



Lepton masses and mixing from modular S_4 symmetry

J.T. Penedo^{a,*}, S.T. Petcov^{a,b,1}

^a *SISSA/INFN, Via Bonomea 265, 34136 Trieste, Italy*

^b *Kavli IPMU (WPI), University of Tokyo, 5-1-5 Kashiwanoha, 277-8583 Kashiwa, Japan*

Received 24 July 2018; received in revised form 19 November 2018; accepted 15 December 2018

Available online 18 December 2018

Editor: Tommy Ohlsson

Abstract

We study models of lepton masses and mixing based on broken modular invariance. We consider invariance under the finite modular group $\Gamma_4 \simeq S_4$ and focus on the minimal scenario where the expectation value of the modulus is the only source of symmetry breaking, such that no flavons need to be introduced. After constructing a basis for the lowest weight modular forms, we build two minimal models, one of which successfully accommodates charged lepton masses and neutrino oscillation data, while predicting the values of the Dirac and Majorana CPV phases.

© 2018 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

It is a challenge to understand the origin of the flavour structure of quarks and leptons. Data obtained in neutrino oscillation experiments (see, e.g., [1]) has revealed a pattern of two large and one small mixing angles in the lepton sector, which is being probed with increasing precision. Additionally, a preference for both a neutrino mass spectrum with normal ordering (NO) and for a Dirac CP violation (CPV) phase close to maximal have been found in recent global fits (see, e.g., [2]).

* Corresponding author.

E-mail address: jpenedo@sissa.it (J.T. Penedo).

¹ Also at: Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria.

A popular approach to explaining the observed mixing patterns consists in assuming the presence of a spontaneously broken discrete flavour symmetry. Typically, however, such breaking requires the introduction of a plethora of scalar fields (flavons) with a set of particularly aligned vacuum expectation values (VEVs). Arranging for such an alignment usually calls for the construction of rather “baroque” scalar potentials.

An interesting framework for the construction of flavour models has recently been put forward in Ref. [3] and further explored in Ref. [4]. In this setup, invariance under finite subgroups of the modular group shapes the relevant mass matrices. These models generalise existing approaches to the flavour puzzle, which can be seen as the particular cases of setting modular weights to zero. In the most economical versions of these models, the VEV of the modulus τ is the only source of symmetry breaking, bypassing the need for flavon fields. Leading-order predictions may be derived by treating τ as a spurion.

In the present paper we investigate the consequences of the presence of modular invariance in the lepton sector. We focus on the action of the finite modular group Γ_4 , which is isomorphic to the group of permutations of four objects S_4 . After reviewing the necessary formalism, we explicitly construct the generators of modular forms of level $N = 4$. We then investigate two minimal models where neutrino masses arise from the dimension 5 Weinberg operator and where no flavons are introduced.

2. The setup

Following Ref. [3], we consider the infinite groups $\Gamma(N)$,

$$\Gamma(N) \equiv \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \wedge \det \gamma = 1 \wedge \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \quad (1)$$

where N is a positive integer. The group $\Gamma(1) \simeq \text{SL}(2, \mathbb{Z})$ is the modular group and $\Gamma(N > 1)$ are normal subgroups of $\Gamma(1)$. Taking the quotient of $\Gamma(1)$ and $\Gamma(2)$ by $\{\mathbb{1}, -\mathbb{1}\}$ we obtain the groups of linear fractional transformations, $\overline{\Gamma}(N) \equiv \Gamma(N)/\{\mathbb{1}, -\mathbb{1}\}$ for $N = 1, 2$, and $\overline{\Gamma}(N > 2) \equiv \Gamma(N)$. Elements of $\overline{\Gamma}(N)$ act on a complex variable τ as:

$$\tau \rightarrow \gamma \tau = \frac{a\tau + b}{c\tau + d}, \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}(N), \quad (2)$$

and it can be shown that the upper half-plane $\{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ is mapped to itself under this action. The complex variable τ is henceforth restricted to have positive imaginary part.

We are interested in studying physical actions which are invariant under transformations of the finite modular groups Γ_N . These discrete groups are obtained from the quotient of two of the aforementioned infinite groups, namely $\Gamma_N \equiv \overline{\Gamma}(1)/\overline{\Gamma}(N)$. The group $\overline{\Gamma}(1)$ is generated by two elements S and T acting on τ as

$$S: \tau \rightarrow -1/\tau, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3)$$

$$T: \tau \rightarrow \tau + 1, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (4)$$

and satisfying

$$S^2 = (ST)^3 = \mathbb{1}. \quad (5)$$

The presentation of the groups Γ_N can be obtained from that of $\bar{\Gamma}(1)$ by extending it with the condition $T^N = \mathbb{1}$, see Eq. (1). Thus, the generators S and T of Γ_N obey

$$S^2 = (ST)^3 = T^N = \mathbb{1}. \tag{6}$$

We consider modular-invariant $\mathcal{N} = 1$ global supersymmetric (SUSY) actions [5,6],

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} K(\chi_i, \bar{\chi}_i; \tau, \bar{\tau}) + \int d^4x d^2\theta W(\chi_i; \tau) + \text{h.c.}, \tag{7}$$

where χ_i denotes the set of matter chiral superfields of the theory.² The physical action \mathcal{S} is required to be invariant under the action of Γ_N . Under an element of this group, one has the following transformations of τ and of the fields χ_i :

$$\begin{cases} \tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d} \\ \chi_i \rightarrow (c\tau + d)^{-k_i} \rho_i(\gamma) \chi_i \end{cases}, \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N, \tag{8}$$

where ρ_i are unitary representation matrices and the k_i are integers. The fields χ_i are said to carry weight $-k_i$. We require that the superpotential W remains invariant under Γ_N and that the Kähler potential is changed at most by a Kähler transformation. To satisfy this last condition, we work with the Kähler:

$$K(\chi_i, \bar{\chi}_i; \tau, \bar{\tau}) = -h \Lambda_0^2 \log(-i(\tau - \bar{\tau})) + \sum_i \frac{|\chi_i|^2}{(-i(\tau - \bar{\tau}))^{k_i}}, \tag{9}$$

with $h > 0$ and Λ_0 a mass parameter. After τ develops a VEV, it gives rise to kinetic terms for the matter fields,

$$\mathcal{L} \supset \sum_i \frac{\partial_\mu \bar{\chi}_i \partial^\mu \chi_i}{(2 \text{Im}(\tau))^{k_i}}. \tag{10}$$

These terms can be made canonical by rescaling the fields χ_i , which in practice amounts to a redefinition of superpotential parameters. The superpotential reads

$$W(\chi_i; \tau) = \sum_n \sum_{\{i_1, \dots, i_n\}} (Y_{\{i_1, \dots, i_n\}}(\tau) \chi_{i_1} \dots \chi_{i_n})_{\mathbf{1}}, \tag{11}$$

and should remain unchanged under Γ_N .

