TORUS EQUIVARIANT K-STABILITY

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Abstract. It is conjectured that to test the K-polystability of a polarised variety it is enough to consider test-configurations which are equivariant with respect to a torus in the automorphism group. We prove partial results towards this conjecture. We also show that it would give a new proof of the K-polystability of constant scalar curvature polarised manifolds.

1. Introduction

The Yau-Tian-Donaldson conjecture for Fano manifolds [26, 23, 7] predicts that a smooth Fano M admits a Kähler-Einstein metric if and only if it is K-polystable, a purely algebro-geometric condition expressed through the positivity of a certain limit of GIT weights (the Donaldson-Futaki weight or invariant). There are by now several proofs, in different degrees of generality (i.e. allowing M to have mild singularities, a boundary in the MMP sense, and/or slightly modifying the notion of K-stability), using different methods.

For an arbitrary polarised manifold (X, L) the most natural generalisation of a Kähler-Einstein metric is a constant scalar curvature Kähler (cscK) metric representing the first Chern class of L. If such a metric exists, (X, L) is called a cscK manifold.

A Kähler-Einstein metric, or more generally a cscK metric, if it exists, can always be taken invariant under the action of a compact group of automorphisms of M. From the GIT point of view, when the point whose stability we would like to investigate has a non-trivial reductive stabiliser H, the Hilbert-Mumford Criterion can be strengthened: it is enough to consider one-parameter subgroups which commute with H [10]. These facts suggest the following folklore conjecture (all the notions required in the rest of this introduction will be recalled in Section 3.)

Conjecture 1. *Let (X, L) be a polarised variety and let G be a reductive subgroup of $\mathrm{Aut}(X, L)$. Then (X, L) is K-polystable if and only if for every G-equivariant test-configuration the Donaldson-Futaki invariant is greater than or equal to zero, with equality if and only if the normalisation of the test-configuration is a product.*

An analytic proof in the case of Fano manifolds is given in [6], relying on an alternative approach to the Yau-Tian-Donaldson conjecture. An algebro-geometric proof in the Fano case and when G is a torus is given in [12].

Recall that a cscK manifold has reductive automorphism group, so K-polystable varieties are expected to have a reductive automorphism group as well; this problem is studied in [5]. Because of this it is natural to formulate Conjecture 1 just for reductive subgroups of $\mathrm{Aut}(X, L)$.

There is a general expectation that for the existence of a cscK metric one actually needs some enhancement of the original notion of K-stability. Quite a few different notions have been proposed. In this paper we focus on the generalisation of K-stability based on (possibly non-finitely generated) filtrations of the coordinate ring of (X, L) (see Definition 24). This notion has been proposed by G. Székelyhidi in [21], building on the work of D. Witt Nyström [25]; in [22], it is called $\hat{K}$-stability. In [21], it is shown that, given a cscK manifold (X, L), if the connected component of the identity of $\mathrm{Aut}(X, L)$ is equal to $\mathbb{C}^*$, then (X, L) is $\hat{K}$-stable. Importantly for us [21] also discusses a variant of $\hat{K}$-stability which replaces the Donaldson-Futaki invariant of a filtration with the asymptotic Chow weight Chow_∞, and proves that the $\hat{K}$-stability result remains true for this variant (the two notions coincide when dealing with classical test-configurations, corresponding to finitely generated filtrations).

Our main result is a step towards a proof of Conjecture 1 in the general case, or possibly of a variant of Conjecture 1 in the $\hat{K}$-stability setup.

Theorem 2. *Let (X, L) be a polarised variety. Fix a complex torus $T \subset \mathrm{Aut}(X, L)$ and let $(\mathcal{X}, \mathcal{L})$ be a test-configuration with Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$. Then we can associate to $(\mathcal{X}, \mathcal{L})$ a T-equivariant filtration χ of the coordinate ring of (X, L) whose asymptotic Chow weight satisfies $\mathrm{Chow}_\infty(\chi) \leq \mathrm{DF}(\mathcal{X}, \mathcal{L})$. If moreover χ is finitely generated, then it corresponds to a T-equivariant test-configuration which is a flat one-parameter limit of $(\mathcal{X}, \mathcal{L})$, and in particular has the same Donaldson-Futaki invariant and L^2 norm.*

Theorem 2 follows at once from Lemma 29, Lemma 30 and Theorem 31, proved in Section 4. Theorem 31 shows that given a generalised test-configuration in the sense of [21], corresponding to a possibly non-finitely generated filtration χ, we can specialise it to a T-invariant filtration $\bar{\chi}$ with $\mathrm{Chow}_\infty(\bar{\chi}) \leq \mathrm{Chow}_\infty(\chi)$. In the Appendix we show that non-finitely generated filtrations can actually arise in Theorem 2.

In Section 5 we show that Conjecture 1 combined with ideas from [17, 19] naturally leads to a proof that cscK manifolds are K-polystable. K-polystability of cscK manifolds is proved in [2] using completely different methods.

Notation. In this paper a polarised variety (X, L) is a complex projective variety X endowed with a very ample and projectively normal line bundle L. For the purposes of this paper one may always replace L with a positive tensor power, so these assumptions are not restrictive.

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2. Some results on filtrations in finite dimensional GIT

In this section we discuss some preliminary notions in a finite dimensional GIT context.

Let V be a finite dimensional complex vector space. Pick an increasing filtration $F = \{F_i V\}_{i\in\mathbb{Z}}$ of V by complex subspaces (with index set $\mathbb{Z}$) and a $\mathbb{C}^*$-action λ on V.

Definition 3. *The specialisation $\bar{F}$ of F via λ is the filtration given by*

$$\bar{F}_i V = \lim_{\tau\to 0} \lambda(\tau)\cdot F_i V,$$

where the limit is taken in the appropriate Grassmannian.

Equivalently $\bar{F}_i V$ is the subspace spanned by the vectors $\bar{v}$ as v varies in $F_i V$, where $\bar{v}$ denotes the lowest weight term with respect to the action of λ. The filtration $\bar{F}$ is λ-equivariant by construction, that is each $\bar{F}_i V$ is preserved by λ.

Let G be a reductive group acting on V, and assume that the kernel of the action is a finite group.

Definition 4. *Let γ be a one-parameter subgroup of G acting on V as above. The weight filtration of γ is the increasing filtration $F = \{F_i V\}_{i\in\mathbb{Z}}$ given by*

$$F_i V = \bigoplus_{j\geq -i} V_j$$

where V_j is the weight j eigenspace for the action of γ.

Let $\mathcal{P}(\gamma)$ be the parabolic subgroup of G associated to the one-parameter subgroup γ. By definition this is the subgroup preserving the flag F.

Suppose that λ is an additional one-parameter subgroup of G. We wish to characterise the specialisation of the weight filtration F of γ via the action of λ. For this we recall that the intersection of parabolic subgroups $\mathcal{P}(\lambda)\cap\mathcal{P}(\gamma)$ contains a maximal torus $\mathcal{T}$ of G (see e.g. [4] Proposition 4.7). Moreover all maximal tori in a parabolic subgroup are conjugated by elements of the parabolic, hence there exists a one-parameter subgroup χ of $\mathcal{T}$ such that

χ is conjugate to γ via an element in $\mathcal{P}(\gamma)$, so that the weight filtration associated to χ is still F. Let

$$\bar{\gamma}(t) = \lim_{\tau \to 0} \lambda(\tau)\chi(t)\lambda(\tau)^{-1}.$$

This limit exists because χ lies in the parabolic $\mathcal{P}(\lambda)$, see [13] section 2.2.

