Vorticity in analogue spacetimes

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Analogue spacetimes can be used to probe and study physically interesting spacetime geometries by constructing, either theoretically or experimentally, some notion of an effective Lorentzian metric $[g_{\mathrm{eff}}(g, V, \Xi)]_{ab}$. These effective metrics generically depend on some physical background metric g_{ab}, often flat Minkowski space η_{ab}, some "medium" with 4-velocity V^a, and possibly some additional background fields and parameters Ξ. (These might include signal propagation speeds and the like.) Analogue spacetimes based on electromagnetic media date back to Gordon's work in the 1920s, analogue spacetimes based on acoustics in fluids date back to Unruh's work in the 1980s, and BEC-based analogue spacetimes date back to various authors in the 1990s. The analogue spacetimes based on acoustic propagation in bulk fluids have perhaps the most rigorous mathematical formulation, and these acoustics-based analogue models really work best in the absence of vorticity, when the medium has an irrotational flow. This physical restriction makes it difficult to mimic the particularly interesting case of rotating astrophysical spacetimes, spacetimes with nonzero angular momentum, and in the current article we explore the extent to which one might hope to be able to develop an analogue model for astrophysical spacetimes with angular momentum (thereby implying vorticity in the 4-velocity of the medium). We shall focus on two particular analogue models: (1) the use of a charged BEC as the background medium, where new results concerning the interplay between healing length and London penetration depth are a key technical improvement, and (2) new results regarding the Gordon metric associated with an isotropic fluid medium.

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I. INTRODUCTION

Analogue spacetimes have an almost century-long, complex, turbulent, and quite checkered history; see [1–4]. With hindsight, Gordon's 1923 paper [5], wherein he explored what would now be called analogue spacetimes based on electro-magnetic media, was considerably more important and insightful than it may initially have seemed at the time. The idea of electromagnetic analogue spacetimes—precisely what distribution of permittivity ϵ and permeability μ (and possibly magnetoelectric ζ effects) can be used to mimic a classical gravitational field—subsequently became one of the exercises in the Landau-Lifshitz volume on classical field theory [6]. Scientific interest in these electromagnetic-based analogue spacetimes is both extensive and ongoing; see for instance [7–19], and references therein.

In counterpoint, in 1981 Unruh developed acoustics-based analogue spacetimes (subsequently called dumb holes, "dumb" in the sense of "mute") [20], with further developments due to one of the present authors [21–23]. While the early acoustic models were based on ordinary barotropic fluid mechanics, much subsequent work was based on more general fluids, and in particular on the "Madelung fluid" interpretation of a quantum condensate wave-function—typically (though not always) in a non-relativistic or relativistic BEC [24–32].

Many other models of analogue spacetimes have subsequently been developed, see for instance the surveys [1–4]. Analogue spacetime models have been applied in mimicking several interesting spacetimes and phenomena therein—up to and including the emission of Hawking quanta from analogue horizons.

However, angular momentum in the physical spacetime to be mimicked corresponds to vorticity in the flow of the medium used in setting up the analogue. This can be established in a number of ways, ranging from explicit computations in toy models (such as $2+1$-dimensional BTZ black holes), or considering equatorial slices of $3+1$-dimensional Kerr black holes, to working in the far-field limit where the Lense-Thirring metric is a good approximation. See for instance the specific Refs. [33–41] and the more general background Refs. [1–4].

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This raises the question of just how one might introduce vorticity, and so angular momentum, into analogue spacetimes. This question is considerably trickier than one might naively think, especially if one demands a mathematically and physically clean and well-controlled formalism.

Certainly there is no difficulty at the level of ray optics or ray acoustics where the "propagation cones" are quite flexible:

$$(g_{\mathrm{eff}})_{ab} \propto g_{ab} + [1 - c^2_{\mathrm{propagation}}]V_a V_b. \tag{1}$$

Here V^a is the 4-velocity of the medium, while $c_{\mathrm{propagation}}$ is the propagation speed of whatever signal one is interested in. Because "propagation cones," at best, describe the effective metric up to an unknown and *unknowable* (even in principle) conformal factor, in the ray optics or ray acoustics limit one can at best determine an effective metric up to proportionality [1–4]. (In this eikonal limit, see also [42].) However, significant subtleties arise when one wishes to derive a wave equation suitable for investigating wave optics or wave acoustics [1–4].

