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Ph.D. Thesis

Differential Calculus for Jordan algebras and connections for Jordan modules

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Chapter 1

Introduction

The goal of this thesis is to establish the basis for the theory of differential calculus for Jordan algebras and connections for Jordan modules.
We are also going to present some preliminary attempts of applications to quantum physics, more specifically to the Standard Model of particle physics.

Jordan algebras are a class of nonassociative algebras introduced during the third decade of the 20th century as a suitable algebraic structure for the set of observables of any quantum system.
However it is a classical result that almost all the Jordan algebras relevant for physical applications can be obtained by endowing the linear subspace of self-adjoint elements of an associative $*$ algebra with the anticommutator and then considering a closed subalgebra with respect to the anticommutator product. The only relevant exception to this construction is the Albert algebra J_3^8, whose elements are three by three matrices with entries in the algebra of octonions.
For this reason, and due to the fact that no existing physical quantum system with observables in J_3^8 was known, Jordan algebras almost faded into oblivion for what regards their applications in quantum physics in favour of C* algebras for which Gelfand theory provides a direct interpretation as algebras of bounded operators on a separable Hilbert space, hence connecting in a natural way with the canonical formalism of Quantum Mechanics.
In [24] (and later in [29]) the authors made a proposal to give a mathematical explanation of the existence of three generations of particles in the Standard Model by allowing the observables of the inner quantum space of each particle to lie in the Albert algebra J_3^8.
This suggestion constituted the main motivation to undertake the research in the theory of differential calculus for Jordan algebras and connections for Jordan modules as my PhD project. In fact if we admit that the kinematic setting of Standard Model can be described by Jordan algebras, then it is reasonable to expect that a

suitable dynamical theory will make use of the related theory of differential calculus.

The thesis is structured as follows.
In the first chapter we introduce Jordan algebras, focusing on Euclidean Jordan algebras which are the ones truly relevant for Quantum Mechanics. In particular we describe some properties and enunciate certain classical results that will be useful in the subsequent chapters.
In the second chapter we present the representation theory for nonassociative algebras, specializing it to Jordan algebras and their modules. We give an original contribution by presenting a theorem that completly characterizes homomorphism between free Jordan modules, a result that was already established in [9].
In the third chapter we study the theory of differential calculi for Jordan algebras and we prove a universal property for the derivation-based differential calculus over J_3^8, which is an original result that was published in [9]. One other original contribution of this thesis is the proof that the derivation-based differential calculus for finite dimensional Euclidean Jordan algebras is associative up to homotopy.
In the fourth chapter we present the theory of connections for Jordan modules and we make use of the results of the second chapter to characterize the theory of connections for free Jordan modules. Moreover we define a base flat connection for any module over any Jordan algebra which has only inner derivations, as done in the last section of [9], and we prove new results for this object.
Finally in chapter five we overview the two proposals made in [24] and [29] and we specialize the results presented in the chapters before to these two cases. This is done by considering Yang-Mills theories where the main degrees of freedom are connections on Jordan modules. This thesis is the first work in which this approach is taken in the context of Jordan algebras and further investigation will be carried in future works.

The last chapter deals with a different topic, that is the spin geometry of the rational noncommutative torus. The philosophy underlying this last chapter is basically the same as for the previous ones: here the "quantumness" of the space we are investigating is completely described by a mere finite dimensional noncommutative inner space, whereas in the first part of the thesis the same phenomenon was encoded by allowing for a finite dimensional quantum space described by Jordan algebras. We realize three different constructions of the canonical spectral triple of the noncommutative torus with rational phase for which we also analyze different double coverings and curved geometries. All the presented results were originally pubblished in [8].

This thesis is based on two pubblished papers ([9] and [8]) and a third paper is in preparation.

Chapter 2

Jordan algebras

In this first chapter we shall give a brief overview of the theory of Jordan algebras.

The first section aims to give a historical outline, as we shall present the physical motivations that led to the introduction of Jordan algebras in the last century.

In section (2.2) we shall provide some useful theorems in the context of Jordan algebras, with particular regard to some structural properties that will be used later on to investigate their differential calculus and the theory of connections for Jordan modules.

Section (2.3) consists of a succinct review of the theory of associative envelopings for Jordan algebras, a widely studied topic in the last century, most notably by Jacobson ([42]).
We will also study a quite new enveloping algebra that is the complex $*$ enveloping introduced by Dubois-Violette and Todorov [23], for which we prove that it is a functorial construction. The motivations for introducing this object lie in the physical applications that we are going to present in chapter (6).

In section (2.4) we provide some definitions in the context of Jordan superalgebras that we are going to exploit in chapters (4) and (5).

Finally in the last section we cover the topic of JB-algebras, that provide a suitable algebraic and topological structure to define the observables of a quantum theory in infinite dimensional cases or even for a quantum field theory.

Through this chapter and the rest of the thesis, the term "algebra" without further specification will mean a non necessarily associative and non necessarily commu-

tative algebra, hence it will just stand for the datum of a vector space A on a ground field $\mathbb{K}$ endowed with a bilinear product $A \times A \to A$ which is distributive with respect to the sum. Algebras will not in general contain a unit, an algebra with a unit will be called unital.
When not differently specified all the terms that we are going to use will have the same meaning as in the context of associative algebras. So, for example, an algebra will be called simple if it contains no non-trivial two-sided ideals and semisimple if its radical (the maximal ideal composed by nilpotents) is zero.

We will mostly work with real algebras,that is over the field $\mathbb{K} = \mathbb{R}$ if not differently specified, even though most of the results we shall present stand true for Jordan algebras over fields with characteristic different from 2 or 3.

2.1 Origins of Jordan algebras: observables in quantum mechanics

In the Copenaghen interpretation of Quantum Mechanics, states of an isolated microscopic system are represented by rays (vectors of length one) of a suitable complex Hilbert space $\mathcal{H}$. The observables on such system are encoded as elements of a vector space of self-adjoint elements inside the algebra of bounded operators on $\mathcal{H}$ in the case of finite dimensional $\mathcal{H}$ and otherwise also unbounded ones (with only partially defined product though), while the possible outcomes of the measurement of an observable are the eigenvalues of the corresponding operator.
The reason for the choice of self-adjoint operators lies in the request that observables quantities shall take real values and that, due to spectral calculus, the exponentiation of any selfadjoint operator provides a one-parameter family of unitary maps on $\mathcal{H}$ (hence encoding continuous transformations on the physical system, such as its evolution in time). When dealing with a system with a finite number of degrees of freedom (such as the spin states of an elementary fermion) this picture is considerably simplified by taking as quantum observable a certain linear space of hermitian matrices of proper dimension.

It is evident though that this setting for quantum observables is somehow redundant, since the main operations defining a C^* algebra A do not correspond to any manipulation which can be performed on a physical system. In fact, if x, y are self adjoint elements in A representing two quantum observables one finds that

- For $\lambda \in \mathbb{C}$, λx is not self-adjoint, having

$$(\lambda x)^* = \overline{\lambda} x \neq \lambda x$$

whenever λ is not real.

- The product xy given by operator composition (or matrix product) is not self-adjoint, since:
$$(xy)^* = y^* x^* = yx \neq xy$$
whenever x and y do not commute.

- By definition, when restricted to self-adjoint elements the conjugation * is simply the identity.

For this reason, the physicist P. Jordan proposed a program to find out the most general algebraic setting that allows for the description of finite dimensional quantum systems.
The aim of the program was to identify the intrinsic algebraic properties of hermitian matrices without referring to any of the "metaphysical" operations of the corresponding matrix C^* algebra.
Then the more general algebraic structure enjoying said properties would be the one which encodes the structure of a quantum theory.
In order to achieve this goal, he pointed out the following as meaningful operations:

- The sum of two hermitian matrices $x + y$.

- The multiplication of a hermitian matrix by a real scalar $\alpha x,\ \alpha \in \mathbb{R}$.

- Taking powers of hermitian matrices by any natural number $x^n,\ n \in \mathbb{N}$.

- Multiplying a hermitian matrix by the identity matrix $x\mathbb{1} = \mathbb{1}x = x$.

By the first two properties one deduces that the set of quantum observables for the given physical system should be endowed with the structure of real vector space, thus allowing for the construction of new observables as linear combination of known ones.
For what regards the third property, some observations have to be made: let us define the product $x \circ y$ from

$$x \circ y = \frac{1}{2}\left[(x+y)^2 - x^2 - y^2\right]$$

for every pair of hermitian matrices x and y; in view of quantum mechanical applications it is natural to require the composition law for polynomials in one variable with respect to the product $\circ$, that is if the polynomial $r(t)$ can be written as composition of two polynomial $p(t)$ and $q(t)$, that is $r(t) = p(q(t))$, one asks

that for every hermitian matrix x it stands $r(x) = p(q(x))$. As matter of fact this request is equivalent to the power associativity law

$$x^n \circ x^m = x^{n+m}$$

for every $n, m \in \mathbb{N}$.

One other important property of hermitian matrices is formal reality, that is

$$x^2 + y^2 = 0 \Rightarrow x = y = 0$$

for every $x, y \in J$. Formal reality is in fact crucial if we want J to represent the algebraic structure of observables for a finite quantum system as we shall comment in the following section.
One has the following:

Theorem 2.1.1 (Jordan 1932)**.** *Let J be a real vector space such that*

$$x \in J \Rightarrow \exists\, x^n \in J \;\forall n \in \mathbb{N}.$$

and let J be formally real. Define the product $x \circ y = \frac{1}{2}\left[(x+y)^2 - x^2 - y^2\right]$, then the following identities are equivalent:

1. $x^n \circ x^m = x^{n+m}$ $\forall n, m \in \mathbb{N}$.
2. $x \circ (y \circ x^2) = (x \circ y) \circ x^2$.

The second identity, which is a weaker version of the associative law for the product $\circ$, is known as Jordan identity.

2.2 Jordan algebras

The considerations made in the previous section lead us to the following definition.

Definition 2.2.1. *A (real) Jordan algebra $(J, \circ)$ is a (real) vector space J endowed with a bilinear product $\circ : J \times J \to J$ such that*

$$x \circ y = y \circ x \tag{2.1}$$
$$x^2 \circ (y \circ x) = (x^2 \circ y) \circ x \tag{2.2}$$

for every $x, y \in J$, with $x^2 = x \circ x$.

Jordan identity (2.2) can also be presented as a linear expression as follows: for any $x \in J$, let $L_x : J \to J$ be the left multiplication by x, that is

$$L_x(y) = x \circ y$$

for every $y \in J$, then we have

$$[L_{x^2}, L_x] = 0 \tag{2.3}$$

for every $x \in J$.

Remark: Since Jordan algebras are commutative, it is quite pointless to specify that L_x acts by multiplying on the left rather then right. Nevertheless this notation will turn useful in the following sections when we will deal with Jordan superalgebras.

Introducing the left multiplication, we can present a linearized version of the Jordan identity that is

$$[L_x, L_{y\circ z}] + [L_y, L_{z\circ x}] + [L_z, L_{x\circ y}] = 0 \tag{2.4}$$

for any $x, y, z \in J$.
In the following we are going to focus on Euclidean Jordan algebra, since they correspond to algebras of observables for quantum systems.
Recall the following.

Definition 2.2.2. *A nilpotent in a Jordan algebra is an element $x \in J$ such that $x^n = 0$ for some $n \in \mathbb{N}$.*

The following theorem holds

Theorem 2.2.3. *Let J be a Jordan algebra with no nonzero nilpotent, then J is unital.*

Definition 2.2.4. *A Jordan algebra J is called Euclidean (or equivalently "formally real") if*

$$x^2 + y^2 = 0 \Rightarrow x = y = 0 \tag{2.5}$$

for every $x, y \in J$.

If a Jordan algebra J is formally real then it is unital in view of Theorem 2.2.3.

Definition 2.2.5. *Let J be a Jordan algebra, an idempotent is an element e such that $e^2 = e$, if e_1 and e_2 are two idempotents of J, they are said to be mutually orthogonal if $e_1 \circ e_2 = 0$.*

Definition 2.2.6. *If J is a unital Jordan algebra with unit 1_J, we say J has capacity n if there exist n orthogonal idempotents $\{e_i\}_{i=1}^n$ such that $e_1+...+e_n = 1_J$*

The following theorem provides an equivalent definition of Euclidean Jordan algebra.

Theorem 2.2.7. *Let J be a real Jordan algebra of dimension n, the following conditions are equivalent:*

1. *J is formally real.*
2. *$\forall x \in J$ there exist $\{e_i\}_{i=1}^n \subset J$ idempotents such that $x = \sum_{i=1}^n x^i e_i$ with $\{x^i\}_{i=1}^n \subset \mathbb{R}$.*

Let us review some examples of Jordan algebras.

Example 2.2.8. *Let A be an associative algebra, we denote its product simply by juxtaposition of two elements. We build the Jordan algebra $A^+ = (A, \circ)$ whose vector space is the same as A and as algebra is endowed with the anticommutator, that is*

$$x \circ y = \frac{1}{2}(xy + yx)$$

for every $x, y \in A$.

The example above leads to the following definition.

Definition 2.2.9. *If a Jordan algebra J is isomorphic to a subalgebra of A^+ for some associative algebra A, we call J a special Jordan algebra. If a Jordan algebra is not special we call it an exceptional Jordan algebra.*

Example 2.2.10. *Let $(A, *)$ be an associative algebra with involution, we denote by A_{sa} the subspace of self-adjoint elements of A with respect to the involution $*$. When endowed with the anticommutator, A_{sa} becomes a special Jordan algebra since we have*

$$(x \circ y)^* = \frac{1}{2}(xy + yx)^* = \frac{1}{2}(yx + xy) = x \circ y$$

for every $x, y \in A_{sa}$.
In particular, for $n \geq 2$ and $i \in \{1, 2, 4\}$ we denote by J_n^i the special Jordan algebra of hermitian $n \times n$ matrices with real, complex or quaternionic entries for $i = 1, 2, 4$ respectively. Jordan algebras of the series J_n^i are Euclidean.

Example 2.2.11. *Denote by $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the standard scalar product on $\mathbb{R}^n$. We define $JSpin_n$ as the vector space $\mathbb{R} \oplus \mathbb{R}^n$ with the product given by*

$$(\alpha, v)(\beta, w) = (\alpha\beta + \langle v, w \rangle, \alpha w + \beta v)$$

for every $\alpha, \beta \in \mathbb{R}$ *and* $v, w \in \mathbb{R}^n$. *Spin factors are Euclidean special Jordan algebras. For* $n = 1$, *the corresponding spin factor is not a simple Jordan algebra, having*

$$JSpin_1 = \mathbb{R}^2$$

while for every $n \geq 2$ *the corresponding spin factors are simple Jordan algebras and in particular the following isomorphisms hold:*

$$JSpin_2 \simeq J_2^1, \quad JSpin_3 \simeq J_2^2, \quad JSpin_5 \simeq J_2^4, \quad JSpin_9 \simeq J_2^8$$

where J_n^i *are defined as in Example 2.2.10 for* $i = 1, 2, 4$ *and* J_2^8 *denotes the Jordan algebra of two by two hermitian matrices with entries in the Cayley algebra* $\mathbb{O}$ *of octnions.*

Example 2.2.12. *Denote by* J_3^8 *the vector space*

$$J_3^8 = \{x \in M_3(\mathbb{O}) \mid x_{ij} = \overline{x}_{ji}\}$$

endowed with the anticommutator

$$x \circ y = \frac{1}{2}(xy + yx)$$

for every $x, y \in J_3^8$. *We call* J_3^8 *the Albert algebra, it is a simple Jordan algebra and the following classical result was proved by Albert ([1]).*

Theorem 2.2.13 (Albert, 1934)**.** J_3^8 *is an exceptional Jordan algebra.*

The examples shown above are indeed quite exhaustive since in 1934 Jordan, von Neumann and Wigner proved the following classification theorem for finite dimensional Euclidean Jordan algebras [43].

Theorem 2.2.14. *(Jordan, von Neumann, Wigner* 1934*) Any finite dimensional Euclidean Jordan algebra is a finite direct sum of simple Euclidean finite-dimensional Jordan algebras. Any finite-dimensional simple Euclidean Jordan algebra is isomorphic to one of the following:*

$$\begin{gathered} \mathbb{R},\ JSpin_{n+2}, \\ J_{n+3}^1,\ J_{n+3}^2,\ J_{n+3}^4,\ J_3^8 \end{gathered}$$

for any $n \in \mathbb{N}^0$.

We notice that the Jordan algebras of the series J_n^i are classified by their capacity n and that for n greater then 3 their elements can have coefficients either in $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, while for $n \leq 3$ also the Cayley algebra of octonions $\mathbb{O}$ is allowed.

In the list above, the only exceptional simple Jordan algebra is J_3^8.

Remark. Even when one takes in consideration infinite dimensional Jordan algebras, any exceptional Jordan algebra will necessarily contain J_3^8 as direct summand ([59]).

2.2.1 Tensor products of Jordan algebras

If we look at Jordan algebras as the suitable algebraic structures to describe quantum systems, it is quite natural to seek for a definition of tensor product between Jordan algebras since it would be natural to think of the tensor product between two Jordan algebras as a composite quantum system.
The definition of tensor product between Jordan algebras has been attempted by many authors which provided a fair ammount of alternative constructions (e.g. [33] [34], [57] and more recently [4]). However none of these definitions is satisfyingly general, meaning that these various products cannot be defined for arbitrary Jordan algebras in a consistent way.

We shall not review all of these proposals, however it is important for us to recall a result obtained in [57] to show how one fails when trying to define the tensor product in the naivest possible way.
In chapter 3 we are going to use this result to prove a property of the derivation-based differential calculus over J_3^8.

Definition 2.2.15. *Let J_1 and J_2 be two Jordan algebras, denote by $J_1 \otimes J_2$ the tensor product of the underlying vector spaces, we call Kronecker product of J_1 and J_2 the vector space $J_1 \otimes J_2$ endowed with the following product*

$$(x \otimes y) \circ (x' \otimes y') = x \circ x' \otimes y \circ y'$$

for $x, x' \in J_1$, $y, y' \in J_2$.

Proposition 2.2.16. *Let J_1 and J_2 be two Jordan algebras,the Kronecker product of J_1 and J_2 is a Jordan algebra if and only if either J_1 or J_2 is associative.*

Remark The proposition above was slightly different when firstly enunciated by Wulfhson in theorem 2 of [57]. In fact he was admitting also for either J_1 or J_2 to be a spin factor. Nevertheless it is quite easy to work out a counterxample to prove that this does not hold true.

2.2.2 Center and derivations

Let A be an algebra, whose product we denote just by juxtaposition, let us define the associator of three elements as

$$[x, y, z] = (xy)z - x(yz)$$

for any $x, y, z \in A$.

Definition 2.2.17. *The center of A, denoted by $Z(A)$, is the associative and commutative subalgebra of elements $z \in A$ satisfying*

$$[x, z] = 0, \ [x, y, z] = [x, z, y] = [z, x, y] = 0$$

for any $x, y \in A$.

The following simple result will be useful later on

Proposition 2.2.18. *Let A be a commutative algebra and let $z \in A$, then $z \in Z(A)$ if and only if*

$$[x, y, z] = 0 \tag{2.6}$$

for all $x, y \in A$.

Proof. The condition $[x, z] = 0$ is trivial for any $x, z \in A$ since we have taken the algebra A to be commutative.
If the condition $[x, y, z] = 0$ holds, then for every $x, y \in A$ one has:

$$0 = [x, y, z] - [y, x, z] = [y, z, x] \tag{2.7}$$

and

$$0 = -[x, y, z] = [z, x, y] \tag{2.8}$$

for any $x, y \in A$, in view of the commutativity. □

In particular, the proposition above holds true for all Jordan algebras. Moreover for every simple Euclidean Jordan algebras the center is $\mathbb{R}$.

Definition 2.2.19. *A derivation of an algebra A is a linear endomorphism X of A, such that one has*

$$X(xy) = X(x)y + xX(y)$$

for all $x, y \in A$.

Proposition 2.2.20. *The vector space $Der(A)$ of derivations of an algebra A has the following properties:*

1. *$Der(A)$ is a Lie algebra with respect to the commutator of endomorphisms.*

2. *$Der(A)$ is a module over the center $Z(A)$.*

3. *The center of A is stable with respect to derivations, that is $X(z) \in Z(A)$ for all $X \in Der(A)$ and for any $z \in Z(A)$.*

4. *The following formula holds:*

$$[X_1, zX_2] = X_1(z)X_2 + z[X_1, X_2]$$

for all $X_1, X_2 \in A$ and $z \in Z(A)$.

Proof. (1), (2) and (4) are trivial, we have only to prove stability of the center. Let $z \in Z(A)$ and $X \in Der(A)$, we have:

$$\begin{aligned}
&[x, y, X(z)] = (xy)X(z) - x(yX(z)) = \\
&= X\left((xy)z\right) - X(xy)z - \left(xX(yz) - x\left(X(y)z\right)\right) = \\
&= X\left((xy)z\right) - X(xy)z - X(x(yz)) + X(x)(yz) + x\left(X(y)z\right) = \\
&= X\left([x, y, z]\right) - [X(x), y, z] - [x, X(y), z] = 0
\end{aligned} \tag{2.9}$$

for any $x, y \in A$. Similarly one proves that $[x, X(z), y] = [X(z), x, y] = 0$ and $[x, X(z)] = 0$. □

Thus the pair $(Z(A), Der(A))$ form a Lie-Rinehart algebra (see e.g. [38],[53] for references).

If J is a Jordan algebra and $\{(x_i, y_i)\} \subset J$ are pairs of elements in J, then the linear map

$$\begin{aligned}
&X : J \to J \\
&z \mapsto X(z) = \sum [x_i, z, y_i]
\end{aligned}$$

is a derivation for J and the following classical result, due to Jacobson and Harris ([41],[35]), is the equivalent of Witehead's first lemma for Lie algebras in the context of Jordan algebras

Theorem 2.2.21. *Let J be a finite dimensional semi-simple Jordan algebra, let $X \in Der(J)$. There exist a finite number of couples of elements $x_i, y_i \in J$ such that one has*

$$X(z) = \sum [x_i, z, y_i] \tag{2.10}$$

for any $z \in J$.

Moreover we have the following.

Proposition 2.2.22. *Let J be a Jordan algebra, let $z \in J$ be such that $X(z) = 0$ for every $X \in Der(J)$, then $z \in Z(J)$*

Proof. For every $x, y \in J$ one has

$$[x, z, y] = 0$$

and from (2.2.18) we conclude that $z \in Z(J)$. □

Example 2.2.23. *The list of derivations for the finite dimensional non-exceptional simple Euclidean Jordan algebras covers the list of the non exceptional simple Lie algebras, i.e. the Lie algebras denoted by $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$ and $\mathfrak{d}_n$ in the Cartan classification, as summed up in the following table*

Derivations for simple Euclidean Jordan algebras

J	$Der(J)$
$JSpin_n$	$\mathfrak{so}(n)$
J_n^1	$\mathfrak{so}(n)$
J_n^2	$\mathfrak{su}(n)$
J_n^4	$\mathfrak{sp}(n)$

while for the exceptional Jordan algebra J_3^8 the algebra of derivations is given by the exceptional Lie algebra $\mathfrak{f}_4$ as shown in the following example.

Example 2.2.24. *The Lie algebra of derivations of the exceptional Jordan algebra J_8^3 is the exceptional Lie algebra $\mathfrak{f}_4$ (see e.g. [58]). Introduce the standard basis of the exceptional Jordan algebra*

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_1^j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon_j \\ 0 & \overline{\epsilon}_j & 0 \end{pmatrix}, \; F_2^j = \begin{pmatrix} 0 & 0 & \overline{\epsilon}_j \\ 0 & 0 & 0 \\ \epsilon_j & 0 & 0 \end{pmatrix}, \; F_3^j = \begin{pmatrix} 0 & \epsilon_j & 0 \\ \overline{\epsilon}_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where ϵ_j $j \in \{0, ..., 7\}$ are a basis of the octonions, so $\epsilon_0 = 1$, $\epsilon_j^2 = -1$ for $j \neq 0$ and the multiplication table of octonions holds (see e.g. on [3]).

As vector space, $\mathfrak{f}_4$ admits a decomposition

$$\mathfrak{f}_4 = \mathfrak{D}_4 \oplus \mathfrak{M}^-$$

given as follows. $\mathfrak{D}_4$ *is the subspace of derivations which annihilates the diagonal of any element in* J_3^8*, that is*

$$\delta E_i = 0 \quad i \in \{1, 2, 3\}$$

for any $\delta \in \mathfrak{D}_4$*. An interesting and concrete characterization of* $\mathfrak{D}_4$ *is given by the following theorem (see e.g. chapter* 2 *of [58]).*

Theorem 2.2.25. *The algebra* $\mathfrak{D}_4$ *is isomorphic to* $\mathfrak{so}(8) = \mathfrak{d}_4$*. The isomorphism is given via the equality:*

$$\delta \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3 x_3 & \overline{D_2 x_2} \\ \overline{D_3 x_3} & 0 & D_1 x_1 \\ D_2 x_2 & \overline{D_1 x_1} & 0 \end{pmatrix}$$

where $\delta \in \mathfrak{D}_4$ *and* $D_1, D_2, D_3 \in \mathfrak{so}(8)$*.* D_2, D_3 *are determined by* D_1 *from the principle of infinitesimal triality*

$$(D_1 x)y + x(D_2)y = \overline{D_3(\overline{xy})}$$

for any $x, y \in \mathbb{O}$*.*

Elements of the vector space $\mathfrak{M}^-$ *are* 3×3 *antihermitian octonion matrices with every element on the diagonal equal to zero. Every* $M \in \mathfrak{M}^-$ *defines a linear endomorphism* $\tilde{M} : J_3^8 \to J_3^8$*, via the commutator*

$$\tilde{M}(x) = Mx - xM$$

where in the expression above juxtaposition is understood as the usual row by column matrix product.

