

About gravitational-wave generation by a three-body system

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Abstract. We highlight some subtleties that affect naive implementations of quadrupolar and octupolar gravitational waveforms from numerically-integrated trajectories of three-body systems. Some of those subtleties arise from the requirement that the source be contained in its “coordinate near zone” when applying the standard PN formulae for gravitational-wave emission, and from the need to use the non-linear Einstein equations to correctly derive the quadrupole emission formula. We show that some of these subtleties were occasionally overlooked in the literature, with consequences for published results. We also provide prescriptions that lead to correct and robust predictions for the waveforms computed from numerically-integrated orbits.

1. Introduction

While the two-body problem is completely solvable in Newtonian theory, no general exact solution to it is known in General Relativity (GR). As a result, the dynamics of a binary system in GR can only be obtained by solving perturbatively the field equations, or through numerical techniques on a computer (“numerical relativity” [1–5]). Perturbative schemes, valid in different regimes, include the Post-Newtonian (PN) approximation [6], which consists in expanding the dynamics in powers of $v/c \ll 1$ † (v being the relative velocity of the binary, c the speed of light in vacuum) and the self-force formalism, which instead relies upon an expansion in the binary’s mass ratio, assumed to be small [7]. The detection of gravitational waves (GWs), indirectly from binary pulsars [8] and directly from systems of two merging black holes (BHs) [9–11], provides an excellent benchmark to test the general-relativistic two-body dynamics. As a matter of fact, these observations are to date in perfect agreement with the GR predictions [10–15].

† In the standard PN book-keeping a term suppressed by a factor $(v/c)^{2n}$ with respect to the leading (i.e., Newtonian) order is said to be of n PN order.

Unsurprisingly, the three-body problem, which already in Newtonian theory does not admit any general closed form solution and, moreover, gives rise to chaotic dynamics, becomes considerably harder in GR. However, its general-relativistic dynamics can be obtained within the PN approximation scheme, just like in the two-body case. Indeed, one can write a PN-expanded, time-dependent Hamiltonian [16–19] that describes the conservative dynamics of a system of three non-spinning bodies up to the 2PN order (i.e., through order $(v/c)^4$ beyond the leading order Newtonian dynamics), as well as its dissipative dynamics (i.e., the back-reaction due to GW emission) to leading order in v/c , which corresponds to a contribution of 2.5PN order, or $\mathcal{O}(v/c)^5$, in the equations of motion.

The PN dynamics of three body systems is not just an academic curiosity. Kozai-Lidov resonances [20–23], first discovered in Newtonian triplets, are believed to be a relevant astrophysical mechanism for the formation of binary systems of stellar-mass BHs (observable by ground-based GW detectors) in dense stellar environments [24, 25] or in isolation [26]. It may also play an important role in the formation and evolution of binaries of supermassive BHs [27–31], whose GW signal is targeted by existing pulsar-timing arrays [32–41] and by the future space-borne interferometer LISA [42]. It turns out that the PN corrections are crucial to assess the efficiency of the Kozai-Lidov mechanism, as they can destroy the resonance on which it relies. Indeed, the coherent piling up of the perturbation induced by the third body may be disrupted due to relativistic precession effects appearing at 1PN order and beyond [22, 27, 43]. Finally, the GW emission from systems of three BHs has been studied in detail by means of numerical techniques [19, 31, 44–46] in the event that they form and radiate in sufficiently large number to provide a sizable population for GW detectors. For particular configurations, they have also been investigated analytically [47, 48] so as to gain some insight on their dynamics.

The GW emission from binary systems with relative velocities $v \ll c$ can be modeled, at leading order, through the Einstein quadrupole formula [6, 49, 50]. Next-to-leading order corrections are given by the mass-octupole and current-quadrupole contributions [6, 51]. A key requirement implicit in the derivation of the corresponding formulae is that the binary must be contained in its “Near Coordinate Zone” (NCZ), i.e., a region (centered on the origin of the coordinates) of radius comparable to (but smaller than) the GW wavelength λ . This requirement comes about because the PN formalism for GW generation, which can only be legitimately applied as long as the source is much smaller than λ , is based on a systematic multipole expansion of the gravitational field outside the source. In order to ensure an overlap between the domain of validity of this expansion (say $|\boldsymbol{x}| \gtrsim r_{\min}$) and the near zone (where the dynamics of the source is computed neglecting retardation effects), one must clearly have $r_{\min} \sim [\text{size of the near zone}] \sim \lambda$, so the coordinate origin and the source cannot be more than one wavelength apart[‡]. Indeed, these formulae are usually applied in the reference frame of the binary’s CoM. In that frame, in the PN regime, the existence of a NCZ containing the binary is guaranteed, since the size of the system — its separation a — is negligible relative to the wavelength $\lambda \sim a/(v/c)$.

[‡] Nevertheless, the exact choice of where the NCZ is centered is a matter of definition. The important point is that it must contain both the whole source and the origin of the coordinates. In fact, one may alternatively think in terms of the binary’s near zone, which is defined to be (roughly) centered on the center of mass (CoM) of the binary. In that case, a proper derivation of the quadrupole formula would require choosing the origin within the near zone. The adoption of this point of view would not alter any of the discussions of this paper.

For a triple system with relative velocities $v \ll c$, it would seem natural to apply the very same formulae in the reference frame of the CoM of the three-body system. However, by doing so, one obtains unphysical results such as those reported in fig. 18 of [19], as we will now explain. Indeed, we have reproduced the same behavior by applying the quadrupole and “quadrupole-octupole” formulae in the CoM reference frame of a series of triple systems with mass ratios $m_2/m_1 = 0.5$ and $m_3/(m_1 + m_2) = 0.05$, whose trajectories are computed with the code of [52] (which includes the 1PN and 2PN conservative triple dynamics, and the leading order dissipative dynamics). The “inner binary” (comprised of m_1 and m_2) of these hierarchical triplets has zero initial eccentricity and an initial separation $a_{\text{in}} = 150Gm_t/c^2$, where G is Newton’s constant and m_t the total mass of the triplet (throughout the paper we instead reserve the symbol m to indicate the total mass of binary systems, i.e., $m = m_1 + m_2$). The “outer binary” (comprised of m_3 and the CoM of the inner binary) has instead initial separation varying in the range $a_{\text{out}} \in [625, 10000]Gm_t/c^2$, and zero initial eccentricity. The results are displayed in fig. 1, where one can observe, paradoxically, that the effect induced by the third body grows as it gets farther away from the inner binary. We will analyze this situation in detail in this paper, and show that the problem is connected to the fact that a NCZ region centered on the CoM of the triplet and having size comparable to the minimum gravitational wavelength excited by the system does *not* include the whole triplet, unlike what happens for a binary system.

This, however, is just one example of the subtleties one should be aware of when computing GW emission from binary or triple systems in a too naive fashion. Another interesting apparent paradox arises, e.g., if one tries to compute the gravitational waveforms of a binary (or triple) system by directly integrating the equations for the linear perturbations $h_{\mu\nu}$ over a background Minkowski space-time (endowed with a flat metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and coordinates $\{x^\mu\}_{\mu=0,1,2,3}$).

In the harmonic gauge, which is defined by the condition $\partial_\mu \bar{h}^{\mu\nu} = 0$, where ∂_μ is the flat four-dimensional derivative and $\bar{h}^{\mu\nu} = h^{\mu\nu} - 1/2 \eta^{\mu\nu} h^\alpha{}_\alpha$ represents the trace-reversed metric perturbation[§], the linearized Einstein equations read (see, e.g., [53] §1.1, [54] §35.1)

$$\square_{\text{flat}} \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}, \quad (1)$$

where the d’Alembert operator $\square_{\text{flat}} = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is computed with the background Minkowski metric and $T^{\mu\nu}$ is the source stress-energy tensor. These equations can be integrated exactly by using the (retarded) Green function of \square_{flat} . The resulting waveforms (obtained from the transverse trace-free part of the spatial components) may then be compared to those predicted by the quadrupole formula (and its higher-order corrections that we have mentioned above).

The comparison between the GW amplitudes obtained with the two procedures for various binaries is shown in fig. 2. As can be seen, there appears to be a factor ~ 2 discrepancy (this factor becomes exactly 2 for binary circular orbits). Similar discrepancies arise when integrating eq. 1 for triple systems. This puzzling difference will be discussed in more details. It is related to the fact, often mentioned but rarely

[§] In our conventions, space-time Greek indices are raised or lowered with the metric $\eta_{\mu\nu}$ or its inverse $\eta^{\mu\nu}$, whereas space Latin indices are raised or lowered with the Euclidean metric δ_{ij} or its inverse δ^{ij} . In particular: $h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$ and $h^\alpha{}_\alpha = \eta^{\alpha\beta} h_{\alpha\beta}$.

