Scuola Internazionale Superiore di Studi Avanzati

Ph.D. Course in Mathematical Analysis, Modelling, and Applications

Some fine properties of SBV and SBD functions with applications to elastodynamics in domains with growing cracks

Ph.D. Thesis

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Introduction

In the framework of Griffith's theory of fracture mechanics, the energy used to produce a new crack is assumed to be proportional to the crack surface. If the medium under consideration is hyperelastic and brittle, i.e., the elastic deformation outside the fracture surface minimizes an elastic energy independent of the crack and the energy dissipated to produce the crack does not depend on its opening, then the equilibrium configuration can be obtained by solving a minimum problem with suitable boundary conditions. The form of the energy functional is

$$E(u,K) := \int_{\Omega\setminus K} W(\nabla u)\,dx + \mathcal{H}^{n-1}(K), \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ is the reference configuration, $K \subset \Omega$ is the crack surface, $u\colon \Omega \to \mathbb{R}^n$ is the displacement, ∇u is its gradient, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The bulk energy density W takes into account the elastic deformation outside the crack while the surface term $\mathcal{H}^{n-1}(K)$ encodes the energy released to produce the crack K.

The functional (0.1) makes sense in a classical way if K is closed and $u \in C^1(\Omega \setminus K)$. Since the set K, which represents the discontinuity set of u, is free to move inside Ω, this explain why a problem of this type is called a *free discontinuity problem*. However, such a formulation is not easy to handle, since in general it is not easy to find a topology on closed sets which guarantees compactness of minimising sequences (u_k, K_k) with $\sup_k E(u_k, K_k) < \infty$. Therefore it is convenient to formulate the problem in a weaker sense. Following the idea of De Giorgi [28] and De Giorgi and Ambrosio [30] it is possible to introduce a class of discontinuous functions, namely the space of special functions of bounded variation $SBV(\Omega)$, which is defined as those $u \in BV(\Omega)$ with no Cantor's part of the gradient, so that the distributional gradient of u is the sum of the absolutely continuous part and of the part concentrated on the jump set (see Subsection 1.4.1). In its simplest form, i.e. when $W(\cdot) = |\cdot|^2$, the weak definition of (0.1) in the SBV context is the following

$$E(u) := \int_{\Omega} |\nabla u|^2\,dx + \mathcal{H}^{n-1}(J_u), \tag{0.2}$$

where ∇u is the *approximate gradient* of u and J_u is the *jump set* of u (see Definition 1.4.3 and Definition 1.4.4). The crack surface is now identified with the jump of u, which is an $(n-1)$-dimensional set and does not need to be closed. The advantage of this formulation is due to the fact that minimising sequences (u_k) such that $\sup_k E(u_k) < \infty$ with the additional hypothesis $\sup_k \|u\|_{L^\infty} < \infty$ are relatively sequentially compact in $SBV(\Omega)$ ([2, Theorem 4.8]). Nevertheless, since even in presence of lower order terms the energy (0.2) does not allow to deduce L^∞ bounds for minimizing sequences, a more general class of functions, namely $GSBV(\Omega)$ the space of *generalised functions of special bounded variation*, has been introduced in [2]. In this space it is still possible to consider an approximate gradient ∇u and an $(n-1)$-dimensional jump set J_u, while at the same time it is possible to prove a compactness result ([2, Theorem 4.36]) which ensures that minimizing sequences, under the hypothesis $\sup_k E(u_k) < \infty$ and $\sup_k \|u\|_{L^1} < \infty$, are relatively sequentially compact in $GSBV(\Omega)$. This allows to solve minimisation problems relative to the functional (0.2) in presence of suitable lower order terms.

In the framework of linearised elasticity, the variational models for fracture mechanics originated by the seminal paper [41] have a sound mathematical weak formulation in the space $SBD(\Omega)$ which is composed of functions having bounded deformation, whose distributional symmetric gradient Eu has no Cantor's part (see Subsection 1.4.2). The common feature of these models is that the main energy term in its simplest form is the following

$$G(u) := \int_\Omega |\mathcal{E}u|^2 \, dx + \mathcal{H}^{n-1}(J_u), \tag{0.3}$$

where $\mathcal{E}u$ is the *symmetric approximate gradient* of u (see (1.12)). Whenever (u_k) is a minimising sequence such that $\sup_k G(u_k) < \infty$ with the additional hypothesis $\sup_k \|u\|_{L^\infty} < \infty$, one can apply a compactness result proved in [5] which ensures that (u_k) is relatively sequentially compact in $SBD(\Omega)$. As in the $GSBV$ case, the drawback of this result is that it is difficult to obtain a priori bounds on the L^∞-norm for minimising sequences even in presence of lower order terms. To overcome this difficulty, following the seminal paper of Dal Maso [20] it is possible to introduce the more general space $GSBD(\Omega)$ of *generalised functions of special bounded deformation* (see Definition 1.4.20). In [20], by performing a one dimensional slicing argument, the author investigates the fine properties of $GSBD$-functions (more in general of GBD-functions). In this context it is still possible to consider a notion of symmetric approximate gradient $\mathcal{E}u$ and of $(n-1)$-dimensional jump set J_u. Moreover it holds true the following compactness result for $GSBD(\Omega)$: if (u_k) is a minimising sequence such that $\sup_k G(u_k) < \infty$ and $\sup_k \|u_k\|_{L^1} < \infty$, then (u_k) is relatively sequentially compact in $GSBD(\Omega)$. As before, this allows to solve minimisation problem relative to the functional (0.3) in presence of lower order terms.

Summarizing, we can say that the spaces $SBV(\Omega)$ and $SBD(\Omega)$, together with their generalised version $GSBV(\Omega)$ and $GSBD(\Omega)$, have been introduced to solve variational problems for the functionals 0.2 and 0.3. The fine properties of functions belonging to these spaces have played an important role to prove the existence results cited above. This thesis is devoted to investigate further fine properties of $GSBV$ and $GSBD$ functions in order to study minimum problems related to functionals of the form 0.2 and 0.3 with the addition of further terms. As an application we shall study an evolution problem in fracture mechanics.

Fine properties of SBV and SBD functions

In the first part of the thesis we study some new fine properties of SBV and SBD functions. In Chapter 1 we begin by summarizing some known results about the main spaces which will be used in the thesis: sets of finite perimeter, $BV(\Omega)$, $SBV(\Omega)$, $GSBV(\Omega)$, $BD(\Omega)$, $SBD(\Omega)$, and $GSBD(\Omega)$.

In the framework of functions with bounded deformation, we study the continuity properties of the trace operator. We present a new result of continuity for this operator acting on functions that might jump on a prescribed $(n-1)$-dimensional set Γ with the only hypothesis of being rectifiable and of finite measure. Using this result it is possible to study a variational model of linear elasticity with cracks and with non homogeneous prescribed Neumann boundary conditions. Moreover, the same idea can be used to show the existence of a solution to the system of the corresponding elastodynamic problem in presence of prescribed growing cracks and coupled with Dirichlet and Neumann boundary conditions.

Chapter 2

The content of this chapter is based on the results obtained in [70].

In the study of the elastodynamic problem with prescribed crack (see Section 4.2 and 4.3) we are led to study minimum problems of functionals of the form

$$F(u) := G(u) + \int_\Omega |u-g|^2\, dx, \tag{0.4}$$

where G is defined in (0.3) and $g\colon \Omega \to \mathbb{R}^n$ is some square integrable function. The study of minimisation problem related to (0.4) leads us to introduce the following subspace of $GSBD(\Omega)$

$$GSBD^2_2(\Omega) := \{u \in GSBD(\Omega) \mid u \in L^2(\Omega;\mathbb{R}^n),\, \mathcal{E}u \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\}.$$

A useful notion of convergence is the following:

$$\begin{cases} \sup_k \left(\|u_k\|_{L^2} + \|\mathcal{E}u_k\|_{L^2} + \mathcal{H}^{n-1}(J_{u_k})\right) \leq C \\ u_k \to u, \text{ in } L^1(\Omega) \\ \mathcal{E}u_k \rightharpoonup \mathcal{E}u, \text{ weakly in } L^1(\Omega) \end{cases} \tag{0.5}$$

Indeed, as already mentioned, some compactness theorems (see Theorem 1.4.27) can be applied to obtain the convergence (in the sense of (0.5)) of minimising sequences (u_k) such that $\sup_k G(u_k) < \infty$.

If we would like to consider some minimum problems for functional F with prescribed Dirichlet and Neumann boundary condition, we have to study the behavior of the trace operator in the SBD context. When Ω is regular enough, the trace operator $Tr\colon BD(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1})$ is well defined, and continuous with respect to the strong topology in $BD(\Omega)$ (see [3, 68, 73]). Unfortunately, even if we consider the space $SBD^2_2(\Omega) \subset BD(\Omega)$ defined as

$$SBD^2_2(\Omega) := \{u \in SBD(\Omega) \mid u \in L^2(\Omega;\mathbb{R}^n),\, \mathcal{E}u \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\},$$

then the trace operator

$$Tr\colon SBD^2_2(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{n-1}),$$

is *not continuous* with respect to the convergence requirements in (0.5). This lack of continuity is due to the fact that a sequence in SBD may have jump sets getting infinitesimally close to the boundary of Ω. Having this in mind, one can easily produce counterexamples to continuity which lead to a free discontinuity problem with no solution. For example, in the one dimensional version of the functional (0.4) the space SBD reduces to the space SBV. If we consider the minimisation problem with $g = 0$ and with Dirichlet boundary condition of the form

$$\min_{\substack{u\in SBV^2_2((0,1))\\ u(0)=\lambda}} \int_0^1 |u'|^2\, dx + \mathcal{H}^0(J_u) + \int_0^1 |u|^2\, dx, \tag{0.6}$$

it is easy to see that for sufficiently large value of λ, any admissible function pays strictly more then 1 in (0.6), while there exists a minimizing sequence for which the functional (0.6) converges to 1 in the limit.

To bypass this problem, it seems convenient to fix an $(n-1)$-dimensional set Γ, and to study the trace properties of functions whose jump sets are contained in Γ. So we introduce

$$GSBD(\Omega;\Gamma) := \{u \in GSBD(\Omega) \mid J_u \subseteq \Gamma\},$$

and

$$GSBD^p_p(\Omega;\Gamma) := \{u \in GSBD(\Omega;\Gamma) \mid u \in L^p(\Omega;\mathbb{R}^n), \mathcal{E}u \in L^p(\Omega;\mathbb{R}^n)\}, \ (p \geq 1).$$

endowed with the the notion of convergence analogous to (0.5), in this case for general p

$$\begin{cases} \sup_k \left(\|u_k\|_{L^p} + \|\mathcal{E}u_k\|_{L^p} + \mathcal{H}^{n-1}(J_{u_k})\right) \leq C \\ u_k \to u, \ \text{in } L^1(\Omega) \\ \mathcal{E}u_k \rightharpoonup \mathcal{E}u, \ \text{weakly in } L^1(\Omega). \end{cases} \tag{0.7}$$

Note that the bound in the first line of (0.7), when $p > 1$, ensures compactness with respect to this notion of convergence.

The main result of this chapter is that there exists a function Θ such that

$$Tr\colon GSBD^p_p(\Omega;\Gamma) \to L^q(\partial\Omega, \Theta\mathcal{H}^{n-1}), \ (p > 1), \tag{0.8}$$

is continuous for every $1 \leq q < p$ when we consider the convergence introduced in 0.7 on $GSBD^p_p(\Omega;\Gamma)$ and the strong topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$. The result holds also for the case $q = p$ when we consider the weak topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$. The weight function Θ *is* $\mathcal{H}^{n-1}$*-a.e. strictly positive* and depends *only on the geometry* of Γ (see Theorem 2.3.1). We also show that $q = p$ cannot be reached in (0.8) when one considers the strong topology, by exhibiting a counterexample.

When Γ is a closed subset of Ω then $GSBD^p_p(\Omega;\Gamma)$ is equivalent to the space $LD^p_p(\Omega \setminus \Gamma)$ which is the space of all functions $u\colon \Omega\setminus\Gamma \to \mathbb{R}^n$ which are L^p-integrable and whose symmetric distributional gradient is L^p-integrable. Moreover, if Γ is regular enough, by means of Korn's inequality the space $LD^p_p(\Omega \setminus \Gamma)$ is equivalent to the Sobolev space $W^{1,p}(\Omega \setminus \Gamma)$ and the Sobolev's embedding Theorem tells us $u \in L^{p^*}(\Omega)$. If Γ is not regular, we cannot deduce that $u \in GSBD^p_p(\Omega;\Gamma)$ implies $u \in L^{p^*}(\Omega)$, but if we assume $u \in L^{p^*}(\Omega)$, then we can improve our summability results on the exponent q appearing in (0.8). More precisely, we prove that the trace operator is continuous

$$Tr\colon GSBD^p_p(\Omega;\Gamma) \cap L^{p*}(\Omega) \to L^q(\partial\Omega, \Theta\mathcal{H}^{n-1}) \ (p > 1),$$

for every $1 \leq q < p(n-1)/(n-p)$ when we consider the convergence defined by 0.7 in $GSBD^p_p(\Omega;\Gamma) \cap L^{p^*}(\Omega)$ with the addition requirement that $\|u_k\|_{L^{p^*}}$ remains bounded, and the strong topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$. For $q = p(n-1)/(n-p)$ the result holds only in the weak topology on $L^q(\partial\Omega, \Theta\mathcal{H}^{n-1})$. Notice that $p(n-1)/(n-p)$ is the usual critical exponent for the trace of Sobolev functions in $W^{1,p}(\Omega)$.

Looking at the definition of Θ, it is easy to see that when $\overline{\Gamma} \subset\subset \Omega$ then $\Theta \geq \operatorname{dist}(\Gamma, \partial\Omega) > 0$. In the paper we give a weaker property for Γ (an adaptation of the classical *cone condition*), which guarantees that $\operatorname{ess\,inf}_{\partial\Omega} \Theta > 0$. This allows us to deduce the classical continuity properties of the trace without the use of weights (see Proposition 2.1.17 and Remark 2.1.18).

An alternative way to obtain a trace estimate without weights on $\partial\Omega$ is to consider a suitable weight Ψ defined on Ω. More precisely we have proved that there exists Ψ such that, if in addition to the convergence conditions in (0.7) we add the uniform bound on the $L^p(\Omega, \Psi\mathcal{L}^n)$ norm, we have the continuity:

$$Tr\colon GSBD^p_p(\Omega;\Gamma) \cap L^p(\Omega, \Psi\mathcal{L}^n) \to L^q(\partial\Omega, \mathcal{H}^{n-1}) \ (p > 1),$$

for $1 \leq q < p$ if we consider the strong topology on $L^q(\partial\Omega, \mathcal{H}^{n-1})$, and also for $q = p$ if we consider the weak topology on $L^q(\partial\Omega, \mathcal{H}^{n-1})$; here Ψ is a weight function defined on Ω, locally integrable, and that depends only on the geometry of Γ (see Theorem 2.3.1 and Remark 2.3.2). A refined version of this result allows us to prove the following inclusions (see Theorem 2.1.19 and Remark 2.1.21):

$$GSBD^p_p(\Omega;\Gamma) \cap L^p(\Omega, \Psi\mathcal{L}^n) \subset SBD^p_p(\Omega;\Gamma),\ (p > 1),$$

which can be considered as an improvement of the obvious inclusions $GSBD^p_p(\Omega;\Gamma) \cap L^\infty(\Omega) \subset SBD^p_p(\Omega;\Gamma)$.

All the results mentioned above are true in the context of BV-functions, with $GSBD(\Omega)$ replaced by $GSBV(\Omega)$. Moreover, they are true not only for the trace of u on the boundary of Ω, but also for both traces $u^\pm$ on Γ.

As it will be clear from Chapter 4, the reason why we studied the trace operator in these spaces comes from the theory of elasticity with cracks, when we consider a traction applied to some part of the boundary $\partial_N\Omega \subseteq \partial\Omega$. In the weak formulation of the problem this leads to a linear term of the form

$$\int_{\partial_N\Omega} F \cdot Tr(u)\ d\mathcal{H}^{n-1},$$

where F represents the traction force acting on the Neumann part of the boundary. It is clear that the continuity properties of traces for $u \in GSBD^2_2(\Omega;\Gamma)$ can be used to obtain continuity result of this linear form (see Subsection 2.3.3).

In the literature, the problem of the integrability of the trace of BV functions has been studied for example by Maz'ja in [64, Chapter 6], where the trace was defined for open set Ω of finite perimeter. The main results were obtained under the assumption of connectedness of Ω and that normals in the sense of Federer exist almost everywhere on the boundary. The results were generalised to the class of open and connected sets Ω with the only hypothesis that its topological boundary is an $(n-1)$-rectifiable set, by Burago, Kosovskiĭ in [7]. Both works rely on the fact that the *Coarea Formula* holds true for u, and so the distributional gradient of u, as measure, can be reconstructed by averaging the perimeter of each level set of u. In this case, under some *more* regularity conditions on the boundary, one can control the L^1 norm of the trace of u with the *full* norm in BV times a constant that depends only on Ω (see [64, Section 6.6.4.]).

In Temam [72] some continuity properties of the trace operator are studied in the space $BD(\Omega)$, with $\Omega \subset \mathbb{R}^n$ open set with smooth boundary. Here $BD(\Omega)$ is endowed with the norm given by the sum of the L^1-norm of the function with the *total variation* of its symmetric distributional derivative. In this case, he introduces a notion of convergence, where substantially our hypothesis of fixing the jump sets of some sequences $(u_k) \subset GSBD(\Omega)$, is substituted by asking that the total variation of the symmetric distributional gradient $|Eu_k|(\Omega)$ converges to the total variation $|Eu|(\Omega)$ of the limit u. Under this notion of convergence, it is possible to show the continuity of the trace in $L^1(\partial\Omega, \mathcal{H}^{n-1})$.

To underline the difference between our results and those of the papers mentioned above, we have to notice that we work with a notion of convergence that do not take into account the jump part $|u^+ - u^-| \cdot \mathcal{H}^{n-1} \llcorner J_u$ of the total variation measure $|Eu|$. On the other hand we have to impose the constraint $J_u \subset \Gamma$. This leads us to introduce proper weights in order to have continuity results of the trace, but on the other hand we do not make any regularity assumptions nor on Γ neither on Ω (except to be respectively $(n-1)$-rectifiable with finite $\mathcal{H}^{n-1}$-measure, and to be an open set of finite perimeter, respectively). Moreover, we can develop a theory in the SBD (even $GSBD$) context, where the Coarea formula is not available.

A second fine property of functions of bounded variation studied in the thesis, concerns the blow-up behavior near a single point. Precisely, given $p > 1$ we study the blow-up of functions $u \in GSBV$, whose approximate gradient is p-th power summable, under suitable geometric assumptions of the jump set. In analogy with the theory of p-capacity in the context of Sobolev spaces, we prove that the blow-up of u converges up to a set of Hausdorff dimension less than or equal to $n-p$. Moreover, we are able to prove a convergence result, which in the analogue in $GSBV$ of the following property of $W^{1,p}(\Omega)$ functions: whenever u_k strongly converges to u, then up to subsequences, u_k pointwise converges to u except on a set whose Hausdorff dimension is at most $n-p$.