2.3 Enveloping algebras

It is often interesting to study whether a Jordan algebra can "fit nicely" in any associative algebra. This because, both from the point of view of algebraic structure and from the point of view of representation theory, associative algebras are well known and tame objects.
There are several ways to achieve this goal and one special role in this context is played by universal envelopings, which are a concrete examples of universal objects in the sense of category theories.

In this section we are always going to denote the product inside a Jordan algebra with the symbol $\circ$, while we use juxtaposition to denote product inside associative

algebras.

The first enveloping algebra we shall present is the universal associative enveloping of Jordan algebra.

Definition 2.3.1. *Let J be a Jordan algebra, let A be an associative algebra, a linear map $\sigma : J \to A$ such that*

$$\sigma(x \circ y) = \frac{1}{2}(xy + yx)$$

for every $x, y \in J$ is called an associative specialization of J in A. If J is unital with unit 1_J we require that also A is unital and

$$\sigma(1_J) = 1_A$$

where 1_A is the unit of A.

Definition 2.3.2. *For a fixed Jordan algebra J we denote by $\boldsymbol{AssSp}_J$ the category of associative specializations of J defined in the following way*

- *The objects of $\boldsymbol{AssSp}_J$ are pairs (A, σ) where A is an associative algebra and $\sigma : J \to A$ is an associative specialization of J in A.*
- *If (A, σ) and (A', σ') are two associative specializations of J, a morphism $\phi : (A, \sigma) \to (A', \sigma')$ in $\boldsymbol{AssSp}_J$ is a morphism of associative algebras $\phi : A \to A'$ such that the following commutative diagram*

$$\begin{tikzcd} J \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & A \arrow[d, "\phi"] \\ & A' \end{tikzcd}$$

  holds.

For any Jordan algebra J, there exist a unique (up to isomorphism) object $(S_{ass}(J), \tau) \in \mathbf{AssSp}_J$ such that for any other $(A, \sigma) \in \mathbf{AssSp}_J$ there exists a unique morphism $(S_{ass}(J), \tau) \to (A, \sigma)$. In other words $(S_{ass}(J), \tau)$ is a initial object in the category $\mathbf{AssSp}_J$.

Definition 2.3.3. *For a Jordan algebra J, we call the associative algebra $S_{ass}(J)$ the universal associative enveloping of J.*

An explicit construction of $S_{ass}(J)$ is given in the following.

Proposition 2.3.4. *Let J be a Jordan algebra, denote by $T(J)$ the free associative tensor algebra over the vector space of J, then one has*

$$S_{ass}(J) = \frac{T(J)}{R}$$

where R is the two-sided ideal of $T(J)$ generated by the relation

$$x \otimes y + y \otimes x - 2x \circ y$$

for all $x, y \in J$.

In particular a Jordan algebra is special if and only if the enveloping map $\tau : A \to S_{ass}(J)$ is injective (see e.g.[42]).
On the other side, taking $J = J_3^8$, it is a known result (see e.g. [42]) that $S_{ass}(J_3^8) = \{0\}$.

Another way to envelop a Jordan algebra into an associative algebra is to give a multiplicative specialization.

Definition 2.3.5. *Let J be a Jordan algebra and A an associative algebra, a multiplicative specialization of J in A is a linear map $\rho : J \to A$ such that*

$$\left[L_{\rho(x)}, L_{\rho(x^2)}\right] = 0$$
$$2\rho(x)\rho(y)\rho(x) + \rho(x^2 \circ y) = 2\rho(x \circ y)\rho(x) + \rho(x \circ (x \circ y))$$

for any $x, y \in J$, where L_a denotes the left multiplication operator by $a \in A$.

Definition 2.3.6. *For a fixed Jordan algebra J we denote by $\boldsymbol{MulSp}_J$ the category of multiplicative specializations of J defined in the following way*

- *The objects of $\boldsymbol{MulSp}_J$ are pairs (A, ρ) where A is an associative algebra and $\sigma : J \to A$ is an multiplicative specialization of J in A.*
- *If (A, ρ) and (A', ρ') are two multiplicative specializations of J, a morphism $\phi : (A, \rho) \to (A', \rho')$ in $\boldsymbol{MulSp}_J$ is a morphism of associative algebras $\phi : A \to A'$ such that the following commutative diagram*

$$\begin{tikzcd} J \arrow[r, "\rho"] \arrow[dr, "\rho'"'] & A \arrow[d, "\phi"] \\ & A' \end{tikzcd}$$

holds.

As for the case of associative specializations, for any Jordan algebra J there exists a unique (up to isomorphism) initial object $(S_{mul}(J), \xi) \in \mathbf{MulSp}_J$. Analogously to what is done for the universal associative enveloping, an explicit construction of the universal multiplicative enveloping of a fixed Jordan algebra is the following.

Proposition 2.3.7. *Let J be a Jordan algebra, denote by $T(J)$ the free tensor algebra over the vector space of J, then one has*

$$S_{mul}(J) = \frac{T(J)}{R}$$

where R is the bilateral ideal of $T(J)$ generated by the relations

$$[L_x, L_{x^2}]$$
$$2x \otimes y \otimes x + x^2 \circ y = 2(x \circ y) \otimes x + x \circ (x \circ y)$$

for all $x, y \in J$.

Both $S_{ass}(J)$ and $S_{mul}(J)$ can be equipped with a unique involution $*$ with the property

$$(\tau(x))^* = \tau(x)$$

and

$$(\rho(x))^* = \rho(x)$$

for any $x \in J$.

Both the envelopings we have defined are generated in degree one by elements of J and both of them are functorial constructions (see e.g. [42] for more details):

Theorem 2.3.8. *Denote by **Ass** and **Jord** the categories of associative and Jordan algebras over a fixed field, then the associative enveloping defines a functor $S_{ass} : \boldsymbol{Jord} \to \boldsymbol{Ass}$ defined as follows*

- *To any Jordan algebra $J \in \boldsymbol{Jord}$ it associates its associative enveloping $S_{ass}(J)$.*
- *For every $J_1, J_2 \in \boldsymbol{Jord}$ and morphism $\phi : J_1 \to J_2$ in **Jord**, define the morphism $S_{ass}(\phi) : S_{ass}(J_1) \to S_{ass}(J_2)$ in **Ass** from $S_{ass}(\phi)(x) = \phi(x)$ for every $x \in J_1$.*

In the same way the multiplicative enveloping defines a functor $S_{mul} : \boldsymbol{Jord} \to \boldsymbol{Ass}$ such that

- *To any Jordan algebra $J \in \boldsymbol{Jord}$ it associates its multiplicative enveloping $S_{mul}(J)$.*
- *For every $J_1, J_2 \in \boldsymbol{Jord}$ and morphism $\phi : J_1 \to J_2$ in $\boldsymbol{Jord}$, define the morphism $S_{mul}(\phi) : S_{mul}(J_1) \to S_{mul}(J_2)$ in $\boldsymbol{Ass}$ from $S_{mul}(\phi)(x) = \phi(x)$ for every $x \in J_1$.*

There is one more enveloping that can be defined for real Jordan algebras which introduced in [23] and seems quite natural if one has in mind the analogy with the quantization of a set of classical real observables as operators on a complex Hilbert space.

Definition 2.3.9. *Let J be a real unital Jordan algebra, denote by $T_{\mathbb{C}}(J)$ the free complex tensor algebra on J. The complex $*$ enveloping of J is the complex associative $*-$algebra $S_{\mathbb{C}}(J)$ given by*

$$S_{\mathbb{C}}(J) = \frac{T_{\mathbb{C}}(J)}{R}$$

where R is the bilateral ideal of $T_{\mathbb{C}}(J)$ generated by the following relations

$$\begin{gathered} xy + yx - 2x \circ y \\ x^* = x \\ \mathbb{1}_J = \mathbb{1}_{S_{\mathbb{C}}(J)} \end{gathered}$$

for every $x, y \in J$.

We prove the following

Theorem 2.3.10. *Denote by $\boldsymbol{Jord}_{\mathbb{R}}$ the category of real Jordan algebras and by $\boldsymbol{Ass}_{\mathbb{C}}$ the category of complex $*$ algebras. The complex $*$ enveloping defines a functor $S_{\mathbb{C}} : \boldsymbol{Jord}_{\mathbb{R}} \to \boldsymbol{Ass}_{\mathbb{C}}$.*

Proof. Let us fix some notation: for two Jordan algebra J_1 and J_2 we denote their products as $\circ_1$ and $\circ_2$ respectively, similarly we denote by $\cdot_1$ and $\cdot_2$ the products in the correspondent complex star envelopings and by $*_1$ and $*_2$ the respective involutions.
Let $\varphi : J_1 \to J_2$ be a morphism of Jordan algebras, that is

$$\varphi(x \circ_1 y) = \varphi(x) \circ_2 \varphi(y)$$

We denote by $\phi^{\mathbb{C}} := S_{\mathbb{C}}(\varphi)(x)$ by

$$\phi^{\mathbb{C}}(x) = \phi(x)$$

for every $x \in J_1$. Now we have

$$\begin{aligned}\phi^{\mathbb{C}}(x \cdot_1 y + y \cdot_1 x) = \\ &= 2\phi^{\mathbb{C}}(x \circ_1 y) = \\ &= 2\phi^{\mathbb{C}}(x) \circ_2 \phi^{\mathbb{C}}(y) = \\ &= \phi^{\mathbb{C}}(x) \cdot_2 \phi^{\mathbb{C}}(y) + \phi^{\mathbb{C}}(x) \cdot_2 \phi^{\mathbb{C}}(y)\end{aligned}$$

for every $x, y \in J_1$. Thus the image of $\phi^{\mathbb{C}}$ respects the relations inside $S_{\mathbb{C}}(J_2)$ and, since $S_{\mathbb{C}}(J_1)$ and $S_{\mathbb{C}}(J_2)$ are generated in degree one by elements of J_1 and J_2 respectively, the map $\phi^{\mathbb{C}}$ is extended as a well posed map of complex associative algebras. Moreover we have

$$\phi^{\mathbb{C}}(x)^{*_2} = \phi^{\mathbb{C}}(x) = \phi^{\mathbb{C}}(x^{*_1})$$

for every $x \in J_1$, thus we have

$$\begin{aligned}\phi^{\mathbb{C}}(xy)^{*_2} = \\ &= \left(\phi^{\mathbb{C}}(x) \cdot_2 \phi^{\mathbb{C}}(y)\right)^{*_2} \\ &= \phi^{\mathbb{C}}(y)^{*_2} \cdot_2 \phi^{\mathbb{C}}(x)^{*_2} \\ &= \phi^{\mathbb{C}}(y^{*_1}) \cdot_2 \phi^{\mathbb{C}}(x^{*_1}) \\ &= \phi^{\mathbb{C}}((xy)^{*_1})\end{aligned}$$

for every $x, y \in J_2$. Thus $\phi^{\mathbb{C}}$ is a map of complex associative $*$ algebras and functoriality of $S_{\mathbb{C}}$ follows by using composition of homomorphisms of associative algebras. □

As mentioned before, the associative enveloping $S_{ass}(J)$ of a real Jordan algebra J is uniquely endowed with an involution $*$ respect to which the elements of J are self-adjoint. From the complexification of $S_{ass}(J)$, one obtains $S_{ass}(J) \otimes \mathbb{C}$ on which $* : S_{ass}(J) \to S_{ass}(J)$ can be extended by skew linearity to an involution of complex $*$ algebra. Then it would be tempting to conclude that $S_{ass}(J) \otimes \mathbb{C}$ is isomorphic to $S_{\mathbb{C}}(J)$ as complex $*$ algebra; however this is not always the case as shown in the following example.

Example 2.3.11. *Let J be the real special Jordan algebra $M_n(\mathbb{C})^+$ for $n \geq 2$ and fix $x, y, z \in J$ such that*

$$\begin{aligned}2x \circ y &= i\mathbb{1} \\ 2x \circ z &= a \\ 2y \circ z &= b\end{aligned}$$

for some $a, b \in J$.
In order to compare $S_{ass}(J) \otimes \mathbb{C}$ and $S_{\mathbb{C}}(J)$, we consider their elements as equivalence classes of elements inside the complex tensor algebra $T_{\mathbb{C}}(J)$ then we show that the equivalence class of $iz \in T_{\mathbb{C}}(J)$ with respect to the relations of $S_{\mathbb{C}}(J)$ contains more elements than the equivalence class it represents in $S_{ass}(J) \otimes \mathbb{C}$. In the following we denote by $\tau_{\mathbb{C}} : J \to S_{\mathbb{C}}(J)$ the map which assigns to every elements of J its equivalence class inside $S_{\mathbb{C}}(J)$.
Let us write iz for $\tau(z) \otimes i \in S_{ass}(J) \otimes \mathbb{C}$, and $i\tilde{z} = i\tau_{\mathbb{C}}(z)$. Since the complexification map $\otimes\mathbb{C} : S_{ass}(J) \to S_{ass}(J) \otimes \mathbb{C}$ is an injective map of real algebras, the equivalence class of iz inside $T_{\mathbb{C}}(J)$ is completely characterized by the equivalence class of $\tau(z) \in S_{ass}(J)$. On the other side, using the relations which define $S_{\mathbb{C}}(J)$, one has

$$i\tilde{z} = 4(xy + yx)\tilde{z} = 4\tilde{z}\,(2xb - 2ay + zxy + 2ya - 2xb + zyx) =: \tilde{z}K$$

where $K \in T_{\mathbb{C}}(J)$. The elements iz and $\tilde{z}K$ represent the same element in $S_{\mathbb{C}}(J)$ which is not true in $S_{ass}(J) \otimes \mathbb{C}$.

2.4 Jordan superalgebras

When studying differential calculi on Jordan algebras and the theory of connections for Jordan modules we are going to make great use of $\mathbb{Z}_2-$graded Jordan algebras, also known as Jordan superalgebras.
In this section we are going to use juxtaposition to mean the product inside a Jordan superalgebra.

Definition 2.4.1. *A Jordan superalgebra $\Omega = \Omega^0 \oplus \Omega^1$ is a $\mathbb{Z}_2-$graded vector space with a graded commutative product, meaning:*

$$xy = (-1)^{|x||y|} yx$$

for all $x, y \in \Omega$ and such that this product respects the Jordan identity.

For a Jordan superalgebra the following property of the associator holds:

$$[x, y, z] = (-1)^{|y||z|}\,[z, y, x]$$

for all $x, y, z \in \Omega$. If we introduce the graded commutator of x and y as

$$[x, y]_{gr} = xy + (-1)^{|x||y|} yx$$

the (super) Jordan identity is equivalent to the following operator identity:

$$\begin{aligned}&(-1)^{|x||z|}\,[L_{xy}, L_z]_{gr} + (-1)^{|z||y|}\,[L_{zx}, L_y]_{gr} + \\ &+ (-1)^{|y||x|}\,[L_{yz}, L_x]_{gr} = 0\end{aligned}$$

for all $x, y, z \in \Omega$. Real finite dimensional Jordan superalgebras have been classified by Kac and Kantor ([44],[45]).

In chapter 3 and 4 we are going to deal with $\mathbb{N}$-graded Jordan superalgebras.

Definition 2.4.2. *Let $\Omega = \oplus_{\mathbb{N}} \Omega^n$ be an $\mathbb{N}-$graded algebra that is*

$$xy \in \Omega^{m+n}$$

for all $x \in \Omega^m$ and $y \in \Omega^n$. Suppose Ω is Jordan superalgebra with respect to the $\mathbb{Z}_2-$grading induced by the decomposition in even and odd components, then we say that Ω is $\mathbb{N}$-graded Jordan superalgebra and we shall denote respectively as Ω^+ and Ω^- its even and odd part respectively.

2.5 JB-algebras

In chapter 5 of this thesis we are going to make use of a finite dimensional Jordan algebra to describe the internal degrees of freedom of fundamental particles of the Standard Model.
The spirit of this approach is very similar to the one followed by Connes in the formalization of the Standard Model by mean of an almost commutative manifold (there is a vast litterature regarding the topic, see for example [10], [11], [14],[15],[21], [55]), meaning that the algebraic data of the Standard Model are going to be given by considering the space-time degrees of freedom as encoded by an algebra of functions on $\mathbb{R}^4$ and coupling it with the algebra of observables of a quantum internal space, that in our conception is going to be described by a Jordan algebra rather then a noncommutative associative $*$ algebra.

In order to provide a consistent description also for the infinite dimensional case, it is crucial for us to specify what is the analogue of C* algebras in the context of Jordan algebras, namely to say what kind of interplay we are going to require between the algebraic and the analytic structure for our observables.
The structure we are going to rely on has been introduced in 1978 by Alfsen, Schulz and Stormer ([2]), who proved an analogue of Gelfand-Neimark theorem for what they called Jordan-C*-algebras and are nowadays known as Jordan-Banach algebras or JB-algebras in short.

Definition 2.5.1. *Let $(J, \circ)$ be a real Jordan algebra and let the vector space J be a Banach space with respect to a norm $|| \cdot ||$ we say that J is a Jordan-Banach algebra or a real JB-algebra if the following hold:*

1. $|| x \circ y || \leq || x || \, || y ||$

2. $\| x^2 \| = \| x \|^2$

3. $\| x^2 \| \leq \| x^2 + y^2 \|$

for every $x, y \in J$.

For the finite dimensional case we have the following.

Theorem 2.5.2. *A finite dimensional Jordan algebra is a JB algebra if and only if it is formally real.*

It is always true that any JB algebra is formally real, while the converse is not necessarily true: there exist formally real infinite dimensional Jordan algebras which are complete in norm and satisfy properties 1 and 2 but do not satisfy property 3 and thus they are not JB algebras.

A complex version of JB algebras is given by the following

Definition 2.5.3. *Let* $(J, \circ)$ *be a complex Jordan algebra and let the vector space* J *be a Banach space with respect to a norm* $\| \cdot \|$ *we say that* J *is a complex JB* algebra if there exist an antilinear map* $* : J \to J$ *and the following hold:*

1. $\| x \circ y \| \leq \| x \| \| y \|$

2. $\| \{x, x^*, x\} \| = \| x \|^3$ *where* $\{x, y, z\} = x \circ (y \circ z) + (x \circ y) \circ z - y(x \circ z)$

3. $\| x^* \| = \| x \|$

The categories of real JB algebras and of complex JB* algebras are equivalent. In fact, for every complex JB* algebra its selfadjoint part is a real JB algebra, while it can be shown that if J is a real JB algebra, then its complexification with $*$ defined by

$$(x + iy)^* = x - iy$$

for every $x, y \in J$, admits a norm that makes it a JB* algebra.

Example 2.5.4. *Let* X *be a compact Haursdoff topological space, denote by* $C(X)$ *the algebra of complex valued continuous functions on* X *with pointwise conjugation and product. Equip* $C(X)$ *with the norm*

$$\| f \| = \sup_{x \in X} | f(x) |$$

for every $f \in C(X)$. *Then* $C(X)$ *is a complex JB* algebra.*

Example 2.5.5. *Let* J *be a finite dimensional Euclidean Jordan algebra, let* X *be a compact Haursdoff topological space, then* $C(X, J)$ *is a complex JB* algebra.*

There is an analogue of Gelfand theorem for complex JB* algebras (that is for complexifications of real JB algebras).

Theorem 2.5.6. *If a complex unital JB* algebra J is associative, then there exist a topological Haursdoff space X such that J is isomorphic to $C(X)$.*

While the analogue of Gelfand-Neimark theorem for JB algebras is enunciated as follows.

Theorem 2.5.7. *Any JB* algebra J admits an isometric embedding*

$$\pi : J \to B(\mathcal{H}) \oplus C(X, J_3^8)$$

where $B(\mathcal{H})$ is the algebra of bounded operators on a separable Hilbert space $\mathcal{H}$, and $C(X, J_3^8)$ denotes the algebra of continuous functions from some compact topological space X into the Albert algebra J_3^8.

Remark: The theorem above shows that not all the complex JB* algebras can be realized as Jordan algebras of self-adjoint operators on a Hilbert space and that the presence of some exceptional factor J_3^8 is the typical obstruction in doing this.

Chapter 3

Jordan modules

In the present chapter we are going to study the representation theory of Jordan algebras. This topic was initialy adressed by Jacobson ([40],[41], [42]) while more recently a classification program for Jordan bimodules has been carried on in [46] by making use of quiver techniques and in [47] by taking advantage of the Tits-Kantor-Koecher construction.
In the following, due to our interest in the theory of connections for Jordan bimodules, we are going to focus on the characterization of morphism between Jordan modules, where we consider as morphisms between modules the linear maps which commute with the action of the algebra. In particular we fully characterize the theory of morphisms between free Jordan modules.

In the first section we introduce the category of modules for any fixed category of algebras defined by relations. This constitutes a generalization of the widely known representation theory for associative algebras and can be found, formulated in the slightly different language of varieties of algebras, in chapter 2 of [42].

In the second section we specialize to the context of Jordan algebras and provide some concrete examples of Jordan bimodules.

In the third section we focus on the full subcategory of free modules over some fixed Jordan algebra J and we establish an isomorphism between this category and the category of free modules over the associative algebra $Z(J)$. The results presented in this chapter has been firstly published in [9].

In the following, if M and N are two modules over an algebra A we are going to denote by $Hom(M, N)$ (respectively $End(M)$) the vector space of linear transformations between the vector spaces M and N (respectively the algebra of linear endomorphisms of M), while $Hom_A(M, N)$ (respectively $End_A(M)$) will denote

the subspace of $Hom(M, N)$ of module homomorphisms between M and N (respectively the subalgebra of $End(M)$ of endomorphisms of M as A-module).

3.1 Modules in categories of algebras

The familiar definition of a module M over an associative algebra A, that is a linear map

$$\rho : A \to End(M)$$

where M is a vector space, such that

$$\rho(xy)m = \rho(x)(\rho(y)m) \tag{3.1}$$

is not suitable for nonassociative algebras such as Jordan algebras.
Indeed, it would be desirable to recover the notion of the regular action of an algebra on itself given by multiplication and it is evident that imposing condition (3.1) to this specific case would be equivalent to requiring associativity of the algebra.
A more suitable definition can be given by making use of categories of algebras ([24],[30],[46]. See [42] for the same definition in the context of varieties of algebras).

In the following we denote by **Alg** the category of all algebras over a fixed field with homomorphisms of algebras.
If R is a set of relations, we denote by **R-Alg** the full subcategory of **Alg** whose objects are algebras which respect relations in R.
Examples of categories of algebras defined by relations are:

- **Ass**, the category of associative algebras.
- **Lie**, the category of Lie algebras.
- **Jord**, the category of Jordan algebras.

Definition 3.1.1. *Let **R-Alg** be a category of algebras over a field $\mathbb{K}$. Let $A \in$ **R-Alg** and let M be a vector space over $\mathbb{K}$. Suppose there is a pair of maps*

$$A \otimes M \to M \quad x \otimes m \mapsto xm$$
$$M \otimes A \to M \quad m \otimes x \mapsto mx$$

We call split null extension of A by M the vector space $A \oplus M$ endowed with product

$$(x, m)(x', m') = (xx', xm' + m'x)$$

for every $x, x' \in A$ and $m, m' \in J$.
*If the split null extension of A by M is itself an object in **R-Alg** we say that M is an **R-Alg**-bimodule over A.*

In particular if the category of algebras happens to be **Ass**, **Lie** or **Jord** we shall refer to M as associative, Lie or Jordan bimodule.

Example 3.1.2. *If $A \in \boldsymbol{Ass}$, M is an associative bimodule for A if and only if the relations*

$$(xm)y = x(my), \quad (xy)m = x(ym), \quad m(xy) = (mx)y$$

hold for every $x, y \in A$ and $m \in M$.

Example 3.1.3. *Let $A \in \boldsymbol{Lie}$, we denote with square brackets $[\cdot, \cdot]$ both the multiplication in A and the action of elements in A on vectors in M, then M is a Lie bimodule for A if and only if the relations*

$$[x, m] = -[m, x], \quad [x, [y, m]] + [y, [m, x]] + [m, [x, y]] = 0$$

hold for every $x, y \in A$ and $m \in M$.

If M is a bimodule for an algebra A, the following

$$\begin{aligned} L : A \to End(M) \qquad & x \mapsto (m \mapsto xm) \\ R : A \to End(M) \qquad & x \mapsto (m \mapsto mx) \end{aligned}$$

define a pair of linear maps on M.
The fact that M is an **R-Alg**-bimodule for A can be equivalently reformulated by listing some properties of the maps L, R, as in the following examples

Example 3.1.4. *If $A \in \boldsymbol{Ass}$, then it is easily seen that a pair of map $R, L : A \to End(M)$ endow M with the structure of associative bimodule if and only if*

$$[L_x, R_y] = 0, \quad [L_{xy}, L_x L_y] = 0, \quad [R_{xy}, R_y R_x] = 0$$

for any $x, y \in A$.

Example 3.1.5. *If $A \in \boldsymbol{Lie}$, then a pair of maps (L, R) as before define a Lie module if and only if*

$$L_x = -R_x \quad L_{[x,y]} = [L_x, L_y]$$

for every $x, y \in A$.

Remark: The characterization of a module for any category of algebras in terms of the pair of maps (L, R) can be found in the second chapter of [42]. We are not going to discuss this topic in full generality, since this would go beyond the aim of this thesis, however in the next section we shall present such characterization for the category of bimodules over Jordan algebras.