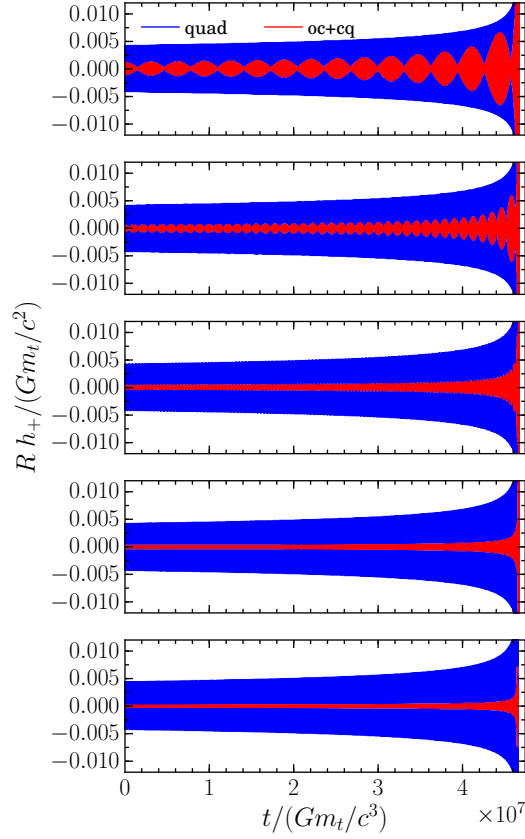


Figure 1: Waveforms from five triple systems with relative inclination $i = 0$, inner separation $a_{\text{in}} = 150 Gm_t/c^2$, inner eccentricity $e_{\text{in}} = 0$, outer eccentricity $e_{\text{out}} = 0$, and mass ratios $m_2/m_1 = 0.5$ and $m_3/(m_1 + m_2) = 0.05$. From top to bottom $a_{\text{out}} = [10000, 5000, 2500, 1250, 625] Gm_t/c^2$. The observer is located in the xz plane of a fixed spatial frame (x, y, z) , with spherical coordinates $\theta = \pi/4$, $\phi = 0$. To be compared with [19], fig. 18. As in [19], the orbits are obtained by integrating numerically the Hamilton equations for the triple systems, through the 2PN order in the conservative dynamics and at the leading (Newtonian) order in the dissipative one.

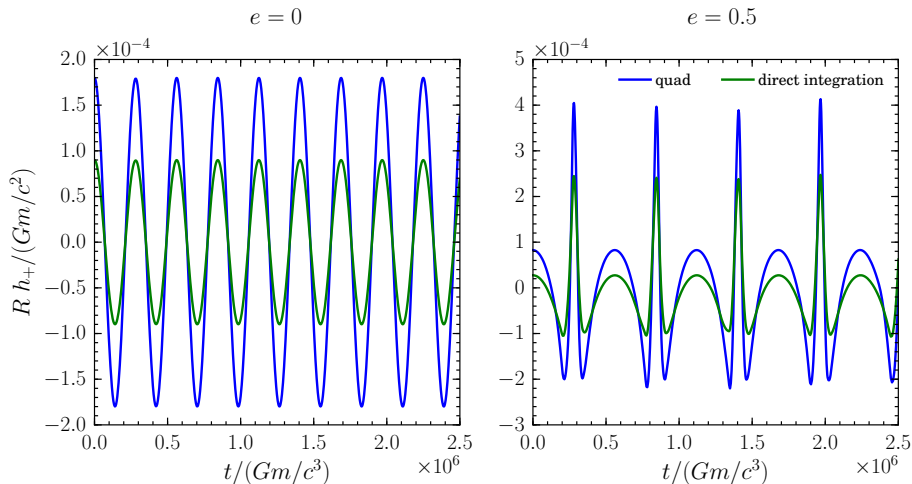


Figure 2: Quadrupole waveforms from two simulations of circular binaries with masses $m_1 = 0.9m$, $m_2 = 0.1m$. Blue lines are obtained with the quadrupole formula (see eq. 2); green lines are computed by direct integration of eq. 1. *Left panel:* circular case. *Right panel:* $e = 0.5$.

illustrated in introductory GR textbooks (see however [53,54]), that a naive derivation of the quadrupole formula based on eq. 1 is wrong. It is because that equation (via the harmonic gauge condition) implies that $\partial_\mu T^{\mu\nu} = 0$, which is clearly not verified for a binary system since it entails that bodies move along straight lines.

The focus of this paper is thus pedagogical. We will discuss the two problems mentioned above as well as other subtleties that we have encountered when computing gravitational waveforms from numerically-integrated orbits of triple systems. More precisely, the organization is as follows: in Section 2 we will illustrate, tackle and solve the problems that arise when applying the standard quadrupole formula (and its higher-order corrections) to a triple system. This will provide a solution to the discrepancy demonstrated in fig. 1, which, as already mentioned, will turn out to be due to the source not being contained in its NCZ when the latter is centered on the triplet’s center of mass. In Section 3 we will further comment about the inconsistent derivation of eq. 1, highlighting the need to use the *non-linear* Einstein equations to compute GW emission self-consistently. Finally, in Section 4 we will draw our conclusions. Throughout the paper we use the $(-, +, +, +)$ signature convention.

2. Emission of gravitational waves in hierarchical triplets

The leading-order contribution to the GW signal observed at space position \mathbf{x} and time t is given, in an appropriate “radiative” gauge, by the quadrupole formula (see [54] §36.10, [53] §3.3, [6] §2.5)

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2G}{Rc^4} \Lambda_{ijkl}(\mathbf{n}) \frac{d^2 M^{kl}(t_{\text{ret}})}{dt^2} + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (2)$$

where $R = |\mathbf{x}| \equiv \sqrt{x^i x_i}$ is the distance of the observer (assumed to be very far from the source compared with the wavelength λ of the emitted GWs), $t_{\text{ret}} = t - R/c$ is the retarded time of the background space-time[†], $\Lambda_{ijkl}(\mathbf{n})$ denotes the projector on the transverse-traceless (TT) gauge (see [Appendix A](#) for the explicit definition), while

$$M^{ij}(t) = \int d^3\mathbf{x}' c^{-2} T^{00}(t, \mathbf{x}') \left(x'^i x'^j - \frac{1}{3} \delta^{ij} x'^k x'_k \right) \quad (3)$$

represents the mass quadrupole moment of the source. As mentioned in the introduction, implicit in the derivation of the quadrupole formula (eq. 2) is the assumption that the source be contained in its NCZ (see, e.g., [53, 54, 57]), i.e., the reference frame in which the quadrupole moment (eq. 3) is evaluated must be such that the source be contained within a region of size $\sim \lambda$ centered on the origin of the coordinates. Finding a frame satisfying this property is always possible for slowly moving binary systems, since λ is related to the system's typical (relative) velocity v and its typical separation a by $\lambda \sim a/(v/c)$.

The most natural reference frame to describe the dynamics of an N -body system is that of the CoM, where the equations of motion take their simplest form. (Note that the usual Newtonian expression of the CoM position in terms of the body locations $\mathbf{x}_1, \mathbf{x}_2$ for a binary, namely $\mathbf{r}_0 = (m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2)/m$, is modified beyond the leading order[‡].) In the case under consideration here (i.e., a three-body system), a particularly interesting configuration is that of the so-called hierarchical triplet. The latter is comprised of an inner close binary (m_1, m_2), supplemented by a third body m_3 at larger distance. In practice, the system in this configuration can be regarded as formed by two separate binaries: an inner binary on the one hand, and an outer binary, formed by the third outer body and the CoM of the inner binary, on the other hand. From the point of view of the dynamics, the choice of the reference frame is of course irrelevant. Indeed, the Hamiltonian (both at the Newtonian order and when including the PN corrections) depends only upon the relative separations of the three bodies, hence the dynamics of the system is frame-independent (see e.g., [60]). However, caution must be exercised when applying the quadrupole formula (eq. 2) (and its higher-order generalizations including octupolar corrections, etc.) to the orbits resulting from numerical integrations of the equations of motion (see fig. 1 and related discussion).