Lemma 5. *Suppose that F is the weight filtration of γ. The specialisation $\bar{F}$ of F via λ coincides with the weight filtration of $\bar{\gamma}$. It follows in particular that $\bar{F}$ is induced by a one-parameter subgroup of G.*

Note that the filtration $\bar{F}$ is uniquely defined, but $\bar{\gamma}$ is not (for example, it depends on the choice of T).

Proof. The key remark is that the weight j eigenspace of $\lambda(\tau)\chi(t)(\lambda(\tau))^{-1}$ is $\lambda(\tau) \cdot V_j$. Now for every $v \in V$ we have

$$\bar{\gamma}(t)(v) = \lim_{\tau \to 0} \lambda(\tau)\chi(t)(\lambda(\tau))^{-1}(v)$$

so v is a weight j eigenvector for $\bar{\gamma}$ if and only if v belongs to

$$\lim_{\tau \to 0} \lambda(\tau) \cdot V_j$$

where the limit is taken in the appropriate Grassmannian. □

Definition 6. *The Hilbert-Mumford weight of a vector $v \in V$ with respect to the one-parameter subgroup γ is*

$$\mathrm{HM}(v, \gamma) = \min_i \{v \in F_i V\}$$

where F is the weight filtration of γ.

This depends only on the weight filtration of γ and we will also denote it by $\mathrm{HM}(v, F)$ rather than $\mathrm{HM}(v, \gamma)$ if we wish to emphasise this fact. But notice that a general filtration of V will not come from a one-parameter subgroup of the fixed reductive group G.

Remark 7. With our sign convention $\mathrm{HM}(v, \gamma)$ is the weight of the induced action of γ on the fibre $\mathcal{O}_{\mathbb{P}(V)}(1)_{[v]_0}$ of the hyperplane line bundle on $\mathbb{P}(V)$ over $[v]_0 = \lim_{\tau \to 0} \lambda(\tau) \cdot [v]$. Thus for example the Hilbert-Mumford Criterion says that $[v]$ is GIT semistable if and only if $\mathrm{HM}(v, \gamma) \geq 0$ for all one-parameter subgroups γ.

Proposition 8. *Let λ be a one-parameter subgroup of the stabiliser of $[v] \in \mathbb{P}(V)$. The we have*

$$\mathrm{HM}(v, \bar{F}) \leq \mathrm{HM}(v, \gamma)$$

where $\bar{F}$ is the specialisation via λ of the weight filtration F of γ.

Recall that by Lemma 5 the filtration $\bar{F}$ is the weight filtration of a one-parameter subgroup of G.

Proof. We only need to show that $v \in F_i V$ implies $v \in \bar{F}_i V$. This follows from the fact that v is an eigenvector of λ, so it is equal to its lowest weight term $\bar{v}$ with respect to the action of λ. □

It is easy to produce examples where the inequality of Proposition 8 is strict.

Example 9. We choose $G = SL(2,\mathbb{C})$ with its standard action on $V = \mathbb{C}^2$, and

$$v = e_2,\, \gamma(t) = \begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix},\, \lambda(\tau) = \begin{pmatrix} \tau^h & 0 \\ \tau^h - \tau^{-h} & \tau^{-h} \end{pmatrix}$$

for fixed $h, k > 0$. Note that λ stabilises $[v] \in \mathbb{P}(V)$. One checks that γ is not contained in the parabolic $\mathcal{P}(\lambda)$. But conjugating γ with a suitable element in $\mathcal{P}(\gamma)$ gives

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^k & t^{-k} - t^k \\ 0 & t^{-k} \end{pmatrix} = \chi \in \mathcal{P}(\gamma) \cap \mathcal{P}(\lambda).$$

A straightforward computation gives

$$\lim_{\tau \to 0} \lambda(\tau)\chi(\lambda(\tau))^{-1} = \begin{pmatrix} t^{-k} & 0 \\ t^{-k} - t^k & t^k \end{pmatrix} = \bar{\gamma},$$

so we have

$$\mathrm{HM}(v, \bar{\gamma}) = -k < \mathrm{HM}(v, \gamma) = k.$$

It is important to realise that even if γ does not stabilise $[v] \in \mathbb{P}(V)$ its specialisation $\bar{\gamma}$ with respect to a λ in the stabiliser could well lie in the stabiliser (so abusing the K-stability terminology which will be recalled in the next section, in the present finite-dimensional setup and without imposing further restrictions, we can end up with a "product test-configuration").

Example 10. Let V, γ, λ be as in the previous example. We choose $v = e_1 + e_2$. Then $[v] \in \mathbb{P}V$ is stabilised by λ and by $\bar{\gamma}$, but not by γ. Note that in this case we have $\mathrm{HM}(v, \gamma) = \mathrm{HM}(v, \bar{\gamma}) = k$.

Let F, F' be filtrations of V with index set $\mathbb{Z}$. We say that F is included in F' if $F_i V \subset F'_i V$ holds for all i. The following observation follows immediately from the definition of the Hilbert-Mumford weight and will be useful in later applications.

Lemma 11. *Let F, F' be the weight filtrations of some one-parameter subgroups. If F is included in F' then we have*

$$\mathrm{HM}(v, F') \le \mathrm{HM}(v, F)$$

for all v in V.

3. Filtrations, test-configurations, approximations

Let (X, L) be a polarised variety. One of the main objects of study in this paper are test-configurations of (X, L). Let us briefly recall their definition.

Definition 12. *Let $\mathbb{C}^*$ act in the standard way on $\mathbb{C}$. A test-configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) with exponent r is a $\mathbb{C}^*$-equivariant flat morphism $\pi\colon \mathcal{X} \to \mathbb{C}$, together with a π-ample line bundle $\mathcal{L}$ and a linearisation of the action of $\mathbb{C}^*$ on $\mathcal{L}$, such that the fibre over 1 is isomorphic to $(X, L^{\otimes r})$. We say that $(\mathcal{X}, \mathcal{L})$ is*

- very ample*, if $\mathcal{L}$ is π-very ample;*
- *a* product*, if it is isomorphic to $(X \times \mathbb{C}, L^{\otimes r} \boxtimes \mathcal{O}_{\mathbb{C}})$, where the action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ is induced by a one-parameter subgroup λ of $\mathrm{Aut}(X, L)$ by $\lambda(\tau) \cdot (x, t) = (\lambda(\tau) \cdot x, \tau t)$;*
- trivial*, if it is a product and, moreover, λ is trivial;*
- normal*, if the total space $\mathcal{X}$ is normal;*
- equivariant with respect to a subgroup $H \subset \mathrm{Aut}(X, L)$*, if the action of $\mathbb{C}^*$ can be extended to an action of $\mathbb{C}^* \times H$ such that the action of $\{1\} \times H$ is the natural action of H on $(X, L^{\otimes r})$;*
- *in the Fano case, a test-configuration is a* special degeneration *if $\mathcal{X}$ is normal, all the fibres are klt and a positive rational multiple of $\mathcal{L}$ equals $-K_{\mathcal{X}}$ (this notion is due to Tian* [23]*, see also* [11] *Definition 1).*

The normalisation of a test-configuration is the normalisation of $\mathcal{X}$ endowed with the natural induced line bundle and $\mathbb{C}^*$ action (or $\mathbb{C}^* \times H$ action). A test-configuration is a product if and only if the central fibre $\mathcal{X}_0$ is isomorphic to X: by standard theory in this case there is a trivialisation $\mathcal{X} \cong X \times \mathbb{C}$ and the $\mathbb{C}^*$-action on $\mathcal{X}$ corresponds to a $\mathbb{C}^*$-action on $X \times \mathbb{C}$ preserving $X \times \{0\}$, which must then be induced by a $\mathbb{C}^*$-action on X as above.