3.2 Jordan bimodules

Let us specialize the definition of module given in the previous section to the context of Jordan algebras obtaining the following

Definition 3.2.1. *Let J be a Jordan algebra, a Jordan bimodule over J is a vector space M together with two bilinear maps*

$$\begin{aligned} J \otimes M \to M \quad & x \otimes m \mapsto xm \\ M \otimes J \to M \quad & m \otimes x \mapsto mx \end{aligned}$$

such that $J \oplus M$, endowed with the product

$$(x, m)(x', m') = (xx', xm' + mx')$$

is a Jordan algebra.

This definition is equivalent to require the following properties for the product in the split null extension $J \oplus M$

$$\begin{aligned} & mx = xm \\ & x(x^2 m) = x^2(mx) \\ & (x^2 y)m - x^2(ym) = 2((xy)(xm) - x(yxm)) \\ & \mathbb{1}_J m = m \end{aligned} \tag{3.2}$$

for any $x, y \in J$ and $m \in M$.

Notice that from the first of relations above we have

$$L_x = R_x$$

for any $x \in J$ and for any bimodule over J.
Thus in the representation theory of Jordan algebras every left module has to be also a right module with respect to the same action of the algebra and for this reason when dealing with Jordan algebras we are going to use the words "bimodule" and "module" as mutually interchangeable.

The second and third relations in (3.2) are two distinct presentations of the (semi-linearized) Jordan rule in the split null extension of J by M. They can alternatively be written as

$$[L_x, L_{x^2}] = 0$$

and

$$L_{x^2 y} - L_{x^2} L_y - 2L_{xy} L_x + 2L_x L_y L_x = 0$$

which is equivalently written as the two conditions

$$L_{x^3} - 3L_{x^2}L_x + 2L_x^3 = 0$$
$$[[L_x, L_y], L_z] + L_{[x,y,z]} = 0$$

for every $x, y, z \in J$.

For special Jordan algebras the following holds (see chapter 2 of [42]).

Theorem 3.2.2. *Let J be a Jordan algebra and M a Jordan bimodule for J, then J is special if and only if the split null extension $J \oplus M$ is special.*

Now we are going to list some examples of Jordan modules over the simple finite dimensional Jordan algebras we have introduced so far.

Example 3.2.3. *It follows from Definition 3.2.1 that any Jordan algebra J is a module over itself, with action given by multiplication. We are going to refer to this as regular representation or, equivalently, by saying that J is a free module over itself. More generally, let J be a finite dimensional Jordan algebra, a free $J-$module M is of the form*

$$M = J \otimes E$$

where E is a finite dimensional vector space and the action of J on M is given by multiplication on the first component of M. As we shall point out later, it turns out that, when J is the Albert algebra, any finite module over J is a free module [42].

Example 3.2.4. *Let A be an associative algebra, let $J \subseteq A^+$ be a special Jordan algebra as in Example 2.2.8. Any element $x \in J$ is also an element of A and A is endowed with J-module structure by setting*

$$L_x a = x \circ a = \frac{1}{2}(ax + xa)$$

for any $x \in J$ and $a \in A$. In the two following examples the same construction is explicitely given for the antihermitian real, complex and quaternionic matrices as a module over hermitian matrices and for the Clifford algebras $Cl(\mathbb{R}^n)$ as modules over the spin factors $JSpin_n$.

Example 3.2.5. *Denote by $A_n^i (i = 1, 2, 4)$ the vector space of antihermitian matrices with real, complex and quaternionic entries respectively. A_n^i is a module over the special Jordan algebra J_n^i with action given by the matrix anticommutator:*

$$L_x a = x \circ a = \frac{1}{2}(ax + xa)$$

for any $x \in J_n^i$ *and* $a \in A_n^i$*. Moreover, taking* J_n^i *as a free module over itself we have:*

$$J_n^i \oplus A_n^i = M_n^i$$

which is the $J-$*module of* $n \times n$ *real, complex or quaternionic matrices with action of* J *defined as above by* (3.2.5)*. In particular for the complex case, we have*

$$M_n^2 = J_n^2 \oplus iJ_n^2$$

where i *is the imaginary unit. Since the anticommutator of a hermitian matrix with an antihermitian matrix is again antihermitian we find that* M_n^2 *is a free Jordan module over* J_n^2*.*

Example 3.2.6. *The Clifford algebra*

$$Cl\left(\mathbb{R}^n\right) = \frac{T\left(\mathbb{R}^n\right)}{\left(\{x \otimes x = ||x||^2,\ \forall x \in \mathbb{R}^n\}\right)}$$

is a module over the Jordan algebra $JSpin_n = \mathbb{R} \oplus \mathbb{R}^n$ *with action given by*

$$L_x[y] = \frac{1}{2}\left([x \otimes y] + [y \otimes x]\right)$$

for any $x \in \mathbb{R}^n$ *and* $[y] \in Cl(\mathbb{R}^n)$*.*

If J is a Jordan algebra, and $\rho : J \to End(M)$ is a multiplicative specialization of J into the linear endomorphisms for some vector space M, then it is clear that M is Jordan module for J.
In particular, in the following definition let J be a unital Jordan algebra with unit 1_J, we denote its image into the free tensor algebra $T(J)$ by the same symbol; recall the universal multiplicative enveloping $S_{mul}(J)$ defined in Section 2.3 and denote by $\mathbb{1}_S$ the unit of $S_{mul}(J)$.

Definition 3.2.7. *The quotient of* $S_{mul}(J)$ *by the ideal generated by* $1_J - \mathbb{1}_S$ *is called the universal unital multiplicative enveloping of* J *and we denote it by* $S^1_{mul}(J)$*.*

The following theorem holds (see e.g. [46]).

Theorem 3.2.8. *The category of unital Jordan bimodules over* J *with linear maps is isomorphic to the category of left associative bimodules over* $S^1_{mul}(J)$ *with linear maps.*

Finally, for what concerns the modules over the finite dimensional Euclidean simple Jordan algebras of the series J_n^i with $i = 1, 2, 4$ (also 8 for $n \leq 3$) the following construction leads to the definition of a Jordan module over J.
Let $(A, *)$ be a unital associative algebra with involution, denote by $J_n(A)$ the Jordan algebra of n by n hermitian matrices with entries in A.

Definition 3.2.9. *An associative bimodule V for A is called a bimodule with involution if there exists linear map $* \in End(V)$ such that*

$$(v^*)^* = v$$
$$(xv)^* = v^* x^*$$

for any $x \in A$ and $v \in V$.

In the following we are going to denote with same symbol $*$ both the involution on the associative algebra A and the involution on a bimodule with involution.
If V is a bimodule with involution for A, the split null extension $W = A \oplus V$ is endowed in a natural way with the structure of associative $*$ algebra.
We let $M_n(W)$ be the associative $*$ algebra of $n \times n$ matrices with entries in W and we endow it with the hermitian conjugation, that is the composition of the matrix transpose with component-wise application of $*$, then the subspace of self-adjoint elements in $M_n(W)$ with the anticommutator as product is a special Jordan algebra that we denote by $J_n(W)$.
The Jordan algebra $J_n(W)$ contains $M = M_n(V) \cap J_n(V)$ that is the subspace whose elements are hermitian $n \times n$ matrices with entries in $(W, *)$ taking values in $(V, *)$, as ideal and then $J_n(V)$ is a Jordan bimodule over the Jordan algebra $J_n(A)$ whose action is given by multiplication inside $J_n(W)$.
This construction for $n \geq 4$ has been used ([46]) to classify of all Jordan bimodules over J_n^i for $i = 1, 2, 4$. the correspondence between the representation theory of the associative $*$ algebra $(V, *)$ and the Jordan algebra $J_n(V)$ is functorial and for $n \geq 3$ it provides an isomorphism between the categories of associative $*-$bimodules for A and the category of Jordan bimodules for $J_n(A)$.
For $n = 3$ the same result can be slightly extended by allowing A to be an alternate $*$ algebra, thus taking in account also the case of modules over the exceptional simple Jordan algebra J_3^8.

Example 3.2.10. *The Example 3.2.5 can be reconsidered in the light of the construction provided above. In fact, taking as A one of the associative division algebras one can consider the regular associative module $V = A$ equipped with $*_V = -*_A,$. Then the construction presented above leads to the space A_n^i of anti-hermitian matrices with values in A as Jordan bimodule over the Jordan algebra J_n^i.*

Finally, let us introduce the notion of morphism between bimodules over a fixed Jordan algebra as follows

Definition 3.2.11. *Let J be a Jordan algebra, let M and N be two modules over J, then a module homomorphism between M and N is a linear map $\varphi : M \to N$ such that*

$$x\varphi(m) = \varphi(xm)$$

for all $m \in M$ and $x \in J$.

In the next sections, we are going to assume that the category of Jordan bimodule over a fixed Jordan algebra J is endowed with module homomorphism defined above as morphisms. In the following section we will study more deeply homomorphisms between free Jordan modules.

3.3 Morphisms between free Jordan modules

In few of the following sections we are going to simplify the notation by dropping the symbol $\circ$ to denote the product in a Jordan algebra and the action of a Jordan algebra on a bimodule and we are going to simply use juxtaposition to denote both.

The following characterization for morphism between free Jordan modules over a fixed Jordan algebra holds.

Theorem 3.3.1. *Let J be a unital and simple Euclidean Jordan algebra, let $M = J \otimes E$ and $N = J \otimes F$, where E and F are two finite dimensional vector spaces, be free modules over J. Then every module homomorphism $\varphi : M \to N$ is of the form*

$$\varphi(x \otimes v) = x \otimes Av \quad x \in J, v \in E$$

where $A : E \to F$ is a linear map.

Proof. For sake of simplicity, start by taking $M = N = J$, then a module homomorphism is a linear map $\varphi : J \to J$ such that:

$$x\varphi(y) = \varphi(xy)$$

for any $x, y \in J$. In particular:

$$\varphi(x) = x\varphi(1) = xA$$

for some $A \in J$, such that $A = \varphi(1)$.
Now, from definition of module homomorphism, we have:

$$\varphi(xy) = (xy)A = x\phi(y) = x\,(yA) \Rightarrow [x, y, A] = 0$$

for all $x, y \in J$, hence $A \in Z(J)$ that is $A \in \mathbb{R}$, since we have taken J simple and Euclidean.

More generally let $M = J \otimes E$ and $N = J \otimes F$, denote by e_α and f_α a basis of E and F respectively. We have

$$\varphi(1 \otimes e_\alpha) = A_\alpha^\lambda \otimes f_\lambda$$

for some $A_\alpha^\lambda \in J$. With the same argument as above, we get:

$$\varphi(xy \otimes e_\alpha) = (xy)\varphi(1 \otimes e_\alpha) = (xy)A_\alpha^\lambda \otimes f_\lambda$$
$$\varphi(xy \otimes e_\alpha) = (x\varphi(y \otimes e_\alpha)) = x(y\varphi(1 \otimes e_\alpha)) = x(yA_\alpha^\lambda) \otimes f_\lambda$$

and so every $A_\alpha^\lambda \in Z(J)$ and it is a real number. From the definition of tensor product of vector spaces one has

$$A_\alpha^\lambda \otimes f_\lambda = 1 \otimes A_\alpha^\lambda f_\lambda$$

and the statement follows by taking as map A from E into F the linear transformation defined by $A(e_\alpha) = A_\alpha^\lambda f_\lambda$. □

Remark: In the theorem above the only role of the formal reality for J is to simplify the statement since in this case every element in the center of J is a real number and hence we are allowed to write equation (3.3.1). However the same kind of result holds true if one drops the formal reality condition as we are going to do in the following.

If one considers a finite dimensional Jordan algebra J, not necessarily simple, the above theorem is generalized as follows.

Lemma 3.3.2. *Let J be a finite dimensional unital Jordan algebra, let $M = J \otimes E$ and $N = J \otimes F$ be free modules over J, with E, F finite dimensional vector space of dimension m and n respectively. Then if $f : M \to N$ is homomorphism of J modules, there exist $\alpha_k \in Z(J)$ and $f_k \in M_{m\times n}$ such that:*

$$f(1 \otimes e) = \sum_k \alpha_k \otimes f_k(e) \tag{3.3}$$

for any $e \in E$.

Remark: The proof of the lemma above does not really rely on the finite dimension of the Jordan algebra. In fact, the same proof can be followed for the infinite dimensional cases paying attention to require convergence of the series in (3.3).

The results above can be summarized in the following way by taking an equivalence of categories.

Theorem 3.3.3. *Let J be a finite dimensional unital Jordan algebra with center $Z(J)$. Denote by $FMod_J$ the category of free Jordan modules over J with homomorphisms of Jordan modules and as $FMod_{Z(J)}$ the category of free modules over the associative algebra $Z(J)$ with homomorphisms of modules over associative algebras. Then the following functor is an isomorphism of categories:*

$$\begin{aligned} \mathcal{F}: \quad & J \otimes E \mapsto Z(J) \otimes E \\ & (\varphi : J \otimes E \to J \otimes F) \mapsto (\varphi_{Z(J)} : Z(J) \otimes E \to Z(J) \otimes F) \end{aligned} \tag{3.4}$$

where $\varphi_{Z(J)}$ is the restriction of φ to $Z(J) \otimes E$.

Proof. We begin by checking functoriality of $\mathcal{F}$.
Obviously the image of the identity of $FMod_J$ is the identity of $FMod_{Z(J)}$.
Let $\varphi : J \otimes E \to J \otimes F$ and $\phi : J \otimes F \to J \otimes H$ be two homomorphisms of free modules over J, then we have:

$$\mathcal{F}(\phi \circ \varphi) = (\phi \circ \varphi)_{Z(J)}$$

From Theorem 3.3.1 we know that $\varphi(Z(J) \otimes E) \subseteq Z(J) \otimes F$, and so:

$$\mathcal{F}(\phi \circ \varphi) = (\phi \circ \varphi)_{Z(J)} = \phi_{Z(J)} \circ \varphi_{Z(J)} = \mathcal{F}(\phi) \circ \mathcal{F}(\varphi)$$

which proves that $\mathcal{F}$ is a functor. Define $\mathcal{F}^{-1}$ as:

$$\begin{aligned} \mathcal{F}^{-1}: \quad & Z(J) \otimes E \to J \otimes E \\ & (\varphi : Z(J) \otimes E \to Z(J) \otimes F) \mapsto (\varphi_J : J \otimes E \to J \otimes F) \end{aligned}$$

where φ_J is defined by regarding the elements of $Z(J)$ as elements of J and setting:

$$\varphi_J(x \otimes e) := x\varphi(1 \otimes e)$$

for any $x \in J$ and $e \in E$. □

Chapter 4

Differential calculi for Jordan algebras

In this chapter we are going to review the general theory for differential calculi over Jordan algebras.
We focus on the derivation-based differential calculus, which serves both as prototype for all differential calculi over a fixed Jordan algebra and as base for the theory of derivation-based connections over Jordan modules that is going to play a main role in the physical applications to the Standard Model in chapter 5.

In the second section we relate the cohomology for finite dimensional simple Euclidean Jordan algebras to the cohomology of their respective simple compact Lie algebras of derivations.

Finally in the last two paragraph we show that the derivation-based differential calculus for the simple Euclidean special Jordan algebras is associative up to homotopy of differential chain complexes.

4.1 Differential calculi

The following definition holds

Definition 4.1.1. *A differential graded Jordan algebra is an* $\mathbb{N}$ *graded Jordan superalgebra* Ω *equipped with a differential, which is a an odd derivation* d *of degree* 1 *and with square zero, that is one has*

$$d\Omega^n \subset \Omega^{n+1}$$
$$d^2 = 0$$

and

$$d(xy) = (dx)y + (-1)^{|x|}xd(y)$$

for all $x, y \in \Omega$.

In the well-established spirit of quantum geometry according to which Jordan algebras will provide a dual description of some quantum space, such differential graded Jordan algebras are going to be our models for generalizing differential forms.
In particular when $\Omega^0 = J$ we say that (Ω, d) is a differential calculus over the Jordan algebra J (this terminology is inspired by [56]).

A prototypical model of differential calculus over a Jordan algebra is the derivation-based differential calculus which has been introduced in [24] (differential calculi for noncommutative algebras was introduced in [22] and it generalizes differential forms as defined in [49]).
Its construction starts with the definition of the first order derivation-based differential calculus, which provides the generalizations of one forms.

Definition 4.1.2. *Let* $\Omega^1_{Der}(J)$ *be the* $J-$*module of* $Z(J)-$*homomorphisms from* $Der(J)$ *into* J. *We define a derivation* $d_{Der} : J \to \Omega^1_{Der}(J)$ *by setting:*

$$(d_{Der}x)\,(X) := X(x) \tag{4.1}$$

for any $x \in J$ *and* $X \in Der(J)$. *We refer to the pair* $(\Omega^1_{Der}(J), d_{Der})$ *as the derivation-based first order differential calculus over* J.

Then we extend d as follows, which is reminiscent of how $n-$forms are defined in differential geometry.

Definition 4.1.3. *Let* $\Omega^n_{Der}(J)$ *be the* J *module of antisymmetric maps* $n-$*linear over* $Z(J)$ *of* $Der(J)$ *into* J, *that is any* $\omega \in \Omega^n_{Der}(J)$ *is a* $Z(J)$ *linear map* ω *from* $\wedge^n_{Z(J)} Der(J)$ *into* J.
Then $\Omega_{Der}(J) = \oplus_{n\geq 0}\Omega^n_{Der}(J)$, *is an* $\mathbb{N}-$*graded Jordan superalgebra with respect to wedge product of linear maps.*
Extend d_{Der} *to a linear endomorphism of* $\Omega_{Der}(J)$ *by making use of the Koszul formula*

$$(d_{Der}\omega)(X_0, ..., X_n) = \sum_{0\leq k\leq n} (-1)^k X_k \left(\omega\left(X_0, ..., \widehat{X_k}, ...X_n\right)\right) +$$
$$+ \sum_{0\leq r<s\leq n} (-1)^{r+s}\omega\left([X_r, X_s], X_0, ..., \widehat{X_r}, ..., \widehat{X_s}, ...X_n\right)$$

for any $\omega \in \Omega_n(J)$, where the symbol $\widehat{X_i}$ means that X_i is omitted.
This extension of d_{Der} is an odd derivation and $d_{Der}^2 = 0$. Thus $\Omega_{Der}(J)$ endowed with d_{Der} is a differential graded Jordan superalgebra with $\Omega_0 = J$.
We refer to the couple $(\Omega_{Der}(J), d_{Der})$ as the derivation-based differential calculus over J.

Clearly, one can restrict the derivation-based differential calculus over J by considering any Lie subalgebra of $Der(J)$ which is also a $Z(J)-$submodule as follows.

Definition 4.1.4. *Let J be a Jordan algebra, $\mathfrak{g} \subset Der(J)$ be a Lie subalgebra and a $Z(J)-$submodule. With the notations above, the restricted derivation based differential calculus $\Omega_{\mathfrak{g}}(J)$ is defined as the superalgebra of antisymmetric $Z(J)-$linear maps from $\mathfrak{g}$ to J with the differential given by the restriction of the differential on $\Omega_{der}(J)$.*

In the applications that will follow, we are going to refer as derivation-based differential calculus (or when, there will be no possibility of misunderstanding just as differential calculus) both to the full derivation-based differential calculus and to any of its restrictions.

Finally, when a Jordan algebra J is fixed, the following definition is straightforward.

Definition 4.1.5. *Let J be a Jordan algebra, we define the category $\boldsymbol{DCal}_J$ of differential calculi over J as follows:*

- *Objects of $\boldsymbol{DCal}_J$ are differential calculi over J, (Ω, d).*
- *If (Ω_1, d_1) and (Ω_2, d_2) are differential calculi over J, a morphism $\Phi : (\Omega_1, d_1) \to (\Omega_2, d_2)$ in $\boldsymbol{DCal}_J$ is given by a set $\{\phi_i : \Omega_1^i \to \Omega_2^i\}_{i\in\mathbb{N}}$ of $J-$bimodule maps of degree 0 such that*

$$d_2 \circ \phi_i = \phi_{i+1} \circ d_1$$

for every degree $i \in \mathbb{N}$.

4.1.1 Derivation-based differential calculus for J_3^8

In general, the derivation-based differential calculus does not play any privileged role in the theory of differential calculus over a given Jordan algebra.
Howewer in the case of exceptional Jordan algebra J_3^8, the derivation-based differential calculus is characterized up to isomorphism by the following universal property ([24], see [9] for the proof).

Theorem 4.1.6. *Let (Ω, d) be a differential graded Jordan algebra and let $\phi : J_3^8 \to \Omega^0$ be a homomorphism of unital Jordan algebras. Then ϕ has a unique extension $\tilde{\phi} : \Omega_{Der}(J_3^8) \to \Omega$ as homomorphism of differential graded Jordan algebras.*

Remark. Notice that this theorem can be restated by saying that $\Omega_{Der}(J_3^8)$ is an initial object (unique up to isomorphims) in the category $\mathbf{DCal}_{J_3^8}$.

In order to prove this theorem we need to prove the following lemma before.

Lemma 4.1.7. *Let Γ be a Jordan superalgebra, then $J_3^8 \otimes \Gamma = \oplus_{n\in\mathbb{N}} J_3^8 \otimes \Gamma^n$ is a Jordan superalgebra if and only if Γ is an associative superalgebra.*

The proof of this lemma will use the following results proved in [60] (Lemma 2 and Lemma 3 in [60]).

Lemma 4.1.8. *Let $\Gamma = \Gamma^+ \oplus \Gamma^-$ be a unital Jordan superalgebra whose even component Γ^+ is associative, then either one of the two equalities*

$$[\Gamma^-, \Gamma^+, \Gamma^+] = 0 \tag{4.2}$$

or

$$[\Gamma^-, \Gamma^+, \Gamma^+] = \Gamma^- \tag{4.3}$$

holds.

Lemma 4.1.9. *Let Γ be as above and such that $[\Gamma^-, \Gamma^+, \Gamma^+] = 0$, then one of the two equalities*

$$[\Gamma^+, \Gamma^-, \Gamma^-] = 0 \tag{4.4}$$

or

$$[\Gamma^+, \Gamma^-, \Gamma^-] = \Gamma^+ \tag{4.5}$$

holds.

Proof of Lemma 4.1.7. Let $\xi = \sum_i a_i \otimes b_i$ and $\eta = x \otimes y$ be elements in $J_3^8 \otimes \Gamma$ we have to find whenever $[\xi^2, \eta, \xi] = 0$, that is

$$\begin{aligned}
&\left[\left(\sum_i a_i \otimes b_i\right)^2, x \otimes y, \sum_j a_j \otimes b_j\right] = \\
&\left[\sum_{i,j<i} a_i a_j \otimes b_i b_j + (-1)^{|b_i||b_j|} a_i a_j \otimes b_i b_j, x \otimes y, \sum_k a_k \otimes b_k\right] + \\
&+ \left[\sum_i a_i^2 \otimes b_i^2, x \otimes y, \sum_{k\neq i} a_k \otimes b_k\right] = 0
\end{aligned} \tag{4.6}$$

$\Gamma = \Gamma^+ \oplus \Gamma^-$ is a Jordan superalgebra and in particular Γ^+ is a graded subalgebra of Γ and one knows ([57]) that the algebra $J_3^8 \otimes \Gamma^+$ is a Jordan graded algebra if and only if Γ^+ is associative. We must then assume Γ^+ associative. In expression (4.6) let us take

$$\begin{aligned} \xi &= a_{-1} \otimes 1 + a_0 \otimes e + \sum_i a_i \otimes o_i, \\ \xi^2 &= a_{-1}^2 \otimes 1 + a_0^2 \otimes e^2 + a_{-1} \otimes e + \\ &\quad + 2 \sum_i a_{-1} a_i \otimes o_i + 2 \sum_i a_o a_i \otimes \tilde{o}_i, \\ \eta &= x_0 \otimes y_e + x_1 \otimes y_o \end{aligned} \tag{4.7}$$

where a_i and $x_i \in J_3^8$, e and $y_e \in \Gamma^+$, o_i and $y_o \in \Gamma^-$ and finally we set $\tilde{o}_i = eo_i \in \Gamma^-$. Then one has

$$\begin{aligned} \left[\xi^2, \eta, \xi\right] &= \left[a_0^2 \otimes e^2, x \otimes y, a_j \otimes o_j\right] + \left[a_{-1} a_0 \otimes e, x \otimes y, a_0 \otimes e + a_j \otimes o_j\right] + \\ &+ \left[a_{-1} a_i \otimes o_i, x \otimes y, a_0 \otimes e\right] + \left[a_0 a_i \otimes \tilde{o}_i, x \otimes y, a_0 \otimes e\right] + \\ &+ \left[a_0 a_i \otimes \tilde{o}_i, x \otimes y, a_0 a_j \otimes \tilde{o}_j\right] = 0 \end{aligned} \tag{4.8}$$

for all $a_i \in J_3^8$. We can choose elements a_i's and x in J_3^8 whose associator is different from zero, so that the above equation leads to the following condition on elements of Γ

$$\begin{aligned} &\left[e^2, y_e, o_j\right] + \left[e^2, y_o, o_j\right] + \left[e, y_o, e\right] + \left[e, y_e, o_j\right] + \left[e, y_o, o_j\right] + \left[o_i, y_e, e\right] + \\ &+ \left[o_i, y_o, e\right] + \left[\tilde{o}_i, y_e, e\right] + \left[\tilde{o}_i, y_o, e\right] + \left[\tilde{o}_i, y_e, \tilde{o}_j\right] + \left[\tilde{o}_i, y_o, \tilde{o}_j\right] = 0 \end{aligned} \tag{4.9}$$

thus, varying elements in Γ, we see that condition above implies

$$\left[\Gamma^-, \Gamma^+, \Gamma^+\right] + \left[\Gamma^-, \Gamma^-, \Gamma^+\right] + \left[\Gamma^-, \Gamma^-, \Gamma^-\right] = 0 \tag{4.10}$$

then, combining Lemma 4.1.8 with Lemma 4.1.9, we see that the equality above can hold only if all the summands above are identically zero, hence $\Gamma = \Gamma^+ \oplus \Gamma^-$ must be an associative superalgebra. □

The proof of Theorem 4.1.6 proceeds as in proposition 4 of [24], as we shall recall for sake of completeness.