To illustrate this point, let us consider a circular binary located “far” from the origin of the coordinates. In triple systems, this happens for the inner binary when $m_3 a_{\text{out}} \gg (m_1 + m_2 + m_3) a_{\text{in}}$ since, in this case, the inner binary is located far away

[†] Note that t as appearing in eq. 2 should rigorously be replaced by the radiative time $T = t - 2GM/c^3 \ln[R/(cb)]$, with M being the total Arnowitt-Deser-Misner energy-mass [55] and b representing some reference time. It is crucial to do so at future radiative infinity. However, since $2GM/c^3 \ln[R/(cb)] \ll R/c$, we may write $T \approx t$ for sufficiently large R if t remains bounded. See [56] for more details, in particular on how the logarithmic term is connected to the tail contribution to the waveform.

[‡] The fact that the time derivative of the CoM position must be constant implies that the latter variable must be constructed in relation to a Noetherian current. In special relativity, this current, $\mathbf{r}_0 - t\dot{\mathbf{r}}_0$, is nothing but the conserved quantity associated with the invariance of the dynamics under Lorentz boosts. Thus, the usual Newtonian definition has to be modified, the masses being replaced, notably, by the total energies of the bodies (see, e.g., [50], Vol II, §14). A similar extension of the Newtonian concept of CoM applies in GR (see [50], Vol II, §96). We refer to [58, 59] for an explicit construction in the case of binary systems.

from the CoM of the triplet chosen as the origin of the coordinates. Setting the GW source in the xy plane and the observer along the z axis, eq. 2 takes the simplest

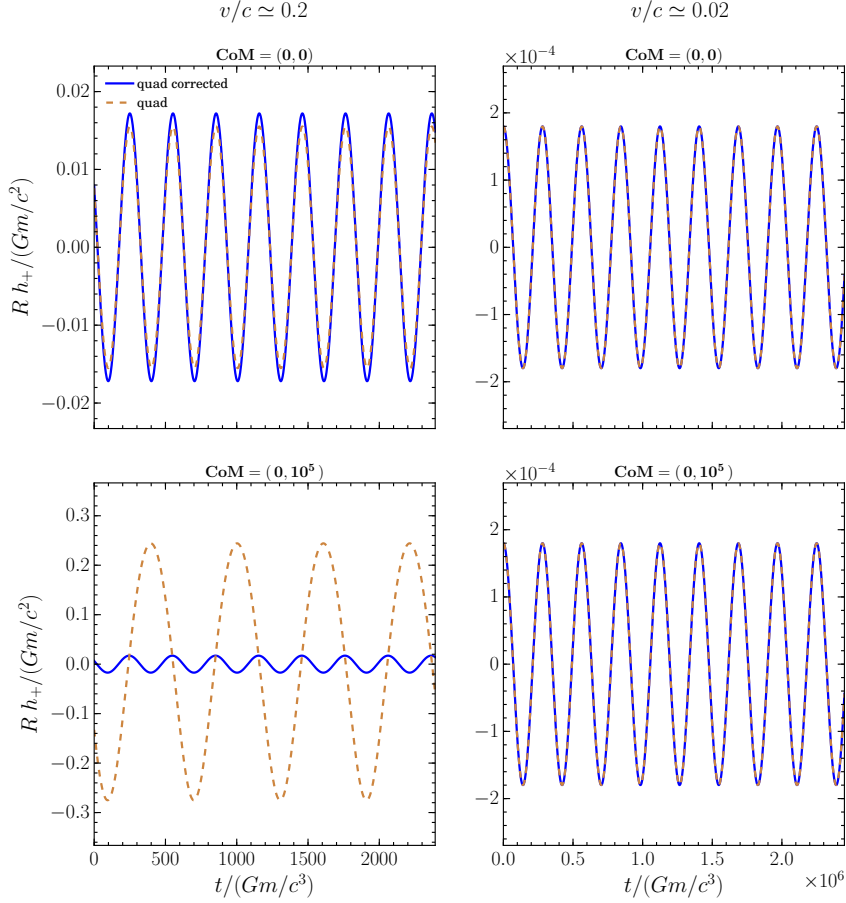


Figure 3: Quadrupole waveforms from four simulations of circular binaries with masses $m_1 = 0.9m$, $m_2 = 0.1m$ evolving according to the 1PN dynamics. *Left panels:* highly relativistic regime $v/c \simeq 0.2$. *Right panels:* mildly relativistic regime $v/c \simeq 0.02$. *Upper panels:* The binary’s center of mass is placed in the origin of coordinates. *Lower panels:* The binary’s center of mass is located at distance $10^5 \times Gm/c^2$ from the origin. The dashed lines represent quadrupole waveforms computed by simply inserting the trajectories of our simulations in eq. 2, while the solid blue lines are waveforms obtained from an “amended” quadrupole formula (see text for details). The “standard” quadrupole formula fails in the most relativistic and shifted binary case, whereas the amended one provides the correct result in all cases (note the different y -axis scales in the two left panels).

possible form, i.e.,

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2G}{Rc^4} \begin{pmatrix} \frac{1}{2}[\ddot{M}_{11}(t_{\text{ret}}) - \ddot{M}_{22}(t_{\text{ret}})] & \ddot{M}_{12}(t_{\text{ret}}) & 0 \\ \ddot{M}_{12}(t_{\text{ret}}) & -\frac{1}{2}[\ddot{M}_{11}(t_{\text{ret}}) - \ddot{M}_{22}(t_{\text{ret}})] & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

where the dots placed over the quadrupole components M_{ij} represent time derivatives. The two independent polarizations of a propagating GW, referred to as the “plus” and “cross” polarizations, are (in the situation considered here) simply the diagonal and off-diagonal part of eq. 4:

$$\begin{aligned} (h_+)_{\text{quad}} &= \frac{G}{Rc^4}(\ddot{M}_{11} - \ddot{M}_{22}), \\ (h_\times)_{\text{quad}} &= \frac{2G}{Rc^4}\ddot{M}_{12}, \end{aligned} \quad (5)$$

where all M_{ij} components are evaluated at retarded time. Explicitly, for a binary these expressions become (see, e.g., Problem 3.2 of [53])

$$\begin{aligned} (h_+)_{\text{quad}} &= \frac{4G^2 m_1 m_2}{aRc^4} \cos(2\omega t_{\text{ret}}), \\ (h_\times)_{\text{quad}} &= \frac{4G^2 m_1 m_2}{aRc^4} \sin(2\omega t_{\text{ret}}), \end{aligned} \quad (6)$$

a denoting the separation of the binary and ω its orbital frequency.

Let us now consider two different circular binaries, both with $m_1 = 0.9m$ and $m_2 = 0.1m$ but representative of two different regimes: a rather relativistic binary with separation $a = 20Gm/c^2$, which corresponds to a relative orbital velocity $v/c \simeq 0.2$, and a mildly-relativistic one, with $a = 2000Gm/c^2$ (corresponding to $v/c \simeq 0.02$). For these two systems, eqs. 6 gives $Rc^2/(Gm)(h_+)_{\text{quad}} \simeq 1.8 \times 10^{-2}$ and $Rc^2/(Gm)(h_\times)_{\text{quad}} \simeq 1.8 \times 10^{-4}$, respectively. We then evolve them numerically in two different frames: (1) one with the origin coinciding with the CoM, and (2) one with the origin shifted by 10^5 gravitational radii (i.e., $10^5 \times Gm/c^2$) from the CoM. Next, we compute the waveforms directly via eqs. 5 from the numerical trajectories. Results are reported in fig. 3 as dashed lines. The mildly-relativistic case is consistent with the analytic predictions of eqs. 6. Instead, for the relativistic binary, $(h_+)_{\text{quad}}$ given by eqs. 5 is more than one order of magnitude higher than the prediction from eqs. 6 when the origin of coordinates is far away from the CoM. Indeed, we have checked that applying eqs. 5 directly to numerically-integrated trajectories yields results that are coordinate-dependent. The discrepancy with eqs. 6 grows with the binary’s relative velocity.