Chapter 3

The content of this chapter is contained in [69].

The following result concerning the Lebesgue points of a Sobolev function is well known (see [34, 36, 48, 64, 75]): given $1 < p < n$, if $u \in L^1_{loc}(\mathbb{R}^n)$ and its first order distributional derivatives are p-th power locally summable, then there exists a set A with $\dim_{\mathcal{H}}(A) \leq n-p$, namely with Hausdorff dimension at most $n-p$, such that every $x \in \mathbb{R}^n \setminus A$ is a Lebesgue point for u. More precisely, for every $x \in \mathbb{R}^n \setminus A$ there exists a real number a such that:

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r(x)} |u(y) - a| \, dy = 0. \tag{0.9}$$

By a change of variables, if we call u_x the function constantly equal to a, the convergence in (0.9) can be rephrased by saying that $u_{r,x}(y) := u(x+ry)$, namely the *blow-up* of u at x, converges in $L^1(B_1(0))$ to u_x, i.e.

$$\lim_{r \to 0^+} \int_{B_1(0)} |u(x+ry) - u_x(y)| \, dy = 0. \tag{0.10}$$

Roughly speaking, (0.10) says in a precise way that the values of u near x are close to a single constant. The aim of this chapter is to investigate the local behavior of functions when we introduce also a jump discontinuity set.

Given $\Omega \subset \mathbb{R}^n$ an open set, for every function u which belongs to the space $GSBV(\Omega)$ and whose approximate gradient ∇u belongs to $L^1(\Omega; \mathbb{R}^n)$, by using the general theory developed in [2] one can deduce that at every point x it holds

$$u_{r,x} \to u_x \text{ in measure in } B_1(0), \text{ as } r \to 0^+,$$

except on a set A with $\mathcal{H}^{n-1}(A) = 0$. Furthermore, if x is a Lebesgue point then u_x is a constant function, while if $x \in J_u$ then u_x assumes two different values on two disjoint subsets of $B_1(0)$ separated by an $(n-1)$-dimensional hyperplane passing through the origin. In this situation u_x may assume from one or two values.

In this chapter we focus our attention on the space $GSBV^p(\Omega)$ when $1 < p \le n$. Precisely, we investigate under which hypothesis on the jump set, the p-th power summability of the approximate gradient guarantees $\dim_{\mathcal{H}}(A) \le n - p$.

To illustrate the result we are going to prove, let us consider the following example. Consider $\Gamma_0 \subset \mathbb{R}^2$ the union of three half lines starting from the origin. Let $\Gamma \subset \mathbb{R}^3$ be defined by $\Gamma_0 \times \mathbb{R}$ and let l be the straight line $\{(0,0,t) \mid t \in \mathbb{R}\}$. The set Γ disconnects $\mathbb{R}^3 \setminus \Gamma$ into three connected components $\Omega_1, \Omega_2, \Omega_3$. Thanks to a well known property of locally integrable $GSBV$-functions, ∇u coincides with the distributional gradient in each open sets Ω_i; then by using Poincaré-Wirtinger inequality on balls, it is easy to prove that every $u \in GSBV^p(\mathbb{R}^3) \cap L^1_{loc}(\mathbb{R}^3)$ with $J_u \subset \Gamma$ satisfies $u \llcorner \Omega_i \in W^{1,p}_{loc}(\Omega_i)$ for $i = 1, 2, 3$. Using a reflection argument, through an obvious modification of the result in [36], there exists a set A with $\dim_{\mathcal{H}}(A) = 3 - p$ such that if $x \in \mathbb{R}^3 \setminus A$ then the blow-up of u at x converges. In addition, on the points $x \in l \setminus A$ the limit u_x can assume three different values α_i each on the set $\Omega_i \cap B_1(0)$, $i = 1, 2, 3$. Therefore, the family of all possible limits u_x is richer than the previous cases.

Nevertheless, the p-th power summability of the approximate gradient is in general not enough to guarantee the convergence of the blow-up at every point except on a set of Hausdorff dimension $(n - p)$. Consider for example $u := \mathbb{1}_E$, the characteristic function of a set with finite perimeter. Clearly ∇u is p-summable for every $p \ge 1$, but from the theory of sets of finite perimeter, we know that the blow-up of u in general converges only up to an $\mathcal{H}^{n-1}$-negligible set. Precisely, it is possible to construct a set $E \subset \mathbb{R}^2$ with finite perimeter and such that, by setting $u = \mathbb{1}_E$, the set of points x where $u_{r,x}$ does not converge has Hausdorff dimension exactly equal to 1 (see Section 3.6). Therefore, it is reasonable to think that the geometry of the jump set affects the local behavior of the functions.

In Definition 3.2.11, for every $1 < p \le n$ we introduce the class $\mathcal{J}_p$ of all admissible jump sets, for which the following two main results hold true.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $\Gamma \in \mathcal{J}_p$ $(1 < p \le n)$. If $u \in GSBV^p(\Omega; \Gamma)$, then there exists a set A_u with Hausdorff dimension at most $n - p$, such that for every $x \in \Omega \setminus A_u$ there exists a function $u_x(\cdot)\colon B_1(0) \to \mathbb{R}$*

$$u_{r,x} \to u_x, \ \text{in measure in } B_1(0), \tag{0.11}$$

as $r \to 0^+$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be open and let $\Gamma \in \mathcal{J}_p$ with $(1 < p \le n)$. Suppose $(u_k)_{k=1}^{\infty} \subset GSBV^p(\Omega; \Gamma) \cap L^p(\Omega)$ is such that*

$$\|u_k - u\|_{L^p} + \|\nabla u_k - \nabla u\|_{L^p} \to 0, \ \text{as } k \to \infty.$$

Then there exists a subsequence $(k_j)_j$, such that for every $x \in \Omega$ except a on set with Hausdorff dimension at most $n - p$ we have

$$(u_{k_j})_x \to u_x \ \text{in measure in } B_1(0), \ \text{as } j \to \infty, \tag{0.12}$$

where in (0.12) $(u_k)_x$ is the one given by (0.11) where u is replaced by u_k.

Theorem 1 can be seen as the analogous of the result (0.10) mentioned above. In the context of Sobolev spaces this is obtained through the theory of capacity, by exploiting the well known fact that smooth functions are dense in $W^{1,p}(\Omega)$. However, it is not known whether there exist dense subspaces of $GSBV^p(\Omega;\Gamma)$ made of regular functions u with the additional constraint $J_u \subset \Gamma$ (see Remark 3.4.23). For this reason, we decide to perform a different analysis based on Geometric Measure Theory techniques. In particular we prove a weak version of Poincaré's inequality on balls, which guarantees that the L^0-distance of u from a particular piecewise constant function can be controlled in terms of the L^p-norm of its approximate gradient plus a small volume error (see Theorem 3.3.3). This tool, together with a fine analysis of the blow-up of u permits us to obtain the conclusion of Theorem 1. The dimension $n-p$ is optimal, since in the $W^{1,p}(\Omega)$ setting, i.e. when $\Gamma=\emptyset$, we already know that it is sharp (see Remark 3.3.14).

Theorem 2 is reminiscent of the following result in the context of Sobolev space: if a sequence u_k in $W^{1,p}(\Omega)$ strongly converges to u, then, up to subsequences, the precise value of $u_k(x)$ defined by (0.10) converges to the precise value of $u(x)$, except on a set of zero p-capacity (see for example [48, Lemma 4.8]). In order to prove Theorem 2, we use a suitable notion of capacity (see Definition (3.93)), which allows us to deduce the convergence (0.12) for every x except on a set of capacity zero. The relation between this novel notion of capacity and the Hausdorff measure (see Theorem 3.4.17) enables us to deduce Theorem 2.

The class $\mathcal{J}_p$ is composed of all $(\mathcal{H}^{n-1}, n-1)$-rectifiable sets (see [35, Subsection 3.2.14]) with finite $\mathcal{H}^{n-1}$-measure, which satisfy a suitable geometric condition at every points except on a set with Hausdorff dimension $n-p$ (see Definition 3.2.11). For example, finite union of $(n-1)$-dimensional manifolds of class C^1 belong to $\mathcal{J}_p$ for every $1<p\leq n$. More in general, finite unions of graphs of Sobolev functions in $W^{2,p}$ belong to $\mathcal{J}_p$ (see Example 3.5.3). As pointed out in Remark 3.5.2, whenever $n>2p+1$, the graph of a $W^{2,p}$-function might have topological closure with arbitrarily large n-dimensional Lebesgue measure. This shows that a generic set in $\mathcal{J}_p$ *does not need to be essentially closed*. In addition, in Example 3.5.5 we are able to construct a set in $\mathbb{R}^2$ which cannot be written as a finite union of graphs, but nevertheless it belongs to $\mathcal{J}_p$ for every $1<p\leq 2$.

In order to define the property which characterizes the sets in $\mathcal{J}_p$, we make use of the theory of indecomposable sets, for which we introduce a geometric quantity called *upper isoperimetric profile* (see (3.13)). This quantity plays a similar role to that of the *Cheeger's constant* in the context of Riemannian manifolds. Roughly speaking, if $\Gamma\in\mathcal{J}_p$ then for every x up to a set of Hausdorff dimension $n-p$, the set $B_1(0)\setminus\Gamma_{r,x}$ can be overrun by N_x indecomposable sets (possibly depending on x), say $(F_{r,i})_{i=1}^{N_x}$, in such a way that the upper isoperimetric profile of the sets $F_{r,i}$ does not vanish as $r\to 0^+$. We call this property *non vanishing upper isoperimetric profile* (see Definition 3.2.7). This property *is optimal* in view of Theorem 1. More precisely, we construct a counterexample to Theorem 1 which shows that, essentially, *the notion of non vanishing upper isoperimetric profile cannot be weakened.*

Elastodynamics in domains with growing cracks

The mathematical models for the evolution of brittle cracks are based on *Griffith's criterion* and have been developed so far mainly in the *quasi-static* case. In this case the boundary conditions and the loads vary slowly in time, so that the material can be assumed to be always in elastic equilibrium. The precise mathematical formulation for quasi-static fracture evolution has been developed in [40] together with [26] and then significant advances were obtained in [4, 6, 12, 13, 21, 24, 26, 40, 43, 54]. A natural step to move forward in the study of crack evolution, is to consider the *dynamic* case. The theory of dynamic fracture mechanics contains basically three principles that can be stated as follows

(1) elastodynamics away from the cracks;

(2) energy-dissipation balance which includes also the surface energy dissipated by the crack;

(3) a principle dictating when a crack must grow.

The first principle simply states the elastic material must obey the wave equation outside the cracks. The second principle is based on Griffith's idea (see [46]) that the crack growth is determined by the competition between the elastic energy of the body and the work needed to produce a new crack, or extending an existing one. The third principle is discussed in some more details in [55], where a maximal dissipation condition is proposed.

In Chapter 4 of the thesis we shall study the problem of elastodynamics out of a prescribed time-dependent crack set.

Chapter 4

This chapter contains the results obtained in [71].

This chapter is devoted to the study of the elastodynamics system in domains with growing cracks. Precisely we consider $\Omega \subset \mathbb{R}^n$ a regular domain as reference configuration, a *fixed* family of growing-in-time (irreversibility assumption) crack sets $\Gamma(t)$ contained in Ω, and $u(t, x)$ the displacement which might be essentially discontinuous for $x \in \Gamma(t)$. Then, given initial conditions on Ω at $t = 0$, and mixed Dirichlet-Neumann boundary condition on $\partial\Omega$, we want to find a solution to the system of (possibly damped) *elastodynamics*

$$\ddot{u}(t) - \mathrm{div}\,[\mathbb{C}\,\mathcal{E}u(t)] - \gamma\mathrm{div}\,[\mathbb{B}\,\mathcal{E}\dot{u}(t)] = f(t), \text{ in } \Omega \setminus \Gamma(t) \tag{0.13}$$

where $\mathbb{C}$ and $\mathbb{B}$ are the elasticity and the viscosity tensors, respectively, $\mathcal{E}u$ denotes the symmetric part of the gradient of u, div denotes the spatial divergence operator, $f(t)$ is a vector field representing the volume force, and at each time t the system (0.13) is complemented with homogeneous Neumann condition on the crack $\Gamma(t)$. This last condition reflects the fact that no external forces are acting on the crack lips (traction free case). The parameter γ can take value only in $\{0, 1\}$, and in particular for $\gamma = 1$ the system is called damped and corresponds to Kelvin-Voigt model for viscoelasticity (see [27, Chapter XIV, Section 3]), while for $\gamma = 0$ the system is called undamped.

In the corresponding quasi-static models, all the known existence results for the coupled problem $(u(t), \Gamma(t))$ without a-priori assumptions on $\Gamma(t)$, are obtained by minimizing a weak form of the Griffith's energy on function spaces with no regularity on the jump sets except the $(n-1)$-rectifiability (see [26] [40] [24] [21]). The existence of a solution with $\Gamma(t)$ closed is obtained only in particular cases through a regularity argument (see [16] [11]). Therefore, also in the dynamic case we expect that in dealing with any general existence results, no a-priori regularity assumptions on the crack sets $\Gamma(t)$ should be assumed. For this reason we suppose only that the prescribed cracks $\Gamma(t)$ are $(n-1)$-rectifiable with $\mathcal{H}^{n-1}(\Gamma(t)) < \infty$. In this chapter we prove that, both in the undamped and damped case, a solution actually exists.

One of the first issue is to give a weak formulation to the system written in (0.13). Due to the presence of the cracks the system has to be solved on the time-dependent domain $\Omega \setminus \Gamma(t)$. Therefore we need to introduce suitable function spaces V_t, containing for each time t the solution $u(t)$ as well as the test functions. Under no regularity assumptions on the crack set, the scalar case has been treated by Dal Maso, Larsen in [22]. Since the structure of the equation implies no bound on the amplitude of the jump of u, but only on the L^2-norm of the gradient, they defined the problem in the context of $GSBV(\Omega)$ (see Definition 1.4.6). Precisely in [22] it has been shown the existence of a solution $u(t)$ living at each time t in the space $GSBV_2^2(\Omega;\Gamma(t))$ (see Definition 1.4.9).

In our case the structure of the equation leads us to an estimate of

$$\int_{\Omega\setminus\Gamma(t)} |\mathcal{E}u(t)|^2\, dx.$$

Hence V_t needs to include all the displacements in $L^2(\Omega,\mathbb{R}^n)$ whose jump discontinuities are contained in $\Gamma(t)$ and with square integrable symmetric gradient away form the cracks. Since we assumed no regularity on the cracks, in this general context a Korn's type inequality is not true. This means that we cannot control the L^2-norm of the gradient of $u(t)$ with the L^2-norm of its symmetric part. As a consequence we are forced to formulate our problem in the context of BD functions, and precisely to define $V_t := GSBD_2^2(\Omega;\Gamma(t))$ (see Definition 1.4.30) and $V_t^* := GSBD_2^2(\Omega;\Gamma(t))^*$ its dual. Note that if $\Gamma(t)$ are closed sets in Ω, then $GSBD_2^2(\Omega;\Gamma(t))$ reduces to the space of square integrable vector fields, whose symmetric gradient in the sense of distribution on $\Omega\setminus\Gamma(t)$ is square integrable.

The weak formulation of the system is

$$\langle\ddot{u}(t),\phi\rangle_t^* + \langle\mathbb{C}\mathcal{E}u(t),\mathcal{E}\phi\rangle_{H_n} + \langle\mathbb{B}\mathcal{E}\dot{u}(t),\mathcal{E}\phi\rangle_{H_n} = \langle f(t),\phi\rangle_H \ \forall\phi\in V_t$$

for a.e. $t\in[0,T]$, where $\langle\cdot,\cdot\rangle_t^*$ denotes the duality pairing between V_t and V_t^*, $\langle\cdot,\cdot\rangle_H$ $\langle\cdot,\cdot\rangle_{H_n}$ denote the scalar product in $L^2(\Omega,\mathbb{R}^n)$ and in $L^2(\Omega,\mathbb{M}^{n\times n}_{sym})$, respectively.

We want to emphasize that one of the most serious mathematical issues arises because these spaces are varying (increasingly) in time, so that test functions at some time t are not necessarily admissible test functions for times $s<t$. Moreover, since $u(t)$ lives on each time t in different spaces V_t, we need to give a meaning to the second derivative in time $\ddot{u}(t)$ as an element of V_t^*.

While in [22] only homogeneous Neumann boundary condition was considered, we consider also non-homogeneous mixed Dirichlet-Neumann boundary conditions on $\partial\Omega$. This introduces another difficulty when the crack sets approach the boundary, and as a consequence possible problems may occur with the boundary conditions. Indeed, in presence of non-homogeneous Neumann boundary conditions on a part of the boundary, we might think that a serious problem may occur when the elastic material between this part and the crack set is infinitesimally small. From a mathematical point of view, the difficulties are due to the lack of continuity of the trace operator acting on functions having jump sets close to the boundary. In order to solve this problem, we make use of the results obtained in Chapter 2, which allow us to restrict our attention to a suitable space of traction forces F.

In order to show the existence of a solution we follow a time discretisation scheme, passing to the limit when the time step goes to zero (see Theorem 4.2.5). More precisely to define the discrete approximate solution u_k in the time interval $(t_k^i,t_k^{i+1}]$, suppose that we have already defined u_k for $t\le t_k^i$, and let u_k^{i+1} be the minimiser in $V_{t_{i+1}}$ of

$$u\mapsto\left\|\frac{u-u_k^i}{\tau_k}-\frac{u_k^i-u_k^{i-1}}{\tau_k}\right\|_{L^2}^2+\langle\mathbb{C}\,\mathcal{E}u,\mathcal{E}u\rangle_{L^2}+\frac{1}{\tau_k}\langle\mathbb{B}\,(\mathcal{E}u-\mathcal{E}u_k^i),\mathcal{E}u-\mathcal{E}u_k^i\rangle_{L^2}-2\langle f_k^i,u\rangle^*,$$

where $u_k^i=u_k(t_k^i)$, f_k^i is a suitable discrete approximation of f and τ_k is the time step. We define u_k on $(t_k^i,t_k^{i+1}]$ as the linear interpolation between u_k^i and u_k^{i+1}.

We also show an energy balance and uniqueness for the damped problem. The energy balance we are able to prove in the damped case, is a conservation of kinetic plus elastic plus dissipated energy due to the damping.

For the undamped problem the energy balance, where only the kinetic plus the elastic energy are considered, is clearly false. This can be seen using the results of Chapter 5. In this case the uniqueness issue is still an open problem.

In Chapter 5 we focus our attention on the so called *energy-dissipation balance* for dynamic fracture.

Chapter 5

The result of this chapter is obtained in collaboration with Caponi M. and Lucardesi I., and can be found in [9].