The following result summarises observations of Ross-Thomas [16] and Odaka [14].

Proposition 13. *For all sufficiently large r there is a bijective correspondence between increasing filtrations of $H^0(X, L^{\otimes r})^\vee$ (with index set $\mathbb{Z}$) and very ample test-configurations of exponent r. Such a test-configuration is a product if and only if the corresponding filtration is the weight filtration of a one-parameter subgroup of $\mathrm{Aut}(X, L)$, and it is equivariant with respect to a reductive subgroup $H \subset \mathrm{Aut}(X, L)$ if and only if the corresponding filtration is preserved by H.*

Proof. An arbitrary increasing filtration of $H^0(X, L^{\otimes r})^\vee$ is induced by the weight filtration of a one-parameter subgroup of $GL(H^0(X, L^{\otimes r})^\vee)$, so we can associate to a filtration the (very ample) test-configuration induced by this one-parameter subgroup. If two one-parameter subgroups induce the same filtration then the corresponding test-configurations are isomorphic, see [14] Theorem 2.3 and its proof. Conversely, by [16, Proposition 3.7], for all sufficiently large r a very ample test-configuration of exponent r is always induced by a one-parameter subgroup of $GL(H^0(X, L^{\otimes r})^\vee)$, and this gives the filtration. The other claims are straightforward. □

One can act on a test-configuration $(\mathcal{X}, \mathcal{L})$ in two basic ways (see e.g. [8] section 2). Firstly we can pull-back $(\mathcal{X}, \mathcal{L})$ via a base-change $t \mapsto t^p$. The effect on the corresponding filtration is to multiply all the indices of the filtration by p. Equivalently the weights of the corresponding one-parameter subgroup are multiplied by p. Secondly we can rescale the linearisation of the action on $\mathcal{L}$ by a constant factor. The effect on the corresponding filtration is to shift all indices by some integer k. Equivalently we are composing the corresponding one-parameter subgroup with a one-parameter subgroup in the the center of $GL(H^0(X, L^{\otimes r})^\vee)$, which corresponds in turn to adding k to all the weights.

Combining the two operations above we can modify the weights to get a filtration with only positive indices, or alternatively to get a filtration induced by a one-parameter subgroup of $SL(H^0(X, L^{\otimes r})^\vee)$.

There is a more global correspondence between filtrations and test-configurations, which avoids fixing the exponent. We introduce the homogeneous coordinate ring

$$R = R(X, L) = \bigoplus_{k \geq 0} R_k = \bigoplus_{k \geq 0} H^0(X, L^{\otimes k}).$$

We focus on filtrations of R of a special type.

Definition 14. *We define a* filtration χ *of* R *to be sequence of vector subspaces*

$$H^0(X, \mathcal{O}) = F_0 R \subset F_1 R \subset \cdots$$

which is

(i) exhaustive*: for every* k *there exists a* $j = j(k)$ *such that* $F_j R_k = H^0(X, L^{\otimes k})$*,*
(ii) multiplicative*:* $(F_i R_l)(F_j R_m) \subset F_{i+j} R_{l+m}$*,*
(iii) homogeneous*: if* f *is in* $F_i R$ *then each homogeneous piece of* f *also lies in* $F_i R$*.*

We denote by χ_k *the filtration of* $H^0(X, L^{\otimes k})$ *induced by* χ*.*

Note that when considering filtrations of R we restrict to those which only have non-negative indices; let us also notice that describing χ is equivalent to describe χ_k for every k. There are two basic algebraic objects attached to a filtration as above.

Definition 15. *Let* χ *be a filtration. The corresponding* Rees algebra *is*

$$\mathrm{Rees}(\chi) = \bigoplus_{i \geq 0} F_i R\, t^i$$

The graded modules are

$$\mathrm{gr}_i(H^0(X, L^{\otimes k})) = F_i(H^0(X, L^{\otimes k}))/F_{i-1}(H^0(X, L^{\otimes k}))$$

The graded algebra is

$$\mathrm{gr}(\chi) = \bigoplus_{k,i \geq 0} \mathrm{gr}_i(H^0(X, L^k))$$

The Rees algebra is a subalgebra of $R[t]$, and by the following elementary result, whose proof relies on the projective normality of L, it is possible to reconstruct χ from it.

Lemma 16. *Let A be a $\mathbb{C}$-subalgebra of $R[t]$. We define a filtration χ_A of R as follows*

$$F_i R = \{s \in R \,|\, t^i s \in A\}$$

The filtration χ_A satisfies the conditions of Definition 14 if and only if A satisfies the conditions

- *$A \cap R = H^0(X, \mathcal{O}_X)$;*
- *for every $s \in H^0(X, L)$ there exists an i such that $t^i s \in A$;*
- *if $t^i f$ is in A, then, for each of the homogenous component f_k of f, $t^i f_k$ is also in A.*

A filtration χ equals χ_A, where A is the Rees algebra of χ. There is an inclusion of filtrations $\chi_1 \subset \chi_2$ (i.e. an inclusion of filtered pieces) if and only if there is a corresponding inclusion of the Rees algebras $\mathrm{Rees}(\chi_1) \subset \mathrm{Rees}(\chi_2)$.

The following notion is crucial for us.

Definition 17. *A filtration is called finitely generated if its Rees algebra is finitely generated.*

Let us review the relation between finitely generated filtrations and test-configurations, as developed by Witt Nyström [25] and Székelyhidi [21] (see [3, Proposition 2.15] for a precise statement).

Let χ be a finitely generated filtration. The Rees algebra $\mathrm{Rees}(\chi)$ is a finitely generated flat $\mathbb{C}[t]$-module; this means that the associated relative Proj with its natural $\mathcal{O}(1)$ is a test-configuration $(\mathcal{X}, \mathcal{L})$. The central fibre is the Proj of the graded algebra $\mathrm{gr}(\chi)$; the $\mathbb{C}^*$-action on the central fibre is given by *minus* the i-grading of $\mathrm{gr}(\chi)$.

On the other hand let $(\mathcal{X}, \mathcal{L})$ be an exponent r test-configuration. Consider the filtration F of $H^0(X, L^{\otimes r})$ associated to it by Proposition 13. Up to base-change and scaling of the linearisation we can assume that all the weights are positive. Denote by N the length of this filtration. Let A be the $\mathbb{C}$-subalgebra of $R[t]$ generated by

$$H^0(X, L)t^N \oplus \bigoplus_{i=1}^{N} F_i H^0(X, L^{\otimes r}) t^i$$

Then the filtration associated to A via Lemma 16 is the filtration of R induced by $(\mathcal{X}, \mathcal{L})$ (the second assumption in Lemma 16 holds because L is projectively normal, i.e. R is generated in degree 1).

Suppose that χ is a not necessarily finitely generated filtration. Following [21] Section 3.2 we can define finitely generated approximations $\chi^{(r)}$ as follows. Let F be the filtration induced by χ on $H^0(X, L^{\otimes r})$, this corresponds to an exponent r test-configuration $(\mathcal{X}, \mathcal{L})$, then $\chi^{(r)}$ is the finitely generated filtration corresponding to $(\mathcal{X}, \mathcal{L})$. Note that this construction also makes sense when χ is finitely generated and corresponds to $(\mathcal{X}, \mathcal{L})$, in which case $\chi^{(r)}$ corresponds to $(\mathcal{X}, \mathcal{L}^{\otimes r})$.