Let us consider now the next-to-leading order contributions to the waveform, comprised of a mass octupole and a current quadrupole term. When these terms are taken into account, eq. 2 becomes (see [53] §3.4)

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2G}{Rc^4} \Lambda_{ijkl}(\mathbf{n}) \left[\frac{d^2 M^{kl}}{dt^2} + \frac{n_m}{3c} \left(\frac{d^3 M^{klm}}{dt^3} + 2 \frac{d^2 S^{klm}}{dt^2} \right) \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (7)$$

where $\mathbf{n} = \mathbf{x}/R$, and M^{klm} , S^{klm} are respectively the Newtonian octupole and current quadrupole moments (evaluated at t_{ret} [see Appendix A]). Again, for a binary in the

xy plane and an observer in the same plane[§], the above equation implies (see Problem 3.3 of [53] and the corresponding erratum)

$$(h_+)_{\text{oc+cq}} = \frac{1}{R} \frac{G^{5/2} m_1 m_2 \delta m}{4(ma^3)^{1/2} c^5} [5 \cos(\omega t_{\text{ret}}) - 9 \cos(3\omega t_{\text{ret}})],$$

$$(h_\times)_{\text{oc+cq}} = 0, \quad (8)$$

with $\delta m = m_1 - m_2$. For the two binaries considered above, we obtain $Rc^2/(Gm)(h_+)_{\text{oc+cq}} \simeq 2.2 \times 10^{-3}$ in the relativistic case, and $Rc^2/(Gm)(h_+)_{\text{oc+cq}} \simeq 2.2 \times 10^{-6}$ in the mildly relativistic one. From fig. 4 (dashed lines), we can see that

[§] We choose the observer in the xy plane rather than in the z direction because the mass octupole and current quadrupole corrections vanish with the latter choice.

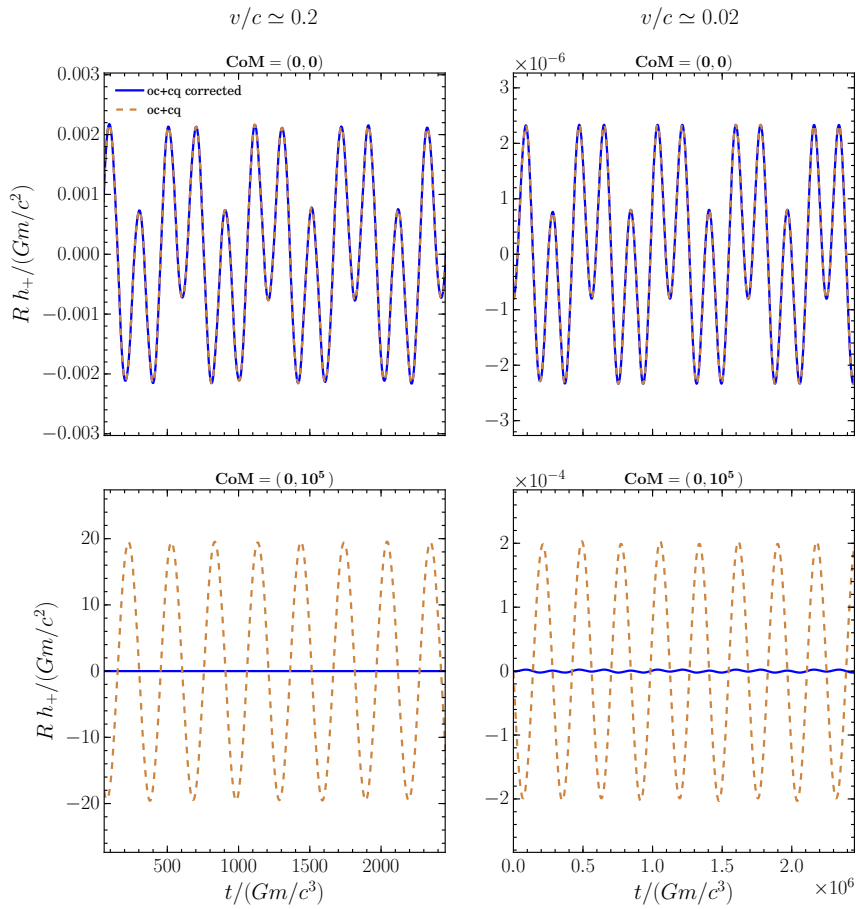


Figure 4: Same as in fig. 3, except that we report the waveforms computed with the mass octupole and current quadrupole corrections (at the 0.5PN order) whereas the observer is located on the y axis. Again, note the different y -axis scales between the upper and lower panels for each value of v/c .

if the origin of the coordinates coincides with the CoM, then $(h_+)_{\text{oc+cq}}$ as given by eq. 7 applied to the numerically-integrated orbits agrees well with the analytic result of eqs. 8. Conversely, if one shifts significantly the origin of the coordinates, $(h_+)_{\text{oc+cq}}$ computed from the numerical trajectories no longer agrees with eqs. 8, even for the mildly-relativistic binary.

In the next two sections, we will analyze the reasons behind these discrepancies and explain how they can be avoided, first for the Newtonian quadrupole formula (sec. 2.1), next for the 0.5PN quadrupole formula with octupolar corrections (sec. 2.2).

2.1. Quadrupole waveform

If we explicitly compute the second-order time derivatives in eq. 2, for a binary system, we obtain

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2G}{Rc^4} \Lambda_{ijkl}(\mathbf{n}) \left(2m_1 \dot{x}_1^k \dot{x}_1^l + 2m_2 \dot{x}_2^k \dot{x}_2^l + m_1 (\ddot{x}_1^k x_1^l + x_1^k \ddot{x}_1^l) + m_2 (\ddot{x}_2^k x_2^l + x_2^k \ddot{x}_2^l) \right) + \mathcal{O}\left(\frac{1}{c^5}\right). \quad (9)$$

Now, the position vectors of the two masses can be expressed in terms of the CoM position \mathbf{r}_0 and the relative separation vector $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ as

$$\mathbf{x}_1 = \mathbf{r}_0 + \frac{m_2}{m} \mathbf{r}, \quad \mathbf{x}_2 = \mathbf{r}_0 - \frac{m_1}{m} \mathbf{r}, \quad (10)$$

so that eq. 9 takes the form

$$h_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2G}{Rc^4} \Lambda_{ijkl}(\mathbf{n}) \left[\frac{2}{m} (m_1 \dot{x}_1^k + m_2 \dot{x}_2^k) (m_1 \dot{x}_1^l + m_2 \dot{x}_2^l) + \frac{2m_1 m_2}{m} (\dot{x}_1^k - \dot{x}_2^k) (\dot{x}_1^l - \dot{x}_2^l) + r_0^k (m_1 \ddot{x}_1^l + m_2 \ddot{x}_2^l) + r_0^l (m_1 \ddot{x}_1^k + m_2 \ddot{x}_2^k) + \frac{m_1 m_2}{m} \left(r^k (\ddot{x}_1^l - \ddot{x}_2^l) + r^l (\ddot{x}_1^k - \ddot{x}_2^k) \right) \right] + \mathcal{O}\left(\frac{1}{c^5}\right). \quad (11)$$

Since the CoM absolute coordinates \mathbf{r}_0 explicitly appears in this expression, it would seem that the GW amplitude should depend on the choice of the origin of the coordinate system. However, because eq. 11 is only correct at leading order in PN theory, it is actually sufficient to compute the accelerations $\ddot{\mathbf{x}}_1$ and $\ddot{\mathbf{x}}_2$ at leading (i.e., Newtonian) order. If one does so, the identity $m_1 \ddot{\mathbf{x}}_1 + m_2 \ddot{\mathbf{x}}_2 = 0$ holds for an isolated system, hence the dependence on the position of the center of mass (and thus on the location of the origin) disappears from eq. 11. Similarly, $m_1 \dot{x}_1^k + m_2 \dot{x}_2^k$ is constant and independent of the location of the origin.

One may want, however, to integrate the binary's equations of motion to higher PN order, either analytically or numerically. For instance, the code of Ref. [52] integrates the PN Hamiltonian for binary or triple systems through the 2.5PN order including the dissipative effects of radiation reaction. Now, when one includes these PN corrections, $m_1 \ddot{\mathbf{x}}_1 + m_2 \ddot{\mathbf{x}}_2 \neq 0$ already at 1PN order, so that the dependence on the location of the origin does *not* disappear. This is the reason of the unphysical behavior visible in fig. 3 (and partly in fig. 1, see also the next section).

A first solution can be therefore to avoid the use of the numerical trajectories to compute the accelerations in eq. 11, but instead evaluate them directly from the positions of the two bodies by using the Newtonian dynamics (i.e., Newton's second law, which ensures $m_1 \ddot{\mathbf{x}}_1 + m_2 \ddot{\mathbf{x}}_2 = 0$). Alternatively, one can note that the

combination $m_1\ddot{\mathbf{x}}_1 + m_2\ddot{\mathbf{x}}_2$ is simply (at Newtonian order) the time derivative of the total linear momentum $\mathbf{P}_N = m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2$. Thus, the identity $m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2 = 0$ just reflects the conservation of \mathbf{P}_N , which is an automatic consequence of the Newtonian dynamics. Beyond it, when PN corrections are included, $m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2$ does not vanish, because the Newtonian linear momentum $\mathbf{P}_N = m_1\dot{\mathbf{x}}_1 + m_2\dot{\mathbf{x}}_2$ is no longer a conserved quantity. This is what causes the dependence on the choice of \mathbf{r}_0 observed in fig. 3. However, one can exploit the fact that there exists a conserved PN linear momentum $\mathbf{P}_{n\text{PN}}$ generalizing \mathbf{P}_N at the n PN order. In practice, replacing \mathbf{P}_N with $\mathbf{P}_{n\text{PN}}$ is equivalent to computing the accelerations appearing in eq. 11 as $\ddot{\mathbf{x}}_i = \boldsymbol{\pi}_i/m_i$, with $i = 1, 2$ and $\boldsymbol{\pi}_i$ denoting the conjugate momentum of each body entering the Hamilton equations. Then, the combination $m_1\ddot{\mathbf{x}}_1 + m_2\ddot{\mathbf{x}}_2$ always vanishes, even if PN corrections are included in the Hamiltonian dynamics.