For instance, the simplest of the nonrelativistic [20–22], and relativistic [23] acoustic models, (which is where we have the best mathematical control over the formalism), were explicitly constructed to be vorticity free. In contrast the nonrelativistic vorticity-supporting acoustic model developed in [43] was somewhat more complex, requiring the use of Clebsch potentials. A more recent 2015 model [44,45] used charged BECs (both nonrelativistic and relativistic, in the usual acoustic limit of the Gross-Pitaveskii equation, where the quantum potential is neglected). In these charged BEC models we shall show below that there is a subtle and nontrivial interplay between the healing length and the London penetration depth.

That one might strongly desire to add vorticity to a wide class of analogue spacetimes is driven by both theoretical and experimental issues: Certainly sound propagates on physical vortex flows, and it would be highly desirable to have a suitable well-controlled mathematically precise wave-equation that goes beyond the Pierce approximation [43,46]. As we shall see later on, the Pierce approximation allows one to derive an approximate wave equation for sound propagation in an inhomogeneous fluid by assuming that the characteristic length and time scales for the ambient medium are larger than the corresponding scales for the acoustic disturbance. Under these conditions, which impose a "separation of scales", the system can be described by an approximate wave equation that is correct to first order in the derivatives of the ambient background quantities.

Certainly the Kerr congruence in the physically important rotating Kerr black hole has nonzero vorticity [33–41], so any analogue model of the Kerr spacetime (or indeed any spacetime with nonzero angular momentum) will need to include some notion of vorticity. In view of these comments, key novel points of the current article will be to better understand the notion of vorticity both in the charged BEC models [44,45] and in Gordon's 1923 model [5]. Indeed, below we shall explicitly verify that the Gordon metric associated with an isotropic fluid medium can support relativistic vorticity.

II. THE GORDON METRIC

Gordon's 1923 effective metric is

$$(g_{\mathrm{eff}})_{ab} = \eta_{ab} + [1 - n^{-2}]V_a V_b. \tag{2}$$

In Gordon's original article, (a condensed-matter model), the refractive index n (and ϵ and μ) and the 4-velocity of the medium V^a were typically (but not always) taken to be position-independent constants. The refractive index n (and ϵ and μ) were always taken to be isotropic. Furthermore, Gordon was often working in the ray optics (eikonal) limit, and mostly assuming a flat Minkowski space as background. Somewhat oddly, Gordon did not seem to recognize the need for the now well-known consistency condition: $\epsilon = \mu = n$, (plus additionally setting the magnetoelectric effect ζ to zero) [13,17–19]. Various generalizations to Gordon's model that one might consider include: (i) Fully general position-dependent $n(x)$ and $V^a(x)$; with considerable hindsight this is implicit but not really explicit in Gordon's 1923 article. (ii) Introducing nonzero vorticity for the background flow. (iii) Going beyond the eikonal (ray) approximation to consider wave physics. (iv) Introducing a nontrivial background metric (going beyond Minkowski space).

III. THE UNRUH METRIC

The Unruh 1981 metric for acoustic perturbations in an irrotational barotropic inviscid fluid can be given in ADM-like form [20]:

$$(g_{\mathrm{eff}})_{ab} = \frac{\rho_0}{c_s}\left[\begin{array}{c|c} -[c_s^2 - v^2] & -v_j \\ \hline -v_i & \delta_{ij} \end{array}\right]. \tag{3}$$

Here v_i is the 3-velocity of the fluid, ρ_0 is its density, and c_s is the speed of sound. The irrotational condition $\nabla \times v = 0$ (and hence $v = \nabla\Phi$) is built in as the very first equation of [20]. (This vorticity-free assumption continues to hold in Refs. [21,22].)