Proof of Theorem 4.1.6. For all $n \in \mathbb{N}$, Ω^n is a Jordan module over J_3^8 and from general theory of J_3^8 modules we know that every module over J_3^8 is a free module, hence we have

$$\Omega^n = J_3^8 \otimes \Gamma^n \tag{4.11}$$

where Γ^n is a vector space. Any differential graded Jordan superalgebra over J_3^8 is then written as

$$\Omega = \oplus_{n\in\mathbb{N}} J_3^8 \otimes \Gamma^n = J_3^8 \otimes \Gamma$$

where $\Gamma = \oplus_{n\in\mathbb{N}}\Gamma^n$ is a Jordan superalgebra. Consider the J_3^8-module $\Omega^1 = J_3^8 \otimes \Gamma^1$, and let $\{e^\alpha\} \subset \Gamma^1$ be a basis of Γ^1. Let $\{\partial_k\}$ be a basis of $Der\,(J_3^8)$ with dual basis $\{\theta_k\}$ such that $\theta_k\,(\partial_j) = \delta_{kj}$. We have

$$dx = \partial_k x \otimes c_\alpha^k e^\alpha \tag{4.12}$$

for all $x \in J$ and for some real constants c_α^k's. Define the linear map $\tilde{\phi}$ from Ω^1_{Der} into Ω^1 by

$$\tilde{\phi}\left(x \otimes \theta^k\right) = x \otimes c_\alpha^k e^\alpha. \tag{4.13}$$

and extend it as homomorphism of superalgebras. We have $\tilde{\phi} \circ d_{Der} = d \circ \tilde{\phi}$, and uniqueness of $\tilde{\phi}$ follows from $d^2 = 0$ and the Leibniz rule. □

We stress that this statement holds true only for the exceptional Jordan algebra and it is a direct consequence of the fact that only irreducible module over J_3^8 is J_3^8 itself. In fact, this theorem holds true for any Jordan algebra J if one constrains the considered differential calculi to be built on a superalgebra of free modules over the Jordan algebra J.
Remark: If J is a unital special Jordan algebra, it can be embedded as (sub)space of self-adjoint elements of a unital associative algebra A. It is a well known fact that for any unital associative algebra A the universal differential calculus is generated in degree one by

$$\Omega_1(A) = ker\,(m : A \otimes A \to A)$$

where m denotes the multiplication in A, with differential given by

$$da := 1 \otimes a - a \otimes 1$$

for every $a \in A$. The universal differential calculus of A constitutes a differential calculus for the Jordan algebra J that cannot be obtained as a quotient of the derivation-based differential calculus $\Omega_{Der}(J)$.

4.2 Cohomology

As it is done for complexes over associative algebra, we define the cohomology of a differential calculus (Ω, d) over J.

Definition 4.2.1. *Let (Ω, d) be a differential calculus over a Jordan algebra J, let $Z^i(\Omega) = Ker\,\{d : \Omega^i \to \Omega^{i+1}\}$ be the $i-th$ subspace of cocycles in Ω and*

$B^i(\Omega) = Im\{d : \Omega^{i-1} \to \Omega^i\}$ be the $i-th$ subspace of coboundaries in Ω, then we define the $i-th$ cohomology group as

$$H^i(\Omega) = \frac{Z^i(\Omega)}{B^i(\Omega)}$$

and we call $H(\Omega) = \oplus_{i\in\mathbb{N}} H^i(\Omega)$ the cohomology of the differential calculus Ω.

In order to state few useful results about the cohomologies of the derivation-based differential calculus for finite dimensional Euclidean Jordan algebras, we need some definitions and results from the cohomological theory for Lie algebras, in particular of the Chavelley-Eilenberg complexes for a Lie algebra (see e.g. [**?**]).

In the following, let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbb{K}$, let M be a module for $\mathfrak{g}$.

Definition 4.2.2. *The cochains from $\mathfrak{g}$ with values in M are elements of Chavelley-Eilenberg complex $Hom_{\mathbb{K}}(\wedge^\bullet \mathfrak{g}, M)$.*

Recall that $Hom_{\mathbb{K}}(\wedge^\bullet \mathfrak{g}, M)$ is canonically isomorphic to $M \otimes_{\mathbb{K}} \mathfrak{g}^*$. We are not going to cover the general case of the cohomological theory for any module over $\mathfrak{g}$, rather we focus on the case $M = \mathbb{K}$ with the trivial action of $\mathfrak{g}$, which is going to be relevant for us.
Consider the Lie bracket $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$, its transpose map $d_{\mathfrak{g}} : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ is defined by

$$(d_{\mathfrak{g}} f)(x_1, x_2) = f([x_1, x_2]) \tag{4.14}$$

for every $x_1, x_2 \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$. Then $d_{\mathfrak{g}}$ is extend to a differential on the cochains from $\mathfrak{g}$ with values in $\mathbb{K}$ by using the Koszul formula

$$\begin{aligned}(d_{\mathfrak{g}} f)(x_0, ..., x_n) = &\sum_{0\le k\le n} (-1)^k x_k f(x_0, ..., \widehat{x_k}, ...x_n) + \\ &+ \sum_{0\le r<s\le n} (-1)^{r+s} f([x_r, x_s], x_0, ..., \widehat{x_r}, ..., \widehat{x_s}, ...x_n)\end{aligned}$$

for every $f \in \wedge^n\mathfrak{g}^*$.

Definition 4.2.3. *For $\mathbb{K} = \mathbb{R}$ we refer to the cohomology of the Chavelley-Eilenberg complex $Hom_{\mathbb{R}}(\wedge^\bullet \mathfrak{g}, \mathbb{R})$ as the cohomology of the Lie algebra $\mathfrak{g}$.*

Now let J be a Jordan algebra, recall that $Der(J)$ is a Lie algebra, call it $\mathfrak{g}$ and consider the derivation-based differential calculus over J. By direct check, one has

$$\begin{aligned}d_{Der}(x \otimes f)(X_0, ..., X_k) &= [(d_{Der}x)(1 \otimes f) + x d_{Der}(1 \otimes f)](X_0, ..., X_k) \\ &= [(d_{Der}x)(1 \otimes f) + x(1 \otimes d_{\mathfrak{g}} f)](X_0, ..., X_k)\end{aligned}$$

for every $x \in J$ and $f \in \wedge^k \mathfrak{g}^*$.

By direct computation one can check that the cohomology of the derivation-based differential calculus for all the finite dimensional Euclidean simple Jordan algebras coincide with the cohomology of the corresponding simple Lie algebras of their derivations.
In what follows we are going to use this result to prove that the derivation-based differential calculus for finite dimensional Euclidean special Jordan algebras is in fact associative up to homotopy. In order to do this we have to recall some definitions in the context of homological algebra, such as the concept of associativity up to homotopy.

4.3 Associativity up to homotopy

Let us recall the definition of homotopy for maps between two differential chain complexes (see e.g. chapter IV of [36] for more details).

Definition 4.3.1. *Let*

$$\mathbf{A} = ... \to A_0 \to A_1 \to A_2 \to ...$$

and

$$\mathbf{B} = ... \to B_0 \to B_1 \to B_2 \to ...$$

be two differential chain complexes. For sake of simplicity we denote by the same symbol d both the differential on $\mathbf{A}$ and $\mathbf{B}$. Let $\phi, \psi : \mathbf{A} \to \mathbf{B}$ be two chain maps of degree 0, that is $\phi(A_n) \subseteq B_n$ for every $n \in \mathbb{N}$, and similarly for ψ. We say that the ϕ and ψ are homotopic if it exists a map K of degree -1 such that

$$Kd + dK = \phi - \psi. \tag{4.15}$$

Definition 4.3.2. *If a map $\psi : \mathbf{A} \to \mathbf{B}$ is homotopic to the zero map, we say that ψ is contractible.*

Let A be a differential graded algebra, let $A^{\otimes n}$ denote the n-th power of the tensor product of A with itself. Then $A^{\otimes n}$ has a structure of graded module over A given as follows: let $a_1, ..., a_n$ be elements of homogeneous degree in A, then the elements of homogeneous degree in $A^{\otimes n}$ are tensor products of homogeneous elements in A, $a_1 \otimes ... \otimes a_n$ and their degree is given by the sum of degrees of their factors:

$$\mid a_1 \otimes ... \otimes a_n \mid = \mid a_1 \mid + ... + \mid a_n \mid \tag{4.16}$$

where we denote the degree of $a_i \in A$ as $\mid a_i \mid$.
The differential $d_{A^{\otimes n}} : A^{\otimes n} \to A^{\otimes n}$ is given by lifting the differential on A as follows

$$d_{A^{\otimes n}} = \sum_{0 \leq s \leq n} 1^{\otimes s} \otimes d \otimes 1^{\otimes n-s} \tag{4.17}$$

or more explicitly

$$d_{A^{\otimes n}}(a_1 \otimes ... \otimes a_n) = da_1 \otimes ... \otimes a_n + ... + a_1 \otimes ... \otimes da_n \tag{4.18}$$

for all $a_1 \otimes ... \otimes a_n \in A^{\otimes n}$. In the following we shall suppress the subscript $A^{\otimes n}$ and will just write d when no confusion arises.

The multiplication map $m_2 : A \otimes A \to A$ has degree 0, this justify the following definition.

Definition 4.3.3. *Let A be a graded algebra, if the associator $[\cdot, \cdot, \cdot] : A^{\otimes 3} \to A$ defined as*

$$[\cdot, \cdot, \cdot] := m_2(1 \otimes m_2) - m_2(m_2 \otimes 1) \tag{4.19}$$

is contractible, we say that A is associative up to homotopy.

In the following section, we are going to show how the derivation-based differential calculus for simple Euclidean special Jordan algebras falls into this definition.

4.4 Associativity up to homotopy for Euclidean Jordan algebras

In this section we simplify the notation by writing $(\Omega(J), d)$ to denote the derivation-based differential calculus over a Jordan algebra J.
The homotopy map $K : \Omega(J) \to \Omega(J)$ that we are going to build has degree -1 and, in order to define it, it is useful to study Hodge theory of $\Omega(J)$.
Recall that if B^i and Z^i denotes respectively the subspaces of coboundaries and cocycles in Ω^i, there exist subspaces of Ω^i, that we denote by H^i and L^i, such that

$$B^i = Z^i \oplus H^i$$

and

$$\Omega^i = B^i \oplus L^i = Z^i \oplus H^i \oplus L^i$$

for every $i \in \mathbb{Z}$. In what follows we identify the $i - th$ cohomology group $H^i(\Omega)$ with the subspace H^i and taking $H = \oplus_{i \in \mathbb{Z}} H^i$ we denote by $P_H : \Omega \to H$ the projection from Ω onto its subspace H, to which we shall refer as projection onto

the cohomology.

On $\Omega(J) = J \otimes \Lambda\mathfrak{g}^*$ one has the tensor scalar product

$$\langle \omega_1, \omega_2 \rangle = \langle x \otimes \alpha, y \otimes \beta \rangle = \langle x, y \rangle_J \cdot \langle \alpha, \beta \rangle_{\Lambda\mathfrak{g}^*} \tag{4.20}$$

for all $\omega_1 = x \otimes \alpha, \omega_2 = y \otimes \beta \in \Omega$.
Let us denote by d^* the adjoint operator of d, that is

$$\langle d^*\omega_1, \omega_2 \rangle = \langle \omega_1, d\omega_2 \rangle \tag{4.21}$$

for all ω_1 and $\omega_2 \in \Omega$. It is clear that d^* is a map of degree -1 and one defines the Laplacian as the map $\Delta : \Omega \to \Omega$ of degree 0 given by

$$\Delta = dd^* + d^*d \tag{4.22}$$

which is clearly self-adjoint.
The Laplacian Δ is positive definite, indeed one has

$$\langle \omega, \Delta\omega \rangle = || \, d\omega \, ||^2 + || \, d^*\omega \, ||^2 \tag{4.23}$$

for all $\omega \in \Omega$. Moreover one has

$$H(\Omega) = ker(\Delta) = ker(d) \bigcap ker(d^*) \tag{4.24}$$

as implied by (4.23).
The laplacian Δ is invertible on Ω/H and one has

$$\Delta^{-1}(dd^* + d^*d) = d\Delta^{-1}d^* + \Delta^{-1}d^*d = I - P_H \tag{4.25}$$

where the first equality is justified by the fact that any function of Δ commutes both with d and d^*.

Now, going back to the problem of finding a contracting homotopy for the associator in $\Omega(J)$, we take as $K : \Omega(J)^{\otimes 3} \to \Omega(J)$ the map

$$K = \Delta^{-1} \circ d^* \circ [\cdot, \cdot, \cdot] \tag{4.26}$$

whose degree is -1 and one has the following:

Theorem 4.4.1. *Let J be an Euclidean simple special Jordan algebra, then $\Omega(J)$ is associative up to homotopy, with homotopy map $K : \Omega(J) \to \Omega(J)$ given as above.*

Proof. We have

$$\begin{aligned} &dK + Kd = \\ &= d\Delta^{-1}d^* \circ [\cdot,\cdot,\cdot] + \Delta^{-1}d^*[\cdot,\cdot,\cdot]d_A^{\otimes 3} \\ &= \Delta^{-1}dd^* \circ [\cdot,\cdot,\cdot] + \Delta^{-1}d^*d[\cdot,\cdot,\cdot] = \\ &= (I - P_H)[\cdot,\cdot,\cdot] \end{aligned} \tag{4.27}$$

and we need to prove that

$$P_H[\cdot,\cdot,\cdot] = 0 \tag{4.28}$$

that is

$$P_H[\alpha,\beta,\gamma] = 0 \tag{4.29}$$

for all $\alpha,\beta,\gamma \in \Omega(J)$. Let us write

$$\alpha = x \otimes g_1, \quad \beta = y \otimes g_2 \quad \gamma = z \otimes g_3 \tag{4.30}$$

with $x,y,z \in J$ and $g_1,g_2,g_3 \in \mathfrak{g}$. We have seen that any element in P_H is an element of $1 \otimes \mathfrak{g}$, thus it is sufficient to prove that $[\alpha,\beta,\gamma] \in (1 \otimes \mathfrak{g})^\perp$. We are going to show that

$$[x,y,z] \in 1^\perp \tag{4.31}$$

for all $x,y,z \in J$. The condition above is equivalent to

$$Tr([x,y,z]) = 0, \tag{4.32}$$

where Tr denotes the trace of a matrix. Since J is special, we write

$$\begin{aligned} &x \circ (y \circ z) - (x \circ y) \circ z = \\ &(xz - zx)y - y(xz - zx) = \\ &[[x,z],y], \end{aligned} \tag{4.33}$$

where in the first of the expressions above we explicit wrote the symbol $\circ$ to denote the Jordan product, while in the second expression we consider the product inside the associative algebra A from which J is obtained by specialization.
Now one has

$$Tr([x,y,z]) = Tr([[x,z],y])) = 0 \tag{4.34}$$

since the trace of a commutator of elements in A is zero by definition. □

We remark that the same kind of argument is true by direct computation for the exceptional Jordan algebra J_3^8, thus we can summarize our results by writing

Theorem 4.4.2. *Let J be an Euclidean simple Jordan algebra, then $\Omega(J)$ is associative up to homotopy, with homotopy map $K : \Omega(J) \to \Omega(J)$ given by*

$$K = \Delta^{-1} \circ d^* \circ [\cdot,\cdot,\cdot]$$

Remark: It has been suggested ([24]) that this result might lead to a structure of A_∞ algebra for the derivation-based differential calculus over Euclidean Jordan algebras (see e.g. [48] for definitions). However it is not clear nowadays how one can generalize the technique used above in order to build the products of order four and higher.

Chapter 5

Connections for Jordan modules

In this section we resume the theory of connections for modules over Jordan algebras.
In particular we make use of Theorem 3.3.1 in chapter 2 in order to present the full theory of connections for free Jordan modules.
There are two equivalent definitions of derivation-based connections for modules of Jordan algebras and correspondingly two equivalent definitions of curvature. While the first definition is only suited for derivation-based connections, the second definition can be generalized to more general cases.
Both approaches have been introduced in [24] (see also [9]) and both are inspired by the dual definition of connections in differential geometry.

In the following, for a module M over a Jordan algebra J, we denote by $End(M)$ the algebra of linear endomorphisms of M as vector space, while we use the notation $End_J(M)$ to denote the endomorphisms of M as $J-$module.

5.1 Derivation-based connections for Jordan modules

The first definition of derviation-based connection for a Jordan module is the following

Definition 5.1.1. *Let J be a Jordan algebra, a derivation-based connection on a module M over J is a linear map $\nabla : Der(J) \to End(M)$, $\nabla : X \mapsto \nabla_X$ such that*

$$\nabla_X(xm) = X(x)m + x\nabla_X m \tag{5.1}$$

and

$$\nabla_{zX}(m) = z\nabla_X(m) \tag{5.2}$$

for any $x \in J$, $m \in M$ *and* $z \in Z(J)$.

From the first property it follows that if ∇ and ∇' are two connections on the same Jordan module M, then $\nabla_X - \nabla'_X$ is an element of $End_J(M)$

Definition 5.1.2. *Let* ∇ *be a derivation-based connection on a Jordan module* M. *The curvature of* ∇ *is defined as*

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{5.3}$$

for all $X, Y \in Der(J)$.

It follows that, for fixed $X, Y \in Der(J)$, $R_{X,Y}$ is a $J-$module endomorphism.

Definition 5.1.3. *A connection will be called flat if its curvature is identically zero that is*

$$R_{X,Y}(m) = 0 \tag{5.4}$$

for all $X, Y \in Der(J)$ *and* $m \in M$.

Remark: In view of applications to particle physics, and in particular to Yang-Mills models, we are interested in classifying flat connections for Jordan modules. In fact, according to a standard semi-heuristic argument (see e.g. [23],[28],[26],[27]), any flat connection corresponds to a different ground state of the theory and the specification of the latter leads to different physical dynamics.

The second definition of derivation-based connections is more suitable to be generalized to connections not based on derivations.
Let J be a Jordan algebra, let M be a module over J and denote by $\Omega^n_{Der}(M)$ the $J-$module of all antisymmetric mappings of $Der(J)$ into M and $n-$linear over $Z(J)$, then $\Omega_{Der}(M) = \oplus\Omega^n_{Der}(M)$ is a module over $\Omega_{Der}(J)$ in the following way: for $\omega \in \Omega^n_{Der}(J)$ and $\Phi \in \Omega^l_{Der}(M)$, the action of ω on Φ is given by

$$(\omega\Phi)\,(X_1, ..., X_{n+l}) = \frac{1}{(n+l)!} \sum_{\sigma(i_1,...,i_{n+l})} (-1)^{|\sigma|}\omega\,(X_{i_1}, ...X_{i_n})\,\Phi(X_{i_{n+1}}, ..., X_{i_{n+l}})$$

where σ denotes a permutation of the indices $(i_1, ..., i_{n+l})$ and $\mid \sigma \mid$ denotes the parity of the permutation σ.

Definition 5.1.4. *Let* J *be a Jordan algebra, let* M *be a module over* J. *A derivation-based connection on* M *is a linear endomorphism* ∇ *of* $\Omega_{Der}(M)$ *such that*

$$\nabla(\Phi) \in \Omega^{l+1}_{Der}(M) \tag{5.5}$$

for all $\Phi \in \Omega^l_{Der}(M)$ *and*

$$\nabla\,(\omega\Phi) = d(\omega)\Phi + (-1)^n\omega\nabla\Phi. \tag{5.6}$$

for all $\omega \in \Omega^n_{Der}(J)$ *and* $\Phi \in \Omega^l_{Der}(M)$.

From (5.6) we see that if ∇ and ∇' are two different connections, then their difference is an endomorphism of $\Omega_{Der}(M)$ as as a module over $\Omega_{Der}(J)$. In this case the curvature of a connection is defined as $R = \nabla^2$. Definitions 5.1.1 and 5.1.4 are equivalent, in fact if ∇ is a connection as in the second definition, one defines a map from $Der(J)$ into $End(M)$ by setting

$$\nabla_X(m) = (\nabla(m))(X) \tag{5.7}$$

and the map $X \mapsto \nabla_X$ is a connection in the sense of 5.1.1.
On the other hand, if $\nabla : X \mapsto \nabla_X$ is a connection according to the first definition, one sets

$$\begin{aligned}\nabla(\Phi)\left(X_0, ..., X_n\right) = &\sum_{0\leq k\leq n} (-1)^k \nabla_{X_p}\left(\Phi\left(X_0, ..., \widehat{X_k}, ...X_n\right)\right) + \\ &+ \sum_{0\leq r<s\leq n} (-1)^{r+s}\Phi\left([X_r, X_s], X_0, ..., \widehat{X_r}, ..., \widehat{X_s}, ...X_n\right)\end{aligned} \tag{5.8}$$

for all $\Phi \in \Omega^n_{Der}(M)$ and $X_p \in Der(J)$ and ∇ is now a connection according to Definition 5.1.4.

In the following examples the term "connection" will stand for derivation-based connection.

Example 5.1.5. *Let J be a finite dimensional and unital Jordan algebra, let $M = J \otimes E$ be a free $J-$module. On M we have a base connection $\nabla^0 = d \otimes Id_E : J \otimes E \to \Omega^1_{Der} \otimes E$. As map from $Der(J)$ into $End(M)$, ∇^0 is the lift of the differential on J, that is*

$$\nabla^0_X\left(x \otimes e\right) = \left(dx\right)\left(X\right) \otimes e \tag{5.9}$$

for any $X \in Der(J)$ and $x \otimes e \in M$. It is easy to check that ∇^0 respects properties (5.1) *and* (5.2)*. Moreover, this connection is gauge invariant whenever the center of J is trivial, meaning that*

$$[d, A] = 0$$

for every module endomorphism $A : M \to M$.

Proposition 5.1.6. *Let J be a finite dimensional Jordan algebra, let $M = J \otimes E$ be a free module over J, where E is a real vector space. Then any connection on M is of the form*

$$\nabla = \nabla^0 + \mathcal{A} \tag{5.10}$$

where $\mathcal{A}$ is a linear map $\mathcal{A} : Der(J) \to Z(J) \otimes End(E)$. If $X \in Der(J)$ the action of $\mathcal{A}(X)$ is given by

$$\mathcal{A}(X)\left(x \otimes e\right) = \alpha_x \circ x \otimes A_X e \tag{5.11}$$

for every $x \otimes e \in J \otimes E$, where A_X is a linear endomorphism of E and $\alpha_X \in Z(J)$ acts on elements of J via multiplication.

Proof. From the definition of connection, it has to be

$$\nabla - \nabla^0 = A \in End_J(M) \tag{5.12}$$

and from Theorem 3.3.1, it follows $A(X) \in Z(J) \otimes End(E)$. □

Remark: In particular if J is an Euclidean Jordan algebra, the centre of J is trivial and so (5.11) *is simplified as*

$$\mathcal{A}(X)\,(x \otimes e) = x \otimes A_X e \tag{5.13}$$

for every $x \otimes e \in J \otimes E$.

For what concerns flat connections, the following result, very similar to its counterpart in the context of Lie algebras, holds.

Proposition 5.1.7. *Let $M = J \otimes E$ be a free module over a simple Jordan algebra J, then flat connections on M are in one to one correspondence with Lie algebra homomorphisms $\mathcal{A} : Der(J) \to End(E)$. That is, for a basis $\{X_\mu\} \subset Der(J)$ with structure constants $c^\tau_{\mu\nu}$ one has*

$$[\mathcal{A}(X_\mu), \mathcal{A}(X_\nu)] = c^\tau_{\mu\nu}\mathcal{A}(X_\tau). \tag{5.14}$$

where $[X_\mu, X_\nu] = c^\tau_{\mu\nu} X_\tau$.

Proof. By direct computation one can check that if a given connection $\nabla = \nabla^0 + \mathcal{A}$ is flat then (5.14) must hold.
On the converse, if $\mathcal{A} : Der(J) \to End(E)$ is such that (5.14) holds on a basis $\{X_\mu\} \subset Der(J)$, then $\nabla = \nabla^0 + \mathcal{A}$ is a flat connection on M. □

Summarizing, all the derivation-based differential calculus for free modules over Jordan algebras is resumed by the following proposition.

Proposition 5.1.8. *Let J be a unital Jordan algebra, let $M = J \otimes E$ be a free module over J then*

1. *$\nabla^0 = d \otimes \mathbb{I}_E : J \otimes E \to \Omega^1(J) \otimes E$ defines a flat connection on M which is gauge invariant whenever the center of J is trivial.*

2. *Any other connection ∇ on M is defined by*

$$\nabla = \nabla^0 + A : J \otimes E \to \Omega^1(J) \otimes E \tag{5.15}$$

where A is a module homomorphism of $J \otimes E$ into $\Omega^1(J) \otimes E$.

3. *For a derivation-based connection ∇ the curvature is given by*

$$\nabla^2(X,Y) = R_{X,Y} = X(A(Y)) - Y(A(X)) + [A(X), A(Y)] - A([X,Y]) \quad (5.16)$$

for any $X, Y \in Der(J)$.