In conclusion, either of these two workarounds (which give rise to the ‘‘amended’’ waveforms represented by blue solid lines in fig. 3) is sufficient to eliminate the unphysical dependence on the origin of the coordinates.

2.2. Octupole and Current Quadrupole waveforms

Let us now examine what happens to the contribution of the mass octupole and current quadrupole moments to the waveform under a change of reference frame of the form of eq. 10. After expanding the time derivatives appearing in the term $n_m/(3c)(\ddot{M}^{klm} + 2\ddot{S}^{klm})$ of eq. 7 by means of the Leibniz rule, the dependence on the CoM location does not cancel out in the waveform, but gives instead a contribution^{||}

$$\delta_{r_0} h_{ij}^{\text{TT}} = \Lambda_{ijkl}(\mathbf{n}) \frac{n_m r_0^m}{3c} \sum_A m_A \left(3 \ddot{x}_A^k x_A^l + 3 x_A^k \ddot{x}_A^l + 9 \ddot{x}_A^k \dot{x}_A^l + 9 \dot{x}_A^k \ddot{x}_A^l \right). \quad (12)$$

This may be rewritten as

$$\delta_{r_0} h_{ij}^{\text{TT}} = \Lambda_{ijkl}(\mathbf{n}) \frac{d^3 M^{kl}}{dt^3} \frac{\delta R}{c}, \quad (13)$$

with $\delta R = \mathbf{n} \cdot \mathbf{r}_0 = n_m r_0^m$. Hence, the terms proportional to \mathbf{r}_0 produced by the mass octupole and current quadrupole moments can be reabsorbed in a shift $\delta R/c$ of the retarded time at which the (quadrupole) waveform is evaluated:

$$\begin{aligned} \Lambda_{ijkl}(\mathbf{n}) \left[\frac{d^2 M^{kl}(t - R/c)}{dt^2} + \frac{d^3 M^{kl}(t - R/c)}{dt^3} \frac{\delta R}{c} \right] \\ = \Lambda_{ijkl}(\mathbf{n}) \left[\frac{d^2 M^{kl}(t - R/c + \delta R/c)}{dt^2} \right] + \mathcal{O}\left(\frac{[\delta R]^2}{c}\right). \end{aligned} \quad (14)$$

This time shift simply enforces the invariance of the waveform under translations of the reference frame in which it is computed. Indeed, the retarded time is always, by definition, $t_{\text{ret}} = t - |\mathbf{x}|/c$ in the generic frame (t, \mathbf{x}) (assuming radiative coordinates), where $R = |\mathbf{x}|$ is the distance between the observer at position \mathbf{x} and the origin. This implies in particular that $t_{\text{ret}} - t_{\text{ret}}^{\text{CoM}} = |\mathbf{x}^{\text{CoM}}|/c - |\mathbf{x}|/c = -\delta R/c + \mathcal{O}([\delta R]^2/c)$, where the quantities labeled with the superscript CoM are referring to the CoM frame. On

^{||} Note that we have neglected terms $\propto \delta^{ij}$ in the sum as they disappear when they are TT-projected since $\Lambda_{ijkl}(\mathbf{n})\delta^{kl} = 0$ (see Appendix A).

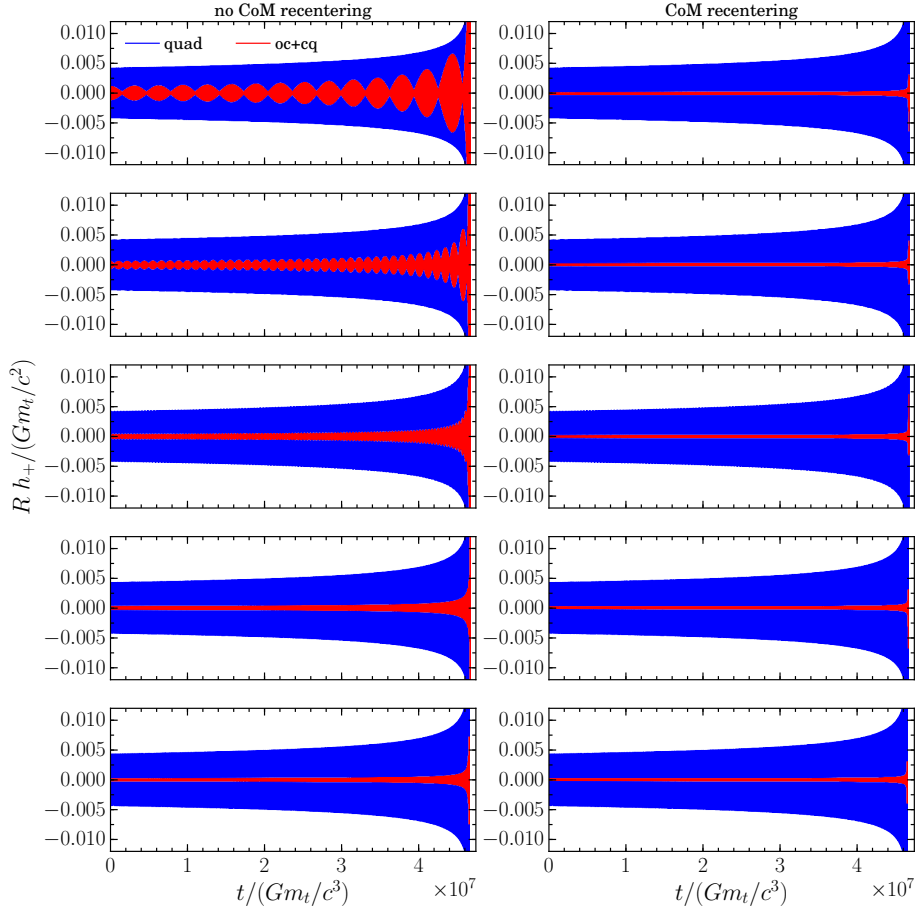


Figure 5: Same triplets as in fig. 1. *Left*: Waveform computed in the frame of the triplet’s CoM. *Right*: Waveform computed after shifting the origin to the CoM of the inner binary. Again to be compared to [19], fig. 18.

the other hand, the expressions of the multipole moments when $\mathbf{r}_0 \neq \mathbf{0}$ differ from their standard forms in CoM coordinates. Obviously, the modifications of t_{ret} and those of the multipole moments must (and do!) compensate each other so that the GW signal remains invariant, irrespective of the choice of the origin of the coordinates. This main conclusion remains true for linearly propagating waves when other multipole moments are taken into account[¶]:

$$h_{ij}^{\text{TT}}[\{M_{ab}, M_{abc}, \dots, S_{abc}, \dots\}, t, \mathbf{x}] = h_{ij}^{\text{TT}}[\{M_{ab}^{\text{CoM}}, M_{abc}^{\text{CoM}}, \dots, S_{abc}^{\text{CoM}}, \dots\}, t, \mathbf{x}^{\text{CoM}}]. \quad (15)$$

However, the analytic “resummations” needed for the above argument to work,

[¶] Beyond linear order, one must replace the source moments $M_{ab}, M_{abc}, \dots, S_{abc}$ in eq. 15 by the so-called “radiative moments” which parametrize the gravitational waveform.