We consider as reference configuration a bounded open set Ω of $\mathbb{R}^2$ with Lipschitz boundary. We work in the anti-plane case, hence the displacement u and the body force f are scalar valued. We fix a time interval $[0,T]$ and we prescribe a boundary deformation on a portion of $\partial\Omega$. We assume that, in response to the external loads, the material breaks along a fixed $C^{3,1}$ curve $\Gamma \subset \Omega$ with end-points on $\partial\Omega$. In this case, the crack set $\Gamma(t)$ at time t is determined by the crack-tip position on Γ, described by its arc-length parameter $s(t)$ along Γ. Here we assume $t \mapsto s(t)$ non decreasing (irreversibility assumption) and of class $C^{3,1}([0,T])$. Far from the crack set, the material undergoes an elastic deformation: since we are in the anti-plane case the displacement u satisfies a wave equation of the form

$$\ddot{u}(t) - \operatorname{div}(A\nabla u(t)) = f(t) \quad \text{in } \Omega \setminus \Gamma(t)\,, \tag{0.14}$$

where A is a suitable matrix field (satisfying the usual ellipticity conditions); the equation is supplemented by boundary conditions, that we choose to be Neumann homogeneous on $\Gamma(t)$ (traction free case), and initial conditions.

The well-posedness of (0.14) in a time-dependent domain has been widely investigated for example in [18, 19, 33, 37, 38, 39, 60, 61, 65]. However, the common feature of all these works is to deal with space-time domains whose boundary is a sufficiently regular manifold (C^1 for example). Unfortunately, this does not fit with our problem, since the domain

$$\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \in (0,T),\ x \in \Omega \setminus \Gamma(t)\}$$

is clearly not regular. In the presence of moving cracks we limit ourselves to cite the papers [22] and [66]: in the former, the authors work under the sole assumption of finite measure of the crack set, provide a notion of solution, and show its existence, using a variational time-discretisation approach; in the latter, the authors work under stronger regularity assumptions and, following a change of variables approach, prove existence of solutions in a suitable weak sense. Later, in [25], the regular case has been resumed: following the same approach of [66], the authors obtain uniqueness of solutions and their continuous dependence on the data. These results have been extended to the vector case in [8].

In this paper we move the natural step forward in the study: the computation of kinetic plus elastic energy and its relation with the crack growth. This computation has a crucial role, in view of the so called energy-dissipation balance which underlies the dynamics (see, e.g., [42, 46]): the kinetic + elastic energy released during the elastodynamics and the energy dissipated to create the fracture (the latter proportional to the crack surface increment) balance the work done by the external loads. In formulas, denoting by $\mathcal{E}(t)$ the energy

$$\mathcal{E}(t) := \frac{1}{2}\int_{\Omega\setminus\Gamma(t)} \left[|\dot{u}(t)|^2 + A\nabla u(t)\cdot\nabla u(t)\right]\,dx\,, \tag{0.15}$$

and fixing homogeneous Dirichlet-Neumann boundary conditions on $\partial\Omega$, the energy-dissipation balance states that, for every time $t\in[0,T]$,

$$\mathcal{E}(t) - \mathcal{E}(0) + \mathcal{H}^1(\Gamma(t)\setminus\Gamma(0)) = \int_0^t \langle f(\tau), \dot{u}(\tau)\rangle_{L^2}\,d\tau\,. \tag{0.16}$$

The difficulty of computing (0.15) is twofold: on one hand, the displacement has a singular behavior near the tip; moreover, the domain of integration appearing in (0.15) is irregular and varies in time. To handle the first issue, a representation result for u is in order: we prove that, if the initial conditions satisfy suitable properties (see (5.35) and (5.36)), then for every time t, the displacement is of class H^1 in a neighborhood of the tip and of class H^2 far from it, namely u has the form

$$u(t) = u^R(t) + \zeta(t)k(t)S(\Phi(t))\,, \tag{0.17}$$

where $u^R(t)\in H^2(\Omega\setminus\Gamma(t))$, $\zeta(t)$ is a cut–off function supported in a neighborhood of the moving tip of $\Gamma(t)$, $k(t)\in\mathbb{R}$, $S\in H^1(\mathbb{R}^2\setminus\{x_1\geq 0\})$, and $\Phi(t)$ is a diffeomorphism of Ω (constructed in a suitable way, according to the properties of Γ, A, and s). Once fixed ζ, S, and Φ, the function u^R and the constant k are uniquely determined. Actually, the coefficient k only depends on A, Γ, and s (see Theorem 5.2.10 and Remark 5.2.11). In addition, we provide another decomposition for u which is more explicit and better explains the behavior of the singular part (see §5.2.4). The second issue is technical and we overcome it exploiting Geometric Measure Theory techniques (see Section 5.3). The computation leads to the following formula:

$$\mathcal{E}(t) - \mathcal{E}(0) + \frac{\pi}{4}\int_0^t k^2(\tau)a(\tau)\dot{s}(\tau)\,d\tau = \int_0^t \langle f(\tau), \dot{u}(\tau)\rangle_{L^2}\,d\tau\,, \tag{0.18}$$

where a is a positive function which depends on A, Γ, and s, and is equal to 1 when A is the identity matrix; see Theorem 5.3.7 for the proof of (0.18) when $A=I$, and Remark 5.3.9 for the general case. By comparing (0.16) and (0.18), we deduce the following necessary and sufficient condition on the crack evolution (in the class of smooth cracks), in order to guarantee the energy-dissipation balance: during the crack opening, namely when $\dot{s}(t)>0$, the function $k(t)$, often called *dynamic stress intensity factor*, has to be equal to $2/\sqrt{\pi a(t)}$.

We mention that a direct computation for a horizontal crack $\Gamma(t)=\Omega\cap\{y=0\,,\ x\leq ct\}$ moving with constant velocity c (+ a suitable boundary datum) can be found in [23, §4].

The representation result stated in (0.17) extends that of [66] for straight cracks (near the tip) and A the identity matrix. Here we adapt their proof to the case of a curved crack and a constant (in time) operator A, possibly non homogeneous; moreover, we remove a restrictive assumption on the acceleration $\ddot{s}$ (see Remark 5.1.2). The main steps in the proof of (0.17) are the following: performing four changes of variables, we reduce problem (0.14) to a second order PDE of the form

$$\ddot{v}(t) - \operatorname{div}(\widetilde{A}(t)\nabla v(t)) + l.o.t. = \widetilde{f}(t) \quad \text{in } \widetilde{\Omega}\setminus\widetilde{\Gamma}_0\,, \tag{0.19}$$

with $\widetilde{\Omega}$ Lipschitz planar domain and $\widetilde{\Gamma}_0$ a $C^{3,1}$ curve straight near its tip. The matrix field $\widetilde{A}$ has time-dependent coefficients but at the tip of $\widetilde{\Gamma}_0$ it is constantly equal to the identity. Finally, the decomposition result for v, solution to (0.19), obtained via semi-group theory, leads to the one for u, solution to the original problem (0.14).

Chapter 1

Notation and preliminary results

Contents

1.1 Notation

Basic notation.

$\alpha \wedge \beta$ / $\alpha \vee \beta$	minimum between α and β / maximum between α and β;
$a \cdot b$	scalar product between $a, b \in \mathbb{R}^n$;
$\overline{\mathbb{R}}$	extended real line $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$;
$O(n)$	group of $n \times n$ orthogonal matrices;
$\mathbb{M}^{n\times n}_{sym}$	space of $n \times n$ symmetric matrices;
$a \odot b$	symmetrised tensor product between $a, b \in \mathbb{R}^n$;
$\lvert \cdot \rvert$	modulus, Euclidean norm of vectors, Frobenius norm of matrices;
$\mathbb{S}^{n-1}$	$(n-1)$-dimensional sphere in $\mathbb{R}^n$;
$B_r(x)$	open ball centered at x with radius r;
$Q_r(x)$	open cube centered at x, sides parallel to the orthogonal axis of length r;
graph(u) / S_u^* / S_u^-	graph / epigraph / hypograph of the function u.

Functions spaces: Let $\Omega \subset \mathbb{R}^n$ be an open set, let H be an Hilbert space, and let V be a

Banach space.

$C^0_c(\Omega)$ / $C^0_c(\Omega;\mathbb{R}^n)$	$\mathbb{R}/\mathbb{R}^n$-valued continuous functions with compact support;
$C^0_0(\Omega)$ / $C^0_0(\Omega;\mathbb{R}^n)$	closure with respect to the sup-norm of the space $C^0_c(\Omega)$ / $C^0_c(\Omega;\mathbb{R}^n)$;
$C^k_c(\Omega)$ / $C^k_c(\Omega;\mathbb{R}^n)$	k-times differentiable $\mathbb{R}/\mathbb{R}^n$-valued functions with compact support;
$L^p(\Omega)$ / $L^p(\Omega;\mathbb{R}^n)$	functions $u\colon \Omega\to\mathbb{R}$ / $u\colon\Omega\to\mathbb{R}^n$ with $\|u\|_{L^p}<\infty$;
$L^p(\Omega;\mathbb{M}^{n\times n}_{sym})$	functions $u\colon\Omega\to\mathbb{M}^{n\times n}_{sym}$ with $\|u\|_{L^p}<\infty$;
$\langle\cdot,\cdot\rangle_{L^2}$	scalar product in $L^2(\Omega)$ or $L^2(\Omega;\mathbb{R}^n)$ or $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$;
$W^{1,p}(\Omega)$	Sobolev space;
$H^1(\Omega)$	Sobolev space $W^{1,2}(\Omega)$;
$\langle\cdot,\cdot\rangle_H$	scalar product of H;
V^*	dual of V;
$\langle\cdot,\cdot\rangle_{V^*}$	duality pairing between V and V^*;
$L^p(a,b;V)$	Böchner-measurable functions $u\colon(a,b)\to V$ with $\|u\|_{L^p((a,b);X)}<\infty$;
$W^{1,p}(a,b;V)$	Böchner-Sobolev space.

Measure theory: Let $\Omega\subset\mathbb{R}^n$ be an open set.

$\mathcal{L}^n$	Lebesgue outer measure in $\mathbb{R}^n$;
$\lvert A\rvert$	Lebesgue measure of the set A;
$\mathcal{H}^k$	k-dimensional Hausdorff measure;
$\mathcal{M}_b(\Omega;\mathbb{R}^n)$	space of $\mathbb{R}^n$-valued finite Radon measures on Ω;
$\lvert\mu\rvert$	total variation of the measure μ;
$\mu\llcorner A$	restriction of μ to the set $A\subset\Omega$;
$\Theta^\alpha(\mu,x)$	α-dimensional density of μ at x;
$\Theta^{*\alpha}(\mu,x)$	α-dimensional upper density of μ at x;
$\Theta^\alpha_*(\mu,x)$	α-dimensional lower density of μ at x;
A^δ	point of density $0\le\delta\le 1$ of the set A;

Sets of finite perimeter: Let $\Omega\subset\mathbb{R}^n$ be an open set and let $E\subset\Omega$ be a set of finite perimeter.

$E_{r,x}$	blow-up of E at x;
$P(E;\Omega)$	perimeter of E in Ω;
∂^*E	reduced boundary of E;
$\nu_E(x)$	theoretic inner normal of E at $x\in\partial^*E$;

GSBV functions and *GSBD* functions: Let $\Omega\subset\mathbb{R}^n$ be an open set and let $\Gamma\subset\Omega$ with

$\mathcal{H}^{n-1}(\Gamma) < \infty$.

$BV(\Omega)$	functions of bounded variation;
$GBV(\Omega)$	generalised functions of bounded variation;
$GSBV(\Omega)$	generalised special functions of bounded variation;
Du	distributional gradient of u;
∇u	approximate gradient of u;
$J_u \,/\, \nu_u$	jump set of u / normal to J_u;
S_u	singular set of u;
u^+ , u^- / $[u]$	traces of u on J_u / jump of u given by $[u] = u^+ - u^-$;
$\tilde{u}$	precise representative of u
$GSBV(\Omega;\Gamma)$	functions in $GSBV(\Omega)$ with $J_u \subset \Gamma$;
$GSBV^p(\Omega)$	functions in $GSBV(\Omega)$ with $\nabla u \in L^p(\Omega;\mathbb{R}^n)$;
$GSBV^p(\Omega;\Gamma)$	functions in $GSBV(\Omega;\Gamma)$ with $\nabla u \in L^p(\Omega;\mathbb{R}^n)$;
$BD(\Omega)$	functions of bounded deformation;
Eu	symmetric part of the distributional gradient of u;
$\mathcal{E}u$	symmetric approximate gradient of u;
$GBD(\Omega)$	generalised functions of bounded deformation;
$GSBD(\Omega)$	generalised special functions of bounded deformation;
$GSBD(\Omega;\Gamma)$	functions in $GSBD(\Omega)$ with $J_u \subset \Gamma$;
$GSBD^p(\Omega)$	functions in $GSBD(\Omega)$ with $\mathcal{E}u \in L^p(\Omega;\mathbb{M}^{n\times n}_{sym})$;
$GSBD^p(\Omega;\Gamma)$	functions in $GSBD(\Omega;\Gamma)$ with $\mathcal{E}u \in L^p(\Omega;\mathbb{M}^{n\times n}_{sym})$;

Slicing: Let $E \subset \mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be an open set.

$\xi^\perp$	hyperplane orthogonal to $\xi \in \mathbb{S}^{n-1}$;
E^ξ_y	$t \in \mathbb{R} \mid y + t\xi \in E$, where $\xi \in \mathbb{S}^{n-1}$ and $y \in \xi^\perp$;
u^ξ_y	slice of $u \colon \Omega \to \mathbb{R}$ defined by $u^\xi_y(t) := u(y + t\xi)$.

1.2 Measures

Given an open set $\Omega \subset \mathbb{R}^n$ we denote by $\mathcal{M}_b(\Omega;\mathbb{R}^n)$ the space of all bounded $\mathbb{R}^n$-valued Radon measures on Ω. Whenever $\mu \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$ we denote by $|\mu|$ its total variation. Given $A \subset \Omega$ a μ-measurable set, the restriction of μ to A is denoted by $\mu \llcorner A$ and it is defined by $(\mu \llcorner A)(B) := \mu(A \cap B)$ for every μ-measurable set $B \subset A$. We denote by $\mathcal{M}_b(\Omega)$ the space of bounded scalar Radon measures, and by $\mathcal{M}^+_b(\Omega)$ the space of all positive measures in $\mathcal{M}_b(\Omega)$.

By means of Riesz's representation Theorem the space $\mathcal{M}_b(\Omega;\mathbb{R}^n)$ can be seen as the dual of the space $C^0_0(\Omega;\mathbb{R}^n)$ equipped with the sup-norm. In this case we can consider the weak notion of convergence induced by this duality. Given $(\mu_k) \subset \mathcal{M}_b(\Omega;\mathbb{R}^n)$ and $\mu \in \mathcal{M}_b(\Omega;\mathbb{R}^n)$, we say that $\mu_k \rightharpoonup \mu$ weakly in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$ if and only if for every $\varphi \in C^0_0(\Omega;\mathbb{R}^n)$ we have

$$\lim_{k\to\infty} \int_\Omega \varphi \, d\mu_k = \int_\Omega \varphi \, d\mu.$$

Analogously, given $(\mu_k) \subset \mathcal{M}_b(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega)$, we say that $\mu_k \rightharpoonup \mu$ weakly in $\mathcal{M}_b(\Omega)$ if and only if for every $\varphi \in C_0^0(\Omega)$ we have

$$\lim_{k\to\infty} \int_\Omega \varphi \, d\mu_k = \int_\Omega \varphi \, d\mu.$$

Following the notation in [35, Subsection 2.10.19], given $x \in \Omega$, whenever $0 \le \alpha \le n$, we denote the α-dimensional upper and lower densities of μ at x, respectively, as

$$\Theta^{*\alpha}(\mu, x) := \limsup_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha},$$

$$\Theta_*^{\alpha}(\mu, x) := \liminf_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}.$$

If the upper and lower density coincide, the α-dimensional density of μ at x is defined as

$$\Theta^{\alpha}(\mu, x) := \lim_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}.$$

Given $0 \le \delta \le 1$ and given a set $A \subset \Omega$ we denote the point of density δ of A as

$$A^{(\delta)} := \{x \in \Omega \mid \Theta^n(\mathcal{L}^n \llcorner A, x) = \delta\},$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue outer measure.

1.3 Sets of finite perimeter

In this section we summarize all the properties of sets with finite perimeter which will be useful in Chapter 3. A particular attention is devoted to the concept of *indecomposability*.

Given Ω an open set of $\mathbb{R}^n$ we recall that an $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ has finite perimeter in Ω if

$$P(E; \Omega) := \sup_{\substack{\varphi \in C_c^1(\Omega;\mathbb{R}^n) \\ \|\varphi\|_\infty \le 1}} \int_E \operatorname{div} \varphi \, dx < \infty,$$

where div denotes the divergence operator defined as usual, i.e. $\operatorname{div} \varphi := \sum_{i=1}^n \frac{\partial \varphi^i}{\partial x_i}$. If $\Omega = \mathbb{R}^n$ we simply write $P(E)$ to denote $P(E; \mathbb{R}^n)$. Whenever E has finite perimeter, by means of Riesz's representation Theorem, we know that the distributional gradient of the characteristic function of E, i.e. $D\mathbb{1}_E$, can be represented as a measure in $\mathcal{M}_b(\Omega; \mathbb{R}^n)$. In particular, by denoting the total variation of $D\mathbb{1}_E$ as $|D\mathbb{1}_E|$, then for every Borel set $B \subset \Omega$ the relative perimeter of E in B is defined as

$$P(E; B) := |D\mathbb{1}_E|(B).$$

We denote by $\partial^* E$ the reduced boundary of E, defined as those $x \in \Omega$ for which there exists $\nu_E(x) \in \mathbb{S}^{n-1}$ such that

$$\lim_{r\to 0^+} \frac{D\mathbb{1}_E(B_r(x))}{|D\mathbb{1}_E|(B_r(x))} = \nu_E(x). \tag{1.1}$$

The unitary vector $\nu_E(x)$ is the *theoretical inner normal* of E at x.

1.3.1 Structure properties

De Giorgi's structure Theorem holds true (see for example [2, Theorem 3.59]).

Theorem 1.3.1. *Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ with $P(E;\Omega) < \infty$. Then $\partial^* E$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable (see [35, Subsection 3.2.14]) and*

$$|D\mathbb{1}_E| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

In addition, for every $x \in \partial^* E$ the following properties hold

(a) the sets $(E - x)/r$ locally converge in measure in $\mathbb{R}^n$ as $r \to 0^+$ to the halfspace H orthogonal to $\nu_E(x)$ and containing $\nu_E(x)$;

(b) $\Theta^{(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = 1$.

We shall make use of the following two results in Chapter 3. The first is due to Federer and concerns the structure of sets having finite perimeter. The second can be seen as a sort of Leibniz's formula for the intersection of two sets of finite perimeter.

Theorem 1.3.2. *Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ with $P(E;\Omega) < \infty$. Then*

- $\mathcal{H}^{n-1}(E^{(1/2)} \Delta \partial^* E) = 0$;
- $\mathcal{H}^{n-1}(\Omega \setminus [E^{(1)} \cup E^{(1/2)} \cup E^{(0)}]) = 0$.