Definition 18. *We introduce two "weight functions" attached to χ, given by*

$$w_\chi(k) = w(k) = \sum_i (-i) \dim \mathrm{gr}_i(H^0(X, L^{\otimes k})),$$

respectively

$$d_\chi(k) = d(k) = \sum_i i^2 \dim \mathrm{gr}_i(H^0(X, L^{\otimes k})).$$

If χ is a finitely generated filtration (corresponding to a test-configuration $(\mathcal{X}, \mathcal{L})$) then by equivariant Riemann-Roch we have, for all sufficiently large k,

$$\begin{aligned} h(k) &= h^0(X, L^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + \cdots \\ w(k) &= b_0 k^{n+1} + b_1 k^n + \cdots \\ d(k) &= c_0 k^{n+2} + c_1 k^{n+1} + \cdots \end{aligned}$$

Definition 19. *Let χ be a finitely generated filtration (which thus corresponds to a test-configuration). One defines the r-th Chow weight, Donaldson-Futaki weight (or invariant) and the L^2 norm as*

$$\mathrm{Chow}_r(\chi) = \mathrm{Chow}_r(\mathcal{X}, \mathcal{L}) = r\frac{b_0}{a_0} - \frac{w(r)}{d(r)},$$

$$\mathrm{DF}(\chi) = \mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{a_1 b_0 - a_0 b_1}{a_0^2},$$

$$||\chi||_{L^2}^2 = ||(\mathcal{X}, \mathcal{L})||_{L^2} = c_0 - \frac{b_0^2}{a_0}.$$

Note that a straightforward computation shows that we have

$$\lim_{r\to\infty} \mathrm{Chow}_r(\mathcal{X}, \mathcal{L}^{\otimes r}) = \mathrm{DF}(\mathcal{X}, \mathcal{L}).$$

Definition 20. *A polarised variety (X, L) is K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every test-configuration $(\mathcal{X}, \mathcal{L})$.*

Given a subgroup H of $\mathrm{Aut}(X, L)$, we say that (X, L) is H-equivariantly K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every H-equivariant test-configuration $(\mathcal{X}, \mathcal{L})$.

Definition 21. *A normal polarised variety (X, L) is K-polystable if for every test-configuration $(\mathcal{X}, \mathcal{L})$ with normal total space we have* $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$*, with equality if and only if $(\mathcal{X}, \mathcal{L})$ is a product.*

Given a subgroup H of $\mathrm{Aut}(X, L)$*, (X, L) is H-equivariantly K-polystable if for every H-equivariant test-configuration $(\mathcal{X}, \mathcal{L})$ with normal total space we have* $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$*, with equality if and only if $(\mathcal{X}, \mathcal{L})$ is a product.*

Following [21] (Definition 3 and Equation (33)) we also define the following two invariants of a non-finitely generated filtration.

Definition 22. *The Donaldson-Futaki and asymptotic Chow weights of a filtration χ are given by*

$$\mathrm{DF}(\chi) = \liminf_{r\to\infty} \mathrm{DF}(\chi^{(r)}) \,,$$

respectively

$$\mathrm{Chow}_{\infty}(\chi) = \liminf_{r\to\infty} \mathrm{Chow}_r(\chi^{(r)}) \,.$$

Note that $\chi^{(r)}$ is an exponent r test configuration, so it is natural to consider its r-th Chow weight. Let us also emphasise that, when χ is finitely generated, both these invariants coincide with the classical Donaldson-Futaki weight, see [21, Section 3.2]. In general these two invariants differ, see [21, Example 4]; we do not know if there is an inequality relating them.

Definition 23. *The L^2 norm of a filtration χ is given by*

$$||\chi||_2 = \liminf_{r\to\infty} ||\chi^{(r)}|| \,.$$

In [21, Lemma 8] it is shown that the above liminf is actually a limit.

Definition 24. *A polarised variety is $\hat{K}$-semistable if for any filtration χ of $R(X, L)$ we have*

$$\mathrm{DF}(\chi) \geq 0.$$

It is $\hat{K}$-stable if the equality holds if and only if $||\chi||_2 = 0$. One can make parallel definitions replacing $\mathrm{DF}(\chi)$ *with the asymptotic Chow weight* $\mathrm{Chow}_{\infty}(\chi)$*.*

One easily checks that $\hat{K}$-semistability is equivalent to K-semistability. On the other hand $\hat{K}$-stability is (at least a priori) stronger than K-stability, and just as K-stability it implies that the automorphism group of (X, L) has no nontrivial one-parameter subgroups.

Székelyhidi [21] (Theorem 10 and Proposition 11) proves that if (X, L) is cscK with trivial automorphisms then it is $\hat{K}$-stable, including the variant notion using the Chow_{∞} weight.

At present we do not know a good candidate for the notion of $\hat{K}$-polystability (i.e. allowing $\mathrm{Aut}(X, L)/\mathbb{C}^*$ to be non-finite, where by $\mathbb{C}^*$ we mean the central one parameter subgroup which acts as the identity on X and scales L).

4. Specialisation of a test-configuration

In the classical situation of a torus T acting on a projective variety one can specialise a point p to a fixed point $\bar{p}$ for the action of T: one picks a generic one-parameter subgroup λ of T and the specialisation is $\bar{p} = \lim_{\tau\to 0} \lambda(t)\cdot p$. This specialisation does depend on λ and when we need to emphasise this dependence we will denote it by $\bar{p}_\lambda$. In this section we first generalise this construction to test-configurations, and then prove some basic facts which imply our main result Theorem 2.

Definition 25. *Let $(\mathcal{X},\mathcal{L})$ be an exponent r test-configuration and F be the corresponding filtration of $H^0(X,L^{\otimes r})^\vee$ given by Proposition 13. Let T be a torus in* $\mathrm{Aut}(X,L)$*, and $\bar{F}$ the specialisation of F via a generic one-parameter subgroup λ of T. Then the specialisation $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ of $(\mathcal{X},\mathcal{L})$ is the T-equivariant exponent r test-configuration corresponding to $\bar{F}$.*

The specialisation depends on the choice of r and λ, but we will mostly suppress this in the notation.

We make a brief digression in order to discuss Definition 25. Recall that by Proposition 13 an exponent r test-configuration for (X,L) is obtained by embedding $\iota\colon X\hookrightarrow \mathbb{P}H^0(X,L^{\otimes r})^\vee$ with the complete linear system $|rL|$ and by taking the flat closure of $\iota(X)$ under the action of a one-parameter subgroup γ of $GL(H^0(X,L^{\otimes r})^\vee)$. The corresponding test-configuration $(\mathcal{X},\mathcal{L})$ is a closed subscheme of $\mathbb{P}H^0(X,L^{\otimes r})^\vee\times\mathbb{C}$ (in fact it can be canonically completed to a closed subscheme of $\mathbb{P}H^0(X,L^{\otimes r})^\vee\times\mathbb{P}^1$ by gluing with the trivial family at infinity). If λ is a one-parameter subgroup of $\mathrm{Aut}(X,L)$ one could attempt to define the λ-specialisation of $(\mathcal{X},\mathcal{L})$ by taking its flat closure as a closed subscheme of $\mathbb{P}H^0(X,L^{\otimes r})^\vee\times\mathbb{C}$ under the action of λ. We give a simple example showing that such a flat closure is not preserved by γ in general, so it is not a λ-equivariant test-configuration in a natural way. In fact we also show that in general the total space of the flat closure cannot support a test-configuration, and compute the corresponding specialisation $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ in the sense of Definition 25 in the example.

