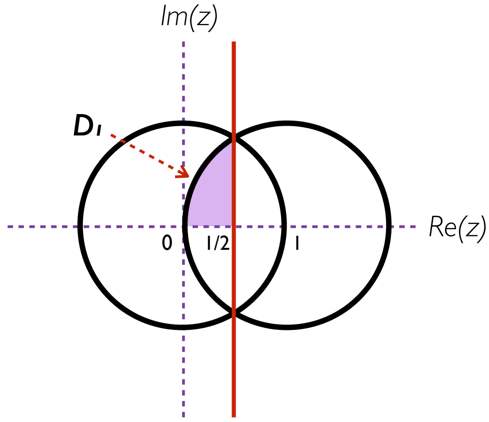


Figure 3.1. Integration in the $(z, \bar{z})$ plane. The fundamental domain D_1 is the violet region. The regions D_2, D_3 and D_4 are defined in (3.45) and are easily recognizable in the figure.

where $g_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}}$ are conformal blocks corresponding to the exchange of an operator $\mathcal{O}'$ with dimension $\Delta_{\mathcal{O}'}$ and spin $l_{\mathcal{O}'}$ (with $l_{\mathcal{O}'}$ even, as in the OPE of two identical scalars only operators with even spin appear). The identity operator contribution cancels the $1/x^{2d}$ divergent contribution in eq. (3.43).

Let us now define the following quantities

$$G_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}} = \frac{\pi^{\frac{d-1}{2}}}{2\Gamma\left(\frac{d-1}{2}\right)} \int_D d^2z \left(\frac{z-\bar{z}}{2i}\right)^{d-2} \frac{g_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}}(z, \bar{z})}{|z|^{2d}}, \ \Delta > d\ , \tag{3.47}$$

$$G_{\Delta_{\mathcal{O}'}, 0} = \frac{\pi^{\frac{d-1}{2}}}{2\Gamma\left(\frac{d-1}{2}\right)} \int_D d^2z \left(\frac{z-\bar{z}}{2i}\right)^{d-2} \left(\frac{g_{\Delta_{\mathcal{O}'}, 0}(z, \bar{z})}{|z|^{2d}} - \frac{1}{|z|^{2d-\Delta}} - \frac{1}{|1-z|^{2d-\Delta}} - |z|^{-\Delta}\right), \quad \Delta < d\ , \tag{3.48}$$

$$A = \frac{\pi^{\frac{d-1}{2}}}{2\Gamma\left(\frac{d-1}{2}\right)} \int_D d^2z \left(\frac{z-\bar{z}}{2i}\right)^{d-2} \left(\frac{1}{|1-z|^{2d}} + 1\right)\ , \tag{3.49}$$

where $G_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}}$ are *integrated conformal blocks* (note that, for $\Delta < d$, that is eq. (3.48), only scalar operators are above the unitarity bound) and A is a positive, dimension-dependent number, which in, *e.g.*, $d = 4$ dimensions reads

$$A = \frac{\pi}{24}\left(9\sqrt{3} + 16\pi\right)\ . \tag{3.50}$$

Using all above definitions, eq. (3.43) can be rewritten as the following sum rule

$$\sum_{\mathcal{O}'} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'} G_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}} = A\ . \tag{3.51}$$

Note that now the contribution of the identity operator is excluded from the sum.

Equation (3.88) is valid in d dimensions, and can be evaluated using known expressions for conformal blocks. Focusing, again, on $d=4$, they read

$$g_{\Delta,l}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(K_{\Delta+l}(z)K_{\Delta-l-2}(\bar{z}) - K_{\Delta+l}(\bar{z})K_{\Delta-l-2}(z)\right), \tag{3.52}$$

where K_β is given in terms of hypergeometric functions, $K_\beta(x) = x^{\beta/2}{}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta;x\right)$. From these, one can then compute integrated conformal blocks $G_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}}$ defined in eqs. (3.47) and (3.49). In figure (3.2), integrated conformal blocks as functions of dimensions Δ and spin l are provided. Relevant scalar operators have negative integrated conformal blocks and therefore give a negative contribution to the sum rule (3.88). The opposite holds for irrelevant scalar operators which give instead a positive contribution. All other operators display an alternating behavior: contributions are positive for $l=4$, 8, ... and negative for $l=2$, 6, ... (our numerics suggests this behavior to hold for arbitrary values of l). One can repeat the above analysis in spacetime dimensions other than four, and it turns out that exactly the same pattern holds.

A point worth stressing is that the sum rule (3.88) is not unique. For one thing, it depends upon the choice of the integration domain D. More generally, this ambiguity comes from crossing symmetry. Indeed, the crossing symmetry equation for a marginal operator is given by

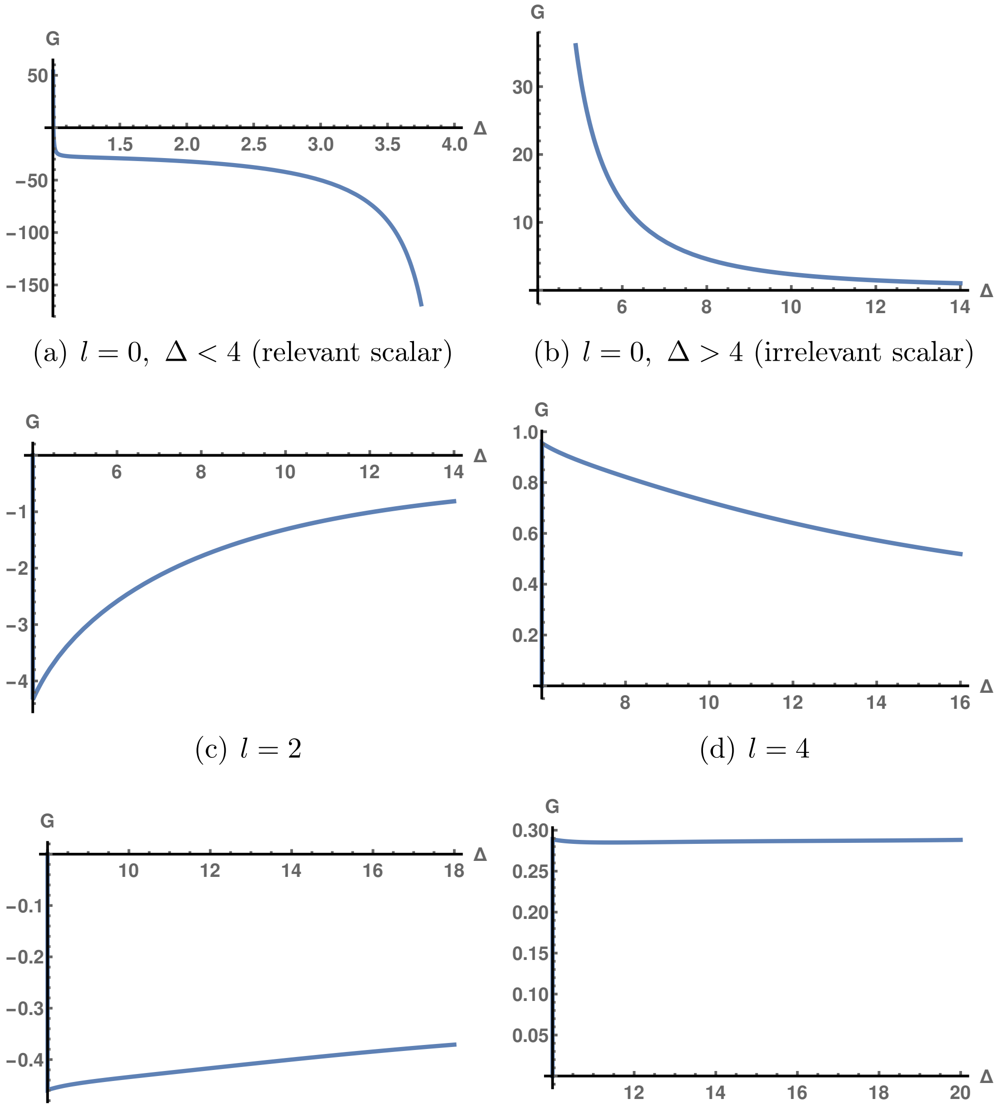
$$\sum_{\mathcal{O}'} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}\left(v^d\ g_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}}(u,v) - u^d\ g_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}}(v,u)\right) = 0\ , \tag{3.53}$$

where u and v are conformal cross-ratios which, in our case, are $u=z\bar{z}$ and $v=(1-z)(1-\bar{z})$. For any point $z,\bar{z}$ this gives a sum of the same form as eq. (3.88) but with a zero on the r.h.s. . Any such sum, or linear combinations thereof, can be added to eq. (3.88), modifying the coefficients in front of $C_{\mathcal{O}\mathcal{O}\mathcal{O}'}$'s without changing the r.h.s., hence giving, eventually, a different sum rule. It would be interesting to see whether there exists a choice which makes all terms in the l.h.s. of (3.88) being positive definite. From such a sum rule it would be possible to get very stringent constraints on CFT data as, *e.g.*, a lower bound on the central charge of the theory. We were not able to find such linear combination for arbitrary d, if it exists at all.

For the sake of what we will do in later sections, let us finally notice that if there are no relevant scalar operators in the $\mathcal{O}\mathcal{O}$ OPE, eq. (3.42) simplifies to

$$\int d^dx \langle \mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(e)\mathcal{O}(\infty)\rangle_c = 0\ , \tag{3.54}$$

and integrated conformal blocks in eq. (3.49), hence contributions as in figure (3.2(a)), would not contribute to (3.88). Still, this would not change the alternate sign behavior of the sum rule (3.88), since also operators with $l = 2 \bmod 4$ contribute with a negative sign.



