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Generating Functions for K-Theoretic Donaldson Invariants and their Refinements

Candidate:
Runako A. Williams

Supervisor:
Prof. Lothar Göttsche

Thesis submitted in partial fulfilment of the requirements for the degree of "Doctor Philosophiæ" in Geometry and Mathematical Physics

Academic Year 2017/2018

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Abstract. In this thesis, we study K-theoretic Donaldson invariants, which are the holomorphic Euler characteristics of Donaldson line bundles on moduli spaces of sheaves on algebraic surfaces. Let S be a smooth projective surface with $b_1(S) = 0$ and $p_g(S) > 0$. Let $M_S^H(c_1, c_2)$ denote the moduli space of rank 2 torsion-free Gieseker H-semistable coherent sheaves on S. If L is a line bundle on S, we denote the corresponding Donaldson line bundle on $M = M_S^H(c_1, c_2)$ by $\mu(L)$. The Donaldson invariants of S are the (virtual) intersection numbers of powers of the first Chern class $c_1(\mu(L))$ on M. The corresponding K-theoretic Donaldson invariants are instead the holomorphic Euler characteristics $\chi^{vir}(M_S^H(c_1, c_2), \mu(L))$.

The famous Witten conjecture gives a generating function for the Donaldson invariants in terms of the Seiberg-Witten invariants of S. We formulate a K-theoretic version of the Witten conjecture, namely a conjectural generating function for the K-theoretic Donaldson invariants of S, again in terms of the Seiberg-Witten invariants of S. By replacing the (virtual) holomorphic Euler characteristics by the χ_{-y}-genus with values in $\mu(L)$ we obtain a refinement of this invariant for which we also find a conjectural formula.

The conjectures were obtained and verified by the use of Mochizuki's Formula, which reduce the computation of the invariants to integrals over products of Hilbert schemes, which we compute via Atiyah-Bott equivariant localization. We define a partition function (from which the invariants can be computed), and show that it satisfies a universality and multiplicativity property which allows us to reduce the computation to 11 specific cases. We use this to verify our conjectures in many explicit cases.

Introduction

Moduli spaces of sheaves on algebraic surfaces have been studied for a long time (see e.g. [**13**],[**25**],[**26**] and also [**21**] and references therein). A particular source of interest became the Donaldson invariants [**2**], which are invariants of differentiable 4-manifolds X, defined via moduli spaces of anti-self-dual connections on principal $SU(2)$ and $SO(3)$ bundles on X. It was quite difficult to make explicit computations with these moduli spaces. However in the case that the 4 manifold X is an algebraic surface S they can be computed as intersection numbers on moduli spaces of rank 2 torsion free sheaves $M_S^H(c_1, c_2)$ on S with Chern classes c_1, c_2 ([**24**],[**31**],[**28**]). In fact they can be computed as top self-intersection numbers of the first Chern classes of so-called Donaldson line bundles $\mu(L)$ on $M_S^H(c_1, c_2)$, associated to line bundles L on S.

The subject was revolutionized with the advent of the Seiberg-Witten invariants ([**35**],[**32**]). These are new differentiable invariants of 4-manifolds defined via moduli spaces of monopoles, which are much easier to handle than moduli spaces of anti-self-dual connections.

Let $b_+(M)$ be the number of positive eigenvalues of the intersection form on the middle cohomology of the 4-manifold M. Under the assumption that $b_+(M) > 1$, Witten conjectured in [**35**] an explicit formula which expresses the Donaldson invariants in terms of the Seiberg-Witten invariants. There has been a lot of work trying to prove this conjecture. In particular in a series of papers (e.g. [**7**],[**8**],[**9**],[**10**]) Feehan and Leness work towards a proof of this Witten conjecture for differentiable 4-manifolds. They prove it in a number of cases and they give a general proof modulo some technical conjectures.

If S is an algebraic surface, then $b_+(S) = 2p_g(S)+1$, where $p_g(S) = \dim(H^0(S, K_S))$ is the geometric genus of S. The Mochizuki formula of [**27**] allows to compute intersection numbers on moduli spaces $M_S^H(c_1, c_2)$ of rank 2 sheaves on S in terms of Seiberg-Witten invariants and intersection numbers on Hilbert schemes of points. Using this result a complete proof of the Witten conjecture for algebraic surfaces was given in [**19**].

The aims of this thesis are the following.

(1) Formulate a K-theoretic version of the Witten conjecture as a formula for the generating function for the (virtual) holomorphic Euler characteristics of Donaldson line bundles $\mu(L)$ on the moduli spaces $M_S^H(c_1, c_2)$.

(2) Formulate a refinement of this K-theoretic Witten conjecture, where the (virtual) holomorphic Euler characteristics are replaced by the χ_{-y}-genus with values in $\mu(L)$.

(3) Give a number of consequences of these conjectures and relate them to other results and conjectures in the field.

(4) Show these conjectures in many cases, thus giving ample evidence for its validity in general.

To be able to state our results, we first briefly review the definition of the Donaldson invariants and the statement of the Witten conjecture in the case of algebraic surfaces. We assume for simplicity that $M_S^H(c_1, c_2)$ only consists of stable sheaves (see Chapter 1, Section 3) and there is a universal sheaf $\mathcal{E}$ on $S \times M_S^H(c_1, c_2)$. Then $M_S^H(c_1, c_2)$ has an obstruction theory of virtual dimension

$$\mathrm{vd} = \mathrm{vd}(S, c_1, c_2) = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$$

and a virtual fundamental class $[M_S^H(c_1, c_2)]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M_S^H(c_1, c_2), \mathbb{Z})$ (see Chapter 1, Section 5). Let

$$p : S \times M_S^H(c_1, c_2) \to M_S^H(c_1, c_2), \quad q : S \times M_S^H(c_1, c_2) \to S$$

be the projections. For a class $\alpha \in H^2(S, \mathbb{Z})$, let

$$\nu(\alpha) := (c_2(\mathcal{E}) - c_1(\mathcal{E})^2/4)/PD(\alpha) \in H^2(M_S^H(c_1, c_2), \mathbb{Q}).$$

The corresponding Donaldson invariant is

$$\Phi_{S,c_1}^H(\alpha^{\mathrm{vd}}) = \int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} \nu(\alpha)^{\mathrm{vd}}.$$

Now assume that $b_1(S) = 0$ and $p_g(S) > 0$. Let $SW : H^2(S, \mathbb{Z}) \to \mathbb{Z}$ be the Seiberg-Witten invariants (see Chapter 1, Section 8.1). A class $w \in H^2(S, \mathbb{Z})$ is called a Seiberg-Witten class if $SW(w) \neq 0$. We will use Mochizuki's convention for Seiberg-Witten invariants, and denote by $\widetilde{SW}$ the standard convention for Seiberg-Witten invariants. Then $SW(w) = \widetilde{SW}(2w - K_S)$. There are only finitely many Seiberg-Witten classes on S.

We write $(\alpha)^2 := \int_S \alpha^2$, $(w\alpha) := \int_S w\alpha$. Then a slightly simplified version of the Witten conjecture (by ignoring the point class) for algebraic surfaces proved in [**19**] is the following.

Theorem 1 (Witten conjecture for algebraic surfaces). *Let S be a smooth projective algebraic surface with $b_1(S) = 0$ and $p_g(S) > 0$. Then*

$$\Phi_{S,c_1}^H\Big(\frac{\alpha^{\mathrm{vd}}}{\mathrm{vd}!}\Big) = \mathrm{Coeff}_{x^{\mathrm{vd}}}\Bigg[2^{2-\chi(\mathcal{O}_S)+K_S^2}(-1)^{\chi(\mathcal{O}_S)+c_1(c_1-K_S)/2} \exp\Big(\frac{(\alpha)^2}{2}x^2\Big)$$
$$\cdot \sum_{w \in H^2(S,\mathbb{Z})} (-1)^{c_1 w} SW(w) \exp\Big(((2w - K_S)\alpha)x\Big)\Bigg].$$

If $\alpha = c_1(L)$ for a line bundle $L \in \mathrm{Pic}(S)$, the Donaldson line bundle $\mu(L) \in \mathrm{Pic}(M_S^H(c_1, c_2))$ associated to L is a line bundle with $c_1(\mu(L)) = \nu(L)$. A K-theoretic version of the Witten conjecture will therefore be a formula for the generating function of the (virtual) holomorphic Euler characteristic $\chi^{\mathrm{vir}}(M_S^H(c_1,c_2)),\mu(L))$.

We obtain the following conjecture.

Conjecture 1. *Let S be a smooth projective surface with $p_g(S)>0$, $b_1(S)=0$, and $L\in \operatorname{Pic}(S)$. Let H,c_1,c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1,c_2. Then $\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L))$ equals the coefficient of x^{vd} of*

$$\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\frac{(L-K_S)^2}{2}+\chi(\mathcal{O}_S)}}\sum_{w\in H^2(S,\mathbb{Z})} SW(w)\,(-1)^{wc_1}\left(\frac{1+x}{1-x}\right)^{\left(\frac{K_S}{2}-w\right)(L-K_S)}.$$

Using the virtual Riemann-Roch Theorem 5 from [**1**], one can show that

$$\lim_{t\to\infty}\left(\frac{1}{t^{\mathrm{vd}}}\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(tL))\right)=\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}}\frac{c_1(\mu(L))^{\mathrm{vd}}}{\mathrm{vd}!},$$

and it is not difficult to check from this that Conjecture 1 implies the Witten conjecture (Theorem 1) for algebraic surfaces.

If the canonical linear system $|K_S|$ contains a smooth connected curve, the only Seiberg-Witten classes of S are 0 and K_S. In this case we obtain a simplified version of the above conjecture.

Proposition 1. *Let S be a smooth projective surface satisfying $p_g(S)>0$, $b_1(S)=0$, $K_S\neq 0$, and such that its only Seiberg-Witten basic classes are 0 and K_S. Let $L\in\operatorname{Pic}(S)$ and let H,c_1,c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1,c_2. Suppose Conjecture 1 holds in this setting. Then $\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L))$ is given by the coefficient of x^{vd} of*

$$2^{3-\chi(\mathcal{O}_S)+K_S^2}\frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}}.$$

Now we come to our refined version of the K-theoretic Witten conjecture, which also generalizes the refinement of [**15**] of the Vafa-Witten conjecture of [**34**].

A natural refinement of the holomorphic Euler characteristic $\chi(X,L)$ of a line bundle L on a smooth projective variety X of dimension d is $\chi_{-y}(X,L)$, the χ_{-y}-genus with values in L. The χ_{-y}-genus of X is

$$\chi_{-y}(X)=\sum_{p=0}^{d}(-y)^p\chi(X,\Omega_X^p),$$

and the χ_{-y}-genus with values in L is defined by

$$\chi_{-y}(X,L)=\sum_{p=0}^{d}(-y)^p\chi(X,\Omega_X^p\otimes L).$$

We replace the χ_{-y}-genus and the $\chi_{-y}-$genus with values in L by their virtual versions $\chi_{-y}^{\mathrm{vir}}(X)$ and $\chi_{-y}^{\mathrm{vir}}(X,L)$.

To state our result we first review the refined Vafa-Witten formula of [**15**]. Consider the following theta function and the normalized Dedekind eta function.

$$\theta_3(x,y)=\sum_{n\in\mathbb{Z}} x^{n^2}y^n,\quad \overline{\eta}(x)=\prod_{n=1}^\infty(1-x^n). \tag{0.1}$$

Then the main conjecture of [**15**] is the following.

Conjecture 2. *Let S be a smooth projective surface with $b_1(S)=0$ and $p_g(S)>0$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves with Chern classes c_1, c_2, and let $M:=M_S^H(c_1,c_2)$. Then $y^{-\frac{\mathrm{vd}}{2}}\chi_{-y}^{\mathrm{vir}}(M)$ equals the coefficient of $x^{\mathrm{vd}(M)}$ of*

$$4\left(\frac{1}{2}\prod_{n=1}^\infty\frac{1}{(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{\chi(\mathcal{O}_S)}\left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x,y^{\frac12})}\right)^{K_S^2}$$
$$\cdot\sum_{a\in H^2(S,\mathbb{Z})}SW(a)(-1)^{c_1a}\left(\frac{\theta_3(x,y^{\frac12})}{\theta_3(-x,y^{\frac12})}\right)^{aK_S}.$$

Our refined K-theoretic Witten conjecture interpolates between Conjecture 2 and Conjecture 1.

Conjecture 3. *Let S be a smooth projective surface with $p_g(S)>0$, $b_1(S)=0$, and $L\in\mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Then $y^{-\frac{\mathrm{vd}}{2}}\chi_{-y}^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L))$ equals the coefficient of x^{vd} of*

$$4\left(\frac{1}{2}\prod_{n=1}^\infty\frac{1}{(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{\chi(\mathcal{O}_S)}\left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x,y^{\frac12})}\right)^{K_S^2}$$
$$\cdot\left(\prod_{n=1}^\infty\left(\frac{(1-x^{2n})^2}{(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{n^2}\right)^{\frac{L^2}{2}}\left(\prod_{n=1}^\infty\left(\frac{1-x^{2n}y^{-1}}{1-x^{2n}y}\right)^n\right)^{LK_S}$$
$$\cdot\sum_{a\in H^2(S,\mathbb{Z})}(-1)^{c_1a}SW(a)\left(\frac{\theta_3(x,y^{\frac12})}{\theta_3(-x,y^{\frac12})}\right)^{aK_S}$$
$$\cdot\left(\prod_{n=1}^\infty\left(\frac{(1-x^{2n-1}y^{\frac12})(1+x^{2n-1}y^{-\frac12})}{(1-x^{2n-1}y^{-\frac12})(1+x^{2n-1}y^{\frac12})}\right)^{2n-1}\right)^{\frac{L(K_S-2a)}{2}}.$$

We see that if $L=\mathcal{O}_S$, then Conjecture 3 specializes to Conjecture 2. On the other hand, specializing at $y=0$, we have

$$\chi_{-y}^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L))\,\Big|_{y=0}=\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L)),$$

and Conjecture 3 specializes to Conjecture 1.

We outline how we derive these conjectures and check them in many cases. Our main tool is Mochizuki's formula, which we review in Chapter 1, Section 8. It allows to express virtual intersection numbers on moduli spaces of sheaves $M_S^H(c_1,c_2)$ as a sum over contibutions of Seiberg-Witten classes a, where each contribution is the coefficient of s^0 of a Laurent series $\widetilde{\Psi}(L,a,c_1-a,n_1,n_2)$ in a variable s

whose coefficients are expressions in intersection numbers on Hilbert schemes of points $S^{[n_1]} \times S^{[n_2]}$. We apply it to the K-theoretic Donaldson invariants in Chapter 2.

So the task is reduced to evaluating the $\widetilde{\Psi}(L, a, c_1 - a, n_1, n_2)$. After pulling out an elementary factor (the perturbation part), we organize the $\widetilde{\Psi}(L, a, c_1 - a, n_1, n_2)$ into a generating function $Z_S(L, a, c_1, s, y, q)$ (the partition function) over all n_1, n_2. $Z_S(L, a, c_1, s, y, q)$ is the partition function used for the χ_{-y}-genera $\chi_{-y}^{\mathrm{vir}}(M, \mu(L))$, and

$$Z_S(L, a, c_1, s, q) := Z_S(L, a, c_1, s, y, q) \big|_{y=0}$$

is the partition function used for the holomorphic Euler characteristics $\chi^{\mathrm{vir}}(M, \mu(L))$.

In Chapter 3 we show that these partition functions satisfy two crucial properties: cobordism invariance and multiplicativity. Cobordism invariance says that the coefficient of any monomial in s, y, q of $Z_S(L, a, c_1, s, y, q)$ is given by a universal polynomial in the 11 intersection numbers L^2, La, a^2, ac_1, c_1^2, Lc_1, LK_S, aK_S, K_S^2, $\chi(\mathcal{O}_S)$. Mulitplicativity says that more precisely there are 11 universal power series $A_1(y), \ldots, A_{11}(y) \in \mathbb{Q}[y, s^{-1}][[s, q]]$, such that

$$\begin{aligned}Z_S(L, a, c_1, s, y, q) = {} & A_1(y)^{L^2} A_2(y)^{La} A_3(y)^{a^2} A_4(y)^{ac_1} A_5(y)^{c_1^2} A_6(y)^{Lc_1} A_7(y)^{LK_S} \\ & \cdot A_8(y)^{aK_S} A_9(y)^{c_1 K_S} A_{10}(y)^{K_S^2} A_{11}(y)^{\chi(\mathcal{O}_S)}.\end{aligned} \tag{0.2}$$

Again we put $A_i = A_i(y) \big|_{y=0}$, and we clearly get

$$Z_S(L, a, c_1, s, q) = A_1^{L^2} A_2^{La} A_3^{a^2} A_4^{ac_1} A_5^{c_1^2} A_6^{Lc_1} A_7^{LK_S} A_8^{aK_S} A_9^{c_1 K_S} A_{10}^{K_S^2} A_{11}^{\chi(\mathcal{O}_S)}.$$

The next step is the reduction to the case of toric surfaces. The 11 power series $A_1, \ldots, A_{11}$ are now determined by computing $Z_S(L, a, c_1, s, y, q)$ for 11 quadruples (S, L, a, c_1) of a surface and 3 line bundles on S, such that the corresponding vectors

$$v_{(S,L,a,c_1)} = (L^2, La, a^2, ac_1, c_1^2, Lc_1, LK_S, aK_S, c_1K_S, K_S^2, \chi(\mathcal{O}_S)$$

in $\mathbb{Q}^{11}$ are linearly independent. We choose all the surfaces to be the toric surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ and the corresponding line bundles to be equivariant line bundles with respect to the natural $\mathbb{C}^* \times \mathbb{C}^*$ action on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

Now we can use localization to compute the power series A_i. For $S = \mathbb{P}^2$ and $S = \mathbb{P}^1 \times \mathbb{P}^1$, the $\mathbb{C}^* \times \mathbb{C}^*$ action on S has finitely many fixpoints, and it lifts to a $\mathbb{C}^* \times \mathbb{C}^*$-action on the Hilbert schemes of points $S^{[n]}$, still with finitely many fixpoints. Furthermore $\widetilde{\Psi}$ can be expressed in terms of the Chern classes of universal and tautological sheaves on these Hilbert schemes of points. Now the Bott-residue formula (or Atiyah-Bott-localization) expresses $\widetilde{\Psi}$ as a sum over contributions at the fixpoints, where each fix point contribution is expressed in terms of the weights of the $\mathbb{C}^* \times \mathbb{C}^*$-action on the fibres of the universal and tautological sheaves and the tangent space of the Hilbert schemes over the fixpoints.

The fixpoints on the Hilbert scheme $S^{[n]}$ are parametrized by tuples of partitions, and the weights on the fibres of the universal and tautological sheaves and the tangent space can be expressed in terms of the combinatorics of partitions. Thus $Z_S(L, a, c_1, s, y, q)$ and $Z_S(L, a, c_1, s, y, q)$ can be computed as a sum over partitions.

This sum has been computed by a program in PARI/GP. We determined the universal series $A_1, \ldots, A_{11}$, $A_1(y), \ldots, A_{11}(y)$ to the following orders:

- For $A_1, \ldots, A_{11}$, we computed the coefficients of $s^{l-3n}q^n$ for all $n \leq 10$, $l \leq 49$. (Recall: A_i and $A_i(y)$ are *Laurent* series in s.)
- For $A_1(y), \ldots, A_{11}(y)$, we computed the coefficients of $s^{l-5n}y^m q^n$ for all $n \leq 6$, $m \leq 9$, $l \leq 30$.

Finally this allows us to prove Conjecture 1 and Conjecture 3 for many surfaces S up to relatively high expected dimension vd: The knowledge of all the $A_i(y)$ modulo a suitable power of q, by formula (0.2), gives us $Z_S(L, a, c_1, s, y, q)$ modulo the same power of q by just substituting in the values of the intersection numbers for (S, L, c_1, a). In the same way we get $Z_S(L, a, c_1, s, q)$ from the knowledge of the A_i. Putting back the perturbation parts, summing over the Seiberg-Witten classes of S, and taking the coefficient of s^0 we get the (refined) K-theoretic Donaldson invariants of S, for all moduli spaces up to a certain virtual dimension vd, which also depends on the intersection numbers of S.

We have done the computation in many cases: double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, complete intersections in $\mathbb{P}^n$, elliptic surfaces and blowups of any of the above. In all these cases Conjecture 1 and 3 are confirmed. The result of this thesis are part of [**16**].

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Contents

CHAPTER 1

Background

1. Conventions

1.1. K-groups. On a Noetherian scheme X we denote $K^0(X)$ the Grothendieck group generated by locally free sheaves, and $K_0(X)$ the Grothendieck group generated by coherent sheaves. $K^0(X)$ is naturally an algebra, $K_0(X)$ is a module over $K^0(X)$, with addition induced by direct sum and multiplication induced by tensor product of locally free sheaves, i.e. it by taking the tensor product $\otimes^L$ of locally free resolutions. Denote by $[F]$ the class of a sheaf F in $K^0(X)$ or $K_0(X)$. For a proper morphism $f : X \to Y$ we have the pushforward homomorphism

$$f_! : K_0(X) \to K_0(Y), [F] \mapsto \sum_i (-1)^i [R^i f_* F].$$

For any morphism $f : X \to Y$ we have the pullback homomorphism $f^! : K^0(Y) \to K^0(X)$, which is given by $[F] \mapsto [f^* F]$ for a locally free sheaf F on Y. Finally if X is smooth, the natural homomorphism $K^0(X) \to K_0(X)$ (induced by the inclusion of locally-free sheaves inside coherent sheaves) is an isomorphism. It's inverse is obtained by taking a locally free resolution.

2. Hilbert Schemes

Let S be a smooth projective surface. We denote by $S^{(n)} = Sym^n(S)$ the n^{th} symmetric power of S. That is $S^{(n)} = S^n/\mathfrak{S}_n$ where $\mathfrak{S}_n$ is the symmetric group in n letters, acting by permuting the factors. We denote by $S^{[n]}$ the Hilbert scheme of n points:

Definition 1. *A flat family of subschemes of S parametrized by a scheme T is a closed subscheme $Z \subset S \times T$ such that the induced morphism $Z \to T$ is flat. For $t \in T$ we denote the fiber of Z over t by Z_t.*

Let $\mathcal{H}ilb^n_S : (Schemes)^{opp} \to (Sets)$ be the functor

$$\mathcal{H}ilb^n_S(T) = \{Z \subset S \times T \mid Z \text{ flat family of subschemes with } h^0(Z_t, \mathcal{O}_{Z_t}) = n \ \forall\ t \in T\}$$

Theorem 2. *(Grothendieck, Fogarty) There is a scheme, which we denote $S^{[n]}$, which represents the functor $\mathcal{H}ilb^n_S$. $S^{[n]}$ is projective, nonsingular and of dimension $2n$.*

Set theoretically, $S^{[n]}$ is the set of 0-dimensional subschemes of S of length n. We have a natural map

$$\pi : S^{[n]} \to S^{(n)}, \quad [Z] \mapsto \sum_{p \in S} (\dim \mathcal{O}_{S,p})p$$

called the Hilbert-Chow morphism. If $Z \subset S$ is a 0-dimensional subscheme of length n, then the support of Z is a finite set. Thus the sum above is a finite sum and the map is well-defined. It can be shown that π is indeed a morphism. By abuse of notation we can also write this map as $[Z] \mapsto \mathrm{supp}(Z)$, where $\mathrm{supp}(Z)$ is the support with multiplicities.

As $S^{[n]}$ represents the functor $\mathcal{H}ilb_S^n$, there is a universal subscheme $Z_n(S) \subset S \times S^{[n]}$, corresponding to the identity morphism $id : S^{[n]} \to S^{[n]}$. Set-theoretically it can just be described as the incidence scheme

$$Z_n(S) = \{(p, Z) \in S \times S^{[n]} \mid p \in Z\}.$$

Let $q : Z_n(S) \to S$ and $p : Z_n(S) \to S^{[n]}$ be the two projections. Then by definition $p : Z_n(S) \to S^{[n]}$ is flat of degree n. We denote by $\mathcal{I}_{Z_n(S)}$ its ideal sheaf in $\mathcal{O}_{S \times S^{[n]}}$, and for a line bundle L on S we denote $\mathcal{I}_{Z_n(S)}(L) := \mathcal{I}_{Z_n(S)} \otimes q^*(L)$.

Definition 2. *Let V be a vector bundle of rank r on S. The tautological vector bundle $V^{[n]}$ on $S^{[n]}$ is defined by*

$$V^{[n]} = p_*(q^*(V)).$$

As $Z_n(S)$ is flat of degree n over $S^{[n]}$ it follows that $V^{[n]}$ is a vector bundle of degree rn over $S^{[n]}$. By definition the fibre of V over a subscheme $Z \in S^{[n]}$ is $V^{[n]}(Z) = H^0(Z, V \otimes \mathcal{O}_Z)$. If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles on S, then it is easy to see that we get an induced exact sequence $0 \to E^{[n]} \to F^{[n]} \to G^{[n]} \to 0$, thus the definition of $V^{[n]}$ extends to a map $K^0(S) \to K^0(S^{[n]})$.

3. Moduli Spaces of Sheaves

Most of this section is based upon [**21**]. Let S be a projective algebraic surface, and let H be an ample divisor on S. Let E be a torsion free coherent sheaf on S. We define the Hilbert Polynomial of E by

$$P_H(E; m) := \chi(S, E \otimes \mathcal{O}_S(mH))$$

by the Riemann-Roch theorem this is a polynomial in m. Let $rk(E)$ be the rank of E. We define the *reduced Hilbert polynomial* of E by $p_H(E; m) := P_H(E; m)/rk(E)$.

Definition 3. *A torsion free sheaf E on S is Gieseker H-semistable if for all proper subsheaves $F \subsetneq E$ we have*

$$p_H(F; m) \leq p_H(E; m), \quad \textit{for} \quad m >> 0$$

E is called Gieseker H-stable if the inequality is strict for all $m >> 0$. In future we will just write H-semistable and H-stable.

We briefly recall from [**21**, Chap.4] the definition of the moduli space $M_S^H(r, c_1, c_2)$ of rank r torsion free H-semistable sheaves on S with Chern classes c_1 and c_2, it is a scheme which corepresents a suitable functor.

DEFINITION 4. *Let $\mathcal{M}' : Schemes/\mathbb{C} \to Sets$ be the contravariant functor which associates to a scheme T the set of isomorphism classes of T-flat families of H-semistable sheaves on S, of rank r and with Chern classes c_1, c_2, and to $f : T' \to T$ the pullback via $f \times 1_S$. Let $\mathcal{M} = \mathcal{M}'/\simeq$ be the quotient functor by the equivalence relation given for $F, F' \in \mathcal{M}'(\mathcal{T})$ by $F \simeq F'$ if and on only $F \simeq F' \otimes p^*L$, where $p : S \times T \to T$ is the projection and $L \in Pic(T)$.*

THEOREM 3 (Gieseker-Maruyama). *See* [**21**, Thm. 4.3.3, Thm. 4.3.4]

(1) *There is a projective scheme $M_S^H(r, c_1, c_2)$ that corepresents the functor $\mathcal{M}$.*
(2) *There is an open subscheme $M_S^H(r, c_1, c_2)^s$ parametrizing equivalence classes of stable sheaves.*

Roughly speaking that $\mathcal{M}$ is corepresented by $M_S^H(r, c_1, c_2)$ means that there is a map ϕ which associates to any T-flat family $\mathcal{E}$ of H-semistable sheaves on S of rank r and with Chern classes c_1, c_2, a morphism $\phi(\mathcal{E}) : T \to M_S^H(r, c_1, c_2)$, and this is compatible with pullback: $\phi((1_S \times f)^*\mathcal{E}) = \phi(\mathcal{E}) \circ f$. If W is a scheme and ψ is another such map $\mathcal{E} \mapsto (\psi(\mathcal{E}) : T \to W)$ compatible with pullback, then ψ factors through a morphism $M_S^H(r, c_1, c_2) \to W$. Note that this means in particular that $M_S^H(r, c_1, c_2)$ is unique up to unique isomorphism.

Corepresenting a functor is a weaker notion than representing it, in particular there will not always be a universal sheaf on $S \times M_S^H(r, c_1, c_2)$. In the sequel we will restrict our attention to the case of rank 2 sheaves and will denote $M_S^H(c_1, c_2) := M_S^H(2, c_1, c_2)$.

4. Obstruction Theory

Let $M \hookrightarrow X$ be an embedding of schemes with X smooth, take for example M a projective variety. Let $\mathfrak{I} = \mathfrak{I}_{M/X}$ be the corresponding ideal sheaf. The complex

$$\mathfrak{I}/\mathfrak{I}^2 \xrightarrow{d} \Omega_{X/M}$$

is called the *truncated cotangent complex* of M in X

DEFINITION 5. *A* 1*-perfect obstruction theory on M is a complex of vector bundles $E^\bullet = [E^{-1} \to E^0]$ on M with a morphism of complexes*

$$\begin{tikzcd} E^{-1} \arrow[r] \arrow[d, "\phi^{-1}"'] & E^0 \arrow[d, "\phi^0"] \\ \mathfrak{I}/\mathfrak{I}^2 \arrow[r, "d"] & \Omega_{X/M} \end{tikzcd}$$

such that

(1) ϕ^0 *induces an isomorphism on the* 0*-th cohomology of the complexes*

(2) ϕ^{-1} *is surjective on* -1*-th cohomology.*

To every 1-perfect obstruction theory $E^\bullet = [E^{-1} \to E^0]$ on M we associate a dual complex $E_\bullet = [E_0 \to E_1]$ by setting $E_i := (E^{-i})^\vee$ for $i = 0, 1$.