Given the field transformations in Eq. (8), the functions $Y_{\{i_1, \dots, i_n\}}(\tau)$ in (11) must transform under Γ_N as (we omit the indices i_1, \dots, i_n):

$$\begin{cases} \tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d} \\ Y(\tau) \rightarrow Y(\gamma\tau) = (c\tau + d)^{2k_Y} \rho_Y(\gamma) Y(\tau) \end{cases}, \quad \text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N, \tag{12}$$

where the unitary matrix $\rho_Y(\gamma)$ and the non-negative integer k_Y , as will shortly be shown, must satisfy specific conditions.

Holomorphic functions $f(\tau)$ satisfying $f(\gamma\tau) = (c\tau + d)^{2k} f(\tau)$ with $\gamma \in \Gamma(N)$ and $k \in \mathbb{N}_0$ are said to be modular forms of weight $2k$ and level N . For $k = 0$, the modular forms are constant

² As in Ref. [3], we turn off gauge interactions and treat τ as a dimensionless spurion.

functions, while for $k < 0$ modular forms do not exist. Modular forms are important objects in the present construction since, under Γ_N , modular forms of weight $2k_Y$ and level N transform in the way we require $Y(\tau)$ in Eq. (12) to transform. The requirement of invariance of the superpotential then implies that the functions $Y(\tau)$ are modular forms of level N . Their weights must cancel those of the fields in Eq. (11), $2k_Y - k_{i_1} - \dots - k_{i_n} = 0$. Additionally, the tensor product $\rho_Y \otimes \rho_{i_1} \otimes \dots \otimes \rho_{i_n}$ should contain at least one singlet. The subscript **1** in Eq. (11) indicates a sum (with independent coefficients) of all possible singlets one can extract from this product.

The effects of invariance under Γ_N for $N = 2$ and $N = 3$ have been studied in Refs. [3,4]. The group Γ_2 is isomorphic to S_3 while $\Gamma_3 \simeq A_4$. In what follows we consider invariance under the group $\Gamma_4 \simeq S_4$, whose generators satisfy the following presentation rules:

$$S^2 = (ST)^3 = T^4 = \mathbb{1}. \tag{13}$$

For a generic value of $\langle \tau \rangle$, Γ_4 will be fully broken. It is important to remark that the leading-order results obtained with this approach are susceptible to corrections from a small number of sources, namely from SUSY breaking and corrections to the Kähler potential. In generalisations of our approach where flavons are introduced, additional corrections are expected from vacuum (mis)alignment.

2.1. Generators of modular forms of level $N = 4$

The functions $Y(\tau)$ are modular forms of level $N = 4$ and weight $2k_Y$. The dimension of the space of modular forms of level 4 and weight $2k$ is $4k + 1$. Thus, the space of (level 4) forms which carry the lowest nontrivial weight, $2k = 2$, has dimension $4k + 1 = 5$. It proves useful to explicitly find a basis $\{Y_1(\tau), \dots, Y_5(\tau)\}$ of this lowest weight space, since modular forms of higher weights can be constructed from homogeneous polynomials in these five modular forms Y_i ($i = 1, \dots, 5$).

A starting point in this search is the recognition of certain properties of the Dedekind eta function $\eta(z \in \mathbb{C})$, defined as:

$$\eta(z) \equiv q^{1/24} \prod_{k=1}^{\infty} (1 - q^k), \quad \text{with } q = e^{2\pi i z}. \tag{14}$$

The η function satisfies $\eta(z + 1) = e^{i\pi/12} \eta(z)$ and $\eta(-1/z) = \sqrt{-iz} \eta(z)$, as well as the identity (see, e.g., [7]):

$$\eta\left(z + \frac{1}{2}\right) = e^{i\pi/24} \frac{\eta^3(2z)}{\eta(z)\eta(4z)}. \tag{15}$$

One sees that the set

$$\{\eta_i\} = \left\{ \eta\left(\tau + \frac{1}{2}\right), \eta(4\tau), \eta\left(\frac{\tau}{4}\right), \eta\left(\frac{\tau + 1}{4}\right), \eta\left(\frac{\tau + 2}{4}\right), \eta\left(\frac{\tau + 3}{4}\right) \right\} \tag{16}$$

respects a certain notion of closure under the action of Γ_4 generators S and T , since under their action one has:

$$S : \left\{ \begin{array}{l} \eta\left(\tau + \frac{1}{2}\right) \rightarrow \frac{1}{\sqrt{2}} \sqrt{-i\tau} \eta\left(\frac{\tau+2}{4}\right) \\ \eta(4\tau) \rightarrow \frac{1}{2} \sqrt{-i\tau} \eta\left(\frac{\tau}{4}\right) \\ \eta\left(\frac{\tau}{4}\right) \rightarrow 2 \sqrt{-i\tau} \eta(4\tau) \\ \eta\left(\frac{\tau+1}{4}\right) \rightarrow e^{-i\pi/6} \sqrt{-i\tau} \eta\left(\frac{\tau+3}{4}\right) \\ \eta\left(\frac{\tau+2}{4}\right) \rightarrow \sqrt{2} \sqrt{-i\tau} \eta\left(\tau + \frac{1}{2}\right) \\ \eta\left(\frac{\tau+3}{4}\right) \rightarrow e^{i\pi/6} \sqrt{-i\tau} \eta\left(\frac{\tau+1}{4}\right) \end{array} \right. , \tag{17}$$

and

$$T : \left\{ \begin{array}{l} \eta\left(\tau + \frac{1}{2}\right) \rightarrow e^{i\pi/12} \eta\left(\tau + \frac{1}{2}\right) \\ \eta(4\tau) \rightarrow e^{i\pi/3} \eta(4\tau) \\ \eta\left(\frac{\tau}{4}\right) \rightarrow \eta\left(\frac{\tau+1}{4}\right) \\ \eta\left(\frac{\tau+1}{4}\right) \rightarrow \eta\left(\frac{\tau+2}{4}\right) \\ \eta\left(\frac{\tau+2}{4}\right) \rightarrow \eta\left(\frac{\tau+3}{4}\right) \\ \eta\left(\frac{\tau+3}{4}\right) \rightarrow e^{i\pi/12} \eta\left(\frac{\tau}{4}\right) \end{array} \right. . \tag{18}$$