(a) $l = 0,\ \Delta < 4$ (relevant scalar)

(b) $l = 0,\ \Delta > 4$ (irrelevant scalar)

(c) $l = 2$

(d) $l = 4$

(e) $l = 6$

(f) $l = 8$

Figure 3.2. Integrated conformal blocks G as a function of operator dimensions for $l = 0,\ 2,\ 4,\ 6,\ 8$ spin in $d = 4$ dimensions.

3.2.2 Constraints and bounds on CFT data

The alternating sign behavior in the sum (3.88) makes it impossible to get straight bounds on $C_{\mathcal{O}\mathcal{O}\mathcal{O}'}$ coefficients, as one might have hoped. Nevertheless, one can still extract useful information out of (3.88) , as we are going to discuss below.

Estimating the tail. The fact that A in eq. (3.88) is a positive number implies that the $\mathcal{O}\mathcal{O}$ OPE must contain at least one operator with positive integrated conformal block. From the results reported in figure (3.2) it follows that at least an irrelevant scalar operator or else a spinning operator with $l = 4$ mod 4 must be present. In principle, this can be interesting since to date numerical bootstrap results are less powerful as far as OPE of operators of dimension $\Delta \gtrsim d$ are concerned. When a marginal operator $\mathcal{O}$ exists, instead, one gets constraints also about the spectrum of other such operators. This can be seen as follows.

Let us consider a given value $\Delta = \Delta_*$ and divide the sum (3.88) as

$$\sum_{\mathcal{O}':\Delta<\Delta_*} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}\, G_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}} + \sum_{\mathcal{O}':\Delta>\Delta_*} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}\, G_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}} = A \ . \tag{3.55}$$

Since the series is expected to converge, there should exist (large enough) values of Δ_* for which

$$\sum_{\mathcal{O}':\Delta>\Delta_*} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}\, G_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}} < A \ . \tag{3.56}$$

This means that

$$\sum_{\mathcal{O}':\Delta<\Delta_*} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}\, G_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}} > 0 \ , \tag{3.57}$$

which implies, in turn, that among the operators with dimension $\Delta < \Delta_*$, at least one operator with positive integrated conformal block should exist. If Δ_* is parametrically large this is something not very informative. If Δ_* is not too large, instead, one can get interesting constraints on the spectrum of low dimension operators.

One can try to give an estimate of the values of $\Delta = \Delta_*$ for which (3.56) is satisfied, *e.g.*, using the approach of [71, 72], where the question of convergence of OPE expansion was addressed, and an estimate of the tail was given. For example, for $d = 4$ this takes the form

$$\sum_{\mathcal{O}':\Delta>\Delta_*} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'} g_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}}(z,\bar{z}) \lesssim \frac{2^{16}\Delta_*^{16}}{\Gamma(17)} \left|\frac{z}{(1+\sqrt{1-z})^2}\right|^{\Delta_*} . \tag{3.58}$$

One can then define

$$\Sigma(\Delta_*) \equiv \pi \int_D d^2 z \, \left(\frac{z-\bar{z}}{2i}\right)^2 \frac{2^{16}\Delta_*^{16}}{\Gamma(17)|z|^8} \left|\frac{z}{(1+\sqrt{1-z})^2}\right|^{\Delta_*} , \tag{3.59}$$

which means that

$$\sum_{\mathcal{O}':\Delta>\Delta_*} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'}\, G_{\Delta_{\mathcal{O}'},l_{\mathcal{O}'}} \lesssim \Sigma(\Delta_*) \ . \tag{3.60}$$

The function $\Sigma(\Delta_*)$ is shown in figure (3.3). In principle, the estimate (3.58) is valid only asymptotically, namely in the limit $\Delta_* \to \infty$. Moreover, the actual

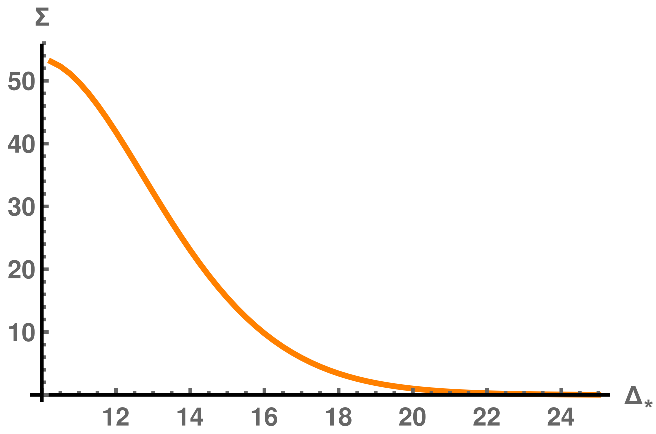


Figure 3.3. The estimate $\Sigma(\Delta_*)$ as a function of Δ_*.

value above which the error one is making can be neglected is theory-dependent. Therefore, one should be careful using (3.58) for too low values of Δ_* and/or to make generic predictions. In fact, numerical bootstrap results suggest that a value of, say, $\mathcal{O}(10)$, can already be in a safe region for a large class of CFTs (see [73] for a discussion on this point).

Looking at (3.57), it is clear that the lower Δ_* the more stringent the constraints on low dimension operators. Requiring the l.h.s. of eq. (3.56) to saturate the inequality, which is the best one can do, and evaluate it using (3.60), we get that $\Sigma(\Delta_*) = A$ for $\Delta_* = 16.3$. This is already a large enough value for which the estimate (3.58) can be trusted, for a large class of CFTs [73]. Looking at figure (3.2) we then conclude that in the OPE of an exactly marginal scalar operator there must be either an irrelevant scalar operator and/or some spin $l = 4, 8, 12$ operators with dimensions $\Delta \lesssim 16$ (recall that the unitarity bound is $\Delta = d - 2 + l$).

In all above discussion we have been focusing, for definiteness, on $d = 4$ dimensions, but similar conclusions can be drawn in any dimensions d.

Let us finally note, in passing, that the same approach used here could more generally be used to constrain the spectra of a CFT whenever the two loop β function coefficient is known.

3.3 Conformal manifolds and holography

In this section we want to focus our attention on CFTs admitting a gravity dual description. These can be characterized as CFTs which admit a large-N expansion and whose single-trace operators with spin greater than two have a parametrically large dimension [63]. More precisely, in the large-N limit the CFT reduces to a subset of operators having small dimension (*i.e.*, a dimension Δ that does not scale with N), and whose connected n-point functions are suppressed by powers of $1/N$. This implies, in particular, that for $N \to \infty$ the four-point function factorizes and hence the connected four-point function vanishes, like for free operators. However, unlike the latter, these operators, also known as generalized free fields, do not saturate the unitarity bound (see [24] for a nice review).

Scalar operators are dual to scalar fields in the bulk. From the mass/dimension relation, which (for scalars and in units of the AdS radius) reads

$$m^2 = \Delta(\Delta - d) \ , \tag{3.61}$$

it follows that in order for the dual operator $\mathcal{O}$ to be marginal, one needs to consider a massless scalar in the bulk. Its non-normalizable mode acts as a source for $\mathcal{O}$, and thus corresponds to a deformation in the dual field theory described by eq. (3.1) (in other words, the non-normalizable mode is dual to the coupling g). The conformal manifold $\mathcal{M}_c$ is hence mapped into the moduli space $\mathcal{M}$ of AdS vacua of the dual gravitational theory, *i.e.*, AdS solutions of bulk equations of motion parametrized by massless, constant scalar fields [17].

The duality between $\mathcal{M}_c$ and $\mathcal{M}$ makes it manifest the difficulty to have conformal manifolds in absence of supersymmetry. A non-supersymmetric CFT is dual to a non-supersymmetric gravitational theory. Differently from supersymmetric moduli spaces, non-supersymmetric moduli spaces are expected to be lifted at the quantum level. Quantum corrections in the bulk are weighted by powers of $1/N$. Hence, one would expect that a moduli space of AdS vacua existing at the classical level, would be lifted at finite N.

For theories with a gravity dual description, this is the simplest argument one can use to argue that conformal manifolds without supersymmetry are something difficult to achieve. In this respect, it is already interesting to find non-supersymmetric conformal manifolds persisting at first non-planar level. One of our aims, in what follows, is to show that this is not an empty set.

We will consider the simplest model one can think of, namely a massless scalar field ϕ minimally coupled to gravity. This corresponds to CFTs which, as far as single-trace operators are concerned, in the large-N limit reduce to a single low-dimension scalar operator $\mathcal{O}$, dual to ϕ.[12]

[12] A CFT must include the energy-momentum tensor. Our toy-model could be thought of as a sector of an AdS compactification in which there is a self-interacting scalar in the approximation that gravity decouples, as in *e.g.* [63]. Most of what we will do, does not depend on this approximation.

3.3.1 Conformal perturbation theory and the 1/N expansion

Our first goal is to discuss how the two perturbative expansions we have to deal with in the CFT, that is, conformal perturbation theory, which is an expansion in g, and the $1/N$ expansion, are related to one another from a holographic dual perspective.

Let us consider a bulk massless scalar ϕ having polynomial interactions of the form

$$\sum_n \lambda_n \left[\phi^n\right] \ , \tag{3.62}$$

where $n \geq 3$ and $[\phi^n]$ stands for Lorentz invariant operators made of n fields ϕ's. For the time being, we do not need to specify their explicit form, which can also include derivative couplings.

Let us consider the one-loop coefficient β_1, eq. (3.38). In order to compute it holographically, one needs to evaluate Witten diagrams [74] with three external lines. Witten diagrams are weighted with different powers of $1/N$, corresponding to tree-level and loop contributions in the bulk. As shown in figure (3.4), at tree level only the cubic vertex can contribute to the three-point function. At higher loops, instead, also couplings with $n > 3$ may contribute to β_1.