Proof. See for example [2, Theorem, 3.61]. □

Proposition 1.3.3 (Leibniz's formula)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $E, F \subset \mathbb{R}^n$ with $P(E;\Omega), P(F;\Omega) < \infty$. Then $P(E \cap F;\Omega) < \infty$ and moreover*

$$\begin{aligned} \mathcal{H}^{n-1} \llcorner \partial^*(E \cap F) = \mathcal{H}^{n-1} \llcorner \partial^* E \cap F^{(1)} + \mathcal{H}^{n-1} \llcorner \partial^* F \cap E^{(1)} \\ + \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\} \end{aligned} \tag{1.2}$$

Proof. See [62, Theorem 16.3] □

1.3.2 Caccioppoli's partition and indecomposable sets

In Chapter 3 we shall make use of the concepts of indecomposable set. For this reason we dedicate this subsection to recall some notions and useful results regarding the indecomposability property. First of all let us recall the definition of Caccioppoli's partitions (see [2] for a reference).

Definition 1.3.4 (Caccioppoli's partition)**.** *Let Ω be an open set of $\mathbb{R}^n$. We say that an $\mathcal{L}^n$-measurable partition $(E_i)_{i=1}^{\infty}$ of Ω is a Caccioppoli's partition if*

$$\sum_{i=1}^{\infty} P(E_i;\Omega) < \infty.$$

Moreover we say that a Caccioppoli's partition is ordered if $|E_i| \geq |E_j|$ whenever $i \leq j$.

Definition 1.3.5 (Indecomposability)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $F \subset \Omega$ with $P(F;\Omega) < \infty$. We say that F is* indecomposable *if for every set E satisfying*

$$E \subset F, \quad P(F;\Omega) = P(E;\Omega) + P(F \setminus E;\Omega), \tag{1.3}$$

then $|E| = 0$ or $|E \Delta F| = 0$.

Remark 1.3.6. *The notion of indecomposability can be found for example in [52] and it is in perfect agreement with the following fact (see [32, Proposition 2.12]): the set F is indecomposable if and only if any $u \in BV(\Omega)$ with $|Du|(F) = 0$ is necessarily constant on F.*

In particular this tells us that every connected open set $U \subset \Omega$ with finite perimeter is indecomposable. ■

Remark 1.3.7. *For every set $E \subset F$ it holds $P(F;\Omega) \leq P(E;\Omega) + P(F \setminus E;\Omega)$. This means that condition (1.3) is equivalent to*

$$E \subset F, \quad P(F;\Omega) \geq P(E;\Omega) + P(F \setminus E;\Omega).$$

Moreover, condition (1.3) can be equivalently stated for a countable family $(E_i)_{i=1}^{\infty}$. This means that F is indecomposable if and only if the following conditions

$$\bigcup_{i=1}^{\infty} E_i = F, \quad |E_i \cap E_j| = 0 \ (i \neq j), \quad \sum_{i=1}^{\infty} P(E_i;\Omega) = P(F;\Omega), \tag{1.4}$$

imply that there exists i_0 such that $E_{i_0} = F$ and $E_i = \emptyset$ for $i \neq i_0$.

Indeed condition (1.4) clearly implies (1.3). While if F is indecomposable, by setting $E := E_1$, (1.4) tells us

$$P(E;\Omega) + P(F \setminus E;\Omega) \leq P(F;\Omega),$$

which implies

$$P(E;\Omega) + P(F \setminus E;\Omega) = P(F;\Omega).$$

By the indecomposability of F we deduce that one between E or $F \setminus E$ has zero Lebesgue measure. If $|F \Delta E| = 0$ we are done. Otherwise $|E| = 0$ and we can proceed as before by defining $E := E_2$. Clearly, if this procedure does not stop, then $|F| = 0$ and we are done. Otherwise if it stops at $i_0 \in \mathbb{N}$ this means that $|F \Delta E_{i_0}| = 0$ and we are done. ■

We end up this subsection with two technical propositions.

Proposition 1.3.8. *Let Ω be an open set of $\mathbb{R}^n$ and let $F \subset \Omega$ be indecomposable. Suppose $E \subset \Omega$ is a set having finite perimeter in Ω and such that*

$$|E \cap F| > 0 \quad \textit{and} \quad |F \setminus E| > 0. \tag{1.5}$$

then

$$\mathcal{H}^{n-1}(\partial^* E \cap F^{(1)}) > 0.$$

Proof. We can consider the measurable partition of F given by $F = (E \cap F) \cup (F \setminus E)$. By hypothesis $|E \cap F|, |F \setminus E| > 0$. Using Leibniz's formula (1.2) we can write

$$\partial^*(E \cap F) = [\partial^* E \cap F^{(1)}] \cup [\partial^* F \cap E^{(1)}] \cup [\{\nu_E = \nu_F\}], \quad (\mathcal{H}^{n-1}\text{-a.e.}),$$

and

$$\partial^*(F \setminus E) = [\partial^* E \cap F^{(1)}] \cup [\partial^* F \cap E^{(0)}] \cup [\{\nu_E = -\nu_F\}], \quad (\mathcal{H}^{n-1}\text{-a.e.}).$$

Since $\partial^* F \cap E^{(1)}$, $\{\nu_E = \nu_F\}$, $\partial^* F \cap E^{(0)}$ and $\{\nu_E = -\nu_F\}$ are pairwise disjoints subsets of $\partial^* F$, if $\mathcal{H}^{n-1}(\partial^* E \cap F^{(1)}) = 0$ then

$$\begin{aligned} P(E \cap F;\Omega) + P(F \setminus E;\Omega) &= \mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^* F \cap E^{(0)}) \\ &\quad + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\}) + \mathcal{H}^{n-1}(\{\nu_E = -\nu_F\}) \\ &\leq P(F;\Omega), \end{aligned}$$

which by Remark 1.3.7 implies (1.3) and this together with (1.5) is in contradiction with the indecomposability of F. □

Proposition 1.3.9. *Let Ω be an open set of $\mathbb{R}^n$ and let $E, E' \subset \Omega$ with $P(E;\Omega), P(E';\Omega) < \infty$ and such that $\partial^* E' \subseteq \partial^* E$. Let $F \subset E$ be an indecomposable set. Then one and only one of the following conditions holds true*

1. $|F \setminus E'| = 0$
2. $|F \setminus (E \setminus E')| = 0$.

Proof. It is enough to show that $|F \cap E'| \neq 0$ implies $|F \setminus E'| = 0$.

Suppose not. Then $|F \cap E'| > 0$ and also $|F \setminus E'| > 0$. By Leibniz's formula both $F \cap E'$ and $F \setminus E'$ are sets having finite perimeter in Ω. Moreover, by Proposition 1.3.8 we would have also $\mathcal{H}^{n-1}(\partial^* E' \cap F^{(1)}) > 0$. But since $F \subset E$ then $F^{(1)} \subset E^{(1)}$, and this implies $\mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}) > 0$ which is in contradiction with the hypothesis $\partial^* E' \subset \partial^* E$. This proves the proposition. □

1.4 Functions with jump discontinuity

1.4.1 $GSBV$ functions

For the general theory concerning the space of generalised functions of bounded variation $GBV(\Omega)$, we refer to [2]. In this subsection we limit ourselves to state the results that we shall use in Chapter 3.

The space $GBV(\Omega)$ arises in the study of minimisation problems related to functionals like (0.2), where no constraints on the L^∞-norm along minimizing sequences are available. We briefly recall that a function $u \in L^1(\Omega)$ is of bounded variation, namely $u \in BV(\Omega)$, if its distributional gradient Du belongs to $\mathcal{M}_b(\Omega;\mathbb{R}^n)$. Moreover it can be decomposed into three orthogonal measures as follows (see [2, Section 3.9])

$$Du = \nabla u\, \mathcal{L}^n + [u]\, \nu\, \mathcal{H}^{n-1} \llcorner J_u + D^c u,$$

where $\nabla u \in L^1(\Omega;\mathbb{R}^n)$ is the approximate gradient, $[u] \in L^1(J_u;\mathcal{H}^{n-1})$ is the jump, J_u is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set called the jump set, $\nu\colon J_u \to \mathbb{S}^{n-1}$ is an orientation of J_u, and $D^c u$ is the so called Cantor part. Whenever $D^c u = 0$ the function u is said to be of special bounded variation, namely $u \in SBV(\Omega)$ (see [2, Section 4.1]). A function u belongs to $BV_{loc}(\Omega)$ or $SBV_{loc}(\Omega)$ if for every open set $U \subset\subset \Omega$ then $u \in BV(U)$ or $u \in SBV(U)$, respectively.

In order to give a precise meaning of jump set and of approximate gradient in the context of GBV functions, we need to recall the notion of approximate limit ([2, Section 4.5]).

Definition 1.4.1 (Upper and lower approximate limit)**.** *Let Ω be an open set of $\mathbb{R}^n$. Given an $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}$ and $x \in \mathbb{R}^n$ such that $\Theta^{*n}(\mathcal{L}^n \llcorner \Omega; x) > 0$, then the upper approximate limit of u at x is defined as*

$$u^+(x) := \text{ap-}\limsup_{y\to x} u(y) := \inf\{t \in \mathbb{R} \mid \Theta^n(\mathcal{L}^n \llcorner \{u > t\}, x) = 0\}$$

while the lower approximate limit of u at x is defined as

$$u^-(x) := \text{ap-}\liminf_{y\to x} u(y) := \sup\{t \in \mathbb{R} \mid \Theta^n(\mathcal{L}^n \llcorner \{u < t\}, x) = 0\}.$$

In addition, we say that u admits an approximate limit equal to $a \in \overline{\mathbb{R}}$ at x, and we write

$$\operatorname*{ap-lim}_{y \to x} u(y) = a,$$

if $u^+(x) = u^-(x) = a$ (the case $a = \pm\infty$ are not excluded).

Definition 1.4.2 (Approximate continuity)**.** *Let Ω be an open set of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}$ we define the approximate continuity set as the set of points $x \in \Omega$ for which there exists $a \in \overline{\mathbb{R}}$ such that*

$$\operatorname*{ap-lim}_{y \to x} u(y) = a.$$

The approximate discontinuity set S_u, is defined as the complement in Ω of the approximate continuity set, i.e.

$$S_u := \{x \in \Omega \mid u^-(x) < u^+(x)\}.$$

When $x \in \Omega \setminus S_u$ we denote the approximate limit of u at x as $\tilde{u}(x)$.

We are now in position to remind the definitions of jump set and of approximate gradient in the context of GBV-functions.

Definition 1.4.3 (Jump set)**.** *Let Ω be an open set of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}$ we define the approximate jump set J_u, as the set of point $x \in \Omega$ for which there exists $a, b \in \overline{\mathbb{R}}$ with $a < b$ and $\nu \in \mathbb{S}^{n-1}$ such that*

$$\operatorname*{ap-lim}_{\substack{(y-x)\cdot\nu>0 \\ y \to x}} v(y) = a \quad \text{and} \quad \operatorname*{ap-lim}_{\substack{(y-x)\cdot\nu<0 \\ y \to x}} v(y) = b.$$

If $x \in J_u$ clearly we have $a = u^+(x)$ and $b = u^-(x)$. The vector ν, uniquely determined by this condition, is denoted by $\nu_u(x)$. The jump of u is the function $[u]\colon J_u \to \mathbb{R}$ defined by $[u](x) := u^+(x) - u^-(x)$ for every $x \in J_u$.

Definition 1.4.4 (Approximate differentiability)**.** *Let $u\colon \Omega \to \mathbb{R}$ be an $\mathcal{L}^n$-measurable function and $x \in \Omega \setminus S_u$. Then u is approximately differentiable at x if $\tilde{u}(x) \in \mathbb{R}$ and there exists a linear map $L\colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$\operatorname*{ap-lim}_{\substack{y \to x \\ y \neq x}} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{|y-x|} = 0.$$

In this case the approximate gradient of u is defined as $\nabla u(x) := L$.

Definition 1.4.5 (GBV functions)**.** *Let Ω be an open set of $\mathbb{R}^n$. We say that a function $u\colon \Omega \to \mathbb{R}$ belongs to $GBV(\Omega)$, if for every $M \in \mathbb{N}$ the truncated function $u^M := (u \vee M) \wedge -M$ belongs to $BV_{loc}(\Omega)$.*

Definition 1.4.6 ($GSBV$ functions)**.** *Let Ω be an open set of $\mathbb{R}^n$. We say that a function $u\colon \Omega \to \mathbb{R}$ belongs to $GSBV(\Omega)$, if for every $M \in \mathbb{N}$ the truncated function $u^M := (u \vee M) \wedge -M$ belongs to $SBV_{loc}(\Omega)$.*

Now we recall the main result about the fine properties of GBV functions (see [2, Theorem 4.34]).

Theorem 1.4.7 (Fine properties)**.** *Let $u \in GBV(\Omega)$, let $M \in \mathbb{N}$. Then*

1. $S_u = \bigcup_{M\in\mathbb{N}} S_{u^M}$ *and*

$$u^+(x) = \lim_{M\to+\infty} (u^M)^+(x), \quad u^-(x) = \lim_{M\to+\infty} (u^M)^-(x);$$

2. S_u *is countably* $\mathcal{H}^{n-1}$*-rectifiable,* $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ *and*

$$Tan(S_u, x) = (\nu_u(x))^\perp, \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_u;$$

3. u *is weakly approximate differentiable* $\mathcal{L}^n$*-a.e. in* Ω *and*

$$\nabla u(x) = \nabla u^M(x), \text{ for } \mathcal{L}^n\text{-a.e. } x \in \{|u| \le M\}.$$

We shall use several times the following compactness result in the sequel.

Theorem 1.4.8. *Let Ω be an open set of $\mathbb{R}^n$ and let $(u_k)_{k=1}^\infty$ be a sequence of functions in $GSBV(\Omega)$. Suppose that there exist $p > 1$ such that*

$$\sup_{k\in\mathbb{N}} \big(\|u\|_{L^p} + \|\nabla u\|_{L^p} + \mathcal{H}^{n-1}(J_u)\big) < \infty.$$

Then there exists $u \in GSBV(\Omega)$ such that, up to passing through a subsequence, we have

$$\lim_{k\to\infty} u_k(x) = u(x), \quad \mathcal{L}^n\text{-a.e.} \quad \text{and} \quad \nabla u_k \rightharpoonup \nabla u, \text{ weakly in } L^1(\Omega), \text{ as } k \to \infty,$$

and

$$\liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}) \ge \mathcal{H}^{n-1}(J_u).$$

Proof. It is a particular case of [2, Theorem 4.36]. □

Finally, we introduce suitable subspaces of $GSBV(\Omega)$.

Definition 1.4.9. *Given $\Gamma \subset \Omega$ a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$, we define for every $p \ge 1$*

- $GSBV^p(\Omega) := \{u \in GSBV(\Omega) \mid \nabla u \in L^p(\Omega; \mathbb{R}^n)\}$;
- $GSBV^p_p(\Omega) := \{u \in GSBV(\Omega) \mid u \in L^p(\Omega), \ \nabla u \in L^p(\Omega; \mathbb{R}^n)\}$;
- $GSBV(\Omega; \Gamma) := \{u \in GSBV(\Omega) \mid J_u \subseteq \Gamma\}$;
- $GSBV^p(\Omega; \Gamma) := \{u \in GSBV^p(\Omega) \mid J_u \subset \Gamma\}$;
- $GSBV^p_p(\Omega; \Gamma) := \{u \in GSBV^p_p(\Omega) \mid J_u \subset \Gamma\}$.

Proposition 1.4.10. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then, for every $p > 1$ the space $GSBV^p_p(\Omega; \Gamma)$ endowed with the norm*

$$\|u\|_p := \|u\|_{L^p} + \|\nabla u\|_{L^p}$$

is a Banach space.

Proof. Thanks to [21, Proposition 2.3] it is a real vector space.

To prove the completeness, suppose that $(u_k) \subset GSBV^p_p(\Omega;\Gamma)$ is a Cauchy sequence. In particular the sequence (u_k) converges strongly in $L^p(\Omega,\mathbb{R}^n)$ to some u . Since

$$\sup_{k\in\mathbb{N}} \left(\|u_k\|_{L^p} + \|\nabla u_k\|_{L^p} + \mathcal{H}^{n-1}(J_{u_k})\right) < \infty,$$

by Theorem 1.4.8, passing through a subsequence, we know that there exists $v \in GSBV(\Omega)$ such that $u_k \to v$ pointwise a.e. and $\nabla u_k \rightharpoonup \nabla v$ weakly in $L^1(\Omega)$, as $k \to \infty$. This implies $u = v$, and thanks to the lower semicontinuity of the L^p norm with respect to the weak convergence also

$$\|\nabla u\|_{L^p} \leq \liminf_{k\to\infty} \|\nabla u_k\|_{L^p},$$

hence $u \in GSBV^p_p(\Omega)$.

It remains to prove that $J_u \subseteq \Gamma$. By using Theorem 1.4.8 on every open set $U \subset \Omega$, we have

$$\mathcal{H}^{n-1}(J_u \cap U) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k} \cap U) \ .$$

Since the measure $\mathcal{H}^{n-1} \llcorner \Gamma$ is inner regular, for every $\epsilon > 0$ we can find a compact set $K \subset \Gamma$, such that $\mathcal{H}^{n-1}(\Gamma \setminus K) \leq \epsilon$, and so

$$\mathcal{H}^{n-1}(J_u \setminus K) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k} \setminus K) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(\Gamma \setminus K) \leq \epsilon.$$

For the arbitrariness of ϵ, we conclude that $\mathcal{H}^{n-1}(J_u \setminus \Gamma) = 0$. □

1.4.2 *GSBD* functions

For the space of *generalised functions of bounded deformation* $GBD(\Omega)$ we always refer to the seminal paper [20]. In this subsection we limit ourselves to recall some notation and the main properties that will be used in Chapters 2 and 4.

The space of generalised function of bounded deformation arises in the framework of variational problems in linearized elasticity with fracture, when no L^∞-bounds on minimizing sequences are available. In this situation it is not possible to approach the problem through a truncation argument as in the case of $GSBV$-functions. This is due to the fact that the symmetric part of the gradient is not preserved after truncation. This issue can be solved by looking at the behavior of the one dimensional slices of the functions. We want to recall that a function $u \in L^1(\Omega;\mathbb{R}^n)$ is said to be of bounded deformation, namely $u \in BD(\Omega)$, if its *symmetric* distributional gradient Eu belongs to $\mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{sym})$. Moreover it can be decomposed into three orthogonal measures as follows (see [1])

$$Eu = \mathcal{E}u\,\mathcal{L}^n + [u] \odot \nu\,\mathcal{H}^{n-1} \llcorner J_u + E^c u,$$

where $\mathcal{E}u \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ is the symmetric approximate gradient, $[u] \in L^1(J_u;\mathcal{H}^{n-1})$ is the jump, J_u is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set called the jump set, $\nu\colon J_u \to \mathbb{S}^{n-1}$ is an orientation of J_u, $\odot$ denotes the symmetric tensor product, and $E^c u$ is the so called Cantor part. Whenever $E^c u = 0$ the function u is said to be of special bounded deformation, namely $u \in SBD(\Omega)$ (see [5]).