Example 26. Embed $\iota\colon\mathbb{P}^1\hookrightarrow\mathbb{P}^2$ via Veronese $[s_0:s_1]\mapsto[s_0^2:s_0s_1:s_1^2]$ and act with the one-parameter subgroup γ of $SL(3,\mathbb{C})$ given by $\mathrm{diag}(t^{-1},t^2,t^{-1})$. This gives a test-configuration $(\mathcal{X},\mathcal{L})$ of exponent 2 for $(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))$ with total space $\mathcal{X}\subset\mathbb{P}^2\times\mathbb{C}$ which is the variety $V(xz-t^6y^2)$. Now choose

$$\lambda=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}\tau^h&0\\0&\tau^{-h}\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}\in SL(2,\mathbb{C})=\mathrm{Aut}(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1)).$$

The induced one-parameter subgroup in $SL(3,\mathbb{C})$, which we still denote by λ, is given by

$$\lambda=\begin{pmatrix}\tau^{2h}&1-\tau^{2h}&(\tau^{-h}-\tau^h)^2\\0&1&-2(1-\tau^{-2h})\\0&0&\tau^{-2h}\end{pmatrix}.$$

One computes

$$\lambda(\tau)\cdot\mathcal{X} = V(\tau^{2h}x((\tau^{-h}-\tau^{h})^2x-2(1-\tau^{-2h})y+\tau^{-2h}z)-t^6((1-\tau^{2h})x+y)^2).$$

Since $\lambda(\tau)\cdot\mathcal{X}\subset\mathbb{P}^2\times\mathbb{C}$ is a familiy of divisors it is straightforward to take the flat limit at $\tau\to 0$. For $h>0$ one finds

$$\lim_{\tau\to 0}\lambda(\tau)\cdot\mathcal{X} = V(x(x+2y+z)-t^6(x+y)^2) =: \bar{\mathcal{X}}. \tag{4.1}$$

The central fibre $V(x(x+2y+z))$ is not preserved by γ, so the flat limit $\bar{\mathcal{X}}$ is not the total space of a test-configuration in a natural way. In this specific case, we can still find a non-canonical $\mathbb{C}^*$-action on $\bar{\mathcal{X}}$ which turns it into a λ-equivariant test-configuration. On the other hand, for $h<0$, we find that the flat limit $\bar{\mathcal{X}}$ is given by the divisor

$$\lim_{\tau\to 0}\lambda(\tau)\cdot\mathcal{X} = V(x^2(t^6-1)).$$

This may be thought of as the product, thickened test-configuration $V(x^2)$ glued to six copies of $\mathbb{P}^2$, and clearly it cannot be the total space of a test-configuration for $\mathbb{P}^1$.

We can also consider the specialisation $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ of $(\mathcal{X},\mathcal{L})$ in the sense of Definition 25. The conjugate one-parameter subgroup $\lambda(\tau)\gamma(t)(\lambda(\tau))^{-1}$ is given by

$$\begin{pmatrix} t^{-1} & -t^{-1}(-1+\tau^{2h})(-1+t^3) & -2t^{-1}(-1+\tau^{2h})^2(-1+t^3) \\ 0 & t^2 & 2t^{-1}(-1+\tau^{2h})(-1+t^3) \\ 0 & 0 & t^{-1}\end{pmatrix},$$

so γ lies in the parabolic $\mathcal{P}(\lambda)$ if and only if $h>0$. In this case $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ is obtained by acting on $V(xz-y^2)$ with $\bar{\gamma}=\lim_{\tau\to 0}\lambda(\tau)\gamma(t)(\lambda(\tau))^{-1}$. The resulting test-configuration is precisely (4.1). The central fibre $\bar{\mathcal{X}}_0 = V(x(2(x+y)+z))$ is preserved by $\bar{\gamma}$ and λ and we obtain a λ-equivariant test-configuration in a canonical way.

For $h<0$ we have $\gamma\notin\mathcal{P}(\lambda)$ and we must first conjugate γ by some element $g\in\mathcal{P}(\gamma)$ to obtain $\chi\in\mathcal{P}(\lambda)$. A direct computation shows that one can choose

$$g=\begin{pmatrix}1 & 0 & -1\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \chi=\begin{pmatrix} t^{-1} & 0 & 0\\ t^{-1}-t^2 & t^2 & t^{-1}-t^2\\ 0 & 0 & t^{-1}\end{pmatrix}$$

yielding

$$\bar{\gamma}=\lim_{\tau\to 0}\lambda(\tau)\chi(t)(\lambda(\tau))^{-1}=\begin{pmatrix} t^2 & t^{-1}-t^2 & -t^{-1}+t^2\\ 0 & t^{-1} & 0\\ 0 & 0 & t^{-1}\end{pmatrix}.$$

The corresponding test-configuration $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ is given by

$$V(t^3x(x+2y+z)-(x+y)^2)$$

endowed with the action of $\bar{\gamma}$, which commutes with λ. Diagonalising $\bar{\gamma}$ (which is of course compatible with diagonalising λ) we see that $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ is isomorphic to the test-configuration induced by $\mathrm{diag}(t^{-1}, t^{-1}, t^2)$ given by $V(t^3xz - y^2)$.

Finally note that the test-configuration $(\mathcal{X}', \mathcal{L}')$ (isomorphic to $(\mathcal{X}, \mathcal{L})$) defined by χ is

$$V((x + y)(y + z) - t^3y(x + 2y + z)).$$

Taking the flat closure of $(\mathcal{X}', \mathcal{L}')$ under the action of λ gives the one-parameter family of divisors of $\mathbb{P}^1 \times \mathbb{C}$ parametrised by τ

$$(x + y)^2 - t^3x(x + 2y + z) + \tau^{-2h}(1 - t^3)(x + y)(x + 2y + z).$$

This is a flat one-parameter family taking $(\mathcal{X}', \mathcal{L}')$ to $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$.

We explain next an alternative approach to specialising test-configurations which is more global, i.e. independent of the exponent, and is based on filtrations of the homogeneous coordinate ring. Let χ be the filtration of $R = R(X, L)$ corresponding to $(\mathcal{X}, \mathcal{L})$, and T a torus in $\mathrm{Aut}(X, L)$.

Definition 27. *Let $\lambda\colon \mathbb{C}^* \to T$ be a one-parameter subgroup. The specialisation $\bar{\chi}$ of χ with respect to λ is given by $\bar{\chi}_k = \lim_{\tau\to 0} \lambda(\tau) \cdot \chi_k$, where the limit is taken in the appropriate Grassmannian; the specialization depends on λ, but we suppress it from the notation. If the image of λ is generic in T (i.e. it avoids finitely many hyperplanes in the lattice of 1PS's of T), then $\bar{\chi}$ is T equivariant, and we call it a speicalization of χ with repsect to T.*

It is straightforward to check that $\bar{\chi}$ is still a filtration of R in the sense of Definition 14. The limit filtration $\bar{\chi}$ can also be described as follows. Let $\mathrm{Rees}(\chi) \subset R$ be the Rees algebra of the finitely generated filtration χ. A one-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{Aut}(X, L)$ acts on R and on $R[t]$ (trivially on t) and we may define a $\mathbb{C}[t]$-subalgebra $\mathrm{Rees}^\lambda(\chi) \subset R$ by

$$\mathrm{Rees}^\lambda(\chi) = \{\lim_{\tau\to 0} \lambda(\tau)(s) \,:\, s \in \mathrm{Rees}(\chi)\}.$$

Then $\bar{\chi}$ is precisely the filtration of R whose Rees algebra is $\mathrm{Rees}^\lambda(\chi)$, i.e.

$$\bar{F}_i R_k = \{s \in R_k \,:\, t^i s \in \mathrm{Rees}^\lambda(\chi)\}.$$

The crucial difficulty with this more global approach lies in the fact that the Rees algebra of $\bar{\chi}$ is not finitely generated in general. This is a well-known phenomenon in commutative algebra and an explicit example is given in the Appendix.