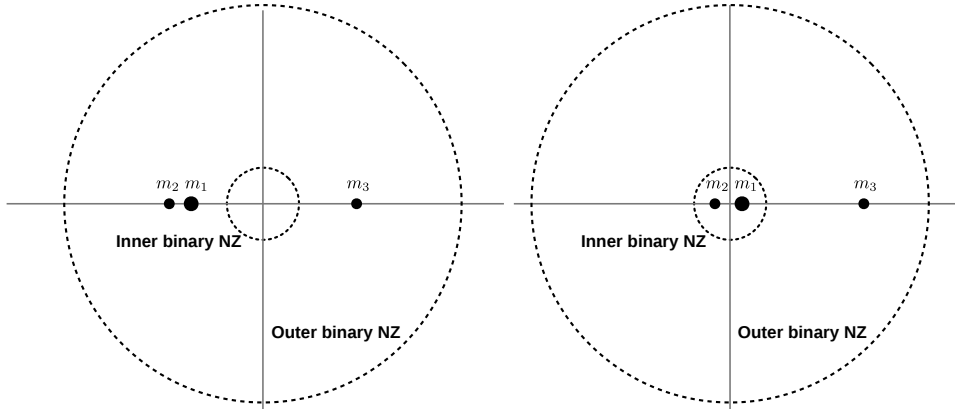


Figure 6: Cartoon representation of the change of reference frame needed to fix the unphysical spurious behavior shown in fig. 1. *Left panel:* the origin of the reference frame coincides with the CoM of the triple system. *Right panel:* the origin instantaneously coincides with the inner binary’s CoM. The latter choice allows both the inner and outer binaries to lie well within their respective NCZs.

such as the one in eq. 14, are based on a Taylor expansion. Therefore, one has to implicitly assume that terms like those in eq. 12 are “small” or, more precisely, that the displacement $|\mathbf{r}_0|$ of the CoM from the origin of the coordinates is much smaller than the wavelength of the quadrupole waveform, $\lambda = \pi c/\omega$. This is in fact very natural because, as already mentioned, one of the assumptions implicit in the derivation of the quadrupole/octupole formulae is that the source be well contained in a NCZ of size $\sim \lambda$ centered on the origin of the coordinates, where retardation effects are negligible. If the generalized quadrupole formula is applied to systems for which the source is *not* well contained in its NCZ, one will *not* be able to resum the terms of eq. 12 into a time shift. This is the origin of the discrepancy shown in fig. 4 for binary systems.

This observation also highlights the reason of the unphysical behavior shown in fig. 5 (left panels) for *triple* systems. In fact, for a weakly/mildly relativistic binary ($v \lesssim c$) with separation a , one has $\lambda \sim a/(v/c) \gtrsim a$, i.e., if one chooses the origin of the coordinates to coincide with the binary’s CoM, the NCZ will always contain the binary. For a hierarchical triple system, instead, if one sets the origin at the location of the triplet’s CoM, the inner binary will be *outside* its NCZ provided that the separation of the outer binary is sufficiently large. This may be understood by noting that there are actually two NCZs for a hierarchical triplet, i.e., an inner-binary NCZ with size $\lambda_{\text{in}} \sim a_{\text{in}}/(v_{\text{in}}/c)$, and an outer-binary NCZ with size $\lambda_{\text{out}} \sim a_{\text{out}}/(v_{\text{out}}/c)$, as illustrated in fig. 6. Clearly, while the outer binary will always be contained in its NCZ if $v_{\text{out}}/c \lesssim 1$, the inner binary will eventually be outside its NCZ if a_{out} is sufficiently large. Indeed, as a_{out} increases, the inner binary’s CoM ends up leaving its own NCZ (which is centered on the origin of the coordinates, i.e. on the triplet’s CoM).

A simple fix to this issue, as shown in fig. 6, is thus to evaluate the multipole moments in an *inertial* reference frame with origin instantaneously coinciding with the

CoM of the *inner binary*⁺, which allows both the inner and outer binaries to lie within their respective NCZs. In the right panels of fig. 5, we show that this eliminates the unphysical behavior of fig. 1 and left panels of fig. 5. Therefore, the problem exhibited by those figures (and the corresponding results of fig. 18 in [19]) was simply that the waveforms were evaluated in the reference frame of the CoM of the triple system. The very same solution applies to the simpler binary cases reported in fig. 4. Here, since the CoM does not move, only a single transformation is needed. Once that transformation has been performed, the waveforms are given by the solid blue lines and reproduce the correct predicted result.

3. Green’s solution in linearized theory

As pointed out in the introduction, another often overlooked problem arises when gravitational waveforms are computed by direct integration of eq. 1 with the help of the retarded Green function. To understand it, let us go back to the textbook derivation of the quadrupole formula for GW generation. We start from the Einstein equations relaxed by the condition of harmonic coordinates (see, e.g., [54] §20.3, §36.9, or [6] §6.3 and [61] for the complete derivation):

$$\square_{\text{flat}} H^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta}, \quad (16)$$

where the pseudo-tensor $H^{\alpha\beta}$ is defined in terms of the Minkowski metric $\eta^{\alpha\beta}$ and the space-time (inverse) metric $g^{\alpha\beta}$ as

$$H^{\alpha\beta} \equiv \eta^{\alpha\beta} - (-g)^{1/2} g^{\alpha\beta}, \quad (17)$$

and satisfies the harmonic gauge condition, i.e., $\partial_\beta H^{\alpha\beta} = 0$. Moreover, the “effective” stress-energy pseudo-tensor $\tau^{\alpha\beta}$ is comprised of a contribution from the stress-energy tensor of matter, and a contribution $\Lambda^{\alpha\beta}$ from the non-linearities of the gravitational field, i.e.,

$$\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \frac{c^4}{16\pi G} \Lambda^{\alpha\beta}, \quad (18)$$

where

$$\Lambda^{\alpha\beta} = \frac{16\pi G}{c^4} (-g) t_{\text{LL}}^{\alpha\beta} + (\partial_\nu H^{\alpha\mu} \partial_\mu H^{\beta\nu} - \partial_\mu \partial_\nu H^{\alpha\beta} H^{\mu\nu}), \quad (19)$$

with $t_{\text{LL}}^{\alpha\beta}$ denoting here the Landau-Lifshitz pseudo-tensor [50]. As a consequence of the harmonic gauge condition, $\tau^{\alpha\beta}$ is also flat-space conserved, i.e., $\partial_\beta \tau^{\alpha\beta} = 0$.

By using the retarded Green function, we can now integrate eq. 16 and obtain its formal solution:

$$H^{\alpha\beta}(t, \mathbf{x}) = \frac{4G}{c^4} \int \frac{d^3 \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}'). \quad (20)$$

If the field point \mathbf{x} lies very far from the source, $|\mathbf{x} - \mathbf{x}'| \approx |\mathbf{x}| \equiv R$, and one can neglect the differences in retarded time among the source components by considering a single global retardation $t_{\text{ret}} = t - R/c$. This yields[†]

$$H^{\alpha\beta}(t, \mathbf{x}) \approx \frac{4G}{Rc^4} \int d^3 \mathbf{x}' \tau^{\alpha\beta}(t_{\text{ret}}, \mathbf{x}'). \quad (21)$$

⁺ Clearly, this reference frame cannot be co-moving with the CoM of the inner binary (which has a non-zero acceleration) and one has to consider a different inertial frame at each step of the system’s evolution.

[†] When neglecting the \mathbf{x}' term in the temporal dependency of the integral in eq. 20 and replacing the source $\tau^{\alpha\beta}$ by its PN expansion, the integral on the right-hand side of eq. 21 becomes formally

Let us now expand the metric up to $1/c^2$ corrections,

$$\begin{aligned} g_{00} &= -1 - 2\frac{\phi}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \\ g_{0i} &= \mathcal{O}\left(\frac{1}{c^3}\right), \\ g_{ij} &= \left(1 - 2\frac{\phi}{c^2}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{c^4}\right), \end{aligned} \quad (22)$$

so that the ij components are given at the 1PN order while the 00 and $0i$ components are Newtonian. The gravitational potential ϕ in eqs. 22 must satisfy the Poisson equation $\nabla^2\phi = 4\pi G\rho$, with ρ being the mass density, in order for ϕ to be a solution of the relaxed Einstein equations. We can see that, at this accuracy level, the metric is linear in the source, which implies

$$H^{\mu\nu} = \bar{h}^{\mu\nu} + \mathcal{O}\left(\frac{1}{c^4}, \frac{1}{c^3}, \frac{1}{c^4}\right), \quad (23)$$

where the three remainders in the arguments of the Landau symbol refer to the 00, $0i$ and ij components, respectively. From the flat-space conservation of $\tau^{\mu\nu}$, it then follows that (see [54] §36.10)‡

$$\partial_0\partial_0(\tau^{00}x^jx^k) = \partial_l\partial_m(\tau^{lm}x^jx^k) - 2\partial_l(\tau^{jl}x^k + \tau^{kl}x^j) + 2\tau^{jk}, \quad (24)$$

which allows one to recast the spatial part of eq. 21 as (see [54] §36.10)

$$\bar{h}^{ij}(t, \mathbf{x}) \approx \frac{2G}{Rc^4} \frac{d^2}{dt^2} \int d^3\mathbf{x}' c^{-2} \tau^{00}(t_{\text{ret}}, \mathbf{x}') x'^i x'^j + \mathcal{O}\left(\frac{1}{c^5}\right). \quad (25)$$