Before giving the definition of $GBD(\Omega)$ we need to recall some notations. For every $\xi \in \mathbb{S}^{n-1}$ let $\xi^\perp := \{y \in \mathbb{R}^n \mid y \cdot \xi = 0\}$ to be the hyperplane orthogonal to ξ passing through the origin, and let $\pi^\xi \colon \mathbb{R}^n \to \xi^\perp$ be the orthogonal projection. For every set $B \subset \mathbb{R}^n$ and for every $y \in \xi^\perp$ we define

$$B^\xi_y := \{t \in \mathbb{R} \mid y + t\xi \in B\}.$$

Moreover, for every function $u : B \to \mathbb{R}^n$ we define the function $\hat{u}^\xi_y : B^\xi_y \to \mathbb{R}$ by

$$\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi.$$

If $u : B \to \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$ the jump set of $\hat{u}^\xi_y$ is denoted by $J_{\hat{u}^\xi_y}$. Moreover we set

$$J^1_{\hat{u}^\xi_y} := \{t \in J_{\hat{u}^\xi_y} \mid |(\hat{u}^\xi_y)^+(t) - (\hat{u}^\xi_y)^-(t)| \geq 1\},$$

where $(\hat{u}^\xi_y)^-(t)$ and $(\hat{u}^\xi_y)^+(t)$ are the approximate right and left limits of $\hat{u}^\xi_y$ at t.

Definition 1.4.11. *Let Ω be an open set of $\mathbb{R}^n$. Let $v : \Omega \to \mathbb{R}^m$ be an $\mathcal{L}^n$-measurable function and let $x \in \mathbb{R}^n$ be such that $\Theta^{*n}(\mathcal{L}^n \llcorner \Omega; x) > 0$. We say that $a \in \mathbb{R}^m$ is the approximate limit of v as $y \to x$, and we write*

$$\operatorname{ap-}\lim_{y \to x} v(y) = a \tag{1.6}$$

if for every $\epsilon > 0$

$$\Theta^n(\mathcal{L}^n \llcorner \{|v - a| > \epsilon\}, x) = 0.$$

Remark 1.4.12. *Let Ω, v, x and a be as in the previous definition, and let ψ be a homeomorphism between $\mathbb{R}^m$ and a bounded open subset of $\mathbb{R}^m$. It is easy to prove that (1.6) holds if and only if*

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_\rho(x)} |\psi(v(y)) - \psi(a)| dy = 0.$$

In particular if v is $\mathcal{L}^n$-measurable , then v admits an approximate limit $\mathcal{L}^n$-a.e.. ■

Remark 1.4.13. *Notice that in the definition of approximate limit in the scalar case (Definition 1.4.1) one allows a to be equal to $\pm\infty$, while in this case $a \in \mathbb{R}^m$.* ■

Definition 1.4.14 (Approximate continuity)**.** *Let Ω be an open set of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $v : \Omega \to \mathbb{R}^m$ we define the approximate continuity set as the set of points $x \in \Omega$ for which there exists $a \in \mathbb{R}^m$ such that*

$$\operatorname{ap}\lim_{y \to x} v(y) = a.$$

The vector a is uniquely determined and is denoted by $\tilde{v}(x)$. The approximate discontinuity set S_v is defined as the complement in U of the approximate continuity set.

Definition 1.4.15 (Jump set)**.** *Let Ω be an open subset of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $v : \Omega \to \mathbb{R}^m$ we define the approximate jump set J_v as the set of point $x \in \Omega$ for which there exist $a, b \in \mathbb{R}^m$ with $a \neq b$, and $\nu \in \mathbb{S}^{n-1}$ such that*

$$\operatorname{ap-}\lim_{\substack{(y-x)\cdot\nu > 0 \\ y \to x}} v(y) = a \text{ and } \operatorname{ap-}\lim_{\substack{(y-x)\cdot\nu < 0 \\ y \to x}} v(y) = b.$$

The triplet (a, b, ν) is uniquely determined up to a permutation of a and b and a change of sign of ν and is denoted by $(v^+(x), v^-(x), \ \nu_v(x))$. The jump of v is the function $[v] : J_v \to \mathbb{R}^m$ defined by $[v](x) := v^+(x) - v^-(x)$ for every $x \in J_v$. Finally we define

$$J^1_v := \{x \mid |[v](x) \mid \ \geq 1\}.$$

Remark 1.4.16. *By [20, Proposition 2.6] we have that* S_v, J_v *and* J_v^1 *are Borel sets and* $\tilde{v} : U \backslash S_v \to \mathbb{R}^m$*, defined as* $\tilde{v}(x) = \text{ap-}\lim_{y\to x} v(y)$*, is a Borel function. Moreover, for every* $x \in J_v$*, we can choose the sign of* $\nu(x)$ *in such a way that* $v^+ : J_v \to \mathbb{R}^m, v^- : J_v \to \mathbb{R}^m$*, and* $\nu_v : J_v \to \mathbb{S}^{n-1}$ *are Borel functions.* ■

Definition 1.4.17. *We define* $\mathcal{T}$ *as the space of all functions* τ *of class* C^1*, defined on the real line* $\mathbb{R}$*, such that* $-\frac{1}{2} < \tau < \frac{1}{2}$ *and with bounded derivative* $|\tau'| < 1$.

Following [20, Definition 4.1], we are now in position to define the space $GBD(\Omega)$. In what follows Ω is an open set of $\mathbb{R}^n$.

Definition 1.4.18. *The space* $GBD(\Omega)$ *of generalised functions of bounded deformation is the space of all* $\mathcal{L}^n$*-measurable functions* $u : \Omega \to \mathbb{R}^n$ *with the following property: there exists* $\lambda \in \mathcal{M}_b^+(\Omega)$ *such that the following equivalent (see [20, Theorem 3.5]) conditions hold for every* $\xi \in \mathbb{S}^{n-1}$*:*

(a) for every $\tau \in \mathcal{T}$ *the directional derivative* $D_\xi(\tau(u\cdot\xi))$ *belongs to* $\mathcal{M}_b(\Omega)$ *and its total variation satisfies*

$$|D_\xi(\tau(u\cdot\xi))|(B) \le \lambda(B) \tag{1.7}$$

for every Borel set $B \subset \Omega$*;*

(b) for $\mathcal{H}^{n-1}$*-a.e.* $y \in \xi^\perp$ *the function* $\hat{u}_y^\xi$ *belongs to* $BV_{loc}(\Omega_y^\xi)$ *and*

$$\int_{\xi^\perp} (|D\hat{u}_y^\xi|(B_y^\xi \setminus J^1_{\hat{u}_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{\hat{u}_y^\xi}))d\mathcal{H}^{n-1}(y) \le \lambda(B) \tag{1.8}$$

for every Borel set $B \subset \Omega$.

Remark 1.4.19. *Following [20, Definition 4.16] and [20, Proposition 4.17], for every* $u \in GBD(\Omega)$*, there exists a measure* $\hat{\mu}_u \in \mathcal{M}_b^+$ *which is the smallest measure* λ *that satisfies (a) and (b) of the previous definition.* ■

Definition 1.4.20. *The space* $GSBD(\Omega)$ *of generalised function of special bounded deformation is the space of functions* $u \in GBD(\Omega)$ *such that for every* $\xi \in \mathbb{S}^{n-1}$ *and for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \xi^\perp$ *the function* $\hat{u}_y^\xi$ *belongs to* $SBV_{loc}(\Omega_y^\xi)$

In view of the results that we are going to present in Chapter 2 we need to recall the following facts.

Theorem 1.4.21. *(Traces on regular submanifold) Let* $u \in GBD(\Omega)$ *and let* $M \subset \Omega$ *be an (n-1)-dimensional manifold of class* C^1 *with unit normal* ν*. Then for* $\mathcal{H}^{n-1}$ *-a.e.* $x \in M$ *there exist* $u_M^+(x)$, $u_M^-(x) \in \mathbb{R}^n$ *such that*

$$\underset{\substack{\pm(y-x)\cdot\nu(x)>0 \\ y\to x}}{\text{ap-}\lim} u(y) = u_M^\pm(x). \tag{1.9}$$

Moreover for every $\xi \in \mathbb{S}^{n-1}$ *and for* $\mathcal{H}^{n-1}$*-a.e.* $y \in \xi^\perp$ *we have*

$$u_M^\pm(y+t\xi)\cdot\xi = \underset{\substack{\pm\sigma_y^\xi(t)(s-t)>0 \\ s\to t}}{\text{ap-}\lim} \hat{u}_y^\xi(s) \text{ for every } t \in M_y^\xi,$$

where $\sigma : M \to \{-1,+1\}$ *is defined by* $\sigma(x) := sign(\xi\cdot\nu(x))$*. Finally, the functions* $u_M^\pm : M \to \mathbb{R}^n$ *are* $\mathcal{H}^{n-1}$*-measurable.*

Proof. See [20, Theorem 5.2] for a detailed proof. □

Definition 1.4.22. *Let $u \in GBD(\Omega)$ and let $M \subset \Omega$ be an $(n-1)$-dimensional manifold of class C^1 oriented by ν. The $\mathbb{R}^n$-valued $\mathcal{H}^{n-1}$-measurable functions u_M^+ and u_M^-, defined $\mathcal{H}^{n-1}$-a.e. on M and satisfying (1.9) , are called the traces of u on the two sides of M.*

Here we recall a fundamental theorem about the jump set of functions in $GBD(\Omega)$ (see [20, Theorem, 8.1]). In particular, this result tells us that the jump set can be reconstructed by the jump points of the one dimensional slices.

Theorem 1.4.23. *(Slicing of the jump set). Let $u \in GBD(\Omega)$, then J_u is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set. Moreover let $\xi \in \mathbb{S}^{n-1}$ and let*

$$J_u^\xi := \{x \in J_u \mid (u^+(x) - u^-(x)) \cdot \xi \neq 0\}.$$

Then for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$ we have

$$(J_u^\xi)_y^\xi = J_{\hat{u}_y^\xi}, \tag{1.10}$$

$$u^\pm(y + t\xi) \cdot \xi = (\hat{u}_y^\xi)^{\pm\sigma_y^\xi(t)}(t) \text{ for every } t \in (J_u)_y^\xi, \tag{1.11}$$

where $\sigma : M \to \{-1, +1\}$ is defined by $\sigma(x) := \operatorname{sign}(\xi \cdot \nu_u(x))$, and $\nu_{\hat{u}_y^\xi} = 1$.

Remark 1.4.24 (Integrable jump implies BD)**.** *The previous theorem says that the jump set J_u can be reconstructed through the jump points of the one-dimensional restriction $J_{\hat{u}_y^\xi}$ for every direction ξ in $\mathbb{S}^{n-1}$. In particular if $u \in GBD(\Omega)$ has integrable jump, i.e. $[u] \in L^1(J_u, \mathcal{H}^{n-1})$, then u is actually a function in $BD(\Omega)$. Indeed, by definition of $BD(\Omega)$ (see [1]), we need only to check that for every $\xi \in \mathbb{S}^{n-1}$:*

$$\int_{\xi^\perp} |D\hat{u}_y^\xi|(\Omega_y^\xi)\; d\mathcal{H}^{n-1}(y) < \infty.$$

But relation (1.11) implies in particular that $[u \cdot \xi](y + t\xi) = [\hat{u}_y^\xi](t)$ for every $t \in J_{\hat{u}_y^\xi}$ and $\mathcal{H}^{n-1}$-a.e. y, so that we can write:

$$\begin{aligned}\int_{\xi^\perp} |D\hat{u}_y^\xi|(\Omega_y^\xi)\; d\mathcal{H}^{n-1}(y) &\leq \int_{\xi^\perp} |D\hat{u}_y^\xi|(\Omega_y^\xi \setminus J_{\hat{u}_y^\xi}) + \sum_{t \in J_{\hat{u}_y^\xi}} |[u \cdot \xi](y + t\xi)|\; d\mathcal{H}^{n-1}(y) \\ &\leq \lambda(\Omega \setminus J_u) + \int_{J_u} |[u]|\; d\mathcal{H}^{n-1},\end{aligned}$$

and we are done. ■

Every $u \in GBD(\Omega)$ admits $\mathcal{L}^n$-almost everywhere an *approximate symmetric gradient* $\mathcal{E}u$, which is a map $\mathcal{E}u : \Omega \to \mathbb{M}^{n\times n}_{sym}$ such that

$$\operatorname*{ap-lim}_{\substack{y \to x \\ y \neq x}} \frac{\big(u(y) - u(x) - \mathcal{E}u(x)(y - x)\big) \cdot (y - x)}{|y - x|^2} = 0. \tag{1.12}$$

Formula (1.12) says that the approximate symmetric gradient is unique. The following theorem proves that $\mathcal{E}u$ is an L^1-function.

Theorem 1.4.25. *Let $u \in GBD(\Omega)$. Then there exists a function $\mathcal{E}u \in L^1(\Omega; \mathbb{M}^{n\times n}_{sym})$ such that (1.12) holds for $\mathcal{L}^n$-a.e. $x \in \Omega$. Moreover for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}-$a.e. $y \in \xi^\perp$ we have (see [20, Theorem, 9.1])*

$$(\mathcal{E}u)^\xi_y \xi \cdot \xi = \nabla \hat{u}^\xi_y,$$

$\mathcal{L}^1$-a.e. on Ω^ξ_y.

Remark 1.4.26. *One of the main differences between GBD-functions and GBV-functions, is that in the first case the symmetric approximate gradient $\mathcal{E}u$ is always an L^1-function, while in the second case the approximate gradient ∇u is in general only $\mathcal{L}^n$-measurable.* ■

The following compactness result holds true.

Theorem 1.4.27. *Let Ω be an open set of $\mathbb{R}^n$ and let $(u_k)_{k=1}^\infty$ be a sequence of functions in $GSBD(\Omega)$. Suppose that there exist $p > 1$ such that*

$$\sup_{k\in\mathbb{N}} \big(\|u\|_{L^p} + \|\mathcal{E}u\|_{L^p} + \mathcal{H}^{n-1}(J_u)\big) < \infty.$$

Then there exists $u \in GSBD(\Omega)$ such that up to passing through a subsequence

$$\lim_{k\to\infty} u_k(x) = u(x), \quad \mathcal{L}^n\text{-a.e.} \quad \text{and} \quad \mathcal{E}u_k \rightharpoonup \mathcal{E}u, \text{ weakly in } L^1(\Omega), \text{ as } k \to \infty,$$

and

$$\liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}) \geq \mathcal{H}^{n-1}(J_u).$$

Proof. It is a particular case of [2, Theorem 4.36]. □

Let us introduce the following spaces.

Definition 1.4.28. *Let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. We define for every $p \geq 1$*

- $GSBD^p(\Omega) := \{u \in GSBD(\Omega) \mid \mathcal{E}u \in L^p(\Omega; \mathbb{M}^{n\times n}_{sym})\}$;
- $GSBD^p_p(\Omega) := \{u \in GSBD(\Omega) \mid u \in L^p(\Omega; \mathbb{R}^n), \mathcal{E}u \in L^p(\Omega; \mathbb{M}^{n\times n}_{sym})\}$;
- $GBD(\Omega; \Gamma) := \{u \in GBD(\Omega) \mid J_u \subseteq \Gamma\}$;
- $GSBD(\Omega; \Gamma) := \{u \in GSBD(\Omega) \mid J_u \subseteq \Gamma\}$;
- $GSBD^p(\Omega; \Gamma) := \{u \in GSBD(\Omega; \Gamma) \mid \mathcal{E}u \in L^p(\Omega; \mathbb{M}^{n\times n}_{sym})\}$;
- $GSBD^p_p(\Omega; \Gamma) := \{u \in GSBD^p_p(\Omega) \mid J_u \subseteq \Gamma\}$.

Proposition 1.4.29. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then, for every $p > 1$ the space $GSBD^p_p(\Omega; \Gamma)$ endowed with the norm*

$$\|u\|_p := \|u\|_{L^p} + \|\mathcal{E}u\|_{L^p},$$

is a Banach space.

Proof. Thanks to [20, Remark 4.6] we know that $GSBD(\Omega)$ is a real vector space, and as a consequence $GSBD^p_p(\Omega)$ is a real vector space too. The fact that $GSBD^p_p(\Omega;\Gamma)$ is also a real vector space follows once we prove that given $u, v \in GSBD(\Omega)$ then $J_{u+v} \subset J_u \cup J_v$ $\mathcal{H}^{n-1}$-a.e.. To see this, fix Ξ an orthonormal basis of $\mathbb{R}^n$, say $\{\xi_1, \dots, \xi_n\}$ and consider the directions in $\mathbb{S}^{n-1}$ defined by

$$C(\Xi,\delta) := \{x \in \mathbb{S}^{n-1} \mid |x \cdot \xi_i| > (1/\sqrt{n} - \delta)|x|, \quad \text{for every } \xi_i \in \Xi\},$$

where δ is any real number in $(0, 1/\sqrt{n})$ (see also Remark 2.1.7). We claim that

$$A := \{x \in J_{u+v} \mid \nu_{u+v}(x) \in C(\Xi,\delta)\} \subset J_u \cup J_v, \quad \mathcal{H}^{n-1}\text{-a.e.} \tag{1.13}$$

Notice that for every $\xi \in \mathbb{S}^{n-1}$ and for every $y \in \xi^\perp$

$$[\mathbb{R}^n \setminus (J_u \cup J_v)]^\xi_y \cap (J_u)^\xi_y = \emptyset, \quad \text{and} \quad [\mathbb{R}^n \setminus (J_u \cup J_v)]^\xi_y \cap (J_v)^\xi_y = \emptyset,$$

and thanks to (1.10) we deduce that for every $\xi \in \mathbb{S}^{n-1}$

$$[\mathbb{R}^n \setminus (J_u \cup J_v)]^\xi_y \cap J_{\hat{u}^\xi_y} = \emptyset, \quad \text{and} \quad [\mathbb{R}^n \setminus (J_u \cup J_v)]^\xi_y \cap J_{\hat{v}^\xi_y} = \emptyset, \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \xi^\perp. \tag{1.14}$$

Since the one dimensional slices of u and v are SBV_{loc}-functions, by (1.14) we deduce that for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$ the sets $[\mathbb{R}^n \setminus (J_u \cup J_v)]^\xi_y$ is contained in the set of Lebesgue points of $\hat{u}^\xi_y + \hat{v}^\xi_y$ which in turn is contained in the sets of Lebesgue point of $((u+v)\cdot\xi)^\xi_y$. By using again (1.10) this means that for every $\xi \in \mathbb{S}^{n-1}$ we have

$$[\mathbb{R}^n \setminus (J_u \cup J_v)]^\xi_y \cap (J^\xi_{u+v})^\xi_y = \emptyset, \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \xi^\perp. \tag{1.15}$$

Now suppose that (1.13) does not hold. This means that there exists a set $A' \subset A$ with $\mathcal{H}^{n-1}(A') > 0$ but $\mathcal{H}^{n-1}((J_u \cup J_v) \cap A') = 0$. Since Ξ is a basis of $\mathbb{R}^n$, then there must exists a $\xi_i \in \Xi$ such that

$$\mathcal{H}^{n-1}(A' \cap \{((u+v)^+ - (u+v)^-) \cdot \xi_i \neq 0\}) > 0.$$

By using also that for $\mathcal{H}^{n-1}$-a.e. $x \in A' \cap \{((u+v)^+ - (u+v)^-)\cdot\xi_i \neq 0\}$ we have $\nu_{u+v}(x)\cdot\xi_i > 0$ (simply by definition of A), by using Coarea Formula applied on the projection map

$$\pi^\xi \colon A' \cap \{((u+v)^+ - (u+v)^-) \cdot \xi_i \neq 0\} \to \xi^\perp$$

we deduce that if we set $A'_i := A' \cap \{((u+v)^+ - (u+v)^-) \cdot \xi_i \neq 0\}$, then $\mathcal{H}^{n-1}(\pi^\xi(A'_i)) > 0$ and

$$\mathcal{H}^0([A' \cap \{((u+v)^+ - (u+v)^-) \cdot \xi_i \neq 0\}]^\xi_y) > 0, \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \pi^\xi(A'_i).$$

But since $A'_i \subset J^{\xi_i}_{u+v}$, by (1.15) this means that also

$$\mathcal{H}^0((J_u \cup J_v)^\xi_y) > 0, \quad \mathcal{H}^{n-1}\text{-a.e. } y \in \pi^\xi(A'_i),$$

which is a contradiction since we assumed that $\mathcal{H}^{n-1}((J_u \cup J_v) \cap A') = 0$ and proves our claim. Finally, thanks to the arbitrariness of Ξ, relation (1.13) proves exactly that $J_{u+v} \subset J_u \cup J_v$ $\mathcal{H}^{n-1}$-a.e..