Definition 6. *To a pair* $(M, E^\bullet)$ *we define the virtual dimension as*

$$\mathrm{vd}(M, E^\bullet) = rk(E_\bullet) = rk(E_1) - rk(E_0).$$

Theorem 4. [**1, 23**] *Let* M *be a scheme with a* 1*-perfect obstruction theory* $E^\bullet$*. Then* M *has*

(1) *a virtual fundamental class* $[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}(M,E^\bullet)}(M)$ *(*[**1**]*),*

(2) *a virtual structure sheaf* $\mathcal{O}_M^{\mathrm{vir}} \in K_0(M)$ *in the sense defined in* [**23**]*,* [**6**, 3.2]*.*

Definition 7. *We define the virtual tangent sheaf by*

$$T_M^{\mathrm{vir}} := E_0 - E_1 \in K^0(M).$$

Let $V \in K^0(M)$, then we define the *virtual holomorphic Euler characteristic* via

$$\chi^{\mathrm{vir}}(M, V) := \chi(M, V \otimes \mathcal{O}_M^{\mathrm{vir}}),$$

and we have the virtual Riemann-Roch theorem, which is a virtual analogue of the Hirzebruch-Riemann-Roch formula.

Theorem 5. [**1**, Thm.3.3]

$$\chi^{\mathrm{vir}}(M, V) = \int_{[M]^{\mathrm{vir}}} ch(V) td(T_M^{\mathrm{vir}})$$

The χ_{-y}-genus $\chi_{-y}(X)$ of a smooth projective variety X is defined by

$$\chi_{-y}(X) = \sum_{p \geq 0} (-y)^p \chi(X, \Omega_X^p).$$

Similarly for an element $V \in K^0(X)$ the χ_{-y}-genus of X with values in V is defined as

$$\chi_{-y}(X, V) = \sum_{p \geq 0} (-y)^p \chi(X, \Omega_X^p \otimes V).$$

By definition $\chi_{-y}(X, V)\big|_{y=0} = \chi(X, V)$. Now following [**6**], we extend these definitions to their virtual versions.

Definition 8. [**6**] *Let M be a scheme with a 1-perfect obstruction theory $E^\bullet$. The virtual χ_{-y}-genus of M is*

$$\chi_{-y}^{\mathrm{vir}}(M) := \sum_{p\geq 0}(-y)^p\chi^{\mathrm{vir}}(M, \Lambda^p(T_M^{\mathrm{vir}})^\vee),$$

and the virtual χ_{-y}-genus of M with values in $V \in K^0(M)$ is

$$\chi_{-y}^{\mathrm{vir}}(M,V) := \sum_{p\geq 0}(-y)^p\chi^{\mathrm{vir}}\big(M, \Lambda^p(T_M^{\mathrm{vir}})^\vee\otimes V\big).$$

In [**6**] it is shown that $\chi_{-y}^{\mathrm{vir}}(M)$ and $\chi_{-y}^{\mathrm{vir}}(M,V)$ are polynomials in y of degree at most $\mathrm{vd}(M)$. Furthermore it is easy to see from the definitions that $\chi_{-y}^{\mathrm{vir}}(M,V)\big|_{y=0} = \chi^{\mathrm{vir}}(X,V)$.

5. Obstruction Theory for the Moduli Space of Sheaves

In this section we assume that all the sheaves in $M_S^H(c_1,c_2)$ are stable. In [**27**, Chapter 5] T. Mochizuki introduced and studied a perfect obstruction theory on $M_S^H(c_1,c_2)$ with

$$T^{\mathrm{vir}} = R\pi_* R\mathcal{H}om(\mathcal{E},\mathcal{E})_0[1], \tag{5.1}$$

where $\mathcal{E}$ denotes the universal sheaf on $M\times S$, $\pi : M\times S\to M$ is projection, and $(\cdot)_0$ denotes the trace-free part. Although $\mathcal{E}$ may only exist étale locally, $R\pi_*R\mathcal{H}om(\mathcal{E},\mathcal{E})_0$ exists globally [**21**, Sect. 10.2]. We will not need the details of the construction or deeper properties of the obstruction theory in the sequel.

By definition we have

$$\begin{aligned}
\mathrm{vd}(M_S^H(c_1,c_2)) &= \mathrm{rk}(R\pi_*R\mathcal{H}om(\mathcal{E},\mathcal{E})_0[1]) = -\mathrm{rk}(R\pi_*R\mathcal{H}om(\mathcal{E},\mathcal{E})_0))\\
&= -\mathrm{rk}(R\pi_*R\mathcal{H}om(\mathcal{E},\mathcal{E})) + \mathrm{rk}(\pi_*(\mathcal{O}_{S\times M}))\\
&= -\chi(E,E) + \chi(\mathcal{O}_S)\\
&= 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S).
\end{aligned}$$

Here in the second to last step E is a sheaf in $M_S^H(c_1,c_2)$ and

$$\chi(E,E) = \hom(E,E) - \mathrm{ext}^1(E,E) + \mathrm{ext}^2(E,E),$$

and in the last step we have applied the Riemann-Roch Theorem on S. In future we write

$$\mathrm{vd} = \mathrm{vd}(S,c_1,c_2) = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$$

for the virtual dimension of the Moduli space $M_S^H(c_1,c_2)$ and $[M_S^H(c_1,c_2)]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M_S^H(c_1,c_2),\mathbb{Z})$ for the virtual fundamental class.

6. Determinant bundles and Donaldson line bundles

We review the determinant line bundles on the moduli spaces $M_S^H(c_1, c_s)$ (see e.g. [**20**, 2.1]), for more details we refer to [**21**, Chap. 8]. Let c be an element in $K^0(S)$, which is the class of a coherent rank 2 sheaf with Chern classes c_1, c_2. We write $M_S^H(c) := M_S^H(c_1, c_2)$. We assume that all sheaves in $M_S^H(c)$ are H-stable.

Let $\mathcal{E}$ be a flat family of coherent sheaves of class $c \in K^0(S)$ on S parametrized by a scheme T; then $\mathcal{E} \in K^0(S \times T)$. Let

$$p : S \times T \to T, \qquad q : S \times T \to S$$

be the projections. Define $\lambda_{\mathcal{E}} : K^0(S) \to \mathrm{Pic}(T)$ as the composition of the following homomorphisms:

$$K^0(S) \xrightarrow{q^*} K^0(S \times T) \xrightarrow{\cdot[\mathcal{E}]} K^0(S \times T) \xrightarrow{p_!} K^0(T) \xrightarrow{det^{-1}} \mathrm{Pic}(T), \tag{6.1}$$

Notice that $p_!([\mathcal{F}]) \in K^0(T)$ for $\mathcal{F}$ T-flat by [**21**, Prop. 2.1.10].

In general there is no universal sheaf $\mathcal{E}$ over $S \times M_S^H(c)$, and even if it exists, it is well-defined only up to tensoring with the pullback of a line bundle from $M_H^X(c)$. Define

$$K_c := c^{\perp} = \big\{ v \in K^0(S) \bigm| \chi(S, v \otimes c) = 0 \big\}.$$

Then we have a well-defined morphism $\lambda \colon K_c \to \mathrm{Pic}(M_S^H(c)^s)$ such that, for $\mathcal{E}$ any flat family of stable sheaves on S of class c parametrized by T, and all $v \in K_c$, we have $\phi_{\mathcal{E}}^*(\lambda(v)) = \lambda_{\mathcal{E}}(v)$ with $\phi_{\mathcal{E}} : T \to M_S^H(c)^s$ the classifying morphism.

If S is simply connected, the determinant line bundle $\lambda(v)$ only depends on the rank and the Chern classes of v. Let $K(S)$ be the Grothendieck group of coherent sheaves over S. Let L be a line bundle on S and assume that $c_1(L)c_1$ is even. Then we put

$$v(L) := (1 - L^{-1}) + \big(\frac{c_1(L)}{2} \cdot (c_1(L) + K_X - c_1)\big)[\mathcal{O}_x] \in K^0(S), \tag{6.2}$$

where $\mathcal{O}_x$ is the structure sheaf of a point in S. Note that $v(L)$ is of rank 0 and first Chern class L. The determinant line bundle

$$\mu(L) := \lambda(v(L)) \in \mathrm{Pic}(M_S^H(c_1, c_2))$$

associated to $v(L)$ is called the *Donaldson line bundle* associated to L.

By [**21**, Prop.8.3.1] we have

$$c_1(\mu(L)) = p_* \left(q^* c_1(L) \cdot (c_2(\mathcal{E}) - \frac{1}{4} c_1(\mathcal{E})^2) \right) = \big(c_2(\mathcal{E}) - \frac{1}{4} c_1(\mathcal{E})^2\big) / PD(c_1(\mu(L))) \tag{6.3}$$

7. Equivariant Cohomology and Localization

We briefly introduce equivariant cohomology and localization for the case of the action of a torus $T=(\mathbb{C}^*)^m$.

Definition 9. *Let X a complex algebraic variety with a left T-action. Let $\mathbb{E}T$ be a contractible space with a free right T-action. We consider the quotient*

$$\mathbb{E}T\times^T X := \mathbb{E}T\times X/\big((e\cdot t,x)\sim(e,t\cdot x),\ \forall t\in T\big)$$

The equivariant cohomology $H^*_T(X,\mathbb{Q})$ of X with respect to T is the cohomology of $\mathbb{E}T\times^T X$. That is,

$$H^*_T(X,\mathbb{Q}):=H^*(\mathbb{E}T\times^T X,\mathbb{Q})$$

Let $\pi:V\to X$ be a T-equivariant vector bundle, i.e. V has a T-action, such that $\pi(t\cdot e)=t\cdot\pi(e)$ for all $t\in T$. Then $\mathbb{E}T\times^T V$ is naturally a vector bundle over $\mathbb{E}T\times^T X$. We define the T-equivariant Chern classes of V by

$$c_i^T(V)=c_i(\mathbb{E}T\times^T V)$$

Later we will usually drop the T in the notation, and call them just the Chern classes of the equivariant bundle V.

We write $t\in T$ as $t=(t_1,\ldots,t_m)$. If p is a point with trivial action of $T=\mathbb{C}^m$, then $H^*_T(p,\mathbb{Q})=\mathbb{Q}[\varepsilon_1,\ldots,\varepsilon_m]$, where the variables ε_i are in $H^2_T(p,\mathbb{Q})$. Here $\varepsilon_i=c_1^T(\mathfrak{t}_i)$ where $\mathfrak{t}_i$ is the equivariant vector bundle on p given by the vector space $\mathbb{C}$, viewed as trivial vector bundle on p with the action given by $t\cdot v=t_iv$. More generally an equivariant vector bundle V of rank r on p is just a representation of T. Then V has a basis $v_1,\ldots,v_r$ of common eigenvectors for the T action, in fact there are monomials $M_i=t_1^{m_{i,1}}\ldots t_n^{m_{i,n}}$, such that $t\cdot v_i=M_iv_i$ for all i. Then we call $w(v_i):=m_{i,1}\varepsilon_1+\ldots+m_{i,r}\varepsilon_r$ the weight of v_i (or of the equivariant line bundle $\mathbb{C}v_i$), and we call the $w(v_i)$: the weights of V. Then the total equivariant Chern class of V is

$$c(V)=c^T(V)=\sum_{i=1}^r c_i^T(V)=\prod_{i=1}^r(1+w(v_i)),$$

More generally if $p\big(c_1^T(V),\ldots,c_r^T(V)\big)$ is a polynomial in the Chern classes we have

$$p(c_1^T(V),\ldots,c_r^T(V))=p\big(\sigma_1(w(v_1),\ldots,w(v_r)),\ldots,\sigma_r(w(v_1)\ldots,w(v_r))\big),$$

where the σ_i are the elementary symmetric functions.

If $X\xrightarrow{\phi}X'$ is a T-equivariant map (i.e. $\phi(t\cdot x)=t\cdot\phi(x)$) then ϕ induces a canonical map $\mathbb{E}T\times^T X\to\mathbb{E}T'\times^{T'}X'$. This induces a pullback

$$\phi^*:H^*_T(X')\to H^*_T(X)$$

We denote by X^T the fixed point locus of the action of T on X. Let $p \in X^T$ be a fixed point and denote $\iota_p : p \to X$ the inclusion map. For an element V in the Grothendieck group of T-vector bundles on X or a class $\alpha \in H^*_T(X)$, we write $V(p) := \iota^*(V)$, (which is an element in the Grothendieck group of T representations) and $\alpha(p) := \iota_p^*(\alpha) \in H^*_T(p, \mathbb{Q})$.

By definition $H^*_T(X, \mathbb{Q})$ is a module over $H^*_T(p, \mathbb{Q})$ with a forgetful map $f : H^*_T(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$. We call a class $\alpha \in H^i_T(X, \mathbb{Q})$ a lift of a class $\overline{\alpha} \in H^i(X, \mathbb{Q})$, if $f(\alpha) = \overline{\alpha}$. The most important case for us is that, if V is an equivariant vector bundle on S, then the equivariant Chern classes $c_i^T(V)$ are lifts of the Chern classes $c_i(\overline{V})$, where $\overline{V}$ is the underlying vector bundle of the equivariant vector bundle V.

The main result we want to use in this paper is the Bott residue formula (or Atiyah-Bott-localization). We will only need it in the case of isolated fixpoints.

Theorem 6. *Let X be a smooth projective variety, with an action of $T = \mathbb{C}^m$ with finitely many fixed points $p_1, \ldots, p_e$. Let $\alpha \in H^*(X, \mathbb{Q})$, and let $\widetilde{\alpha} \in H^*_T(X, \mathbb{Q})$ be an equivariant lift of α. Then*

$$\int_X \alpha = \sum_{i=1}^{e} \frac{\iota_{p_i}^*(\widetilde{\alpha})}{c_n^T(T_{X,p_i})}\Big|_{\varepsilon_1=\ldots=\varepsilon_m=0}.$$

With the conventions above we can also write this as

$$\int_X \alpha = \sum_{i=1}^{e} \frac{\widetilde{\alpha}(p_i)}{c_n(T_{X,p_i})}\Big|_{\varepsilon_1=\ldots=\varepsilon_m=0}.$$

8. Mochizuki's Formula

Let S be an algebraic surface with $b_1(S) = 0$, $p_g(S) > 0$. As we have seen above, $M_S^H(c_1, c_2)$ has an obstruction theory of virtual dimension

$$\mathrm{vd} = \mathrm{vd}(S, c_1, c_2) = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S).$$

Thus it has in particular a virtual fundamental class

$$[M_S^H(c_1, c_2)]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M_S^H(c_1, c_2), \mathbb{Z}).$$

Mochizuki's formula expresses virtual intersection numbers

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} \alpha, \qquad \alpha \in H^*(M_S^H(c_1, c_2), \mathbb{Q})$$

on the moduli spaces of sheaves in terms of intersection numbers on Hilbert schemes of points $S^{[n]}$ on S. It is the most important tool in our work, because it allows us to work on the more accessible Hilbert schemes of points instead of on the intractable moduli spaces of higher rank sheaves.

8.1. Seiberg-Witten Invariants. Before we can write down Mochizuki's Formula, we need to write down some facts about Seiberg-Witten invariants. Most of this is based on [**29**],[**11**]. Let M be a closed Riemannian 4-manifold with metric g. For simplicity, we assume that $H^2(M;\mathbb{Z})$ has no 2-torsion. Let L be a characteristic complex line bundle on M. Then we may construct the Seiberg-Witten moduli space $\mathcal{M}(L)$. The Seiberg-Witten invariant is then defined to be a function $\widetilde{SW}_{M,g}$ which assigns to such an L an integer defined as an intersection product on $\mathcal{M}(L)$. If L is such that $SW_{M,g}(L) \neq 0$, then L will be called a Seiberg-Witten **basic class**.

We consider smooth projective algebraic surfaces S with $b_1 = 0$ and $p_g > 0$. We write $\widetilde{SW}_S = \widetilde{SW}_{S,g}$ for g the Fubini-Study metric, with respect to a projective embedding. Note that under the assumptions $b_1 = 0$ and $p_g > 0$ the Seiberg-Witten invariants are independent of g. Mochizuki uses a somewhat nonstandard convention regarding Seiberg-Witten invariants, which we will follow in this work: for $a \in H^2(S,\mathbb{Z})$ we will write

$$SW_S(a) = \widetilde{SW}_S(K_S - 2a)$$

and will in future call a a Seiberg-Witten class if $SW_S(a) \neq 0$. We will often drop the index S if S is understood. We will denote $SW(S) \subset H^2(S,\mathbb{Z})$ the set of Seiberg-Witten classes of S.

We have the following results which we will use.

THEOREM 7. *Assume either of the following.*

(1) *S is a minimal surface of general type,*

(2) *the linear system $|K_S|$ contains a smooth irreducible curve.*

then the Seiberg-Witten basic classes are 0, K_S, with

$$SW(0) = 1 \quad \textit{and} \quad SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}.$$

PROOF. In case (1) this is shown in [**29**, Thm. 7.4.1], and in case (2) for instance in [**15**, Sect. 6.3]). ☐

THEOREM 8. [**29**, Thm. 7.4.6] *Let $\hat{S} \xrightarrow{\pi} S$ be the blowup of a surface S with $p_g > 0$ and $b_1 = 0$. If E is the exceptional divisor, then the set of Seiberg-Witten basic classes of $\hat{S}$ is*

$$SW(\hat{S}) = \{\pi^*(a) \mid a \in SW(S)\} \cup \{\pi^*(a) + E \mid a \in SW(S)\}$$

and furthermore $SW_{\hat{S}}(\pi^*(a)) = SW_{\hat{S}}(\pi^*(a) + E) = SW_S(a)$ for all $a \in SW(S)$.

Now we come to the case of elliptic surfaces. Let $S \xrightarrow{\pi} \mathbb{P}^1$ be a minimal elliptic surface such that there are only finitely many singular fibers of π, no multiple fibres, and the singular fibres are all nodal curves. Then it can be shown that there are exactly $12n$ singular fibres where $n = \chi(\mathcal{O}_S)$. Let F be the class of a fibre of π. Then $K_S = (n-2)F$.

PROPOSITION 2. *Under the above assumptions we have*

$$SW(S) = \{kF \mid 0 \leq k \leq n-2\}$$

and

$$SW(kF) = (-1)^k \binom{n-2}{k}$$

PROOF. This is [**12**, Proposition 4.2], and is also explained after Proposition 2.4 in [**11**]. □

8.2. Descendent Insertions. Let again $M := M_S^H(c_1, c_2)$ be the moduli space of torsion free coherent sheaves on S. We now define some natural cohomology classes on M. Let $\alpha \in H^i(S, \mathbb{Q})$ and $k \geq 0$. Furthermore we denote by $\mathcal{E}$ the universal sheaf on $M \times S$. Then we define **descendent insertions** $\tau_k(\alpha) \in H^{2k-4+i}(M)$ by the following formula

$$\tau_k(\alpha) := \mathrm{ch}_k(\mathcal{E})/PD(\alpha).$$

Here $PD(\alpha) \in H_{4-i}(S, \mathbb{Q})$ is the Poincaré dual of α, and

$$/ : H^q(S \times M, \mathbb{Q}) \times H_p(S, \mathbb{Q}) \to H^{q-p}(M, \mathbb{Q})$$

is the slant product. See e.g. [**33**, Chapter 6] for its definition and properties, for instance it is easy to see that if M is nonsingular then

$$\beta/PD(\alpha) = \pi_{M*}(\beta \pi_S^* \alpha).$$

Here we write by abuse of notation $\pi_{M*} : H^*(S \times M, \mathbb{Q}) \to H^*(M, \mathbb{Q})$ for the pushforward in cohomology, given by $PD^{-1} \circ \pi_{M*} \circ PD$, where now π_{M*} is the pushforward in homology and $PD : H^* \to H_*$ the Poincaré duality map.

8.3. The Formula. We now state Mochizuki's Formula [**27**, Thm. 1.4.6]. It involves certain integrals of universal sheaves over Hilbert schemes of points that we will now introduce. On $S \times S^{[n_1]} \times S^{[n_2]}$, we have the universal subschemes

$$\mathcal{Z}_1,\ \mathcal{Z}_2 \subset S \times S^{[n_1]} \times S^{[n_2]}$$

with

$$\mathcal{Z}_i = \left\{ (x, Z_1, Z_2) \in S \times S^{[n_1]} \times S^{[n_2]} \;\middle|\; x \in Z_i \right\},$$

and their corresponding ideal sheaves $\mathcal{I}_1 := \mathcal{I}_{\mathcal{Z}_1}$, $\mathcal{I}_2 := \mathcal{I}_{\mathcal{Z}_2}$. For any line bundle $L \in \mathrm{Pic}(S)$ we denote by $L^{[n_i]}$ the tautological vector bundle on $S^{[n_1]} \times S^{[n_2]}$ defined by

$$L^{[n_i]} := p_* q^* L,$$

with $p : \mathcal{Z}_i \to S^{[n_1]} \times S^{[n_2]}$, $q : \mathcal{Z}_i \to S$ the projections from the universal subscheme $\mathcal{Z}_i \subset S \times S^{[n_1]} \times S^{[n_2]}$.

We consider $S^{[n_1]} \times S^{[n_2]}$ to be endowed with a trivial $\mathbb{C}^*$-action and we denote the generator of the character group by $\mathfrak{s}$. Moreover we write s for the generator of

$$H^*(B\mathbb{C}^*, \mathbb{Q}) = H^*_{\mathbb{C}^*}(pt, \mathbb{Q}) \cong \mathbb{Q}[s].$$

In other words $\mathfrak{s}$ can be viewed as a trivial line bundle on $S^{[n_1]} \times S^{[n_2]}$, with a $\mathbb{C}^*$-action given by $t \cdot v = tv$, and $s = c_1^{\mathbb{C}^*}(\mathfrak{s})$ is its equivariant first Chern class.

Let $L \in \mathrm{Pic}(S)$. Let $P(\mathcal{E})$ be any polynomial in descendent insertions $\tau_k(\sigma)$, which arises from a polynomial in Chern numbers of T_M^{vir} and $c_1(\mu(L))$ (see below). For any $a_1, a_2 \in A^1(S)$ and $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, Mochizuki defines a class $\Psi(a_1, a_2, n_1, n_2) \in H^*(S^{[n_1]} \times S^{[n_2]}, \mathbb{Q})$ by the following formula

$$\Psi(a_1, a_2, n_1, n_2) := \mathrm{Coeff}_{s^0}\left(\frac{P(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s})}{Q(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1}, \mathcal{I}_2(a_2) \otimes \mathfrak{s})} \frac{\mathrm{Eu}(\mathcal{O}(a_1)^{[n_1]}) \, \mathrm{Eu}(\mathcal{O}(a_2)^{[n_2]} \otimes \mathfrak{s}^2)}{(2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}\right). \tag{8.1}$$

We explain the notations. Here $\mathrm{Eu}(\cdot)$ denotes the $\mathbb{C}^*$-equivariant Euler class and Coeff_{s^0} refers to taking the coefficient of s^0. The notation $\mathcal{I}_i(a_i)$ is short-hand for $\mathcal{I}_i \otimes \pi_S^*\mathcal{O}(a_i)$.

Furthermore, for any $\mathbb{C}^*$-equivariant sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ on $S \times S^{[n_1]} \times S^{[n_2]}$ flat over $S^{[n_1]} \times S^{[n_2]}$, Mochizuki defines

$$Q(\mathcal{E}_1, \mathcal{E}_2) := \mathrm{Eu}(-R\mathcal{H}om_\pi(\mathcal{E}_1, \mathcal{E}_2) - R\mathcal{H}om_\pi(\mathcal{E}_2, \mathcal{E}_1)),$$

where $\pi : S \times S^{[n_1]} \times S^{[n_2]} \to S^{[n_1]} \times S^{[n_2]}$ denotes projection and

$$R\mathcal{H}om_\pi(\cdot, \cdot) := R\pi_* R\mathcal{H}om(\cdot, \cdot).$$

Finally for $\sigma \in H^i(S, \mathbb{Z})$ we define $\tau'_k(\sigma) \in H^{2k-4+i}(S^{[n_1]} \times S^{[n_2]}, \mathbb{Q})$ by

$$\begin{aligned}\tau'_k(\sigma) &:= \mathrm{ch}_k(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s})/PD(\sigma) \\ &= \pi_{S^{[n_1]} \times S^{[n_2]} *}(\mathrm{ch}_k(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s}) \pi_S^* \sigma),\end{aligned}$$

and we define $P(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s})$ as the expression obtained by formally replacing each $\tau_k(\sigma)$ in $P(\mathcal{E})$ by $\tau'_k(\sigma)$. We define

$$\widetilde{\Psi}(a_1, a_2, n_1, n_2, s)$$

by expression (8.1) but *without* applying Coeff_{s^0}. Thus

$$\widetilde{\Psi}(a_1, a_2, n_1, n_2, s) \in H^*(S^{[n_1]} \times S^{[n_2]}, \mathbb{Q})[s^{-1}][[s]].$$

Let c_1, c_2 be a choice of Chern classes and $c \in K^0(S)$ be the class of sheaf in $M_S^H(c_1, c_2)$. We denote $\mathrm{ch}(c) = (2, c_1, \frac{1}{2}c_1^2 - c_2)$ its Chern character. For any decomposition $c_1 = a_1 + a_2$ with $a_1, a_2 \in A^1(S)$, we define following Mochizuki

$$\mathcal{A}(a_1, a_2, c_2) := \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]}\times S^{[n_2]}} \Psi(a_1, a_2, n_1, n_2). \tag{8.2}$$

We denote by $\widetilde{\mathcal{A}}(a_1, a_2, c_2, s)$ the same expression with Ψ replaced by $\widetilde{\Psi}$.

THEOREM 9 (Mochizuki). *Let S be a smooth projective surface with $b_1(S) = 0$ and $p_g(S) > 0$. Let H, c_1, c_2 be chosen such that there exist no rank 2 strictly Gieseker H-semistable sheaves with Chern classes c_1, c_2. Denote $c \in K^0(S)$ the class of an element of $M_S^H(c_1, c_2)$. Suppose a universal sheaf $\mathcal{E}$ exists on $S \times M_S^H(c_1, c_2)$. Furthermore suppose that the following conditions hold:*

1. *$\chi(\mathrm{ch}(c)) > 0$, where $\chi(\mathrm{ch}(c)) := \int_S \mathrm{ch}(c) \cdot td(S)$ and $\mathrm{ch}(c) = (2, c_1, \frac{1}{2}c_1^2 - c_2)$.*
2. *$p_c > p_{K_S}$, where p_c and p_{K_S} are the reduced Hilbert polynomials associated to the class $c \in K^0(S)$ and to K_S.*
3. *For all Seiberg-Witten basic classes a_1 satisfying $a_1H \leq (c_1 - a_1)H$ the inequality is strict.*

Let $P(\mathcal{E})$ be any polynomial in descendent insertions which arises from a polynomial in Chern numbers of T_M^{vir} and $c_1(\mu(L))$. Then

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} P(\mathcal{E}) = -2^{1-\chi(\mathrm{ch}(c))} \sum_{\substack{c_1 = a_1 + a_2 \\ a_1H < a_2H}} SW(a_1)\, \mathcal{A}(a_1, a_2, c_2).$$

REMARK 1. *The assumption that the universal sheaf $\mathcal{E}$ exists on $S \times M_S^H(c_1, c_2)$ is unnecessary. The virtual tangent sheaf $T_M^{\mathrm{vir}} = -R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{E})_0$ always exists globally, and also the definition of $\mu(L)$ and thus of $c_1(\mu(L))$ is independent of the existence of a universal sheaf. So the left-hand side of Mochizuki's formula always makes sense, and the statement of the theorem holds. Moreover, Mochizuki* [**27**] *works over the Deligne-Mumford stack of oriented sheaves, which always has a universal sheaf. This can be used to show that global existence of $\mathcal{E}$ on $S \times M$ can be dropped from the assumptions. In fact, when working on the stack, P can be* any *polynomial in descendent insertions defined using the universal sheaf on the stack.*

8.4. Strong form of Mochizuki's formula. The following **strong form of Mochizuki's formula** was conjectured in [**19**].

CONJECTURE 4. [**19**] *Let S be a smooth projective surface with $b_1(S) = 0$ and $p_g(S) > 0$. Let H, c_1, c_2 be chosen such that there exist no rank 2 strictly Gieseker H-semistable sheaves with Chern classes c_1, c_2. Denote $c \in K^0(S)$ the class of an element $M_S^H(c_1, c_2)$. Suppose a universal sheaf $\mathcal{E}$ exists on $S \times M_S^H(c_1, c_2)$. Suppose that $\chi(\mathrm{ch}(c)) > 0$, where $\chi(\mathrm{ch}(c)) := \int_S \mathrm{ch}(c) \cdot td(S)$ and $\mathrm{ch}(c) = (2, c_1, \frac{1}{2}c_1^2 - c_2)$. Let $P(\mathcal{E})$ be any polynomial in descendent insertions which arises from a polynomial in Chern numbers* *of* T_M^{vir} *and* $c_1(\mu(L))$. *Then*

$$\int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} P(\mathcal{E}) = -2^{1-\chi(\mathrm{ch}(c))} \sum_{c_1=a_1+a_2} SW(a_1)\,\mathcal{A}(a_1,a_2,c_2).$$

In other words conjecturally assumptions (2) and (3) can be dropped from Theorem 9 and the sum can be replaced by a sum over *all* Seiberg-Witten basic classes. Often, assuming this conjecture, our computations can be applied more widely, computing the K-theoretic Donaldson invariants for more examples of up to higher virtual dimension of the moduli spaces.

CHAPTER 2

Application of the Mochizuki formula to K-theoretic Donaldson invariants

In this chapter we want to apply Mochizuki's formula to the computation of K-theoretic Donaldson invariants. The first step is to show that Mochizuki's formula applies to this computation. As Mochizuki's formula computes integrals of polynomials in descendent insertions on moduli spaces of sheaves, we have to express the K-theoretic Donaldson invariants in terms of descendent insertions. This is done in the first section.

The next step is to give the explicit form of the Mochizuki formula in the case of (refined) K-theoretic Donaldson invariants. Then we introduce the partition function $Z_S(L,a_1,c_1,s,y,q)$, which is up to an elementary factor the generating function for the functions $\widetilde{\Psi}(a_1,a_2,n_1,n_2,s)$ (or equivalently $\widetilde{\mathcal{A}}(a_1,a_2,c_2,s)$) which occur in Mochizuki's formula, for the case of the K-theoretic Donaldson invariants. $Z_S(L,a_1,c_1,s,y,q)$ is a power series in q starting with 1, whose coefficient of q^n is the contribution of the Hilbert schemes $S^{[n_1]}\times S^{[n_2]}$ with $n_1+n_2=n$.