4. *If J is a simple Jordan algebra, then ∇ defines a flat connection if and only if the map $A : Der(J) \to End(E)$ is a Lie algebra homomorphism.*

Example 5.1.9. *Consider again A_n^i as a module over J_n^i as in Example 3.2.5. We can provide a base connection for this module. From Theorem 2.2.21 we know that for any $X \in Der(J_n^i)$ there exists a finite number of couples of $x_i, y_i \in J_n^i$ such that*

$$X(z) = \sum_i (x_i \circ z) \circ y_i - x_i \circ (j \circ y_i) \quad (5.17)$$

for any $z \in J_n^i$ and where we have explicitly written $\circ$ to design matrix anticommutator. Let $X_i = [x_i, y_i]$, where the commutator is taken with respect to the standard row by column product, then the expression above can also be written as:

$$X(z) = \sum_i [X_i, z] \quad (5.18)$$

for any $z \in J_n^i$.
Recall that the commutator of two hermitian matrices is an antihermitian matrix, then a good base connection on the Jordan module A_n^i is given by

$$\nabla_X(a) = \sum_i [X_i, a] \quad (5.19)$$

for all $a \in A_n^i$, indeed:

$$\begin{aligned} &\nabla_X(z \circ a) = \sum_i [X_i, z \circ a] = \\ &[X_i, z] \circ a + \sum_i [X_i, a] \circ z = X(z) \circ a + z \circ \nabla_X(a) \end{aligned} \quad (5.20)$$

for all $z \in J_n^i$ and $a \in A_n^i$. Moreover this base connection is flat, indeed:

$$\begin{aligned} &\left([\nabla_x, \nabla_Y] - \nabla_{[X,Y]}\right)(a) = [X\,[Y,a]] - [Y\,[X,a]] - [[X,Y]\,,a] = \\ &[[a,Y]\,,X] + [[X,a]\,,Y] + [[Y,X]\,,a] = 0 \end{aligned} \quad (5.21)$$

in view of the Jacobi identity in the Lie algebra $M_n(\mathbb{R})$.

Due to commutativity, for any Jordan algebra J it holds

$$[x, z, y] = -[L_x, L_y]\, z \tag{5.22}$$

for all $x, y, z \in J$. Hence the commutator $[L_x, L_y]$ defines an inner derivation for J. In fact, formula (5.17) is a consequence of this in the particular case of special Jordan algebras.

The example above can be generalized to the case of a module M over any finite dimensional, semisimple Jordan algebra. In fact, in view of Theorem 2.2.21 all the derivations of such algebras are inner and one has the following.

Proposition 5.1.10. *Let J be a finite dimensional semisimple Jordan algebra so that $\forall X \in Der(J)$ there exist a finite number of couples of elements $x_i, y_i \in J$ such that*

$$X(z) = \sum [x_i, z, y_i] \tag{5.23}$$

for every $z \in J$. Then the map

$$\begin{aligned} \nabla :& Der(J) \to End(M) \\ & X \mapsto \nabla_X = \sum [x_i, \cdot, y_i] \end{aligned} \tag{5.24}$$

is a connection on M.

Proof. Let $X = \sum [x_i, \cdot, y_i] \in Der(J)$, it extends to a derivation $\tilde{X}$ on the split null extension $J \oplus M$ given by

$$\tilde{X}(z, m) = \sum [(x_i, 0), (z, m), (y_i, 0)] \tag{5.25}$$

for all $(z, m) \in J \oplus M$.
If we identify M with elements of the form $(0, m)$ in $J \oplus M$, we see that $\tilde{X}$ restricts to a linear endomorphism on M. Then ∇ is a $Z(J)$ linear map from $Der(J)$ into $End(M)$ and from Leibniz rule applied to $\tilde{X} \in Der(J \oplus M)$ we have $\nabla_X(zm) = X(z)m + z\nabla_X m$. □

Moreover the following hold

Proposition 5.1.11. *Let J be a finite dimensional Jordan algebra, let M be a module over J, denote by ∇ the connection given by* (5.24)*, then*

1. *If M is a free module over J, ∇ coincides with the lift of the differential over J to M.*

2. *∇ is a flat connection.*

Proof. To prove 1, we write $M = \oplus_{i=1}^{n} J$ and by direct computation we have

$$\nabla_X m = \begin{pmatrix} X(x_1) \\ . \\ . \\ . \\ X(x_n) \end{pmatrix} = \begin{pmatrix} dx_1(X) \\ . \\ . \\ . \\ dx_n(X) \end{pmatrix}$$

for every $m \in M$, written as a column of elements $x_1, ..., x_n$ inside J and $X \in Der(J)$.

To prove 2, we start by taking J simple for sake of simplicity, the proposition will follow by taking in account than any finite dimensional Jordan algebra is a finite direct sum of simple Jordan algebras.

If $J = J_3^8$, then M has to be a free module over J and thus ∇ coincides with the lift of the differential to M as shown above and hence the proposition is trivial.

Suppose then that J is a special Jordan algebra, then the split null extension $J \oplus M$ is a special Jordan algebra, that is there exist an associative algebra A such that $J \oplus M \subseteq A^+$. We are going to denote the product inside $J \oplus M$ by the symbol $\circ$ and the product inside A by juxtaposition, alas any associator is meant to be taken with respect to the Jordan product in $J \oplus M$, while commutators are considered with respect to the associative product in A.
With notations of Proposition 5.1.10 we have

$$\begin{aligned} \nabla_X m &= \sum [x_i, m, y_i] \\ &= \sum [[x_i, y_i], m] \end{aligned}$$

for every $m \in M$ and $X \in Der(J)$.
Let us compute the curvature of ∇ by taking two derivatives $X = \sum_i [x_i, \cdot, y_i]$ and $Y = \sum_i [\tilde{x}_i, \cdot, \tilde{y}_i]$, we write

$$X^i = [x_i, y_i]$$

and

$$Y^i = [\tilde{x}_i, \tilde{y}_i]$$

in order to find

$$R_{X,Y} m = \sum_i \sum_j \left[X^i \left[Y^j, m\right]\right] + \left[Y^j, \left[X^i, m\right]\right] + \left[\left[X^i, Y^j\right], m\right]$$

for every $m \in M$. Thus $R_{X,Y} m = 0$ in view of the Jacobi identity of the commutator for the associative algebra A. $\square$

Connection (5.24) can be defined for every Jordan module over a Jordan algebra for which all derivations are inner in the sense of Theorem 2.2.21 and such that the extension of derivations of the algebra to derivations on the split null extension is unique. The set of Jordan algebras for which all derivations are inner contains all finite dimensional semi-simple Jordan algebra over a field of characteristic zero but it is in fact much wider, for example from theorem 2 of [35] we see that this request holds true for finite dimensional and separable Jordan algebras on any field of characteristic different from 2.

5.2 General connections for Jordan modules

The second definition of derivation-based connection for a Jordan is generalized as follows.

Definition 5.2.1. *Let $\Omega = \oplus_{\mathbb{N}}\Omega^n$ be a differential graded Jordan algebra and let $\Gamma = \oplus_{\mathbb{N}}\Gamma^n$ be a graded Jordan module over Ω, a connection on Γ is a linear endomorphism $\nabla : \Gamma \to \Gamma$ such that*

$$(\nabla\Phi) \in \Gamma^{l+1} \tag{5.26}$$

$$\nabla(\omega\Phi) = d(\omega)\Phi + (-1)^n\omega\nabla(\Phi) \tag{5.27}$$

for all $\omega \in \Omega^n$ and $\Phi \in \Gamma^l$.

In particular when $\Omega^0 = J$ and $\Gamma^0 = M$ one obtains the definition of $\Omega-$connection over the $J-$module M from Definition 5.1.4.

Chapter 6

Jordan modules and the Standard Model

In this chapter we are going to review two possible approaches to the mathematical formalization of the Standard Model of particle physics, that were proposed respectively in [24] and [29]. Both the approaches aim to make use of Jordan algebras in order to present the particle content of the Standard Model as emerging from purely mathematical structure.
In particular, in the formulations of both the proposals the exceptional Jordan algebra J_3^8 is deeply involved, hence these two models cannot be just some rephrasing of any approach that makes use of associative algebras but rather constitute a purely novel point of view that makes sense only in the context of Jordan algebras.

In Connes' almost commutative spectral triples description of the Standard Model (see for example [10],[11], [14],[15],[21], [55]), one makes use of a commutative algebra of functions over a 4 dimensional manifold to describe the space-time degrees of freedom in the dynamics of a particle, while the internal degrees of freedom, which deal with charges, chirality and other "inner" properties of quantum particles, are described by a finite quantum space whose geometry is treated dually by making use of a finite dimensional noncommutative algebra.
In the same spirit, the two approaches that we are going to consider make use of a finite dimensional Euclidean Jordan algebra to take in account the inner degrees of freedom of fundamental fermions.

One striking feature of Connes' scheme is that most of the physical peculiarities of the Standard Model, in particular the existence of the Higgs boson and the sew-saw mechanism which confers masses to the particles, can be derived simply by following the hypothesis of Connes' reconstruction theorem for commutative spectral triples as prescriptions to pick the right finite dimensional spectral triple

to describe the quantum geometry of physical particles.
Indeed, almost every datum from the Standard Model can be derived from this frame with three significant exceptions

- The unimodularity of $SU(3)$, which is the fact that colour gauge group is taken by considering only unitary matrices with determinant equal to one rather then just unitary three by three complex matrices.
  Notice that the same problem does not stand for the $SU(2)$ gauge group, since this group is actually isomorphic to $U(1, \mathbb{H})$, the group of unitary quaternions.

- The quark-lepton symmetry, which is the hard experimental fact that to any existing quark there exist a corresponding lepton.

- The existence of three generations of particles, whose content is perfectly identical for number of particles and charges but differ heavily for what regards the mass scale.
  This is well resumed by the following table where each column represent one generation, hence one mass scale, of particles and the particles on the same row have the same charges.

$Q = 2/3(Quarks)$	u	c	t
$Q = 0\ (Leptons)$	ν_e	ν_μ	ν_τ
$Q = -1/3(Quarks)$	d	s	b
$Q = -1(Leptons)$	e	μ	τ

  We are going to refer to this property as "triality of the Standard Model" or just "triality."

All these three data have to be put by hand in Connes' formulation while, as it will be shown, they all emerge quite naturally if one allows to describe quantum observables by making use of the exceptional Jordan algebra J_3^8.
The two different approaches can be distinguished by the respective criterion from which the triality of the Standard Model is derived, more precisely:

- In the first approach, the three generations of particles correspond to the three octonions appearing in every matrix inside J_3^8.

- In the second approach, each generation of particle correspond to a J_2^8 subalgebra which sits inside J_3^8.

In the first section we will present the first approach that was originally taken in [24], which starts from the observation the internal colour space of elementary

particles $\mathbb{C} \oplus \mathbb{C}^3$ is endowed with the same algebraic structure as the algebra of octonions $\mathbb{O}$ when using the operations which preserves the $SU(3)$ action, namely the scalar product and the cross product.

In the second approach, presentend in the second section, the starting observation that all the particle content of each generation can fit inside a copy of $J_2^8 \simeq JSpin_9$ and that the subgroup of automorphisms of J_2^8 which preserves the decomposition of each octonion in $\mathbb{C} \oplus \mathbb{C}^3$ is exactly the Standard Model group

$$G_{SM} = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}.$$

Then in the third section we are going to attempt some comparisons between the two models.

Finally we are going to use all the machinery of differential calculus and the theory of connections in order to study the Yang-Mills kind of Lagrangians when the objects involved are connections over Jordan modules.
It is important to stress again that in this framework the classification of flat connections corresponds to the classification of different minima for these kind of Lagrangians, hence the study carried in the previous chapter is going to be relevant in some future work when we will be studying mechanisms of spontaneous symmetry breaking for this formulation of the Standard Model.

Both the discussions here and in the following section deal with "internal" degrees of freedom of elementary particles, that is we are going to describe the internal quantum geometry of particles by making use of finite dimensional algebras of observables.

6.1 First approach

In this section we are going to give a review of the model proposed in [24].
Let $E = \mathbb{C}^3$ as Hilbert space equipped with the standard hermitian product denoted as $\langle \cdot, \cdot \rangle : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$. Let $\{e_k\}_{k=1}^3$ be a basis for E and write $X = X^k e_k$ for any vector in E. We define a volume form on E by

$$v(x, y, z) = \epsilon_{ijl} X^i Y^j Z^l$$

for every $X, Y, Z \in E$. This volume form is $SU(3)$ invariant and we can make use of it to define a vector product $\times : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ from

$$v(X, Y, Z) = \langle X \times Y, Z \rangle$$

for every $X, Y, Z \in E$. One has

$$(X \times Y)_l = \epsilon_{ijl} \overline{X}^i Y^j$$

for every $X, Y \in E$. Notice that the cross product is nonassociative and it is $SU(3)$ invariant. We can equip $\mathbb{C}$ with the trivial representation of $SU(3)$:

$$(g, z) \mapsto z$$

for every complex number z and $g \in SU(3)$. We combine the scalar and the crossed product on E as a product on $A = \mathbb{C} \oplus \mathbb{C}^3$ which is going to be written as

$$(x, X)(y, Y) = (\alpha \langle x, y \rangle, \beta X \times Y)$$

for some $\alpha, \beta \in \mathbb{C}$. Now if we endow $\mathbb{C} \oplus \mathbb{C}^3$ with its natural structure of Hilbert space one has

$$|| \, (0, X)(0, Y) \, ||^2 = (| \, \alpha \, |^2 - | \, \beta \, |^2) \, || \, \langle X, Y \rangle \, || + | \, \beta \, |^2 || \, X \, || \, || \, Y \, ||$$

so that by choosing the normalization

$$| \, \alpha \, | = | \, \beta \, | = 1$$

one has

$$|| \, (0, X)(0, Y) \, ||^2 = || \, (0, X) \, || \, || \, (0, Y) \, ||^2$$

and

$$|| \, (x, 0)(y, 0) \, ||^2 = || \, (x, 0) \, || \, || \, (y, 0) \, ||^2$$

for every $x, y \in \mathbb{C}$ and $X, Y \in E$. It is then natural to require that

$$|| \, (x, X)(y, Y) \, ||^2 = || \, (x, X) \, || \, || \, (y, Y) \, ||^2$$

for the product on A.
One choice of the constants α and β which realizes this condition is given by taking the following product

$$(x, X)(y, Y) = (xy - \langle X, Y' \rangle, \, \overline{x} Y - y X + i X \times Y) \, . \tag{6.1}$$

for all (x, X) and $(y, Y) \in A$. This product is unique up to renormalization of α and β. Notice that product above is $SU(3)$ invariant and bilinear with respect to multiplication by real scalars but not with respect to the multiplication by complex scalars. It turns out that A equipped with this product is a nonassociative real algebra and the following theorem holds (see e.g. [58]):

Proposition 6.1.1. *The algebra A is isomorphic to the algebra of octonions $\mathbb{O}$. The subgroup of the exceptional Lie group G_2 which preserves the decomposition $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ is isomorphic to $SU(3)$.*

In what follows we are going to interpret $\mathbb{C}^3$ as the Hilbert space of colour degrees of freedom for quarks and $\mathbb{C}$ as the Hilbert space of internal colour degrees of freedom of leptons (the fact that such Hilbert space is one dimensional means that leptons do not carry any significant colour charge, as it should be).

In this approach the unimodularity condition emerges naturally from a mathematical virtue, since here $SU(3)$ appears as the subgroup of G_2 that preserves the splitting of A in its leptonic and quark part.
Thus in this setting the quark-lepton symmetry and the unimodularity condition are deeply connected features. Now we show how this binds to the existence of three generations of particles.

Consider the exceptional Jordan algebra J_3^8, its element are hermitian matrices with octonion entries. The decomposition of $\mathbb{O}$ as $\mathbb{C} \oplus \mathbb{C}^3$ induces a decomposition of J_3^8 as $J_3^2 \oplus M^2(\mathbb{C})$ given as follows

$$\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_3 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & z_3 & \overline{z}_2 \\ \overline{z}_3 & \xi_2 & z_1 \\ z_2 & \overline{z}_3 & \xi_3 \end{pmatrix} \oplus (Z_1, Z_2, Z_3)\,, \tag{6.2}$$

where the diagonal elements ξ_i's are real numbers and, as consequence of the isomorphism above, we write every octonion as $x_i = (z_i, Z_i)$ with $z_i \in \mathbb{C}$, $Z_i \in \mathbb{C}^3$. Recall that the group of automorphism of J_3^8 is isomorphic to the exceptional Lie group F_4 (see again [58]), and, following the same scheme as before we have the following.

Proposition 6.1.2. *The subgroup of F_4 which preserves the decomposition of J_3^8 as $J_3^2 \oplus M_3(\mathbb{C})$ is isomorphic to $(SU(3) \times SU(3))/\mathbb{Z}_3$.*

Let $(U, V) \in (SU(3) \times SU(3))/\mathbb{Z}_3$, then the action mentioned in the proposition above is given by

$$H \mapsto VHV^*, \quad M \mapsto UMV^* \tag{6.3}$$

for any $(H, M) \in J_3^2 \oplus M_3(\mathbb{C})$.

So far we have pointed out how J_3^8 might play the role of quantum algebra for the Standard Model and, in order to complete the cinematic picture of this approach, we have to select the right J_3^8-module on which such algebra is going to

be represented. As said in chapter 2, every J_3^8-module has to be a free module over J_3^8, namely we have

$$M = J_3^8 \otimes \mathbb{R}^N$$

and we have to select the right N in such a way that all the particle content of the Standard model can be suitably fitted inside the module M.
Since for each generation of particles there are two families, that are irreducible representations of the $SU(2)$ symmetry of standard model, the most natural choice is to consider the module $M = J_3^8 \oplus J_3^8$, for which the following particle assignment can be taken:

$$J^u = \begin{pmatrix} \alpha_1 & \nu_\tau & \overline{\nu}_\mu \\ \overline{\nu}_\tau & \alpha_2 & \nu_e \\ \nu_\mu & \overline{\nu}_e & \alpha_3 \end{pmatrix} + (u, c, t) \quad J^d = \begin{pmatrix} \beta_1 & \tau & \overline{\mu} \\ \overline{\tau} & \beta_2 & e \\ \mu & \overline{e} & \beta_3 \end{pmatrix} + (d, s, b)\,. \tag{6.4}$$

where, taking in account the decomposition of J_3^8 as $J_3^2 \oplus M_3(\mathbb{C})$ and the interpretation of the aforementioned quark-lepton symmetry, every lepton is collocated in the position of a complex entry inside J_3^2 and every quark corresponds to a column vector inside $\mathbb{C}^3$.

We notice that this model is slightly superabundant, since there are six free real parameters α_i's and β_i's that cannot ruled out a priori and which do not correspond to any known particle of the Standard Model.

Finally, by looking at the expression above and to the action of $(SU(3) \times SU(3))/\mathbb{Z}_3$, we see that the action of $U \in SU(3)$ mixes different colours while the action of $V \in SU(3)$ mixes different generations of leptons.

6.2 Second approach

The exceptional Jordan algebra J_3^8 contains three copies of $J_2^8 \simeq JSpin_9$ as subalgebras, given simply by taking

$$J_2^{8^{(1)}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x}_1 & \xi_3 \end{pmatrix}, \quad J_2^{8^{(2)}} = \begin{pmatrix} \xi_1 & 0 & \overline{x}_2 \\ 0 & 0 & 0 \\ x_2 & 0 & \xi_3 \end{pmatrix}, \quad J_2^{8^{(3)}} = \begin{pmatrix} \xi_1 & x_3 & 0 \\ \overline{x}_3 & \xi_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

inside J_3^8. Just to fix the ideas let us concentrate on the first copy $J_2^{8^{(1)}}$, the discussion is analogous for the two remains copies.

The group of automorphisms of F_4 which preserves J_2^8 is isomorphic to $Spin(9)$ and it coincides with the subgroup of elements in F_4 which fix the projector E_1.

The subgroup of automorphisms of J_2^8 which preserves the decomposition of the octonions in $\mathbb{C} \oplus \mathbb{C}^3$ is the intersection of the subgroup $\frac{SU(3)\times SU(3)}{\mathbb{Z}_3}$ with $Spin(9)$

$$\frac{SU(3) \times SU(3)}{\mathbb{Z}_3} \cap Spin(9) = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}.$$

which coincides with the gauge group of the Standard Model G_{SM}.
The action of G_{SM} is given as follows: let us decompose an element in J_2^8 by writing

$$\begin{pmatrix} \xi_1 & x \\ \overline{x} & \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & z \\ \overline{z} & \xi_2 \end{pmatrix} + Z$$

for $\xi_1, \xi_2 \in \mathbb{R}$, $x \in \mathbb{O}$, $z \in \mathbb{C}$ and $Z \in C^3$.

Let $(U, V) \in U(3) \times SO(3) = U(3) \times SU(2)/\mathbb{Z}_2$, we have

$$\begin{pmatrix} \xi_1 & z \\ \overline{z} & \xi_2 \end{pmatrix} \mapsto V \begin{pmatrix} \xi_1 & z \\ \overline{z} & \xi_2 \end{pmatrix} V^*, \quad Z \mapsto UZ.$$

The action of the quantum chromodynamic subgroup $SU(3) \times U(1)$ of G_{SM} corresponds to the action of $U \in U(3)$, while the electroweak interaction of the subgroup $SU(2)/\mathbb{Z}_2$ is given by the action of $V \in SO(3)$.

The complex $*$ enveloping of the real Jordan algebra J_2^8 is given by

$$S_{\mathbb{C}}(J_2^8) = M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C})$$

whose minimal injective representation as algebra of bounded operators is given by considering the finite dimensional Hilbert space

$$\mathbb{C}^{16} \oplus \mathbb{C}^{16} = \mathbb{C}^{32}$$

that is the space of internal degrees of freedom for one fermion in each generation. Moreover $\mathbb{C}^{32}$ is the Hilbert space obtained when considering the diagonal part of the subspace of self-adjoint elements inside $S_{\mathbb{C}}(J_2^8)$, given by

$$S_{\mathbb{C}}(J_2^8)_{sa} = J_{16}^2 \oplus J_{16}^2$$

which is itself a Jordan algebra and also a Jordan module for J_2^8 which realizes the quantum inner space for the particles of one generation of the standard model.

To relate the analysis done so far to the triality of the Standard Model, we consider each generation of fermions as corresponding to each one of $J_2^{8^{(i)}}$ subalgebras of J_3^8.

The subgroup of F_4 which preserves the decomposition of the octonions as $\mathbb{C} \oplus \mathbb{C}^3$ is given by

$$\frac{SU(3)_c \times SU(3)}{\mathbb{Z}_3}$$

where the first factor plays the role of the colour group of the Standard Model as it was in the previous approach.
For what regards the second $SU(3)$ factor, the analysis we have carried above shows that, after one has passed to a single generation of particles by considereding the intersection of $\frac{SU(3)_c \times SU(3)}{\mathbb{Z}_3}$ with $Spin(9)$, it reduces to the electroweak gauge group $U(2)$ of the Standard Model. Thus we can consider the second factor as generalized electroweak symmetry and we write

$$\frac{SU(3)_c \times SU(3)_{ew}}{\mathbb{Z}_3}$$

for the symmetry group of the whole theory.

6.3 Discussion and comparison of the two approaches

The two approaches we have studied are quite different, even though both take advantage of the exceptional algebra J_3^8 to describe the internal structure of particles of the Standard Model, while the external degrees of freedom are going to be described by the algebra of smooth functions on the space-time.

The first approach is quite similar, from a formal point of view, to what is done in the noncommutative approach to the Standard Model via Connes' spectral triples, even though it is not clear to the actual state of art how to replicate all the machinery of noncommutative geometry in the context of Jordan algebras.
A crucial obstacle is that the analogue of Dirac operator, and even a satisfactory definition of first order operators between Jordan modules are lacking so far.

Contrary to what happens with the second approach, by interpreting each octonion inside the matrices J_3^8 as one generation of the Standard Model we have an immediate explanation of the quark-lepton symmetry as consequence of the splitting $\mathbb{O} \simeq \mathbb{C} \oplus \mathbb{C}^3$. What seems to be a flaw of this approach is the fact that the it accounts only for the quantum chromodynamic degrees of freedom of fundamental fermions. While in the noncommutative approach this problem would be easily overcome by making use of the tensor product with a suitable algebra, this cannot be done in the context of Jordan algebras, as pointed out in subsection 2.2.1 of

chapter 1.

In the second approach, following the prescription of taking J_3^8 as the algebra of functions for the inner quantum space of elementary particles, one starts by looking at F_4 as the analogue of the group of diffeomorphisms of space-time, while the subgroup plays the same role played by the Poincaré group (the group of inhomogeneous Lorentz transformations) for the external degrees of freedom.
This means that in this approach, in order to consider a physically meaningful model, one has to endow the internal quantum space with some additional structure (given by the splitting $\mathbb{O} \simeq \mathbb{C} \oplus \mathbb{C}^3$) and look to the group of automorphisms which preserve said structure. This is completely analogue to what is done in general relativity where the physical space-time is described by a 4 dimensional Lorentzian manifold M and by allowing as only physically admissible change of coordinates the ones described by elements in the Poincaré group which is the one that preserves the Lorentzian structure of M.

As a final remark, let us notice that both the approaches might allow for a small step beyond the standard model. In the first approach this could be done by giving physical meaning to the real fields α_i's and β_i's appearing on the diagonal part of (6.4). Since they might correspond to $\frac{1}{2}$ spinors with real internal degrees of freedom, it has been suggested in [24] that they might in fact correspond to some Majorana particles.
In the second approach, the gauge group of the full theory is given by

$$\frac{SU(3)_c \times SU(3)_{ew}}{\mathbb{Z}_3}$$

and since the $SU(3)_{ew}$ is not observed at the scales of energies that can be reached nowadays, it has to be broken via a Higgs mechanism and it is reasonable to expect that the corresponding Higgs field emerges from the inner directions of the connection as we are going to explain in the following section.