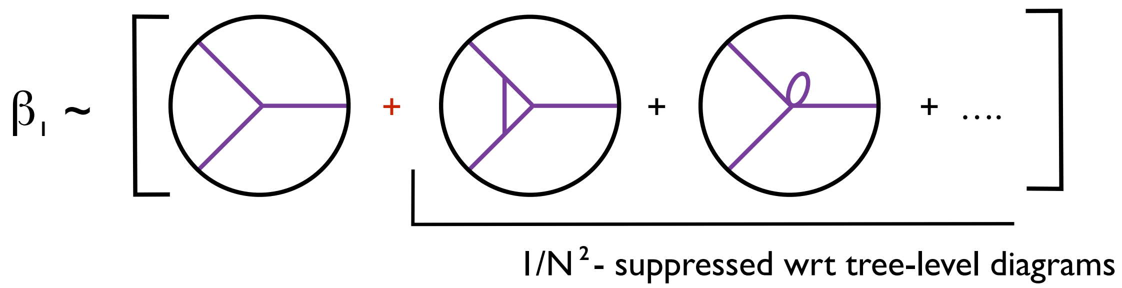


Figure 3.4. Witten diagrams contributing to $C_{\mathcal{OOO}}$. Violet lines correspond to propagation of ϕ fields and may have spacetime derivatives acting on them, depending on the specific structure of the operators (3.62). At tree-level, only cubic couplings can contribute to the three-point function. At loop level, also couplings with $n > 3$ can contribute, *e.g.*, the quintic coupling shown in the figure.

A similar story holds for the two-loop coefficient β_2 (note that in our one-field model eq. (3.42) simplifies just to the integral of the four-point function, eq. (3.54)).

To leading order, there are two contributions. The contact quartic interaction and the cubic scalar exchange, as shown in figure (3.5). Again, at higher-loops in the bulk coupling, one can get contributions also from operators with $n > 4$.

The analysis applies unchanged to the three-loop coefficient β_3 and higher. In particular, only operators $[\phi^n]$ with $n \leq m$ can contribute to the m-point function of $\mathcal{O}$ at tree level. Conversely, at loop level, also operators with $n > m$ may contribute.

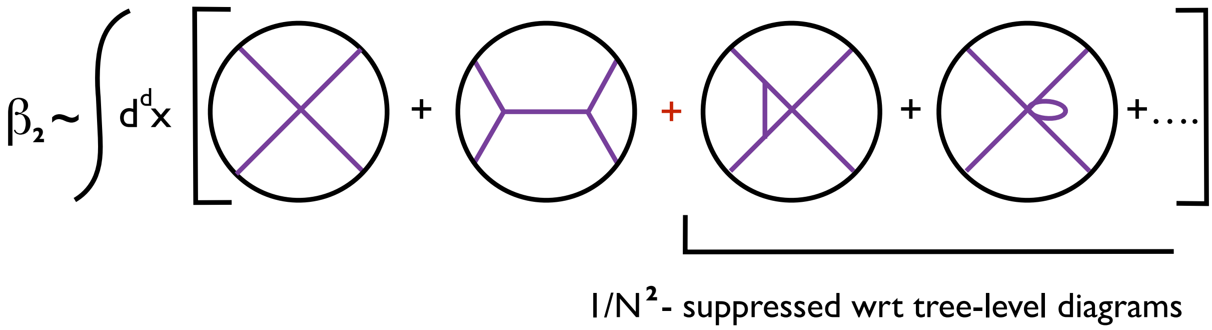


Figure 3.5. Structure of Witten diagrams contributing to the two-loop coefficient of $\beta(g)$, after integration in d^dx. Conventions are as in figure (3.4).

What we would like to emphasize with this discussion is that by doing tree-level computations in the bulk, one can extract the leading, planar contribution to $\beta(g)$ at all loops in g. In other words, classical gravity provides an exact answer, in conformal perturbation theory, to the existence of a conformal manifold, at leading order $1/N$. To get this, rather than computing Witten diagrams, it is clearly much simpler to solve bulk equations of motion and see which constraints on the structure of the operators (3.62) does the existence of AdS solutions with constant ϕ impose. This is what we will do, first. Then, we will compute explicitly tree-level Witten diagrams contributing to β_1 and β_2, and check that the constraints one gets by requiring them to vanish, are in agreement with those coming from equations of motion analysis.

A non-trivial question one can ask is whether the vanishing of $\beta(g)$ at two-loops leaves some freedom in the scalar couplings compared to the equation of motion analysis. And, if this is the case, at which loop order in CPT one should go, to fix such freedom. The answer turns out to be rather simple: admissible operators of the form $[\phi^n]$ will be fully determined by imposing the vanishing of the β-function coefficient β_{n-2}, and no higher orders will be needed. The toy model we are going to discuss has operators with $n = 3, 4$ only, and, consistently, we will see that the constraints coming just from the vanishing of β_1 and β_2, will provide the full gravity answer.

Another interesting question is which further constraints the vanishing of the one and two-loop coefficients of $\beta(g)$ put on the CFT taking into account $1/N$ corrections, that is, going beyond planar level. As already emphasized, one does not expect exact conformal manifolds to survive at finite N, without supersymmetry. However, one can ask whether non-trivial CFTs with non-supersymmetric conformal manifolds persisting at first non-planar level could exist. That this can be, it is not obvious, and this is what we will address next.

3.3.2 Scalar fields in AdS

We want to compare the holographic analysis with CPT at two-loops, which, as such, involves at most four-point functions, eqs. (3.41) and (3.54). Therefore, for simplicity, we will focus on models with cubic and quartic couplings, only. The bulk action reads

$$S = \frac{1}{2\kappa_{d+1}^2}\int d^{d+1}x\sqrt{-g}\left(R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - 2\Lambda + \left[\phi^3\right] + \left[\phi^4\right]\right)\ , \tag{3.63}$$

where Λ is the (negative) cosmological constant and the last two terms represent cubic and quartic interactions. The absence of a mass term for ϕ guarantees that the dual operator $\mathcal{O}$ is marginal, *i.e.* $\Delta_{\mathcal{O}} = d$. We would like to constrain the explicit form of cubic and quartic couplings by requiring the existence of a conformal manifold under a deformation parametrized by ϕ itself. We take $\kappa_{d+1} \sim N^{-1}$ to match holographic correlators with CFT correlation functions in the large-N limit. In the above normalization, the two point function $\langle\mathcal{O}\mathcal{O}\rangle$ scales as N^2. Such unusual normalization has the advantage to treat democratically all Witten diagrams (as well as the dual n-point functions, and so the β-function coefficients β_n), in the sense that, regardless the number of external legs, they all scale the same with N, at any fixed order in the bulk loop expansion.[13] This is the most natural choice that avoids mixing-up the expansion in $1/N$ with that in g.

From the action (3.63) one can derive the equations of motion, which read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\; \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\,(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + 2\Lambda) - \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\sqrt{-g}\,\left(\left[\phi^3\right] + \left[\phi^4\right]\right) \tag{3.64}$$

$$\Box_g\,\phi \;=\; -\frac{\delta}{\delta\phi}\left(\left[\phi^3\right] + \left[\phi^4\right]\right)\ , \tag{3.65}$$

[13] The interested reader can explicitly check this statement, after having properly chosen the normalization of the bulk-to-boundary propagator.

where $\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)$.

We need to look for pure AdS solutions with constant scalar profile. In absence of interactions, that is in the strict generalized free-field limit, the large-N CFT reduces to a massless free scalar ϕ propagating in a rigid AdS background. The equations of motion admit a solution with AdS metric and constant scalar field $\phi = \phi_0$ which, in Poincaré coordinates, reads

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dx_i dx^i\right) \tag{3.66}$$
$$\phi = \phi_0 \tag{3.67}$$

with $L = \sqrt{d(1-d)\Lambda}$ being the AdS radius and the AdS boundary sitting at $z = 0$. The modulus ϕ_0 parametrizes the dual conformal manifold, described by the deformation $g\int d^dx\,\mathcal{O}$. Eqs. (3.41) and (3.54) are trivially satisfied: since ϕ is a free field, Witten diagrams vanish identically (in particular, in eq. (3.54) the integrand itself vanishes).

Let us now consider possible cubic and quartic interactions. From eqs. (3.64)-(3.65) it follows that couplings compatible with solutions with AdS metric and a constant scalar profile are couplings where spacetime derivatives appear (note that, due to Lorentz invariance, only even numbers of derivatives are allowed). Schematically, acceptable operators look like

$$\nabla\nabla\ldots\phi\;\nabla\nabla\ldots\phi\;\nabla\nabla\ldots\phi\nabla\nabla\ldots\phi\ldots\;, \tag{3.68}$$

where full contraction on Lorentz indexes is understood and some (but not all) naked ϕ's, that is ϕ's without derivatives acting on them, can appear. Therefore, at the classical level, *i.e.* to leading order in $1/N$, the requirement of existence of a conformal manifold under the deformation (3.1) rules out the non-derivative couplings ϕ^3 and ϕ^4, only.[14]

As anticipated, we want to compare the above analysis with a direct computation of three and (integrated) four-point functions, which are related to the one and two-loop coefficients of $\beta(g)$ via eqs. (3.38)-(3.39), by means of tree-level Witten diagrams. This could be seen as a simple AdS/CFT self-consistency check, but one can in fact learn from it some interesting lessons, which could be useful when considering more involved models, as well as when taking into account loop corrections in the bulk.

Tree-level Witten diagrams

Let us consider the one-loop coefficient β_1, which is proportional to $C_{\mathcal{OOO}}$. To leading order at large N, this corresponds to the Witten diagram shown in figure

[14] One can consider the more general structure (3.62) and the same conclusion holds. Any coupling $[\phi^n]$ with (an even number of) derivatives is allowed, classically.

(3.6), to which only cubic couplings $[\phi^3]$ can contribute.