At the end of this chapter we use Mochizuki's formula to give an explicit formula that expresses the (refined) K-theoretic Donaldson invariants in terms of the partition function. In Chapters 3 and 4 we will give an explicit computation (up to a certain power in q) of the partition function $Z_S(L,a_1,c_1,s,y,q)$ for all S,L,a_1,c_1. These two results together allow us all our explicit computations and verifications of our conjectures in the last chapter.

1. Expression in terms of descendent insertions

We still keep our standard assumption that S is a smooth projective surface with $b_1(S)=0$ and $p_g(S)>0$, and that we have chosen c_1,c_2,H in such a way that the moduli space $M_S^H(c_1,c_2)$ consists only of stable sheaves. Let $L\in \mathrm{Pic}(S)$. We assume for simplicity that $\mathcal{E}$ is a universal sheaf on $S\times M_S^H(c_1,c_2)$. However using Remark 1 of Chapter 1, we see that our results are independent of the existence of a universal sheaf.

Proposition 3. *Let S,H,c_1,c_2 be as above.*

(1) *There exists a polynomial expression $P(\mathcal{E})$ in certain descendent insertions $\tau_\alpha(\sigma)$ and y such that*

$$\chi_{-y}^{\mathrm{vir}}(M_S^H(r,c_1,c_2),\mu(L))=\int_{[M_S^H(r,c_1,c_2)]^{\mathrm{vir}}}P(\mathcal{E}).$$

(2) *There exists a polynomial expression* $P_0(\mathcal{E})$ *in certain descendent insertions* $\tau_\alpha(\sigma)$ *such that*

$$\chi^{\mathrm{vir}}(M_S^H(r,c_1,c_2),\mu(L)) = \int_{[M_S^H(r,c_1,c_2)]^{\mathrm{vir}}} P_0(\mathcal{E}).$$

The proof of this result is an adaptation of the proof of [**15**, Prop. 2.1]. We know from the definitions that $\chi^{\mathrm{vir}}_{-y}(M,\mu(L))\big|_{y=0} = \chi^{\mathrm{vir}}(M,\mu(L))$. Therefore (2) follows from (1), by putting $P_0(\mathcal{E}) = P(\mathcal{E})\big|_{y=0}$. Thus we will only prove (1).

We start by reviewing some properties of the virtual χ_{-y}-genus that we will use. We write $M = M_S^H(c_1,c_2)$. Consider the K-group $K^0(M)$ generated by locally free sheaves on M. For any rank r vector bundle on M, define

$$\Lambda_y V := \sum_{i=0}^{r}[\Lambda^i V]y^i \in K^0(M)[[y]], \qquad \mathrm{Sym}_y V := \sum_{i=0}^{\infty}[\mathrm{Sym}^i V]y^i \in K^0(M)[[y]].$$

These expressions can be extended to complexes in $K^0(M)$ by setting $\Lambda_y(-V) = \mathrm{Sym}_{-y}V$ and $\mathrm{Sym}_y(-V) = \Lambda_{-y}V$. For any complex $E \in K^0(M)$, we define

$$\mathsf{X}_y(E) := \mathrm{ch}(\Lambda_y E^\vee)\, td(E). \tag{1.1}$$

Since $\Lambda_y(E \oplus F) = \Lambda_y E \otimes \Lambda_y F$, we obtain

$$\mathsf{X}_y(E \oplus F) = \mathsf{X}_y(E)\,\mathsf{X}_y(F).$$

Furthermore, for any $L \in \mathrm{Pic}(M)$

$$\mathsf{X}_{-y}(L) = \frac{L(1-ye^{-L})}{1-e^{-L}}.$$

Therefore, given a formal splitting $c(V) = \prod_{i=1}^r (1+x_i)$, we have

$$\mathsf{X}_{-y}(V) = \prod_{i=1}^r \frac{x_i(1-ye^{-x_i})}{1-e^{-x_i}}.$$

Lemma 1. *Let* S, H, c_1, c_2 *and* $M := M_S^H(r,c_1,c_2)$ *be as above. Let* $L \in \mathrm{Pic}(S)$*. Then*

$$\chi^{\mathrm{vir}}_{-y}(M,\mu(L)) = \int_{[M]^{\mathrm{vir}}} \mathsf{X}_{-y}(T_M^{\mathrm{vir}})\, e^{c_1(\mu(L))}.$$

PROOF. This is a direct application of the virtual Hirzebruch-Riemann-Roch theorem [**6**, Cor. 3.4]. By definition and the virtual Riemann-Roch theorem we have

$$\begin{aligned}\chi^{\mathrm{vir}}_{-y}(M,\mu(L)) &= \sum_{p\geq 0}(-y)^p\chi^{\mathrm{vir}}\big(M,\Lambda^p\big((T_M^{\mathrm{vir}})^\vee\big)\otimes\mu(L)\big)\\ &= \chi^{\mathrm{vir}}\big(M,\big(\Lambda_{-y}(T_M^{\mathrm{vir}})^\vee\big)\otimes\mu(L)\big)\\ &= \int_{[M]^{\mathrm{vir}}}\mathrm{ch}\big(\Lambda_{-y}(T_M^{\mathrm{vir}})^\vee\big)\mathrm{ch}(\mu(L))td(T_M^{\mathrm{vir}})\\ &= \int_{[M]^{\mathrm{vir}}}\mathsf{X}_{-y}(T_M^{\mathrm{vir}})\,e^{c_1(\mu(L))}.\end{aligned}$$

□

To finish the proof of Proposition 3 we therefore just need to show the following Lemma.

LEMMA 2. *There exists a polynomial expression $P(\mathcal{E})$ in certain descendent insertions $\tau_\alpha(\sigma)$ and y such that*

$$\int_{[M]^{\mathrm{vir}}}\mathsf{X}_{-y}(T_M^{\mathrm{vir}})\,e^{c_1(\mu(L))}=\int_{[M]^{\mathrm{vir}}}P(\mathcal{E}).$$

PROOF. By definition $\mathsf{X}_{-y}(T_M^{\mathrm{vir}})$ is $\mathbb{Q}$-linear combination of monomials in the virtual Chern classes $c_i(T_M^{\mathrm{vir}})$ and y, and thus $\mathsf{X}_{-y}(T_M^{\mathrm{vir}})\,e^{\mu(c_1(L))}$ is a $\mathbb{Q}$-linear combination of monomials in the $c_i(T_M^{\mathrm{vir}})$, $c_1(\mu(L))$ and y.

In the proof of [**15**, Prop. 2.1] it is shown that each $c_i(T_M^{\mathrm{vir}})$ is a polynomial in descendent insertions. In the course of the proof it is shown that every expression of the form

$$\pi_{M*}\big(\mathrm{ch}_i(\mathcal{E})\mathrm{ch}_j(\mathcal{E})\pi_S^*\sigma\big)=\big(\mathrm{ch}_i(\mathcal{E})\mathrm{ch}_j(\mathcal{E})\big)/PD(\sigma), \tag{1.2}$$

is a polynomial in descendent insertions.

On the other hand we have by (6.3) in Chapter 1

$$\begin{aligned}c_1(\mu(L)) &= \big(c_2(\mathcal{E})-\frac{1}{4}c_1(\mathcal{E})^2\big)/PD(c_1(L))\\ &= -\mathrm{ch}_2(\mathcal{E})/PD(c_1(L))+\frac{1}{4}\mathrm{ch}_1(\mathcal{E})^2/PD(c_1(L)).\end{aligned}$$

Thus $c_1(\mu(L))$ can be expressed in the form of (1.2) and thus is a polynomial in descendent insertions. The result follows. □

2. Explicit form of Mochizuki's formula

We still assume that S is a smooth projective surface with $b_1(S)=0$ and $p_g(S)>0$, and that c_1,c_2,H are such that $M_S^H(c_1,c_2)$ only consists of H-stable sheaves. Finally let $L\in\mathrm{Pic}(S)$. We again write $M:=M_S^H(c_1,c_2)$. Our aim is to evaluate Mochizuki's formula in two situations:

(1) for

$$\chi^{\mathrm{vir}}_{-y}(M,\mu(L)) = \int_{[M_S^H(c_1,c_2)]^{\mathrm{vir}}} \mathsf{X}_{-y}(T_M^{\mathrm{vir}})\exp(c_1(\mu(L)),$$

(2) for

$$\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L)) = \chi^{\mathrm{vir}}_{-y}(M,\mu(L))\big|_{y=0}.$$

Clearly it is enough to do this for case (1). At the end we will indicate how the formula simplifies in case (2).

We denote

$$\widetilde{\Psi}_P(a_1,a_2,n_1,n_2,s)$$

the $\widetilde{\Psi}(a_1,a_2,n_1,n_2,s)$ of Chapter 1 (8.1) for the specific choice

$$P(\mathcal{E}) := \mathsf{X}_{-y}(T_M^{\mathrm{vir}})\exp(c_1(\mu(L)).$$

As a first step we give an explicit formula for $\widetilde{\Psi}_P(a_1,a_2,n_1,n_2,s)$.

We will use the following well-known Lemma, (see e.g. [Lemma 3.1][**15**]).

Lemma 3. *Let $\pi : S\times S^{[n]}\to S^{[n]}$ denote projection. Then*

$$-R\mathcal{H}om_\pi(\mathcal{I},\mathcal{I})_0 \cong \mathcal{E}xt^1_\pi(\mathcal{I},\mathcal{I})_0 \cong T_{S^{[n]}},$$

where $\mathcal{I}$ denotes the universal ideal sheaf and $T_{S^{[n]}}$ denotes the tangent bundle.

Definition 10. *We formally define*

$$\mathrm{ch}(\mathcal{I}_j(\alpha)) = \mathrm{ch}(\mathcal{I}_j)\exp(q^*\alpha) \qquad \textit{for } \alpha\in H^2(S,\mathbb{Q}).$$

We put $\xi := a_2-a_1$ and define a class $\nu(L,\xi)\in H^2(S^{[n_1]}\times S^{[n_2]},\mathbb{Q})$ by

$$\nu(L,\xi) := \pi_*\left(-\mathrm{ch}_2\left(\mathcal{I}_1\left(-\frac{\xi}{2}\right)\otimes\mathfrak{s}^{-1}\right)q^*(c_1(L))\right) + \pi_*\left(-\mathrm{ch}_2\left(\mathcal{I}_2\left(\frac{\xi}{2}\right)\otimes\mathfrak{s}\right)q^*(c_1(L))\right).$$

Lemma 4. *On $S^{[n_1]}\times S^{[n_2]}$ we have*

$$\nu(L,\xi) = -(\xi L)s + \pi_*\big(([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L)\big),$$

where (ξL) is the intersection number on S.

Proof. As $\mathcal{Z}_1$, $\mathcal{Z}_2$ have codimension 2 in $X := S\times S^{[n_1]}\times S^{[n_2]}$ and are generically reduced, we have for $i=1,2$ that $\mathrm{ch}(\mathcal{I}_i)) = 1-[\mathcal{Z}_i]$, modulo $H^{>4}(X,\mathbb{Q})$, where $[\mathcal{Z}_i]$ is the Poincaré dual cohomology class to the fundamental class of the universal subscheme $\mathcal{Z}_i \subset S \times S^{[n_1]} \times S^{[n_2]}$. Using that we can write

$$\begin{aligned}\nu(L,\xi) &= \pi_*\left(\left[(-1+[\mathcal{Z}_1])e^{-q^*(\xi/2)-s} + (-1+[\mathcal{Z}_2])e^{q^*(\xi/2)+s}\right]_2 q^*(c_1(L))\right)\\ &= \pi_*\big(\big(-(q^*(\xi/2)+s)^2/2 + [\mathcal{Z}_1] - (q^*(\xi/2)+s)^2/2 + [\mathcal{Z}_2]\big)q^*(L)\big)\\ &= \pi_*\big(\big(-(q^*(\xi/2)+s)^2/2 + [\mathcal{Z}_1] - (q^*(\xi/2)+s)^2/2 + [\mathcal{Z}_2]\big)q^*(L)\big)\\ &= -(\xi L)s + \pi_*\big([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L)\big)\end{aligned}$$

Here $[\cdot]_2$ means the part in $H^4(X,\mathbb{Q})$ and (ξL) is the intersection number on S. □

LEMMA 5.

$$\begin{aligned}&\widetilde{\Psi}_P(a_1,a_2,n_1,n_2,s)\\ &\quad= \frac{\mathsf{X}_{-y}\big(T_{S^{[n_1]}\times S^{[n_2]}} - R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2(\xi)\otimes\mathfrak{s}^2) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_1(-\xi)\otimes\mathfrak{s}^{-2})\big)\exp\left(\nu(L,\xi)\right)}{\mathrm{Eu}\big(-R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2(\xi)\otimes\mathfrak{s}^2) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_1(-\xi)\otimes\mathfrak{s}^{-2})\big)}\\ &\qquad\qquad\cdot\frac{\mathrm{Eu}\big(\mathcal{O}(a_1)^{[n_1]}\big)\mathrm{Eu}\big(\mathcal{O}(a_2)^{[n_2]}\otimes\mathfrak{s}^2\big)}{(2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}.\end{aligned}$$

PROOF. Let $\mathcal{F} := \mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1}\oplus\mathcal{I}_2(a_2)\otimes\mathfrak{s}$. By definition we have

$$\widetilde{\Psi}_P(a_1,a_2,n_1,n_2,s) = \frac{P(\mathcal{F})\mathrm{Eu}\big(\mathcal{O}(a_1)^{[n_1]}\big)\mathrm{Eu}\big(\mathcal{O}(a_2)^{[n_2]}\otimes\mathfrak{s}^2\big)}{Q(\mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1},\oplus\mathcal{I}_2(a_2)\otimes\mathfrak{s})(2s)^{n_1+n_2-\chi(\mathcal{O}_S)}},$$

with

$$\begin{aligned}Q(\mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1},\mathcal{I}_2(a_2)\otimes\mathfrak{s}) &= \mathrm{Eu}\left(-R\mathcal{H}om_\pi(\mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1},\mathcal{I}_2(a_2)\otimes\mathfrak{s}) - R\mathcal{H}om_\pi(\mathcal{I}_2(a_2)\otimes\mathfrak{s},\mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1})\right)\\ &= \mathrm{Eu}\left(-R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2(\xi)\otimes\mathfrak{s}^2) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_1(-\xi)\otimes\mathfrak{s}^{-2})\right)\end{aligned}$$

By definition we have $P(\mathcal{E}) = P_1(\mathcal{E})P_2(\mathcal{E})$ with

$$\begin{aligned}P_1(\mathcal{E}) &= \mathsf{X}_{-y}(T_M^{\mathrm{vir}}) = \mathsf{X}_{-y}(-R\mathcal{H}om_\pi(\mathcal{E},\mathcal{E})_0),\\ P_2(\mathcal{E}) &= \mathrm{ch}(\mu(L)) = \exp\left(\left(c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2\right)/PD(c_1(L))\right).\end{aligned}$$

Thus we see

$$c_2(\mathcal{F}) - \frac{1}{4}c_1(\mathcal{F})^2 = -\mathrm{ch}_2\left(\mathcal{I}_1\left(-\frac{\xi}{2}\right)\otimes\mathfrak{s}^{-1}\right) - \mathrm{ch}_2\left(\mathcal{I}_2\left(\frac{\xi}{2}\right)\otimes\mathfrak{s}\right),$$

Thus

$$\begin{aligned}P_2(\mathcal{F}) &= \exp\left(-\mathrm{ch}_2\left(\mathcal{I}_1\left(-\frac{\xi}{2}\right)\otimes\mathfrak{s}^{-1}\right)/PD(c_1(L)) - \mathrm{ch}_2\left(\mathcal{I}_2\left(\frac{\xi}{2}\right)\otimes\mathfrak{s}\right)/PD(c_1(L))\right)\\ &= \exp\left(\nu(L,\xi)\right).\end{aligned}$$

Now we compute $P_1(\mathcal{F})$. We have

$$T_M^{\mathrm{vir}} = -R\mathcal{H}om_\pi(\mathcal{E},\mathcal{E})_0 = -R\mathcal{H}om_\pi(\mathcal{E},\mathcal{E}) + R\pi_*(\mathcal{O}),$$

where we write $\mathcal{O} = \mathcal{O}_{S\times S^{[n_1]}\times S^{[n_2]}}$. We get

$$\begin{aligned}
&-R\mathcal{H}om_\pi(\mathcal{F},\mathcal{F}) + R\pi_*(\mathcal{O})\\
&\quad= -R\mathcal{H}om_\pi\left(\mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1}\oplus\mathcal{I}_2(a_2)\otimes\mathfrak{s}, \mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1}\oplus\mathcal{I}_2(a_2)\otimes\mathfrak{s}\right) + R\pi_*(\mathcal{O})\\
&\quad= -R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_1) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_2) - R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2(\xi)\otimes\mathfrak{s}^2) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_1(-\xi)\otimes\mathfrak{s}^{-2}) + R\pi_*(\mathcal{O})\\
&\quad= T_{S^{[n_1]}\times S^{[n_2]}} - R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2(\xi)\otimes\mathfrak{s}^2) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_1(-\xi)\otimes\mathfrak{s}^{-2}) - R\pi_*(\mathcal{O})
\end{aligned}$$

Here we use that by Lemma 3

$$\begin{aligned}
T_{S^{[n_1]}\times S^{[n_2]}} &= -R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_1)_0 - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_2)_0\\
&= -R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_1) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_2) + 2R\pi_*(\mathcal{O}).
\end{aligned}$$

Note that $R\pi_*(\mathcal{O}) = (\mathcal{O}_{S^{[n_1]}\times S^{[n_2]}})^{\oplus\chi(\mathcal{O}_S)}$, in particular $\mathsf{X}_{-y}(R\pi_*(\mathcal{O})) = 1$. Therefore we get

$$\begin{aligned}
P_1(\mathcal{F}) &= \mathsf{X}_{-y}(-R\mathcal{H}om_\pi(\mathcal{E},\mathcal{E})_0)\\
&= \mathsf{X}_{-y}\big(T_{S^{[n_1]}\times S^{[n_2]}} - R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2(\xi)\otimes\mathfrak{s}^2) - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_1(-\xi)\otimes\mathfrak{s}^{-2})\big).
\end{aligned}$$

□

3. Partition function

We now introduce the following partition function, which will be the main object of our study in the rest of the paper: we will show in the rest of this chapter that the K-theoretic Donaldson invariants can be expressed in terms of this partition function. On the other hand, we will show in the following 2 chapters how to compute the partition function.

DEFINITION 11. *For any a in the Chow group $A^1(S)$ we abbreviate $\chi(a) := \chi(\mathcal{O}_S(a))$. We will write $\mathcal{O} = \mathcal{O}_{S^{[n_1]}\times S^{[n_2]}}$, and furthermore $\chi(a)\otimes\mathcal{O}$ for $\mathcal{O}^{\oplus\chi(a)}$. For any $a_1, c_1 \in A^1(S)$, we put $\xi = c_1 - 2a_1$ and define the* partition function *by*

$$Z_S(L,a_1,c_1,s,y,q) :=$$

$$\sum_{n_1,n_2\geq 0} q^{n_1+n_2}\int_{S^{[n_1]}\times S^{[n_2]}} \frac{\mathsf{X}_{-y}(E_{n_1,n_2})\exp\big((\pi_*([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L))\big)\,\mathrm{Eu}(\mathcal{O}(a_1)^{[n_1]})\,\mathrm{Eu}(\mathcal{O}(c_1-a_1)^{[n_2]}\otimes\mathfrak{s}^2)}{\mathrm{Eu}\big(E_{n_1,n_2} - T_{S^{[n_1]}\times S^{[n_2]}}\big)}.$$

Here we denote

$$E_{n_1,n_2} := -R\mathcal{H}om_\pi(\mathcal{F},\mathcal{F})_0 + \chi(\mathcal{O}_S)\otimes\mathcal{O} + \chi(\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^2 + \chi(-\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2}$$

with

$$\mathcal{F} = \mathcal{I}_1(a_1)\otimes\mathfrak{s}^{-1}\oplus\mathcal{I}_2(a_2)\otimes\mathfrak{s}.$$

This can be rewritten as

$$\begin{aligned} E_{n_1,n_2} = T_{S^{[n_1]}\times S^{[n_2]}} &+ \chi(\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^2 - R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2(\xi))\otimes\mathfrak{s}^2 \\ &+ \chi(-\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2} - R\mathcal{H}om_\pi(\mathcal{I}_2,\mathcal{I}_1(-\xi))\otimes\mathfrak{s}^{-2}. \end{aligned} \tag{3.1}$$

One checks directly by Riemann-Roch that the complex E_{n_1,n_2} has rank $4n_1+4n_2$. If $n_1=n_2=0$, we see that

$$\begin{aligned} E_{0,0} &= \chi(c_1-2a_1)\otimes\mathcal{O}\otimes\mathfrak{s}^2 - R\mathcal{H}om_\pi(\mathcal{O}_S(a_1),\mathcal{O}_S(c_1-a_1))\otimes\mathfrak{s}^2 \\ &+ \chi(2a_1-c_1)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2} - R\mathcal{H}om_\pi(\mathcal{O}_S(c_1-a_1),\mathcal{O}_S(a_1))\otimes\mathfrak{s}^{-2} = 0, \\ T_{S^{[0]}\times S^{[0]}} &= 0, \quad \nu(L,\xi)=0, \quad \mathcal{O}(a)^{[0]}=0. \end{aligned}$$

Thus the coefficient of q^0 of $Z_S(L,a_1,c_1,s,y,q)$ is 1, and we can see from the definitions that

$$Z_S(L,a_1,c_1,s,y,q)\in 1+q\,\mathbb{Q}[y]((s))[[q]]. \tag{3.2}$$

We write

$$\widetilde{\mathcal{A}}_P(a_1,c_1-a_2,c_2,s) = \sum_{n_1+n_2=c_2-a_1(c_1-a_1)}\int_{S^{[n_1]}\times S^{[n_2]}}\widetilde{\Psi}_P(a_1,c_1-a_1,n_1,n_2,s). \tag{3.3}$$

for $\widetilde{\mathcal{A}}(L,a_1,c_1-a_2,c_2,s)$ in case $P(\mathcal{E})=\mathsf{X}_{-y}(T_M^{\mathrm{vir}})$. Now we express $\widetilde{\mathcal{A}}_P(a_1,c_1-a_2,c_2,s)$ in terms of the partition function $Z_S(L,a_1,c_1,s,y,q)$.

Corollary 1. *Suppose we have a decomposition $c_1=a_1+a_2$. Then*

$$\begin{aligned} &\sum_{c_2\in\mathbb{Z}}\widetilde{\mathcal{A}}_P(a_1,c_1-a_1,c_2,s)\,q^{c_2} \\ &= e^{-(\xi L)s}(2s)^{\chi(\mathcal{O}_S)}\left(\frac{1-e^{-2s}}{1-ye^{-2s}}\right)^{\chi(\xi)}\left(\frac{1-e^{2s}}{1-ye^{2s}}\right)^{\chi(-\xi)} q^{c_1(c_1-a)}Z_S\Big(L,a_1,c_1,s,y,\frac{q}{2s}\Big). \end{aligned} \tag{3.4}$$

PROOF. We put the formula for $\widetilde\Psi_P(a_1,c_1-a_1,n_1,n_2,s)$ of Lemma 5 into (3.3) and compare the result to Definition 11. Thus we get

$$\begin{aligned}
&\sum_{c_2\in\mathbb{Z}}\widetilde{\mathcal{A}}_P(a_1,c_1-a_1,c_2,s)\,q^{c_2}=\sum_{c_2\in\mathbb{Z}}\ \sum_{n_1+n_2=c_2-a_1(c_1-a_1)}\int_{S^{[n_1]}\times S^{[n_2]}}\widetilde\Psi_P(a_1,c_1-a_1,n_1,n_2,s)\,q^{c_2}\\
&=q^{c_1(c_1-a)}\sum_{n_1,n_2\ge 0}\int_{S^{[n_1]}\times S^{[n_2]}}\widetilde\Psi_P(a_1,c_1-a_1,n_1,n_2,s)q^{n_1+n_2}\\
&=q^{c_1(c_1-a)}\sum_{n_1,n_2\ge 0}q^{n_1+n_2}\int_{S^{[n_1]}\times S^{[n_2]}}\Bigg[\frac{\mathsf{X}_{-y}(F_{n_1,n_2})\exp\big(\pi_*([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L)\big)}{\mathrm{Eu}\big(F_{n_1,n_2}-T_{S^{[n_1]}\times S^{[n_2]}}\big)(2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}\\
&\qquad\qquad\cdot\mathrm{Eu}(\mathcal{O}(a_1)^{[n_1]})\mathrm{Eu}(\mathcal{O}(c_1-a_1)^{[n_2]}\otimes\mathfrak{s}^2)\Bigg]\\
&=\frac{\mathsf{X}_{-y}\big(-\chi(\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^2-\chi(-\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2}\big)}{\mathrm{Eu}\big(-\chi(\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^2-\chi(-\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2}\big)}q^{c_1(c_1-a)}\\
&\quad\cdot\sum_{n_1,n_2\ge 0}q^{n_1+n_2}\int_{S^{[n_1]}\times S^{[n_2]}}\Bigg[\frac{\mathsf{X}_{-y}(E_{n_1,n_2})\exp\big(-(\xi L)s+\pi_*([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L)\big)}{\mathrm{Eu}\big(E_{n_1,n_2}-T_{S^{[n_1]}\times S^{[n_2]}}\big)(2s)^{n_1+n_2-\chi(\mathcal{O}_S)}}\\
&\qquad\qquad\cdot\mathrm{Eu}(\mathcal{O}(a_1)^{[n_1]})\mathrm{Eu}(\mathcal{O}(c_1-a_1)^{[n_2]}\otimes\mathfrak{s}^2)\Bigg]\\
&=e^{-(\xi L)s}(2s)^{\chi(\mathcal{O}_S)}\left(\frac{1-e^{-2s}}{1-ye^{-2s}}\right)^{\chi(\xi)}\left(\frac{1-e^{2s}}{1-ye^{2s}}\right)^{\chi(-\xi)}q^{c_1(c_1-a)}\\
&\quad\cdot\sum_{n_1,n_2\ge 0}\left(\frac{q}{2s}\right)^{n_1+n_2}\int_{S^{[n_1]}\times S^{[n_2]}}\Bigg[\frac{\mathsf{X}_{-y}(E_{n_1,n_2})\exp\big(\pi_*([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L)\big)}{\mathrm{Eu}\big(E_{n_1,n_2}-T_{S^{[n_1]}\times S^{[n_2]}}\big)}\\
&\qquad\qquad\cdot\mathrm{Eu}(\mathcal{O}(a_1)^{[n_1]})\mathrm{Eu}(\mathcal{O}(c_1-a_1)^{[n_2]}\otimes\mathfrak{s}^2)\Bigg]\\
&=e^{-(\xi L)s}(2s)^{\chi(\mathcal{O}_S)}\left(\frac{1-e^{-2s}}{1-ye^{-2s}}\right)^{\chi(\xi)}\left(\frac{1-e^{2s}}{1-ye^{2s}}\right)^{\chi(-\xi)}q^{c_1(c_1-a)}Z_S(L,a_1,c_1,s,y,\frac{q}{2s}).
\end{aligned}$$

Here in the third line we have put

$$F_{n_1,n_2}=E_{n_1,n_2}-\chi(\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^2-\chi(-\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2}.$$

In the fourth line we use Lemma 4. The fifth line follows by noticing that by definition of X_{-y} and Eu we have

$$\mathsf{X}_{-y}(\mathfrak{s}^2)=\frac{1-e^{-2s}}{2s(1-ye^{-2s})},\quad \mathrm{Eu}(\mathfrak{s}^2)=c_1(\mathfrak{s}^2)=2s.$$

and therefore

$$\frac{\mathsf{X}_{-y}\big(-\chi(\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^2-\chi(-\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2}\big)}{\mathrm{Eu}\big(-\chi(\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^2-\chi(-\xi)\otimes\mathcal{O}\otimes\mathfrak{s}^{-2}\big)}=\left(\frac{1-e^{-2s}}{1-ye^{-2s}}\right)^{\chi(\xi)}\left(\frac{1-e^{2s}}{1-ye^{2s}}\right)^{\chi(-\xi)}$$

$\square$

Now we specialize this result to the non-refined K-theoretic Donaldson invariants. We put $P_0(\mathcal{E}) = P(\mathcal{E})\big|_{y=0}$, so that by Proposition 3

$$\chi^{\mathrm{vir}}_{-y}(M^H_S(r,c_1,c_2),\mu(L)) = \int_{[M^H_S(r,c_1,c_2)]^{\mathrm{vir}}} P_0(\mathcal{E}).$$

We put

$$\widetilde{\mathcal{A}}_{P_0}(a_1,c_1-a_2,c_2,s) := \widetilde{\mathcal{A}}_P(a_1,c_1-a_2,c_2,s)\big|_{y=0},$$
$$Z_S(L,a_1,c_1,s,q) := Z_S(L,a_1,c_1,s,y,q)\big|_{y=0}.$$

Then by specializing $y=0$ in Corollary 1 we immediately get the following.

Corollary 2. *Suppose we have a decomposition $c_1 = a_1 + a_2$. Then*

$$\begin{aligned}&\sum_{c_2\in\mathbb{Z}} \widetilde{\mathcal{A}}_{P_0}(a_1,c_1-a_1,c_2,s)\,q^{c_2}\\ &\quad= e^{-(\xi L)s}(2s)^{\chi(\mathcal{O}_S)}\left(1-e^{-2s}\right)^{\chi(\xi)}\left(1-e^{2s}\right)^{\chi(-\xi)} q^{c_1(c_1-a)} Z_S\Big(L,a_1,c_1,s,\frac{q}{2s}\Big).\end{aligned} \tag{3.5}$$

For future reference we note that the definitions of $Z_S(L,a,c_1,s,y,q)$, $Z_S(L,a,c_1,s,q)$ make sense for any *possibly disconnected* smooth projective surface S and $L,a,c_1\in A^1(S)$.

4. The K-theoretic Donaldson invariants in terms of the partition function

In this section we will prove a formula that expresses the refined and nonrefined K-theoretic Donaldson invariants in terms of the partition functions $Z_S(L,a_1,c_1,s,y,q)$ and $Z_S(L,a_1,c_1,s,q)$, and Seiberg-Witten invariants.