Let $(\mathcal{X}, \mathcal{L})$ be a very ample test-configuration of exponent r. Given a generic one-parameter subgroup of $T \subset \mathrm{Aut}(X, L)$ we can perform two basic constructions. On the one hand we can specialise $(\mathcal{X}, \mathcal{L})$ to $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ in the sense of Definition 25. This specialisation corresponds to a finitely generated filtration η. The Veronese filtration $\eta^{(j)}$ corresponds to the Veronese test-configuration $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$ with exponent jr. On the other hand $(\mathcal{X}, \mathcal{L})$ corresponds to a finitely generated filtration χ of R via the construction described at the end of the previous section. We may specialise χ to $\bar{\chi}$ and consider a finitely generated approximation $\bar{\chi}^{(j)}$, corresponding to a test-configuration of exponent jr: by definition this is in fact $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$. Since $\bar{\chi}$ is not finitely generated (in general), the filtrations $\eta^{(j)}$, $\bar{\chi}^{(j)}$ will differ for infinitely many j, that is the test-configurations $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$ and $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$ differ for infinitely many j. However we can establish a simple comparison.

Proposition 28. *The filtration of $H^0(X, L^{\otimes jr})$ induced by $\bar{\chi}$ (or equivalently by $\bar{\chi}^{(j)}$ or $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$) is included in the filtration of the same vector space induced by $\eta^{(j)}$, i.e. by the filtration corresponding to $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$.*

Proof. The result follows at once from the fact that the Rees algebra of $\bar{\chi}$ contains all the generators of the Rees algebra of η, by construction. □

Let us show that when $\bar{\chi}$ is finitely generated then $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is in fact a flat limit of $(\mathcal{X}, \mathcal{L})$ under a $\mathbb{C}^*$-action, and in particular the filtrations $\bar{\chi}^{(j)}$, $\eta^{(j)}$ coincide for all j, that is $(\bar{\mathcal{X}}, \bar{\mathcal{L}}^{\otimes j})$ and $(\bar{\mathcal{X}}, \overline{\mathcal{L}^{\otimes j}})$ coincide. In order to simplify the notation (without loss of generality) we assume in the following result that $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ has exponent 1 and χ is the corresponding finitely generated filtration.

Lemma 29. *Suppose that $\mathrm{Rees}(\bar{\chi}) = \mathrm{Rees}^{\lambda}(\chi)$ is a finitely generated $\mathbb{C}[t]$-subalgebra of $R[t]$. Then there exist an embedding $\iota\colon \mathcal{X} \to \mathbb{P}^N \times \mathbb{C}$ and a 1-parameter subgroup $\widehat{\lambda}\colon \mathbb{C}^* \to GL(N+1, \mathbb{C})$ such that*

- *$\iota^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{L}^{\otimes r}$ for some $r \geq 1$,*
- *$\widehat{\lambda}$ acting on $\mathbb{P}^N$ preserves $\iota(\mathcal{X}_1) \cong X$ and restricts to the induced action of λ on it,*
- *the 1-parameter flat family of subschemes of $\mathbb{P}^N \times \mathbb{C}$ induced by $\widehat{\lambda}$ (acting trivially on the second factor) has central fibre isomorphic to $\bar{\mathcal{X}} := \mathrm{Proj}(\mathrm{Rees}(\bar{\chi}))$ endowed with its natural Serre line bundle $\mathcal{O}(r)$.*

In particular it follows that the central fibre $(\bar{\mathcal{X}}_0, \mathcal{L}_0'^{\otimes r})$ is a flat 1-parameter degeneration of the central fibre $(\mathcal{X}_0, \mathcal{L}_0^{\otimes r})$ (as closed subschemes of $\mathbb{P}^N$).

Proof. If $\mathrm{Rees}(\bar{\chi}) = \mathrm{Rees}^{\lambda}(\chi) \subset R[t]$ is a finitely generated $\mathbb{C}[t]$-subalgebra there exists a finite set of elements σ_i of $\mathrm{Rees}(\chi)$ such that the limits $\lim_{\tau\to 0} \lambda(\tau)\cdot\sigma_i$ generate $\mathrm{Rees}(\bar{\chi})$. Since $\lambda(\tau)$ is $\mathbb{C}[t]$-linear and we have $\lambda(\tau)\cdot(s_1+s_2) = \lambda(\tau)\cdot s_1 + \lambda(\tau)\cdot s_2$ and $\lambda(\tau)\cdot(s_1 s_2) = (\lambda(\tau)\cdot s_1)(\lambda(\tau)\cdot s_2)$ for all $s_1, s_2 \in R$, we can then choose our σ_i of the special form $\sigma_i = t^{p(i)} s_i$ where the s_i are homogeneous elements of R. Moreover, enlarging the collection of σ_i's, we can assume that the elements $t^{p(i)} s_i$, $i = 0, \dots, N$ generate $\mathrm{Rees}(\chi)$. For a suitable $r \geq 1$ the monomials $\tilde{s}_j$ in our elements s_i of homogenous degree r generate the Veronese algebra $\tilde{R} = \bigoplus_{k\geqslant 0} R_{kr}$ (which is thus generated in degree 1) and so the corresponding elements $t^{p(j)}\tilde{s}_j$ generate the Veronese algebra $\bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr}$ and their limits $t^{p(j)}\lim_{\tau\to 0}\lambda(\tau)\cdot\tilde{s}_j$ generate the Veronese algebra $\bigoplus_{k\geqslant 0}(\bar{F}_{kr}\tilde{R})t^{kr}$.

With these assumptions we define a surjective morphism of $\mathbb{C}[t]$-algebras

$$\phi\colon \mathbb{C}[\xi_0,\dots,\xi_N][t] \to \bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr}$$

by $\phi(t) = t$, $\phi(\xi_i) = t^{p(i)}\tilde{s}_i$. Suppose that the action of λ is given by $\lambda(\tau)\cdot\tilde{s}_i = \sum_j a_{ij}(\tau)\tilde{s}_j$. We define a one-parameter subgroup $\widehat{\lambda}: \mathbb{C}^* \to GL(\mathbb{C}_1[\xi_0,\dots,\xi_N])$, acting on degree 1 elements by $\widehat{\lambda}(\tau)\cdot\xi_i = \sum_j a_{ij}(\tau)\xi_j$, and extend its action trivially on t. The morphism ϕ induces the required embedding

$$\iota\colon \mathcal{X} = \operatorname{Proj}_{\mathbb{C}[t]}\bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr} \to \operatorname{Proj}_{\mathbb{C}[t]}\mathbb{C}[\xi_0,\dots,\xi_N][t],$$

which intertwines the actions of λ and $\widehat{\lambda}$. By construction the limit as $\tau \to 0$ of the flat family of closed subschemes of $\mathbb{P}^N\times\mathbb{C}$ given by

$$\widehat{\lambda}(\tau)\cdot\iota\big(\operatorname{Proj}_{\mathbb{C}[t]}\bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr}\big)$$

is isomorphic to $\operatorname{Proj}_{\mathbb{C}[t]}\bigoplus_{k\geqslant 0}(\bar{F}_{kr}\tilde{R})t^{kr}$ and so it gives a copy of $\bar{\mathcal{X}}$ embedded in $\mathbb{P}^N\times\mathbb{C}$ as a flat 1-parameter degeneration of $\mathcal{X}$.