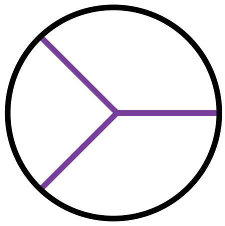


Figure 3.6. Witten diagram contributing to β_1 at leading order in $1/N$.

The pure non-derivative coupling ϕ^3 provides a non-vanishing contribution to $C_{\mathcal{OOO}}$. Therefore, it is excluded. The first non-trivial couplings are then two-derivative interactions. In principle, the following interaction terms are allowed

$$\phi\nabla_\mu\phi\nabla^\mu\phi \quad , \quad \phi^2\,\nabla_\mu\nabla^\mu\phi \; . \tag{3.69}$$

Upon using integration by parts and the equation of motion which, at lowest order in the couplings, is just $\nabla_\mu\nabla^\mu\phi = 0$, these interactions are either total derivatives or vanish on-shell. Therefore, they do not contribute to $C_{\mathcal{OOO}}$ (this is to be contrasted with the case of a massive scalar, where these interactions are proportional to ϕ^3).

Next, one can consider interactions with four spacetime derivatives, that is

$$\phi\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi \quad , \quad \nabla_\mu\phi\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi \quad , \quad \phi^2\nabla_\mu\nabla_\nu\nabla^\mu\nabla^\nu\phi \; . \tag{3.70}$$

These terms are also either vanishing on-shell or total derivatives, and do not provide any contribution to the three-point function $\langle\mathcal{OOO}\rangle$, at leading order. Let us briefly see this. Using integration by parts, the second term in (3.70) can be written as

$$\int \nabla_\mu\phi\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi = -\frac{1}{2}\int \nabla_\nu\nabla^\nu\phi\;\nabla_\mu\phi\nabla^\mu\phi \; , \tag{3.71}$$

which vanishes upon using the equation of motion. As for the other two terms in (3.70), using the identity $[\Box, \nabla_\mu]\phi = -d\;\nabla_\mu\phi$, they can be re-written, respectively, as

$$\int \phi\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi = \int \left(\frac{1}{2}\;\nabla_\nu\nabla^\nu\phi\;\nabla_\mu\phi\nabla^\mu\phi - \frac{d}{2}\;\phi^2\nabla_\nu\nabla^\nu\phi\right) \; , \tag{3.72}$$

$$\phi^2\nabla_\mu\nabla_\nu\nabla^\mu\nabla^\nu\phi = \phi^2\nabla_\mu\nabla^\mu\nabla_\nu\nabla^\nu\phi - d\phi^2\nabla_\mu\nabla^\mu\phi \; . \tag{3.73}$$

Again, both terms vanish upon using the equation of motion, and hence provide no contribution to $C_{\mathcal{OOO}}$. One can proceed further, and consider couplings with an increasing number of derivatives, with structures that generalize (3.70). Using previous results and proceeding by induction, one can prove that contributions vanish for any number of derivatives. The upshot is that all operators with two or more derivatives either vanish or can be turned into total spacetime derivatives, and hence give a vanishing contribution to the Witten diagram in figure (**??**) and, in turn, to $C_{\mathcal{O}\mathcal{O}\mathcal{O}}$.

Although derivative couplings provide a vanishing contribution to cubic Witten diagrams, they can provide non-vanishing contribution to the four-point function by exchange Witten diagrams like the one depicted in figure (3.7) (which include, in the dual CFT, the exchange of double-trace operators). Therefore, these interactions can potentially contribute to the two-loop coefficient β_2.

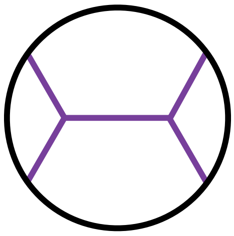


Figure 3.7. Exchange Witten diagram contributing to β_2, after integration in $\int d^d x$. https://ru.sharelatex.com/project/5b583ccf59168a3e1bc42c45

The pure non-derivative coupling ϕ^3 is already excluded by previous analysis (and it would also contribute to the Witten diagram in figure (3.7), in fact). Let us then start considering contributions from operators having one field ϕ not being acted by derivatives, *i.e.* the first ones in (3.69) and (3.70) and generalizations thereof, that is operators of the form

$$\phi \nabla\nabla \dots \phi \nabla\nabla \dots \phi \, . \tag{3.74}$$

There are two possible types of exchange Witten diagrams: (a) diagrams where all external lines are acted by derivatives, (b) diagrams where at least one external line is free of derivatives. Focusing, for definiteness, on two-derivative couplings, contributions of type (a) and (b) correspond to the following integrals, respectively

$$\int d^d x_1 \int d^d w_1 dz_1 \int d^d w_2 dz_2 \; \nabla_\mu K(z_1, w_1 - x_1) \nabla^\mu K(z_1, w_1 - x_2) G(z_1 - z_2, w_1 -, w_2) \\ \nabla_\nu K(z_2, w_2 - x_3) \nabla^\nu K(z_2, w_2 - x_4) \tag{3.75}$$

$$\int d^dx_1 \int d^dw_1dz_1 \int d^dw_2dz_2\ K(z_1,w_1-x_1)\nabla^\mu K(z_1,w_1-x_2)\nabla^{(1)}_\mu\nabla^{(2)}_\nu G(z_1-z_2,w_1-,w_2)$$
$$K(z_2,w_2-x_3)\nabla^\nu K(z_2,w_2-x_4)\ . \tag{3.76}$$

$K(z,w-x_i)$ is the bulk-to-boundary propagator which, for massless scalars, reads

$$K(z,w-x)=\left(\frac{z}{z^2+(w-x)^2}\right)^d\ , \tag{3.77}$$

and satisfies the equation $\nabla_\mu\nabla^\mu K(z,w-x)=\Box_g K(z,w-x)=0$. $G(z_1-z_2,w_1-w_2)$ is instead the bulk-to-bulk propagator which, for massless scalars, reads

$$G(z_1-z_2,w_1-w_2)=\frac{2^{-d}C_d}{d}\xi^d F\left(\frac{d}{2},\frac{d}{2}+\frac{1}{2};\frac{d}{2}+1;\xi^2\right)\ ,\qquad C_d=\frac{\Gamma(d)}{\pi^{d/2}\Gamma(d/2)}\ , \tag{3.78}$$

where ξ is the geodesic distance between the two points in the bulk where interactions occur, (z_1,w_1) and (z_2,w_2),

$$\xi=\frac{2z_1z_2}{z_1^2+z_2^2+(w_1-w_2)^2}\ . \tag{3.79}$$

The bulk-to-bulk propagator satisfies the equation $\Box_g G(z_1,w_1;z_2,w_2)=\frac{1}{\sqrt{g}}\delta\left(z_1-z_2,w_1-w_2\right)$.

Diagrams of type (a) vanish because the integrated bulk-to-boundary propagator $K(z,w-x)$ is independent of z and w, namely

$$\int d^dx\ K(z,w-x)=\frac{\pi^{d/2}\Gamma(d/2)}{\Gamma(d)}\ , \tag{3.80}$$

and, plugging (3.80) into (3.75), one gets

$$\int d^dx_1\nabla_\mu K(z_1,w_1;x_1)=0\ . \tag{3.81}$$

Diagrams of type (b), after x-integration, also vanish. Indeed, the integral (3.76) becomes

$$\frac{\pi^{d/2}\Gamma(d/2)}{\Gamma(d)}\int d^dw_1dz_1\int d^d,w_2dz_2\ \nabla^\mu K(z_1,w_1-x_2)\nabla^{(1)}_\mu\nabla^{(2)}_\nu G(z_1-z_2,w_1-w_2)$$
$$K(z_2,w_2-x_3)\nabla^\nu K(z_2,w_2-x_4)\ , \tag{3.82}$$

and, integrating by parts, one can transfer the covariant derivative $\nabla^{(1)}_\mu$ acting on the bulk-to-bulk propagator onto $K(z_1,w_1-x_2)$, getting

$$-\frac{\pi^{d/2}\Gamma(d/2)}{\Gamma(d)}\int d^dw_1dz_1\int d^dw_2dz_2\ \Box_g K(z_1,w_1-x_2)\nabla^{(2)}_\nu G(z_1-z_2,w_1-w_2)$$

$$K(z_2, w_2 - x_3)\nabla^\nu K(z_2, w_2 - x_4) \ , \qquad (3.83)$$

which vanishes because $\Box_g K = 0$. This computation can be repeated for terms with four or more derivatives, just replacing single derivatives acting on the propagators in (3.75) and (3.76) with multiple derivatives. The end result can again be shown to be zero.

The second possible cubic vertexes which could contribute to the exchange Witten diagram are those with derivatives acting on one field only, schematically

$$\phi^2 \, \nabla\nabla\nabla \ldots \phi \ . \qquad (3.84)$$

Using properties of Ricci and Riemann tensors in AdS, one can show that these couplings can be re-written as sums of terms of the form $\phi^2 \, \Box^p \phi$, with p an integer. Due to the property $\Box_g K = 0$, if derivatives are acting on at least one external line, the result is zero. If not, namely if derivatives act only on the bulk-to-bulk propagator, then the corresponding diagram is a special instance of a (b)-type diagram previously discussed and, following similar steps as in eqs. (3.82)-(3.83), one gets again a vanishing result.

Finally, let us consider shift-symmetric couplings, that is couplings without naked ϕ's. This kind of couplings give rise to diagrams of type (a), very much like (3.75), where all external lines (in fact any line) contain derivatives. Therefore, they do not contribute to exchange Witten diagrams, either.

This ends our analysis of cubic operators, which fully agrees with equations of motion analysis.

Let us emphasize that while all cubic couplings but ϕ^3 do not contribute at the level of three-point functions, they do, in general, as far as exchange Witten diagrams are concerned. There, it matters that, in computing the two-loop coefficient β_2, integration in $d^d x$ is required, and this plays a crucial role in providing a vanishing result, in the end.