6.4 Connections and Yang-Mills models

In this section we are going to review some dynamical aspects of the two models considered through the chapter. We will mainly follow the same approach as [23] and [25] where the author presents the Yang-Mills functionals for certain cases in which the connections have to be evaluated on some quantum directions.

We consider complete field theories, hence we will take in account both the internal and external degrees of freedom of fundamental particles. Thus, if we take

the algebra of the observables for the inner quantum space to be a Jordan algebra J, represented on some Jordan module M, then the full algebra of observables that we take is given by

$$\mathcal{J} = C^{\infty}\left(\mathbb{R}^4\right) \otimes J$$

and as the full module on which the algebra is represented we take

$$\mathcal{M} = C^{\infty}\left(\mathbb{R}^4\right) \otimes M$$

where $= C^{\infty}\left(\mathbb{R}^4\right)$ is taken as a free module over itself.

Remark: More generally we could have considered

$$\mathcal{M} = \Gamma^{\infty}\left(\mathbb{R}^4, \Sigma\right) \otimes M$$

where Σ is some vector bundle over $\mathbb{R}^4$ and by $\Gamma^{\infty}\left(\mathbb{R}^4, \Sigma\right)$ we denote the $C^{\infty}\left(\mathbb{R}^4\right)$ module of section of Σ. In particular we could have taken as Σ the spinor bundle on $\mathbb{R}^4$. However, in order to simplify notations, we take the simplest possible module for the space-time degrees of freedom, given by smooth functions on $\mathbb{R}^4$ since the analysis that we are going to carry is completely analogous for the general case. In particular, we expect that a realistic theory of Standard Model within our framework would require the use of the spinor bundle over space-time both to take in account relativity and to exhibit a Lagrangian for the Higgs field that allows for spontaneous symmetry-breaking mechanism.

In the following we denote by $\{\partial/\partial x^{\mu}\}_{\mu=1}^{4}$ the natural basis for derivations on $C^{\infty}\left(\mathbb{R}^4\right)$ and as $\{e_i\}$ a fixed basis of derivations for J. The Lie algebra of derivations on $\mathcal{J}$ is then given by

$$Der\left(\mathcal{J}\right) = span_{C^{\infty}(\mathbb{R}^4)}\left\{\frac{\partial}{\partial x^{\mu}} \otimes 1; 1 \otimes e_i\right\}$$

and in what follows we always use greek letters to refer to external directions and latin letters refer to the inner directions. In particular if ∇ is a connection over $\mathcal{M}$ we simplify our notations by setting $\nabla_{\mu} := \nabla_{\frac{\partial}{\partial x^{\mu}} \otimes 1}$ and $\nabla_i := \nabla_{1 \otimes e_i}$.
In what follows, we assume the following expression for ∇_i

$$\nabla_i = \nabla_i^0 + A_i(x)$$

where ∇_i^0 is a base connection which is known to be flat when restricted to the $J-$module M and $A_i(x) \in C^{\infty}(\mathbb{R}^4)$.

The curvature of a connection ∇ on $\mathcal{M}$ is then given by the following expression

$$\begin{aligned} F =& \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) dx^\mu dx^\nu + \\ & \left(\partial_\mu A_k + [A_\mu, A_k]\right) dx^\mu \theta^k + \\ & \frac{1}{2}\left([A_k, A_l] + c^m_{lk} A_m\right) \theta^k \theta^l \end{aligned}$$

where $\{\theta^i\}$ is the dual basis of $\{e_i\}$.
The Yang-Mills functional, which governs the propagation of gauge fields when there is no matter, is then written as follows

$$\begin{aligned} \| F \|^2 = & \int d^{s+1} tr \left\{ \frac{1}{4} \sum \left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)^2 + \right. \\ & \left. + \frac{1}{2} \sum \left(\partial_\mu A_i + [A_\mu, A_i]\right)^2 + \frac{1}{4} \sum \left([A_i, A_j] - c^k_{ij} A_k\right)^2 \right\} \end{aligned} \tag{6.5}$$

and it is evident that the minima for this functional, which are naturally interpreted as admissible vacua states for the physical theories, corresponds one to one to flat connections on $\mathcal{M}$.

Moreover the expression (6.5) is the sum of three non negative components, hence it is zero if and only each summand is zero.
In other words, minimizing (6.5) is equivalent to solve the following system of differential equations

$$\begin{cases} \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0 \\ \partial_\mu A_i + [A_\mu, A_i] = 0 \\ [A_i, A_j] - c^k_{ij} A_k = 0 \end{cases} \tag{6.6}$$

for every $\mu \in \{1, 2, 3, 4\}$ and for every $1 \leq k \leq dim(Der(J))$.

The third set of equations in system (6.6) requires that the inner component ∇_k's of the connection define a flat connection when restricted on the $J-$module M. If one chooses a non trivial flat connection on M, meaning that not all A_i are equal to zero, the solution of the second set of equations in (6.6) will describe a set of propagating massive fields $A_i(x)$.
In general all the mass spectrum of the field theory will depend upon the choice of a flat connection for the inner degrees of freedom, which is the exact phenomenon at basis of the Higgs mechanism.

Finally, let us comment how the construction made above specialize to the two models presented through the chapter.

1. In the first approach the inner degrees of freedom of elementary particles are represented as elements of $M = J_3^8 \otimes \mathbb{R}^2$. Hence, in view of Proposition 5.1.8, the inner part of any connection on $\mathcal{M}$ is going to be written as

$$\nabla_i = d \otimes \mathbb{1} + 1_J \otimes A_i$$

where A_i is a linear map from $\mathfrak{f}_4 = Der(J_3^8)$ into $C^\infty(\mathbb{R}^4, M_2(\mathbb{R}))$ and in view of (5.14) every minimum of the Yang-Mills functional corresponds to any of such map which is a Lie homomorphism.

2. In the spirit of the second approach, the quantum degrees of freedom of the particles of the Standard Model are represented as diagonal part $M = J_{16}^2 \oplus J_{16}^2$ which is a module over the Jordan algebra $J = J_2^8 = JSpin_9$. According to the discussion we have presented in the previous section, all the relevant physical dynamics has to preserve the splitting of J_2^8 as $J_2^2 \oplus \mathbb{C}^3$. Following this prescription, we are going to consider connections of the following kind

$$\nabla : \mathfrak{g}_{SM} \to End(M)$$

where $\mathfrak{g}_{SM} = \mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ is the Lie subalgebra of the Lie group G_{SM}. Then if $X \in \mathfrak{g}_{SM}$ is a derivation on J we extend it as derivation on $S_{\mathbb{C}}(J) = M_{16}(\mathbb{C}) \oplus M_{16}(\mathbb{C})$, defining in this way a connection $\tilde{\nabla}^0 : \mathfrak{g}_{SM} \to End(S_{\mathbb{C}}(J))$ and we take as the base connection $\nabla^0 : \mathfrak{g}_{SM} \to End(M)$ the corestriction of $\tilde{\nabla}^0$ to the space of linear endomorphisms of the $J-$submodule M of $S_{\mathbb{C}}(J)$.
Let us take the following basis of $JSpin_9$

$$\left\{\mathbb{1}, \gamma^i\right\}_{i=1}^9$$

where

$$\gamma^i \circ \gamma^j = \delta^{ij}\mathbb{1}.$$

We denote by the same symbol the image of γ^i in the complex $*$ enveloping M and in order to study the module endomorphisms of M we introduce the following basis:

$$\left\{\mathbb{1}, i^{\frac{k(k-1)}{2}}\gamma^{i_1,\dots,i_k}\right\}_{k=1}^9$$

where $\gamma^{i_1,\dots,i_k} = \gamma^{i_1}...\gamma^{i_k}$ and again $i_1 < ... < i_k$ (in what follows,we might consider elements of the above basis up to a sign).
Let $f : M \to M$ be a linear endomorphism of M, we write

$$f(x) = a_0(x)1 + i^{\frac{k(k-1)}{2}} a_I(x)\gamma^I \tag{6.7}$$

where all the a's are real coefficients ans I is an ordered multindex

$$I = i_1, ..., i_k \quad i_1 < ... < i_k$$

and when evaluated on an element basis we are going to write

$$a_I^J := a_I(\gamma^J)$$

and similar, for any multi index $J = i_1, ..., i_k$.
Let us point out that the action of $JSpin_9$ on homogeneous elements inside M is either of degree $+1$ or -1 and, due to the presence of the $i^{\frac{k(k-1)}{2}}$ factors and the fact that M is a real vector space, the only admissible changes of degree are the following

$$0 \leftrightarrow 1, \quad 2 \leftrightarrow 3, \quad 4 \leftrightarrow 5, \quad 6 \leftrightarrow 7, \quad 8 \leftrightarrow 9 \tag{6.8}$$

and, in order to characterize the module homorphisms of M, we have to evaluate (6.7) only for the kinds of steps listed in (6.8).
We use the following notation: for a multindex $I = i_1, ..., i_k$ we denote by $\check{I}^l$ the multindex $i_1, ..., \check{i}_l, ..., i_k$ obtained from I by removing i_l, similarly we denote by I_+^l the multindex $i_1, ..., i_k, i_l$ obtained by adding i_l to I.
Let $f : M \to M$ be a linear endomorphism for M, we are going to study which constraint it must respect in order to be a Jordan module endomorphism.

- For the step $0 \leftrightarrow 1$ we evaluate (6.7) for $x = 1$ and $x = \gamma^i$ and we have the system given by the two set of equations

$$\begin{cases} \gamma^i f(\gamma^i) = f(1) \\ \gamma^j f(\gamma^i) = 0 \end{cases} \tag{6.9}$$

  where we take $j \neq i$. From the second set of equations above we have
  - $a_0^i = 0$ for all $i \in \{1, ..., 9\}$.
  - $a_j^i = 0$.
  - $a_I^i = 0$ for $I = i_1, ..., i_k$ when k is even and every $i_k \neq j$.
  - $a_I^i = 0$ for $I = i_1, ..., i_l, ..., i_k$ when k odd and I contains one index $i_l = j$.

  From the last two conditions, ranging on all possible values for $j \neq i$, we find that $a_I^i = 0$ for every I with degree greater then 1. Using the first set of equations we find that

$$a_0(1) = a_i^i =: \lambda^1$$

  for some $\lambda_1 \in \mathbb{R}$ and for every $i = 1, ..., 9$. Summarizing, we have

$$\begin{aligned} f(1) &= \lambda^1 1 \\ f(\gamma^i) &= \lambda^1 \gamma^i. \end{aligned} \tag{6.10}$$

- For the step $2 \leftrightarrow 3$ we have

$$\begin{cases} \gamma^{i_3} f(\gamma^{i_1 i_2 i_3}) = f(\gamma^{i_1 i_2}) \\ \gamma^{i_4} f(\gamma^{i_1 i_2 i_3}) = 0 \end{cases} \tag{6.11}$$

  from the second set of equations we get
  - $a^{i_1 i_2 i_3}_{j_1 j_2 j_3} = 0$ for $j_1 j_2 j_3 \neq i_1 i_2 i_3$.
  - $a^i_I = 0$ for $I = i_1, ..., i_k$ when k is even and every $i_k \neq j$.
  - $a^i_I = 0$ for $I = i_1, ..., i_l, ..., i_k$ when k odd and different from 3 and I contains one index $i_l = j$.

  Again, from the last two conditions, ranging on all possible values for $j \neq i$ and we find that $a^i_I = 0$ for every I with degree different from 3.

After some analogues computations we find the following result.

Proposition 6.4.1. *Let M be the Jordan module $J^2_{16} \oplus J^2_{16}$ over $JSpin_9$ and denote by M_I the eigenspaces of the action of $JSpin_9$ where $I = (i, j)$ is a double index which take the values listed in* (6.8)*. If $f : M \to M$ is a Jordan homomorphism, we have $f = \sum_I \lambda^I id_I$ where id_i denotes the identity on the subspace M_I.*

Chapter 7

Spin geometry of the rational noncommutative torus

In this last chapter we are going to study a fairly different topic with respect to the differential calculus for Jordan algebras and modules that was addressed in the previous part of the thesis through chapters from 1 to 5.
Namely we are going to review the content of [8] and will talk about the spin geometry of the rational noncommutative torus.
It is worth to point out that, even if stated in the completely different context of Connes' spectral triples, the noncommutative approach taken here is deeply similar to what we have done in the previous chapters, namely we are going to study the geometry of a quantum space for which the discrepancy with respect to a classical space is given by a mere finite dimensional quantum space whose only available description is provided by a dual algebra of noncommutative functions.

Contrary to the convention taken in the previous part of the thesis, when using the term "algebra" in this chapter, we are always going to assume associativity and to consider consider complex $*$ algebras.

This chapter is structured as follows. In the first section we will recall the notion of spectral triples as noncommutative generalizations of Rienmannian spin manifolds. In the second section we are going to introduce the rational noncommutative torus by presenting its spectral triple built from an abstract universal C* algebra; then in the subsequent subsections we provide two isomorphic presentations of the same object as

- A spectral triple over an algebra bundle on the classical torus $\mathbb{T}^2$.
- A spectral triple over a subalgebra of invariants of the tensor product of $C^\infty(\mathbb{T}^2)$ with a finite dimensional algebra with respect to the action of a

finite group.

In the third section, we make use of a suitable definition of double covering for noncommutative spaces in order to build several spectral triples on the noncommutative torus and then study their counterparts in the light of the isomorphisms presented in the previous section.

Finally in the last section we briefly address the subject of curved geometries for the noncommutative torus, again using the isomorphism of second section to delucidate the geoemtrical insights of the topic.

7.1 Spectral triples

The geometry of a compact Riemannian spin manifold M can be encoded ([11],[5]) in terms of its canonical spectral triple, which consists of the algebra of smooth complex functions on M, the Hilbert space $L^2(M, \Sigma)$ of square integrable Dirac spinors on M and the Dirac operator on M. More generally the following definition holds

Definition 7.1.1. *A spectral triple is the datum $(A, \mathcal{H}, D)$ of a unital $*$-algebra A, a Hilbert space $\mathcal{H}$ carrying a faithful unitary representation $\pi : A \to B(\mathcal{H})$, and a selfadjoint operator D on $\mathcal{H}$ with compact resolvent, such that the commutators $[D, \pi(a)]$ are bounded operators for any $a \in A$.*

A spectral triple is called *even* if there is a $\mathbb{Z}/2$-grading operator χ commuting with $\pi(a)$ for any $a \in A$ and anticommuting with D. Furthermore, it is called *real* if there is a $\mathbb{C}$-antiunitary operator J, such that $[a, JbJ] = 0$ for $a, b \in A$, $J^2 = \epsilon$, $JD = \epsilon' DJ$ and $J\chi = \epsilon'' \chi J$. The three signs $\epsilon, \epsilon', \epsilon''$ determine the so called KO-dimension of the spectral triple.

Definition 7.1.2. *We call two spectral triples $(A_1, \mathcal{H}_1, D_1)$ and $(A_2, \mathcal{H}_2, D_2)$ isomorphic if and only if there exist an isomorphism of algebras $\gamma : A_1 \to A_2$ together with unitary operator $T : H_1 \to H_2$ such that*

$$Ad_T \pi_1 = \pi_2 \gamma \tag{7.1}$$

and

$$TD_1 = D_2 T \tag{7.2}$$

In case of spectral triples that are even, real or both, we require in addition that

$$T\chi_1 = \chi_2 T, \tag{7.3}$$

$$TJ_1 = J_2 T, \tag{7.4}$$

or both.

Note that (7.2) can be equivalently stated in terms of $\check{\pi} : A \times \mathcal{H} \to \mathcal{H}$, $\check{\pi}(a,h) = \pi(a)h$, as

$$T\check{\pi}_1 = \check{\pi}_2(\gamma \times T) \tag{7.5}$$

Definition 7.1.3. *The product of two (even) spectral triples $(A_1, \mathcal{H}_1, D_1)$ with $(A_2, \mathcal{H}_2, D_2)$ is given by*

$$(A_1 \otimes A_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D_1 \otimes 1 + \chi_1 \otimes D_2), \tag{7.6}$$

where we assume that the algebraic tensor product of algebras can be suitably completed, and we use the usual tensor product of Hilbert spaces. Equivalently one can take $D_1 \otimes \chi_2 + 1 \otimes D_2$ as Dirac operator. Furthermore, in the case of even real spectral triples the grading is $\chi_1 \otimes \chi_2$, and the real structure is $J_1 \otimes J_2$.

In the case that both the spectral triples are commutative, the product of spectral triples coincides with a spectral triple built on the product of the corresponding Riemannian manifolds.

By almost-commutative spectral triple (see e.g. [11],[55]), we mean the product of the canonical spectral triple $(C^\infty(M), L^2(M,\Sigma), \not{D})$ with a finite spectral triple, i.e. one with finite dimensional Hilbert space $\mathcal{H}_2 = \mathbb{C}^n$.
Thus, taking advantage of the isomorphism $L^2(M,\Sigma) \otimes \mathbb{C}^n \approx L^2(M, \Sigma \times (M \times \mathbb{C}^n))$ we see that the Hilbert space consists of n-copies of Dirac spinors (globally), and the algebra of smooth A_2-valued functions on M.

A generalization of this notion is given when one allows the Hilbert space to consist of L^2 sections of the product of Σ with a (locally trivial) vector bundle with a typical fiber $\mathbb{C}^n$, and correspondingly the algebra to consist of smooth sections of some (locally trivial) bundle of finite dimensional $*$-algebras.

Definition 7.1.4. *A topologically non-trivial almost-commutative manifold is a spectral triple of the form $(C^\infty(M,F), L^2(M, \Sigma \otimes E), D)$, where F is an algebra sub-bundle of endomorphisms of a (locally trivial) finite rank hermitian vector bundle E on M and the operator D has locally an almost product form*

$$D = \not{D} \otimes \mathrm{id}_E + \chi \otimes D_E, \tag{7.7}$$

where $D_E \in \mathrm{End(E)}$.

In particular, in what follows, we are going to be interested in the case where, even though F is trivial from the topological point of view, it is globally non-trivial

as algebra bundle.

A special case of the above definition occurs when M is a quotient of another Riemannian spin manifold $\tilde{M}$ by a group of isometries G (for simplicity assumed to be discrete, or even finite, and preserving the spin structure). Then our spectral triples can be built from the canonical spectral triple $(C^\infty(\tilde{M}), L^2(\Sigma), \not{D})$ on $\tilde{M}$ and a finite G-equivariant noncommutative spectral triple (A, H, D). More precisely, the relevant algebra will be given by the G-invariant subalgebra of $C^\infty(\tilde{M}) \otimes A$, the Hilbert space given by the G-invariant Hilbert subspace of $L^2(\Sigma) \otimes H$ and the Dirac-type operator given by $\not{D} \otimes 1 + \chi \otimes D$, where χ is the chiral grading of $L^2(\Sigma)$.

7.2 Rational noncommutative torus

In this section, we establish an isomorphism between the standard spectral triple on the rational noncommutative torus, that is $\left(C^\infty(\mathbb{T}^2_{p/q}), L^2(\mathbb{T}^2_{p/q}) \otimes \mathbb{C}^2, D_{p/q}\right)$, and two other spectral triples.

The first one is $(\Gamma^\infty(F), L^2(F) \otimes \mathbb{C}^2, D_F)$, where F is an algebra bundle of $q \times q$ matrices over the torus $\mathbb{T}^2$, and D_F is certain differential operator on F. This construction originates from a known (see e.g. [5]) isomorphism between the algebras $C(\mathbb{T}^2_{p/q})$ and the continuous sections of F, which as a bundle of algebras is not a product bundle. It is an example of Definition 7.1.4 with F regarded as self-endomorphisms consisting of fiber-wise left multiplication.

The second one, denoted as $\left(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}, \mathcal{D}_{p/q}\right)$, is built on $\mathcal{A}_{p/q}$ which is the subalgebra of $\mathbb{Z}_q \times \mathbb{Z}_q$-invariants of $C(\mathbb{T}^2) \otimes M_q$, $\mathcal{H}_{p/q}$ is a Hilbert subspace of $\mathbb{Z}_q \otimes \mathbb{Z}_q$-invariants in $L^2(\mathbb{T}^2) \otimes M_q \otimes \mathbb{C}^2$, and $D_{p/q}$ is the canonical Dirac operator $\not{D}$ on $\mathbb{T}^2$ tensor the identity on M_q.

For both the spectral triples the Hilbert spaces are given as completions of the corresponding two algebras with respect to certain norms, while the Dirac operators are defined in such a way that their action on the respective spaces satisfy (7.1).

7.2.1 The standard spectral triple

We recall the definition of the standard spectral triple on the noncommutative torus $\mathbb{T}_\theta$ with a parameter θ.

The algebra

Definition 7.2.1. *Let U, V be two unitary generators with the commutation relation*

$$UV = \lambda VU, \tag{7.8}$$

where $\lambda = e^{2\pi i\theta}$, $0 \leq \theta \leq 1$. The algebra $C^\infty(\mathbb{T}^2_\theta)$ of smooth complex valued functions on the noncommutative torus consists of the series:

$$a = \sum_{(m,n)\in\mathbb{Z}_2} a_{mn} U^m V^n, \tag{7.9}$$

where the double sequence of $a_{mn} \in \mathbb{C}$ satisfies

$$\| a \|_k := \sup_{(m,n)\in\mathbb{Z}^2} (1 + m^2 + n^2)^k \mid a_{mn} \mid^2 < \infty, \quad \forall k \in \mathbb{N}. \tag{7.10}$$

Clearly, when $\theta = 0$ the algebra $C^\infty(\mathbb{T}^2_0)$ is isomorphic to the algebra $C^\infty(\mathbb{T}^2)$ of smooth complex functions on the classical torus $\mathbb{T}^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = 1 = |z_2|\}$, generated by the identity functions on the two factors $S^1 \subset \mathbb{C}$ denoted (with a slight abuse of notation) by z_1 and z_2 and called coordinate functions on $\mathbb{T}^2$. It should be clear from the context if we regard z_1 and z_2 as numbers or as functions.

The Hilbert space $L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2$

Denote by $\mathfrak{t}$ the following tracial state on $C^\infty(\mathbb{T}^2_\theta)$:

$$\mathfrak{t}\left(\sum a_{mn} U^m V^n\right) = a_{00}\ , \tag{7.11}$$

where a_{00} is the coefficient of 1. We will refer to $\mathfrak{t}$ as the *trace*.
The trace defines a sesquilinear form on $C^\infty(\mathbb{T}^2_\theta)$ by

$$\langle a \mid b \rangle = \mathfrak{t}(a^* b) \tag{7.12}$$

and a norm:

$$|| a || = \sqrt{\mathfrak{t}(a^* a)}. \tag{7.13}$$

Denote by $L^2(\mathbb{T}^2_\theta)$ the Hilbert space obtained by completion of $C^\infty(\mathbb{T}^2_\theta)$ with respect to this norm. It carries a $*-$representation of $C^\infty(\mathbb{T}^2_\theta)$ by left multiplication:

$$\pi(a) : b \mapsto ab, \tag{7.14}$$

i.e. $L^2(\mathbb{T}^2_\theta)$ is a left $*$-module over $C^\infty(\mathbb{T}^2_\theta)$. The elements of $L^2(\mathbb{T}^2_\theta)$ are analogues of Weyl spinors on the noncommutative torus.

As the full Hilbert space of analogues of Dirac spinors on the noncommutative torus we take $L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2$, with the diagonal $*$-module structure over $C^\infty(\mathbb{T}^2_\theta)$.

The Dirac operator D_θ

The trace (7.11) is invariant under the actions of the torus group $\mathbb{T}^2$ on $C^\infty(\mathbb{T}^2_\theta)$ by translations defined as

$$U \mapsto z_1 U, \quad V \mapsto z_2 V, \quad \forall (z_1, z_2) \in \mathbb{T}^2. \tag{7.15}$$

These actions are infinitesimally generated by the two commutating derivations

$$\delta_1 U = iU, \quad \delta_1 V = 0 \tag{7.16}$$

$$\delta_2 U = 0, \quad \delta_2 V = iV. \tag{7.17}$$

The canonical flat Dirac operator on the Hilbert space $L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2$ is a contraction of derivations δ_ℓ with Pauli matrices σ_ℓ (Clifford multiplication):

$$D_\theta = i\left(\sigma_1 \delta_1 + \sigma_2 \delta_2\right) := \begin{pmatrix} 0 & i\delta_1 + \delta_2 \\ i\delta_1 - \delta_2 & 0 \end{pmatrix}. \tag{7.18}$$

Recall that the spectral triple $\left(C^\infty(\mathbb{T}^2_\theta), L^2(\mathbb{T}^2_\theta) \otimes \mathbb{C}^2, D_\theta\right)$ is even, with the grading $\chi_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ that commutes with every $a \in C^\infty(\mathbb{T}^2_\theta)$ and anticommutes with D_θ.
This spectral triple is also real, by taking as real structure

$$J_\theta = -iJ^0_\theta \otimes (\sigma_2 \circ c.c.), \tag{7.19}$$

where $J^0_\theta : \mathcal{H}^0_\theta \to \mathcal{H}^0_\theta$ is the Tomita conjugation:

$$J^0_\theta(a) = a^*. \tag{7.20}$$

It is immediately seen that for $\theta = 0$ the derivations δ_ℓ become the coordinate derivatives that can be expressed also as $\partial_\ell = z_\ell \frac{\partial}{\partial z_\ell}$. Furthermore the standard spectral triple described above is just the canonical spectral triple and in particular

$$D_0 = i\left(\sigma_1 \partial_1 + \sigma_2 \partial_2\right) = \begin{pmatrix} 0 & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & 0 \end{pmatrix} \tag{7.21}$$

is the Dirac operator constructed from the (flat) Levi-Civita connection. It should be however mentioned that this corresponds to a particular choice of a spin structure on the noncommutative torus; we will describe the other spin structures in Section 7.3.