To prove the completeness we proceed exactly as in Proposition 1.4.10. Suppose that $(u_k) \subset GSBD^p_p(\Omega;\Gamma)$ is a Cauchy sequence. In particular (u_k) converges strongly in $L^p(\Omega,\mathbb{R}^n)$ to some u . Since

$$\sup_{k\in\mathbb{N}} \big(\|u_k\|_{L^p} + \|\mathcal{E}u_k\|_{L^p} + \mathcal{H}^{n-1}(J_{u_k})\big) < \infty,$$

by Theorem 1.4.27, passing through a subsequence, we know that there exists $v \in GSBD(\Omega)$ such that $u_k \to v$ pointwise a.e. on Ω and $\mathcal{E}u_k \rightharpoonup \mathcal{E}v$ weakly in $L^1(\Omega)$. This implies $u = v$ and thanks to the lower semicontinuity of the L^p norm with respect to the weak convergence also

$$\|\mathcal{E}u\|_{L^p} \leq \liminf_{k\to\infty} \|\mathcal{E}u_k\|_{L^p},$$

hence $u \in GSBD^p_p(\Omega)$.

It remains to prove that $J_u \subseteq \Gamma$. Using [20, Theorem, 11.3], for every open set $U \subset \Omega$ we have

$$\mathcal{H}^{n-1}(J_u \cap U) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k} \cap U)\ .$$

Since the measure $\mathcal{H}^{n-1} \llcorner \Gamma$ is inner regular, for every $\epsilon > 0$ we can find a compact set $K \subset \Gamma$, such that $\mathcal{H}^{n-1}(\Gamma \setminus K) \leq \epsilon$, and so

$$\mathcal{H}^{n-1}(J_u \setminus K) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k} \setminus K) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(\Gamma \setminus K) \leq \epsilon.$$

For the arbitrariness of ϵ, we conclude that $\mathcal{H}^{n-1}(J_u \setminus \Gamma) = 0$. □

Remark 1.4.30. *Thanks to the previous proposition, the space $GSBD^2_2(\Omega;\Gamma)$ endowed with the scalar product*

$$\langle u, v\rangle_2 := \langle u, v\rangle_{L^2} + \langle \mathcal{E}u, \mathcal{E}v\rangle_{L^2}$$

is actually an Hilbert space ■

Chapter 2

Trace operator on $GSBV(\Omega)$ and $GSBD(\Omega)$

Contents

2.1 Integrability of the trace in $GBD(\Omega)$

In the context of the Sobolev spaces, when Ω is an open set whose boundary forms an external cusp, it is not possible to have a trace inequality of the form

$$\int_\Omega |Tr(u)|^p \, d\mathcal{H}^{n-1} \leq C\bigg(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx\bigg), \quad u \in W^{1,p}(\Omega), \tag{2.1}$$

for some $C = C(\Omega, p) > 0$. Nevertheless, it is possible to introduce a weight function in front of the $\mathcal{H}^{n-1}$-measure which makes inequality (2.1) holds true. This is the starting point of our analysis.

2.1.1 The weight function Θ

Given $\Gamma \subset \Omega$ a countably $(\mathcal{H}^{n-1},\ n-1)$-rectifiable with $\mathcal{H}^{n-1}(\Gamma) < \infty$, we want to introduce a family of functions $(\theta^\xi)_{\xi \in \mathbb{S}^{n-1}}$, $\theta^\xi : \mathbb{R}^n \to \mathbb{R}^+$, called *one sectional distance*, which will play a fundamental role in the integrability of the trace of a GBD function. Before doing this, let us recall a property of rectifiable sets with finite measure.

Remark 2.1.1. *Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1},\ n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Choose any $\xi \in \mathbb{S}^{n-1}$ then*

$$\mathcal{H}^0(\Gamma^\xi_y) < \infty \text{ for a.e. } y \in \xi^\perp.$$

This fact is a simple consequence of the Coarea formula applied to the projection map π^ξ from $\mathbb{R}^n$ onto $\xi^\perp$ restricted on Γ. ■

Definition 2.1.2. *(One sectional distance) Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$, and let $\xi \in \mathbb{S}^{n-1}$. Writing $x \in \mathbb{R}^n$ as $x = y + t\xi$ (for $(y,t) \in \xi^\perp \times \mathbb{R}$), we define $\theta^\xi \colon \mathbb{R}^n \to \mathbb{R}^+$ in such a way that:*

$$\theta^\xi(y + t\xi) = \begin{cases} |t_{i+1} - t_i| \wedge 1 & \text{if } 1 < \mathcal{H}^0(\Gamma^\xi_y) < \infty \text{ and } t \in (t_i, t_{i+1}) \\ 1 & \text{otherwise,} \end{cases}$$

where $(t_i)_{i=1}^{\mathcal{H}^0(\Gamma^\xi_y)}$ are the elements of the set Γ^ξ_y ordered so that $t_1 < \ldots < t_i < \ldots < t_{\mathcal{H}^0(\Gamma^\xi_y)}$.

Definition 2.1.3. *(Orientation) Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1},\ n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. We call an orientation of Γ any map $\nu : \Gamma \to \mathbb{S}^{n-1}$ which is $\mathcal{H}^{n-1}$-measurable and such that $\nu(x)$ is orthogonal to the tangent space of Γ at x for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.*

Proposition 2.1.4. *Let $\Gamma \subset \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1},\ n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$, and choose $\xi \in \mathbb{S}^{n-1}$. Then the function θ^ξ of Definition 2.1.2 is $\mathcal{L}^n$-measurable.*

Proof. By [36, Theorem 3.2.29] Γ is contained in a countable union of C^1 submanifolds of $\mathbb{R}^n$ say $(M_k)_{k=1}^\infty$ up to a $\mathcal{H}^{n-1}$-negligible set. If we define $\Gamma^\xi := \{x \in \Gamma \mid \nu_\Gamma(x) \cdot \xi \neq 0\}$ and $M_k^\xi := \{x \in M_k \mid \nu_{M_k}(x) \cdot \xi \neq 0\}$, where $\nu_\Gamma(\cdot)$ and $\nu_{M_k}(\cdot)$ are respectively an orientation of Γ and of M_k in the sense of Definition 2.1.3, then

$$\mathcal{H}^{n-1}(\Gamma^\xi \setminus \bigcup_k M_k^\xi) = 0.$$

For each k, M_k^ξ can be covered by countably many $(n-1)$-dimensional submanifolds of class C^1, say $(\Sigma_{k,i})_{i\in\mathbb{N}}$, which are the graph of C^1 functions, say $(f_{k,i})_{i\in\mathbb{N}}$, defined on some open subset of $\xi^\perp$ (using Lindëlof property and the Implicit Function Theorem). Hence, possibly re-enumerating the $(\Sigma_{k,i})_{(k,i)\in\mathbb{N}^2}$ as $(\Sigma_i)_{i=1}^\infty$ (and respectively the $(f_{k,i})_{(k,i)\in\mathbb{N}^2}$ as $(f_i)_{i=1}^\infty$), we have

$$\mathcal{H}^{n-1}(\Gamma^\xi \setminus \bigcup_i \Sigma_i) = 0. \tag{2.2}$$

For any couple of indices $(i_1, i_2) \in \mathbb{N}^2$, define $\theta^\xi_{i_1,i_2}$ to be the one sectional distance relative to the rectifiable set $\Sigma_{i_1} \cup \Sigma_{i_2}$. Suppose for a moment that we already know that $\theta^\xi_{i_1,i_2}$ is $\mathcal{L}^n$-measurable for any $(i_1,\ i_2)$. In this case we can define

$$\tilde{\theta}^\xi_{i_1,i_2}(y + t\xi) := \begin{cases} \theta^\xi_{i_1,i_2}(y + t\xi) & \text{if } \ y \in \pi^\xi(\Gamma \cap \Sigma_{i_1}) \cap \pi^\xi(\Gamma \cap \Sigma_{i_2}) \\ 1 & \text{otherwise} \end{cases} \tag{2.3}$$

Clearly $\tilde{\theta}^\xi_{i_1,i_2}$ is $\mathcal{L}^n$-measurable because the set $\pi^\xi(\Gamma \cap \Sigma_{i_1}) \cap \pi^\xi(\Gamma \cap \Sigma_{i_2})$ is $\mathcal{H}^{n-1}$-measurable and we use Fubini's theorem on the product space $\xi^\perp \times \mathbb{R}$ (see [34, Section 1.4]) to deduce that the set $\big(\pi^\xi(\Gamma \cap \Sigma_{i_1}) \cap \pi^\xi(\Gamma \cap \Sigma_{i_2})\big) \times \mathbb{R}$ is $\mathcal{L}^n$-measurable. With (2.3) it is easy to see that

$$\theta^\xi(x) = \inf_{(i_1,i_2)\in\mathbb{N}^2} \tilde{\theta}^\xi_{i_1,i_2}(x), \tag{2.4}$$

for any $x = y + t\xi$ such that $(\Gamma^\xi)^\xi_y \subset \bigcup_{i=0}^{\infty}(\Sigma_i)^\xi_y$. Thanks to (2.2), the previous inclusion holds for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$, hence (2.4) holds for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. This gives that θ^ξ is $\mathcal{L}^n$-measurable.

Finally it remains to prove the measurability of $\theta^\xi_{i_1,i_2}$. It is enough to notice that on the set of point where $f_{i_1} < f_{i_2}$:

$$\theta^\xi_{i_1,i_2}(y+t\xi) = \begin{cases} |f_{i_2}(y) - f_{i_1}(y)| \wedge 1 & \text{if } \; y \in \pi^\xi(\Sigma_{i_1}) \cap \pi^\xi(\Sigma_{i_2}) \;,\; f_{i_1}(y) < t < f_{i_2}(y) \\ 1 & \text{otherwise,} \end{cases} \tag{2.5}$$

while on the set of points where $f_{i_1} > f_{i_2}$:

$$\theta^\xi_{i_1,i_2}(y+t\xi) = \begin{cases} |f_{i_2}(y) - f_{i_1}(y)| \wedge 1 & \text{if } \; y \in \pi^\xi(\Sigma_{i_1}) \cap \pi^\xi(\Sigma_{i_2}) \;,\; f_{i_2}(y) < t < f_{i_1}(y) \\ 1 & \text{otherwise,} \end{cases} \tag{2.6}$$

□

Remark 2.1.5. *The one sectional distance θ^ξ of a rectifiable set Γ with $\mathcal{H}^{n-1}(\Gamma) < \infty$, has finite total variation in the direction ξ. In fact it can be easily proved that:*

$$|D_\xi \theta^\xi|(\mathbb{R}^n) \le \int_\Gamma |\nu(x) \cdot \xi| d\mathcal{H}^{n-1}(x) \le \mathcal{H}^{n-1}(\Gamma). \tag{2.7}$$

So given any countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set $\Gamma \subset \mathbb{R}^n$ with finite measure, by [20, Theorem 5.1], we can talk about the trace of θ^ξ on the set $\{x \in \Gamma \mid \nu_\Gamma(x) \cdot \xi \neq 0\}$. ■

Definition 2.1.6. *Let $\xi \in \mathbb{S}^{n-1}$ and let $0 < L < 1$.*

We define the cone with axis ξ and opening L as

$$C(\xi, L) := \{x \in \mathbb{R}^n \setminus \{0\} \mid |\xi \cdot x| > L|x|\}.$$

We define the upper half cone with axis ξ and opening L as:

$$C^+(\xi, L) := \{x \in \mathbb{R}^n \setminus \{0\} \mid \xi \cdot x > L|x|\},$$

and analogously the lower half cone cone with axis ξ and opening L as:

$$C^-(\xi, L) := \{x \in \mathbb{R}^n \setminus \{0\} \mid \xi \cdot x < -L|x|\}.$$

Remark 2.1.7. *Consider $\Xi := \{\xi_1, \dots, \xi_n\}$ an orthonormal basis of $\mathbb{R}^n$ and let δ be a real number such that $0 < \delta < 1/\sqrt{n}$. Define:*

$$C(\Xi,\ \delta) := \bigcap_{i=1}^{n} C(\xi_i,\ 1/\sqrt{n} - \delta) \cap \mathbb{S}^{n-1}. \tag{2.8}$$

Notice that $C(\Xi, \delta)$ is open in the relative topology of $\mathbb{S}^{n-1}$ and contains for example the vector $\sum_{i=1}^n \xi_i/\sqrt{n}$. This means that the family $\Lambda :=\{\ C(\Xi, \delta) \mid \Xi$ orthonormal basis$\}$ is an open covering of $\mathbb{S}^{n-1}$, and so by compactness we can always extract a finite subcovering from Λ.

We denote by $N(\delta)$ the minimum number of elements of Λ that needs to cover $\mathbb{S}^{n-1}$. $N(\delta)$ is a constant that depends only on the dimension n and on δ. ■

2.1.2 Definition of the trace operator

We want to extend the notion of trace operator for an arbitrary open set of $\mathbb{R}^n$ having finite perimeter. We start with the following extension property: let Ω be an open set of finite perimeter and let $u \in GBD(\Omega)$, then it is possible to extend u to a function defined on the whole of $\mathbb{R}^n$ which still belongs to $GBD(\mathbb{R}^n)$. Before doing this, we need the following proposition concerning an extension property of BV functions of one variable.

Proposition 2.1.8. *Let $E = \bigcup_{k=1}^{M} I_k$ where $I_k = (a_k,\ b_k) \subset \mathbb{R}$ are open intervals (possibly unbounded) and pairwise disjoint. If $u \in BV(E)$ then the function defined by:*

$$v(t) := \begin{cases} u(t) & \text{if } t \in E, \\ 0 & \text{otherwise.} \end{cases}$$

belongs to $BV(\mathbb{R})$. Moreover

$$Dv = \sum_{k=0}^{M} (u^-(b_k)\delta_{b_k} - u^+(a_k)\delta_{a_k}) + Du(E), \tag{2.9}$$

where $\delta_{(\cdot)}$ denotes the Dirac's delta, and

$$|Dv| = \sum_{k=0}^{M} (|u^-(b_k)| + |u^+(a_k)|) + |Du|(E). \tag{2.10}$$

Proof. It is a simple application of the theory of BV functions in one variable. □

Proposition 2.1.9. *(Extension of GBD functions) Let $\Omega \subset \mathbb{R}^n$ be an open set of finite perimeter and let $u \in GBD(\Omega)$. If we define:*

$$\underline{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise,} \end{cases}$$

then $\underline{u} \in GBD(\mathbb{R}^n)$. Moreover we have:

(a) $J_{\underline{u}} \subset J_u \cup \partial^*\Omega$;

(b) for every Borel set $B \subset \mathbb{R}^n$ and every $\xi \in \mathbb{S}^{n-1}$ the following inequality holds true:

$$\int_{\xi^\perp} \left(|D\hat{\underline{u}}_y^\xi|(B_y^\xi \setminus J^1_{\hat{\underline{u}}_y^\xi}) + \mathcal{H}^0(B_y^\xi \cap J^1_{\hat{\underline{u}}_y^\xi})\right) d\mathcal{H}^{n-1}(y) \leq \hat{\mu}_u(B) + \mathcal{H}^{n-1}(\partial^*\Omega \cap B)\ , \tag{2.11}$$

where $\hat{\mu}_u$ is the smallest measure relative to u that satisfies conditions (1.7) and (1.8) (see Remark 1.4.19);

(c) the approximate symmetric gradient of $\underline{u}$ is such that:

$$\mathcal{E}\underline{u}(x) = \begin{cases} \mathcal{E}u(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

(d) if $u \in GSBD(\Omega)$ then $\underline{u} \in GSBD(\mathbb{R}^n)$.

Proof. First we show that (1.7) holds true. Fix $\xi \in \mathbb{S}^{n-1}$ and $\tau \in \mathcal{T}$. By [2, Theorem 3.103] we have

$$|D_\xi \tau(\underline{u}\cdot\xi)|(B) = \int_{\xi^\perp} |D(\tau(\underline{u}\cdot\xi)^\xi_y)|(B^\xi_y)dy, \tag{2.13}$$

for any Borel set $B \subset \mathbb{R}^n$, and

$$\int_{\Pi\xi} |D\mathbb{1}_{\Omega^\xi_y}|(\mathbb{R})\, dy = |D_\xi \mathbb{1}_\Omega|(\mathbb{R}^n) \le \mathcal{H}^{n-1}(\partial^*\Omega) < \infty. \tag{2.14}$$

It follows that for $\mathcal{H}^{n-1}$-a.e. $y \in \xi^\perp$, Ω^ξ_y has finite perimeter. By the characterisation of sets of finite perimeter in $\mathbb{R}$, we know that for those $y \in \xi^\perp$, Ω^ξ_y is equivalent to a finite union of open pairwise disjoint intervals. Notice that

$$\tau(\underline{u}\cdot\xi) = \tau(u\cdot\xi)\mathbb{1}_\Omega + \tau(0)\mathbb{1}_{\Omega^c}. \tag{2.15}$$

Now for each $y \in \xi^\perp$ such that $|D\mathbb{1}_{\Omega^\xi_y}| < \infty$, we can apply Proposition 2.1.8 to the one dimensional sections $t \mapsto \tau(\hat{u}^\xi_y)\mathbb{1}_{\Omega^\xi_y} + \tau(0)\mathbb{1}_{(\Omega^c)^\xi_y}$, and by using also (2.13) and the Coarea formula we have that:

$$\begin{aligned} |D_\xi \tau(\underline{u}\cdot\xi)|(B) &= \int_{\xi^\perp} |D\big(\tau(\hat{u}^\xi_y)\mathbb{1}_{\Omega^\xi_y} + \tau(0)\mathbb{1}_{(\Omega^c)^\xi_y}\big)|(B^\xi_y)\, dy \\ &\le \int_{\xi^\perp} \Big(|D\tau(\hat{u}^\xi_y)|(\Omega^\xi_y \cap B^\xi_y) + \sum_{t\in\partial^*\Omega^\xi_y\cap B^\xi_y} |\tau(\hat{u}^\xi_y(t))^{\sigma^\xi_y(t)} - \tau(0)|\Big)\, dy \\ &\le |D_\xi\tau(u\cdot\xi)|(B\cap\Omega) + \mathcal{H}^{n-1}(\partial^*\Omega\cap B) \\ &\le \hat{\mu}_u(B\cap\Omega) + \mathcal{H}^{n-1}(\partial^*\Omega\cap B), \end{aligned} \tag{2.16}$$

for every Borel set $B \subseteq \mathbb{R}^n$, where $\sigma(x) = \mathrm{sign}(\nu_\Omega(x)\cdot\xi)$ and ν_Ω denotes the measure theoretic inner unit normal. Let $\eta := \hat{\mu}_u + \mathcal{H}^{n-1} \llcorner \partial^*\Omega$ then

$$|D_\xi \tau(\mathbb{1}_\Omega u\cdot\xi)|(B) \le \eta(B), \tag{2.17}$$

for every $\tau \in \mathcal{T}$ and for every $\xi \in \mathbb{S}^{n-1}$. This is exactly (1.7), and we deduce that $\underline{u} \in GBD(\mathbb{R}^n)$.