Corollary 3. *Suppose S satisfies $b_1(S)=0$ and $p_g(S)>0$. Let H,c_1,c_2 be chosen such that there exist no rank 2 strictly Gieseker H-semistable sheaves with Chern classes c_1,c_2. Let $L\in\mathrm{Pic}(S)$. Assume furthermore that:*

(i) $c_2 < \frac{1}{2}c_1(c_1-K_S)+2\chi(\mathcal{O}_S)$.

(ii) *$p_c > p_{K_S}$, where p_c and p_{K_S} are the reduced Hilbert polynomials associated to the class $c\in K^0(S)$ of an element in $M^H_S(2,c_1,c_2)$ and K_S.*

(iii) *For all SW basic classes a_1 satisfying $a_1H\le(c_1-a_1)H$ the inequality is strict.*

Denote $\mathrm{vd} = 4c_2-c_1^2-3\chi(\mathcal{O}_S)$ the expected dimension of $M^H_S(2,c_1,c_2)$. Put $\xi := c_1-2a_1$ (note that therefore in the sums below ξ depends on a_1). Then we have

(1)

$$\chi_{-y}^{\rm vir}(M_S^H(2,c_1,c_2),\mu(L)) = -2^{1-\chi(c)}\mathrm{Coeff}_{s^0}\mathrm{Coeff}_{x^{\rm vd}}\Bigg[\sum_{\substack{a_1\,\in\,H^2(S,\mathbb{Z})\\ a_1H\,<\,(c_1-a_1)H}}\bigg(SW(a_1)e^{-(\xi L)s}(2s)^{\chi(\mathcal{O}_S)}$$
$$\cdot\left(\frac{1-e^{-2s}}{1-ye^{-2s}}\right)^{\chi(\xi)}\left(\frac{1-e^{2s}}{1-ye^{2s}}\right)^{\chi(-\xi)}x^{3c_1^2-4c_1a_1-3\chi(\mathcal{O}_S)}Z_S\Big(L,a_1,c_1,s,y,\frac{x^4}{2s}\Big)\bigg)\Bigg],$$

(2)

$$\chi^{\rm vir}(M_S^H(2,c_1,c_2),\mu(L)) = -2^{1-\chi(c)}\mathrm{Coeff}_{s^0}\mathrm{Coeff}_{x^{\rm vd}}\Bigg[\sum_{\substack{a_1\,\in\,H^2(S,\mathbb{Z})\\ a_1H\,<\,(c_1-a_1)H}}SW(a_1)e^{-(\xi L)s}(2s)^{\chi(\mathcal{O}_S)}$$
$$\cdot\left(1-e^{-2s}\right)^{\chi(\xi)}\left(1-e^{2s}\right)^{\chi(-\xi)}x^{3c_1^2-4c_1a_1-3\chi(\mathcal{O}_S)}Z_S\Big(L,a_1,c_1,s,\frac{x^4}{2s}\Big)\Bigg].$$

PROOF. The Mochizuki formula Theorem 9, says that under the assumptions of the corollary we have

$$\chi_{-y}^{\rm vir}(M_S^H(c_1,c_2),\mu(L)) = -2^{1-\chi(c)}\mathrm{Coeff}_{s^0}\mathrm{Coeff}_{q^{c_2}}\sum_{\substack{a_1\,\in\,H^2(S,\mathbb{Z})\\ a_1H\,<\,(c_1-a_1)H}}SW(a_1)\sum_{c_2\in\mathbb{Z}}\widetilde{\mathcal{A}}_P(a_1,c_1-a_1,c_2,s)q^{c_2}.$$

Replacing the inner sum

$$\sum_{c_2\in\mathbb{Z}}\widetilde{\mathcal{A}}_P(a_1,c_1-a_1,c_2,s)q^{c_2}$$

with the right hand side of the formula 3.4 from Corollary 1 and using that $\mathrm{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$, the result for $\chi_{-y}^{\rm vir}(M_S^H(c_1,c_2),\mu(L))$ follows immediately. The proof for $\chi^{\rm vir}(M_S^H(c_1,c_2),\mu(L))$ is the same, using Corollary 2 instead of Corollary 1. □

REMARK 2. *Assuming the strong form of the Mochizuki formula Conjecture 4 of Chapter 1, we get a simpler (conjectural) version of this corollary with wider applicability. In fact, assuming Conjecture 4, we get that Corollary 3 holds without assuming (ii) and (iii) and with*

$$\textit{the sum}\quad\sum_{\substack{a_1\,\in\,H^2(S,\mathbb{Z})\\ a_1H\,<\,(c_1-a_1)H}}\quad\textit{replaced by}\quad\sum_{a_1\in H^2(S,\mathbb{Z})}.$$

We will also refer to this formula as the strong from of Mochizuki's formula.

Corollary 3 reduces the determination of the K-theoretic Donaldson invariants to the computation of the partition functions $Z_S(L,a_1,c_1,s,y,q)$ and $Z_S(L,a_1,c_1,s,q)$. In the next two chapters we will compute these two partition functions for any quadruple (S,L,a_1,c_1) of a surface and 3 elements of $\mathrm{Pic}(S)$ up to a certain power in q. This allows us to compute in Chapter 6 the (refined) K-theoretic Donaldson invariants for many surfaces, by just plugging the partition functions, the intersection numbers of the (S, L, a_1, c_1) and the Seiberg-Witten invariants of S into Corollary 3.

CHAPTER 3

Universality and Multiplicativity of K-theoretic Donaldson Invariants

In this chapter and the next we want to compute the partition functions $Z_S(L, a, c_1, s, y, q)$ and $Z_S(L, a, c_1, s, q)$ for any quadruple (S, L, a_1, c_1) of a surface and 3 elements of $\mathrm{Pic}(S)$, up to a suitable power of q. In this chapter we will establish two crucial properties of $Z_S(L, a, c_1, s, y, q)$.

(1) Universality: $Z_S(L, a, c_1, s, y, q)$ depends only on the 11 intersection numbers

$$L^2,\ La,\ a^2,\ ac_1,\ c_1^2,\ Lc_1,\ LK_S,\ aK_S,\ c_1K_S,\ K_S^2,\ \chi(\mathcal{O}_S).$$

More precisely the coefficient of any monomial in q, y, s of $Z_S(L, a_1, c_1, s, y, q)$ is given by a universal polynomial in the above 11 intersection numbers.

(2) Multiplicativity: furthermore there are 11 universal power series $A_1(y), \ldots, A_{11}(y) \in \mathbb{Q}((s))[y][[q]]$ such that

$$\begin{aligned} Z_S(L, a, c_1, s, y, q) = A_1(y)^{L^2} A_2(y)^{La} A_3(y)^{a^2} A_4(y)^{ac_1} A_5(y)^{c_1^2} A_6(y)^{Lc_1} \\ A_7(y)^{LK_S} A_8(y)^{aK_S} A_9(y)^{c_1K_S} A_{10}(y)^{K_S^2} A_{11}(y)^{\chi(\mathcal{O}_S)}. \end{aligned}$$

As $Z_S(L, a, c_1, s, q) = Z_S(L, a, c_1, s, y, q)\big|_{y=0}$ it is clear that universality and multiplicativity also hold for $Z_S(L, a, c_1, s, q)$ with $A_1(y), \ldots, A_{11}(y)$ replaced by $A_1, \ldots, A_{11}$, where $A_i = A_i(y)\big|_{y=0}$.

These results will allow us in the next chapter to reduce our computation to the case that $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$ and use localization to compute the power series $A_1(y), \ldots, A_{11}(y)$ and $A_1, \ldots, A_{11}$ up to suitable powers of q, and thus also $Z_S(L, a, c_1, s, y, q)$ and $Z_S(L, a, c_1, s, q)$ for any (S, L, a, c_1).

Universality results for intersection numbers on Hilbert schemes of points and multiplicativity of their generating functions have been first proved in [**4**] for Chern numbers of the Hilbert schemes and intersection numbers of tautological sheaves. These arguments have been adapted and refined successively in [**17**],[**18**],[**19**],[**15**] to deal with the intersection numbers necessary for the wall-crossing of Donaldson invariants, the proof of the Witten conjecture and (refined) Vafa-Witten invariants. We have to further adapt these arguments in order to deal with the partition function $Z_S(L, a, c_1, s, y, q)$. These results and methods are scattered over several papers, so to make the logic of the arguments clear and make this work more self-contained, we will not just indicate the additional changes necessary to the above papers, but will give a complete proof only using results of [**4**], which we cite precisely.

1. Universality

1.1. Case of Hilbert schemes. We start by reviewing some of the intermediate results in the universality proof in [**4**]. Let S be a smooth projective surface. Let $p : S^{[n]} \times S \to S^{[n]}$, $q : S^{[n]} \times S \to S$ be the projections. Let $Z_n(S) \subset S^{[n]} \times S$ be the universal subscheme, and $\mathcal{I}_{Z_n(S)}$ its ideal sheaf.

$$\begin{tikzcd} Z_n(S) \arrow[r, hook] & S^{[n]} \times S \arrow[r, "q"] \arrow[d, "p"] & S \\ & S^{[n]} & \end{tikzcd}$$

Let $\sigma : \mathbb{P}(\mathcal{I}_{Z_n(S)}) \to S^{[n]} \times S$ be the natural projection, and denote $\rho := q \circ \sigma$, $\phi := p \circ \sigma$. Let $j := (id, \rho) : \mathbb{P}(\mathcal{I}_{Z_n(S)}) \to P(\mathcal{I}_{Z_n(S)}) \times S$. Following [**4**] we use the following notation.

Notation 1. *For $f : X \to Y$ a morphism, we write $f_S := (f \times 1_S) : X \times S \to Y \times S$.*

In [**4**, Section 1] it is shown that there is a surjective morphism $\psi : \mathbb{P}(\mathcal{I}_{Z_n(S)}) \to S^{[n+1]}$. More precisely the following is shown: Let $S^{[n,n+1]}$ be the incidence variety

$$S^{[n,n+1]} := \{(Z, W) \in S^{[n]} \times S^{[n+1]} \mid Z \text{ is a subscheme of } W\}.$$

Then there is a natural isomorphism $\mathbb{P}(\mathcal{I}_{Z_n(S)}) \to S^{[n,n+1]}$, sending a one dimensional quotient $\alpha : I_Z \to k(x)$ over (x, Z) to (Z, W) with $I_{W/Z} = \ker(\alpha)$. This isomorphism identifies ϕ and ψ with the projections of $S^{[n]} \times S^{[n+1]}$ to the two factors, and $\rho(Z, W) = \mathrm{supp}(I_{W/Z})$. Thus we have the commutative diagrams

$$\begin{tikzcd} S & \mathbb{P}(\mathcal{I}_{Z_n(S)}) \arrow[l, "\rho"'] \arrow[r, "\psi"] \arrow[dl, "\sigma"] \arrow[d, "\phi"] & S^{[n+1]} \\ S^{[n]} \times S \arrow[u, "q"] \arrow[r, "p"] & S^{[n]} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{P}(\mathcal{I}_{Z_n(S)}) \arrow[r, "j"] \arrow[d, "\rho"] & \mathbb{P}(\mathcal{I}_{Z_n(S)}) \times S \arrow[d, "\rho_S"] \arrow[l, bend right, "pr_0"'] \\ S \arrow[r, "\delta"] & S \times S \end{tikzcd}$$

where $\delta : S \to S \times S$ is the diagonal map, and we will also in the future in a product $X \times S^n$ denote by pr_0 the projection to X. We denote $\Delta \subset S \times S$ the diagonal.

We denote $\mathcal{L} := \mathbb{P}(\mathcal{I}_{Z_n(S)})(1)$, and $\ell = c_1(\mathcal{L})$. We have the following Lemma ([**4**, Lemma 1.1]).

Lemma 6.

$$\sigma_*(\ell^i) = (-1)^i c_i(\mathcal{O}_{Z_n(S)}) = (-1)^i c_i(-\mathcal{I}_{Z_n(S)}).$$

The following two identities in $K^0(S^{[n,n+1]})$ from [eqs. (5), (6)][**4**] describe the universal ideal sheaves $\mathcal{I}_{Z_n(S)}$ and the universal structure sheaves $\mathcal{O}_{Z_n(S)}$ inductively.

$$\psi_S^!(\mathcal{I}_{Z_{n+1}(S)}) = \phi_S^!(\mathcal{I}_{Z_n(S)}) - j_*(\mathcal{L}), \tag{1.1}$$

$$\psi_S^!(\mathcal{O}_{Z_{n+1}(S)}) = \phi_S^!(\mathcal{O}_{Z_n(S)}) + j_*(\mathcal{L}). \tag{1.2}$$

Note that by the last diagram we have $j_!(\mathcal{O}_{\mathbb{P}(\mathcal{I}_{Z_n(S)})}) = \rho_S^!(\mathcal{O}_\Delta)$, thus by the projection formula we have

$$j_*(\mathcal{L}) = j_!\big(\mathcal{O}_{\mathbb{P}(\mathcal{I}_{Z_n(S)})} \otimes^L j^!(pr_0^!(\mathcal{L}))\big) = pr_0^!(\mathcal{L}) \otimes^L \rho_S^!(\mathcal{O}_\Delta).$$

Thus we can rewrite these formulas as

$$\psi_S^!(\mathcal{I}_{Z_{n+1}(S)}) = \phi_S^!(\mathcal{I}_{Z_n(S)}) - pr_0^!(\mathcal{L}) \otimes^L \rho_S^!(\mathcal{O}_\Delta), \tag{1.3}$$

$$\psi_S^!(\mathcal{O}_{Z_{n+1}(S)}) = \phi_S^!(\mathcal{O}_{Z_n(S)}) + pr_0^!(\mathcal{L}) \otimes^L \rho_S^!(\mathcal{O}_\Delta). \tag{1.4}$$

Note also that $\mathrm{ch}_2(\mathcal{I}_{Z_{n+1}(S)}) = -[Z_{n+1}(S)]$ where $[Z_{n+1}(S)]$ is the cohomology class Poincaré dual to the fundamental class of $Z_{n+1}(S)$, and similar for $Z_n(S)$. On the other hand we have $\mathrm{ch}_2(M \otimes^L \rho_S^! \mathcal{O}_\Delta) = \rho_S^*[\Delta]$ for any line bundle M on $\mathbb{P}(\mathcal{I}_{Z_n(S)}) \times S$, Thus (1.3) also gives

$$\psi_S^*([Z_{n+1}(S)]) = \phi_S^*([Z_n(S)]) + \rho_S^*([\Delta]) \in H^2(S^{[n,n+1]} \times S, \mathbb{Q}). \tag{1.5}$$

Let F be a vector bundle on S. Let $F^{[n]}$ be the corresponding tautological sheaf on $S^{[n]}$. [**4**, Lemma 2.1] inductively describes the tautological sheaves $F^{[n]}$ on $S^{[n]}$.

Lemma 7. *In $K^0(S^{[n,n+1]}$ we have the relation*

$$\psi^!(F^{[n+1]}) = \phi^!(F^{[n]}) + \mathcal{L} \otimes^L \rho^!(F).$$

Now we describe the tangent bundle of the Hilbert schemes. The first step is [**4**, Proposition 2.2].

Proposition 4. *The class of $T_{S^{[n]}}$ in $K^0(S^{[n]})$ is given by*

$$T_{S^{[n]}} = \chi(\mathcal{O}_S) \otimes^L \mathcal{O} - p_!(I^\vee_{Z_n(S)} \otimes^L I_{Z_n}(S)).$$

Using this, in [**4**, Proposition 2.3] the tangent bundle of the Hilbert schemes is described inductively.

Proposition 5. *In $K^0(S^{[n,n+1]})$ we have the relation*

$$\psi^! T_{S^{[n+1]}} = \phi^! T_{S^{[n]}} + \mathcal{L} \otimes^L \sigma^! I_{Z_n(S)} \otimes^L \rho^!(\omega_S^\vee) - \rho^!(1 - T_S + \omega_S^\vee).$$

In the proof of this proposition the following elementary identities are obtained.

$$(pr_0)_!\big(\rho_S^!(\mathcal{O}_\Delta)\phi_S^!(\mathcal{I}^\vee_{Z_n(S)})\big) = \sigma^!(\mathcal{I}^\vee_{Z_n(S)}) \tag{1.6}$$

$$(pr_0)_!\big(\rho_S^!(\mathcal{O}_\Delta^\vee) \otimes^L \phi_S^!(\mathcal{I}_{Z_n(S)})\big) = \sigma^!(\mathcal{I}_{Z_n(S)}) \otimes^L \rho^!(\omega_S^\vee) \tag{1.7}$$

$$(pr_1)_!(\mathcal{O}_\Delta^\vee \otimes^L \mathcal{O}_\Delta) = \mathcal{O}_S - T_S + \omega_S^\vee. \tag{1.8}$$

1.2. Application to the product. Now we apply this to $S^{[n_1]} \times S^{[n_2]}$. Let

$$\pi_1 : S^{[n_1]} \times S^{[n_2]} \to S^{[n_1]}, \quad \pi_2 : S^{[n_1]} \times S^{[n_2]} \to S^{[n_1]}$$

$$q : S^{[n_1]} \times S^{[n_2]} \times S \to S, \quad p : S^{[n_1]} \times S^{[n_2]} \times S \to S^{[n_1]} \times S^{[n_2]}$$

be the projections. For $\alpha = 1, 2$ we use the following subvarieties, sheaves and maps, which are the pullbacks of the corresponding objects from above section. Let $Z_\alpha := \pi_{\alpha S}^{-1}(Z_{n_\alpha}(S))$, and $\mathcal{I}_\alpha = \mathcal{I}_{Z_\alpha}$, $\mathcal{O}_\alpha = \mathcal{O}_{Z_\alpha}$. We have the two incidence varieties

$$\mathbb{P}(\mathcal{I}_1) = S^{[n_1,n_1+1]} \times S^{[n_2]}, \quad \mathbb{P}(\mathcal{I}_2) = S^{[n_1]} \times S^{[n_2,n_2+1]}.$$

We denote

$$\mathcal{L}_\alpha = \mathbb{P}(\mathcal{I}_\alpha)(1) = \pi^*_{S^{[n_\alpha,n_\alpha+1]}}(\mathcal{L}), \quad \ell_\alpha = c_1(\mathcal{L}^\alpha), \quad F_\alpha^{[n_\alpha]} := \pi_\alpha^*(F^{[n_\alpha]}),$$

$$\sigma_1 = \sigma \times 1_{S^{[n_2]}}, \quad \sigma_2 = 1_{S^{[n_1]}} \times \sigma, \quad \rho_1 := q \circ \sigma_1, \quad \phi_1 := p \circ \sigma_1 = \phi \times 1_{S^{[n_2]}},$$

$$\psi_1 = \psi \times 1_{S^{[n_2]}}, \quad j_1 := (1_{\mathbb{P}(\mathcal{I}_1)}, \rho_1),$$

and similar for $\alpha = 2$. To simplify our notations we always use the following notation.

Notation 2. *We write the products in the order $X \times S$ and view e.g σ_1, σ_2 as maps $\mathbb{P}(\mathcal{I}_\alpha) \to S^{[n_1]} \times S^{[n_2]} \times S$.*

As all the objects are just pullbacks from the objects considered before for $S^{[n]}$, the results above immediately give the following.

$$\sigma_\alpha(\ell_\alpha^i) = (-1)^i c_i(\mathcal{O}_\alpha) = (-1)^i c_i(-\mathcal{I}_\alpha) \tag{1.9}$$

$$(\psi_\alpha)^!_S(\mathcal{I}_\alpha) = (\phi_\alpha)^!_S(\mathcal{I}_\alpha) - p^!(\mathcal{L}_\alpha) \otimes^L (\rho_\alpha)^!_S(\mathcal{O}_\Delta), \quad (\psi_{3-\alpha})^!_S(\mathcal{I}_\alpha) = (\phi_{3-\alpha})^!_S(\mathcal{I}_\alpha), \tag{1.10}$$

$$(\psi_\alpha)^!_S(\mathcal{O}_\alpha) = (\phi_\alpha)^!_S(\mathcal{O}_\alpha) + p^!(\mathcal{L}_\alpha) \otimes^L (\rho_\alpha)^!_S(\mathcal{O}_\Delta), \quad (\psi_{3-\alpha})^!_S(\mathcal{O}_\alpha) = (\phi_{3-\alpha})^!_S(\mathcal{O}_\alpha) \tag{1.11}$$

$$\psi^!_\alpha(F_\alpha^{[n_\alpha+1]}) = \phi^!(F_\alpha^{[n_\alpha]}) + \mathcal{L}_\alpha \otimes^L \rho^!_\alpha(F), \tag{1.12}$$

$$\psi^!_\alpha(T_{S^{[n_\alpha+1]}}) = \phi^!_\alpha(T_{S^{[n_\alpha]}}) + \mathcal{L}_\alpha \otimes^L \sigma^!_\alpha(\mathcal{I}_\alpha) \otimes^L \rho^!_\alpha(\omega_S^\vee) - \rho^!_\alpha(1 - T_S + \omega_S^\vee). \tag{1.13}$$

An additional ingredient of the Mochizuki formula are the

$$p_!(R\mathcal{H}om(\mathcal{I}_1, \mathcal{I}_2 \otimes^L q^!(M))), \quad p_!(R\mathcal{H}om(\mathcal{I}_2, \mathcal{I}_1 \otimes^L q^!(M))) \in K^0(S^{[n_1]} \times S^{[n_2]})$$

for M a line bundle on S. Note that by definition

$$p_!(R\mathcal{H}om(\mathcal{I}_\alpha, \mathcal{I}_{3-\alpha} \otimes^L q^!(M))) = p_!(\mathcal{I}_\alpha^\vee \otimes^L \mathcal{I}_{3-\alpha} \otimes^L q^!(M)).$$

We will now concentrate on the case $\alpha = 1$, the analogous arguments and results hold with the same proof for $\alpha = 2$. These sheaves are inductively determined by the following identities.f

$$(\psi_1)^! p_! (\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(M)) = \phi_1^! p_! (\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(M)) - \sigma_1^! \mathcal{I}_2 \otimes^L \rho_1^! (M \otimes^L \omega_S^\vee) \otimes^L \mathcal{L}_1^\vee, \tag{1.14}$$

$$(\psi_1)^! p_! (\mathcal{I}_2^\vee \otimes^L \mathcal{I}_1 \otimes^L q^*(M)) = \phi_1^! p_! (\mathcal{I}_1^\vee \otimes^L \mathcal{I}_1 \otimes^L q^*(M)) - \sigma_1^! (\mathcal{I}_2^\vee) \otimes^L \rho_1^! (M) \otimes^L \mathcal{L}_1. \tag{1.15}$$

These identities we get by applying (1.3), multiplying out and using the projection formula, where we also denote $p : \mathbb{P}(\mathcal{I}_1) \times S \to \mathbb{P}(\mathcal{I}_1)$, $q : \mathbb{P}(\mathcal{I}_1) \times S \to S$ the projections.

$$\begin{aligned}
(\psi_1)^! p_! (\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(M)) &= p_! (\psi_1)_S^! (\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(M)) \\
&= p_! \Big(\big((\phi_1)_S^! (\mathcal{I}_1^\vee) - \big(p^! (\mathcal{L}_1) \otimes^L (\rho_1)_S^! (\mathcal{O}_\Delta) \big)^\vee \big) \otimes^L \mathcal{I}_2 \otimes^L q^*(M) \Big) \\
&= \phi_1^! \big(p_! (\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(M) \big) - \sigma_1^! \mathcal{I}_2 \otimes^L \rho_1^! (M \otimes^L \omega_S^\vee) \otimes^L \mathcal{L}_1^\vee
\end{aligned}$$

where in the last step we have used (1.7). Similarly

$$\begin{aligned}
(\psi_1)^! p_! (\mathcal{I}_2^\vee \otimes^L \mathcal{I}_1 \otimes^L q^*(M)) &= p_! (\psi_1)_S^! (\mathcal{I}_2^\vee \otimes^L \mathcal{I}_1 \otimes^L q^*(M)) \\
&= \phi_1^! (p_! (\mathcal{I}_2^\vee \otimes^L \mathcal{I}_1 \otimes^L q^*(M)) - \sigma_1^! (\mathcal{I}_2^\vee) \otimes^L \rho_1^! (M) \otimes^L \mathcal{L}_1,
\end{aligned}$$

where in the last step we have used (1.6).

Finally $p_* \big(([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(c) \big) \in H^*(S^{[n_1]} \times S^{[n_2]}, \mathbb{Q})$ for $c \in H^2(S)$ is computed inductively by

$$(\psi_1)_S^* \big(([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(c) \big) = (\phi_1)_S^* \big(([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(c) \big) + (\rho_1)_S^* ([\Delta] pr_1^*(c)),$$

which follows immediately from (1.5), and which gives

$$(\psi_1)^* p_* \big(([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(c) \big) = \phi_1^* p_* \big(([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(c) \big) + \rho_1^*(c), \tag{1.16}$$

1.3. The inductive argument. We will prove the following statement.

PROPOSITION 6. *Let $P(S, A_1, A_2, A_3, A_4, A_5)$ be a polynomial in*

$$p_* \big(([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(A_1) \big),$$

the Chern classes of

$$(A_2)_1^{[n_1]}, \quad (A_3)_2^{[n_2]}, \quad p_! (\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^!(A_4)), \quad p_! (\mathcal{I}_2^\vee \otimes^L \mathcal{I}_2 \otimes^L q^!(A_5)), \quad T_{S^{[n_1]} \times S^{[n_2]}},$$

for tuples $(S, A_1, \dots, A_5)$ of a surface and 5 elements in $\operatorname{Pic}(S)$. Then there exists a polynomial Q in the intersection numbers

$$\int_S A_i A_j, \quad \int_S A_i K_S, \quad i, j = 1, \dots, 5, \qquad \int_S K_S^2, \quad \chi(\mathcal{O}_S),$$

such that

$$\int_{S^{[n_1]}\times S^{[n_2]}} P(S, A_1, A_2, A_3, A_4, A_5) = Q.$$

for all $(S, A_1, \ldots, A_5)$.

As a corollary we get the universality of the partition function.

Corollary 4. *The coefficient of any monomial* $q^n y^l s^k$ *in* $Z_S(L, a, c_1, s, y, q)$ *is a universal polynomial in the* 11 *intersection numbers*

$$L^2,\ La,\ a^2,\ ac_1,\ c_1^2,\ Lc_1,\ LK_S,\ aK_S,\ c_1K_S,\ K_S^2,\ \chi(\mathcal{O}_S).$$

Proof. By definition the coefficient of $q^n y^l s^k$ in $Z_S(L, a, c_1, s, y, q)$ is $\int_{S^{[n_1]}\times S^{[n_2]}} P$ for P a polynomial in $p_*\big(([\mathcal{Z}_1] + [\mathcal{Z}_2])q^*(L)\big)$, the Chern classes of

$$\mathcal{O}(a)_1^{[n_1]},\quad \mathcal{O}(c_1-a)_2^{[n_2]},\quad p_!(\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(c_1-2a)),\quad p_*(\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(2a-c_1)),\quad T_{S^{[n_1]}\times S^{[n_2]}}.$$

Thus the claim follows from Proposition 6 □

We will show Proposition 6 by an inductive argument. We want to relate integrals on $S^{[n_1+1]} \times S^{[n_2]} \times S^m$ and $S^{[n_1]} \times S^{[n_2+1]} \times S^m$ to integrals on $S^{[n_1]} \times S^{[n_2]} \times S^{m+1}$ We will show Proposition 6 using the following inductive statement. Let

$$W_1 := S^{[n_1,n_1+1]} \times S^{[n_2]} \times S^m, \quad W_2 := S^{[n_1]} \times S^{[n_2,n_2+1]} \times S^m.$$

For $\alpha = 1, 2$ let

$$\Psi_\alpha = \psi_\alpha \times 1_{S^m}, \quad \Phi_\alpha = \phi_\alpha \times 1_{S^m}.$$

For any $I \subset \{0, \ldots, m\}$ let pr_I be the projection $X \times S^m$ to the factors indexed by I.