The transformations under S of the elements $\eta(\tau + 1/2)$ and $\eta((\tau + 2)/4)$ can be derived by making use of the relation (15). Up to multiplicative factors, this set is closed under S and T . Furthermore, each element is taken into itself (up to a factor) by the (left-)actions of S^2 , $(ST)^3$ and T^4 . The above suggests that the desired $k = 1$ modular forms can be written as linear combinations of the logarithmic derivatives of the elements of the set $\{\eta_i\}$. We define:

$$\begin{aligned} Y(a_1, \dots, a_6|\tau) &\equiv \frac{d}{d\tau} \left(\sum_{i=1}^6 a_i \log \eta_i(\tau) \right) \\ &= a_1 \frac{\eta'(\tau + 1/2)}{\eta(\tau + 1/2)} + 4a_2 \frac{\eta'(4\tau)}{\eta(4\tau)} + \frac{1}{4} \left[a_3 \frac{\eta'(\tau/4)}{\eta(\tau/4)} \right. \\ &\quad \left. + a_4 \frac{\eta'((\tau + 1)/4)}{\eta((\tau + 1)/4)} + a_5 \frac{\eta'((\tau + 2)/4)}{\eta((\tau + 2)/4)} + a_6 \frac{\eta'((\tau + 3)/4)}{\eta((\tau + 3)/4)} \right]. \end{aligned} \tag{19}$$

The use of logarithmic derivatives allows one to eliminate the multiplicative factors in the transformations (17) and (18) by requiring $\sum_i a_i = 0$. We are thus left with five independent linear combinations of the η'_i/η_i , as expected. Under the action of the generators S and T this function transforms as:

$$S : Y(a_1, \dots, a_6|\tau) \rightarrow Y(a_1, a_2, a_3, a_4, a_5, a_6|-1/\tau) = \tau^2 Y(a_5, a_3, a_2, a_6, a_1, a_4|\tau), \tag{20}$$

$$T : Y(a_1, \dots, a_6|\tau) \rightarrow Y(a_1, a_2, a_3, a_4, a_5, a_6|\tau + 1) = Y(a_1, a_2, a_6, a_3, a_4, a_5|\tau). \tag{21}$$

We find a basis for lowest weight modular forms,

$$\begin{aligned} Y_1(\tau) &\equiv Y(1, 1, \omega, \omega^2, \omega, \omega^2|\tau), \\ Y_2(\tau) &\equiv Y(1, 1, \omega^2, \omega, \omega^2, \omega|\tau), \\ Y_3(\tau) &\equiv Y(1, -1, -1, -1, 1, 1|\tau), \\ Y_4(\tau) &\equiv Y(1, -1, -\omega^2, -\omega, \omega^2, \omega|\tau), \\ Y_5(\tau) &\equiv Y(1, -1, -\omega, -\omega^2, \omega, \omega^2|\tau), \end{aligned} \tag{22}$$

where $\omega \equiv e^{2\pi i/3}$. These five linearly independent forms $Y_i(\tau)$ arrange themselves into two irreducible representations of $\Gamma_4 \simeq S_4$, a doublet $\mathbf{2}$ and a triplet $\mathbf{3}'$,

$$Y_{\mathbf{2}}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \quad Y_{\mathbf{3}'}(\tau) \equiv \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}. \tag{23}$$

The multiplets $Y_{\mathbf{2}}$ and $Y_{\mathbf{3}'}$ transform under Γ_4 as indicated in Eq. (12). In Appendix A.1 we specify our basis choice for the representation matrices $\rho(\gamma)$ of S_4 and we list the Clebsch–Gordan coefficients for this basis in Appendix A.2. In Appendix B we give the q -expansions of (combinations of) the five functions defined in Eq. (22).

Multiplets transforming in the other representations of S_4 can be obtained from tensor products of $Y_{\mathbf{2}}$ and $Y_{\mathbf{3}'}$. The representations $\mathbf{1}$ and $\mathbf{3}$ arise at weight $2k = 4$, while the $\mathbf{1}'$ representation first arises at weight 6. Since we can form 15 combinations $Y_i Y_j$, one may expect 15 independent (level 4) forms at weight $2k = 4$. However, the dimension of the space of these forms is $4k + 1 = 9$. In fact, we find 6 constraints between the several $Y_i Y_j$, which we list in Appendix C. These constraints reduce the 15 potentially independent combinations to 9 truly independent ones, which are organised in the following representations of S_4 :

$$\begin{aligned} Y_{\mathbf{1}}^{(4)} &= Y_1 Y_2 \sim \mathbf{1}, \\ Y_{\mathbf{2}}^{(4)} &= (Y_2^2, Y_1^2)^T \sim \mathbf{2}, \\ Y_{\mathbf{3}}^{(4)} &= (Y_1 Y_4 - Y_2 Y_5, Y_1 Y_5 - Y_2 Y_3, Y_1 Y_3 - Y_2 Y_4)^T \sim \mathbf{3}, \\ Y_{\mathbf{3}'}^{(4)} &= (Y_1 Y_4 + Y_2 Y_5, Y_1 Y_5 + Y_2 Y_3, Y_1 Y_3 + Y_2 Y_4)^T \sim \mathbf{3}'. \end{aligned} \tag{24}$$

3. Phenomenology

To understand how invariance under the subgroup Γ_4 of the modular group may play a role in determining lepton masses and mixing, one needs to specify the S_4 representations ρ_i and the modular weights $-k_i$ of the relevant fields χ_i , which transform as indicated in Eq. (8). In what follows, we search for choices of representations and weights which are in line with a certain notion of minimality.

Table 1

Transformation properties of the relevant MSSM chiral superfields under the gauge group and under the Γ_4 discrete modular symmetry. The ρ_i denote the representations of the fields transforming under S_4 and the k_i correspond to (minus) their modular weights.

	H_u	H_d	L	$E_{1,2,3}^c$
$SU(2)_L \times U(1)_Y$	$(\mathbf{2}, 1/2)$	$(\mathbf{2}, -1/2)$	$(\mathbf{2}, -1/2)$	$(\mathbf{1}, 1)$
$\Gamma_4 \left\{ \begin{array}{l} \rho_i \\ k_i \end{array} \right.$	ρ_d k_u	ρ_u k_d	ρ_L k_L	$\rho_{1,2,3}$ $k_{1,2,3}$

In a minimal approach, the superpotential W includes only the Yukawa interactions of the Minimal Supersymmetric Standard Model (MSSM) and the SUSY Weinberg operator as the origin of (Majorana) neutrino masses:

$$W = \alpha (E^c L H_d f_E(Y_2, Y_3))_1 + \frac{g}{\Lambda} (L H_u L H_u f_W(Y_2, Y_3))_1, \quad (25)$$

where, as mentioned before, different coefficients are implied for different singlet combinations. No flavons are present in the above superpotential. Recall also that the functions $Y_2 = Y_2(\tau)$ and $Y_3 = Y_3(\tau)$ have definite transformation properties under Γ_4 . After the breaking of modular symmetry we obtain:

$$W \rightarrow \lambda_{ij} E_i^c (L_j H_d) + c_{ij} (L_i H_u) (L_j H_u), \quad (26)$$

which leads to the Lagrangian terms

$$\mathcal{L} \supset -\frac{1}{2} (M_\nu)_{ij} \overline{v_{iR}^c} v_{jL} - (M_e)_{ij} \overline{e_{iL}} e_{jR} + \text{h.c.}, \quad (27)$$

written in terms of four-spinors, where³ $M_e \equiv v_d \lambda^\dagger$ and $M_\nu \equiv 2c v_u^2$, with $\langle H_u \rangle = (0, v_u)^T$ and $\langle H_d \rangle = (v_d, 0)^T$.