IV. EXTENSIONS BEYOND UNRUH 1981

Various partial extensions of the acoustic analogue models include:

A. Relativistic barotropic irrotational acoustics

The metric here is best presented in Gordon-like form [23]

$$(g_{\text{eff}})_{ab} = \Omega^2(g_{ab} + [1 - c_s^2]V_a V_b); \qquad V_a = \frac{\nabla_a \Phi}{\|\nabla\Phi\|}. \tag{4}$$

Here c_s is the speed of sound, while g_{ab} is an arbitrary background metric. The conformal factor Ω is a specific known but somewhat messy function of the barotropic equation of state $\rho(p)$, the baryon number density $n(\rho)$, and the speed of sound $c_s^2 = dp/d\rho$. Specifically, in terms of background quantities,

$$\Omega^2 = \frac{n_0^2}{c_s(\rho_0 + p_0)}. \tag{5}$$

The only significant physics restriction on this metric is that the background 4-velocity V_a is irrotational, that is vorticity-free, in the relativistic sense that the 4-velocity satisfies $V_{[a}V_{b,c]} = 0$.

B. Nonrelativistic barotropic rotational acoustics

The central idea here is to introduce vorticity through the use of Clebsch potentials. (See [43] and references therein.) Any 3-vector field can be put in the form

$$v = \nabla\phi + \beta\nabla\gamma. \tag{6}$$

One then decomposes the fluid 3-velocity into background plus (small) perturbation,

$$v = v_0 + v_1, \qquad v_1 = \nabla\psi + \xi, \tag{7}$$

with

$$\psi = \phi_1 + \beta_0\gamma_1, \qquad \xi = \beta_1\nabla\gamma_0 - \gamma_1\nabla\beta_0. \tag{8}$$

The analysis then leads to a *system* of PDEs. The effective metric is again of ADM-like form

$$(g_{\text{eff}})_{ab} = \frac{\rho_0}{c_s}\left[\begin{array}{c|c} -[c_s^2 - v^2] & -v_j \\ \hline -v_i & \delta_{ij} \end{array}\right], \tag{9}$$

with inverse

$$(g_{\text{eff}}^{-1})^{ab} = \frac{1}{\rho_0 c_s}\left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c_s^2\delta_{ij} - v^i v^j \end{array}\right]. \tag{10}$$

(Spatial indices are raised and lowered using the flat Euclidean metric δ_{ij}.) Then the scalar perturbation ψ satisfies a wavelike PDE

$$\Box_g \psi = -\frac{c_s}{\rho_0^2}\nabla\cdot(\rho_0\xi), \tag{11}$$

where $\Box_g$ is the d'Alembertian wave operator for the metric $(g_{\text{eff}})_{ab}$, whereas in terms of the advective derivative, the vector perturbation ξ satisfies the PDE

$$\frac{d\xi}{dt} = \nabla\psi \times \omega_0 - (\xi\cdot\nabla)v_0. \tag{12}$$

Thus the rotational part of the velocity perturbation, the vector ξ, acts as a source for the wave equation for ψ. Conversely the background vorticity, ω_0, helps drive the evolution of the rotational part of the velocity perturbation, ξ.

In this context, Pierce's approximation [46] amounts to asserting that the background velocity gradients ∇v_0 be much smaller than the frequency of the wave, which automatically implies that the background vorticity ω_0 is much smaller than the frequency of the wave. Under those conditions Pierce argues that both the rotational part of the velocity perturbation and its gradient, (both ξ and $\nabla\xi$), will always remain small and can safely be neglected [43,46]. Under those circumstances one obtains an *approximate* wave equation

$$\Box_g \psi \approx 0. \tag{13}$$

Generally though, one simply has to keep track of the extra complications coming from ξ, the rotational part of the velocity perturbation.

In summary, this analogue model certainly exhibits an effective metric, and the effective metric certainly can support background vorticity, but the model is "contaminated" by the presence of the extra field ξ, which complicates any attempt at setting up a clean "fully geometric" interpretation.