Proposition 7. (1) *Let* f *be a polynomial in the Chern classes of the following sheaves on* $S^{[n_1+1]} \times S^{[n_2]} \times S^m$*:*

$$pr_0^* T_{S^{[n_1+1]}\times S^{[n_2+1]}},\quad pr_0^*(A_2)_1^{[n_1+1]},\quad pr_0^*(A_3)_1^{[n_2]}$$

$$pr_{0i}^!(\mathcal{I}_1),\quad pr_{0i}^!(\mathcal{I}_2),\quad pr_{ij}^*(\mathcal{O}_\Delta),\quad pr_i^*(T_S),$$

$$p_!(\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^!(A_4)),\quad p_!(\mathcal{I}_2^\vee \otimes^L \mathcal{I}_2 \otimes^L q^!(A_5)),$$

and the classes

$$p_*(([\mathcal{Z}_1] + [\mathcal{Z}_2])q^*(A_1)),$$

$$pr_i^*(A_j),\quad j = 1, \ldots 5.$$

Then there is a polynomial $\widetilde{f}$ in the analogous classes on $S^{[n_1]} \times S^{[n_2]} \times S^{m+1}$ such that

$$\int_{S^{[n_1+1]}\times S^{[n_2]}\times S^m} f = \int_{S^{[n_1]}\times S^{[n_2]}\times S^{m+1}} \widetilde{f}.$$

(2) *The analogous statement holds for $S^{[n_1]} \times S^{[n_2+1]} \times S^m$*

Proof. We show (1), the proof of (2) is the analogous. We write $W := W_1$, $\Psi := \Psi_1$, $\Phi = \Phi_1$. The morphism $\Psi : W \to S^{[n_1+1]} \times S^{[n_2]} \times S^m$ is generically finite of degree $n_1 + 1$. Therefore

$$\int_{S^{[n_1+1]}\times S^{[n_2]}\times S^m} f = \frac{1}{n_1+1}\int_W \Psi^*(f).$$

As we insert an additional factor S between $S^{[n_1]} \times S^{[n_2]}$ and S^m, we have

$$\Psi^! pr_i^* = \Phi^! pr_{i+1}, \quad \Psi^! pr_{0,i}^* = \Phi^! pr_{0,i+1}, \quad \Psi^! pr_{i,j}^* = \Phi^! pr_{i+1,j+1},$$

in particular we have

$$\Psi^! pr_i^* A_j = \Phi^! pr_{i+1}^* A_j, \quad \Psi^! pr_i^* T_S = \Phi^! pr_{i+1}^* T_S, \quad \Psi^! pr_{i,j}^*(\mathcal{O}_\Delta) = \Phi^! pr_{i+1,j+1}^*(\mathcal{O}_\Delta) \tag{1.17}$$

The formula (1.16) gives

$$\Psi^* p_*\big(([\mathcal{Z}_1] + [\mathcal{Z}_2])q^*(A_1)\big) = \Phi^* p_*\big(([\mathcal{Z}_1] + [\mathcal{Z}_2])q^*(A_1))\big) + pr_1^*(A_1), \tag{1.18}$$

Equations (1.10), (1.12), (1.13) give

$$\Psi^! pr_{0,i}^!(\mathcal{I}_1) = \Phi^! pr_{0,i+1}(\mathcal{I}_1) - pr_0^!(\mathcal{L}_1) \otimes^L pr_{1,i+1}^!(\mathcal{O}_\Delta), \quad \Psi^! pr_{0,i}^!(\mathcal{I}_2) = \Phi^! pr_{0,i+1}(\mathcal{I}_2), \tag{1.19}$$

$$\Psi^!((A_2)_1^{[n_1+1]}) = \Phi^!((A_2)_1^{[n_1]}) + pr_0^! \mathcal{L}_1 \otimes^L pr_1^!(A_2), \quad \Psi^!((A_3)_2^{[n_2]}) = \Phi^!((A_3)_2^{[n_2]}), \tag{1.20}$$

$$\Psi^!(T_{S^{[n_1+1]}\times S^{[n_2]}}) = \Phi^!(T_{S^{[n_1]}\times S^{[n_2]}}) + pr_0^!(\mathcal{L}_1) \otimes^L pr_{0,1}^!(\mathcal{I}_1) \otimes^L pr_1^!(\omega_S^\vee) - pr_1^!(1 - T_S + \omega_S^\vee). \tag{1.21}$$

Finally (1.14) and (1.15) give

(1.22)
$$\Psi^! p_!(\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(A_4)) = \Phi^!(p_!(\mathcal{I}_1^\vee \otimes^L \mathcal{I}_2 \otimes^L q^*(A_4)) - pr_{0,1}^!(\mathcal{I}_2) \otimes^L pr_1^!(A_4 \otimes^L \omega_S^\vee) \otimes^L pr_0^!(\mathcal{L}_1^\vee),$$

(1.23)
$$\Psi^! p_!(\mathcal{I}_2^\vee \otimes^L \mathcal{I}_1 \otimes^L q^*(A_5)) = \Phi^!(p_!(\mathcal{I}_1^\vee \otimes^L \mathcal{I}_1 \otimes^L q^*(A_5)) - pr_{0,1}^!(\mathcal{I}_2^\vee) \otimes^L pr_1^!(A_5) \otimes^L pr_0^!(\mathcal{L}_1).$$

Putting these results together we obtain that there are polynomials f_ν for $\nu \geq 0$, in the analogous classes on $S^{[n_1]} \times S^{[n_2]} \times S^{m+1}$, such that

$$\int_{S^{[n_1+1]}\times S^{[n_2]}\times S^m} f = \frac{1}{n_1+1}\int_W \Psi^*(f) = \int_W \sum_{\nu\geq 0} \Phi^*(f_\nu) \cdot pr_0^*((-c_1(\mathcal{L}_1))^\nu).$$

By (1.9) and the projection formula the last integral equals

$$\int_{S^{[n_1]}\times S^{[n_2]}\times S^{m+1}} \Phi_*\left(\sum_{\nu\geq 0}\Phi^*(f_\nu)\cdot pr_0^*((-c_1(\mathcal{L}_1))^\nu)\right) = \int_{S^{[n_1]}\times S^{[n_2]}\times S^{m+1}} \sum_{\nu\geq 0} f_\nu\cdot\Phi_*\big(pr_0^*((-\ell_1)^\nu)\big)$$
$$= \int_{S^{[n_1]}\times S^{[n_2]}\times S^{m+1}} \sum_{\nu\geq 0} f_\nu\cdot c_\nu(-pr_{0,1}^*(\mathcal{I}_1)).$$

The integrand on the right hand side is the polynomial $\widetilde{f}$. □

Proof of Proposition 6. Given Proposition 7 the proof is now almost identical to the proof of [**4**, Proposition 0.5]. Suppose we are given a polynomial P like in Proposition 6. Applying parts (1) and (2) of Proposition 7 repeatedly, we can write

$$\int_{S^{[n_1]}\times S^{[n_2]}} P = \int_{S^{n_1+n_2}} \widetilde{P}$$

for $\widetilde{P}$ a polynomial, which depends only on P, in the Chern classes of $pr_i^*T_S$, $pr_{ij}^*(\mathcal{O}_\Delta)$ and the classes $pr_i^*A_j,\quad j=1,\dots 5$. Any such expression $\int_{S^{n_1+n_2}}\widetilde{P}$ can be universally reduced to a polynomial expression of integrals over S of polynomials in Chern classes of T_S and the A_i. To see this for the Chern classes $pr_{ij}^*(\mathcal{O}_\Delta)$ we use Riemann-Roch without denominators [**22**]. □

2. Multiplicativity

In this section let S be a possibly disconnected smooth projective surface and let $\mathrm{Pic}(S)$. We consider the partition function $Z_S(L,a_1,c_1,s,y,q)$ from Definition 11.

Remark 3. *Let X be a smooth projective scheme, assume $X = X_1\sqcup X_2$, where each X_i is a union of connected components of X. Let $i_1: X_1\to X$, $i_2: X_2\to X$ be the inclusions.*

Note that any $\alpha\in K^0(X)$ or $\alpha\in\mathrm{Pic}(X)$ or $\alpha\in H^*(X,\mathbb{Q})$ is of the form $\alpha = i_{1*}\alpha_1+i_{2*}\alpha_2=\alpha_1+\alpha_2$, with $\alpha_j=i_j^*\alpha$ (and we have suppressed the pushforward via the inclusion in the notation in the second step).

Definition 12. *Assume $S=S'\sqcup S''$, for smooth projective surfaces S', S'' (each of them can have more than one connected component, but the connected components of S' and S'' do not intersect). Note that a line bundle $L\in\mathrm{Pic}(S)$ is the same as line bundles $L':=L\big|_{S'}\in\mathrm{Pic}(S')$, $L'':=L\big|_{S''}\in\mathrm{Pic}(S'')$.*

If $L,a_1,c_1\in\mathrm{Pic}(S)$ with $L|_{S'}=L',\quad a_1|_{S'}=a_1',\quad c_1|_{S'}=c_1'$ (and similarly for S'', L'', a_1'', c_1''), then we say that (S,L,a_1,c_1) is the sum of (S',L',a_1',c_1') and (S'',L'',a_1'',c_1''), and write

$$(S,L,a_1,c_1)=(S',L',a_1',c_1')+(S'',L'',a_1'',c_1'')$$

Our aim is to prove the following multiplicativity result for the partition function.

PROPOSITION 8. *Assume* $(S, L, a_1, c_1) = (S', L', a_1', c_1') + (S'', L'', a_1'', c_1'')$, *then*

$$Z_S(L, a_1, c_1, s, y, q) = Z_{S'}(L', a_1', c_1', s, y, q) Z_{S''}(L'', a_1'', c_1'', s, y, q).$$

PROOF. We write

$$\Phi_S(L, a_1, c_1, n_1, n_2) := \frac{\mathsf{X}_{-y}(E_{n_1,n_2}) \exp\left(\pi_*([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(L))\right) \mathrm{Eu}(\mathcal{O}(a_1)^{[n_1]}) \mathrm{Eu}(\mathcal{O}(c_1 - a_1)^{[n_2]} \otimes \mathfrak{s}^2)}{\mathrm{Eu}\big(E_{n_1,n_2} - T_{S^{[n_1]} \times S^{[n_2]}}\big)}$$
$$\in H^*(S^{[n_1]} \times S^{[n_2]}, \mathbb{Q})[y]((s)),$$

so that by definition

$$Z_S(L, a_1, c_1, s, y, q) = \sum_{n_1, n_2 \geq 0} q^{n_1+n_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Phi_S(L, a_1, c_1, n_1, n_2).$$

By definition we have

$$S^{[m]} = \coprod_{m_1+m_2=m} (S')^{[m_1]} \times (S'')^{[m_2]},$$

thus we also have

$$S^{[n_1]} \times S^{[n_2]} = \coprod_{n_1'+n_1''=n_1} \coprod_{n_2'+n_2''=n_2} ((S')^{[n_1']} \times (S')^{[n_2']}) \times ((S'')^{[n_1'']} \times (S'')^{[n_2'']}).$$

We put

$$\Phi_{S'}(L', a_1', c_1', n_1', n_2') := \frac{\mathsf{X}_{-y}(E'_{n_1',n_2'}) \exp\left(\pi_*([\mathcal{Z}_1'] + [\mathcal{Z}_2']) q^*(L'))\right) \mathrm{Eu}(\mathcal{O}'(a_1')^{[n_1']}) \mathrm{Eu}(\mathcal{O}'(c_1' - a_1')^{[n_2']} \otimes \mathfrak{s}^2)}{\mathrm{Eu}\big(E'_{n_1',n_2'} - T_{(S')^{[n_1']} \times (S')^{[n_2']}}\big)}$$
$$\in H^*((S')^{[n_1']} \times (S')^{[n_2']}, \mathbb{Q})[y]((s))$$

Here we put $\xi' = c_1' - 2a_1'$, $\mathcal{O}'(a')^{[l]} = \mathcal{O}_{S'}(a')^{[l]}$ for $a \in \mathrm{Pic}(S')$. We write $\mathcal{Z}_1'$ the pullback of the universal subscheme from $S' \times (S')^{[n_1']}$, and by I_1' its ideal sheaf and we put

$$E'_{n_1',n_2'} := -R\mathcal{H}om_\pi(\mathcal{F}', \mathcal{F}')_0 + \chi(\mathcal{O}_{S'}) \otimes \mathcal{O} + \chi(\xi') \otimes \mathcal{O} \otimes \mathfrak{s}^2 + \chi(-\xi') \otimes \mathcal{O} \otimes \mathfrak{s}^{-2}$$

with $\mathcal{F}' = \mathcal{I}_1'(a_1') \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2'(a_2') \otimes \mathfrak{s}$. $\Phi_{S''}(L'', a_1'', c_1'', n_1'', n_2'')$ is defined in the same way.

We introduce the following notation for the rest of the proof. We put

$$X_1 = (S')^{[n_1']} \times (S')^{[n_2']}, \quad X_2 = (S'')^{[n_1'']} \times (S'')^{[n_2'']} \subset S^{[n_1]} \times S^{[n_2]}$$

and let

$$i_{12} : X_1 \times X_2 \hookrightarrow S^{[n_1]} \times S^{[n_2]}$$

be the inclusion, and let

$$\pi_1 : X_1 \times X_2 \to X_1, \quad \pi_2 : X_1 \times X_2 \to X_2$$

be the projections.

For $\alpha' \in H^*(X_1,\mathbb{Q})$, $\alpha'' \in H^*(X_2,\mathbb{Q})$ we denote by $\alpha'\times\alpha'' = \pi_1^*(\alpha')\times\pi_2^*(\alpha'')$ their cross product, so that

$$\int_{X_1\times X_2}\alpha'\times\alpha'' = \left(\int_{X_1}\alpha'\right)\left(\int_{X_2}\alpha''\right).$$

Claim 1.

$$i_{12}^*\big(\Phi_S(L,a_1,c_1,n_1,n_2)\big) = \Phi_{S'}(L',a_1',c_1',n_1',n_2')\times\Phi_{S''}(L'',a_1'',c_1'',n_1'',n_2'').$$

It is easy to see that the claim implies the proposition. In fact we get

$$\begin{aligned}
&Z_S(L,a_1,c_1,s,y,q) = \sum_{n_1,n_2\ge 0} q^{n_1+n_2}\int_{S^{[n_1]}\times S^{[n_2]}}\Phi_S(L,a_1,c_1,n_1,n_2)\\
&= \sum_{n_1,n_2\ge 0} q^{n_1+n_2}\sum_{n_1'+n_1''=n_1}\ \sum_{n_2'+n_2''=n_2}\\
&\qquad\int_{((S')^{[n_1']}\times(S')^{[n_2']})\times((S'')^{[n_1'']}\times(S'')^{[n_2'']})}\Phi_{S'}(L',a_1',c_1',n_1',n_2')\times\Phi_{S''}(L'',a_1'',c_1'',n_1'',n_2'')\\
&= \sum_{n_1',n_1''\ge 0}\ \sum_{n_2',n_2''\ge 0} q^{n_1'+n_1''+n_2'+n_2''}\\
&\qquad\int_{(S')^{[n_1']}\times(S')^{[n_2']}}\Phi_{S'}(L',a_1',c_1',n_1',n_2')\int_{(S'')^{[n_1'']}\times(S'')^{[n_2'']}}\Phi_{S''}(L'',a_1'',c_1'',n_1'',n_2'')\\
&= \sum_{n_1',n_2'\ge 0} q^{n_1'+n_2'}\int_{(S')^{[n_1']}\times(S')^{[n_2']}}\Phi_{S'}(L',a_1',c_1',n_1',n_2')\\
&\quad\cdot\sum_{n_1'',n_2''\ge 0} q^{n_1''+n_2''}\int_{(S'')^{[n_1'']}\times(S'')^{[n_2'']}}\Phi_{S''}(L'',a_1'',c_1'',n_1'',n_2'')\\
&= Z_{S'}(L',a_1',c_1',s,y,q)Z_{S''}(L'',a_1'',c_1'',s,y,q).
\end{aligned}$$

To prove the claim we have to show that

$$\begin{aligned}
i_{12}^*\big(\mathsf{X}_{-y}(E_{n_1,n_2})\big) &= \mathsf{X}_{-y}(E'_{n_1',n_2'})\times\mathsf{X}_{-y}(E''_{n_1'',n_2''})\\
i_{12}^*\big(\exp\left(\pi_*(([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L))\right)\big) &= \exp\left(\pi_*(([\mathcal{Z}_1']+[\mathcal{Z}_2'])q^*(L))\right)\times\exp\left(\pi_*(([\mathcal{Z}_1'']+[\mathcal{Z}_2''])q^*(L))\right)\\
i_{12}^*\big(\mathrm{Eu}(\mathcal{O}(a_1)^{[n_1]})\mathrm{Eu}(\mathcal{O}(c_1-a_1)^{[n_2]}\otimes\mathfrak{s}^2)\big) &= \mathrm{Eu}(\mathcal{O}'(a_1')^{[n_1]})\mathrm{Eu}(\mathcal{O}'(c_1'-a_1')^{[n_2']}\otimes\mathfrak{s}^2)\\
&\quad\times\mathrm{Eu}(\mathcal{O}''(a_1'')^{[n_1']})\mathrm{Eu}(\mathcal{O}''(c_1''-a_1'')^{[n_2'']}\otimes\mathfrak{s}^2)\\
i_{12}^*\big(\mathrm{Eu}\big(E_{n_1,n_2}-T_{S^{[n_1]}\times S^{[n_2]}}\big)\big) &= \mathrm{Eu}\big(E'_{n_1',n_2'}-T_{(S')^{[n_1']}\times(S')^{[n_2']}}\big)\times\mathrm{Eu}\big(E''_{n_1'',n_2''}-T_{(S'')^{[n_1'']}\times S^{[n_2'']}}\big)
\end{aligned}$$

All these are simple verifications. By definition we have

$$(1_S\times i_{12})^*[\mathcal{Z}_j] = (i'\times\pi_1)^*[\mathcal{Z}_j'] + (i''\times\pi_2)^*[\mathcal{Z}_j''],\quad j=1,2, \tag{2.1}$$

where $i' : S' \to S$, $i'' : S'' \to S$ are the inclusions. Therefore it follows from Definition 10 that

$$i_{12}^*\big(\pi_*(([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L))\big) = \pi_1^*(\pi_*([\mathcal{Z}_1']+[\mathcal{Z}_2'])q^*(L))) + \pi_2^*(\pi_*([\mathcal{Z}_1'']+[\mathcal{Z}_2''])q^*(L))),$$

and thus

$$i_{12}^*\big(\exp(\ \pi_*(([\mathcal{Z}_1]+[\mathcal{Z}_2])q^*(L)\big) = \exp(\pi_*([\mathcal{Z}_1']+[\mathcal{Z}_2'])q^*(L))) \times \exp(\ \pi_*([\mathcal{Z}_1'']+[\mathcal{Z}_2''])q^*(L))).$$

Note that for $A' \in K^0(X_1)$, $A'' \in K^0(X_2)$, we have

$$\mathrm{Eu}(\pi_1^*(A')+\pi_2^*(A'')) = \pi_1^*(\mathrm{Eu}(A'))\pi_2^*(\mathrm{Eu}(A'')) = \mathrm{Eu}(A') \times \mathrm{Eu}(A''),$$

$$\mathsf{X}_{-y}(\pi_1^*(A')+\pi_2^*(A'')) = \mathsf{X}_{-y}(A') \times \mathsf{X}_{-y}(A'').$$

It follows directly from the definitions that

$$i_{12}^*\big(\mathcal{O}(a_1)^{[n_1]}\big) = \pi_1^*(\mathcal{O}'(a_1')^{[n_1']}) + \pi_2^*(\mathcal{O}''(a_1'')^{[n_1'']})$$

and similar for $\mathrm{Eu}(\mathcal{O}(c_1-a_1)^{[n_2]} \otimes \mathfrak{s}^2)$. Furthermore we clearly have

$$i_{12}^*\big(T_{S^{[n_1]}\times S^{[n_2]}}\big) = \pi_1^*(T_{(S')^{[n_1']}\times (S')^{[n_2']}}) + \pi_2^*(T_{(S'')^{[n_1'']}\times (S'')^{[n_2'']}}).$$

Thus it finally only remains to show that $i_{12}^*\big(E_{n_1,n_2}\big) = \pi_1^*(E'_{n_1',n_2'}) + \pi_2^*(E''_{n_1'',n_2''})$. Again this follows from the definitions. Putting

$$\widetilde{\mathcal{F}}' := (i' \times \pi_1)^*(\mathcal{F}'), \quad \widetilde{\mathcal{F}}'' := (i'' \times \pi_2)^*(\mathcal{F}''),$$

we see by the relation (2.1) that we also have $(1_S \times i_{12})^*(\mathcal{F}) = \widetilde{\mathcal{F}}' + \widetilde{\mathcal{F}}''$. As $\widetilde{\mathcal{F}}'$ and $\widetilde{\mathcal{F}}''$ have their supports on disjoint components of $S \times X_1 \times X_2$, it follows that

$$R\mathcal{H}om_\pi(\widetilde{\mathcal{F}}', \widetilde{\mathcal{F}}'')_0 = R\mathcal{H}om_\pi(\widetilde{\mathcal{F}}'', \widetilde{\mathcal{F}}')_0 = 0.$$

Thus

$$i_{12}^*\big(-R\mathcal{H}om_\pi(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}})_0\big) = -R\mathcal{H}om_\pi(\widetilde{\mathcal{F}}', \widetilde{\mathcal{F}}')_0 - R\mathcal{H}om_\pi(\widetilde{\mathcal{F}}'', \widetilde{\mathcal{F}}'')_0. \tag{2.2}$$

On the other hand we clearly have

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S'}) + \chi(\mathcal{O}_{S''}), \quad \chi(\pm\xi) = \chi(\pm\xi') + \chi(\pm\xi''). \tag{2.3}$$

Thus putting (2.2) and (2.3) into the definition of E_{n_1,n_2} we get $i_{12}^*(E_{n_1,n_2}) = \pi_1^*(E'_{n_1',n_2'}) + \pi_2^*(E''_{n_1'',n_2''})$, and the claim follows. $\square$

The universality of Theorem 4 and the multiplicativity of Proposition 8 together imply that we can write the partition function $Z_S(L, a_1, c_1, s, y, q)$ for any (S, L, a_1, c_1) as a product of powers of 11 universal power series. This is the main result of this chapter and crucial for the rest of the work.

THEOREM 10. *There are power series $A_1(y), \dots, A_{11}(y) \in \mathbb{Q}[y]((s))[[q]]$ such that for any quadruple (S, L, a_1, c_1) we have*

$$Z_S(L, a, c_1, s, y, q) = A_1(y)^{L^2} A_2(y)^{La} A_3(y)^{a^2} A_4(y)^{ac_1} A_5(y)^{c_1^2} A_6(y)^{Lc_1}$$
$$A_7(y)^{LK_S} A_8(y)^{aK_S} A_9(y)^{c_1K_S} A_{10}(y)^{K_S^2} A_{11}(y)^{\chi(\mathcal{O}_S)}.$$

COROLLARY 5. *With $A_i := A_i(0)$ for $i = 1, \dots, 11$, we get for any quadruple (S, L, a_1, c_1) that*

$$Z_S(L, a, c_1, s, q) = A_1^{L^2} A_2^{La} A_3^{a^2} A_4^{ac_1} A_5^{c_1^2} A_6^{Lc_1} A_7^{LK_S} A_8^{aK_S} A_9^{c_1K_S} A_{10}^{K_S^2} A_{11}^{\chi(\mathcal{O}_S)}.$$

Let K_r be the set of tuples (S, L, a, c_1) such that S is a projective surface (possibly disconnected), $L, c_1, a, \in \mathrm{Pic}(S)$. We define a map

$$\gamma : K_r \to \mathbb{Q}^{11} : \ (S, L, a, c) \mapsto \left(L^2, La, a^2, ac_1, c_1^2, Lc_1, LK_S, aK_S, c_1K_S, K_S^2, \chi(\mathcal{O}_S)\right).$$

REMARK 4. *If $(S, L, a, c) = (S', L', a', c') + (S', L', a', c')$ in notations of Definition 12, then $\gamma(S, L, a, c) = \gamma(S', L', a', c') + \gamma(S', L', a', c')$.*

PROOF. Let $i' : S' \to S$, $i'' : S' \to S$ be the inclusions. Then $K_{S'} = (i')^* K_S$, $K_{S''} = (i'')^* K_S$, As $L', a', c', K_{S'}$ have all disjoint supports of $L'', a'', c'', K_{S''}$, we see that $\alpha' \beta'' = 0$, for $\alpha, \beta \in \{L', a', c', K_{S'}\}$, and therefore $\alpha\beta = \alpha'\beta' + \alpha''\beta''$. It is also clear that $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S'}) + \chi(\mathcal{O}_{S''})$. Thus we get $\gamma(S, L, a, c) = \gamma(S', L', a', c') + \gamma(S', L', a', c')$. $\square$

PROOF OF THEOREM 10. By Theorem 4 we know that $Z_S(L, a, c, y, s, q)$ only depends on $\gamma(S, L, a, c)$, so we can write $Z(\gamma(S, L, a, c)) := Z_S(L, a, c, y, s, q)$. Furthermore we know by Remark 4 and Proposition 8 that

$$Z\big(\gamma(S', L', a', c') + \gamma(S'', L'', a'', c'')\big) = Z(\gamma(S', L', a', c'))Z(\gamma(S'', L'', a'', c'')). \tag{2.4}$$

We choose tuples (S_j, L_j, a_j, c_j) for $j = 1, \dots, 11$ (in Chapter 4, Section 1 we give an example of such tuples) such that the vectors $v_j := \gamma(S_j, L_j, a_j, c_j)$ are all linearly independent and form a basis of $\mathbb{Q}^{11}$. We denote $e_1, \dots, e_{11}$ the standard basis of $\mathbb{Q}^{11}$. Then we can write

$$v_j = \sum_{i=1}^{11} b_{j,i} e_i, \qquad e_i = \sum_{j=1}^{11} a_{i,j} v_j, \qquad \text{for matrices } (b_{j,i})_{j,i=1}^{11}, \quad (a_{i,j})_{i,j=1}^{11} \in \mathbb{Q}^{11 \times 11}$$

which are inverse to each other. We define Let $B_j := Z(v_j)$, and put

$$A_i(y) := \prod_{j=1}^{11} B_j^{a_{i,j}} \in 1 + \mathbb{Q}[y]((s))[[q]], \quad i = 1, \dots, 11. \tag{2.5}$$

Note that then we also have

$$B_j = \prod_{i=1}^{11} A_i(y)^{b_{j,i}}, \quad j = 1, \dots, 11. \tag{2.6}$$

We want to show that Theorem 10 is true with this choice of the $A_i(y)$. Thus we have to show

$$Z(w) = \prod_{i=1}^{11} A_i(y)^{\omega_i}, \qquad \text{for all } w = (\omega_1, \dots, \omega_{11}) \in \gamma(K_r). \tag{2.7}$$

Let

$$\Gamma_+ = \Big\{ \sum_{j=1}^{11} \beta_j v_j \;\Big|\; \beta_j \in \mathbb{Z}_{\geq 0} \Big\}.$$

be the set of nonnegative linear combinations of the v_j. First we show (2.7) for all w in Γ_+. Let $w = \sum_{j=1}^{11} \beta_j v_j \in \Gamma_+$. Write $w = (\omega_1, \dots, \omega_{11})$, then $\omega_i = \sum_{j=1}^{11} \beta_j b_{j,i}$. Thus we have by (2.4) and (2.6) that

$$Z(w) = \prod_{j=1}^{11} B_j^{\beta_j} = \prod_{j=1}^{11} \left(\prod_{i=1}^{11} A_i(y)^{b_{j,i}} \right)^{\beta_j} = \prod_{i=1}^{11} A_i(y)^{\omega_i}. \tag{2.8}$$

We denote $x_1, \dots, x_{11}$ the coordinates of $\mathbb{C}^{11}$. We note that Γ_+ is a positive orthant in a lattice in $\mathbb{R}^{11}$. Therefore it is Zariski dense in $\mathbb{C}^{11}$. Thus if two polynomials $f, g \in \mathbb{C}[x_1, \dots, x_{11}]$ satisfy $f(v) = g(v)$ for all $v \in \Gamma_+$, then $f = g$.