The generic assignments of representations and weights to the MSSM fields present in Eq. (25) are defined in Table 1. We will keep the Higgs sector assignments trivial for simplicity. We will also take lepton doublets (singlets) to transform as three (one) dimensional representations of S_4 , as is customary. Minimal models are then built by adhering to the following guidelines:

- No flavons are introduced,
- Neutrino masses arise from the dimension 5 Weinberg operator,
- Higgs multiplets transform trivially $\rho_{u,d} \sim \mathbf{1}$ under Γ_4 , with $k_{u,d} = 0$,
- Lepton $SU(2)_L$ doublets transform as a triplet $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$ under Γ_4 ,
- Lepton $SU(2)_L$ singlets transform as singlets $\rho_{1,2,3} \sim \mathbf{1}$ or $\mathbf{1}'$ under Γ_4 , and
- Lowest possible weights are chosen such that a rank 3 charged-lepton mass matrix M_e is possible without imposing additional “shaping” symmetries.

³ In the decoupling limit of the MSSM (e.g. when the heavier Higgs scalar states have masses exceeding ~ 1 TeV), the lightest Higgs boson couplings to charged leptons (and, for that matter, also to quarks) differ insignificantly from those of the Standard Model Higgs.

Table 2

Transformation properties of chiral superfields under Γ_4 , for the minimal model with $k_L = 1$ (model I). Both lines of lepton assignments lead to the same results.

	H_u	H_d	L	E_1^c	E_2^c	E_3^c
ρ_i	1	1	3	1'	1	1'
k_i	0	0	1	1	3	3

Given the above conditions, we further expand the superpotential as:

$$W = \sum_i \alpha_i (E_i^c L H_d Y_2^{a_i} Y_3^{b_i})_1 + \frac{g}{\Lambda} (L H_u L H_u Y_2^c Y_3^d)_1, \tag{28}$$

where the integer, non-negative exponents of the modular forms satisfy $2(a_i + b_i) = k_L + k_i + k_d = k_L + k_i$ and $2(c + d) = 2k_L + 2k_u = 2k_L$. In order to obtain some nontrivial structure in the Weinberg operator, we assume that $k_L > 0$. We explore in what follows two minimal choices of weights and representations, corresponding to $k_L = 1$ (model I) and $k_L = 2$ (model II).

3.1. Model I ($k_L = 1$)

The choices $k_L = 1$ and $k_u = 0$ along with the fact that $\rho_L \sim \mathbf{3}$ or $\rho_L \sim \mathbf{3}'$ are enough to determine the structure of the Weinberg operator and hence of the neutrino mass matrix M_ν .

On the charged lepton side, the cancellation-of-weights condition $2(a_i + b_i) = 1 + k_i$ implies that the k_i are odd, $k_i \in \{-1, 1, 3, \dots\}$. To avoid having $\text{rank}(\lambda) < 3$, one must make sure that different singlets can be extracted from each term in W . If two lepton singlet superfields share the same weight and representation under Γ_4 , the rank of the charged-lepton mass matrix is lower than its possible maximum. If some $k_i = -1$, then $a_i = b_i = 0$ and no singlet can be formed for that generation. If instead $k_i = 1$, a singlet can only be formed if $\rho_i \sim \mathbf{1}'$ (**1**) when $\rho_L \sim \mathbf{3}$ (**3'**). We summarise in Table 2 the weight and representation assignments of the minimal model allowing for $\text{rank}(\lambda) = 3$.

Based solely on the weight assignments of Table 2, the superpotential would read:

$$\begin{aligned} W^I = & \alpha_1 (E_1^c L Y_2)_1 H_d + \alpha_2 (E_1^c L Y_3)_1 H_d \\ & + \beta_1 (E_2^c L Y_2^2)_1 H_d + \beta_2 (E_2^c L Y_2 Y_3)_1 H_d + \beta_3 (E_2^c L Y_3^2)_1 H_d \\ & + \gamma_1 (E_3^c L Y_2^2)_1 H_d + \gamma_2 (E_3^c L Y_2 Y_3)_1 H_d + \gamma_3 (E_3^c L Y_3^2)_1 H_d \\ & + \frac{g_1}{\Lambda} (L^2 Y_2)_1 H_u^2 + \frac{g_2}{\Lambda} (L^2 Y_3)_1 H_u^2. \end{aligned} \tag{29}$$

Making use of the Clebsch–Gordan coefficients given in Appendix A.2 and of the $\Gamma_4 \simeq S_4$ representation choices in Table 2 we find that only some terms in Eq. (29) contain non-zero singlets. We are left with

$$W^I = \alpha (E_1^c L Y_3)_1 H_d + \beta (E_2^c L Y_2 Y_3)_1 H_d + \gamma (E_3^c L Y_2 Y_3)_1 H_d + \frac{g_1}{\Lambda} (L^2 Y_2)_1 H_u^2, \tag{30}$$

where $\alpha \equiv \alpha_2$, $\beta \equiv \beta_2 + \beta_3$ and $\gamma \equiv \gamma_2$. It is interesting to note that the constraints listed in Appendix C imply that the singlets of the β_2 and β_3 terms coincide. This superpotential gives rise to the mass matrices:

$$M_\nu^I = \frac{2g_1 v_u^2}{\Lambda} \begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix}, \quad (31)$$

and

$$M_e^I = v_d \begin{pmatrix} \alpha Y_3 & \alpha Y_5 & \alpha Y_4 \\ \beta (Y_1 Y_4 - Y_2 Y_5) & \beta (Y_1 Y_3 - Y_2 Y_4) & \beta (Y_1 Y_5 - Y_2 Y_3) \\ \gamma (Y_1 Y_4 + Y_2 Y_5) & \gamma (Y_1 Y_3 + Y_2 Y_4) & \gamma (Y_1 Y_5 + Y_2 Y_3) \end{pmatrix}^\dagger. \quad (32)$$

Specifying values for the parameters α , β , γ , and τ determines both mass matrices up to global factors,⁴ and hence determines mass ratios as well as lepton mixing.