Let us now consider quartic couplings. At tree level they do not contribute to β_1, but they can contribute to β_2, instead, via contact-terms, as the one depicted in figure (3.8).

The operators we should consider are just obtained by adding an extra field ϕ to all cubic vertexes previously considered. Again, the pure non-derivative coupling ϕ^4 is excluded from the outset, since it clearly gives a non-vanishing contribution. The other operators have the following structures

$$\phi \, \nabla\nabla \ldots \phi \, \nabla\nabla \ldots \phi \, \nabla\nabla \ldots \phi \ \ , \ \ \phi^2 \, \nabla\nabla \ldots \phi \, \nabla\nabla \ldots \phi \ \ , \ \ \phi^3 \, \nabla\nabla \ldots \phi \ , \qquad (3.85)$$

as well as the shift-symmetric one

$$\nabla\nabla \ldots \phi \, \nabla\nabla \ldots \phi \, \nabla\nabla \ldots \phi \, \nabla\nabla \ldots \phi \ . \qquad (3.86)$$

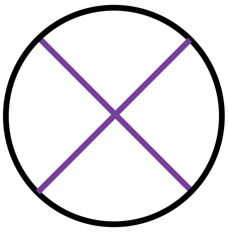


Figure 3.8. Contact Witten diagram contributing to β_2, after integration in $d^d x$.

Given our previous analysis it is not difficult to compute the contribution of these diagrams to the integrated four-point function and hence to the β function two-loop coefficient β_2. Upon integration, the diagram in figure (3.8) either gives zero, when the x-dependence is on a line where bulk derivatives act, see eq. (3.81), or, after x-integration, it reduces to the effective vertex of one of the cubic vertices discussed previously, which vanish. We thus see that all operators (3.85) and (3.86) do not give any contribution to β_2. Note, again, that x-integration plays a crucial role.

To summarize, the constraints on cubic and quartic couplings coming from CPT at two-loops, already capture the (full) gravity answer, as anticipated. From the analysis in section (3.2), it is not difficult to get convinced that operators with n fields ϕ will be univocally fixed by computing tree-level Witten diagrams with n external legs, which contribute to the β function at $n-2$ loop order.

As already emphasized, a CFT must include the energy-momentum tensor in the spectrum of primary operators, which amounts to include dynamical gravity in the bulk. This would contribute to the exchange Witten diagram in figure (3.7), since now also graviton exchange should be considered in the bulk-to-bulk propagator. Still, all graviton-scalar couplings surviving our previous analysis are in fact derivative couplings, and one can see that the corresponding (integrated) exchange Witten diagrams vanish as well. So, our conclusions are unchanged also once gravity is taken into account.[15]

Before closing this section, let us note the following interesting fact. Suppose we add a quartic, non-derivative coupling $\lambda\phi^4$ to the free scalar theory. This lifts the flat direction associated to ϕ. In the dual CFT, a non-vanishing β function for the dual coupling g is generated at two-loops, at leading order in $1/N$ (recall that a one-loop coefficient β_1 cannot be generated by a quartic interaction at tree level in the bulk). In the bulk, the sign of λ matters. In particular, the quartic interaction destabilizes the AdS background for $\lambda < 0$, while it leaves AdS as a stationary point for $\lambda > 0$. One can then try to understand what this instability corresponds to, in the dual CFT. The two-loop coefficient of the β function in

[15]This is at least true for models where scalars are minimally coupled to gravity, as it is the case here.

CPT is proportional to the (integrated) contact Witten diagram of figure (3.8), which in this case is non-vanishing, *i.e.* $\beta_2 = a\lambda$, with a a positive d-dependent number, $a = \pi^d\Gamma(d/2)^4/2\Gamma(d)^3$. Therefore, β_2 has the same sign as λ. This means that for $\lambda > 0$ the operator $\mathcal{O}$ becomes marginally irrelevant, while for $\lambda < 0$ it becomes marginally relevant. Hence, in the latter case, a deformation triggered by $\mathcal{O}$ induces an RG-flow which brings the theory away from the fixed point. On the contrary, for $\lambda > 0$ the deformation is marginally irrelevant and the undeformed CFT remains, consistently, a stable point. Note how different this is from the case of SCFTs. There, marginal operators may either remain marginal or become marginally irrelevant, but never marginally relevant [19], which agrees with the fact that AdS backgrounds are stable in supersymmetric setups.

Loops in AdS

An obvious question is whether one can push the above analysis to higher orders in $1/N$. This corresponds to take into account loop corrections in the bulk. Already at one-loop, this is something very hard to do (see, *e.g.*, [78–80], and, more recently, [64,82], where interesting progress have been obtained from complementary perspectives).

The main issue in this matter is not really to compute loop amplitudes per sé, but to make their relation to tree-level amplitudes precise, and this is something non-trivial to do in AdS. In fact, the question we are mostly interested in, here, is slightly different. Starting from the effective action (3.63), which is valid up to some energy cut-off E, in computing quantum corrections we are not much interested on how the couplings run with the scale but else on which (new) operators would be generated at energies lower than E.[16] More precisely, what we have to do is to pinpoint, between the operators having passed our tree-level bulk analysis, *i.e.*, operators of the form (3.68), those which could induce, at loop level, effective couplings which have instead been excluded at tree-level, that is the pure non-derivative couplings ϕ^3 and ϕ^4, as well as a mass term, which was set to zero from the outset. Such operators would spoil the vanishing of the β function, see figure (3.9) (the generation of a ϕ^2-term would modify the scaling dimension of $\mathcal{O}$, which should instead remain a marginal operator). So, the basic question we have to answer is whether (one and higher) loop analysis still leaves some of the operators (3.68) being compatible with the vanishing of the β function (3.36) and with $\Delta_{\mathcal{O}} = d$.

That this is not an empty set can be easily seen as follows. Out of the full set (3.68), let us consider shift-symmetric operators, only, namely operators which are

[16] In doing so, we can use the intuition from flat-space physics, since we are dealing with local effects in the bulk.

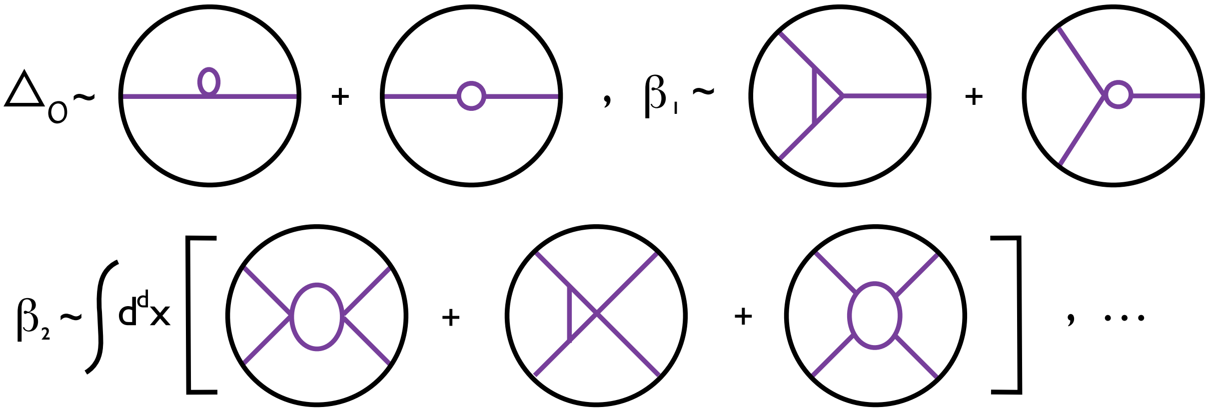


Figure 3.9. One-loop Witten diagrams contributing to $\Delta_{\mathcal{O}}$, β_1 and β_2. Cubic and quartic Witten diagrams should include also those with loop corrections to propagators, but we have not drawn them explicitly.

invariant under the shift symmetry

$$\phi \to \phi + a \ . \tag{3.87}$$

In perturbation theory, such operators cannot generate effective operators not respecting (3.87), hence in particular ϕ^n terms. Therefore, at least perturbatively, a conformal manifold does persist, if only shift-symmetric couplings are allowed in the action (3.63).

Let us now consider all other couplings, those with at least one naked ϕ, which do not respect the shift-symmetry (3.87). Generically, these operators would generate any effective operator of the form ϕ^n, quantum mechanically. In particular, regardless of spacetime dimension, ϕ^2 and ϕ^3 will be generated at one-loop by any (non shift-symmetric) operator of the form (3.68). Operators ϕ^n with $n \geq 4$, instead, will be generated at one-loop or higher, depending on spacetime dimension and the specific operator (3.68) one is considering. In any event, the upshot is that, unless one invokes some unnatural tuning between the a priori independent couplings $_n$, any operator with at least one naked ϕ should be excluded, eventually, by requiring a conformal manifold to persist at finite N. This leaves only shift-symmetric couplings in business, meaning that the shift symmetry (3.87) should be imposed on the bulk action (3.63) altogether.[17]

As already noticed, shift-symmetric couplings would not contribute to (integrated) Witten diagrams not just at one loop but at any loop order in the bulk. Therefore, the final answer we got may be extended as a statement on the existence

[17]Again, the inclusion of a dynamical graviton, hence of the energy-momentum tensor in the low-dimension CFT operators, would not affect this result.

of a conformal manifold generated by $\mathcal{O}$ at all orders in the $1/N$ perturbative expansion.

This apparently strong statement is just due to the axion-like behavior of a field subject to eq. (3.87), which, as such, is expected to be lifted by non-perturbative effects only. The latter are suppressed as, say, e^{-N}. Richer holographic models would behave differently, and not share such perturbative non-renormalization property. Our analysis just aims at showing that, in principle, non-supersymmetric conformal manifolds can exist also beyond planar limit. It would be very interesting to consider models with richer structure. We will offer a few more comments on this issue in the next, concluding section.