In the following two subsections we will focus on the case in when θ is a rational number, so unless stated differently from now on:

$$\theta = p/q, \quad \text{i.e.,} \quad \lambda = e^{2\pi i p/q}, \tag{7.22}$$

where $0 < p < q \in \mathbb{Z}$ are relatively prime. In this case the center $\mathcal{Z}_{p/q}$ of $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ is generated by U^q and V^q, and is just the invariant subalgebra for the finite subgroup $G \approx \mathbb{Z}_q \times \mathbb{Z}_q$ of (7.15) of pairs of qth roots of 1. The center $\mathcal{Z}_{p/q}$ is isomorphic to $C^\infty\left(\check{\mathbb{T}}^2\right)$, where $\check{\mathbb{T}}^2$ is the quotient of $\mathbb{T}^2$ by the free action κ of G given by

$$\kappa_{m,n}(z_1, z_2) = (\lambda^m z_1, \lambda^n z_2). \tag{7.23}$$

Clearly $\mathbb{T}^2$ is a q^2-fold covering of $\check{\mathbb{T}}^2$ (a principal G-bundle), but $\check{\mathbb{T}}^2$ is also diffeomorphic to a torus. We denote by $[z_1, z_2]_\kappa$ the κ-equivalence classes (orbits of κ).

From the metric point of view we will equip $\mathbb{T}^2$ first with the standard flat Riemannian metric, and then also with some other G-invariant ones. They descend to $\check{\mathbb{T}}^2$ so that π is an isometric submersion. Then $\check{\mathbb{T}}^2$ is actually isometric to $\mathbb{T}^2$ when the latter one is equipped with the original metric rescaled by q^2. These metric properties reflect themselves via certain invariance properties of D_0 in expression (7.18). Namely it commutes with the derivations δ_ℓ and with the torus group action (7.15) they generate; thus, in particular, it is invariant under the subgroup G.

7.2.2 First isomorphic spectral triple

The algebra $\Gamma^\infty(F)$

As it is well known the C*-algebra $C\left(\mathbb{T}^2_{p/q}\right)$ of the rational noncommutative torus is isomorphic to the algebra of continuous sections of certain vector bundle F of $q \times q$ matrix algebras, over a 2-torus. The same holds of course also on the smooth level, as we will present now in full detail.

Let M_q be the algebra of $q \times q$ complex matrices, and define $R, S \in M_q$ by

$$R = \begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ & & \lambda^2 & \\ \dots & \dots & \dots & \dots \\ & & & \lambda^{q-1} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & & \\ \dots & \dots & \dots & \dots \\ & & 1 & 0 \end{pmatrix}. \tag{7.24}$$

We have $R^q = S^q = \mathbb{1}$ and $RS = \lambda SR$. Consider another action τ of G on $\mathbb{T}^2 \times M_q$:

$$\tau_{m,n}(z_1, z_2, A) = (\lambda^m z_1, \lambda^n z_2, R^m S^n A S^{-n} R^{-m}), \quad \forall A \in M_q\ .$$

We denote by $[z_1, z_2, A]_\tau$ the τ-equivalence classes (orbits). The space F of orbits of τ forms a vector bundle over $\check{\mathbb{T}}^2$ with typical fiber M_q, and (well defined)

projection

$$\pi_F : F \to \check{\mathbb{T}}^2, \quad \pi_F : [z_1, z_2, A]_\tau \mapsto [z_1, z_2]_\kappa \ .$$

We remark that the bundle F is associated to the principal G-bundle $\mathbb{T}^2$ over $\check{\mathbb{T}}^2$, via the representation $\rho : G \to End(M_q)$, given by

$$\rho_{m,n}(A) = R^m S^n A S^{-n} R^{-m}, \quad A \in M_q \, .$$

Indeed, the assignment

$$F \ni [z_1, z_2, A]_\tau \mapsto [z_1, z_2, A]_\rho \in \mathbb{T}^2 \times_\rho M_q \, ,$$

is well defined since $[\cdot, \cdot, \cdot]_\rho$ are the equivalence classes of the relation

$$(\lambda^m z_1, \lambda^n z_2, A) \sim (z_1, z_2, \rho_{m,n}^{-1} A),$$

and is an isomorphism.

The smooth sections of F form a $*$-algebra $\Gamma^\infty(F)$ with respect to the point-wise multiplication and point-wise hermitian conjugation of matrices. Its obvious completion is the C*-algebra of continuous sections.

Summarizing, one has the following

Lemma 7.2.2. *The map Q defined on the generators by*

$$U \mapsto \xi_U, \, V \mapsto \xi_V, \tag{7.25}$$

where

$$\xi_U : \check{\mathbb{T}}^2 \to F, \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, z_1 S]_\tau \ ,$$
$$\xi_V : \check{\mathbb{T}}^2 \to F, \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, z_2 R^{-1}]_\tau$$

extends to a $*$-isomorphism of algebras $Q : C^\infty\left(\mathbb{T}^2_{p/q}\right) \to \Gamma^\infty(F)$.

Proof. It is straightforward to check that ξ_U and ξ_V are well defined and (7.25) extends to a $*$-isomorphism due to the properties of the Fourier coefficients of a smooth function and the exchange rule $\xi_U \xi_V = \lambda \, \xi_V \xi_U$. □

Note that F is trivial as a vector bundle but it is nontrivial as the bundle of algebras. Indeed the monomials ${\xi_U}^m {\xi_V}^n$ define a basis of $\Gamma^\infty(F)$ over $C^\infty(\check{\mathbb{T}}^2)$ and any $\xi \in \Gamma^\infty(F)$ can be written as

$$\xi = \sum_{m,n=1}^{q} f_{mn} {\xi_U}^m {\xi_V}^n, \tag{7.26}$$

where $f_{mn} \in C^\infty(\check{\mathbb{T}}^2)$, $\forall\, 1 \leq m, n \leq q$. By viewing the coefficients f_{mn} as a $q \times q$ matrix of functions in $C^\infty(\check{\mathbb{T}}^2)$ we can write an isomorphism of vector bundles

$$F \approx \check{\mathbb{T}}^2 \times M_q, \quad \xi([z_1, z_2]_\kappa) \mapsto ([z_1, z_2]_\kappa, f_{mn}([z_1, z_2]_\kappa)\,.$$

However, for $\theta = p/q \neq 0$ this is not an isomorphism of algebra bundles since the multiplication of sections ξ does not correspond to the matrix multiplication of f_{mn}.

The bundle F is the bundle of all vertical endomorphisms of another complex vector bundle E of rank q. Namely, E is the orbit space of another free action of G this time on $\mathbb{T}^2 \times \mathbb{C}^q$, given by

$$(z_1, z_2, r) \mapsto (\lambda^m z_1, \lambda^n z_2, R^m S^n r),$$

where $r \in \mathbb{C}^q$. In fact, the (completed) algebras $C\left(\mathbb{T}^2_{p/q}\right)$ and $C\left(\check{\mathbb{T}}^2\right)$ are strongly Morita equivalent via the $C\left(\check{\mathbb{T}}^2\right) - C\left(\mathbb{T}^2_{p/q}\right)$ bimodule of continuous sections of E. The bundle E won't play however any role in the definition of the Hilbert space representation of $\Gamma^\infty(F)$, for which we shall employ the bundle F itself, with $\Gamma^\infty(F)$ acting on itself by left multiplication.

We notice that the center of $\Gamma^\infty(F)$ is generated by $\xi_{U^q} = \xi_U^q$ and $\xi_{V^q} = \xi_V^q$ and is isomorphic to the center $\mathcal{Z}_{p/q}$ of $C^\infty\left(\mathbb{T}^2_{p/q}\right)$, and thus also to $C^\infty\left(\check{\mathbb{T}}^2\right)$, in turn identified with the G-invariant subalgebra $C^\infty(\mathbb{T}^2)^G$ of $C^\infty(\mathbb{T}^2)$, via the map that sends $\xi_U^q \mapsto z_1^q$ and $\xi_V^q \mapsto z_2^q$. Of course, over this isomorphism $\Gamma^\infty(F)$ and $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ are isomorphic as modules over their centers.

The Hilbert space $L^2(F) \otimes \mathbb{C}^2$

Now we look for a Hilbert space which can serve as a codomain of the isometric extension of the map Q (7.25) to $L^2(\mathbb{T}^2_{p/q})$. Let us define on $\Gamma(F)$ the following tracial state

$$\mathfrak{t}_F := Q\mathfrak{t}Q^{-1}$$

or, more explicitely

$$\mathfrak{t}_F\,(\xi) = \int_{\check{\mathbb{T}}^2} f_{00}$$

for ξ as in (7.26), where $\int_{\check{\mathbb{T}}^2}$ is the normalized integral. The corresponding sesquilinear form reads

$$\langle \xi \mid \xi' \rangle = \mathfrak{t}_F(\xi^* \xi'), \tag{7.27}$$

where

$$\xi^* = \sum_{m,n=1}^{q} \bar{f}_{mn} {\xi_V}^{-n} {\xi_U}^{-m}. \tag{7.28}$$

We define first the Hilbert space $L^2(F)$ as the completion of $\Gamma(F)$ with respect to the norm defined by the scalar product (7.27). It carries a $*-$representation of $\Gamma(F)$ by left multiplication, i.e. it is a $\Gamma(F)-$module (and similarly for $\Gamma^\infty(F)$). Then as the full Hilbert space for the spectral triple we take $L^2(F) \otimes \mathbb{C}^2$.

Taking advantage of the (inverse) isomorphism Q and its Hilbert space amplification we have

Lemma 7.2.3. *The $*$-representation of $\Gamma^\infty(F)$ on $L^2(F) \otimes \mathbb{C}^2$ is unitarily equivalent to the $*$-representation of $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ on $L^2\left(\mathbb{T}^2_{p/q}\right) \otimes \mathbb{C}^2$.*

The Dirac operator D_F

We define the derivations of the algebra $\Gamma^\infty(F)$ by $\partial^F_\ell := Q\delta_\ell Q^{-1}$, so that their actions on the generators of $\Gamma^\infty(F)$ are just

$$\partial^F_1 \xi_U = i\xi_U, \quad \partial^F_1 \xi_V = 0, \quad \partial^F_2 \xi_U = 0, \quad \partial^F_2 \xi_V = i\xi_V. \tag{7.29}$$

The Dirac operator D_F which satisfies (7.1) for $D_1 = D_\theta$ and $D_2 = D_F$ is then

$$D_F = i(\sigma_1 \partial^F_1 + \sigma_2 \partial^F_2)\ . \tag{7.30}$$

We call a (locally defined) M_q-valued function $\widetilde{\xi}$ on $\mathbb{T}^2$ *local components* of ξ when

$$\xi([z_1, z_2]_\kappa) = [z_1, z_2, \widetilde{\xi}(z_1, z_2)]_\tau.$$

In particular, the local components of ξ_U and ξ_V are respectively $z_1 S$ and $z_2 R^{-1}$.

Next, we call $\widetilde{T}$ local components of an operator T on $\Gamma^\infty(F)$ when $\widetilde{T\xi} = \widetilde{T}\,\widetilde{\xi}$, and similarly for operators on $L^2(F) \otimes \mathbb{C}^2$. In particular, the local components of the differential operators ∂^F_ℓ are simply the coordinate derivatives

$$\widetilde{\partial^F_\ell} = \partial_\ell\ . \tag{7.31}$$

Thus, the local components of D_F are

$$\widetilde{D_F} = i(\sigma_1 \partial_1 + \sigma_2 \partial_2). \tag{7.32}$$

Note that $\widetilde{D_F}$ looks quite like the canonical Dirac operator (7.21) on the torus constructed from the (flat) Levi-Civita connection of the standard metric on $\mathbb{T}^2$, and D_F in fact is unitarily equivalent to (7.21).

The isomorphism

By using Lemmata (7.2.2) and (7.2.3) and the above discussion, we obtain:

Proposition 7.2.4. *The spectral triple* $(\Gamma^\infty(F), L^2(F)\otimes\mathbb{C}^2, D_F)$ *is isomorphic to the standard spectral triple* $\left(C^\infty\left(\mathbb{T}^2_{p/q}\right), L^2\left(\mathbb{T}^2_{p/q}\right)\otimes\mathbb{C}^2, D_{p/q}\right)$*, where* $D_{p/q}$ *is given by* (7.18) *for* $\theta = p/q$*.*

Moreover we can equip $(\Gamma^\infty(F), L^2(F)\otimes\mathbb{C}^2, D_F)$ with a grading and real structure and enhance the isomorphism to an isomorphism of even real spectral triples.

The suitable grading χ is just given by id$\otimes$diag(1,-1). Furthermore it is evident that (7.4) holds for the following real structure:

$$J_F = -iJ_F^0 \otimes (\sigma_2 \circ c.c.), \tag{7.33}$$

where J_F^0 acts on a section $\xi : \mathbb{T}^2 \to F$ by hermitian conjugation, that is:

$$J_F^0\left(\sum_{m,n=1}^{q} f_{mn}{\xi_U}^m{\xi_V}^n\right) = \sum_{m,n=1}^{q} \overline{f}_{mn}\lambda^{-mn}{\xi_U}^{-m}{\xi_V}^{-n}. \tag{7.34}$$

Notice that J_F admits a decomposition along the infinite and finite dimensional component of its Hilbert space:

$$J_F = J \otimes h.c. \tag{7.35}$$

where $J : L^2(\mathbb{T}^2,\Sigma)\otimes\mathbb{C}^2 \to L^2(\mathbb{T}^2,\Sigma)\otimes\mathbb{C}^2$ is the charge conjugation on the spinor bundle of the commutative torus and $h.c.$ denotes fiber-wise hermitian conjugation on the matrix algebra M_q.

7.2.3 Second isomorphic spectral triple

The algebra $\mathcal{A}_{p/q}$

Now we pass to another description of $C^\infty\left(\mathbb{T}^2_{p/q}\right)$. The starting point is the natural bijective identification of an arbitrary smooth section of the bundle F with a smooth function $\varphi : \mathbb{T}^2 \to M_q$ that is κ-ρ-equivariant, i.e. $\varphi \circ \kappa_{m,n} = \rho_{m,n} \circ \varphi$, or more explicitly

$$\varphi(\lambda^m z_1, \lambda^n z_2) = R^m S^n \varphi(z_1, z_2) S^{-n} R^{-m}, \tag{7.36}$$

via the algebra isomorphism

$$\varphi \mapsto \xi_\varphi,$$

where

$$\xi_\varphi : \check{\mathbb{T}}^2 \to F, \quad [z_1, z_2]_\kappa \mapsto [z_1, z_2, \varphi(z_1, z_2)]_\tau \ .$$

Next, we observe that a smooth function $\varphi : \mathbb{T}^2 \to M_q$ is κ-ρ-equivariant as in (7.36), exactly when it is invariant under the pullback of the τ-action of G, which is

$$\tau^*_{m,n}(\varphi) := \rho_{m,n} \circ \varphi_{m,n} \circ \kappa_{m,n}.$$

Furthermore, under the standard identification

$$C^\infty\left(\mathbb{T}^2, M_q\right) = C^\infty\left(\mathbb{T}^2\right) \otimes M_q \tag{7.37}$$

the subalgebra $C^\infty\left(\mathbb{T}^2, M_q\right)^\tau$ of τ^*-invariant functions corresponds to the subalgebra

$$\mathcal{A}_{p/q} := \left(C^\infty(\mathbb{T}^2) \otimes M_q\right)^{\kappa\otimes\rho}$$

of invariant elements under the action of the tensor product representation $\kappa \otimes \rho$ of G.

With these observations we can state:

Lemma 7.2.5. *The map T defined on the generators by*

$$U \mapsto u := z_1 \otimes S, \quad V \mapsto v := z_2 \otimes R^{-1}, \tag{7.38}$$

where z_ℓ is the ℓ-th coordinate function on $\mathbb{T}^2$, extends to a $*$-isomorphism from the algebra $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ to the algebra $\mathcal{A}_{p/q} = \left(C^\infty(\mathbb{T}^2) \otimes M_q\right)^{\kappa\otimes\rho}$.

Proof. By a straightforward check using the properties of the Fourier coefficients of a smooth function and noting that $u = z_1 \otimes S$ and $v = z_2 \otimes R^{-1}$ are $\kappa\otimes\rho$-invariant, unitary and satisfy $uv = \lambda vu$.

Next, it is easily seen that any element in $\mathcal{A}_{p/q}$ can be written as:

$$\sum_{r,s=0}^{q-1} f_{rs}(z_1, z_2) u^r v^s, \tag{7.39}$$

where, for any $(r, s) \in (\mathbb{Z}/q)^2$, $f_{rs}(z_1, z_2)$ are Schwartz functions on $\mathbb{T}^2$. Since u and v are invariant such an element is $\kappa \otimes \rho$-invariant if and only if each f_{rs} is κ-invariant, that is defines a function on $\check{\mathbb{T}}^2$. Thus the set $\{u^m v^n\}$ for $m, n \in \mathbb{Z}$ is a basis of $\mathcal{A}_{p/q}$ over $C^\infty\left(\check{\mathbb{T}}^2_{p/q}\right)$. This shows surjectivity of T and concludes the proof. □

Lemma 7.2.5 and its direct proof refine to smooth algebras the *-isomorphism in [37] for the rational rotation algebra, where the classification theory of C*-algebras admitting an ergodic action of $\mathbb{T}^2$ is used.

The Hilbert space $\mathcal{H}_{p/q}$

Let $\mathcal{H}^0_{p/q} := (L^2(\mathbb{T}^2) \otimes M_q)^{\kappa\otimes\rho}$ be the Hilbert subspace of invariant elements in $L^2(\mathbb{T}^2) \otimes M_q$ under the (extension of the bounded) action of the tensor product representation $\kappa \otimes \rho$ of G, which also is the same as the obvious completion of $\mathcal{A}_{p/q}$. For the spectral triple on $\mathcal{A}_{p/q}$ we take as Hilbert space

$$\mathcal{H}_{p/q} = \mathcal{H}^+_{p/q} \oplus \mathcal{H}^-_{p/q} = \mathcal{H}^0_{p/q} \otimes \mathbb{C}^2, \tag{7.40}$$

where the superscripts $+$ and $-$ are just to mark which copy of $\mathcal{H}^0_{p/q}$ is in the ± 1 eigenspace of the grading operator $\gamma_{p/q}$ =diag$(1, -1)$ Then, taking advantage of the (inverse) isomorphism T given by (7.38) and its Hilbert space amplification, it is clear that:

Lemma 7.2.6. *The Hilbert modules* $\left(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}\right)$ *and* $\left(C^\infty(\mathbb{T}^2_{p/q}), L^2(\mathbb{T}^2_{p/q}) \otimes \mathbb{C}^2\right)$ *are* $*-$ isomorphic.

The Dirac operator $\mathcal{D}_{p/q}$

Now we are going to select the Dirac operator $\mathcal{D}_{p/q}$ on $\mathcal{H}_{p/q}$ in such a way that (7.1) is satisfied for $D_1 = D_\theta$ and $D_2 = \mathcal{D}_{p/q}$. For sake of simplicity, we start with Weyl spinors (of grade $+1$) on the noncommutative torus. By linearity, it is enough to check (7.1) on each vector $u^m v^n = T(U^m V^n)$ of the basis, which in view of

$$(\delta_1 + i\delta_2)\, U^m V^n = i\,(m + in)\, U^m V^n \tag{7.41}$$

requires that

$$\begin{aligned} \mathcal{D}_{p/q} u^m v^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= i\,(m + in)\, u^m v^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= i\,(\partial_1 + i\partial_2) \otimes 1\,(z^m w^n \otimes R^m S^n) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{7.42}$$

Thus the appropriate Dirac operator on $\mathcal{H}_{p/q}$ reads (modulo exchange of the tensor factors in $M_q \otimes \mathbb{C}^2$):

$$\mathcal{D}_{p/q} = \not{D} \otimes \mathbb{1}_q, \tag{7.43}$$

that has the usual product form (7.6) though with a vanishing second term and acting not on the full tensor product of Hilbert spaces but only on its subspace of $\kappa \otimes \rho$-invariants.

The isomorphism

Similarly to the treatment of the first isomorphic spectral triple, by means of 7.2.5 and 7.2.6 and the above discussion, we obtain:

Proposition 7.2.7. *The spectral triple* $\left(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}, \mathcal{D}_{p/q}\right)$ *is isomorphic to the standard spectral triple* $\left(C^\infty(\mathbb{T}^2_{p/q}), L^2(\mathbb{T}^2_{p/q}) \otimes \mathbb{C}^2, D_{p/q}\right)$*, where* $D_{p/q}$ *is given by* (7.18) *for a fractional* $\theta = p/q$*.*

Furthermore this isomorphism becomes an isomorphism of even real spectral triples if we equip $\left(\mathcal{A}_{p/q}, \mathcal{H}_{p/q}, \mathcal{D}_{p/q}\right)$ with the grading $\gamma_{p/q}$ as above and a real structure satisfying (7.4) given by the $\mathbb{C}-$antiunitary operator

$$\mathcal{J}_{p/q} = -i\mathcal{J}^0_{p/q} \otimes (\sigma_2 \circ c.c.), \tag{7.44}$$

where $\mathcal{J}^0_{p/q}$ acts by component-wise conjugation:

$$\mathcal{J}^0_{p/q}\,(f \otimes A) = \overline{f} \otimes A^* \ . \tag{7.45}$$

It should be mentioned that the second isomorphic realization as a suitable subalgebra of the tensor product of two algebras of this subsection echoes the splitting homomorphism of theta deformations of the canonical spectral triple [12] (see also [13]), and in particular the spectral triple of Proposition 7.2.7 is a concrete example for the class of spectral triples obtained in theorem 5.14 of [7] through a refinement of the aforesaid splitting homomorphism for the specific case of Connes-Landi deformations of commutative spectral triples when the deformation is performed via a rational parameter θ. Such realization is in fact suited also to infinite dimensional "internal" algebras and overcomes the requirement that in the second description, vector bundles need to be of finite rank.

7.3 Inequivalent spin structures

On the noncommutative torus the inequivalent spin structures correspond in a natural manner to double coverings. This is most easily seen in the commutative case, since $\mathbb{T}^2$ is parallelizable. Thus the structure group of its bundle of oriented orthonormal frames can be reduced to the trivial (one element) group, and so the total space of such a reduced bundle is just a copy of $\mathbb{T}^2$ itself, with a projection on the base being the identity map. In a similar way, the whole spin structure can be reduced: the structure group $Spin(2)$ to its two element subgroup $\mathbb{Z}_2$, the total space of the principal $Spin(2)$-bundle to a double cover of $\mathbb{T}^2$ and the spin structure map to the double covering map. The fully fledged (non-reduced) spin structure can be reconstructed from the double cover as the bundle associated

with the natural action of $\mathbb{Z}_2$ on $Spin(2)$ (as a subgroup). It is also a matter of straightforward checking that two such spin structures are equivalent precisely if and only if the double coverings are equivalent.

In the so reduced setting the Weyl spinors are just sections of the bundle associated with the faithful representation of $\mathbb{Z}_2$ on $\mathbb{C}$, or equivalently $\mathbb{Z}_2$-equivariant complex valued smooth functions on the double cover of $\mathbb{T}^2$, or what is the same, (-1)-eigenfunctions of the generator of $\mathbb{Z}_2$. Then, of course, the Dirac spinors are just two copies of Weyl spinors.

All that makes sense also in the noncommutative realm by working dually in terms of algebras. The appropriate language is actually that of noncommutative double coverings, interpreted as noncommutative principal $\mathbb{Z}_2$-bundles. For our purposes this will essentially mean that we consider C^* algebras that contain $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ as a subalgebra of index 2. More precisely, we formulate it as follows.

Definition 7.3.1. *Let A and B be C^* algebras. We say that B is a noncommutative double coverings of A if B is a graded algebra $B = B^0 \oplus B^1$, such that $B^0B^1 = B^1 = B^{1*}$, $B^1B^1 = B^0$ and B^0 is isomorphically identified with A. Two double noncommutative coverings B and B' of A are said to be equivalent if and only if there is a $*$-isomorphism from B to B' that is identity on A.*

This definition extends easily to suitable pre-C^* algebras of B and of A, and in particular to the case of smooth noncommutative torus.

7.3.1 Inequivalent double coverings

In the classical case, there exist four inequivalent double coverings $c_{j,k}\colon \widetilde{\mathbb{T}}^2_{j,k} \to \mathbb{T}^2$ labelled by a pair of indices j, k that take the values 0 or 1:

Table 1.

j,k	$\widetilde{\mathbb{T}}^2_{j,k}$	$c_{j,k}$
$0,0$	$\mathbb{T}^2 \times \mathbb{Z}_2$	$(z, \pm 1) \mapsto z$
$1,0$	$\mathbb{T}^2$	$(w,z) \mapsto (w^2, z)$
$0,1$	$\mathbb{T}^2$	$(w,z) \mapsto (w, z^2)$
$1,1$	$\mathbb{T}^2/\mathbb{Z}_2^{diag}$	$[(w,z)] \mapsto (w^2, z^2)$

(7.46)

The double coverings of noncommutative torus have been studied in [20]. Just like in the classical case there exist four inequivalent double coverings, and thus four inequivalent spin structures for arbitrary parameter θ. They can be again labelled

by $j,k = 0,1$ and to simplify the notation of the double covers $C^\infty((\widetilde{\mathbb{T}}^2_{j,k})_\theta)$ we shall denote them by $\mathcal{C}_{j,k}$ and correspondingly denote their even and odd part respectively by $\mathcal{C}^0_{j,k}$ and $\mathcal{C}^1_{j,k}$.