Fix integers $l, n, m \in \mathbb{Z}$. By Theorem 4 there is a polynomial $f_{l,n,m} \in \mathbb{Q}[x_1, \dots, x_{11}]$ such that for all $w = (\omega_1, \dots, \omega_{11}) \in \gamma(K_r)$ we have

$$\mathrm{Coeff}_{y^l s^n q^n}(Z(w)) = f_{l,n,m}(\omega_1, \dots, \omega_{11}).$$

On the other hand clearly $\mathrm{Coeff}_{y^l s^n q^n} \prod_{i=1}^{11} A_i(y)^{\omega_i}$ can be written as $g_{l,n,m}(\omega_1, \dots, \omega_{11})$ for some polynomial $g_{l,n,m} \in \mathbb{Q}[x_1, \dots, x_{11}]$, and by (2.8) $f_{l,n,m} = g_{l,n,m}$ on the Zariski dense set Γ_+. Therefore $f_{l,n,m} = g_{l,n,m}$ and the result follows. $\square$

CHAPTER 4

Computation of Mochizuki's formula via localization

Let S be a smooth projective surface with $b_1(S) = 0$, $p_g(S) > 0$ and $c_1, L \in \mathrm{Pic}(S)$. In Chapter 2, Definition 11 we first introduced the partition functions $Z_S(L, a_1, c_1, s, y, q)$ and $Z_S(L, a_1, c_1, s, q) = Z_S(L, a_1, c_1, s, y, q)|_{y=0}$ for $a_1 \in \mathrm{Pic}(S)$ and then showed in Corollary 3 that the refined K-theoretic Donaldson invariants $\chi^{\mathrm{vir}}_{-y}(M^H_S(c_1, c_2), \mu(L))$ can be expressed in terms of the $Z_S(L, a_1, c_1, s, y, q)$, where a_1 runs through the Seiberg-Witten classes of S, and similarly the K-theoretic Donaldson invariants $\chi^{\mathrm{vir}}(M^H_S(c_1, c_2), \mu(L))$ in terms of the $Z_S(L, a_1, c_1, s, q)$. Thus our task is reduced to computing $Z_S(L, a_1, c_1, s, y, q)$ and $Z_S(L, a_1, c_1, s, q)$ for any quadruple (S, L, a_1, c_1). In Chapter 3 Theorem 10 we finally showed that there are universal power series $A_1(y), \dots, A_{11}(y)$ such that

$$\begin{aligned} Z_S(L, a, c_1, s, y, q) = A_1(y)^{L^2} A_2(y)^{La} A_3(y)^{a^2} A_4(y)^{ac_1} A_5(y)^{c_1^2} A_6(y)^{Lc_1} \\ A_7(y)^{LK_S} A_8(y)^{aK_S} A_9(y)^{c_1K_S} A_{10}(y)^{K_S^2} A_{11}(y)^{\chi(\mathcal{O}_S)}. \end{aligned}$$

or any quadruple (S, L, a_1, c_1). Therefore our task is reduced to computing $A_1(y), \dots, A_{11}(y)$. In the proof we considered for each quadruple (S, L, a, c) the tuple

$$\gamma(S, L, a, c) = \left(L^2, La, a^2, ac, c^2, Lc, LK_S, aK_S, cK_S, K_S^2, \chi(\mathcal{O}_S)\right),$$

and we showed that $A_1(y), \dots, A_{11}(y)$ can be computed from $Z_{S_j}(L_j, a_j, c_j, s, y, q)$ for 11 quadruples (S_j, L_j, a_j, c_j), such that $\gamma(S_j, L_j, a_j, c_j)$, $\quad i = 1, \dots, 11$ are linearly independent. In fact in the proof of Theorem 10 we saw explicitly the following: if

$$\gamma(S_j, L_j, a_j, c_j) = (b_{j,1}, \dots, b_{j,11}), \quad B := (b_{j,i})_{j,i=1}^{11}, \quad B^{-1} = (a_{i,j})_{i,j=1}^{11}, \tag{0.1}$$

then we have

$$A_i(y) = \prod_{j=1}^{11} Z_{S_j}(L_j, a_j, c_j, s, y, q)^{a_{i,j}}.$$

1. Reduction to toric surfaces

We have seen that in order to compute $Z_S(L, a, c_1, s, y, q)$ for any surface, we need to compute $Z_{S_j}(L_j, a_j, c_j, s, y, q)$ for 11 tuples (S_j, L_j, a_j, c_j) for which the $\gamma(S_j, L_j, a_j, c_j)$ are linearly independent. While in the application to the (refined) K-theoretic Donaldson invariants we need that $p_g(S) > 0$, here we have no restriction on the choice of these tuples and we can choose them conveniently so that the computation becomes easier. We make the following choice

NOTATION 3. *We let* $(S_i, L_i, a_i, c_i), i = 1, \ldots, 11$ *in the order below be given by*

$$(\mathbb{P}^2, \mathcal{O}, \mathcal{O}, \mathcal{O}), \quad (\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)), \quad (\mathbb{P}^2, \mathcal{O}, \mathcal{O}, \mathcal{O}(1))$$

$$(\mathbb{P}^2, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(3)), \quad (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}, \mathcal{O}), \quad (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}(1), \mathcal{O}(2)),$$

$$(\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}, \mathcal{O}(1)), \quad (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}, \mathcal{O}) \quad (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(0,2)),$$

$$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}, \mathcal{O}(0,1)), \quad (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(0,1), \mathcal{O}, \mathcal{O})$$

The choice of the line bundles would seem to be slightly more complicated than neccessary, however in our computer computations we use $\xi = c - 2a$ *instead of* c*, and then these choices lead to simpler computations.*

REMARK 5. $\gamma(S_i, L_i, a_i, c_i), i = 1, \ldots, 11$ *are linearly independent.*

PROOF. This follows from the relevant intersection products. We have that $A(\mathbb{P}^2) \simeq \mathbb{Z}[h]/h^3$, where h is the class of the hyperplane bundle $\mathcal{O}(1)$ whose self intersection $h \cdot h = 1$. We also have that the canonical bundle is given by $K_{\mathbb{P}^2} = \mathcal{O}(-3)$, and that $\chi(\mathbb{P}^2) = 1$. For the tuples involving $\mathbb{P}^1 \times \mathbb{P}^1$, we have that $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}F \oplus \mathbb{Z}G$ where F, G are the two fiber classes. We denote the associated line bundles by $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$ respectively. Then the intersection product is given by the relations $F^2 = 0 = G^2$, $F \cdot G = 1$. We also have that $K_{\mathbb{P}^1 \times \mathbb{P}^1} = -2F - 2G$ and $\chi(\mathbb{P}^1 \times \mathbb{P}^1) = 1$. This data allows one to compute all the intersection products. In fact the matrix B from (0.1) is readily computed as

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 1 \\
0 & 0 & 1 & 2 & 4 & 0 & 0 & -3 & -6 & 9 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & 9 & 1 \\
0 & 0 & 1 & 3 & 9 & 0 & 0 & -3 & -9 & 9 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 9 & 1 \\
1 & 1 & 1 & 2 & 4 & 2 & -3 & -3 & -6 & 9 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & -3 & 0 & -3 & 9 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -4 & 8 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 8 & 1
\end{pmatrix}.$$

This matrix is invertible. Its inverse that we need to compute the $A_i(y)$ is

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 3/2 & 0 & 0 & -3/2 \\ -1 & -1 & 2 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ -7 & 3 & 6 & -2 & 0 & 0 & 0 & 15/2 & -3/2 & -6 & 0 \\ 5 & -1 & -5 & 1 & 0 & 0 & 0 & -6 & 0 & 6 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 3/2 & 0 & -3/2 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

□

REMARK 6. *We note that the surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are toric surfaces. In particular they are equipped with an action of $T = \mathbb{C}^* \times \mathbb{C}^*$, with finitely many fixpoints. We will see that this action lifts to the Hilbert schemes of points and this will allow us to compute the invariants via Atiyah-Bott equivariant localization.*

2. Action on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

Let $T = \mathbb{C}^* \times \mathbb{C}^*$. We describe the action of T on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. We will need the results and the notation established here when we study the lift of the action to the Hilbert schemes of points.

2.1. Action on $\mathbb{P}^2$. Let X_0, X_1, X_2 be the homogeneous coordinates on $\mathbb{P}^2$. Let $U_i = \mathbb{P}^2 \setminus Z(X_i)$. On U_i we have coordinates x_i, y_i with

$$x_0 = X_1/X_0, \quad y_0 = X_2/X_0, \qquad x_1 = X_0/X_1, \quad y_1 = X_2/X_1, \qquad x_2 = X_0/X_2, \quad y_1 = X_1/X_2.$$

T acts on $\mathbb{P}^2$ via acting on the coordinates by

$$(t_1, t_2) \cdot (X_0 : X_1 : X_2) = (X_0 : t_1 X_1 : t_2 X_2).$$

We see that the fixpoints of the action are

$$p_0 = (1 : 0 : 0), \quad p_1 = (0 : 1 : 0), \quad p_2 = (0 : 0 : 1).$$

The coordinates x_i, y_i are eigenvectors for the T-action, in fact one sees immediately that

$$
\begin{aligned}
(t_1,t_2)\cdot x_0 &= t_1 x_0, \quad (t_1,t_2)\cdot y_0 = t_2 y_0,\\
(t_1,t_2)\cdot x_1 &= t_1^{-1} x_1, \quad (t_1,t_2)\cdot y_1 = \frac{t_2}{t_1} y_1,\\
(t_1,t_2)\cdot x_2 &= t_2^{-1} x_2, \quad (t_1,t_2)\cdot y_2 = \frac{t_1}{t_2} y_2,
\end{aligned}
$$

so that the corresponding weights of the coordinates are

$$
w(x_0)=\varepsilon_1,\ w(y_0)=\varepsilon_2, \quad w(x_1)=-\varepsilon_1,\ w(y_1)=\varepsilon_2-\varepsilon_1, \quad w(x_2)=-\varepsilon_2,\ w(y_2)=\varepsilon_1-\varepsilon_2.
$$

Finally on U_i a trivializing section of $\mathcal{O}_{\mathbb{P}^2}(n)$ is X_i^n, thus the weights w_i of the T-action on the fiber of $\mathcal{O}_{\mathbb{P}^2}(n)$ at the fixpoints p_i are given by

$$
w_0 = 0, \quad w_1 = n\varepsilon_1, \quad w_2 = n\varepsilon_2.
$$

2.2. Action on $\mathbb{P}^1\times\mathbb{P}^1$. Let X_0, X_2, Y_0, Y_1 be the homogeneous coordinates on the two factors of $\mathbb{P}^1\times\mathbb{P}^1$. T acts on $\mathbb{P}^1\times\mathbb{P}^1$ via

$$
(t_1,t_2)\cdot((X_0:X_1),(Y_0,Y_1)) = (X_0:t_1X_1),(Y_0,t_2Y_1)).
$$

The action has 4 fixpoints

$$
p_0=((1:0),(1:0)),\ p_1=((1:0),(0:1)),\ p_2=((0:1),(1:0)),\ p_3=((0:1),(0:1)),
$$

and in an affine neighbourhood U_i of p_i there are coordinates x_i, y_i given by

$$
x_0=x_1=\frac{X_1}{X_0},\ y_0=y_2=\frac{Y_1}{Y_0}, \quad x_2=x_3=\frac{X_0}{X_1},\ y_1=y_3=\frac{Y_1}{Y_0},
$$

which are eigenvectors for the T-action with weights

$$
w(x_0)=w(x_1)=\varepsilon_1,\ w(x_2)=w(x_3)=-\varepsilon_1, w(y_0)=w(y_2)=\varepsilon_2,\ w(y_1)=w(y_3)=-\varepsilon_2.
$$

Finally we see that on U_i the trivializing sections of $\mathcal{O}(n_1,n_2)$ are $X_0^{n_1}Y_0^{n_2}$, $X_0^{n_1}Y_1^{n_2}$, $X_1^{n_1}Y_0^{n_2}$, $X_1^{n_1}Y_1^{n_2}$ for $i=0,1,2,3$ respectively, therefore the weights of the action on the fibres of $\mathcal{O}(n_1,n_2)$ are 0, $n_2\varepsilon_2$, $n_1\varepsilon_1$, $n_1\varepsilon_1+n_2\varepsilon_2$ for $i=0,1,2,3$ respectively.

We also notice that both in the case of $S=\mathbb{P}^2$ and $S=\mathbb{P}^1\times\mathbb{P}^1$ a natural T-equivariant basis of $K_S(p_i)$ is $dx_i\wedge dy_i$ thus the weight of $K_S(p_i)$ is $w(x_i)+w(y_i)$.

3. Action on Hilbert schemes of points

In the following let $S = \mathbb{P}^2$ and $S = \mathbb{P}^1 \times \mathbb{P}^1$ with the action we described in the previous section. Thus S has an action of $T = (\mathbb{C}^*)^2$, with finitely many fixpoints $p_0, \dots, p_e$ (with $e = 2$ or $e = 3$). We lift the action of T on S to an action on $S^{[n]}$. For each $t = (t_1, t_2) \in T$, t defines an automorphism $t : S \to S; p \mapsto t \cdot p$. We define an action on $S^{[n]}$ by $t \cdot Z := t(Z)$. Clearly this defines an action on $S^{[n]}$. Note that by definition, if I_Z is the ideal sheaf of Z in S, then $I_{t(Z)} = (t^{-1})^*(I_Z)$. Thus we can also describe the action on $S^{[n]}$ by its action on ideal sheaves via $t \cdot I_Z = (t^{-1})^*(I_Z)$.

We now want to describe the fixpoints of the T-action on $S^{[n]}$. We have an obvious action of T on the symmetric power $S^{(n)}$ by

$$t \cdot \sum_i n_i q_i = \sum_i n_i (t \cdot q_i).$$

This is clearly compatible with the Hilbert Chow morphism $\pi : S^{[n]} \to S^{(n)}; Z \mapsto \mathrm{supp}(Z)$ of Chapter 1 Section 2, where we denote by $\mathrm{supp}(Z)$ the support of Z with multiplicities, i.e. $t \cdot \mathrm{supp}(Z) = \mathrm{supp}(t \cdot Z)$.

Now let $Z \in (S^{[n]})^T$ be a fixpoint. Then $\mathrm{supp}(Z)$ must be a fixpoint of the T-action on $S^{(n)}$ and it follows that $\mathrm{supp}(Z)$ is a linear combination of the fixpoints $p_1, \dots, p_e$, i.e. we can write

$$\mathrm{supp}(Z) = \sum_{i=0}^{e} n_i p_i, \quad n_i \in \mathbb{Z}_{\geq 0}, \quad \sum_{i=1}^{e} n_i = n.$$

We denote

$$Hilb^n(S, p) := \left\{ Z \in S^{[n]} \;\middle|\; \mathrm{supp}(Z) = np \right\}$$

the Hilbert scheme of points supported at the point $p \in S$. Then for a fixpoint p_i the action of T on $S^{[n_i]}$ restricts to an action on $Hilb^{n_i}(S, p_i)$. and we get by the above a decomposition

$$Z = \coprod_{i=0}^{e} Z_i, \quad Z_i \in (Hilb^{n_i}(S, p_i))^T.$$

Conversely any subscheme of this form is a fixpoint of the T-action on S, so we have shown that the fixpoints $Z \in (S^{[n]})^T$ are precisely the subschemes

$$Z = \coprod_{i=0}^{e} Z_i, \quad Z_i \in Hilb^{n_i}(S, p_i)^T, \quad n_i \in \mathbb{Z}_{\geq 0}, \quad \sum_{i=1}^{e} n_i = n. \tag{3.1}$$

Thus to describe $(S^{[n]})^T$ completely, it is enough to deal with the punctual Hilbert schemes $Hilb^n(S, p_i)^T$ for p_i a fixpoint of the T-action on S. We put $p := p_i$ and write $x = x_i$, $y = y_i$ for the local equivariant coordinates at p from Section 2. We can write the action for $t = (t_1, t_2) \in T$ as

$$t \cdot x = t_1^{u_1} t_2^{u_2} x, \quad t \cdot y = t_1^{v_1} t_2^{v_2} y,$$

and we can see from the results in Section 2 that the weights $w(x) = u_1 \varepsilon_1 + u_2 \varepsilon_2$ and $w(y) = v_1 \varepsilon_1 + v_2 \varepsilon_2$ are linearly independent.

A subscheme $Z \in Hilb^n(S,p)$ is given by an ideal $I_Z \in k[x,y]$ such that $\mathcal{O}_Z = \mathbb{C}[x,y]/I_Z$ has support p and dimension n as a $\mathbb{C}$ vector space. Such a Z will be T-invariant, if and only if I_Z is T-invariant under the action on $\mathbb{C}[x,y]$ given by $t \cdot x = t_1^{u_1} t_2^{u_2} x, \quad t \cdot y = t_1^{v_1} t_2^{v_2} y$. It is easy to see that this is equivalent to the fact that we can write $I_Z = (f_1, \ldots, f_r)$ where the generators $f_1, \ldots, f_r$ are eigenvectors for the T-action. As the weights $w(x)$, $w(y)$ are linearly independent, this implies that $f_1, \ldots, f_r$ are monomials in x, y. Choosing for each power x^i of x the smallest power of y^{a_i} of y such that $x^i y^{a_i} \in I_Z$, we can therefore write

$$I_Z = (x_0 y^{a_0}, x_1 y^{a_1}, \ldots, x^r y^{a_r}, x^{r+1})$$

with $a_0 \geq a_1 \geq \ldots \geq a_r > 0$. Furthermore we have that the monomials

$$\left\{ x^i y^j \;\middle|\; 0 \leq i \leq r, \quad 0 \leq j < a_i \right\}$$

are a basis of $\mathcal{O}_Z$ as $\mathbb{C}$-vector space. Therefore $\sum_{i=0}^r a_i = n$.

Thus we see that the subschemes $Z \in Hilb^n(S,p)^T$ are in a bijection with the partitions of n, via the correspondence $\nu = (a_1, \ldots, a_r) \mapsto Z_\nu(x,y)$, where $Z_\nu(x,y)$ is given by

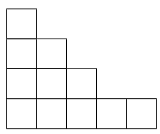
$$I_{Z_\nu(x,y)} = (x_0 y^{a_0}, x_1 y^{a_1}, \ldots, x^r y^{a_r}, x^{r+1}).$$

Finally partitions are in one one correspondence to Young diagrams.

Definition 13. *The Young diagram of a partition $\nu = (a_0, \ldots, a_r)$ is the set*

$$Y(\nu) := \left\{ (i,j) \in \mathbb{Z}^2_{\geq 0} \;\middle|\; i \leq r, \quad j < a_i \right\}.$$

A Young diagram is a subset of the form $Y(\nu)$ for a partition ν.



For Y a Young diagram we write $|Y|$ the number of elements of Y. Clearly $|Y(\nu)| = n$ if ν is a partition of n. For a Young diagram $Y = Y(\nu)$, we define

$$Z(Y; x_i, y_i) := Z_\nu(x_i, y_i) \in Hilb^{|Y|}(S, p_i)^T$$

For an tuple $\overline{Y} := (Y_0, \ldots, Y_e)$ of Young diagrams we write $|\overline{Y}| := |Y_0| + \ldots |Y_e|$.

For a tuple $\overline{Y} := (Y_0, \ldots, Y_e)$ of Young diagrams with $|\overline{Y}| = n$ we put

$$Z(\overline{Y}) := \coprod_{i=0}^{e} Z(Y_i; x_i, y_i) \in (S^{[n]})^T.$$

Summing up, we have shown the following proposition.

PROPOSITION 9. *There is a natural bijection*

$$Z : \left\{ \textit{tuples } \overline{Y} := (Y_0, \dots, Y_e) \textit{ of Young diagrams, with } |\overline{Y}| = n \right\} \to (S^{[n]})^T : \ \overline{Y} \mapsto Z(\overline{Y})$$

REMARK 7. *We also see immediately that under this bijection we can write*

$$H^0(\mathcal{O}_{Z(Y)}) = \bigoplus_{i=0}^{e} H^0(\mathcal{O}_{Z(Y_i;x_i,y_i)})$$

and set $\left\{ x_i^j y_i^k \mid j,k \in Y_i \right\}$ *is a T-equivariant basis of* $H^0(\mathcal{O}_{Z(Y_i;x_i,y_i)})$ *as $\mathbb{C}$-vector space.*

4. The localization formula for the partition function

In this whole section we assume that $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$. Our aim is to compute $Z_S(L, a_1, c_1, s, y, q)$ via localization on the Hilbert schemes $S^{[n_1]} \times S^{[n_2]}$. In this section we write down the localization formula for $Z_S(L, a_1, c_1, s, y, q)$, in terms of the T-action at the fixpoints on certain sheaves. In the next section we will describe this action. By definition

$$Z_S(L, a_1, c_1, s, y, q) = \sum_{n_1, n_2 \geq 0} q^{n_1+n_2}$$
$$\int_{S^{[n_1]} \times S^{[n_2]}} \frac{\chi_{-y}(E_{n_1,n_2}) \exp\big(\pi_*([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(L)\big) \operatorname{Eu}(\mathcal{O}(a_1)^{[n_1]}) \operatorname{Eu}(\mathcal{O}(c_1 - a_1)^{[n_2]} \otimes \mathfrak{s}^2)}{\operatorname{Eu}\big(E_{n_1,n_2} - T_{S^{[n_1]} \times S^{[n_2]}}\big)}.$$

By Proposition 9 the fixpoints of the T-action of $S^{[n_1]} \times S^{[n_2]}$ are the $(Z(\overline{Y}^1), Z(\overline{Y}^2))$ for pairs

$$(\overline{Y}^1 = (Y_0^1, \dots, Y_e^1), \overline{Y}^2 = (Y_0^1, \dots, Y_e^1))$$

of tuples of Young diagrams with $|\overline{Y}^1| = n_1, \ |\overline{Y}_2| = n_2$.

We will use the following Lemma ([**3**, Lemma 3.4]).

LEMMA 8.

$$\pi_*([\mathcal{Z}_1] + [\mathcal{Z}_2]) q^*(L))(Z(\overline{Y}^1), Z(\overline{Y}^2)) = \sum_{i=0}^{e} (|Y_i^1| + |Y_i^2|) L(p_i).$$

NOTATION 4. *In future will just write E for instead E_{n_1,n_2} on $S^{[n_1]} \times S^{[n_2]}$.*

Applying the Atiyah-Bott localization, Chapter 1 Theorem 6, we get

$$Z_S(L, a_1, c_1, s, y, q) = \sum_{(\overline{Y}^1, \overline{Y}_2)} q^{|\overline{Y}_1| + |\overline{Y}_2|} \Bigg(\frac{\chi_{-y}(E(Z(\overline{Y}^1), Z(\overline{Y}^2)))}{\operatorname{Eu}\big(E(Z(\overline{Y}^1), Z(\overline{Y}^2))\big)}$$
$$\exp\Big(\sum_{i=0}^{e} (|Y_i^1| + |Y_i^2|) L(p_i)\Big) \operatorname{Eu}(H^0(\mathcal{O}_{Z(\overline{Y}^1)}(a_i)_{Z(\overline{Y}^1)}))) \operatorname{Eu}(H^0(\mathcal{O}_{Z(\overline{Y}^2)(c_1 - a_1)} \otimes \mathfrak{s}^2)) \Bigg) \Big|_{\varepsilon_1 = \varepsilon_2 = 0}.$$

Here the sum runs over all pairs $(\overline{Y}^1 = (Y_0^1, \dots, Y_e^1), \overline{Y}^2 = (Y_0^1, \dots, Y_e^1))$, and we use the following notations.

NOTATION 5. *For a class F in the K group of T-equivariant sheaves on a variety X and $p \in X$ a fixpoint of the action with $\iota : p \to X$ the inclusion, we denote by $F(p) := \iota^*(F)$ as an element in the representation ring of T. Moreover, let $\mathrm{Eu}(F(p)) \in \mathbb{Q}[\varepsilon_1, \varepsilon_2]$, $\mathsf{X}_{-y}(F(p)) \in \mathbb{Q}[y][[\varepsilon_1, \varepsilon_2]]$ denote respectively the equivariant Euler class and χ_{-y}-genus.*

We can rewrite $Z_S(L, a_1, c_1, s, y, q)$ as a product over the fixpoints on S. This allows to compute $Z_S(L, a_1, c_1, s, y, q)$ as a sum over pairs of partitions and not as a sum over $(2e+2)$-tuples of partitions, which makes for a much more efficient algorithm.

PROPOSITION 10.

$$Z_S(L, a_1, c_1, s, y, q) = \prod_{i=0}^{e} \left(\sum_{Y^1, Y^2} q^{|Y^1|+|Y^2|} \frac{\mathsf{X}_{-y}(E(Z(Y^1; x_i, y_i), Z(Y^2; x_i, y_i)))}{\mathrm{Eu}\big(E(Z(Y^1; x_i, y_i), Z(Y^2; x_i, y_i))\big)} \right.$$
$$\left. \cdot \exp\big((|Y^1| + |Y^2|)L(p_i)\big)\mathrm{Eu}(H^0(\mathcal{O}_{Z(Y^1;x_i,y_i)}(a_i)))\,\mathrm{Eu}(H^0(\mathcal{O}_{Z(Y^2;x_i,y_i)}(c_1 - a_1) \otimes \mathfrak{s}^2)) \right)\Bigg|_{\varepsilon_1=\varepsilon_2=0}.$$

PROOF. For V, W in the K-group of T-equivariant vector bundles on $S^{[n_1]} \times S^{[n_2]}$ we have by definition

$$\mathsf{X}_{-y}(V + W) = \mathsf{X}_{-y}(V)\mathsf{X}_{-y}(W), \quad \mathrm{Eu}(V + W) = \mathrm{Eu}(V)\mathrm{Eu}(W).$$

We will show below (Corollary 6) that

$$E(Z(\overline{Y}_1), Z(\overline{Y}_2)) = \bigoplus_{i=0}^{e} E(Z(Y_i^1; x_i, y_i), Z(Y_i^2 x_i, y_i)).$$

Finally it is clear by definition that

$$H^0(\mathcal{O}_{Z(\overline{Y}^1)}(a_i)) = \bigoplus_{i=0}^{e} H^0(\mathcal{O}_{Z(Y_i^1;x_i,y_i)}(a_i)),$$
$$H^0(\mathcal{O}_{Z(\overline{Y}_2)}(c_1 - a_1) \otimes \mathfrak{s}^2) = \bigoplus_{i=0}^{e} H^0(\mathcal{O}_{Z(Y_i^2;x_i,y_i)}(c_1 - a_1) \otimes \mathfrak{s}^2).$$

Thus the formula follows by just distributing out the product. □

5. Action on the relevant sheaves at the fixpoints

Now $p = p_i \in S^T$. Let $x = x_i$, $y = y_i$ be the equivariant local coordinates at p, with weights $w(x)$, $w(y)$. Let L, a, c be equivariant line bundles on S. Let (Y^1, Y^2) be a pair of Young diagrams with $|Y^1| = n$, $|Y^2| = m$. We denote $L(p)$, $a(p)$, $c(p)$ the 1-dimensional representation of T on the fibre of L, a, c at p, and we denote x, y the one dimensional representations given by the action of T on x, y.

5.1. Tautological sheaves. The tautological sheaves are easy to understand.

Remark 8. *In the Grothendieck ring of T-represenations we have for $k = 1, 2$*

$$\mathcal{O}_{Z(Y^k;x;y)}(a) = \sum_{(i,j)\in Y^k} x^i y^j a(p)$$

and thus

$$\mathrm{Eu}(\mathcal{O}_{Z(Y^k;x;y)}(a)) = \prod_{(i,j)\in Y^k} (iw(x) + jw(y) + w(a(p))).$$

Proof. In the Remark 7 we saw that $\{x^i y^j \mid (i,j) \in Y_k\}$ is an equivariant basis of $H^0(\mathcal{O}_{Z(Y^k;x,y)})$. Thus $\{x^i y^j a(p) \mid (i,j) \in Y_k\}$ is an equivariant basis of $H^0(\mathcal{O}_{Z(Y^k;x,y)}(a))$, and the claim follows. □

5.2. The tangent bundle of the Hilbert scheme. Now we want to describe the action of T on the tangent space $T_{S^{[n]}\times S^{[m]},(Z(Y^1;x,y),Z(Y^2,x,y))}$ in terms of the combinatorics of the partition. We introduce some notation.

Definition 14. *Let Y be a Young diagram, corresponding to a partition $(\lambda_0, \dots, \lambda_r)$ of n. Let $s = (i,j) \in Y$. The arm length of s is $a_Y(s) = \lambda_i - j + 1$. The dual Young diagram to Y is $Y' = \{(j,i) \mid (i,j) \in Y$, and the leg length of $s = (i,j)$ is $l(s) = a_{Y'}(j,i)$.*

By Lemma 3 in Chapter 2 we have $T_{S^{[n]},Z(Y^1)} = Ext^1(I_{Z(Y^1)}, I_{Z(Y^1)})_0$. We have the following theorem, which was proven in [**5**],[**30**].

Theorem 11. *In the Grothendieck group of T-representations we have for $k = 1, 2$*

$$T_{S^{[n]},Z(Y^k;x,y)} = \sum_{s\in Y^k} \left(x^{-l_{Y^k}(s)} y^{a_{Y^k}(s)+1} + x^{l_{Y^k}(s)+1} y^{-a_{Y^k}(s)}\right).$$

In particular we have

$$c(T_{S^{[n]},Z(Y^k)}) = \prod_{s\in Y^k} \big(1 - l_{Y^k}(s)w(x) + (a_{Y^k}(s)+1)w(y)\big)\big(1 + (l_{Y^k}(s)+1)w(x) - a_{Y^k}(s)w(y)\big),$$

$$\mathrm{Eu}(T_{S^{[n]},Z(Y^k)}) = \prod_{s\in Y^k} \big(-l_{Y^k}(s)w(x) + (a_{Y^k}(s)+1)w(y)\big)\big((l_{Y^k}(s)+1)w(x) - a_{Y^k}(s)w(y)\big).$$

5.3. Action on the on Ext-sheaves. Finally we have to determine the fibres of E at the fixpoints of $S^{[n_1]} \times S^{[n_2]}$. We will use the following lemma to reduce the computation to a computation in [**3**]. The lemma should be standard, but we include a proof because we did not find a reference.

Lemma 9. *X be a smooth projective variety with an action of a torus $T = (\mathbb{C}^*)^n$ and let $\mathcal{F}$, $\mathcal{G}$ be coherent T-equivariant sheaves on X. Let V be an equivariant vector bundle on X. Then in the Grothendieck group of T-equivariant vector spaces we have an identity*

$$R\mathrm{Hom}(\mathcal{F}, \mathcal{G}\otimes V) = R\Gamma(R\mathcal{H}om(\mathcal{F},\mathcal{G})\otimes V).$$

Here $R\Gamma$ is the derived functor of the functor of global sections.

Proof. Note that, as V is locally free, we have $R\mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes V) = R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes V$, thus it is enough to show (replacing $\mathcal{G}$ by $\mathcal{G} \otimes V$) that

$$R\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = R\Gamma(R\mathcal{H}om(\mathcal{F}, \mathcal{G})).$$

We use the local to global spectral sequence, with $E_2^{p,q} = H^p(\mathcal{E}xt^q(\mathcal{F}, \mathcal{G}))$ abutting to $\mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$. For all r we write

$$[E_r] = \sum_{p,q} (-1)^{p+q} E_r^{p,q}$$

in the Grothendieck group of equivariant vector spaces. Let $d = \dim(X)$. As X is smooth projective of dimension d, we get that $E_2^{p,q} = 0$ for $(p,q) \notin [0, \dots, d] \times [0, \dots, d]$. It follows that $E_\infty^{p,q} = E_{d+1}^{p,q}$, and the $E_{d+1}^{p,q}$ are the associated graded pieces of a filtration on $\mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$. It follows that

$$[E_{d+1}] = \sum_{p,q} (-1)^{p+q} E_{d+1}^{p,q} = \sum_n (-1)^n \mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) = R\mathrm{Hom}(\mathcal{F}, \mathcal{G}).$$

Thus it is enough to show that $[E_{r+1}] = [E_r]$ for all r. By definition we have the differentials

$$d_r : E_r^{p,q} \to E_{r+1}^{p+r,q-r+1},$$

and

$$E_{r+1}^{p,q} = \frac{ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})}{im(d_r : E_r^{p-r,q-r+1} \to E_r^{p,q})}.$$

Thus, with

$$K_{r+1}^{p,q} := ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}), \quad Q_{r+1}^{p+r,q-r+1} = im(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}),$$

we have for all p, q an exact sequence

$$0 \to K_{r+1}^{p,q} \to E_r^{p,q} \to Q_{r+1}^{p+r,q-r+1} \to 0$$

and $E_{r+1}^{p,q} = K_{r+1}^{p,q}/Q_{r+1}^{p,q}$, thus $E_{r+1}^{p,q} = K_{r+1}^{p,q} - Q_{r+1}^{p,q}$ in the Grothendieck group.

Therefore we get in the equivariant Grothendieck group of vector spaces

$$\begin{aligned}
[E_r] &= \sum_{p,q}(-1)^{p+q}E_r^{p,q} \\
&= \sum_{p,q}(-1)^{p+q}(K_{r+1}^{p,q} + Q_{r+1}^{p+r,q-r+1}) \\
&= \sum_{p,q}(-1)^{p+q}K_{r+1}^{p,q} + \sum_{p,q}(-1)^{p+q}Q_{r+1}^{p+r,q-r+1} \\
&= \sum_{p,q}(-1)^{p+q}K_{r+1}^{p,q} - \sum_{p,q}(-1)^{p+q+1}Q_{r+1}^{p+r,q-r+1} \\
&= \sum_{p,q}(-1)^{p+q}(K_{r+1}^{p,q} - Q_{r+1}^{p,q}) \\
&= \sum_{p,q}(-1)^{p+q}E_{r+1}^{p,q} = [E_{r+1}].
\end{aligned}$$

□

We go back to our assumption that $S = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ with the action of $T = \mathbb{C}^*$ specified above (but the results work for any smooth projective toric surface). We have the following generalization of [**3**, Lemma 3.2].