3.4 Conclusion and Outlook

In the first part of this chapter we have considered constraints on CFT data that theories living on a conformal manifold should satisfy. These constraints come from the condition of vanishing β function of the (putative) exactly marginal operators. This is an infinite set of constraints, with one constraint corresponding to each order in CPT. We were mostly focusing on the one-loop and two-loop constraints only.

One-loop constraint demands that the OPE coefficients between exactly marginal operators must vanish, and this had been known for a long time [66]. Two-loop constraint gives an interesting the sum rule

$$\sum_{\mathcal{O}'} C^2_{\mathcal{O}\mathcal{O}\mathcal{O}'} G_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}} = A \, , \tag{3.88}$$

with the sum over the operators appearing in the OPE of exactly marginal operator with itself, and $G_{\Delta_{\mathcal{O}'}, l_{\mathcal{O}'}}$ are integrated conformal blocks. A peculiar property of this sum rule is that operators with spin $l = 0$ mod 4 provide positive contribution, while operators with $l = 2$ mod 4 provide negative contribution.

Since A is positive definite, one obvious (yet not so powerful) conclusion is that operators with $l = 0$ mod 4 must be present in the OPE. Using also the estimate for OPE tale, we have formulated a stronger claim: at least one such operator must have dimension $\Delta \lesssim 16$. This is not a rigorous bound, but rather an estimate. We expect that some more precise bounds on CFT data, implied by our sum rule, can be extracted by using numerical bootstrap techniques.

Similar considerations of two-loop beta functions constrained were also performed in [75], with the same conclusion about the alternating behaviour of the sum rule. The authors proceeded with studying differential equations, determining the dependence of CFT data on the coordinated on a conformal manifold (also known as evolution equations). Analyzing these equation, they managed to prove, although in the not terribly interesting $d = 1$ setup, the absence of level crossing for operators with the same symmetry properties. Evolution equations were also recently discussed in [76], where it was argued that they can be used for establishing, in a mathematically rigorous way, the existence of non-trivial QFTs in $d = 4$ dimensions. Finally, we would like to mention [77], where the following question was addressed. The authors note that though commonly believed (end was implicitly assumed in thie thesis), *a priori* it is not self-evident that that a deformation by exactly marginal operator necessarily leads to another CFT. The authors check this statement in the CPT framework. computing leading corrections to two- and three-point functions and demonstrating that they have the form, consisting with conformal invariance.

In the second part of this chapter we focused on CFTs with gravity duals and put in contact the β functions constraints, defining conformal manifolds, and conditions for a gravitational theory to have a moduli space, not lifted quantum mechanically. β function constraints are translated to some conditions that Witten diagrams of the gravitational theory should satisfy. Considering a prototypical example of a scalar field with shift symmetry, we checked explicitly that those conditions are indeed satisfied. This can be considered as another check of AdS/CFT correspondence, but probably more fruitful point of view would be to transform the vanishing β function constrains into the statements about Witten diagrams of gavity theories with exact moduli spaces of AdS vacua.

Concluding, we would like to note that at the moment a concrete example of a non-supersymmetric conformal manifold in $d > 2$ dimensions is not known. Still, there is a very intriguing setup which finds itself between the complete absence of supersymmetry and the powerful field of holomorphic theories with four supercharges. This setup is $\mathcal{N} = 1$ supersymmetry in three dimensions, which was the subject of Chapter 2 of the present thesis. Given the intimate relation between moduli spaces and conformal manifolds, and taking into account recent results regarding the existence of exact moduli spaces in theories with two supercharges [49], it seems suggestive to look for examples of conformal manifolds in $\mathcal{N} = 1$ $3d$ SCFTs. Understanding such a set-up could shed light on general aspects regarding the relation between conformal manifolds and supersymmetry which, in turn, could also help in the quest for a no-go theorem for conformal manifolds without supersymmetry in $d > 2$ dimensions, if any.

Appendix A

W-bosons and CS coupling renormalization

Consider a $3d$ theory with spontaneous gauge symmetry breaking from gauge group G to gauge group H. In order to obtain the Wilsonian low energy effective theory for the unhiggsed fields, massive fields must be integrated out. In particular, there are W-bosons, which renormalize propagators of unbroken gauge fields. If this renormalization contains parity-odd contribution, then the corresponding CS coupling is renornalized. In this appendix we consider as an example $SU(2) \to U(1)$ breaking and show that this renormalization does not take place.

Consider $SU(2)_k$ $3d$ gauge theory, coupled to a scalar in the adjoint representation, with the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}F^2 + \frac{ikg^2}{4\pi}\mathrm{Tr}(AdA + \frac{2}{3}A^3) + \frac{1}{2}(D\phi)^2. \tag{A.1}$$

Classically there is a moduli space of vacua, parametrized by the vev of ϕ: we then write around such point $\phi^a = v^a + \chi^a$. Using gauge transformations, we can put v to diagonal form, and so it is clear that gauge group is spontaneously broken from $SU(2)$ to $U(1)$.

Following the usual procedure, we fix the gauge by adding gauge fixing term and ghosts Lagrangian:

$$\mathcal{L}_{gf} = \frac{1}{2\xi}(\partial_\mu A^{a\mu} + \xi f^{abc}\chi^b v^c)^2, \tag{A.2}$$

$$\mathcal{L}_{ghost} = \partial^\mu \bar{c}^a D^{ab}_\mu c^b - \xi f^{abc}\bar{c}^a c^b (v+\chi)^c. \tag{A.3}$$

Propagator for higgsed gauge fields takes the form:

$$\Delta^{broken}_{\mu\nu} = -\frac{k^2+m_W^2}{(k^2+m_+^2)(k^2+m_-^2)}\left(g_{\mu\nu} + \frac{\mu\ \epsilon^{\mu\nu\rho}k_\rho}{k^2+m_W^2}\right) + \frac{(k^2+m_W^2)(1+\xi)+\mu^2\xi}{(k^2+m_+^2)(k^2+m_-^2)(k^2-\xi m_W^2)}k_\mu k_\nu, \tag{A.4}$$

where

$$
\begin{aligned}
\mu &= \frac{kg^2}{4\pi}, \\
m_W &= g^2 v^2, \\
m_\pm &= \pm\frac{1}{2}\mu + \sqrt{m_W^2 + \frac{1}{4}\mu^2}.
\end{aligned} \tag{A.5}
$$

Vertex of the interaction between to heavy bosons and one light boson is given by (all the momenta are outcoming)

$$
V_{\mu,\nu,\rho}(p,q,r) = -i\ g((r-q)_\mu g_{\nu\rho} + (p-r)_\nu g_{\mu\rho} + (q-p)_\rho g_{\mu\nu} - \mu\epsilon_{\mu\nu\rho}). \tag{A.6}
$$

In the unitary gauge, where $\xi = \infty$, unphysical fields as ghosts and Nambu-Goldstone bosons have infinite mass, and so integration out of heavy gauge bosons fields can be understood in the clearest way. In principle, integrating out heavy gauge bosons, CS term can get renormalized. At one loop it is given by the odd part of the light gauge boson self-energy

$$
(p_\mu p_\nu - p^2 g_{\mu\nu})\Pi^E(p^2) + \mu\ \epsilon_{\mu\nu\rho}p^\rho \Pi^O(p^2), \tag{A.7}
$$

coming from the one-loop diagram with heavy bosons running in the loop. Straightforward but tedious computation gives

$$
\Pi^O(0) = \frac{g^2}{m_W^2}\int \frac{d^3q}{(2\pi)^3}\frac{q^2(q^2+\mu^2)^2 + 4m_W^2\ q^2(2q^2+\mu^2) + m_W^4(q^2-\mu^2) - 6m_W^4}{(q^2+m_+^2)^2(q^2+m_-^2)^2}. \tag{A.8}
$$

This integral can be evaluated in dimensional regularization and gives zero. So, we conclude that CS term is not renormalizaed at 1-loop level. It is also not expected to be renormalized at higher loops, since it would violate quantization of CS coupling.

Appendix B

Proof of the combinatorial identity

In this appendix we give a proof of the identity used in section (2.3.5) to prove that Witten index does not jump for $m > 0$. The statement of the identity is

$$\sum_P (-1)^{N-L} \prod_{I=1}^{L} \frac{k!}{S_I!(k-S_I)!} = \frac{(N+k-1)!}{N!(k-1)!}, \tag{B.1}$$

where P denotes the set of compositions of N, L is the length of a composition, and S_I are summonds, such that $N = \sum_I^L S_I$.

We start by introducing the generating functions

$$F_k = -\sum_{i=1}^{k} \binom{k}{i} x^i = 1 - (x+1)^k, \tag{B.2}$$

$$G_k = F_k + F_k^2 + \ \dots \ . \tag{B.3}$$

Expanding the second function as

$$G_k = \sum_m d_k^m \, x^m, \tag{B.4}$$

it is easy to observe that

$$d_k^N = -\binom{k}{N} + \sum_i \binom{k}{N-i}\binom{k}{i} + \ \dots \ = \sum_P (-1)^L \prod_{I=1}^{L} \binom{k}{S_I}. \tag{B.5}$$

Each term in the sum above correspond to certain composition of N.

G_k can be computed explicitely, with the result

$$G_k = \frac{F_k}{1-F_k} = \frac{1-(x+1)^k}{(x+1)^k} = -1 + (1+x)^{-k}. \tag{B.6}$$

Using the standard expansion

$$(1+x)^{-k} = \sum (-1)^j \binom{k+j-1}{j} x^j, \tag{B.7}$$

we deduce that

$$d_k^N = (-1)^N \binom{k+N-1}{N}, \tag{B.8}$$

which proves (B.1).