The concrete form of the algebras $\mathcal{C}_{j,k}$ is shown in the first column of the Table 2 below. The corresponding embedding homomorphisms

$$h_{j,k} : \mathcal{C}_{j,k} \hookleftarrow C^\infty(\mathbb{T}^2_\theta)$$

of $C^\infty(\mathbb{T}^2_\theta)$ as subalgebras of index 2, send the generators $U_\theta, V_\theta \in C^\infty(\mathbb{T}^2_\theta)$ to the elements listed respectively in the third and fourth column, where we also introduce a label on the generators to indicate the parameter of the corresponding noncommutative torus.

Table 2.

j,k	$\mathcal{C}_{j,k}$	$h_{j,k}(U_\theta)$	$h_{j,k}(V_\theta)$
$0,0$	$C^\infty(\mathbb{T}^2_\theta)\otimes\mathbb{C}^2$	$U_\theta\otimes\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$	$V_\theta\otimes\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$
$1,0$	$C^\infty\left(\mathbb{T}^2_{\frac{\theta}{2}}\right)$	$U^2_{\frac{\theta}{2}}$	$V_{\frac{\theta}{2}}$
$0,1$	$C^\infty\left(\mathbb{T}^2_{\frac{\theta}{2}}\right)$	$U_{\frac{\theta}{2}}$	$V^2_{\frac{\theta}{2}}$
$1,1$	$C^\infty\left(\mathbb{T}^2_{\frac{\theta}{4}}\right)^{\mathbb{Z}'_2}$	$U^2_{\frac{\theta}{4}}$	$V^2_{\frac{\theta}{4}}$

(7.47)

In the last row the generator of $\mathbb{Z}'_2$ acts by

$$U^m_{\frac{\theta}{4}}V^n_{\frac{\theta}{4}} \mapsto (-)^{m+n}U^m_{\frac{\theta}{4}}V^n_{\frac{\theta}{4}}$$

and thus the generators of $\mathbb{Z}'_2$-invariant subalgebra are $U^m_{\frac{\theta}{4}}V^n_{\frac{\theta}{4}}$ with $m+n$ even.

It is not difficult to see that the isomorphic images of $C^\infty(\mathbb{T}^2_\theta)$ under the embeddings $h_{j,k}$ are the subalgebras of $\mathbb{Z}_2$-fixed elements

$$(\mathcal{C}_{j,k})^{\mathbb{Z}_2} = C^\infty(\mathbb{T}^2_\theta)\ .$$

Here the generator of $\mathbb{Z}_2$ acts by

$$a\otimes\left(\begin{smallmatrix}w\\z\end{smallmatrix}\right) \mapsto a\otimes\left(\begin{smallmatrix}z\\w\end{smallmatrix}\right),\ \text{if } j,k=0,0\ ,$$

$$U^m_{\frac{\theta}{2}}V^n_{\frac{\theta}{2}} \mapsto (-)^m U^m_{\frac{\theta}{2}}V^n_{\frac{\theta}{2}},\ \text{if } j,k=1,0\ ,$$

$$U^m_{\frac{\theta}{2}}V^n_{\frac{\theta}{2}} \mapsto (-)^n U^m_{\frac{\theta}{2}}V^n_{\frac{\theta}{2}},\ \text{if } j,k=0,1\ ,$$

$$U^m_{\frac{\theta}{4}}V^n_{\frac{\theta}{4}} \mapsto (-)^{mn} U^m_{\frac{\theta}{4}}V^n_{\frac{\theta}{4}},\ \text{if } j,k=1,1\ ,$$

where in the last case $m+n$ is even. Note that although in the maximally twisted case the fourth root of λ is involved in $C^\infty(\mathbb{T}^2_{\frac{\theta}{4}})$, only the square root of λ really matters in $C^\infty(\mathbb{T}^2_{\frac{\theta}{4}})^{\mathbb{Z}_2'}$. Anyhow a kind of 'transmutation' occurs: the more twisted the spin structure is, the more the commutative parameter λ is involved, namely λ, $\lambda^{1/2}$, $\lambda^{1/4}$.

It turns out that the four spin structures described above are inequivalent and the only four possible if assumed projectively $\mathbb{T}^2$-equivariant:

Proposition 7.3.2. *The four spin structures on the noncommutative 2-torus $\mathbb{T}^2_\theta$ for arbitrary θ represented by the noncommutative double coverings $\mathcal{C}_{j,k}$ are pairwise inequivalent in the sense of Definition 7.3.1. They are the only possible if assumed to be equivariant under one of the four double coverings of $\mathbb{T}^2$ as listed in 7.46.*

Proof. It is straightforward to check that the lifts of $\mathbb{T}^2$ as two-parameter group of automorphisms of $\mathcal{C}_{j,k}$ indeed form the four inequivalent usual double coverings $\widetilde{\mathbb{T}}^2_{j,k}$ of $\mathbb{T}^2$, as listed in 7.46. This can be most easily seen on the odd part $\mathcal{C}^1_{j,k}$ of the algebra $\mathcal{C}_{j,k}$. However, these double covers of $\mathbb{T}^2$ would be in fact equivalent if the noncommutative double coverings in question were equivalent, which shows the first statement.

Next assume that a double covering $B = B^0 \oplus B^1$ of the noncommutative torus algebra A admits a continuous projective action of $\mathbb{T}^2$ by automorphisms, or what is the same, a continuous action of some double covering $\widetilde{\mathbb{T}}^2_{j,k}$ of the torus group $\mathbb{T}^2$ (7.15) as listed in (7.46). By the saturation property of B in Definition 7.3.1 it can be seen that each subspace of a fixed $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_2$-grade if $(j,k)=(0,0)$ or $\mathbb{Z}\times\mathbb{Z}$-grade if $(j,k)=(1,0),(0,1)$ or $(1,1)$, is complex one-dimensional. Clearly a non-zero element b in B^1 of $\widetilde{\mathbb{T}}^2_{j,k}$ weight $(0,0,1 \mod 2)$, $(1,0)$, $(0,1)$ or $(1,1)$ respectively when $(j,k)=(0,0)$, $(1,0)$, $(0,1)$ or $(1,1)$ generates B^1 as a A-module. Then, the assignment to b of respectively $1\otimes\binom{1}{1}$, $U_{\frac{\theta}{2}}$, $V_{\frac{\theta}{2}}$ or $U_{\frac{\theta}{4}}V_{\frac{\theta}{4}}$, extends to a $*$-isomorphism with one of the quantum coverings listed in 7.47, which concludes the proof. □

We we note in passing that the inequivalent double coverings of the group $\mathbb{T}^2$ regarded as lifts to inequivalent spin structures of the canonical action of $\mathbb{T}^2$ on $\mathbb{T}^2$ appeared already in [17].

7.3.2 Inequivalent spectral triples

Now we shall construct a spectral triple for each spin structure. In analogy to the commutative case, Dirac operators of these spin structures are obtained by

lifting the Dirac operator on the base space $\mathbb{T}^2_\theta$.

Algebra

For every spectral triple as its algebra datum we take the even part $\mathcal{C}^0_{j,k} \approx C^\infty(\mathbb{T}^2_\theta)$ of the algebra of functions on the covering space. Instead, as discussed above, the smooth Dirac spinors are direct sum of smooth Weyl spinors, which are just those elements of $\mathcal{C}_{j,k}$ that change sign under the action the generator of $\mathbb{Z}_2$, that is the elements of the odd part $\mathcal{C}^1_{j,k}$. Therefore the space of smooth (two-component) Dirac spinors is just $\mathcal{C}^1_{j,k} \otimes \mathbb{C}^2$.

Hilbert space

Next to obtain a suitable Hilbert space we use the completion $\overline{\mathcal{C}_{j,k}}$ of $\mathcal{C}_{j,k}$ with respect to the norm as in (7.13), and take as square integrable Weyl spinors the elements of its odd part $\overline{\mathcal{C}^1_{j,k}}$ while as the Hilbert space of Dirac spinors we take

$$\mathcal{H}_{j,k} := \overline{\mathcal{C}^1_{j,k}} \otimes \mathbb{C}^2.$$

Dirac operator

To construct the Dirac operator $D_{j,k}$ we start by lifting the derivations on the noncommutative torus to its double coverings, or more precisely, extending the derivations δ_ℓ, $\ell = 1, 2$ to derivations $\tilde{\delta}_\ell$ of the algebras $\mathcal{C}_{j,k}$. In other words, we require the commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{C}_{j,k} & \xrightarrow{\tilde{\delta}_\ell} & \mathcal{C}_{j,k} \\ \uparrow & & \uparrow \\ C^\infty(\mathbb{T}^2_\theta) & \xrightarrow{\delta_\ell} & C^\infty(\mathbb{T}^2_\theta)\,. \end{array} \tag{7.48}$$

As easily seen $\tilde{\delta}_\ell$ are nothing but the usual derivations on the noncommutative tori with the modified parameters $\theta, \theta/2, \theta/2$ and $\theta/4$, respectively as in Table 7.47, times a factor $1/2$ every time the ℓth entry of the spin structure label j_k equals to 1. Then, the action of $\tilde{\delta}_\ell$ on $\mathcal{C}^1_{j,k}$ extends to unbounded densely defined operators on the completions $\overline{\mathcal{C}^1_{j,k}}$ and (diagonally in $\mathbb{C}^2$) on the Hilbert spaces $\mathcal{H}_{j,k}$.

Remark. It is not difficult to see that these operators are precisely the infinitesimal generators of the lifted two-parameter groups of automorphisms forming the four inequivalent double coverings of $\mathbb{T}^2$ as in the Proof of Proposition 7.3.2. $\diamond$

Next we contract these operators (extended derivations) with the Pauli matrices (Clifford multiplication) to get an operator which acts on Dirac spinors:

$$D_{j,k} = i\left(\sigma_1\tilde{\delta}_1 + \sigma_2\tilde{\delta}_2\right). \tag{7.49}$$

In the following table in the first column we list all $D_{j,k}$'s in terms of the usual derivations δ_1 and δ_2 defined on each covering noncommutative torus $\mathcal{C}_{j,k}$ (with parameters as in 7.47), while in the second column we report their respective spectra $Spec\left(D_{j,k}\right)$ as operators on $\mathcal{H}_{j,k}$:

Table 3.

j,k	D_{jk}	$Spec\left(D_{j,k}\right)$
$0,0$	$i\sigma_1\delta_1 + i\sigma_2\delta_2$	$\pm\sqrt{m^2+n^2}$
$1,0$	$\frac{i}{2}\sigma_1\delta_1 + i\sigma_2\delta_2$	$\pm\sqrt{\left(m+\frac{1}{2}\right)^2+n^2}$
$0,1$	$i\sigma_1\delta_1 + \frac{i}{2}\sigma_2\delta_2$	$\pm\sqrt{m^2+\left(n+\frac{1}{2}\right)^2}$
$1,1$	$\frac{i}{2}\sigma_1\delta_1 + \frac{i}{2}\sigma_2\delta_2$	$\pm\sqrt{\left(m+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)^2}$

(7.50)

As it should, the spectral triple for the first spin structure agrees with the one given in Section 7.2.1, since in that case both the even and odd subspaces (of functions and of Weyl spinors) are isomorphic with the algebra $C^\infty(\mathbb{T}^2_\theta)$. Note also that for all the four inequivalent spin structures $D_{j,k}$ are isospectral deformations (have the same spectra) of the classical case $\theta = 0$. Furthermore the four spectral triples defined above are even and real with grading and real structure defined as in subsection 7.2.1 for an appropriate value of the parameter θ.

A few remarks are in order. For any (in particular rational) θ we recalled after [20] the four inequivalent spin structures as double coverings of the noncommutative torus in the sense of noncommutative principal bundles. For any spin structure we constructed a corresponding real spectral triple. In fact, these spectral triples have an additional property of being equivariant under the Lie algebra actions of $u(1) \approx i\mathbb{R}$ (c.f. our lifts of the derivations δ_j), or equivalently, under one of the four inequivalent double covers of the automorphism group of $\mathbb{T}^2$. Thus they fit the classification scheme of [52], where it was also shown that up to a unitary equivalence such real spectral triples are the only four possible. Since it is not difficult to see that an equivalence of double coverings would lead to a unitary equivalence of the related equivariant real spectral triples, it provides an independent proof that the four double coverings listed in 7.47 are the only four possible. Note however

that in contrast with [52] it is claimed in [54] that some of the four spin structures can be equivalent precisely in the case of rational parameter θ, however it is unclear to us what is a relation between the notions and classifications in [52] and [54].

7.3.3 Isomorphic spectral triples

We now assume that $\theta = p/q$ and study isomorphic images of these spectral triples under both of the isomorphisms presented in sections 7.2.2 and 7.2.3.

As just mentioned, the case $(j,k) = (0,0)$ is identical to what we told in previous sections. Furthermore, for all the spin structures the algebra of smooth functions is isomorphic to $C^\infty(\mathbb{T}^2_\theta)$ and hence both to $\Gamma(F)$ and $\mathcal{A}_{\frac{p}{q}}$. Thus we shall just take care of the Hilbert spaces and Dirac operators, in the three nontrivial cases when $(j,k) \neq (0,0)$.

In the following we let

$$G^{10} = G^{01} = \mathbb{Z}_{2q} \times \mathbb{Z}_{2q}, \quad \lambda^{10} = \lambda^{01} = e^{2\pi i \frac{p}{2q}},$$
$$G^{11} = \mathbb{Z}_{4q} \times \mathbb{Z}_{4q}, \quad \lambda^{11} = e^{2\pi i \frac{p}{4q}} .$$

We denote by $\check{\mathbb{T}}^2$ the quotient of $\mathbb{T}^2$ by the free action κ of G^{jk}, given by $\kappa_{m,n}(z_1,z_2) = \left(\left(\lambda^{jk}\right)^m z_1, \left(\lambda^{jk}\right)^n z_2\right)$, similarly as in section 7.2.2.

Now let $(j,k) = (1,0)$ and take $R, S \in M_{2q}$ as in (7.24)and define an action τ of G^{10} on $\mathbb{T}^2 \times M_{2q}$ given by

$$\tau_{m,n}(z_1,z_2,A) = (\lambda^m z_1, \lambda^n z_2, R^m S^n A S^{-n} R^{-m}),$$

where $A \in M_{2q}$. We let F' to be the orbit space of τ, that is a vector bundle over $\check{\mathbb{T}}^2$, with typical fibre M_{2q}. The space of functions $\mathcal{C}^+_{1,0}$ is regarded as the $*-$subalgebra of $\Gamma(F')$ generated by ξ^2_U and ξ_V, where

$$\xi_U : \check{\mathbb{T}}^2 \to F', \quad [z_1,z_2]_\kappa \mapsto [z_1,z_2,z_1 S]_\tau ,$$

$$\xi_V : \check{\mathbb{T}}^2 \to F', \quad [z_1,z_2]_\kappa \mapsto [z_1,z_2,z_2 R^{-1}]_\tau .$$

The odd subalgebra $\mathcal{C}^-_{1,0}$ is isomorphic to the linear span of $U^{2m+1}V^n{}_{(m,n)\in\mathbb{Z}^2}$ with coefficients f_{mn} which are smooth functions on the torus. The Hilbert space of Weyl spinors is isomorphic with its closure with respect to the norm defined by the scalar product

$$(g,f) = \sum_{(m,n)\in\mathbb{Z}^2} \int_{\check{\mathbb{T}}^2} \overline{g}_{-m-n} f_{mn}. \tag{7.51}$$

The Hilbert space $\mathcal{H}_{1,0}$ of Dirac spinors is as usual a direct sum of two Hilbert spaces of Weyl spinors.

Finally, with ∂_1^F, ∂_2^F defined in (7.30), the Dirac operator $D_{1,0}$ is unitarily equivalent to:

$$D_{1,0}^F = i(\sigma_1 \partial_1^F + \sigma_2 \partial_2^F). \tag{7.52}$$

Concerning the second isomorphic spectral triple, we have to regard $\mathcal{A}_{\frac{p}{q}}$ as the subalgebra of $\mathcal{A}_{\frac{p}{2q}}$ generated by $u_{1,0} = z_1^2 \otimes S$ and $v_{1,0} = z_2 \otimes R^{-1}$, while the Hilbert space $\mathcal{H}_{10}$ is isomorphic to its orthogonal complement, completed with respect to the scalar product defined by

$$(g, f) = \sum_{(m,n)\in\mathbb{Z}^2} \overline{g}_{-2m-n-1} f_{2m+n+1} \ . \tag{7.53}$$

Then the Dirac operator D_{10} is unitarily equivalent to the restriction of $\mathcal{D}_{p/q} = \not{D} \otimes \mathbb{1}_q$ to $\mathcal{H}_{10} \otimes \mathbb{C}^2$ (modulo the exchange of the tensor factors as in (7.43)).

Both these descriptions are similar for the spin structure $(0, 1)$ provided that one exchanges the roles of U and V, while for the fourth spin structure $(1, 1)$ one has to replace every M_{2q} by M_{4q} and repeat the constructions above.

7.4 Curved rational noncommutative torus

So far we established isomorphisms between spectral triples by selecting Dirac operators, both for the first and second case, which act trivially as the identity on the finite part of the respective Hilbert spaces. In order to push further the analogy with almost-commutative manifolds, hence with standard model of particles, it would be interesting to get a Dirac operator whose action on the "internal" degrees of freedom is non trivial. In quantum field theory, internal degrees of freedom of a single, isolated fermion change whenever it moves in a space-time region with a gauge field (e.g. electromagnetic) whose field strength is different from zero. From a more mathematical point of view ([55],[25],[28]), having a non zero field strength corresponds to consider a Dirac operator which contains a non flat connection acting on the finite part of the Hilbert space. Roughly speaking, to have a Dirac operator which changes internal degrees of freedom of particles, one should consider the one that describes the analogue of a curved geometry of the internal directions.

Thus we are going to investigate some curved geometries on the rational noncommutative torus in view of the generalizations of the spectral triple on $\mathbb{T}^2_{p/q}$ to the case of spectral triples on a non necessarily flat topologically non-trivial almost-commutative manifold. The study of curvature on noncommutative torus was initiated in [16], where conformal rescaling of the the standard Dirac operator are considered, that have been later generalized to arbitrary conformal class in [31]. A different perspective, that we are going to adopt, can be found in [18] and [19], where the kind of perturbations employed preserves the boundedness of the

commutator of the Dirac operators with any element of the algebra and relies on the real structure J of the noncommutative torus.

For simplicity we focus on the case of trivial spin structure but the other ones can be dealt with analogously. In principle we would like to introduce wide class of perturbations that include the rescalings with a conformal factor in $JC^\infty\left(\mathbb{T}^2_\theta\right)J^{-1}$ and the transformations studied in [18] and [19]

$$D^{(k)} = i\sum_{j,\ell=1,2} \sigma^j k_j^\ell\, \delta_\ell\, k_j'^\ell + h.c.\ , \tag{7.54}$$

where $k_j^\ell, k_j'^\ell \in JC^\infty\left(\mathbb{T}^2_{p/q}\right)J^{-1}$ for $j,\ell \in \{1,2\}$ are assumed such that $D^{(k)}$ has compact resolvent. With this choice of $k_j^\ell, k_j'^\ell$ the operator $D^{(k)}$ maintains the bounded commutators with the algebra elements, while the choice of $k_j^\ell, k_j'^\ell$ from the algebra $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ would lead to a spectral triple with twisted commutators.

To see whether for rational $\theta = p/q$ the counterparts of $D^{(k)}$ on $L^2(F)\otimes\mathbb{C}^2$ and $\mathcal{H}_{p/q}$ will present some non trivial action on the finite part of the respective Hilbert spaces we consider in the following two simple examples the special case

$$D^k = i\sigma^1\delta_1 + i\sigma^2 k\delta_2, \quad \text{where} \quad 0 < k \in JAJ. \tag{7.55}$$

Example 7.4.1. *Let $k = J(U+U^*)J^{-1} + t$, where $t > 2$. The action of D^k on the basis of left handed Weyl spinors is given by*

$$D^k\begin{pmatrix} U^mV^n \\ 0\end{pmatrix} = \begin{pmatrix} 0 \\ D^{k^+}(U^mV^n)\end{pmatrix}\ , \tag{7.56}$$

where

$$D^{k^+}(U^mV^n) = mU^mV^n + n(\lambda U^{m+1}V^n + \lambda^{-1}U^{m-1}V^n + tU^mV^n)\ , \tag{7.57}$$

whose isomorphic image under a similarity with the map Q given by (7.25) *is given by*

$$D_F^{k^+}(\xi_U^m\xi_V^n) = m\xi_U^m\xi_V^n + n(\lambda\xi_U^{m+1}\xi_V^n + \lambda^{-1}\xi_U^{m-1}\xi_V^n + t\xi_U^m\xi_V^n). \tag{7.58}$$

Hence the operator D^{k^+} is unitarily equivalent to

$$D_F^{k^+} = i\partial_1^F + A\,\partial_2^F : L^2(F) \to L^2(F)\ , \tag{7.59}$$

where $A \in \Gamma^\infty(End(F))$ has local components given by

$$\widetilde{A}(z_1,z_2) = \lambda z_1 S + \lambda^{-1}z_1^{-1}S^{-1} + t\mathbb{1}_q\ . \tag{7.60}$$

From this Example we see that in general the operators $D_F^{(k)}$ on the Hilbert space $L^2(F)\otimes\mathbb{C}^2$ and $\mathcal{D}_{p/q}^{(k)}$ on $\mathcal{H}_{p/q}$, do not admit the decomposition $D_1\otimes\mathbb{1}_q+\chi_1\otimes D_2$. However this is not a surprise since even classically a product manifold need not have such a structure on the metric level. Indeed in the noncommutative setting both for the description of the spectral triple on the associated vector bundle F and for the description as subalgebra of $C^\infty(\mathbb{T}^2)\otimes M_q$, topologically non-trivial almost-commutative spectral triples have been modelled on the invariant elements of the respective algebras under some action of $\mathbb{Z}_q\times\mathbb{Z}_q$ which intertwines the external and internal degrees of freedom. Thus, for these spectral triples, we are not able to write every admissible Dirac operator by simply joining together a Dirac operator on the standard spectral triple of the commutative torus with a Dirac operator on the finite space M_q.

Next we use an element k in the center of $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ to transform the Dirac operator.

Example 7.4.2. *Let $k=U^q+U^{-q}+t$, where $t>2$. We consider again the action of D^k on a basis of left-handed Weyl spinor as in the previous example and similar computations lead to*

$$D_F^{k+}\xi_U^m\xi_V^n=m\xi_U^m\xi_V^n+n(\xi_U^{m+q}\xi_V^n+\xi_U^{m-q}\xi_V^n+t\xi_U^m\xi_V^n). \tag{7.61}$$

Hence, with the exchange of tensor factors as in (7.43)*, D^k is unitarily equivalent to*

$$D_F^k=i\sigma^1\partial_1^F+A\,\sigma^2\partial_2^F:L^2(F)\otimes\mathbb{C}^2\to L^2(F)\otimes\mathbb{C}^2\ , \tag{7.62}$$

where $A\in\Gamma^\infty(End(F\otimes\mathbb{C}^2))$ has local components given by

$$\widetilde{A}(z_1,z_2)=z_1^q+z_1^{-q}+t\ . \tag{7.63}$$

This Example shows that the situation is different if we assume that in (7.54) the elements k_j^ℓ and $k_j'^\ell$ belong to the center $\mathcal{Z}_{p/q}$ of $C^\infty\left(\mathbb{T}^2_{p/q}\right)$ (so in fact to the center of $JC^\infty\left(\mathbb{T}^2_{p/q}\right)J^{-1}$ too). We prove the following proposition for transformations (7.54) of the isomorphic spectral triple $(\Gamma(F),L^2(F)\otimes\mathbb{C}^2,D_F)$, the proof for the corresponding transformations of the spectral triple $\left(\mathcal{A}_{p/q},\mathcal{H}_{p/q},\mathcal{D}_{p/q}\right)$ is similar. We adopt the notation of Subsection 7.2.2 for the local components of various partial differential operators.

Proposition 7.4.3. *Let k_j^ℓ, $k_j'^\ell\in\mathcal{Z}_{p/q}$ and let $\check{k}_j^\ell$, $\check{k}_j'^\ell\in C^\infty(\mathbb{T}^2)^G$ be the corresponding elements via the composed isomorphism $\mathcal{Z}_{p/q}=C^\infty\left(\check{\mathbb{T}}^2\right)=C^\infty(\mathbb{T}^2)^G$ that sends $U^q\mapsto z_1^q$ and $V^q\mapsto z_2^q$. The isomorphic image $D_F^{(k)}$ of the operator $D^{(k)}$ defined as in* (7.54) *has a local expression with vanishing second term:*

$$\widetilde{D_F^{(k)}}=\not{D}^{(\check{k})}\mathbb{1}_q\ , \tag{7.64}$$

where

$$\not{D}^{(\check{k})} = i \sum_{j,\ell=1}^{2} \sigma^j \check{k}_j^\ell \, \partial_\ell \, \check{k}_j'^\ell + h.c. \tag{7.65}$$

is the accordingly transformed canonical Dirac operator on the classical torus $\mathbb{T}^2$.

Proof. If k_j^ℓ and $k_j'^\ell$ are in the center of $C^\infty\left(\mathbb{T}^2_{p/q}\right)$, then their isomorphic images regarded as (scalar) multiplication operators on $L^2(F) \otimes \mathbb{C}^2$ have local expressions given by $\check{k}_j^\ell$ and $\check{k}_j'^\ell$. The statement then follows by (7.31). □

This result demonstrates that modifications of the Dirac operator on the rational noncommutative torus by the central elements exhibit in the second description (in terms of bundle F) only the 'external' part given by the usual flat differential Dirac operator appropriately modified by the corresponding functions on the classical torus, while the 'internal' part remains trivial. This of course applies also to the third description.

Chapter 8

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