Point (a) can be deduced simply by Theorem 1.4.23.

To show estimate (2.11) it is enough to notice that the two definitions of $GBD(\Omega)$ are equivalent (see Definition 1.4.18).

Point (c) follows from the characterisation of the symmetric approximate gradient given by the formula (1.12).

Finally (d) follows from Proposition 2.1.8 using the same argument as above. □

Remark 2.1.10. *Under the assumptions of Proposition 2.1.9 let $\partial^*\Omega$ be oriented by its measure theoretic inner normal. Then the extended function $\underline{u}$ of the previous proposition, is such that $\underline{u}^- = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^*\Omega$. Roughly speaking, $\underline{u}$ has almost everywhere zero trace from the complement of Ω. Indeed we can consider a finite measurable partition of $\partial^*\Omega$, say $(\Sigma_j)_{j=1}^N$. To each Σ_i there exists an orthonormal basis of $\mathbb{R}^n$ $\{\xi_1,\dots,\xi_n\}$ such that $\nu(x)\cdot\xi_i \neq 0$ for every $x \in \Sigma_i$ and for every $i = 1,\dots,n$ (see Remark 2.1.7). If we call $\sigma(x) = sign(\nu_\Omega(x)\cdot\xi)$, it is easy to see that for any $i = 1,\dots,n$, it holds*

$$(\hat{\underline{u}}^{\xi_i}_y)^{-\sigma^\xi_y(t)}(t) = 0, \text{ for every } t \in J_{\hat{\underline{u}}^{\xi_i}_y}, \text{ and for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^{\xi_i}. \tag{2.18}$$

Since $\partial^*\Omega$ can be covered by countably many $(n-1)$-dimensional manifolds of class C^1, using Theorem 1.4.21 and $\nu(x)\cdot\xi_i \neq 0$, we can conclude

$$\underline{u}^-_{\partial^*\Omega}(x) = 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma_j. \tag{2.19}$$

Because of the fact that $(\Sigma_j)_{j=1}^N$ is a measurable partition of $\partial^*\Omega$ we have

$$\underline{u}^-_{\partial^*\Omega}(x) = 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^*\Omega, \tag{2.20}$$

which is the desired result. ■

Definition 2.1.11. *(Trace operator in $GBD(\Omega)$). Let Ω be an open set of $\mathbb{R}^n$ of finite perimeter, and let $u \in GBD(\Omega;\Gamma)$. We define the trace operator as*

$$Tr(u)(x) := \underline{u}^+_{\partial^*\Omega}(x), \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^*\Omega, \tag{2.21}$$

where $\underline{u}$ is the function extended to 0 outside of Ω given in proposition 2.1.9, and the trace from above $\underline{u}^+$ is considered with respect to the measure theoretic inner normal ν_Ω of the reduced boundary $\partial^*\Omega$.

Moreover in order to simplify the notation, when there is no misunderstanding, we simply write

$$u^+(x) = \begin{cases} u^+_\Gamma(x) & \text{if } x \in \Gamma, \\ Tr(u)(x) & \text{if } x \in \partial^*\Omega, \end{cases} \tag{2.22}$$

and:

$$u^-(x) = \begin{cases} u^-_\Gamma(x) & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in \partial^*\Omega. \end{cases} \tag{2.23}$$

Remark 2.1.12 (Coincidence of Trace)**.** *When Ω is a Lipschitz-regular domain, our definition of trace coincides with the usual one in the space $BD(\Omega)$.*

First of all in this case, the reduced boundary $\partial^*\Omega$ coincides $\mathcal{H}^{n-1}$-a.e. with the topological one. Moreover on the space of regular functions up to the boundary, our definition coincides with the restriction operator on $\partial\Omega$. Then using a density argument together with identities (5.3) and (5.5) in [20], we deduce the coincidence of our notion of trace with the usual one in $BD(\Omega)$. ■

2.1.3 Trace inequalities with weighted surface measure

Now we are in position to prove our main results about the integrability of the trace in $GBD(\Omega;\Gamma)$ and $GSBD^p_p(\Omega;\Gamma)$. As mentioned in the introduction, we will consider the trace on $\partial^*\Omega$ and both traces $u^\pm$ on Γ. We decide to split our results into two theorems, the first concerns the case GBD.

Theorem 2.1.13. *(Trace inequality in $GBD(\Omega)$). Let Ω be an open set of $\mathbb{R}^n$ of finite perimeter, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1},\ n-1)$-rectifiable set, with $\mathcal{H}^{n-1}(\Gamma) < \infty$ and oriented by ν. Then there exist two $\mathcal{H}^{n-1}$-measurable functions $\Theta^\pm : \Gamma \cup \partial^*\Omega \to \mathbb{R}^+$ depending only on the geometry of Γ, its orientation ν, and on Ω, such that denoting with $u^\pm$ the traces of u according to Definition 2.1.11 , we have*

(a) $\mathcal{H}^{n-1}\big(\{\Theta^\pm = 0\}\big) = 0$ and $\Theta^\pm \in L^\infty(\Gamma \cup \partial^*\Omega,\ \mathcal{H}^{n-1})$ (in particular $\|\Theta^\pm\|_\infty \le 1$);

(b) For every $u \in GBD(\Omega;\Gamma) \cap L^1(\Omega,\mathbb{R}^n)$ *we have:*

$$\int_{\Gamma\cup\partial^*\Omega} |u^{\pm}(x)|\Theta^{\pm}(x)d\mathcal{H}^{n-1}(x) \le C(n)\Big(\hat{\mu}_u(\Omega\setminus J_u) + \int_\Omega |u(x)|dx\Big). \tag{2.24}$$

Proof. Let $\underline{u} \in GBD(\mathbb{R}^n;\Gamma\cup\partial^*\Omega)$ be the function extended to 0 outside of Ω as in proposition 2.1.9. In order to simplify the notation, we write Γ to denote $\Gamma\cup\partial^*\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with $\nu_{\partial^*\Omega}$ on $\partial^*\Omega$. By our definition of $u^{\pm}$ (Definition 2.1.11) and by Proposition 2.1.9 (in particular point (b) tells us that $\hat{\mu}_{\underline{u}} \le \hat{\mu}_u + \mathcal{H}^{n-1} \llcorner \partial^*\Omega$), (2.24) can be rewritten as:

$$\int_\Gamma |\underline{u}^{\pm}|\Theta^{\pm}\, d\mathcal{H}^{n-1} \le C(n)\Big(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}(x)|dx\Big). \tag{2.25}$$

So let us prove (2.25) for any function in the space $GBD(\mathbb{R}^n;\Gamma\cup\partial^*\Omega)$.

Consider Λ the covering of $\mathbb{S}^{n-1}$ of Remark 2.1.7 and by compactness define $\big(C(\Xi_i,\delta)\big)_{i=1}^{N(\delta)}$ to be a subcovering of Λ. If we define for any $i = 1,\dots,N$, $\Gamma_i := \nu^{-1}\big(C(\Xi_i,\delta)\big)$ then $(\Gamma_i)_{i=1}^N$ is a finite measurable covering of Γ. By definition of Λ, for any $\xi \in \Xi_i$ and for every $x \in \Gamma_i$, we have $|\xi\cdot\nu(x)| > 1/\sqrt{n} - \delta$.

Now we fix i and $\xi \in \Xi_i$. We write the generic point $x \in \mathbb{R}^n$ as $(y,t) \in \xi^\perp \times \mathbb{R}$, and from now on we will work on the set of points $y \in \pi^\xi(\Gamma_i)$ such that $\hat{\underline{u}}^\xi_y \in BV_{loc}(\mathbb{R})$ and $\mathcal{H}^0(\Gamma^\xi_y) < \infty$; from the Definition 1.4.18 of GBD and Remark 2.1.1 we already know that $\mathcal{H}^{n-1}$ almost all of y have these properties.

We call $(t_k)_{k=1}^{\mathcal{H}^0(\Gamma^\xi_y)}$ the point of the slicing Γ^ξ_y ordered such that $t_k < t_{k+1}$ for any k.

Let $\theta^\xi : E \to \mathbb{R}^+$ be the one sectional distance introduced in Definition 2.1.2. Thanks to Remark 2.1.5, for $x \in \Gamma$ we can consider $\theta^{\xi^\pm}(x)$ the trace respectively from above and from below on Γ_i.

By Theorem 1.4.23:

$$\underline{u}^+(y+t\xi)\cdot\xi = (\hat{\underline{u}}^\xi_y)^+(t) \text{ if } t \in (\Gamma_i)^\xi_y \text{ and } \nu(x)\cdot\xi > 0,$$

and

$$\underline{u}^+(y+t\xi)\cdot\xi = (\hat{\underline{u}}^\xi_y)^-(t) \text{ if } t \in (\Gamma_i)^\xi_y \text{ and } \nu(x)\cdot\xi < 0.$$

Since ξ has been fixed, in order to simplify the notation, we omit the dependence on ξ and write $\Gamma_i^+ := \Gamma_i \cap \{\nu\cdot\xi > 0\}$ and $\Gamma_i^- := \Gamma_i \cap \{\nu\cdot\xi < 0\}$. Let's focus for example on the set Γ_i^+:

$$\hat{\underline{u}}^\xi_y(t) - (\hat{\underline{u}}^\xi_y)^+(t_k) = \int_{t_k}^t dD\hat{\underline{u}}^\xi_y, \text{ for } t_k \in (\Gamma_i^+)^\xi_y \text{ and } t_k < t < t_{k+1}. \tag{2.26}$$

Now at fixed $y \in \pi^\xi(\Gamma_i^+)$ we can integrate again on $t \in (t_k,\ t_{k+1})$ to get

$$(t_{k+1}-t_k)|(\hat{\underline{u}}^\xi_y)^+(t_k)| \le (t_{k+1}-t_k)\int_{t_k}^{t_{k+1}} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(t)|dt, \\ \text{for } t_k \in (\Gamma_i^+)^\xi_y, \text{ and } t_{k+1}-t_k \le 1, \tag{2.27}$$

and

$$|(\hat{\underline{u}}^\xi_y)^+(t_k)| \le \int_{t_k}^{1+t_k} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(t)|\, dt, \\ \text{for } t_k \in (\Gamma_i^+)^\xi_y, \text{ and } t_{k+1}-t_k > 1. \tag{2.28}$$

Using the fact that θ^{ξ^+} is equal to $t_{k+1}-t_k$ or 1 on the set $\{y+t\xi \mid t_k<t<t_{k+1}\}$, we sum on $t_k \in (\Gamma_i^+)^\xi_y$ to get

$$\begin{aligned}\sum_{t_k} |(\hat{\underline{u}}^\xi_y)^+(t_k)|\theta^{\xi^+}(y+t_k\xi) &\leq \sum_{t_k}\left(\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} |\hat{\underline{u}}^\xi_y(t)|\,dt\right)\\ &\leq |D\hat{\underline{u}}^\xi_y|(\mathbb{R}\setminus J_{\hat{\underline{u}}^\xi_y}) + \sum_{t_k}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} |\hat{\underline{u}}^\xi_y(t)|\,dt.\end{aligned} \tag{2.29}$$

The first term in the left hand side of (2.29) is a measurable function of y. In fact thanks to theorem 1.4.23, $(\hat{\underline{u}}^\xi_y)^+(t_k)$ is the trace on Γ_i^+ of $u\cdot\xi$ hence $\mathcal{H}^{n-1}$-measurable, and θ^{ξ^+} is $\mathcal{H}^{n-1}$-measurable as well because trace of a measurable function. Then approximating $|\underline{u}^+\cdot\xi|\theta^{\xi^+}$ by simple functions $(s_m)_{m=0}^\infty$ and applying the Coarea formula with the projection map π^ξ on Γ_i^+, we have in particular that the maps:

$$y \mapsto \sum_{t_k\in(\Gamma_i^+)^\xi_y} (s_m)^\xi_y(t_k),$$

are $\mathcal{H}^{n-1}$-measurable for every $m\in\mathbb{N}$, hence we deduce directly that the term in the left hand-side of (2.29) is $\mathcal{H}^{n-1}$-measurable.
The term $|D\hat{\underline{u}}^\xi_y|(\mathbb{R}\setminus J_{\hat{\underline{u}}^\xi_y})$ is a measurable function of y just by definition of GBD, while the last term in the right hand-side of (2.29) is a measurable function of y once we show that the set:

$$\Lambda_i^{\xi^+} := \big\{(y,t)\in\pi^\xi(\Gamma_i^+)\times\mathbb{R} \mid t_k<t<t_k+\theta^{\xi^+}(y+t_k\xi),\, t_k\in(\Gamma_i^+)^\xi_y\big\},$$

is $\mathcal{L}^n$-measurable. To show this, we notice that since $\Gamma_i^+\subset\{x\in\Gamma \mid \nu(x)\cdot\xi\neq 0\}$, then using the characterisation of rectifiable set (as explained in proposition 2.1.4), $\mathcal{H}^{n-1}$-almost all of Γ_i^+ can be covered by countably many submanifolds of class C^1 say $(\Sigma_j)_{j=0}^\infty$, which are graphs of C^1 functions $(f_j)_{j=0}^\infty$ defined on some open subset of $\xi^\perp$. Clearly if we call $\Lambda_{i,j}^{\xi^+}$ the set of points $(y,t)\in\xi^\perp\times\mathbb{R}$ such that :

$$y\in\pi^\xi(\Sigma_j\cap\Gamma_i^+) \text{ and } f_j(y)<t<f_j(y)+\theta^{\xi^+}(y+f_j(y)\xi),$$

then $\Lambda_{i,j}^{\xi^+}$ is $\mathcal{L}^n$-measurable, simply because both maps appearing in the left hand side and in the right hand side of the previous inequality are restriction of $\mathcal{H}^{n-1}$-measurable functions on a $\mathcal{H}^{n-1}$-measurable set. Finally we notice that:

$$\mathcal{L}^n(\Lambda_i^{\xi^+}\Delta\bigcup_{j=0}^\infty\Lambda_{i,j}^{\xi^+})=0,$$

and we are done.

So we can consider the integral on $\pi^\xi(\Gamma_i^+)$ on both sides of (2.29). By Theorem 1.4.23 $J_{\hat{u}^\xi_y}=(J^\xi_u)^\xi_y$ for a.e. y. Since $(J^\xi_u)^\xi_y\cap\Gamma_i=(J_u)^\xi_y\cap\Gamma_i$, after integration we have:

$$\begin{aligned}\int_{\pi^\xi(\Gamma_i^+)}\sum_{t_k}|(\hat{\underline{u}}^\xi_y)^+(t_k)|\theta^{\xi^+}(y+t_k\xi)\,dy &\leq \int_{\pi^\xi(\Gamma_i^+)}|D\hat{\underline{u}}^\xi_y|(\mathbb{R}\setminus(J_{\underline{u}})^\xi_y)\,dy\\ &+\int_{\pi^\xi(\Gamma_i^+)}\left(\sum_{t_k}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)}|\underline{u}(y+t\xi)|\,dt\right)dy.\end{aligned} \tag{2.30}$$

Analogously we have the same inequality on the set where $\{\nu \cdot \xi < 0\}$:

$$\begin{aligned}\int_{\pi^\xi(\Gamma_i^-)} \sum_{t_k} |(\underline{\hat{u}}_y^\xi)^-(t_k)|\theta^{\xi^+}(y+t_k\xi)\ dy \le \int_{\pi^\xi(\Gamma_i^-)} &|D\underline{\hat{u}}_y^\xi|(\mathbb{R}\setminus(J_{\underline{u}})_y^\xi)\ dy \\ &+ \int_{\mathbb{R}^n} |\underline{u}(x)|\ dx.\end{aligned} \tag{2.31}$$

Summing the two inequalities (2.30) and (2.31), by the relations between the trace of the function and the trace of its slicing (1.10) and (1.11), we have:

$$\begin{aligned}\int_{\pi^\xi(\Gamma_i)} \sum_{t_k\in(\Gamma_i)_y^\xi}^{\mathcal{H}^0((\Gamma_i)_y^\xi)} |\underline{u}^+(y+t_k\xi)\cdot\xi|\theta^{\xi^+}(y+t_k\xi)\ dy \le 2\int_{\pi^\xi(\Gamma_i)} &|D\underline{\hat{u}}_y^\xi|(\mathbb{R}\setminus(J_{\underline{u}})_y^\xi)\ dy \\ &+ 2\int_{\mathbb{R}^n}|\underline{u}(x)|\ dx.\end{aligned} \tag{2.32}$$

Finally Coarea formula on the rectifiable set Γ_i applied to the projection $\xi^\perp$ with the fact that $|\nu(x)\cdot\xi| > 1/\sqrt{n}-\delta$, allows us to write:

$$\begin{aligned}\frac{1-\sqrt{n}\delta}{\sqrt{n}}\int_{\Gamma_i} |\underline{u}^+(x)\cdot\xi|\theta^{\xi^+}(x)\ d\mathcal{H}^{n-1}(x) &\le \int_{\Gamma_i} |\underline{u}^+(x)\cdot\xi|\ |\nu(x)\cdot\xi|\theta^{\xi^+}(x)\ d\mathcal{H}^{n-1}(x) \\ &= \int_{\pi^\xi(\Gamma_i)}\left(\sum_{t_k\in(\Gamma_i)_y^\xi}^{\mathcal{H}^0((\Gamma_i)_y^\xi)} |\underline{\hat{u}}_y^\xi(t_k)|\theta^{\xi^+}(y,t_k)\right) dy \\ &\le 2\int_{\pi^\xi(\Gamma_i)} |D\underline{\hat{u}}_y^\xi|(\mathbb{R}\setminus(J_{\underline{u}})_y^\xi)dy + 2\int_{\mathbb{R}^n}|u(x)|dx \\ &\le 2\hat{\mu}_{\underline{u}}\big((\pi^\xi(\Gamma_i)\times\mathbb{R})\setminus J_{\underline{u}}\big) + 2\int_{\mathbb{R}^n}|\underline{u}(x)|dx.\end{aligned} \tag{2.33}$$