To prove the statement on central fibres we look at the family of closed subschemes of $\mathbb{P}^N$ given by

$$\widehat{\lambda}(\tau)\cdot\iota(\mathcal{X}_0) = \widehat{\lambda}(\tau)\cdot\iota\big(\operatorname{Proj}_{\mathbb{C}[t]}\operatorname{gr}\bigoplus_{k\geqslant 0}(F_{kr}\tilde{R})t^{kr}\big).$$

Taking the flat closure of this 1-parameter family we obtain a closed subscheme $\mathcal{Y}_0\subset\mathbb{P}^N$ whose underlying reduced subscheme $\mathcal{Y}_0^{\text{red}}$ is contained in $\bar{\mathcal{X}}_0\subset\mathbb{P}^N$. By flatness the Hilbert function of $\mathcal{Y}_0$ is the same as that of the central fibre $(\mathcal{X}_0,\mathcal{L}_0^{\otimes r})$ and so the same as that of the general fibre $(X, L^{\otimes r})$. Similarly the Hilbert function of $\bar{\mathcal{X}}_0\subset\mathbb{P}^N$ is the same as that of $(\bar{\mathcal{X}}_0,\bar{\mathcal{L}}_0^{\otimes r})$ and so the same as that of the general fibre $(X, L^{\otimes r})$. As we have $\mathcal{Y}_0^{\text{red}}\subset\bar{\mathcal{X}}_0\subset\mathbb{P}^N$ and $\bar{\mathcal{X}}_0,\mathcal{Y}_0\subset\mathbb{P}^N$ have the same Hilbert functions we must actually have $\mathcal{Y}_0=\bar{\mathcal{X}}_0$ as required. □

The following observation follows immediately from the definitions of the weight functions (Definitions 18, 19) and of the specialisation $\bar{\chi}$ (Definition 27).

Lemma 30. *In the situation of Lemma 29 we have*

$$w_{(\bar{\mathcal{X}},\bar{\mathcal{L}})}(k) = w_{(\mathcal{X},\mathcal{L})}(k),\quad d_{(\bar{\mathcal{X}},\bar{\mathcal{L}})}(k) = d_{(\mathcal{X},\mathcal{L})}(k).$$

for all k. In particular we have

$$\mathrm{DF}(\bar{\mathcal{X}},\bar{\mathcal{L}}) = \mathrm{DF}(\mathcal{X},\mathcal{L}),\quad ||(\bar{\mathcal{X}},\bar{\mathcal{L}})||_{L^2} = ||(\mathcal{X},\mathcal{L})||_{L^2}.$$

Let us now consider the general case.

Theorem 31. *Let χ be a possibly non-finitely generated filtration, and let $\bar{\chi}$ be its specialisation with respect to a torus $T \subset \mathrm{Aut}(X, L)$ in the sense of Definition 27. Then we have*

$$\mathrm{Chow}_{\infty}(\bar{\chi}) \leq \mathrm{Chow}_{\infty}(\chi).$$

Proof. We claim that the inequality $\mathrm{Chow}_r(\bar{\chi}^{(r)}) \leq \mathrm{Chow}_r(\chi^{(r)})$ holds for every r. By Definition 22 this will imply the Theorem.

Before proving the claim, let us recall the relation between the Chow weight and classical GIT, following [16, Section 3], [9, Section 7] and [21, Section 2]. Let $V_r = H^0(X, L^{\otimes r})^{\vee}$, and denote by γ a 1PS of $GL(V_r)$ which induces the test configuration associated to $\chi^{(r)}$. The group $GL(V_r)$ acts on the appropriate Chow variety Z_r, and $X \subset \mathbb{P}(H^0(X, L^{\otimes r})^{\vee})$ gives a point $[X] \in Z_r$. On Z_r we have the classical, ample Chow line bundle, giving a linearisation for the action of $GL(V_r)$. The r-th Chow weight of $\chi^{(r)}$ introduced in Definition 19 is the Hilbert-Mumford weight of the point $[X] \in Z_r$ under γ, computed with respect to a convenient rational rescaling of the ample Chow line bundle (with this normalisation the Chow line bundle becomes an ample $\mathbb{Q}$-line bundle, but this causes no difficulties).

The claim now follows from Proposition 8, i.e. the fact that Hilbert-Mumford weights decrease under specialisation. □

5. Application to cscK polarised manifolds

In this Section we show that Conjecture 1 combined with ideas from [17, 19] implies a new proof that cscK manifolds are K-polystable.

Theorem 32. *Let (X, L) be a cscK manifold and let T be a maximal torus in $\mathrm{Aut}(X, L)$. Then (X, L) is T-equivariantly K-polystable.*

More explicitly, Theorem 32 states that, given a normal T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$, we have

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$$

with equality if and only if $(\mathcal{X}, \mathcal{L})$ is a product.

Proof. Let $(\mathcal{X}, \mathcal{L})$ be a normal T-equivariant test configuration. By a result of Donaldson [8] (X, L) is K-semistable, so it is enough to assume that $(\mathcal{X}, \mathcal{L})$ is not a product and to show that we cannot have $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$. We argue by contradiction assuming $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$.

Denote by α the $\mathbb{C}^*$ action on $(\mathcal{X}, \mathcal{L})$. Let β_i be an orthogonal basis of 1-parameter subgroups β_i of $\mathrm{Aut}(X, L)$ (see [20] for a discussion of the formal inner product on $\mathbb{C}^*$-actions). As $(\mathcal{X}, \mathcal{L})$ is T-equivariant, there are $\mathbb{C}^*$-actions $\tilde{\beta}_i$ on $(\mathcal{X}, \mathcal{L})$, preserving the fibres, commuting with each other and with α, and extending the action of β_i. Fixing i, the total space $(\mathcal{X}, \mathcal{L})$ endowed with the $\mathbb{C}^*$-action $\alpha \pm \tilde{\beta}_i$ is a test-configuration for (X, L), with

Donaldson-Futaki invariant

$$\begin{aligned}\mathrm{DF}(\alpha \pm \tilde{\beta}_i) &= \mathrm{DF}(\alpha) \pm \mathrm{DF}(\tilde{\beta}_i) \\ &= \pm \mathrm{DF}(\tilde{\beta}_i)\end{aligned}$$

(the first equality follows since α, $\tilde{\beta}_i$ are commuting $\mathbb{C}^*$-actions on the same polarised scheme). Since we are assuming that (X, L) is cscK we know it is K-semistable and so we must have $\mathrm{DF}(\tilde{\beta}_i) = 0$ for all i. Let $(\mathcal{X}, \mathcal{L})^{\perp}_{T}$ denote the L^2-orthogonal in the sense of [20], i.e. the test-configuration with total space $(\mathcal{X}, \mathcal{L})$ endowed with $\mathbb{C}^*$-action

$$\alpha - \sum_i \frac{\langle \alpha, \tilde{\beta}_i \rangle}{||\tilde{\beta}_i||^2} \tilde{\beta}_i.$$

Then we see that $\mathrm{DF}(\mathcal{X}, \mathcal{L})^{\perp}_{T} = 0$.