After performing a numerical search, it seems this minimal model does not allow to reproduce known data. As a benchmark, we find the point $\alpha = 0.1$, $\beta = 1$, $\gamma = 2 \times 10^{-4}$, and $\tau = 0.1 + 0.985i$, which accommodates a neutrino mass spectrum with inverted ordering, but does not provide an acceptable value for $\sin^2 \theta_{12}$. For this point, we have

$$\begin{aligned} \frac{m_e}{m_\mu} &\simeq 0.0045, & \sin^2 \theta_{12} &\simeq 0.497, & \delta &\simeq 1.47\pi, \\ \frac{m_\mu}{m_\tau} &\simeq 0.0522, & \sin^2 \theta_{13} &\simeq 0.021, & \alpha_{21} &\simeq 1.00\pi, \\ r &\simeq 0.0308, & \sin^2 \theta_{23} &\simeq 0.496, & \alpha_{31} &\simeq 0.53\pi. \end{aligned} \quad (33)$$

The obtained predictions are to be compared with neutrino oscillation data and information on charged-lepton mass ratios, which we collect in Table 3.

3.2. Model II ($k_L = 2$)

As before, the choices $k_L = 2$ and $k_u = 0$ along with the fact that $\rho_L \sim \mathbf{3}$ or $\rho_L \sim \mathbf{3}'$ are enough to determine the structure of M_ν .

On the charged lepton side, the cancellation-of-weights condition $2(a_i + b_i) = 2 + k_i$ implies that the k_i are even, $k_i \in \{-2, 0, 2, \dots\}$. If some $k_i = -2$, then again $a_i = b_i = 0$ and no singlet can be formed for that generation. If instead $k_i = 0$, a singlet can only be formed if $\rho_i \sim \mathbf{1}'$ ($\mathbf{1}$) when $\rho_L \sim \mathbf{3}$ ($\mathbf{3}'$). We summarise in Table 4 the weight and representation assignments of the minimal model allowing for $\text{rank}(\lambda) = 3$.

With the weight assignments of Table 4, the charged-lepton Yukawa part of the superpotential matches that of the case $k_L = 1$, leading to the same charged-lepton mass matrix as the one given in Eq. (32), i.e. $M_e^{\text{II}} = M_e^{\text{I}}$.

The Weinberg operator part of the superpotential reads instead:

$$W^{\text{II}} \supset \frac{g_1}{\Lambda} (L^2 Y_2^2)_1 H_u^2 + \frac{g_2}{\Lambda} (L^2 Y_2 Y_3)_1 H_u^2 + \frac{g_3}{\Lambda} (L^2 Y_3^2)_1 H_u^2. \quad (34)$$

⁴ The parameters α , β and γ can be made real through the rephasing of the singlet fields E_i^c . One of them may be taken outside of the matrix M_e^{I} as a global factor. It is assumed that the correct charged-lepton mass scale is reproduced by an appropriate choice of this global factor, after v_d has been specified.

Table 3

Best-fit values and 1σ ranges for neutrino oscillation parameters, obtained from the global analysis of Ref. [2], and for charged-lepton mass ratios, given at the scale 2×10^{16} GeV with the $\tan\beta$ averaging described in [3], obtained from Ref. [8]. The parameters entering the definition of r are $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$. The best-fit value and 1σ range of δ did not drive the numerical searches here reported.

Parameter	Best-fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\delta m^2/(10^{-5} \text{ eV})$		$7.34^{+0.17}_{-0.14}$
$ \Delta m^2 /(10^{-3} \text{ eV})$	$2.455^{+0.035}_{-0.032}$	$2.441^{+0.033}_{-0.035}$
$r \equiv \delta m^2/ \Delta m^2 $	0.0299 ± 0.0008	0.0301 ± 0.0008
$\sin^2 \theta_{12}$	$0.304^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2 \theta_{13}$	$0.0214^{+0.0009}_{-0.0007}$	$0.0218^{+0.0008}_{-0.0007}$
$\sin^2 \theta_{23}$	$0.551^{+0.019}_{-0.070}$	$0.557^{+0.017}_{-0.024}$
δ/π	$1.32^{+0.23}_{-0.18}$	$1.52^{+0.14}_{-0.15}$

Table 4

Transformation properties of chiral superfields under Γ_4 , for the minimal model with $k_L = 2$ (model II). Both lines of lepton assignments lead to the same results.

	H_u	H_d	L	E_1^c	E_2^c	E_3^c
ρ_i	1	1	3 3'	1' 1	1 1'	1' 1
k_i	0	0	2	0	2	2

The first term in Eq. (34) contributes with two different non-zero singlets, since $L^2 \sim \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$ with vanishing antisymmetric $\mathbf{3}'$, $Y_2^2 \sim \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}$ with vanishing antisymmetric $\mathbf{1}'$, and singlets may be obtained from both the $\mathbf{1} \otimes \mathbf{1}$ and the $\mathbf{2} \otimes \mathbf{2}$ combinations. The second term contributes with a third singlet, as $Y_2 Y_{3'} \sim \mathbf{3} \oplus \mathbf{3}'$ and thus a singlet is obtained from the $\mathbf{3} \otimes \mathbf{3}$ combination. The third term contains all the three singlets: $\mathbf{1} \otimes \mathbf{1}$, $\mathbf{2} \otimes \mathbf{2}$, and $\mathbf{3} \otimes \mathbf{3}$. Due to the constraints in Appendix C, there are only three independent singlets which enter the Weinberg operator part of W^{II} . Explicitly, using the Clebsch–Gordan coefficients of Appendix A.2:

$$\begin{aligned}
 W^{\text{II}} \supset \frac{1}{\Lambda} \Bigg[& g Y_1 Y_2 (L_1^2 + 2L_2 L_3) \\
 & + g' \left(Y_1^2 (L_2^2 + 2L_1 L_3) + Y_2^2 (L_3^2 + 2L_1 L_2) \right) \\
 & + g'' \left((Y_1 Y_4 - Y_2 Y_5) (L_1^2 - L_2 L_3) + (Y_1 Y_5 - Y_2 Y_3) (L_2^2 - L_1 L_3) \right. \\
 & \left. + (Y_1 Y_3 - Y_2 Y_4) (L_3^2 - L_1 L_2) \right) \Bigg] H_u^2, \tag{35}
 \end{aligned}$$