C. Charged nonrelativistic BECs

The metric here is given in ADM-like form [44,45]:

$$(g_{\text{eff}})_{ab} = \frac{\rho_0}{c_s}\left[\begin{array}{c|c} -[c_s^2 - v^2] & v_j \\ \hline v_i & \delta_{ij} \end{array}\right]; \tag{14}$$

with the 3-velocity of the effective Madelung fluid now defined by

$$v_i = \nabla_i\Phi - \frac{eA_i}{mc}. \tag{15}$$

The BEC wavefunction is $\Psi = \sqrt{\rho_0}e^{i\Phi} = \|\Psi\|e^{i\Phi}$, while c_s is the speed of sound in the condensate, and the purely spatial 3-vector $v \propto \nabla\Phi - eA$ is gauge invariant. The key point is that $\nabla_i\Phi - eA_i$ makes sense only if one has a condensate. This would, in principle, seem to allow the background flow to have some vorticity while keeping the perturbations irrotational. Specifically the vorticity would be

$$\omega = \frac{eB}{mc}. \tag{16}$$

Unfortunately one also has $v \propto j_{\text{London}}$, the so-called London current that is central to the analysis of the

Meissner effect. Indeed, there is widespread agreement within the condensed matter community that *any* charged BEC, (not just a BCS superconductor, where formation of the Cooper pairs, and condensation of the Cooper pairs are essentially simultaneous), will exhibit the Meissner effect—magnetic flux expulsion. See, for instance, [47–49]. This would naively seem to confine any vorticity to a thin layer of thickness comparable to the London penetration depth. However, there is a trade-off between the healing length (which controls the extent to which one can trust the effective metric picture) and the London penetration depth.

Let us be more quantitative about this: The London penetration depth λ and healing length ξ are given by

$$\lambda = \sqrt{\frac{m}{\mu_0 n q^2}}; \qquad \xi = \frac{1}{\sqrt{8\pi n a}}. \tag{17}$$

Here m is the mass of the atoms making up the charged BEC, μ_0 is the magnetic permeability in vacuum, and now n is the number density of atoms in the condensate; $q = Qe$ is the charge of each atom, and a is the scattering length. In particular, for the ratio of penetration depth to healing length the number density n cancels and using $\epsilon_0\mu_0 = 1/c^2$ we have

$$\frac{\lambda}{\xi} = \sqrt{\frac{8\pi a m}{\mu_0 q^2}} = \sqrt{\frac{8\pi\epsilon_0 c^2 a m}{q^2}}. \tag{18}$$

Write $q = Qe$ and $m = Nm_p$, where N is the atomic mass number. Then

$$\frac{\lambda}{\xi} = \frac{\sqrt{N}}{Q}\sqrt{\frac{2am_p c}{\alpha\hbar}}. \tag{19}$$

Then in terms of the Bohr radius, $a_0 = \alpha^{-1}\hbar/(m_e c)$, one has

$$\frac{\lambda}{\xi} = \frac{\sqrt{N}}{Q}\frac{1}{\alpha}\sqrt{\frac{2m_p}{m_e}}\sqrt{\frac{a}{a_0}}. \tag{20}$$

We know that $\sqrt{2m_p/m_e} \approx 60$. For a heavy-atom charged BEC $\sqrt{N}/Q \approx 9$ and typically $a \approx 100a_0$. Then

$$\frac{\lambda}{\xi} \approx 750000. \tag{21}$$

We can, in principle, make this ratio even larger, simply by tuning to a Feshbach resonance to increase the value of the scattering length a:

$$\frac{\lambda}{\xi} \approx 750000\sqrt{\frac{a_{\text{with resonance}}}{a_{\text{without resonance}}}}. \tag{22}$$

So there is a significant separation of scales between healing length and London penetration depth, which can be made even larger by tuning to a Feshbach resonance. Demonstrating this separation of scales is a key novelty in the present article. The net outcome of this discussion is that despite potential problems due to the Meissner effect there is a parameter regime in which we can simultaneously have vorticity penetrate deep into the bulk and still trust the effective metric formalism.

Another interesting and novel feature of this nonrelativistic construction (not commented on previously) is that even if the background has vorticity, the perturbations are vorticity-free. This is a side effect of the Madelung representation, (and the approximation of neglecting the quantum potential). One again takes

$$\psi_{\text{total}} = \Psi(1+\psi_{\text{perturbation}}) = \sqrt{\rho_0}e^{i\Phi}(1+\psi_{\text{perturbation}}), \tag{23}$$

and obtains a d'Alembertian PDE for $\psi_{\text{perturbation}}$. So in contrast to the previous vorticity supporting acoustics model, no extra fields need to be introduced.