Since $\mathcal{X}$ is normal and not isomorphic to $X \times \mathbb{C}$, by [19] section 3 there exists a point $p \in (\mathcal{X}_1, \mathcal{L}_1)$ which is fixed by the maximal torus T, and such that denoting by $\overline{\alpha \cdot p}$ the closure of the orbit of p in $(\mathcal{X}, \mathcal{L})$ we have

$$\begin{aligned}\mathrm{DF}(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})^{\perp}_{T} &= \mathrm{DF}(\mathcal{X}, \mathcal{L})^{\perp}_{T} - C\epsilon^{n-1} + O(\epsilon^n) \\ &= -C\epsilon^{n-1} + O(\epsilon^n)\end{aligned} \tag{5.1}$$

for some constant $C > 0$. Here $(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})$ is the test-configuration for $(\mathrm{Bl}_p X, L - \epsilon E)$ ($E, \mathcal{E}$ denoting the exceptional divisors) induced by blowing up the orbit $\overline{\alpha \cdot p}$ in $\mathcal{X}$ with sufficiently small rational parameter $\epsilon > 0$. Since p is fixed by T there is a natural inclusion $T \subset \mathrm{Aut}(\mathrm{Bl}_p X, L - \epsilon E)$ and then $(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})^{\perp}_{T}$ denotes the L^2 orthogonal to T in the sense of [20].

As explained in [19] Theorem 2.4 a well-known result of Arezzo, Pacard and Singer [1] implies that the polarised manifold $(\mathrm{Bl}_p X, L - \epsilon E)$ admits an extremal metric in the sense of Calabi. The semistability result of [20] shows that we must have $\mathrm{DF}(\mathrm{Bl}_{\overline{\alpha \cdot p}} \mathcal{X}, \mathcal{L} - \epsilon \mathcal{E})^{\perp}_{T} \geq 0$. But this contradicts (5.1), so we must have in fact $\mathrm{DF}(\mathcal{X}, \mathcal{L}) > 0$ as claimed. □

Corollary 33. *If Conjecture 1 holds, then cscK manifolds are K-polystable.*

Proof. Let (X, L) be a cscK manifold, and T a maximal torus in $\mathrm{Aut}(X, L)$. Theorem 32 implies that (X, L) is T-equivariantly K-polystable. Conjecture 1 then implies that (X, L) is K-polystable. □

Remark 34. The proof of the main result of [19] (Theorem 1.4) shows that if (X, L) is extremal and $T \subset \mathrm{Aut}(X, L)$ is a maximal torus then we have $\mathrm{DF}(\mathcal{X}, \mathcal{L})^{\perp}_{T} > 0$ for all T-equivariant test-configurations whose normalisation is not induced by a holomorphic vector field in T (or equivalently, which are not isomorphic to such a product outside a closed subscheme of codimension at least 2). If the assumption is dropped there are counterexamples. Note that Theorem 1.4 in [19] is mistakenly stated without this assumption. See [11] Remark 4 and the note [18] for further discussion.

Appendix

In this appendix we present an example of a test-configuration $(\mathcal{X}, \mathcal{L})$ with a 1-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{Aut}(X, L)$ such that the λ-equivariant filtration $\bar{\chi}$ of Definition 25 is not finitely generated. This is done by adapting a well-known example in the literature on canonical bases of subalgebras, due to Robbiano and Sweedler ([15] Example 1.20).

Consider the polynomial algebra $\mathbb{C}[t][x, y]$ over the ring $\mathbb{C}[t]$ and let A denote the $\mathbb{C}[t]$-subalgebra generated by

$$t(x + y), txy, txy^2, t^2y.$$

Then $A \subset R[t]$ is the Rees algebra of a homogeneous, multiplicative, pointwise left bounded finitely generated filtration χ of the homogeneous coordinate ring $R = \mathbb{C}[x, y]$ of the projective line $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. So $\mathrm{Proj}_{\mathbb{C}[t]} A$ endowed with its natural Serre bundle $\mathcal{O}(1)$ is a test-configuration for $\mathbb{P}^1$. Consider the 1-parameter subgroup $\lambda\colon \mathbb{C}^* \to SL(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)))$ acting by

$$\lambda(\tau) \cdot x = \tau^{-1}x, \quad \lambda(\tau) \cdot y = \tau y.$$

We let $\bar{\chi}$ be the limit of χ under the action of λ as in the proof of Proposition 27.

Proposition 35. *The limit filtration $\bar{\chi}$ is not finitely generated.*

Proof. The 1-parameter subgroup λ induces a term ordering $>$ on the $\mathbb{C}[t]$-algebra $\mathbb{C}[t][x, y]$ which is compatible with the graded $\mathbb{C}[t]$-algebra structure and for which we have $x > y$. Let us denote the initial term of an element $\sigma \in \mathbb{C}[t][x, y]$ by $\mathrm{in}_> \sigma$. The Rees algebra $\mathrm{Rees}(\bar{\chi})$ coincides with the initial algebra of A defined by

$$\mathrm{in}_> A = \{\mathrm{in}_> \sigma \,:\, \sigma \in A\}.$$

We show that $\mathrm{in}_> A$ is not finitely generated. The proof follows closely the original argument in [15] Example 1.20.

Claim 1. The algebra A contains all the monomials of the form $t^{n-1}xy^n$ for $n \geq 3$, and does not contain elements which have a homogeneous component of the form t^kxy^n for $k < n-1$. In particular no element of A can have initial term of the form t^kxy^n for $k < n - 1$. To check the first statement we observe that we have for $n \geq 3$

$$t^{n-1}xy^n = t(x + y)t^{n-2}xy^{n-1} - t(xy)t(t^{n-3}xy^{n-2})$$

and then argue by induction starting from the fact that A contains the monomials $t(x + y), txy, txy^2$. For the second statement it is enough to check that A does not contain t^kxy^n for $k < n - 1$ (since A is a graded subalgebra). This is a simple check.

Claim 2. The algebra A does not contain elements which have a homogeneous component of the form t^ky^j for $k \leq j$. In particular no element of A can have initial term of the form t^ky^j for $k \leq j$. Since A is a graded subalgebra it is enough to show that t^ky^j cannot belong to A if $k \leq j$. All the elements of A are of the form $f(t(x+y), txy, txy^2, t^2y)$ where $f(x_1, x_2, x_3, x_4)$ is a polynomial with coefficients in $\mathbb{C}[t]$. Assuming

$$f(t(x+y), txy, txy^2, t^2y) = t^k y^j$$

and setting $y = 0$ gives $f(tx, 0, 0, 0) = 0$. Similarly setting $x = 0$ gives $f(ty, 0, 0, t^2y) = t^k y^j$. If $k \leq j$ it follows that necessarily $k = j$ and $f(x_1, 0, 0, x_2) = x_1$. Comparing with $f(tx, 0, 0, 0)$ we find $tx = 0$, a contradiction.

Claim 3. $\text{in}_> A$ *is not finitely generated.* Assuming $\text{in}_> A$ is finitely generated we can find a finite set σ_i of elements of A such that $\text{in}_> \sigma_i$ generate $\text{in}_> A$. By finiteness we can choose $m \gg 1$ such that for all i we have $\text{in}_> \sigma_i \neq t^{m-1}xy^m$. On the other hand by Claim 1 we know that for all m we have $t^{m-1}xy^m \in \text{in}_> A$. By the definition of a term ordering we know thus that $t^{m-1}xy^m$ must be a product of powers of initial terms of the elements σ_i. As x appears linearly it follows that there must be two generators σ_i, σ_j with $\text{in}_> \sigma_i = t^p xy^r$, respectively $\text{in}_> \sigma_j = t^q y^s$ with $p + q = m - 1$, $r + s = m$. By Claim 1 we must have $p \geq r - 1$ and by Claim 2 we must have $q > s$. Hence $p + q > r + s - 1 = m - 1$ so $p + q \geq m$, a contradiction. □

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