which gives rise to the mass matrix:

$$M_{\nu}^{\text{II}} = \frac{2g'v_u^2}{\Lambda} \left[\begin{pmatrix} (g/g')Y_1Y_2 & Y_2^2 & Y_1^2 \\ Y_2^2 & Y_1^2 & (g/g')Y_1Y_2 \\ Y_1^2 & (g/g')Y_1Y_2 & Y_2^2 \end{pmatrix} + \frac{1}{2} \frac{g''}{g'} \begin{pmatrix} 2(Y_1Y_4 - Y_2Y_5) & -(Y_1Y_3 - Y_2Y_4) & -(Y_1Y_5 - Y_2Y_3) \\ -(Y_1Y_3 - Y_2Y_4) & 2(Y_1Y_5 - Y_2Y_3) & -(Y_1Y_4 - Y_2Y_5) \\ -(Y_1Y_5 - Y_2Y_3) & -(Y_1Y_4 - Y_2Y_5) & 2(Y_1Y_3 - Y_2Y_4) \end{pmatrix} \right]. \quad (36)$$

Specifying values for the parameters α , β , γ , g/g' , g''/g' and τ determines both mass matrices up to global factors,⁵ and hence determines mass ratios as well as lepton mixing.

Through numerical search, we find that this minimal model is successful in accommodating the data. We find a first benchmark, $\alpha = 0.1$, $\beta = 1$, $\gamma = 2.3 \times 10^{-4}$, $g/g' = -0.99 - 0.52i$, $g''/g' = 0.15 - 0.06i$, and $\tau = 0.04 + 1.11i$, which admits a neutrino mass spectrum with normal ordering, with

$$\begin{aligned} \frac{m_e}{m_{\mu}} &\simeq 0.0048, & \sin^2 \theta_{12} &\simeq 0.288, & \delta &\simeq 0.88\pi, \\ \frac{m_{\mu}}{m_{\tau}} &\simeq 0.0593, & \sin^2 \theta_{13} &\simeq 0.021, & \alpha_{21} &\simeq 1.46\pi, \\ r &\simeq 0.0299, & \sin^2 \theta_{23} &\simeq 0.553, & \alpha_{31} &\simeq 1.09\pi. \end{aligned} \quad (37)$$

These results are in good agreement with the values of Table 3, except in what regards the experimental hint of $\delta \sim 3\pi/2$.

We find a second benchmark, $\alpha = 0.11$, $\beta = 1$, $\gamma = 2.3 \times 10^{-4}$, $g/g' = -6.2 - 1.5i$, $g''/g' = -0.03 + 0.03i$, and $\tau = -0.09 + 0.96i$, also admitting a neutrino mass spectrum with normal ordering, for which the values of δ and of other measured parameters are less than 2σ away from the best-fit values of Table 3:

$$\begin{aligned} \frac{m_e}{m_{\mu}} &\simeq 0.0048, & \sin^2 \theta_{12} &\simeq 0.292, & \delta &\simeq 1.64\pi, \\ \frac{m_{\mu}}{m_{\tau}} &\simeq 0.0560, & \sin^2 \theta_{13} &\simeq 0.021, & \alpha_{21} &\simeq 0.10\pi, \\ r &\simeq 0.0298, & \sin^2 \theta_{23} &\simeq 0.493, & \alpha_{31} &\simeq 1.10\pi. \end{aligned} \quad (38)$$

For this second benchmark, in order to fit the individual mass-squared differences δm^2 and Δm^2 , we set the global factor $2g'v_u^2/\Lambda \simeq 0.0037$ eV. In this case, the neutrino masses read $m_1 \simeq 0.042$ eV, $m_2 \simeq 0.043$ eV, and $m_3 \simeq 0.066$ eV. A distinctive feature of this framework is the prediction of the Dirac and Majorana CPV phases. One is then in a position to extract a prediction for the effective Majorana mass $|\langle m \rangle|$ which controls the rate of neutrinoless double beta decay. Using the values in (38) we find $|\langle m \rangle| \simeq 0.042$ eV. This value can be probed by new-generation experiments which are working towards the $|\langle m \rangle| \sim 10^{-2}$ eV frontier. We note that, for this model, a good fit of the neutrino mass and mixing data is possible also in the case of IO spectrum, which is however strongly disfavoured by the data with respect to the NO spectrum [2].

In this setup, the correlations between pairs of mixing angles, phases and mass ratios are non-trivial, as is suggested, in particular, by comparing the predictions for $\sin^2 \theta_{23}$ and δ in Eqs. (37)

⁵ As before, α , β and γ can be made real and one of them may be taken outside of $M_e^{\text{II}} = M_e^{\text{I}}$ as a global factor.

and (38). The existence and the success of the above benchmark warrants further exploration of the present framework.

4. Summary and conclusions

In the present article we have explored the consequences of the presence of modular invariance in the lepton sector via the action of the finite modular group $\Gamma_4 \simeq S_4$ which is a subgroup of $SL(2, \mathbb{Z})$.

Fields carrying a non-trivial modular weight transform with a scale factor in addition to the usual unitary rotation. To build an invariant theory under such transformations, special functions with the appropriate scaling properties need to be present in order to provide compensating factors. These functions are modular forms. For a fixed scaling (modular) weight, they make up a finite-dimensional space. In section 2.1, we have detailed the construction of a basis $\{Y_1, \dots, Y_5\}$ for the lowest weight modular forms of level 4 (corresponding to $\Gamma_4 \simeq S_4$), necessary to the generation of higher weight forms. We have additionally shown how the weight 2 and 4 forms organise themselves into different multiplets of S_4 , and that they satisfy non-trivial constraints which guarantee the correct dimensionality of higher-weight spaces (see Appendix C).

We have then studied supersymmetric models of lepton masses and mixing based on the breaking of the $\Gamma_4 \simeq S_4$ modular invariance. Focusing on minimality as a guiding principle, we considered models where neutrino Majorana masses have their origin in the Weinberg operator and where no flavons are introduced. The expectation value of the modulus τ is the only source of symmetry breaking.

We study two minimal models, differing in the weight $-k_L$ of the lepton doublet under modular transformations. While the first model (model I, $k_L = 1$) does not seem to be able to reproduce the data, the second model (model II, $k_L = 2$) successfully accommodates charged-lepton masses and neutrino oscillation data. Deviations of individual parameters from best-fit values are found to be below the 2σ level for the benchmark of Eq. (38).