D. Charged relativistic BECs

The metric here is best presented in Gordon-like form [44,45],

$$(g_{\text{eff}})_{ab} = \frac{\rho_0}{c_s}(\eta_{ab} + [1 - c_s^2]V_a V_b), \tag{24}$$

with

$$V_a = \frac{\nabla_a\Phi - eA_a}{\|\nabla\Phi - eA\|}. \tag{25}$$

It is now the RBEC wavefunction that is written in terms of the Madelung representation $\Psi = \sqrt{\rho_0}e^{i\Phi}$, while c_s is the speed of sound in the condensate, and the 4-velocity $V \propto \nabla\Phi - eA$ is gauge invariant. This (formally) allows the background flow to have some vorticity while keeping the perturbations irrotational. Specifically for the 4-vorticity we have

$$\epsilon_{abcd}\omega^d = V_{[a}V_{b,c]} = e\frac{V_{[a}F_{bc]}}{\|\nabla\Phi - eA\|}. \tag{26}$$

Working in the rest frame of the fluid, we see

$$\|\omega\| = \frac{e\|B\|}{\|\nabla\Phi - eA\|}. \tag{27}$$

The key point is that $\nabla_a\Phi - eA_a$ is a gauge invariant 4-vector field that makes sense only if one has a condensate.

The same potentially problematic issue regarding the Meissner effect also arises in this relativistic setting. The 4-velocity now satisfies $V \propto \nabla\Phi - eA \propto J_{\text{London}}$, where this is now the London 4-current $J_{\rm London} = (\rho_{\rm London}, j_{\rm London})$. Naively, the magnetic field (and hence the vorticity) will be confined to a thin transition layer of thickness comparable to the London penetration depth. However the same parameter regime as was considered for the nonrelativistic case will still apply in the full relativistic setting: One can drive the London penetration depth large while holding the healing length constant. This separation of scales is a key novelty in the present article.

V. EXTENSIONS BEYOND GORDON 1923

For linear constitutive ϵ-μ-ζ electrodynamics, a recent fundamental result for the effective metric is [13] that when standard consistency constraints are imposed

$$\epsilon^{ab} = \mu^{ab}; \qquad \zeta^{ab} = 0; \tag{28}$$

the effective metric can be written as

$$(g_{\rm eff})_{ab} = -\sqrt{\frac{-\det(g^{\bullet\bullet})}{\mathrm{pdet}(\epsilon^{\bullet\bullet})}} V_a V_b + \sqrt{\frac{\mathrm{pdet}(\epsilon^{\bullet\bullet})}{-\det(g^{\bullet\bullet})}} [\epsilon^{\bullet\bullet}]^{\#}_{ab}, \tag{29}$$

Here V^a is the 4-velocity of the medium, and one has generalized Gordon 1923 by allowing for nontrivial permittivity and permeability tensors. These are both symmetric and transverse in the sense that $\epsilon^{ab} V_b = 0 = \mu^{ab} V_b$, so that in the rest frame of the medium they reduce to 3×3 symmetric tensors. No constraint is put on the 4-velocity of the medium, apart from the minimal fact that it be timelike in the background metric [13].

Note specifically the pseudodeterminant [13] and Moore-Penrose pseudoinverse appearing above. The pseudodeterminant $\mathrm{pdet}(X)$ is simply the product over nonzero eigenvalues; sometimes one sees notation such as $\det'(X)$. For symmetric matrices X, the Moore-Penrose pseudoinverse $X^{\#}$ simplifies to diagonalizing the matrix, taking the reciprocal of the nonzero eigenvalues, and then undoing the diagonalization. See, for instance, [13–16] and references therein. Specifically, the notation $[\epsilon^{\bullet\bullet}]^{\#}_{ab}$ means that one should take the 4×4 contravariant matrix ϵ^{ab}, which is a singular matrix due to the transversality conditions, and construct its Moore-Penrose pseudoinverse, which is also a singular 4×4 matrix but now with covariant indices.