Definition 15. *For Young diagrams Y^1, Y^2 we write*

$$W_{Y^1,Y^2}(x,y) := \sum_{s\in Y^1}\big(x^{-l_{Y^2}(s)}y^{a_{Y^1}(s)+1} + \sum_{s\in Y^2} x^{l_{Y^1}(s)+1}y^{-a_{Y^2}(s)}\big).$$

Proposition 11. *Let*

$$(\overline{Y}^1 = (Y_0^1,\dots,Y_e^1), \overline{Y}^2 = (Y_0^2,\dots,Y_e^2))$$

be a pair of tuples of Young diagrams. Let M be an equivariant line bundle on S. Then in the representation ring of T we have

$$-RHom(I_{Z(\overline{Y}^1)}, I_{Z(\overline{Y}^2)}\otimes M) = -R\Gamma(S,V) + \sum_{i=0}^{e} W_{Y_i^1,Y_i^2}(x_i,y_i)\cdot M(p_i).$$

Proof. This was proven in [**3**, Lemma 3.2] under the assumption that $\pm M$ is not effective and $\pm M + K_S$ is not effective. Lemma 9 serves to remove this assumption, and using this lemma, one can essentially repeat the arguments from the proof of [**3**, Lemma 3.2]. We will indicate the changes. We write $Y := Z(\overline{Y}^1)$, $Z := Z(\overline{Y}^2)$. The first step in the proof of [**3**, Lemma 3.2] is the proof of the following claim.

Claim: In the representation ring of T we have the identity

$$-R\mathrm{Hom}(I_Y, I_Z\otimes M) = -R\Gamma(S,M) + R\Gamma(\mathcal{E}xt^1(I_Y,I_Z)\otimes M) + H^0(S,\mathcal{O}_Z\otimes M) + H^0(S,\mathcal{H}om(\mathcal{O}_Y,\mathcal{O}_Z)\otimes M).$$

Proof of the claim. By Lemma 9 we have

$$\begin{aligned}-R\mathrm{Hom}(\mathcal{I}_Y,\mathcal{I}_Z\otimes M) &= -R\Gamma(R\mathcal{H}om(I_Y,I_Z)\otimes M)\\ &= H^0(S,\mathcal{E}xt^1(I_Y,I_Z)\otimes M)-R\Gamma(\mathcal{H}om(I_Y,I_Z)\otimes M).\end{aligned}$$

The second line follows because $\mathcal{E}xt^2(I_Y,I_Z)=0$ and $\mathcal{E}xt^1(I_Y,I_Z)$ is supported on the supports of Y and Z. Thus $H^i(S,\mathcal{E}xt^1(I_Y,I_Z)\otimes M)=0$ for $i>0$. We have the exact sequence

$$0\to I_Z\to\mathcal{H}om(I_Y,I_Z)\to\mathcal{H}om(\mathcal{O}_Y,\mathcal{O}_Z)\to 0, \tag{5.1}$$

which is obtained by applying $R\mathcal{H}om(\cdot,I_Z)$ to the exact sequence $0\to I_Y\to\mathcal{O}_S\to\mathcal{O}_Y\to 0$ and noticing that $\mathcal{H}om(\mathcal{O}_Y,I_Z)=0$, $\mathcal{E}xt^1(\mathcal{O}_Y,I_Z)=\mathcal{H}om(\mathcal{O}_Y,\mathcal{O}_Z)$. Tensoring (5.1) by M, applying $R\Gamma$ and using that $H^i(\mathcal{H}om(\mathcal{O}_Y,\mathcal{O}_Z)\otimes M)=0$ for $i>0$ (because $\mathcal{H}om(\mathcal{O}_Y,\mathcal{O}_Z)$ has zero-dimensional support), we get the following identity in the representation ring of T

$$-R\Gamma(\mathcal{H}om(I_Y,I_Z)\otimes M)=-R\Gamma(S,I_Z\otimes M)-H^0(\mathcal{H}om(\mathcal{O}_Y,\mathcal{O}_Z),\otimes M). \tag{5.2}$$

Finally we use the sequence $0\to I_Z\otimes M\to M\to\mathcal{O}_Z\otimes M\to 0$ and the vanishing of $H^i(S,\mathcal{O}_Z\otimes M)$ for $i>0$ to replace $-R\Gamma(S,I_Z\otimes M)$ in (5.2) by $H^0(S,\mathcal{O}_Z\otimes M)-R\Gamma(S,M)$. This shows the claim.

Using the claim, the rest of the proof of [**3**, Lemma 3.2] is unchanged. □

Finally we use this result to describe the action of T on the fibers of the bundle E, which is E_{n_1,n_2} on $S^{[n_1]}\times S^{[n_2]}$.

Corollary 6. *Let*

$$(\overline{Y}^1=(Y_0^1,\dots,Y_e^1),\overline{Y}^2=(Y_0^2,\dots,Y_e^2))$$

be a pair of tuples of Young diagrams. Then

$$E(Z(\overline{Y}^1),Z(\overline{Y}^2))=\prod_{\alpha=1,2}\prod_{\beta=1,2}\sum_{i=0}^{e}W_{Y_i^\alpha,Y_i^\beta}(x_i,y_i)\cdot\xi^{\otimes(\beta-\alpha)}(p_i)\otimes\mathfrak{s}^{\otimes 2(\beta-\alpha)}.$$

Proof. This follows directly from (3.1) in Chapter 2, Theorem 11 and Proposition 11. □

6. Results of the computations

We have carried out the computation of $Z_S(L,a_1,c_1,s,y,q)$ and $Z_S(L,a_1,c_1,s,y,q)$ for each of the 11 cases above. The formulas above were implemented in a Pari/GP program.

We determined the universal series $A_1,\dots,A_{11}$ and $A_1(y),\dots,A_{11}(y)$ of Theorem 10 in Chapter 4 to the following orders:

- For $A_1\dots,A_{11}$, we computed the coefficients of $s^{l-3n}q^n$ for all $n\le 10$, $l\le 49$. (Recall: $A_i,A_i(y)$ are *Laurent* series in s.)

- For $A_1(y), \ldots, A_{11}(y)$, we computed the coefficients of $s^{l-5n} y^m q^n$ for all $n \leq 6$, $m \leq 9$, $l \leq 30$.

CHAPTER 5

Applications

In this section (except for the case of K3 surfaces, where we also deal with the refined invariants), we restrict our attention to non-refined K-theoretic Donaldson invariants and study some applications of Conjecture 1.

(1) We state the formulas of Conjectures 1 and 3 for K3 surfaces. In this case they were proven in [**14**].
(2) We give a simplified formula for minimal surfaces of general type or more generally for surfaces whose only Seiberg-Witten classes are 0 and K_S.
(3) We give an alternative formula for surfaces with disconnected canonical divisor K_S, written in terms of the connected components of K_S.
(4) We formulate a blowup formula, relating the K-theoretic Donaldson invariants of a surface S and its blowup $\widehat{S}$ in a point.
(5) We show that the Witten conjecture is also a consequence of Conjecture 1.

We will start by rewriting Conjecture 1. If S is a smooth projective surface with $b_1(S)=0$, $p_g(S)>0$, and we assume that $M_S^H(c_1,c_2)$ consists only of stable sheaves, Conjecture 1 says that $\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L))=$ $\mathrm{Coeff}_{x^{\mathrm{vd}}}\left[\psi_{S,L,c_1}(x)\right]$, with

$$\psi_{S,L,c_1}(x)=\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\frac{(L-K_S)^2}{2}+\chi(\mathcal{O}_S)}}\sum_{a\in H^2(S,\mathbb{Z})}SW(a)\,(-1)^{ac_1}\left(\frac{1+x}{1-x}\right)^{\left(\frac{K_S}{2}-a\right)(L-K_S)}.$$

We get a different form of Conjecture 1 by rewriting $\psi_{S,L,c_1}(x)$.

Remark 9.

$$\psi_{S,L,c_1}(x)=\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}}\sum_{a\in H^2(S,\mathbb{Z})}SW(a)\,(-1)^{ac_1}(1+x)^{(K_S-a)(L-K_S)}(1-x)^{a(L-K_S)}. \tag{0.1}$$

Proof. Note that

$$\chi(L)=\frac{L(L-K_S)}{2}+\chi(\mathcal{O}_S)=\frac{(L-K_S)^2}{2}+\chi(\mathcal{O}_S)+\frac{K_S(L-K_S)}{2}.$$

Thus

$$\psi_{S,L,c_1}(x)=\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}}\sum_{a\in H^2(S,\mathbb{Z})}SW(a)\,(-1)^{ac_1}\left(\frac{1+x}{1-x}\right)^{\left(\frac{K_S}{2}-a\right)(L-K_S)}\big((1+x)(1-x)\big)^{\frac{K_S}{2}(L-K_S)},$$

and the sumands on the right simplify to $SW(a)\,(-1)^{ac_1}(1+x)^{(K_S-a)(L-K_S)}(1-x)^{a(L-K_S)}$. $\square$

1. K3-surfaces

The formulas of Conjectures 1 and 3 for the K-theoretic Donaldson invariants are sums over contributions of Seiberg-Witten classes. Thus the formula will be simple if the Seiberg-Witten invariants are simple. The simplest case is that of K3-surfaces where the only Seiberg-Witten class is 0 with $SW(0) = 1$.

Let S be a K3 surface. Then Conjecture 3 takes the following attractive and simple form.

Conjecture 5. *Let $L \in \mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Then*

$$y^{-\frac{\mathrm{vd}}{2}}\chi^{\mathrm{vir}}_{-y}(M^H_S(c_1,c_2),\mu(L)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[\prod_{n=1}^{\infty}\frac{\left(\frac{(1-x^{2n})^2}{(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{n^2\frac{L^2}{2}}}{(1-x^{2n})^{20}(1-x^{2n}y)^2(1-x^{2n}y^{-1})^2}\right],$$

and in particular

$$\chi^{\mathrm{vir}}(M^H_S(c_1,c_2),\mu(L)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[\frac{1}{(1-x^2)^{\frac{L^2}{2}+2}}\right].$$

This conjecture was proven in [**14**] as a special case of [**14**, Theorem 1.5]. This is an important general verification (i.e. for all possible Chern classes) of Conjectures 1 and 3.

2. Minimal surfaces of general type

The second simplest possiblity for the Seiberg-Witten invariants of a surface S with $b_1(S) = 0$, $p_g(S) > 0$ is when the only Seiberg-Witten classes are 0 and $K_S \neq 0$.

Remark 10. *This is true in the following two cases.*

(1) *Minimal surfaces of general type satisfying $p_g(S) > 0$ and $b_1(S) = 0$* [**29**, Thm. 7.4.1],
(2) *smooth projective surfaces with $b_1(S) = 0$ and containing an irreducible reduced curve $C \in |K_S|$ (e.g. discussed in* [**15**, Sect. 6.3]*).*

Proposition 12. *Let S be a smooth projective surface satisfying $p_g(S) > 0$, $b_1(S) = 0$, $K_S \neq 0$, and such that its only Seiberg-Witten basic classes are 0 and K_S. Let $L \in \mathrm{Pic}(S)$ and let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Suppose Conjecture 1 holds in this setting. Then $\chi^{\mathrm{vir}}(M^H_S(c_1,c_2),\mu(L))$ is given by the coefficient of x^{vd} of*

$$2^{3-\chi(\mathcal{O}_S)+K_S^2}\frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}}.$$

Proof. Since $SW(0) = 1$, we have $SW(K_S) = (-1)^{\chi(\mathcal{O}_S)}$ [**27**, Prop. 6.3.4]. By Conjecture 1, $\chi^{\mathrm{vir}}(M^H_S(c_1,c_2),\mu(L))$ is given by the coefficient of x^{vd} of (0.1), which simplifies to

$$\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}}\left[(1+x)^{K_S(L-K_S)} + (-1)^{c_1K_S+\chi(\mathcal{O}_S)}(1-x)^{K_S(L-K_S)}\right].$$

Varying over c_2, we put the coefficients of all terms x^{vd} of $\psi_{S,L,c_1}(x)$ into a generating series as follows. Write

$$\psi_{S,L,c_1}(x) := \sum_{n=0}^{\infty} \psi_n x^n.$$

Then for $\mathrm{vd} = \mathrm{vd}(S, c_1, c_2) = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$, we have

$$\begin{aligned}
&\sum_{c_2} \mathrm{Coeff}_{x^{\mathrm{vd}(S,c_1,c_2)}}(\psi_{S,L,c_1}(x))\, x^{\mathrm{vd}(S,c_1,c_2)} = \sum_{n \equiv -c_1^2 - 3\chi(\mathcal{O}_S) \mod 4} \psi_n\, x^n \\
&= \sum_{k=0}^{3} \frac{1}{4} i^{k(c_1^2+3\chi(\mathcal{O}_S))} \psi(i^k x) \\
&= 2^{1-\chi(\mathcal{O}_S)+K_S^2} \left[\frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}} + (-1)^{c_1^2+3\chi(\mathcal{O}_S)} \frac{(1-x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}} \right. \\
&\qquad \left. + i^{c_1^2+3\chi(\mathcal{O}_S)} \frac{(1+ix)^{K_S(L-K_S)}}{(1+x^2)^{\chi(L)}} + (-i)^{c_1^2+3\chi(\mathcal{O}_S)} \frac{(1-ix)^{K_S(L-K_S)}}{(1+x^2)^{\chi(L)}} \right],
\end{aligned}$$

where the third equality uses $c_1 K_S \equiv c_1^2 \mod 2$. Now define

$$\phi_{S,L,c_1}(x) := 2^{3-\chi(\mathcal{O}_S)+K_S^2} \frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}}.$$

Then

$$\begin{aligned}
\sum_{c_2} \mathrm{Coeff}_{x^{\mathrm{vd}(S,c_1,c_2)}}(\phi_{S,L,c_1}(x))\, x^{\mathrm{vd}(S,c_1,c_2)} &= \sum_{n \equiv -c_1^2 - 3\chi(\mathcal{O}_S) \mod 4} \phi_n\, x^n \\
&= \sum_{k=0}^{3} \frac{1}{4} i^{k(c_1^2+3\chi(\mathcal{O}_S))} \phi(i^k x)
\end{aligned}$$

is given by the same expression as above, which proves the proposition. □

Corollary 7. *Let S be a smooth projective surface with $b_1(S) = 0$ and containing a smooth connected curve $C \in |K_S|$ of genus g. Let $L \in \mathrm{Pic}(S)$ and let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Suppose Conjecture 1 holds in this setting. Then $\chi^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L))$ is given by the coefficient of x^{vd} of*

$$2^{3-\chi(\mathcal{O}_C)-\chi(\mathcal{O}_S)} \frac{(1+x)^{\chi(L|_C)}}{(1-x^2)^{\chi(L)}}.$$

Proof. By the adjuction formula we have $K_C = (K_S + C)\big|_C$, thus $2g - 2 = \deg(K_C) = 2K_S^2$. Therefore $g = K_S^2 + 1$ and $\chi(L|_C) = 1 - g + \deg L|_C$ by Riemann-Roch. Therefore the corollary follows from Proposition 12. □

3. Disconnected canonical divisor

If the canonical divisor K_S is the union of disjoint irreducible reduced curves $K_S = C_1 + \ldots + C_m$, then the Seiberg-Witten classes of S are sums of the C_i, and the corresponding Seiberg-Witten invariants can be expressed in terms of the C_i. This allows us to express the formula of Conjecture 1 in terms of the connected components of K_S.

Suppose $C_1,\ldots,C_m$ are irreducible reduced mutually disconnected curves on a smooth projective surface S with $b_1(S)=0$ and $p_g(S)>0$, and let $M:=\{1,\ldots,m\}$. Then for any $I=\{i_1,\ldots,i_k\}\subset M$, we define

$$C_I := \sum_{i\in I} C_i.$$

For $I,J\subset M$ we write $I\sim J$ whenever C_I is linearly equivalent to C_J. This defines an equivalence relation. We denote the equivalence class corresponding to I by $[I]$ and denote its number of elements by $|[I]|$. We denote $N_{C_j/S}$ the normal bundle of $C_j\subset S$. We use the following result.

Lemma 10. [**15**, Lemma 3.1] *Let S be a smooth projective surface with $b_1(S)=0$ and $p_g(S)>0$, and suppose $C_1+\cdots+C_m\in|K_S|$, where $C_1,\ldots,C_m$ are mutually disjoint irreducible reduced curves. Then the Seiberg-Witten basic classes of S are $\{C_I\}_{I\subset M}$ and*

$$SW(C_I)=|[I]|\prod_{i\in I}(-1)^{h^0(N_{C_i/S})}.$$

Our result is the following.

Proposition 13. *Let S be a smooth projective surface with $b_1(S)=0$ and $p_g(S)>0$, and suppose there exists $0\neq C_1+\cdots+C_m\in|K_S|$, where $C_1,\ldots,C_m$ are mutually disjoint irreducible reduced curves. Let $L\in\mathrm{Pic}(S)$ and let H,c_1,c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1,c_2. Suppose Conjecture 1 holds in this setting. Then $\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L))$ is given by the coefficient of x^{vd} of*

$$\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}}\prod_{j=1}^m\left[(1+x)^{\chi(L|_{C_i})}+(-1)^{C_ic_1+h^0(N_{C_i/S})}(1-x)^{\chi(L|_{C_i})}\right],$$

where $N_{C_i/S}$ denotes the normal bundle of $C_i\subset S$.

Proof. By Lemma 10 equation (0.1) becomes

$$\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}}\left(\sum_{[I]}|[I]|\prod_{i\in I}(-1)^{h^0(N_{C_i/S})}\right)(-1)^{C_Ic_1}(1+x)^{C_{M\setminus I}(L-K_S)}(1-x)^{C_I(L-K_S)}$$
$$=\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}}\sum_{I\subset M}\left(\prod_{i\in I}(-1)^{C_ic_1+h^0(N_{C_i/S})}(1-x)^{C_i(L-C_i)}\right)\left(\prod_{i\in M\setminus I}(1+x)^{C_i(L-C_i)}\right),$$

where we used $K_S=C_M$ and the assumption that the curves C_i are mutually disjoint, which implies that

$$C_I^2=\sum_{i\in I}C_i^2=C_IK_S,\qquad C_iK_S=C_i^2.$$

On the other hand expanding the product in the statement of the proposition we get

$$\prod_{i=1}^{m}\Big[(1+x)^{\chi(L|_{C_i})}+(-1)^{C_ic_1+h^0(N_{C_i/S})}(1-x)^{\chi(L|_{C_i})}\Big]$$
$$=\sum_{I\subset M}\left(\prod_{i\in I}(-1)^{C_ic_1+h^0(N_{C_i/S})}(1-x)^{C_i(L-C_i)}\right)\left(\prod_{i\in M\setminus I}(1+x)^{C_i(L-C_i)}\right),$$

and the result follows. $\square$

4. Blow-up formula

A very important role in the understanding of the Donaldson invariants was played by the blowup formulas which relate the Donaldson invariants of a surface S and its blowup $\widehat{S}$ in a point. We show that for the K-theoretic Donaldson invariants a simple blowup formula follows from Conjecture 1.

Proposition 14. *Let S be a smooth projective surface, $\pi:\widehat{S}\to S$ the blow-up of S in a point, and E the exceptional divisor. Let $L,c_1\in\operatorname{Pic}(S)$, $\widehat{c}_1=\pi^*c_1-kE$, and $\widehat{L}=\pi^*L-\ell E$. Then*

$$\psi_{\widehat{S},\widehat{L},\widehat{c}_1}(x)=\frac{1}{2}(1-x^2)^{\binom{\ell+1}{2}}\left[(1+x)^{\ell+1}+(-1)^k(1-x)^{\ell+1}\right]\psi_{S,L,c_1}(x).$$

Thus if S is a smooth projective surface with $b_1(S)=0$, $p_g(S)>0$ and Conjecture 1 is true for $M^H_{\widehat{S}}(\widehat{c}_1,c_2)$, then

$$\chi^{\mathrm{vir}}(M^H_{\widehat{S}}(\widehat{c}_1,c_2),\mu(\widehat{L}))=\operatorname{Coeff}_{x^{\mathrm{vd}(\widehat{S},\widehat{c}_1,c_2)}}\left[\frac{1}{2}(1-x^2)^{\binom{\ell+1}{2}}\left[(1+x)^{\ell+1}+(-1)^k(1-x)^{\ell+1}\right]\psi_{S,L,c_1}(x)\right].$$

Proof. The Seiberg-Witten basic classes of $\widehat{S}$ are π^*a and π^*a+E with corresponding Seiberg-Witten invariant $SW(a)$, where a runs over all Seiberg-Witten basic classes of S [**29**, Thm. 7.4.6]. Using $\chi(\mathcal{O}_{\widehat{S}})=\chi(\mathcal{O}_S)$, $K_{\widehat{S}}=\pi^*K_S+E$, $E^2=-1$, $\chi(\widehat{L})=\chi(L)-\binom{\ell+1}{2}$, the proposition follows at once from equation (0.1) on pg. 67. $\square$

5. Witten conjecture

Let S be a smooth projective surface satisfying $b_1(S)=0$ and $p_g(S)>0$. Now we want to show that Conjecture 1 implies the Witten conjecture (Theorem 1) for algebraic surfaces, which had been proven in [**19**]. This can be be viewed as additional evidence for Conjecture 1, and it also illustrates that Conjecture 1 is indeed a K-theoretic version of the Witten conjecture.

Proposition 15. *Let S be a smooth projective surface satisfying $b_1(S)=0$ and $p_g(S)>0$. Then Conjecture 1 implies*

$$\int_{[M^H_S(c_1,c_2)]^{\mathrm{vir}}}\frac{c_1(\mu(L))^{\mathrm{vd}}}{\mathrm{vd}!}=\operatorname{Coeff}_{x^{\mathrm{vd}}}\Bigg[2^{2-\chi(\mathcal{O}_S)+K_S^2}(-1)^{\chi(\mathcal{O}_S)+c_1(c_1-K_S)/2}\exp\Big(\frac{(c_1(L)^2)}{2}x^2\Big)$$
$$\cdot\sum_{a\in H^2(S,\mathbb{Z})}(-1)^{c_1a}SW(a)\exp\Big(\big((2a-K_S)c_1(L)\big)x\Big)\Bigg],$$

i.e. the Witten conjecture (Theorem 1).

PROOF. By the Virtual Riemann-Roch Theorem 5, putting $M := M_S^H(c_1, c_2)$, we get for all $n \in \mathbb{Z}$ (writing the Picard group additively)

$$\chi^{\mathrm{vir}}(M, \mu(nL)) = \int_{[M]^{\mathrm{vir}}} \exp\big(nc_1(\mu(L))\big) td(T_M^{\mathrm{vir}}).$$

We can write $td(T_M^{\mathrm{vir}}) = 1 + \sum_{i=1}^{\mathrm{vd}} t_i$ with $t_i \in H^{2i}(M, \mathbb{Q})$. Therefore we get that $\chi^{\mathrm{vir}}(M, \mu(nL))$ is a polynomial of degree vd in n whose leading term is $n^{\mathrm{vd}} \int_{[M]^{\mathrm{vir}}} \frac{c_1(\mu(L))^{\mathrm{vd}}}{\mathrm{vd}!}$. Thus we get

$$\lim_{n\to\infty} \frac{1}{n^{\mathrm{vd}}} \chi^{\mathrm{vir}}(M, \mu(nL)) = \int_{[M]^{\mathrm{vir}}} \frac{c_1(\mu(L))^{\mathrm{vd}}}{\mathrm{vd}!}.$$

On the other hand we can also compute this limit from Conjecture 1. It is standard that

$$\lim_{n\to\infty} (1 + \frac{x}{n})^n = \lim_{n\to\infty} \frac{1}{(1-\frac{x}{n})^n} = \exp(x).$$

Thus we get by Conjecture 1 that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^{\mathrm{vd}}} \chi^{\mathrm{vir}}(M, \mu(nL)) \\
&\quad = \lim_{n\to\infty} \mathrm{Coeff}_{x^{\mathrm{vd}}} \left[\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-\frac{x^2}{n^2})^{\frac{(nL-K_S)^2}{2}+\chi(\mathcal{O}_S)}} \sum_{a \in H^2(S,\mathbb{Z})} SW(a)\,(-1)^{ac_1} \left(\frac{1+\frac{x}{n}}{1-\frac{x}{n}}\right)^{\left(\frac{K_S}{2}-a\right)(nL-K_S)} \right] \\
&\quad = \lim_{n\to\infty} \mathrm{Coeff}_{x^{\mathrm{vd}}} \left[\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-\frac{x^2}{n^2})^{\frac{n^2L^2}{2}}} \sum_{a \in H^2(S,\mathbb{Z})} SW(a)\,(-1)^{ac_1} \left(\frac{1+\frac{x}{n}}{1-\frac{x}{n}}\right)^{\left(\frac{K_S}{2}-a\right)(nL)} \right] \\
&\quad = \mathrm{Coeff}_{x^{\mathrm{vd}}} \left[2^{2-\chi(\mathcal{O}_S)+K_S^2} \exp\left(\frac{L^2}{2}x^2\right) \sum_{a \in H^2(S,\mathbb{Z})} SW(a)\,(-1)^{ac_1} \exp((K_S-2a)Lx) \right].
\end{aligned}$$

$\square$

CHAPTER 6

Examples

1. K3 surfaces

Let S be a $K3$ surface. Let H be ample on S and let L be a line bundle on S. We assume that $M_S^H(c_1, c_2)$ consists only of stable sheaves. Note that in this case $M_S^H(c_1, c_2)$ is nonsingular of the expected dimension $\mathrm{vd} = 4c_2 - c_1^2 - 6$, so that

$$\chi^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L)) = \chi(M_S^H(c_1, c_2), \mu(L)), \quad \chi_{-y}^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L)) = \chi_{-y}(M_S^H(c_1, c_2), \mu(L)).$$

As mentioned in Chapter 5 Section 1, Conjectures 1 and 3 have been proven in this case in [**14**].

All the same, we also calculate the numbers $\chi(M_S^H(c_1, c_2), \mu(L))$ directly by applying Corollary 3 on page 35 and our explicit knowledge of the universal functions $A_1, \ldots, A_{11}$, and $A_1(y), \ldots, A_{11}(q)$ as described in Chapter 4, Section 6. The easiest way to satisfy all assumptions of Corollary 3 is by choosing c_1 and H such that $c_1 H > 0$ is odd. Under this assumption we computed $\chi(M_S^H(c_1, c_2), \mu(L))$ for all $L \in \mathrm{Pic}(S)$ for c_1 with $c_1^2 = 0, 2, \ldots, 20$ and c_2 chosen in such a way that $\mathrm{vd}(S, c_1, c_2) < 14$. For the χ_{-y}-genus we did the corresponding calculation for $c_1^2 = 0, 2, \ldots, 20$ and c_2 chosen in such a way that $\mathrm{vd}(S, c_1, c_2) < 11$. In all these cases Conjectures 1 and 3 were confirmed.

2. Blowup of K3 surfaces

Let S be the blow-up of a K3 surface in a point, with exceptional divisor E. Again we choose H and c_1 so that all sheaves in $M_S^H(c_1, c_2)$ are stable. Let $\pi : S \to S_0$ be the blowup map. Under this assumption we computed $\chi^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L))$ for any L and for $c_1 = \pi^* C + rE$ such that $C^2 = -4, -2, \ldots, 10$, $r = -2, -1, \ldots, 2$, and $\mathrm{vd} < 15$. In all cases we get that

$$\chi^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L)) = \mathrm{Coeff}_{x^{\mathrm{vd}}} \left[\frac{(1+x)^{LE+1}}{(1-x^2)^{\frac{L^2 - LE}{2} + 2}} \right].$$

Note that the only basic classes of S are 0 and E with Seiberg-Witten invariants $SW(0) = SW(E) = 1$. As $K_S = E$, $K_S^2 = -1$ and $\chi(\mathcal{O}_S) = 2$, the formula above therefore coincides with the prediction of Conjecture 1.

We also compute $\chi_{-y}^{\mathrm{vir}}(M_S^H(c_1, c_2), \mu(L))$ for $c_1 = \pi^* C + rE$ such that $C^2 = -4, -2, \ldots, 14$, $r = -2, -1, \ldots, 2$, and $\mathrm{vd} < 10$, confirming Conjecture 3.

3. Elliptic surfaces

Let $E_N \to \mathbb{P}^1$ be a non-trivial elliptic surface with section, $12N > 0$ rational nodal fibres, and no other singular fibres. Then the canonical class is given by $K_{E_N} = (N-2)F$, where F denotes the class of the fibre. Note that $\chi(\mathcal{O}_{S_N}) = N$. Choose a section $B \subset S$, then its class satisfies $B^2 = -N$.

We assume $n \geq 2$, then $E(n)$ has a smooth canonical divisor which has $N-2$ connected components F_j; each a smooth elliptic fibre of S. The Seiberg-Witten classes are the lF with $0 \leq l \leq N-2$ and $SW(lF) = (-1)^l\binom{N-2}{l}$.

For $N = \chi(\mathcal{O}_S) = 3, 4, \ldots, 7$, we compute $\chi^{\mathrm{vir}}(M^H_{E_N}(c_1, c_2), \mu(L))$ for $c_1 = mB + nF$ where B is the class of a section, F is the class of a fibre, for $m = -1, 0, 1, 2$, $n = -2, -1, \ldots, 5$, and $\mathrm{vd} < 12$. Conjectures 1 and 3 are confirmed in all these cases.