Since in the present scheme the mass matrices are fully determined by the VEV $\langle \tau \rangle$ and by superpotential parameters – see Eqs. (31), (32) and (36) – the values of the Dirac and Majorana CPV phases can be predicted. For the benchmark of Eq. (38), which corresponds to a neutrino mass spectrum with normal ordering, a prediction for the effective Majorana mass $|\langle m \rangle| \simeq 0.042$ eV is possible. This value can be probed by new-generation experiments working towards the $|\langle m \rangle| \sim 10^{-2}$ eV frontier.

The predictions here derived are expected to hold at leading order. There is a limited number of sources for deviations, namely SUSY breaking and Kähler corrections. The predictive power of the present approach and the existence of a successful benchmark (i.e. a “proof of existence”) make this model building avenue worthy of future study.

Acknowledgements

We would like to thank M. Tanimoto for useful discussions. This work was supported in part by the INFN program on Theoretical Astroparticle Physics (TASP), by the European Union Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grants 674896 and 690575 (J.T.P. and S.T.P.), and by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan (S.T.P.).

Appendix A. S_4 group theory

A.1. Presentation and basis

S_4 is the symmetric group of permutations of four objects. It contains $4! = 24$ elements and admits five irreducible representations $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$ (see, e.g., [9]). While a presentation in terms of three generators is common, it proves convenient to consider here a presentation given in terms of two generators S and T (cf. Eq. (13)),

$$S^2 = (ST)^3 = T^4 = \mathbb{1}. \tag{A.1}$$

We will use the group theoretical results of Ref. [10]. The two S_4 generators therein, which we denote here with primes, satisfy $S'^4 = T'^3 = (S'T'^2)^2 = 1$. We define $S \equiv S'T'^2$, $T \equiv S'$, which imply the inverse relations $S' = T$ and $T' = ST$. Then, S and T furnish the presentation (A.1) of S_4 , useful to the discussion of modular invariance. Making use of this identification and of the results in Appendix A of Ref. [10], we find an explicit basis for the irreducible representations of S_4 , which we employ in our discussion:

$$\mathbf{1}: \quad \rho(S) = 1, \quad \rho(T) = 1, \tag{A.2}$$

$$\mathbf{1}': \quad \rho(S) = -1, \quad \rho(T) = -1, \tag{A.3}$$

$$\mathbf{2}: \quad \rho(S) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A.4}$$

$$\mathbf{3}: \quad \rho(S) = \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}, \tag{A.5}$$

$$\mathbf{3}': \quad \rho(S) = -\frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad \rho(T) = -\frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}, \tag{A.6}$$

where as usual $\omega = e^{2\pi i/3}$.

A.2. Clebsch–Gordan coefficients

After establishing a dictionary between presentations (see previous subsection), we can directly use the Clebsch–Gordan coefficients from Ref. [10], since no change of basis on the representation matrices has been performed. We reproduce the coefficients here for completeness. Entries of each multiplet entering the tensor product are denoted by α_i and β_i .

$$\begin{aligned} \mathbf{1} \otimes \mathbf{r} = \mathbf{r} &\sim \alpha \beta_i \\ \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1} &\sim \alpha \beta \\ \mathbf{1}' \otimes \mathbf{2} = \mathbf{2} &\sim \begin{pmatrix} \alpha \beta_1 \\ -\alpha \beta_2 \end{pmatrix} \\ \mathbf{1}' \otimes \mathbf{3} = \mathbf{3}' &\sim \begin{pmatrix} \alpha \beta_1 \\ \alpha \beta_2 \\ \alpha \beta_3 \end{pmatrix} \\ \mathbf{1}' \otimes \mathbf{3}' = \mathbf{3} &\sim \begin{pmatrix} \alpha \beta_1 \\ \alpha \beta_2 \\ \alpha \beta_3 \end{pmatrix} \end{aligned} \tag{A.7}$$

$$\begin{aligned}
 2 \otimes 2 = 1 \oplus 1' \oplus 2 & \left\{ \begin{array}{l} 1 \sim \alpha_1 \beta_2 + \alpha_2 \beta_1 \\ 1' \sim \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ 2 \sim \begin{pmatrix} \alpha_2 \beta_2 \\ \alpha_1 \beta_1 \end{pmatrix} \end{array} \right. \\
 2 \otimes 3 = 3 \oplus 3' & \left\{ \begin{array}{l} 3 \sim \begin{pmatrix} \alpha_1 \beta_2 + \alpha_2 \beta_3 \\ \alpha_1 \beta_3 + \alpha_2 \beta_1 \\ \alpha_1 \beta_1 + \alpha_2 \beta_2 \end{pmatrix} \\ 3' \sim \begin{pmatrix} \alpha_1 \beta_2 - \alpha_2 \beta_3 \\ \alpha_1 \beta_3 - \alpha_2 \beta_1 \\ \alpha_1 \beta_1 - \alpha_2 \beta_2 \end{pmatrix} \end{array} \right. \\
 2 \otimes 3' = 3 \oplus 3' & \left\{ \begin{array}{l} 3 \sim \begin{pmatrix} \alpha_1 \beta_2 - \alpha_2 \beta_3 \\ \alpha_1 \beta_3 - \alpha_2 \beta_1 \\ \alpha_1 \beta_1 - \alpha_2 \beta_2 \end{pmatrix} \\ 3' \sim \begin{pmatrix} \alpha_1 \beta_2 + \alpha_2 \beta_3 \\ \alpha_1 \beta_3 + \alpha_2 \beta_1 \\ \alpha_1 \beta_1 + \alpha_2 \beta_2 \end{pmatrix} \end{array} \right.
 \end{aligned} \tag{A.8}$$

$$3 \otimes 3 = 3' \otimes 3' = 1 \oplus 2 \oplus 3 \oplus 3' \left\{ \begin{array}{l} 1 \sim \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2 \\ 2 \sim \begin{pmatrix} \alpha_2 \beta_2 + \alpha_1 \beta_3 + \alpha_3 \beta_1 \\ \alpha_3 \beta_3 + \alpha_1 \beta_2 + \alpha_2 \beta_1 \end{pmatrix} \\ 3 \sim \begin{pmatrix} 2\alpha_1 \beta_1 - \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ 2\alpha_3 \beta_3 - \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ 2\alpha_2 \beta_2 - \alpha_1 \beta_3 - \alpha_3 \beta_1 \end{pmatrix} \\ 3' \sim \begin{pmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \end{pmatrix} \end{array} \right. \tag{A.9}$$

$$3 \otimes 3' = 1' \oplus 2 \oplus 3 \oplus 3' \left\{ \begin{array}{l} 1' \sim \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2 \\ 2 \sim \begin{pmatrix} \alpha_2 \beta_2 + \alpha_1 \beta_3 + \alpha_3 \beta_1 \\ -\alpha_3 \beta_3 - \alpha_1 \beta_2 - \alpha_2 \beta_1 \end{pmatrix} \\ 3 \sim \begin{pmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \end{pmatrix} \\ 3' \sim \begin{pmatrix} 2\alpha_1 \beta_1 - \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ 2\alpha_3 \beta_3 - \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ 2\alpha_2 \beta_2 - \alpha_1 \beta_3 - \alpha_3 \beta_1 \end{pmatrix} \end{array} \right. \tag{A.10}$$