The formalism was carefully set up so that there is no constraint on the background 4-velocity; it can, in principle, be arbitrary [13–16]. The formalism was also carefully set up so that there is no constraint on the background 4-geometry; it can, in principle, be arbitrary [13–16]. We now use this generality to specialize to a particularly interesting subcase.

If we now assume an isotropic medium, then

$$\epsilon_{ab} = \epsilon(g_{ab} + V_a V_b); \tag{30}$$

$$\mu_{ab} = \mu(g_{ab} + V_a V_b); \tag{31}$$

$$\zeta_{ab} = \zeta(g_{ab} + V_a V_b). \tag{32}$$

The compatibility condition then reduces to

$$\epsilon = \mu = n; \qquad \zeta = 0; \tag{33}$$

so that

$$(g_{\rm eff})_{ab} = n^{3/2} V_a V_b + n^{1/2}(g_{ab} + V_a V_b). \tag{34}$$

That is:

$$(g_{\rm eff})_{ab} = \sqrt{n}\{g_{ab} + [1 - n^{-2}] V_a V_b\} \tag{35}$$

The $\sqrt{n}$ pre-factor is completely conventional; its presence is simply due to the conformal invariance of electromagnetism in $(3+1)$ dimensions and the convenient demand that $\det[(g_{\rm eff})_{ab}] = \det[g_{ab}]$; see [13].

We could just as well write

$$(g_{\rm eff})_{ab} = \Omega^2\{g_{ab} + [1 - n^{-2}] V_a V_b\}. \tag{36}$$

The conformal factor Ω^2 is arbitrary, the background metric g_{ab} is arbitrary, the refractive index n is also arbitrary (subject only to the consistency condition $\epsilon = \mu = n$), and finally the 4-velocity V_a is arbitrary. Certainly, in principle, any arbitrary nonzero background vorticity is allowed. This settles the main physics issue—we have demonstrated that there is no deep physics obstruction to putting vorticity into the Gordon metric at the level of wave optics.

VI. DISCUSSION

While, as we have seen, introducing vorticity into analogue models at the level of ray optics or ray acoustics (the eikonal limit) is straightforward, even trivial, the situation at the level of wave optics or wave acoustics is considerably more subtle. Fortunately, we have now demonstrated that for wave optics the Gordon metric (now suitably generalized and placed in a more up-to-date context) provides a suitable model. We have also demonstrated that for realistic charged BECs (both nonrelativistic and relativistic) there is a significant separation of scales between the London penetration depth and the healing length, allowing the introduction of vorticity into these charged-BEC-based analogue spacetimes.

There are of course many additional relevant articles on related topics from within the astrophysics, condensed matter, and optics communities. See, for instance, [42] and the extensive list of references in [1]. We have unavoidably had to be somewhat selective in our selection of references. Relatively recent developments include the notions of "quantum vorticity" [50–52] and "holographic vorticity" [53–56].

Taken as a whole, these novel observations collectively give us confidence that it is likely to be possible to mimic the Kerr solution at the wave optics or wave acoustics level—presumably through some "Kerr-Gordon" form of the metric. It is already known that the Schwarzschild metric can be put into Gordon form [45,57]:

$$g_{ab} = \sqrt{n}(\eta_{ab} + [1 - n^{-2}]V_a V_b); \tag{37}$$

$$V_a = (-\sqrt{1 + 2m/r}; \sqrt{2m/r}\hat{r}_i). \tag{38}$$

Here n is an arbitrary position-independent constant, V_a is a 4-velocity, and the parameter m is proportional to the physical mass. The overall conformal factor $\sqrt{n}$ in the metric enforces $\det(g) = -1$. It is easy to check that this metric this is Ricci flat. A similar "Kerr-Gordon" construction for the Kerr spacetime would be very interesting. (So far there has only been limited perturbative progress in the slow-rotation and near-null limits [41].) Indeed looking for such a construction is largely the reason we became interested in the ideas presented in the current article.

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