In fact as mentioned above K_{E_N} is the sum of $N-2$ smooth elliptic fibres F_j and it is easy to see that we have $h^0(N_{F_j/E_N}) = 1$ and $\chi(L|_{F_j}) = LF_j$. We write $Q = L^2$, $w = LF$. Then our results confirm the prediction of Proposition 13, which takes the form

$$\chi^{\mathrm{vir}}(M^H_{E_N}(mB + nF, c_2), \mu(L)) = \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[\frac{2^{2-N}}{(1-x^2)^{\chi(L)}}\prod_{j=1}^{N-2}\left[(1+x)^w - (-1)^m(1-x)^w\right]\right].$$

For instance, if we assume that $N \geq 3$ and $L \cdot F = 0$, then the result simplifies to

$$\chi^{\mathrm{vir}}(M^H_{E_N}(mB + nF, c_2), \mu(L)) = \begin{cases} \mathrm{Coeff}_{x^{\mathrm{vd}}}\left[\frac{1}{(1-x^2)^{L^2/2+N}}\right] & m \text{ odd}, \\ 0 & m \text{ even}. \end{cases}$$

Using our explicit determination of $A_1(y), \ldots, A_{11}(y)$ we also confirmed Conjecture 3 for $\chi^{\mathrm{vir}}(M^H_{E_N}(c_1, c_2), \mu(L))$ for $N = 3, 4, 5$, $c_1 = mB + nF$ with $m = -1, 0, 1, 2$, $n = -2, -1, \ldots, 10$, and $\mathrm{vd} < 9$.

4. Minimal surfaces of general type

We verify Conjectures 1 and 3 for two cases of minimal surfaces S of general type.

(1) S is a double cover of $\mathbb{P}^2$ branched along a smooth octic,
(2) S is a smooth quintic in $\mathbb{P}^3$.

4.1. Double cover of $\mathbb{P}^2$.

Let

$$\pi : S \to \mathbb{P}^2$$

be a double cover branched over a smooth curve C of degree 8. Then $K_S = L$, where L is the pull-back of the class of a line on $\mathbb{P}^2$. These surfaces satisfy $b_1(S) = 0$. It is easy to calculate

$$K_S^2 = 2, \quad \chi(\mathcal{O}_S) = 4.$$

The canonical linear system $|K_S|$ contains smooth connected canonical divisors. Let $c_1 = \epsilon L$. We apply Corollary 3 to the universal functions A_i. The Seiberg-Witten basic classes are $0, K_S \neq 0$ with Seiberg-Witten invariants $SW(0) = 1$, $SW(K_S) = (-1)^{\chi(\mathcal{O}_S)} = 1$. We first take $H = L$ as the polarization on S. Then conditions (ii), (iii) of Corollary 3 require

$$c_1 H = 2\epsilon > 4 = 2K_S H,$$

i.e. $\epsilon > 2$. If $\epsilon = 2k$ is even, we choose c_2 such that

$$\frac{1}{2}c_1(c_1 - K_S) - c_2 = \epsilon(\epsilon - 1) - c_2.$$

Then by [**21**, Rem. 4.6.8] the moduli space $M_S^H(c_1, c_2)$ only consists of stable sheaves.

Now assume that $\epsilon = 2k + 1$ is odd. If L generates the Picard group of S, then there are no rank 2 strictly μ-semistable sheaves with Chern classes ϵL and c_2. In general the Picard group of S can have more generators, but L is still ample and primitive. In this case we take the polarization H general and sufficiently close to L (i.e. of the form $nL + H$ for n sufficiently large), so that conditions (ii) and (iii) of Corollary 3 still hold when $\epsilon > 2$, and so that there are no rank 2 strictly μ-semistable sheaves with Chern classes ϵL and c_2.

We verified Conjecture 1 when $c_1 \cdot K_S = 0, 1, \ldots, 10$, $c_1^2 = 0, 1, \ldots, 30$, and $\mathrm{vd} < 12$. As $|K_S|$ contains a smooth connected curve the result is given by Proposition 1: For $A \in \mathrm{Pic}(S)$, we put $w = LA$, $Q = A^2$; then we have

$$\chi^{\mathrm{vir}}(M_S^H(\epsilon L, c_2), \mu(A)) = \mathrm{Coeff}_{x^{4c_2 - 2\epsilon^2 - 12}} \left[2\frac{(1+x)^{LA-2}}{(1-x^2)^{\frac{Q}{2} - \frac{w}{2} + 4}} \right].$$

As an illustration we write down the formula for a couple of examples. μ-stability is invariant under tensorizing by a line bundle.Therefore we know that $M_S^H(2, L, c_2)$ is isomorphic to $M_S^H(2, (2k+1)L, c_2 + 2(k^2 + k))$.

$$\begin{aligned}
\chi^{\mathrm{vir}}(M_S^H(L,4), \mu(A)) &= w^2 - 6w + Q + 14, \\
\chi^{\mathrm{vir}}(M_S^H(L,5), \mu(A)) &= \frac{1}{360}w^6 - \frac{7}{60}w^5 + (\frac{1}{24}Q + \frac{67}{36})w^4 - (\frac{5}{6}Q + \frac{91}{6})w^3 + (\frac{1}{8}Q^2 + \frac{79}{12}Q + \frac{3038}{45})w^2 \\
&\quad - (\frac{3}{4}Q^2 + \frac{65}{3}Q + \frac{2317}{15})w + \frac{1}{24}Q^3 + 2Q^2 + \frac{185}{6}Q + 154.
\end{aligned}$$

We also verified Conjecture 3 for c_1 such that $c_1 \cdot K_S = -2, -1, \ldots, 2$, $c_1^2 = -16, -15, \ldots, -6$, and $\mathrm{vd} < 9$. In particular we get with the notations above

$$\chi_{-y}^{\mathrm{vir}}(M_S^H(L,4), \mu(A)) = Q + w^2 - 6w(1+y) + (14 + 92y + 14y^2).$$

4.2. Quintic hypersurface in $\mathbb{P}^3$. Let S be a smooth surface of degree 5 in $\mathbb{P}^3$, then

$$K_S = L,$$

where L is the hyperplane section. Moreover S_d is simply connected by the Lefschetz hyperplane theorem. It is easy to compute that

$$K_S^2 = 5, \quad \chi(\mathcal{O}_S) = 5.$$

The hyperplane section H on S is very ample so $|K_S|$ contains smooth connected canonical divisors. We test Conjecture 1 using Corollary 3.

We take H as polarization and put $c_1 = \epsilon H$. We assume that S is very general, i.e. in the complement of countably many closed subvarieties in the projective space of hypersurfaces of degree 5, such that $\mathrm{Pic}(S) = \mathbb{Z}H$ by the Noether-Lefschetz Theorem. For

$$\begin{aligned} &c_1 H = 5\epsilon > 10 = 2K_S H, \\ &c_1 H = 5\epsilon \quad \text{odd, or } S \text{ very general and } \epsilon \text{ odd} \end{aligned}$$

there are no rank 2 strictly μ-semistable sheaves with first Chern class c_1, and conditions (ii), (iii) of Corollary 3 are satisfied. We assume that both ϵ is odd and

$$c_1 H = 5\epsilon > 10 = 2K_S H.$$

Then there are no rank 2 strictly μ-semistable sheaves with first Chern class c_1 and conditions (ii), (iii) of Corollary 3 are satisfied. We consider the case $c_1 = 3H$ and $\mathrm{vd} \leq 8$. Using $(dH)^2 = 5d^2$, $(dH)K_S = 5d$, $\chi(\mathcal{O}_S) = 5$, Conjecture 1 and Proposition 1 give the prediction

$$\chi^{\mathrm{vir}}(M_S^H(3H, c_2), \mu(dH)) = \mathrm{Coeff}_{x^{4c_2-60}} \left[8 \frac{(1+x)^{5(d-1)}}{(1-x^2)^{\frac{5}{2}(d^2-d+2)}} \right],$$

and we get

$$\begin{aligned} \chi^{\mathrm{vir}}(M_S^H(3H, 16), \mu(dH)) &= \frac{1}{3}\big(1450d^4 - 5800d^3 + 10730d^2 - 9860d\big) + 1280, \\ \chi^{\mathrm{vir}}(M_S^H(3H, 40), \mu(dH)) &= \frac{120625}{84}d^8 - \frac{241250}{21}d^7 + \frac{849625}{18}d^6 - 122375d^5 + \frac{7785425}{36}d^4 - \frac{790975}{3}d^3 \\ &\quad + \frac{13599230}{63}d^2 - \frac{757390}{7}d + 25520, \end{aligned}$$

confirming this prediction. Assuming the strong form of Mochizuki's formula holds (Chapter 2, Remark 2), we also verified Conjecture 3 for c_1, c_2 such that $c_1 \cdot K_S = 2, 3, \ldots, 6$, $c_1^2 = -16, -15, \ldots, -3$, and vd < 7. In particular we get the following refinement of the formula for $\chi^{\mathrm{vir}}(M_S^H(3H,16),\mu(dH))$ above

$$\chi^{\mathrm{vir}}_{-y}(M_S^H(3H,16),\mu(dH)) = \frac{1}{3}\Big((1450d^4 - 5800d^3(1+y) + d^2(10730y^2+37700y+10730 \\ - d(9860y^3+60100y^2+60100+9860)\Big) + (1280+11440y+27280y^2+11440y^3+1280y^4).$$

(Recall that by definition $\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L)) = \chi^{\mathrm{vir}}_{-y}(M_S^H(c_1,c_2),\mu(L))\big|_{y=0}$).

5. Blowups of the above surfaces

Finally we deal with the blowups up all the surfaces considered above in a point: For $\chi^{\mathrm{vir}}(M_S^H(c_1,c_2),\mu(L))$ we confirmed Conjecture 1 in the following cases, getting in each case the formula obtained in Proposition 14.

(1) S is the blow-up of a K3 surface in two distinct points, $c_1 = \pi^*C + \epsilon_1 E_1 + \epsilon_2 E_2$ such that $C^2 = -2, 0, \ldots, 6$, $\epsilon_1, \epsilon_2 = 0, 1$, and vd < 10.

(2) S is the blow-up of an elliptic surface of type E_3 (see Section 3) in a point, $c_1 = \pi^*C + \epsilon E$ such that $CK_S = -1, 0, \ldots, 4$, $C^2 = -4, -3, \ldots, 10$, $\epsilon = 0, 1$, and vd < 12.

Assuming the strong form of Mochizuki's formula holds (Chapter 2, Remark 2), we also verified Conjecture 1 in the following cases:

(3) S is the blow-up of a quintic in $\mathbb{P}^3$ in a point, $c_1 = \pi^*C + \epsilon E$ such that $CK_S = -5, -4, \ldots, 5$, $C^2 = -4, -3, \ldots, 8$, $\epsilon = 0, 1$, and vd < 10.

Applying the same method and using our explicit expansions of $A_1(y,q)$, …, $A_{11}(y,q)$, we verified Conjecture 3 in the following cases:

(1) S is the blow-up of a K3 surface in two distinct points, $c_1 = \pi^*C + \epsilon_1 E_1 + \epsilon_2 E_2$ such that $C^2 = -2, 0, \ldots, 6$, $\epsilon_1, \epsilon_2 = 0, 1$, and vd < 10.

(2) S is the blow-up of an elliptic surface of type E_3) in a point, $c_1 = \pi^*C + \epsilon E$ such that $CK_S = -1, 0, \ldots, 4$, $C^2 = -16, -15, \ldots, 0$, $\epsilon = 0, 1$, and vd < 9.

Assuming the strong form of Mochizuki's formula holds (Chapter 2, Remark 2), we also verified Conjecture 3 in the following cases:

(3) S is the blow-up of a smooth quintic in $\mathbb{P}^3$ in a point, $c_1 = \pi^*C + \epsilon E$ such that $CK_S = 0$, $C^2 = -23, -22, \ldots, -14$, $\epsilon = 0, 1$, and vd < 4.

Appendix

In this appendix we will show some explicit computer computations. We restrict our attention to the non-refined K-theoretic Donaldson invariants. First we list the Pari/GP program that computes the instanton part of the partition function $Z_S(L,a,c_1,s,q)$ for the nonrefined K-theoretic Donaldson invariants. In the second part we list the beginning terms of the 11 universal power series $A_1,\ldots,A_{11}$, such that

$$Z_S(L,a,c_1,s,q) = A_1^{L^2} A_2^{La} A_3^{a^2} A_4^{ac_1} A_5^{c_1^2} A_6^{Lc_1} A_7^{LK_S} A_8^{aK_S} A_9^{c_1K_S} A_{10}^{K_S^2} A_{11}^{\chi(\mathcal{O}_S)}.$$

Pari/gp program for K-theoretic Donaldson invariants

```
HH=H+O(H^25);\ps 31;par=vector(31);p=vector(32);r=30;ss=s+O(s^40);


p[1]=1;p[2]=1;p[3]=2;p[4]=3;p[5]=5;p[6]=7;p[7]=11;p[8]=15;p[9]=22;p[10]=30;p[11]=42;p[12]=56;
p[13]=77;p[14]=101;p[15]=135;p[16]=176;p[17]=231;p[18]=297;p[19]=385;p[20]=490;p[21]= 627;
p[22]=792;p[23]=1002;p[24]=1255;p[25]=1575;p[26]=1958;p[27]=2436;p[28]=3010;
p[29]= 3718;p[30]= 4565;p[31]=5604;p[32]=6842;


/*This computes a vector of partitions of n*/
part(n)={global(k);
k=1;
ll=max(31,n);lll=max(5605,p[n+1]);
P=vector(lll);B=vector(ll);
part0(n,n,1,B);print(k);
P};
part0(n,m,i,B)={global(k);
if(n>0,
for(j=1,m,if(j<=n,B[i]=j;part0(n-j,j,i+1,B);if(j==n,k=k+1))),P[k]=B)};
for(n=0,30,par[n+1]=part(n));


/*for ideal (y^(b0),xy^(b1)\ldots) in Hilb^n(A^2,0), given by a partition b, and an action
on cordinates with weights u,v, compute the Equivariant todd genus of tangent space
T_b Hilb^n(A^2)\otimes O(t)*/
```

```
totchi1(u,v,b,t) ={local(fin,bb,s,i,j,a1,a2,ee);
fin=1;
for(i=1,r,bi=b[i];
for(j=i,r,bj=b[j];bje=b[j+1];
for(s=bje,bj-1,
a1=(u*(i-j-1)+v*(bi-s-1)+t);
a2=(u*(j-i)+v*(s-bi)+t);
if(a1==0,fin=fin,fin=fin*a1/(1-exp(-a1)));
if(a2==0,fin=fin,fin=fin*a2/(1-exp(-a2)));)))
;fin};

/*for an ideal sheafs I_a,I_b supported in 0, given by partitions a,b,
with weights of local coordinates u,v, compute
contribution at the fixpoint to
todd(Ext^1_loc(I_a,I_b)\otimes O(t))*/

totchiab(u,v,a,b,t)={local(e1,fin,s);fin=1;
for(i=1,r,bi=b[i];ai=a[i];
for(j=i,r,bj=b[j];aj=a[j];aje=a[j+1];
for(s=aje,aj-1,a1=(u*(i-j-1)+v*(bi-s-1)+t);if(a1==0,fin=fin,fin=fin*a1/(1-exp(-a1))))));
for(i=1,r,bi=b[i];ai=a[i];
for(j=i,r,bj=b[j];aj=a[j];bje=b[j+1];
for(s=bje,bj-1,a1=(u*(j-i)+v*(s-ai)+t);if(a1==0,fin=fin,fin=fin*a1/(1-exp(-a1))))));
fin};

/*for ideal (y^(b0),xy^(b1),\ldots) in Hilb^n(A^2,0), given by a partition b, and an action
on cordinates with weights u,v, compute the Equivariant eulerclass of tangent space
T_b Hilb^n(A^2)*/
denomm(u,v,b)={local(e1,fin,s);fin=1;
for(i=1,r,bi=b[i];
for(j=i,r,bj=b[j];bje=b[j+1];
for(s=bje,bj-1,fin=fin*(u*(i-j-1)+v*(bi-s-1))*(u*(j-i)+v*(s-bi)))));
fin};
```

```
/*for a subscheme Z_a given by partition a at fixpoint with weights of local coordinates
u,v compute Eu(O_{Z_a))\otimes O(t)*/
OZ(u,v,a,t)={local(e1,fin,i,j);fin=1;
for(i=0,r-1,for(j=0,a[i+1]-1,fin=fin*(u*i+v*j+t)));
fin};

denomabt(u,v,a,b,t)={local(e1,fin,s);fin=1;
for(i=1,r,bi=b[i];ai=a[i];
for(j=i,r,bj=b[j];aj=a[j];aje=a[j+1];
for(s=aje,aj-1,fin=fin*(u*(i-j-1)+v*(bi-s-1)+t))));
for(i=1,r,bi=b[i];ai=a[i];
for(j=i,r,bj=b[j];aj=a[j];bje=b[j+1];
for(s=bje,bj-1,fin=fin*(u*(j-i)+v*(s-ai)+t))));
fin};

/*This computes to contribution to the instanton part of the
the partition function for one fixpoint on the surface, where the weights of
the action on the coordinates are u,v*/

ZchiL(u,v,nn,aa1,xi,L,s)={local(e1,e2,fin);fin=0;
 for(N=0,nn,e2=0;print(N);eeL=exp(HH*(N-(xi/2+s)^2/(u*v))*L);/*ch(\mu(L)*/
  for(n=0,N,
  m=N-n;
   for(l1=1,p[n+1],
   PP=par[n+1][l1];
    for(l2=1,p[m+1],
    QQ=par[m+1][l2];
    e1=1;e1=e1*OZ(u,v,PP,aa1)*OZ(u,v,QQ,2*s+xi+aa1)*totchi1(u,v,PP,0)*totchi1(u,v,QQ,0);
    e1=e1*totchiab(u,v,PP,QQ,xi+2*s)*totchiab(u,v,QQ,PP,-xi-2*s);
    e1=e1/(denomm(u,v,PP)*denomm(u,v,QQ)*denomabt(u,v,PP,QQ,xi+2*s)*denomabt(u,v,QQ,PP,-xi-2*s));
    e2=e2+e1)));
 fin=fin+(e2*eeL+O(q^(10*nn+20)))*x^N; );
 fin+O(x^(nn+1))}

/*On P2 compute the instanton part of Mochizuki formula for chi(M(c1),mu(L))
```

```
for SW a1*H, xi=(c1-2a1)*H,line bundle L*H, up to the Hilbert scheme of nn points */
ZchiP2L(a1,xi,L,nn)={local(e1,e2,fin,w0,w1,w2);fin=0;qq=q+O(q^(10*nn+20));;
w0=0;w1=1*qq;w2=19*qq;
fin=ZchiL((w1-w0),(w2-w0),nn,a1*w0,xi*w0,L*w0,ss)
*ZchiL((w2-w1),(w0-w1),nn,(a1*w1),xi*w1,L*w1,ss)
*ZchiL(w0-w2,w1-w2,nn,a1*w2,xi*w2,L*w2,ss);
fin1=0;for(ll=0,15,fin1=fin1+polcoeff(polcoeff(fin,ll,H)+O(q),0,q)*(H^ll)+O(x^(nn+1)));
fin1+O(x^(nn+1))};

/*On P1xP1 compute the instanton part of Mochizuki formula for
\chi(M(c1),\mu(L)) for SW a11*F+a12*G, xi=xi1F+xi2G=(c1-2a11*F+a12*G),
line bundle L1*F+L2G, up to the Hilbert scheme of nn points */
ZchiP11L(a11,a12,xi1,xi2,L1,L2,nn)={local(e1,e2,fin,w0,w1,v0,v1);
fin=0;qq=q+O(q^(10*nn+20));
w0=0;w1=1*qq;v0=0;v1=19*qq;
fin=ZchiL((w1-w0),(v1-v0),nn,a11*w0+a12*v0,xi1*w0+xi2*v0,L1*w0+L2*v0,ss)*
ZchiL((w1-w0),(v0-v1),nn,a11*w0+a12*v1,xi1*w0+xi2*v1,L1*w0+L2*v1,s)*
ZchiL((w0-w1),(v1-v0),nn,a11*w1+a12*v0,xi1*w1+xi2*v0,L1*w1+L2*v0,s)*
ZchiL((w0-w1),(v0-v1),nn,a11*w1+a12*v1,xi1*w1+xi2*v1,L1*w1+L2*v1,s);
fin1=0;for(ll=0,15,fin1=fin1+polcoeff(polcoeff(fin,ll,H)+O(q),0,q)*(H^ll)+O(x^(nn+1)));
fin1+O(x^(nn+1))};

chL=vector(11);
chL[1]=ZchiP2L(0,0,0,10);
chL[2]=ZchiP11L(0,0,0,0,0,0,10));
chL[3]=ZchiP2L(1,0,0,10));
chL[4]=ZchiP2L(0,1,0,10));
chL[5]=ZchiP2L(1,1,0,10));
chL[6]=ZchiP11L(0,1,0,0,0,0,8));
chL[7]=ZchiP11L(0,0,0,1,0,0,8));
chL[8]=subst(ZchiP2L(0,0,1,8),H,1));/*gives L^2*/
chL[9]=subst(ZchiP2L(1,0,1,8),H,1));/*gives a1L*/
chL[10]=subst(ZchiP2L(0,1,1,10),H,1));/*gives xiL*/
chL[11]=subst(ZchiP11L(0,0,0,0,1,0,10),H,1));/*gives LK*/
```

```
/* Change of Basis matrix from computations on P^2, P^1x P^1 to the invariants
a^2, a xi, ,xi^2,aK_S ,xiK_S,L^2,a L,xi L,LK_S,K^2,chi(O_S)*/
AC={[0,0,0,0,0,0,0,0,0,9,1;
    0,0,0,0,0,0,0,0,0,8,1;
    1,0,0,-3,0,0,0,0,0,9,1;
     0,0,1,0,-3,0,0,0,0,9,1;
     1,1,1,-3,-3,0,0,0,0,9,1;
     0,0,0,-2,0,0,0,0,0,8,1;
     0,0,0,0,-2,0,0,0,0,8,1;
     0,0,0,0,0,1,0,0,-3,9,1;
     1,0,0,-3,0,1,1,0,-3,9,1;
     0,0,1,0,-3,1,0,1,-3,9,1;
     0,0,0,0,0,0,0,0,-2,8,1]}
BC=mattranspose(AC)^-1;


/*Compute instanton part of Mochizuki formula for surface and line bundle with
given invariants
a^2, a xi, ,xi^2,aK_S ,xiK_S,L^2,a L,xi L,LK_S,K^2,chi(O_S)
note xi=c1-2*a
*/
ZchiLX(aa,axi,xixi,ak,xik,LL,aL,xiL,LK,kk,xo)={local(e1);
e1=mattranspose(BC*mattranspose([aa,axi,xixi,ak,xik,LL,aL,xiL,LK,kk,xo]));
erg=prod(i=1,11,chL[i]^(e1[i]));
erg}
```

The universal power series

As an illustration of our computations and to give an idea of the shape of the formulas, we give a list of the universal power series in the product formula

$$Z_S(L,a,c_1,s,q) = A_1^{L^2} A_2^{La} A_3^{a^2} A_4^{ac_1} A_5^{c_1^2} A_6^{Lc_1} A_7^{LK_S} A_8^{aK_S} A_9^{c_1K_S} A_{10}^{K_S^2} A_{11}^{\chi(\mathcal{O}_S)}$$

for the instanton part of the partition function for the nonrefined K-theoretical Donaldson invariants. They are listed here only modulo x^4 and only the few lowest order terms in s are written (recall that we computed the coefficients of $s^{l-3n}q^n$ for all $n \leq 10$, $l \leq 49$).)

$$\begin{aligned} A_1 = 1 &+ (-\frac{1}{4}s^{-1} + \frac{1}{12}s - \frac{1}{60}s^3 + \frac{1}{378}s^5 + \ldots)q + (-\frac{1}{16}s^{-4} + \frac{11}{96}s^{-2} - \frac{19}{240} + \frac{1039}{30240}s^2 + \ldots)q^2 \\ &+ (-\frac{3}{64}s^{-7} + \frac{5}{64}s^{-5} - \frac{157}{1920}s^{-3} + \frac{379}{5760}s^{-1} + \ldots)q^3 + O(q^4) \end{aligned}$$

$$\begin{aligned} A_2 = (1 + 2s + 2s^2 + \frac{4}{3}s^3 + \frac{2}{3}s^4 + \frac{4}{15}s^5 + \ldots) &+ (-s^{-2} - 2s^{-1} - 2 - \frac{4}{3}s - \frac{3}{5}s^2 - \frac{2}{15}s^3 + \ldots)q \\ + (-\frac{9}{16}s^{-5} - \frac{5}{8}s^{-4} + \frac{7}{24}s^{-3} + \frac{13}{12}s^{-2} + s^{-1} \ldots)q^2 & \\ + (-\frac{17}{32}s^{-8} - \frac{1}{2}s^{-7} + \frac{7}{16}s^{-6} + \frac{3}{4}s^{-5} + \ldots)q^3 + O(q^4) & \end{aligned}$$

$$\begin{aligned} A_3 = 1 &+ (-\frac{3}{2}s^{-3} - \frac{1}{30}s + \frac{2}{63}s^3 - \frac{1}{90}s^5 + \ldots)q + (-\frac{3}{64}s^{-6} + \frac{9}{16}s^{-4} - \frac{7}{160}s^{-2} - \frac{233}{5040} + \ldots)q^2 \\ &+ (-\frac{9}{128}s^{-9} + \frac{17}{96}s^{-7} - \frac{577}{1920}s^{-5} + \frac{11}{105}s^{-3} + \ldots)q^3 + O(q^4) \end{aligned}$$

$$\begin{aligned} A_4 = 1 &+ (s^{-3} + \frac{1}{15}s - \frac{8}{189}s^3 + \frac{1}{75}s^5 \ldots)q + (\frac{77}{64}s^{-6} - \frac{5}{16}s^{-4} + \frac{47}{480}s^{-2} - \frac{661}{15120} + \ldots)q^2 \\ &+ (\frac{101}{64}s^{-9} - \frac{41}{48}s^{-7} + \frac{311}{960}s^{-5} - \frac{1811}{15120}s^{-3} + \ldots)q^3 + O(q^4) \end{aligned}$$

$$\begin{aligned} A_5 = 1 &+ (-\frac{1}{8}s^{-3} - \frac{1}{40}s + \frac{5}{378}s^3 - \frac{7}{1800}s^5 + \ldots)q + (-\frac{9}{128}s^{-6} + \frac{1}{32}s^{-4} + \frac{1}{320}s^{-2} + \frac{17}{10080} + \ldots)q^2 \\ &+ (-\frac{65}{1024}s^{-9} + \frac{37}{768}s^{-7} - \frac{49}{3072}s^{-5} + \ldots)q^3 + O(q^4) \end{aligned}$$

$$\begin{aligned} A_6 = (1 - s + \frac{1}{2}s^2 - \frac{1}{6}s^3 + \frac{1}{24}s^4 - \frac{1}{120}s^5 + \ldots) &+ (\frac{1}{4}s^{-2} - \frac{1}{4}s^{-1} + \frac{5}{24} - \frac{1}{8}s + \frac{1}{480}s^2 + \ldots)q \\ + (\frac{1}{8}s^{-}5 - \frac{3}{32}s^{-4} - \frac{5}{96}s^{-3} + \frac{19}{192}s^{-2} + \ldots)q^2 & \\ + (\frac{7}{64}s^{-8} - \frac{5}{64}s^{-7} - \frac{5}{64}s^{-6} + \frac{17}{192}s^{-5} + \ldots)q^3 + O(q^4) & \end{aligned}$$

$$A_7 = 1 + (\frac{1}{4}s^{-2} + \frac{1}{2}s^{-1} - \frac{1}{6}s - \frac{1}{60}s^2 + \ldots)q + (\frac{5}{32} * s^{-5} + \frac{5}{32}s^{-4} - \frac{1}{24}s^{-2} - \frac{1}{80}s^{-1} + \ldots)q^2$$
$$+ (\frac{5}{32}s^{-8} + \frac{17}{128}s^{-7} - \frac{7}{128}s^{-6} - \frac{5}{64}s^{-5} + \ldots)q^3 + O(q^4)$$

$$A_8 = 1 + (\frac{1}{2}s^{-3} + s^{-2} + \frac{1}{30}s - \frac{1}{15}s^2 - \frac{4}{189}s^3 + \ldots)q + (\frac{19}{32}s^{-6} + \frac{17}{16}s^{-5} + 14s^{-4} - \frac{5}{12}s^{-3} + \ldots)q^2$$
$$+ (\frac{13}{16}s^{-9} + \frac{45}{32}s^{-8} + \frac{5}{24}s^{-7} - \frac{5}{6}s^{-6} + \ldots)q^3 + O(q^4)$$

$$A_9 = 1 + (-\frac{1}{4}s^{-3} - \frac{1}{4}s^{-2} - \frac{1}{12} - \frac{1}{60}s + \frac{1}{20}s^2 + \frac{2}{189}s^3 + \ldots)q$$
$$+ (-\frac{21}{128}s^{-6} - \frac{1}{16}s^{-5} + \frac{1}{8}s^{-4} + \frac{5}{48}s^{-3} + \ldots)q^2$$
$$+ (-\frac{83}{512}s^{-9} - \frac{19}{512}s^{-8} + \frac{1}{6}s^{-7} + \frac{109}{1536}s^{-6} + \ldots)q^3 + O(q^4)$$

$$A_{10} = 1 + (-\frac{1}{8}s^{-3} - \frac{1}{4}s^{-2} - \frac{1}{6}s^{-1} + \frac{19}{360}s + \frac{1}{60}s^2 + \ldots)q$$
$$+ (-\frac{1}{8}s^{-6} - \frac{1}{8}s^{-5} + \frac{1}{12}s^{-4} + \frac{1}{6}s^{-3} + \frac{19}{360}s^{-2} + \ldots)q^2$$
$$+ (-\frac{77}{512}s^{-9} - \frac{27}{256}s^{-8} + \frac{1}{6}s^{-7} + \frac{31}{192}s^{-6} + \ldots)q^3 + O(q^4)$$

$$A_{11} = 1 + (-\frac{1}{2}s^{-1} + \frac{1}{6}s - \frac{1}{30}s^3 + \frac{1}{189}s^5 - \frac{1}{1350}s^7 + \ldots)q$$
$$+ (\frac{3}{32}s^{-6} - \frac{3}{16}s^{-4} + \frac{37}{120}s^{-2} - \frac{3}{14} + \frac{1619}{16800}s^2 + \ldots)q^2$$
$$+ (\frac{5}{32}s^{-9} - \frac{55}{192}s^{-7} + \frac{59}{192}s^{-5} - \frac{3365}{12096}s^{-3} + \ldots)q^3 + O(q^4)$$

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