Bibliography

[1] R. Haag, J. T. Lopuszanski and M. Sohnius, *All Possible Generators of Supersymmetries of the S Matrix,* Nucl. Phys. B **8**8, 257 (1975).

[2] S. R. Coleman and E. J. Weinberg, *Radiative Corrections as the Origin of Spontaneous Symmetry Breaking,* Phys. Rev. D **7**, 1888 (1973).

[3] M. T. Grisaru, W. Siegel and M. Rocek, *Improved Methods for Supergraphs,* Nucl. Phys. B **159**, 429 (1979).

[4] N. Seiberg, *Naturalness versus supersymmetric nonrenormalization theorems,* Phys. Lett. B **318**, 469 (1993) [hep-ph/9309335].

[5] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, *Instanton Effects in Supersymmetric Theories,* Nucl. Phys. B **229**, 407 (1983).

[6] N. Seiberg, *Electric - magnetic duality in supersymmetric non-Abelian gauge theories,* Nucl. Phys. B **435**, 129 (1995) [hep-th/9411149].

[7] M. A. Shifman and A. I. Vainshtein, *On Gluino Condensation in Supersymmetric Gauge Theories. $SU(N)$ and $O(N)$ Groups,* Nucl. Phys. B **296**, 445 (1988) [Sov. Phys. JETP **66**, 1100 (1987)].

[8] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory,* Nucl. Phys. B **426**, 19 (1994) Erratum: [Nucl. Phys. B **430**, 485 (1994)] [hep-th/9407087].

[9] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD,* Nucl. Phys. B **431**, 484 (1994) [hep-th/9408099].

[10] K. Intriligator and N. Seiberg, *Aspects of 3d N=2 Chern-Simons-Matter Theories,* JHEP **1307**, 079 (2013) [arXiv:1305.1633 [hep-th]].

[11] I. Affleck, J. A. Harvey and E. Witten, *Instantons and (Super)Symmetry Breaking in (2+1)-Dimensions,* Nucl. Phys. B **206**, 413 (1982).

[12] E. Witten, *Constraints on Supersymmetry Breaking,* Nucl. Phys. B **202**, 253 (1982).

[13] J. L. Cardy, *Continuously Varying Exponents and the Value of the Central Charge,* J. Phys. A **20**, L891 (1987).

[14] R. G. Leigh and M. J. Strassler, *Exactly marginal operators and duality in*

four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory, Nucl. Phys. B **447**, 95 (1995) [hep-th/9503121].

[15] O. Aharony, B. Kol and S. Yankielowicz, *On exactly marginal deformations of $\mathcal{N} = 4$ SYM and type IIB supergravity on $AdS(5)xS^5$,* JHEP **0206**, 039 (2002) [hep-th/0205090].

[16] B. Kol, *On conformal deformations,* JHEP **0209**, 046 (2002) [hep-th/0205141].

[17] Y. Tachikawa, *Five-dimensional supergravity dual of a-maximization,* Nucl. Phys. B **733**, 188 (2006) [hep-th/0507057].

[18] B. Kol, *On Conformal Deformations II,* arXiv:1005.4408 [hep-th].

[19] D. Green, Z. Komargodski, N. Seiberg, Y. Tachikawa and B. Wecht, *Exactly Marginal Deformations and Global Symmetries,* JHEP **1006**, 106 (2010) [arXiv:1005.3546 [hep-th]].

[20] A. B. Zamolodchikov, *Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,* JETP Lett. **43**, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. **43**, 565 (1986)].

[21] D. Kutasov, *Geometry on the Space of Conformal Field Theories and Contact Terms,* Phys. Lett. B **220**, 153 (1989).

[22] V. Asnin, *On metric geometry of conformal moduli spaces of four-dimensional superconformal theories,* JHEP **1009**, 012 (2010) [arXiv:0912.2529 [hep-th]].

[23] J. Gomis, P. S. Hsin, Z. Komargodski, A. Schwimmer, N. Seiberg and S. Theisen, *Anomalies, Conformal Manifolds, and Spheres,* JHEP **1603**, 022 (2016) [arXiv:1509.08511 [hep-th]].

[24] S. El-Showk and K. Papadodimas, *Emergent Spacetime and Holographic CFTs,* JHEP **1210**, 106 (2012) [arXiv:1101.4163 [hep-th]].

[25] M. Nakahara, *Geometry, topology and physics,* Boca Raton, USA: Taylor Francis (2003) 573 p

[26] E. Witten, *The "Parity" Anomaly On An Unorientable Manifold,* Phys. Rev. B **94**, no. 19, 195150 (2016) [arXiv:1605.02391 [hep-th]].

[27] N. Seiberg and E. Witten, *Gapped Boundary Phases of Topological Insulators via Weak Coupling,* PTEP **2016**, no. 12, 12C101 (2016) [arXiv:1602.04251 [cond-mat.str-el]].

[28] S. R. Coleman and B. R. Hill, *No More Corrections to the Topological Mass Term in QED in Three-Dimensions,* Phys. Lett. **159B**, 184 (1985).

[29] E. Witten, *Quantum Field Theory and the Jones Polynomial,* Commun. Math. Phys. **121**, 351 (1989).

[30] P. S. Hsin and N. Seiberg, *Level/rank Duality and Chern-Simons-Matter Theories,* JHEP **1609**, 095 (2016) [arXiv:1607.07457 [hep-th]].

[31] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry,* Front. Phys. **58**, 1 (1983) [hep-th/0108200].

[32] J. Wess and J. Bagger, *Supersymmetry and supergravity,* Princeton, USA: Univ. Pr. (1992) 259 p

[33] B. M. Zupnik and D. G. Pak, *Topologically Massive Gauge Theories in Superspace,* Sov. Phys. J. **31**, 962 (1988).

[34] E. A. Ivanov, *Chern-Simons matter systems with manifest $\mathcal{N} = 2$ supersymmetry,* Phys. Lett. B **268**, 203 (1991).

[35] S. J. Gates, Jr. and H. Nishino, *Remarks on the $\mathcal{N} = 2$ supersymmetric Chern-Simons theories,* Phys. Lett. B **281**, 72 (1992).

[36] D. Gaiotto and X. Yin, *Notes on superconformal Chern-Simons-Matter theories,* JHEP **0708**, 056 (2007) [arXiv:0704.3740 [hep-th]].

[37] E. Witten, *Supersymmetric index of three-dimensional gauge theory,* In *Shifman, M.A. (ed.): The many faces of the superworld* 156-184 [hep-th/9903005].

[38] J. Gomis, Z. Komargodski and N. Seiberg, *Phases Of Adjoint QCD_3 And Dualities,* arXiv:1710.03258 [hep-th].

[39] G. V. Dunne, *Aspects of Chern-Simons theory,* hep-th/9902115.

[40] *Integral quadratic forms, Kac-Moody algebras, and fractional quantum Hall effect: An ADE-O classification,* J. Frohlich and E. Thiran, J. Stat. Phys. **76**, 209 (1994).

[41] A. Armoni and T. J. Hollowood, *Sitting on the domain walls of N=1 super Yang-Mills,* JHEP **0507**, 043 (2005) [hep-th/0505213].

[42] A. Armoni and T. J. Hollowood, *Interactions of domain walls of SUSY Yang-Mills as D-branes,* JHEP **0602**, 072 (2006) [hep-th/0601150].

[43] D. Jafferis and X. Yin, *A Duality Appetizer,* arXiv:1103.5700 [hep-th].

[44] A. Kapustin, H. Kim and J. Park, *Dualities for 3d Theories with Tensor Matter,* JHEP **1112**, 087 (2011) [arXiv:1110.2547 [hep-th]].

[45] V. Bashmakov, J. Gomis, Z. Komargodski and A. Sharon, *Phases of $\mathcal{N} = 1$ theories in 2 + 1 dimensions,* JHEP **1807**, 123 (2018) [arXiv:1802.10130 [hep-th]].

[46] C. Choi, M. Rocek and A. Sharon, *Dualities and Phases of 3D N=1 SQCD,* arXiv:1808.02184 [hep-th].

[47] O. Aharony, *Baryons, monopoles and dualities in Chern-Simons-matter theories,* JHEP **1602**, 093 (2016) [arXiv:1512.00161 [hep-th]].

[48] F. Benini and S. Benvenuti, $\mathcal{N}$=1 *dualities in 2+1 dimensions,* arXiv:1803.01784 [hep-th].

[49] D. Gaiotto, Z. Komargodski and J. Wu, *Curious Aspects of Three-Dimensional* $\mathcal{N} = 1$ *SCFTs,* arXiv:1804.02018 [hep-th].

[50] F. Benini and S. Benvenuti, $\mathcal{N} = 1$ *QED in 2+1 dimensions: Dualities and enhanced symmetries,* arXiv:1804.05707 [hep-th].

[51] D. Gaiotto, Z. Komargodski and N. Seiberg, *Time-reversal breaking in* QCD_4*, walls, and dualities in 2 + 1 dimensions,* JHEP **1801**, 110 (2018) [arXiv:1708.06806 [hep-th]].

[52] O. Aharony, S. S. Razamat and B. Willett, *From 3d duality to 2d duality,* JHEP **1711**, 090 (2017) [arXiv:1710.00926 [hep-th]].

[53] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory,*

[54] S. Rychkov, *EPFL Lectures on Conformal Field Theory in* $D \geq 3$ *Dimensions,* arXiv:1601.05000 [hep-th].

[55] D. Simmons-Duffin, *The Conformal Bootstrap,* arXiv:1602.07982 [hep-th].

[56] F. A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion,* Nucl. Phys. B **599**, 459 (2001) [hep-th/0011040].

[57] F. A. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion,* Nucl. Phys. B **678**, 491 (2004) [hep-th/0309180].