A priori, the PN expansion of the stress-energy pseudo-tensor $\tau^{\alpha\beta}$ defined by eq. 18 still involves both the matter term $T^{\alpha\beta}$ and the purely gravitational non-linear source term $\Lambda^{\alpha\beta}$. Modeling the matter system by point particles, i.e., taking (see [68] §2.8, [53] §3.3.5)

$$T^{\alpha\beta}(t, \mathbf{x}) = \sum_A \frac{m_A u_A^\alpha u_A^\beta}{(u_A^0/c) \sqrt{-g}} \delta^3(\mathbf{x} - \mathbf{x}_A(t)), \quad (26)$$

where m_A is the mass of particle A and u_A^α is its four-velocity, one finds

$$\tau^{00}(t, \mathbf{x}) = \sum_A m_A c^2 \delta^3(\mathbf{x} - \mathbf{x}_A(t)) + \mathcal{O}\left(\frac{1}{c^0}\right), \quad (27)$$

$$\tau^{0i}(t, \mathbf{x}) = \sum_A m_A c \dot{x}_A^i \delta^3(\mathbf{x} - \mathbf{x}_A(t)) + \mathcal{O}\left(\frac{1}{c}\right), \quad (28)$$

$$\begin{aligned} \tau^{ij}(t, \mathbf{x}) &= \sum_A m_A \dot{x}_A^i \dot{x}_A^j \delta^3(\mathbf{x} - \mathbf{x}_A(t)) \\ &\quad + \frac{1}{4\pi G} \left(\partial^i \phi \partial^j \phi - \frac{1}{2} \delta^{ij} \partial_k \phi \partial^k \phi \right) + \mathcal{O}\left(\frac{1}{c^2}\right), \end{aligned} \quad (29)$$

where δ^3 is the three-dimensional Dirac delta function and \dot{x}_A^i represents the components of the three-dimensional velocity. We observe that $\tau^{00} \approx T^{00}$ at leading PN order, hence eq. 25 coincides with the ‘‘usual’’ quadrupole formula,

$$\bar{h}^{ij}(t, \mathbf{x}) \approx \frac{2G}{Rc^4} \frac{d^2 M^{ij}}{dt^2}(t_{\text{ret}}) + \mathcal{O}\left(\frac{1}{c^5}\right). \quad (30)$$

divergent. This is one of the main problems that GW generation formalisms have to address. In the Blanchet-Damour-Iyer formalism [62, 63], this particular problem is solved by resorting to a combination of asymptotic matching techniques and a specific regularization procedure that cures those divergences [64–67].

‡ The very same identity holds for $T^{\mu\nu}$ in linearized theory, since $\partial_\mu T^{\mu\nu} = 0$.

By contrast, τ^{ij} contains a direct contribution from the gravitational field at Newtonian order, i.e., a term arising at the same order as T^{ij} . Therefore, the direct integration of eq. 1 will give an *incorrect* result, because the right-hand side is wrong already at the leading PN order.

To be more explicit, let us evaluate eq. 25 in the special case of a binary system ($A, B = 1, 2$) by using eq. 27 and the Newtonian equations of motion

$$\ddot{x}_A^i = - \sum_{B \neq A} \frac{Gm_B}{r_{AB}^2} n_{AB}^i, \quad (31)$$

where $\mathbf{r}_{AB} = \mathbf{x}_A - \mathbf{x}_B$, $r_{AB} = |\mathbf{x}_A - \mathbf{x}_B|$, and $\mathbf{n}_{AB} = \mathbf{r}_{AB}/r_{AB}$. Straightforward algebra yields

$$\bar{h}^{ij}(t, \mathbf{x}) \approx \frac{4G}{Rc^4} \left(m_1 \dot{x}_1^i \dot{x}_1^j + m_2 \dot{x}_2^i \dot{x}_2^j - \frac{Gm_1 m_2 n_{12}^i n_{12}^j}{r_{12}} \right) + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (32)$$

while direct integration of eq. 1 would lead to the different expression

$$\bar{h}^{ij}(t, \mathbf{x}) \approx \frac{4G}{Rc^4} \left(m_1 \dot{x}_1^i \dot{x}_1^j + m_2 \dot{x}_2^i \dot{x}_2^j \right) + \mathcal{O}\left(\frac{1}{c^5}\right). \quad (33)$$

The extra term in eq. 32 that is missing in eq. 33 is related to the purely gravitational part of the right-hand side of eq. 29 (i.e., the part involving the Newtonian potential ϕ), which is prematurely neglected in the source T^{ij} of eq. 1. This is the origin of the factor ~ 2 discrepancy shown in fig. 2.

Nonetheless, one can still obtain the correct expression without resorting to the identity 24. In fact, by substituting eq. 29 into the spatial components of eq. 21, one gets

$$\bar{h}^{ij}(t, \mathbf{x}) \approx \frac{4G}{c^4} \int \frac{d^3 \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \left[T^{ij} + \frac{1}{4\pi G} \left(\partial^i \phi \partial^j \phi - \frac{1}{2} \delta^{ij} \partial_k \phi \partial^k \phi \right) \right] + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (34)$$

which may be rewritten as (see Appendix B for details)

$$\bar{h}^{ij}(t, \mathbf{x}) \approx \frac{4G}{Rc^4} \int d^3 \mathbf{x}' T^{ij} - \frac{2G^2}{Rc^4} \int d^3 \mathbf{y}' d^3 \mathbf{y}'' \rho(\mathbf{y}') \rho(\mathbf{y}'') \frac{\hat{n}^i \hat{n}^j}{|\mathbf{y}' - \mathbf{y}''|} + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (35)$$

where $\hat{n}^k = (y'^k - y''^k)/|\mathbf{y}' - \mathbf{y}''|$. Focusing again on the case of two point particles, eq. 35 gives

$$\bar{h}^{ij}(t, \mathbf{x}) \approx \frac{4G}{Rc^4} \left(m_1 \dot{x}_1^i \dot{x}_1^j + m_2 \dot{x}_2^i \dot{x}_2^j - \frac{Gm_1 m_2 n_{12}^i n_{12}^j}{r_{12}} \right) + \mathcal{O}\left(\frac{1}{c^5}\right). \quad (36)$$

This expression agrees with eq. 32. In conclusion, the discrepancy shown in fig. 2 was simply due to neglecting the purely gravitational part of the source in the relaxed Einstein equations, which would be equivalent to assuming motion along straight lines for the binary components. In other words, the last term in eq. 32 accounts for the fact that motion does not take place on rectilinear trajectories, but instead along curved-spacetime geodesics.

4. Conclusions

In this paper, we have highlighted several subtleties that occur when applying the quadrupole and quadrupole-octupole formulae to numerically-integrated binary and triple systems. We have shown that, as expected, applying these formulae to a binary in a reference frame whose origin is displaced from the CoM of the system leads to unphysical spurious results, if the displacement exceeds the wavelength λ of the emitted GWs. This simply happens because implicit in the derivation of the (generalized) quadrupole formula is the assumption that the multipoles are defined in the NCZ [6], i.e., a region of size $\sim \lambda$ centered on the origin in which the binary is supposed to be contained. The same problem manifests itself in hierarchical triple systems, when the quadrupole and quadrupole-octupole formulae are applied in a reference frame centered on the triplet's CoM. The resulting unphysical behavior, which to the best of our knowledge has gone unrecognized in the literature [19], is in contrast with what happens for a binary system (where it is safe to define the multipoles in the CoM reference frame), but can be understood by bearing in mind that a hierarchical triplet may be decomposed in an inner binary and an outer one. Indeed, as the separation of the outer binary grows, the CoM of the inner binary will eventually move out of its NCZ, thus violating the assumptions on which the derivation of the quadrupole and quadrupole-octupole formulae relies.

We have described two remedies to this problem. When using the leading order quadrupole formula, it suffices to express the waveforms in terms of appropriate conserved quantities (namely the total linear momentum) to eliminate the observed spurious behavior. When using the quadrupole-octupole formula, the simplest approach is instead to compute the multipoles in an inertial frame whose origin instantaneously coincides with the CoM of the inner binary. Remarkably, neither of these two fixes seems to work for a system of *four* bodies§, for which a more sophisticated approach should be developed.