Appendix B. *q*-expansions of lowest weight $N = 4$ modular forms

The five linearly independent modular forms in Eq. (22) admit the expansions:

$$-\frac{8i}{3\pi} Y_1(\tau) = 1 - 24y - 72y^2 + 288y^3 + 216y^4 + \dots, \tag{B.1}$$

$$-\frac{8i}{3\pi} Y_2(\tau) = 1 + 24y - 72y^2 - 288y^3 + 216y^4 + \dots, \tag{B.2}$$

$$\frac{4i}{\pi} Y_3(\tau) = 1 - 8z + 64z^3 + 32z^4 + 192z^5 - 512z^7 + 384z^8 + \dots, \quad (\text{B.3})$$

$$\frac{2i}{\pi} [Y_4(\tau) + Y_5(\tau)] = 1 + 4z - 32z^3 + 32z^4 - 96z^5 + 256z^7 + 384z^8 + \dots, \quad (\text{B.4})$$

$$\frac{i}{\pi} [Y_4(\tau) - Y_5(\tau)] = 2\sqrt{3}z \left(1 + 8z^2 - 24z^4 - 64z^6 + \dots \right), \quad (\text{B.5})$$

where $y \equiv i\sqrt{q/3}$, $z \equiv e^{i\pi/4}(q/4)^{1/4}$, and as usual $q = e^{2\pi i \tau}$.

Appendix C. Forms of higher weight and constraints

Through tensor products of Y_2 and Y_3 , one can find the multiplets:

$$\begin{aligned} Y_1^{(4)} &= Y_1 Y_2 \sim \mathbf{1}, \\ Y_1^{(4)'} &= Y_3^2 + 2Y_4 Y_5 \sim \mathbf{1}, \\ Y_2^{(4)} &= (Y_2^2, Y_1^2)^T \sim \mathbf{2}, \\ Y_2^{(4)'} &= (Y_4^2 + 2Y_3 Y_5, Y_5^2 + 2Y_3 Y_4)^T \sim \mathbf{2}, \\ Y_3^{(4)} &= (Y_1 Y_4 - Y_2 Y_5, Y_1 Y_5 - Y_2 Y_3, Y_1 Y_3 - Y_2 Y_4)^T \sim \mathbf{3}, \\ Y_3^{(4)'} &= (Y_3^2 - Y_4 Y_5, Y_5^2 - Y_3 Y_4, Y_4^2 - Y_3 Y_5)^T \sim \mathbf{3}, \\ Y_3^{(4)''} &= (Y_1 Y_4 + Y_2 Y_5, Y_1 Y_5 + Y_2 Y_3, Y_1 Y_3 + Y_2 Y_4)^T \sim \mathbf{3}'. \end{aligned} \quad (\text{C.1})$$

Not all of these multiplets are expected to be independent. Indeed, from the q -expansions of the $Y_i(\tau)$ given in Appendix B we find 6 constraints between the 15 different $Y_i(\tau)Y_j(\tau)$ products:

$$\begin{aligned} \frac{1}{3} (Y_3^2 + 2Y_4 Y_5) &= Y_1 Y_2, & -\frac{1}{\sqrt{3}} (Y_3^2 - Y_4 Y_5) &= Y_1 Y_4 - Y_2 Y_5, \\ \frac{1}{3} (Y_4^2 + 2Y_3 Y_5) &= Y_2^2, & -\frac{1}{\sqrt{3}} (Y_5^2 - Y_3 Y_4) &= Y_1 Y_5 - Y_2 Y_3, \\ \frac{1}{3} (Y_5^2 + 2Y_3 Y_4) &= Y_1^2, & -\frac{1}{\sqrt{3}} (Y_4^2 - Y_3 Y_5) &= Y_1 Y_3 - Y_2 Y_4. \end{aligned} \quad (\text{C.2})$$

These constraints imply that $Y_1^{(4)}$ and $Y_1^{(4)'}$, $Y_2^{(4)}$ and $Y_2^{(4)'}$, and $Y_3^{(4)}$ and $Y_3^{(4)'}$ in Eq. (C.1) denote the same multiplets, and only one of each pair is kept in our discussion, cf. Eq. (24).

In Ref. [3] it is argued that the presence of a covariant constraint similar to the ones given in Eq. (C.2) signals the non-linear realisation of the discrete symmetry.

References

- [1] K. Nakamura, S.T. Petcov, in: C. Patrignani, et al., Review of particle physics, Chin. Phys. C 40 (2016) 100001, and 2017 update.
- [2] F. Capozzi, E. Lisi, A. Marrone, A. Palazzo, Current unknowns in the three neutrino framework, Prog. Part. Nucl. Phys. 102 (2018) 48, arXiv:1804.09678.
- [3] F. Feruglio, Are neutrino masses modular forms?, in: S. Forte, A. Levy, G. Ridolfi (Eds.), From My Vast Repertoire: The Legacy of Guido Altarelli, World Scientific, 2018, arXiv:1706.08749.

- [4] T. Kobayashi, K. Tanaka, T.H. Tatsuishi, Neutrino mixing from finite modular groups, *Phys. Rev. D* 98 (2018) 016004, arXiv:1803.10391.
- [5] S. Ferrara, D. Lust, A.D. Shapere, S. Theisen, Modular invariance in supersymmetric field theories, *Phys. Lett. B* 225 (1989) 363.
- [6] S. Ferrara, D. Lust, S. Theisen, Target space modular invariance and low-energy couplings in orbifold compactifications, *Phys. Lett. B* 233 (1989) 147.
- [7] G. Köhler, *Eta Products and Theta Series Identities*, Springer Monogr. Math., Springer Berlin Heidelberg, 2011.
- [8] G. Ross, M. Serna, Unification and fermion mass structure, *Phys. Lett. B* 664 (2008) 97, arXiv:0704.1248.
- [9] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, M. Tanimoto, Non-Abelian discrete symmetries in particle physics, *Prog. Theor. Phys. Suppl.* 183 (2010) 1, arXiv:1003.3552.
- [10] F. Bazzocchi, L. Merlo, S. Morisi, Fermion masses and mixings in a $S(4)$ -based model, *Nucl. Phys. B* 816 (2009) 204, arXiv:0901.2086.