[58] T. Banks, M. R. Douglas, G. T. Horowitz and E. J. Martinec, *AdS dynamics from conformal field theory,* hep-th/9808016.

[59] V. Balasubramanian, S. B. Giddings and A. E. Lawrence, *What do CFTs tell us about Anti-de Sitter space-times?,* JHEP **9903**, 001 (1999) [hep-th/9902052].

[60] A. Hamilton, D. N. Kabat, G. Lifschytz and D. A. Lowe, *Local bulk operators in AdS/CFT: A Boundary view of horizons and locality,* Phys. Rev. D **73**, 086003 (2006) [hep-th/0506118].

[61] A. Hamilton, D. N. Kabat, G. Lifschytz and D. A. Lowe, *Holographic representation of local bulk operators,* Phys. Rev. D **74**, 066009 (2006) [hep-th/0606141].

[62] A. Hamilton, D. N. Kabat, G. Lifschytz and D. A. Lowe, *Local bulk operators in AdS/CFT: A Holographic description of the black hole interior,* Phys. Rev. D **75**, 106001 (2007) Erratum: [Phys. Rev. D **75**, 129902 (2007)] [hep-th/0612053].

[63] I. Heemskerk, J. Penedones, J. Polchinski and J. Sully, *Holography from Conformal Field Theory,* JHEP **0910**, 079 (2009) [arXiv:0907.0151 [hep-th]].

[64] O. Aharony, L. F. Alday, A. Bissi and E. Perlmutter, *Loops in AdS from Conformal Field Theory*, arXiv:1612.03891 [hep-th].

[65] S. Benvenuti and A. Hanany, *Conformal manifolds for the conifold and other toric field theories*, JHEP **0508** (2005) 024 [hep-th/0502043].

[66] J. L. Cardy, *Continuously Varying Exponents and the Value of the Central Charge*, J. Phys. A **20** (1987) L891.

[67] Z. Komargodski and D. Simmons-Duffin, *The Random-Bond Ising Model in 2.01 and 3 Dimensions*, J. Phys. A **50** (2017) no.15, 154001 [arXiv:1603.04444 [hep-th]].

[68] C. Behan, L. Rastelli, S. Rychkov and B. Zan, *Long-range critical exponents near the short-range crossover*, Phys. Rev. Lett. **118** (2017) no.24, 241601 [arXiv:1703.03430 [cond-mat.stat-mech]].

[69] C. Behan, L. Rastelli, S. Rychkov and B. Zan, *A scaling theory for the long-range to short-range crossover and an infrared duality*, J. Phys. A **50** (2017) no.35, 354002 [arXiv:1703.05325 [hep-th]].

[70] M. Gillioz, X. Lu and M. A. Luty, *Scale Anomalies, States, and Rates in Conformal Field Theory*, JHEP **1704** (2017) 171 [arXiv:1612.07800 [hep-th]].

[71] D. Pappadopulo, S. Rychkov, J. Espin, R. Rattazzi, *OPE Convergence in Conformal Field Theory*, Phys. Rev. D **86** (2012) 105043 [hep-th/1208.6449]

[72] S. Rychkov, P. Yvernay, *Remarks on the Convergence Properties of the Conformal Block Expansion*, Phys. Lett. **B753** (2016) 682-686 [arXiv:1510.08486v1 [hep-th]]

[73] A. Castedo Echeverri, B. von Harling and M. Serone, *The Effective Bootstrap*, JHEP **1609** (2016) 097 [arXiv:1606.02771 [hep-th]].

[74] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[75] C. Behan, *Conformal manifolds: ODEs from OPEs,* JHEP **1803**, 127 (2018) [arXiv:1709.03967 [hep-th]].

[76] S. Hollands, *Action principle for OPE,* Nucl. Phys. B **926**, 614 (2018) doi:10.1016/j.nuclphysb.2017.11.013 [arXiv:1710.05601 [hep-th]].

[77] K. Sen and Y. Tachikawa, *First-order conformal perturbation theory by marginal operators,* arXiv:1711.05947 [hep-th].

[78] L. Cornalba, M. S. Costa and J. Penedones, *Eikonal approximation in*

AdS/CFT: Resumming the gravitational loop expansion, JHEP **0709** (2007) 037 [arXiv:0707.0120 [hep-th]].

[79] J. Penedones, *Writing CFT correlation functions as AdS scattering amplitudes*, JHEP **1103** (2011) 025 [arXiv:1011.1485 [hep-th]].

[80] A. L. Fitzpatrick and J. Kaplan, *Analyticity and the Holographic S-Matrix*, JHEP **1210** (2012) 127 [arXiv:1111.6972 [hep-th]].

[81] A. L. Fitzpatrick and J. Kaplan, *Unitarity and the Holographic S-Matrix*, JHEP **1210** (2012) 032 [arXiv:1112.4845 [hep-th]].

[82] S. Giombi, C. Sleight and M. Taronna, *Spinning AdS Loop Diagrams: Two Point Functions*, arXiv:1708.08404 [hep-th].

[83] C. Cardona, *Mellin-(Schwinger) representation of One-loop Witten diagrams in AdS*, arXiv:1708.06339 [hep-th].

[84] S. Teber, *Electromagnetic current correlations in reduced quantum electrodynamics*, Phys. Rev. D **86** (2012) 025005 [arXiv:1204.5664 [hep-ph]].

[85] C. P. Herzog and K. W. Huang, *Boundary Conformal Field Theory and a Boundary Central Charge*, arXiv:1707.06224 [hep-th].

[86] A. Dymarsky, I. R. Klebanov and R. Roiban, *Perturbative search for fixed lines in large N gauge theories*, JHEP **0508** (2005) 011 [hep-th/0505099].

[87] A. Dymarsky, I. R. Klebanov and R. Roiban, *Perturbative gauge theory and closed string tachyons* JHEP **0511** (2005) 038 [hep-th/0509132].

[88] E. Pomoni and L. Rastelli, *Large N Field Theory and AdS Tachyons*, JHEP **0904** (2009) 020 [arXiv:0805.2261 [hep-th]].

[89] A. Adams and E. Silverstein, *Closed string tachyons, AdS / CFT, and large N QCD*, Phys. Rev. D **64** (2001) 086001 [hep-th/0103220].

[90] H. Ooguri and C. Vafa, *Non-supersymmetric AdS and the Swampland* arXiv:1610.01533 [hep-th].

[91] C. Angelantonj and A. Armoni, *Nontachyonic type 0B orientifolds, nonsupersymmetric gauge theories and cosmological RG flow*, Nucl. Phys. B **578** (2000) 239 [hep-th/9912257].

[92] P. Liendo, *Orientifold daughter of N=4 SYM and double-trace running*, Phys. Rev. D **86** (2012) 105032 [arXiv:1107.3125 [hep-th]].

[93] Ö. Gürdogan and V. Kazakov, *New Integrable 4D Quantum Field Theories from Strongly Deformed Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory*, Phys. Rev. Lett. **117** (2016) no.20, 201602 Addendum: [Phys. Rev. Lett. **117** (2016) no.25, 259903] [arXiv:1512.06704 [hep-th]].

[94] C. Sieg and M. Wilhelm, *On a CFT limit of planar γ_i-deformed $\mathcal{N}=4$ SYM theory*, Phys. Lett. B **756** (2016) 118 [arXiv:1602.05817 [hep-th]].

[95] O. Mamroud and G. Torrents, *RG stability of integrable fishnet models*, JHEP **1706** (2017) 012 [arXiv:1703.04152 [hep-th]].

[96] G. Arutyunov, S. Frolov and A. Petkou, *Perturbative and instanton corrections to the OPE of CPOs in $\mathcal{N}=4$ SYM$_4$* Nucl. Phys. B **602** (2001) 238 Erratum: [Nucl. Phys. B **609** (2001) 540] [hep-th/0010137].

[97] M. Bianchi, S. Kovacs, G. Rossi, and Y. S. Stanev, *Properties of the Konishi Multiplet in $\mathcal{N}=4$ SYM theory* JHEP **0105** (2001) 042 [hep-th/0104016]

[98] K. A. Intriligator, W. Skiba, *Bonus Symmetry and The Operator Product Expansion of $\mathcal{N}=4$ SYM* Nucl. Phys. **B559** (1999) 165-183 [hep-th/9905020]

[99] V. Bashmakov, M. Bertolini and H. Raj, *Broken current anomalous dimensions, conformal manifolds, and renormalization group flows*, Phys. Rev. D **95** (2017) no.6, 066011 [arXiv:1609.09820 [hep-th]].

Acknowledgements

I wish to express my gratitude to Matteo Bertolini for the guidance and warm support. His unaltered enthusiasm for conducting research and aspiration in making all the steps and ideas as clear and precise as possible will always serve as an example for me.

I would also like to thank Zohar Komargodski, who introduced me to the field of three-dimensional quantum field theories. Collaboration with him was one of the brightest events during my PhD.

The fact that the research process has been so enjoyable is due to my collaborators Himanshu Raj, Lorenzo Di Pietro, Adar Sharon, Francesco Benini and Sergio Benvenuti.

My friends Alessandro Michelangeli, André Benevides, Antonio Shiarappa, Arsenii Titov, Gabriele Spada, Gleb Smirnov, João Penedo, Lorenzo Bordin, Matteo Poggi, Meng-Qi Zhu, Ok Song An and Reyimuaji Yakefu made my stay in Trieste so special and rare days free of Physics so memorable.

I am infinitely indebted to my parents Fedor and Marina, to my grandparents, Victor, Sofia, Victor and Anna, and to my brother Alexei. Their constant support and encouragement throughout my life were crucial for my formation, and their merit presents in everything I am doing.

The last but not the least, I wish to thank Tanya, who has always been nearby, for all her love and patience. Without her this thesis would not be possible.