Finally, we have shown that, if one were to compute the GW emission from a binary or triple system by integrating directly the equations for the linear metric perturbations over flat space, one would obtain GW amplitudes that are wrong by a factor ~ 2 . We have found that this is related to the fact that the derivation of the quadrupole formula is quite subtle and actually requires one to use the *non-linear* Einstein equations.

Acknowledgments

The work of MB was supported in part by the ERC Project No. 267117 (DARK) hosted by Université Pierre et Marie Curie (UPMC) - Paris 6, PI J. Silk. MB acknowledges the CINECA award under the ISCRA initiative, for the availability of high performance computing resources and support. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 690904. For the numerical simulations, we have made use of the Horizon Cluster, hosted by the Institut d'Astrophysique de Paris. We thank Stephane Rouberol for running smoothly this cluster for us. AS is supported by the Royal Society.

§ For a system of two relativistic binaries at sufficiently large mutual separations, even if we set the origin in the CoM of one binary, the NCZ corresponding to the second binary will *not* contain the CoM of that binary, hence the "standard" PN formalism is not applicable.

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Appendix A. Definitions

In this appendix we summarize the explicit leading order expressions for the quadrupole and octupole mass radiation, as well as for the current quadrupole radiation.

The second and third mass moments are defined from the time-time component of the matter stress-energy tensor as

$$\begin{aligned} M^{ij}(t) &= \int d^3\mathbf{x} c^{-2} T^{00}(t, \mathbf{x}) x^{(i} x^{j)}, \\ M^{ijk}(t) &= \int d^3\mathbf{x} c^{-2} T^{00}(t, \mathbf{x}) x^{(i} x^j x^{k)}, \end{aligned} \quad (\text{A.1})$$

where $\langle \rangle$ represents the symmetric trace-free (STF) operator, i.e., (see [53] §3.5.1)

$$x^{(i} x^{j)} = x^i x^j - \frac{1}{3} \delta^{ij} r^2, \quad x^{(i} x^j x^{k)} = x^i x^j x^k - \frac{1}{5} (\delta^{ij} r^2 x^k + \delta^{ik} r^2 x^j + \delta^{jk} r^2 x^i). \quad (\text{A.2})$$

The current quadrupole moment is defined by

$$S^{ijk} = \int d^3\mathbf{x} [x^i j^{kj} + x^j j^{ki}], \quad (\text{A.3})$$

where

$$j^{ij} = \frac{1}{c} (x^i T^{0j} - x^j T^{0i}) \quad (\text{A.4})$$

is the angular momentum density tensor, which is connected to the angular momentum density vector by $j^{ij} = \varepsilon^{ij}{}_k j^k$. Alternatively, one may define the STF quadrupole tensor

$$J^{ij} = \text{STF}_{ij} \varepsilon_{abi} \int d^3\mathbf{x} x^{(j} x^{a)} T^{0b}, \quad (\text{A.5})$$

and replace S^{abk} by $-2\varepsilon_l{}^k{}_a J^{bl}$ in the waveform, using the fact that $S^{abk} n_k$ and $-2\varepsilon_l{}^k{}_a J^{bl} n_k$ have the same transverse trace-free part (with respect to \mathbf{n}). The advantage of working with J^{ij} is that it belongs to an irreducible representation of $\text{SO}(3)$.

The expression of the GW waveform up to the next-to-leading order in the TT gauge is finally given by

$$\begin{aligned} h_{ij}^{\text{TT}}(t, \mathbf{x}) &\approx \frac{2G}{Rc^4} \Lambda_{ijkl}(\mathbf{n}) \left[\frac{d^2 M^{kl}(t_{\text{ret}})}{dt^2} \right. \\ &\quad \left. + \frac{n_m}{3c} \frac{d^3 M^{klm}(t_{\text{ret}})}{dt^3} + \frac{2n_m}{3c} \frac{d^2 S^{klm}(t_{\text{ret}})}{dt^2} \right] + \mathcal{O}\left(\frac{1}{c^6}\right), \end{aligned} \quad (\text{A.6})$$

where the projector tensor $\Lambda_{ijkl}(\mathbf{n})$ is defined in terms of the GW propagation direction \mathbf{n} (see [54] §36.10 and Box 35.1):

$$\Lambda_{ijkl}(\mathbf{n}) = \mathcal{P}_{ik} \mathcal{P}_{jl} - \frac{1}{2} \mathcal{P}_{ij} \mathcal{P}_{kl}, \quad \mathcal{P}_{ij} = \delta_{ij} - n_i n_j. \quad (\text{A.7})$$

Appendix B. Calculations

In this appendix, we perform the explicit calculation of

$$\bar{h}^{ij}(t, \mathbf{x}) = \frac{4G}{c^4} \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \left[T^{ij} + \frac{1}{4\pi G} \left(\partial^i \phi \partial^j \phi - \frac{1}{2} \delta^{ij} \partial_k \phi \partial^k \phi \right) \right] + \mathcal{O}\left(\frac{1}{c^5}\right). \quad (\text{B.1})$$

Since the first term of eq. B.1 is trivial to evaluate because it has a compact support, we focus on the second one, where one is not a priori allowed to approximate $|\mathbf{x} - \mathbf{x}'|$ by R under the integral, since the integration extends up to spatial infinity. We perform the calculation by considering only the term $\partial^i \phi \partial^j \phi$. Indeed, $\delta^{ij} \partial_k \phi \partial^k \phi$, which is simply the trace of $\partial^i \phi \partial^j \phi$ multiplied by a Kronecker delta, disappears when taking the TT projection.

Inserting the expression for the Newtonian gravitational potential, we find

$$\begin{aligned} \frac{1}{4\pi G} \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \partial^i \phi \partial^j \phi &= \frac{G}{4\pi} \int \frac{d^3\mathbf{x}' d^3\mathbf{y}' d^3\mathbf{y}''}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{y}') \rho(\mathbf{y}'') \times \\ &\times \frac{\partial}{\partial x'^i} \left(\frac{1}{|\mathbf{x}' - \mathbf{y}'|} \right) \frac{\partial}{\partial x'^j} \left(\frac{1}{|\mathbf{x}' - \mathbf{y}''|} \right). \end{aligned} \quad (\text{B.2})$$

After transforming the derivative $\partial/\partial x'^i$ that acts on $|\mathbf{x}' - \mathbf{y}'|^{-1}$ into $-\partial/\partial y'^i$ by virtue of the translation invariance of $\mathbf{x}' - \mathbf{y}'$, and similarly for $\partial/\partial x'^j$, we may change the order of integration, so that $\partial^2/(\partial y'^i \partial y''^j)$ can be put outside the integral with respect to \mathbf{x}' . With this trick, eq. B.2 becomes

$$\frac{1}{4\pi G} \int \frac{d^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \partial^i \phi \partial^j \phi = G \int d^3\mathbf{y}' d^3\mathbf{y}'' \rho(\mathbf{y}') \rho(\mathbf{y}'') \frac{\partial^2}{\partial y'^i \partial y''^j} g(\mathbf{x}', \mathbf{y}', \mathbf{y}''), \quad (\text{B.3})$$

where g satisfies the Poisson equation $\Delta g(\mathbf{x}, \mathbf{y}', \mathbf{y}'') = |\mathbf{x} - \mathbf{y}'|^{-1} |\mathbf{x} - \mathbf{y}''|^{-1}$ in the sense of distributions. It is straightforward to check that the relevant solution is $g = \ln(|\mathbf{x} - \mathbf{y}'| + |\mathbf{x} - \mathbf{y}''| + |\mathbf{y}' - \mathbf{y}''|) + \text{constant}$ (see, e.g., p.355 in [69]), from which one infers the asymptotic behavior

$$\frac{\partial^2}{\partial y'^i \partial y''^j} K = -\frac{\hat{n}^i \hat{n}^j - \delta^{ij}}{2R|\mathbf{y}' - \mathbf{y}''|} + \mathcal{O}\left(\frac{1}{R^2}\right). \quad (\text{B.4})$$

At large distance R from the origin, eq. B.1 then reduces to

$$\bar{h}^{ij} = \frac{4G}{c^4 R} \int d^3\mathbf{x}' T^{ij} - \frac{2G^2}{Rc^4} \int d^3\mathbf{y}' d^3\mathbf{y}'' \rho(\mathbf{y}') \rho(\mathbf{y}'') \frac{\hat{n}^i \hat{n}^j}{|\mathbf{y}' - \mathbf{y}''|} + \mathcal{O}\left(\frac{1}{R^2 c^5}\right). \quad (\text{B.5})$$