Repeating the same argument for every $\xi_j \in \Xi_i$ we may write:

$$\begin{aligned}\int_{\Gamma_i}\sum_{\xi_j\in\Xi_i} |\underline{u}^+(x)\cdot\xi_j|\theta^{\xi_j^+}(x)\ d\mathcal{H}^{n-1}(x) &\le \frac{2\sqrt{n}}{1-\sqrt{n}\delta}\sum_{\xi_j\in\Xi_i}\hat{\mu}_{\underline{u}}\big((\pi^{\xi_j}(\Gamma_i)\times\mathbb{R})\setminus J_{\underline{u}}\big) \\ &+ \frac{2n^{3/2}}{1-\sqrt{n}\delta}\int_{\mathbb{R}^n}|\underline{u}(x)|\ dx \\ &\le \frac{2n^{3/2}}{1-\sqrt{n}\delta}\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\ dx\right).\end{aligned} \tag{2.34}$$

Now define $\Theta^+\colon \Gamma_i \to \mathbb{R}^+$ as:

$$\Theta^+(x) := \min\{\theta^{\xi_j^+}(x) \mid \xi_j \in \Xi_i\} \text{ for } x \in \Gamma_i. \tag{2.35}$$

By construction for each $j = 1,\dots,n$ the functions $\theta^{\xi_j^+}$ are strictly greater than zero $\mathcal{H}^{n-1}$-a.e. on Γ_i, hence $\Theta^+(x) > 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_i$ and this gives (b). So by inequality (2.34) and the definition of Θ^+ we can write

$$\int_{\Gamma_i}|\underline{u}^+(x)|\Theta^+(x)\ d\mathcal{H}^{n-1}(x) \le \frac{2n^{3/2}}{1-\sqrt{n}\delta}\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\ dx\right). \tag{2.36}$$

Now summing the last inequality (2.36) for every i, together with the choice $\delta = 1/(2\sqrt{n})$, we get:

$$\int_{\Gamma}|\underline{u}^+(x)|\Theta^+(x)\ d\mathcal{H}^{n-1}(x) \le 4n^{3/2}N(1/2\sqrt{n})\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\ dx\right), \tag{2.37}$$

which is (2.25) for $\underline{u}^+$. Defining $\Theta^-\colon \Gamma_i \to \mathbb{R}^+$ as

$$\Theta^-(x) := \min\{\theta^{\xi_j^-}(x) \mid \xi_j \in \Xi_i\} \text{ for } x \in \Gamma_i, \tag{2.38}$$

using the same argument we can prove (a) for Θ^- and (2.24) for $\underline{u}^-$, and we conclude. □

The following is analogous of Theorem 2.1.13 in the case $GSBD^p_p$:

Theorem 2.1.14. *(Trace inequality in $GSBD^p_p(\Omega)$). Let Ω and Γ be as in Theorem 2.1.13. Then there exist two $\mathcal{H}^{n-1}$-measurable functions $\Theta^\pm : \Gamma \cup \partial^*\Omega \to \mathbb{R}^+$ depending only on the geometry of Γ, its orientation ν, and on Ω, such that denoting with $u^\pm$ the traces of u according to Definition 2.1.11 , we have*

(a) $\mathcal{H}^{n-1}(\{\Theta^\pm = 0\}) = 0$ *and* $\Theta^\pm \in L^\infty(\Gamma \cup \partial^*\Omega, \mathcal{H}^{n-1})$ *(in particular* $\|\Theta^\pm\|_\infty \le 1$*);*

(b) For every $u \in GSBD^p_p(\Omega;\Gamma)$ $(p \ge 1)$ *we have:*

$$\int_{\Gamma\cup\partial^*\Omega} |u^\pm|^p \Theta^\pm \, d\mathcal{H}^{n-1} \le C(n,p)\bigg(\int_\Omega |\mathcal{E}u|^p dx + \int_\Omega |u|^p \, dx\bigg), \tag{2.39}$$

where $C(n,p)$ is a constant depending only on n and p;

(c) Let $p^* = np/(n-p)$ $(1 \le p < n)$ be the usual critical Sobolev exponent, then we have:

$$\int_{\Gamma\cup\partial^*\Omega} |u^\pm|^{\frac{p(n-1)}{n-p}} \Theta^\pm \, d\mathcal{H}^{n-1} \le C'(n,p)\bigg(\int_\Omega |\mathcal{E}u|^p \, dx + \int_\Omega |u|^{p^*} \, dx\bigg), \tag{2.40}$$

for every $u \in GSBD^p_p(\Omega;\Gamma) \cap L^{p^*}(\Omega)$.

Proof. Let $\underline{u} \in GBD(\mathbb{R}^n; \Gamma\cup\partial^*\Omega)$ be the function extended to 0 outside of Ω as in Proposition 2.1.9. In order to simplify the notation, we write Γ to denote $\Gamma \cup \partial^*\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with $\nu_{\partial^*\Omega}$ on $\partial^*\Omega$. By our definition of $u^\pm$ (see Definition 2.1.11) and by proposition 2.1.9, (2.39) and (2.40), can be rewritten as:

$$\int_\Gamma |\underline{u}^\pm|^p \Theta^\pm \, d\mathcal{H}^{n-1} \le C(n,p)\bigg(\int_{\mathbb{R}^n} |\mathcal{E}\underline{u}|^p dx + \int_{\mathbb{R}^n} |\underline{u}|^p \, dx\bigg), \tag{2.41}$$

and

$$\int_\Gamma |\underline{u}^\pm|^{\frac{p(n-1)}{n-p}} \Theta^\pm \, d\mathcal{H}^{n-1} \le C'(n,p)\bigg(\int_{\mathbb{R}^n} |\mathcal{E}\underline{u}|^p \, dx + \int_{\mathbb{R}^n} |\underline{u}|^{p^*} \, dx\bigg). \tag{2.42}$$

We argue similarly to the proof of Theorem 2.1.13: consider $(\Gamma_i)_{i=1}^N$ the partition of Γ given in Theorem 2.1.13, and let Ξ_i be the orthonormal basis of $\mathbb{R}^n$ associated to Γ_i. Fix i and $\xi \in \Xi_i$.

From now on we will work on the points $y \in \pi^\xi(\Gamma_i)$ such that $\hat{u}^\xi_y \in SBV_{loc}(\mathbb{R})$ and $\mathcal{H}^0(\Gamma^\xi_y) < \infty$; from the Definition 1.4.20 of $GSBD$ and Remark 2.1.1 we already know that $\mathcal{H}^{n-1}$-almost all of y have these properties.

We call $(t_k)_{k=1}^{\mathcal{H}^0(\Gamma^\xi_y)}$ the points of the slicing Γ^ξ_y ordered such that $t_k < t_{k+1}$ for any k. Let $\theta^\xi\colon E \to \mathbb{R}^+$ be the one sectional distance introduced in Definition 2.1.2. For $x \in \Gamma$ let $\theta^{\xi^\pm}(x)$ to be the trace of θ^ξ according to ν.
Now we work on Γ_i. By Theorem 1.4.23

$$\underline{u}^+(y+t\xi)\cdot\xi = (\hat{\underline{u}}^\xi_y)^+(t) \text{ if } t \in (\Gamma_i)^\xi_y \text{ and } \nu(x)\cdot\xi > 0, \tag{2.43}$$

and

$$\underline{u}^{+}(y+t\xi)\cdot\xi=(\hat{\underline{u}}^{\xi}_{y})^{-}(t) \text{ if } t\in(\Gamma_i)^{\xi}_{y} \text{ and } \nu(x)\cdot\xi<0.$$

Since ξ has been fixed, in order to simplify the notation, we omit the dependence on ξ and write $\Gamma_i^{+}:=\Gamma_i\cap\{\nu\cdot\xi>0\}$ and $\Gamma_i^{-}:=\Gamma_i\cap\{\nu\cdot\xi<0\}$. Let's focus for example on the set Γ_i^{+}:

$$\hat{\underline{u}}^{\xi}_{y}(t)-(\hat{\underline{u}}^{\xi}_{y})^{+}(t_k)=\int_{t_k}^{t}\nabla\hat{\underline{u}}^{\xi}_{y}(r)dr, \text{ for } t_i\in(\Gamma_i^{+})^{\xi}_{y} \text{ and } t_k<t<t_{k+1},$$

passing to the modulus and raising to the power p:

$$|(\hat{\underline{u}}^{\xi}_{y})^{+}(t_k)|^p\le 2^{p-1}(t_{k+1}-t_k)^{p-1}\int_{t_k}^{t_{k+1}}|\nabla\hat{\underline{u}}^{\xi}_{y}(r)|^p\,dr+2^{p-1}|\hat{\underline{u}}^{\xi}_{y}(t)|^p, \quad \text{for } t_k\in(\Gamma_i^{+})^{\xi}_{y} \text{ and } t_k<t<t_{k+1}.$$

The same holds true for $|(\hat{\underline{u}}^{\xi}_{y})^{-}|$ on the set Γ_i^{-}. Notice that by Theorem 1.4.25 $\nabla\hat{u}^{\xi}_{y}(t)=\mathcal{E}u(y+t\xi)\xi\cdot\xi$ for $\mathcal{H}^{n-1}$-a.e. $y\in\xi^{\perp}$ and $\mathcal{H}^{1}$-a.e. $t\in\Omega^{\xi}_{y}$. So exactly as in Theorem 2.1.13, at fixed y we can integrate on $t\in(t_k,t_{k+1})$ so that we don't touch points of the slicing $(\Gamma_i)^{\xi}_{y}$; then we integrate with respect to $y\in\pi^{\xi}(\Gamma_i)$ and we use Coarea formula with the fact that $|\nu\cdot\xi|>1/\sqrt{n}-\delta$:

$$\begin{aligned}\frac{1-\sqrt{n}\delta}{\sqrt{n}}\int_{\Gamma_i}|\underline{u}^{+}(x)|^p\theta^{\xi^{+}}(x)\,d\mathcal{H}^{n-1}&\le\sum_{t_k\in(\Gamma_i)^{\xi}_{y}}^{\mathcal{H}^0((\Gamma_i)^{\xi}_{y})}\int_{\pi^{\xi}(\Gamma_i)}|\underline{u}^{+}(y+t_k\xi)\cdot\xi|^p\theta^{\xi^{+}}(y,t_k)\,dy\\&\le 2^p\int_{\pi^{\xi}(\Gamma_i)}\Big(\int_{\mathbb{R}}|\mathcal{E}\underline{u}(y+t\xi)\xi\cdot\xi|^p\,dt\Big)\,dy\\&\quad+2^p\int_{\mathbb{R}^n}|\underline{u}(x)|^p\,dx.\end{aligned}\tag{2.44}$$

Summing (2.44) for every $\xi_j\in\Xi_i$ we get:

$$\begin{aligned}\int_{\Gamma_i}\sum_{\xi_j\in\Xi_i}|\underline{u}^{+}\cdot\xi_j|^p\theta^{\xi_j^{+}}\,d\mathcal{H}^{n-1}&\le C(n,p)\sum_{\xi_j\in\Xi_i}\int_{\pi^{\xi_j}(\Gamma_i)}\Big(\int_{\mathbb{R}}|\mathcal{E}\underline{u}(y+t\xi)\xi_j\cdot\xi_j|^p\,dt\Big)\,dy\\&\quad+C(n,p)\int_{\mathbb{R}^n}|\underline{u}(x)|^p\,dx\\&\le C(n,p)\Big(\int_{\mathbb{R}^n}|\mathcal{E}\underline{u}(x)|^p\,dx+\int_{\mathbb{R}^n}|\underline{u}(x)|^p\,dx\Big).\end{aligned}\tag{2.45}$$

Now define for every $i=1,\dots,N$ (where N is the dimensional constant introduced in Remark 2.1.7), exactly as in (2.35):

$$\Theta^{+}(x):=\min\{\theta^{\xi_j^{+}}(x)\mid\xi_j\in\Xi_i\} \text{ for } x\in\Gamma_i,\tag{2.46}$$

so that (2.41) holds for $\underline{u}^{+}$. Now (a) follows exactly as in Theorem 2.1.13. Analogously by defining

$$\Theta^{-}(x):=\min\{\theta^{\xi_j^{-}}(x)\mid\xi_j\in\Xi_i\} \text{ for } x\in\Gamma_i,\tag{2.47}$$

we can prove (a) for Θ^{-} and (2.41) for the trace from below $\underline{u}^{-}$.

To prove (2.42) fix i and $\xi\in\Xi_i$. Then we notice that for $\mathcal{H}^{n-1}$-a.e. $y\in\xi^{\perp}$ we have all the properties mentioned in the first lines of this proof and moreover that $\hat{\underline{u}}^{\xi}_{y}\in L^{p*}(\mathbb{R})$, $\nabla\hat{\underline{u}}^{\xi}_{y}\in L^p(\mathbb{R})$. Then we elevate the one dimensional sections $\hat{\underline{u}}^{\xi}_{y}$ to the power $p(n-1)/(n-p)$

and we notice that for $\mathcal{H}^{n-1}$-a.e. y we have $\underline{u}^\xi_y \in W^{1,p}\big((t_k,t_{k+1})\big)$ so by means of the *chain rule formula* we get

$$
\begin{aligned}
(t_{k+1}-t_k)|(\hat{\underline{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} \leq (t_{k+1}-t_k)\frac{p(n-1)}{n-p}&\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \hat{\underline{u}}^\xi_y(r)|\, dr \\
&+\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\
&\text{for } t_k \in \big(\Gamma^+_i\big)^\xi_y, \text{ and } t_{k+1}-t_k \leq 1,
\end{aligned} \tag{2.48}
$$

and

$$
\begin{aligned}
|(\hat{\underline{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} \leq \frac{p(n-1)}{n-p}&\int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \hat{\underline{u}}^\xi_y(r)|\, dr \\
&+\int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\
&\text{for } t_k \in \big(\Gamma^+_i\big)^\xi_y, \text{ and } t_{k+1}-t_k > 1,
\end{aligned} \tag{2.49}
$$

Hölder's inequality with exponents $p/(p-1)$ and p, and then Young's inequality with the same exponents yield to:

$$
\begin{aligned}
\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \hat{\underline{u}}^\xi_y(r)|\, dr &\leq \left(\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{p^*}\, dr\right)^{\frac{p-1}{p}}\left(\int_{t_k}^{t_{k+1}} |\nabla \hat{\underline{u}}^\xi_y(r)|^{p}\, dr\right)^{\frac{1}{p}} \\
&\leq \frac{p-1}{p}\int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(r)|^{p^*}\, dr + \frac{1}{p}\int_{t_k}^{t_{k+1}} |\nabla \hat{\underline{u}}^\xi_y(r)|^{p}\, dr,
\end{aligned} \tag{2.50}
$$

Now we first integrate on the interval (t_k,t_{k+1}) both inequalities (2.48) and (2.49) using also (2.50), and then we integrate with respect to $y \in \xi^\perp$. Finally we can conclude exactly as before, getting (2.42) for $\underline{u}^+$. The same argument works for $\underline{u}^-$ and we conclude. □

2.1.4 The cone condition

Definition 2.1.15. *Given $\Gamma \subset \mathbb{R}^n$ a countably $(\mathcal{H}^{n-1}, n-1)$- rectifiable set oriented by ν, we say that Γ satisfies the cone condition, if there exist $0 < r \leq 1$, $0 < L < 1$, and two $\mathcal{H}^{n-1}$-measurable maps $\eta^\pm \colon \Gamma \to \mathbb{S}^{n-1}$, such that for every $x \in \Gamma$ we have*

$$\big\{x + C^+(\eta^+(x), L)\big\} \cap B_r(x) \cap \Gamma = \emptyset, \tag{2.51}$$

and

$$\big\{x + C^-(\eta^-(x), L)\big\} \cap B_r(x) \cap \Gamma = \emptyset. \tag{2.52}$$

Remark 2.1.16. *For example if Γ is the boundary of some Lipschitz-regular domain $\Omega \subset \mathbb{R}^n$ then it satisfies the cone condition.* ■

Proposition 2.1.17. *(Trace inequality with no weights). Let Ω and Γ be as in Theorem 2.1.13. Suppose that $\Gamma \cup \partial^*\Omega$ satisfies the cone condition with parameters r and L (see Definition 2.1.15), then we have:*

(a) If $u \in GBD(\Omega;\Gamma) \cap L^1(\Omega)$ then $u^\pm \in L^1(\Gamma \cup \partial^*\Omega, \mathcal{H}^{n-1})$, and moreover there exists a constant $C(n,L,r) > 0$ such that:

$$\int_{\Gamma \cup \partial^*\Omega} |u^\pm|\, d\mathcal{H}^{n-1} \leq C(n,L,r)\Big(\hat{\mu}_u(\Omega \setminus J_u) + \int_\Omega |u|\, dx\Big) \tag{2.53}$$

(b) If $u \in GSBD^p_p(\Omega;\Gamma)$ $(p \geq 1)$ then $u^\pm \in L^p(\Gamma \cup \partial^*\Omega, \mathcal{H}^{n-1})$, and moreover there exists a constant $C(n,L,r,p) > 0$ such that:

$$\int_{\Gamma\cup\partial^*\Omega} |u^\pm|^p \, d\mathcal{H}^{n-1} \leq C(n,L,r,p)\biggl(\int_\Omega |\mathcal{E}u|^p \, dx + \int_\Omega |u|^p \, dx\biggr) \tag{2.54}$$

Proof. We prove (a). The proof of (b) is similar.

Let $\underline{u} \in GBD(\mathbb{R}^n; \Gamma \cup \partial^*\Omega)$ be the function extended to 0 outside of Ω as in Proposition 2.1.9. In order to simplify the notation, we write Γ to denote $\Gamma \cup \partial^*\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with ν_Ω on $\partial^*\Omega$. By our definition of $u^\pm$ (see Definition 2.1.11) and by proposition 2.1.9, (2.53) and (2.54), can be rewritten as:

$$\int_\Gamma |\underline{u}^\pm| \, d\mathcal{H}^{n-1} \leq C(n,L,r)\biggl(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}| \, dx\biggr), \tag{2.55}$$

and

$$\int_\Gamma |\underline{u}^\pm|^p \, d\mathcal{H}^{n-1} \leq C'(n,L,r,p)\biggl(\int_{\mathbb{R}^n} |\mathcal{E}\underline{u}|^p \, dx + \int_{\mathbb{R}^n} |\underline{u}|^p \, dx\biggr). \tag{2.56}$$

We prove (2.55), the proof of (2.56) is similar. Let us focus on the trace from above $\underline{u}^+$: first notice that if $x \in \Gamma$ admits an approximate tangent space [1], say $\mathrm{Tan}(x,\Gamma)$, then it must lies on the set of points $y \in \mathbb{R}^n \setminus C^+(\eta^+(x), L)$: this is simply because by definition of approximate tangent space [2]:

$$\mathcal{H}^{n-1} \llcorner \left(\frac{\Gamma - x}{\lambda}\right) \rightharpoonup \mathcal{H}^{n-1} \llcorner \mathrm{Tan}(x,\Gamma) \text{ as } \lambda \to 0^+,$$

weakly in $\mathcal{M}_b(\mathbb{R}^n)$; by our hypothesis for every $\lambda > 0$, $\frac{\Gamma - x}{\lambda} \cap C^+(\eta^+(x), L) \cap B_{r/\lambda}(0) = \emptyset$, and this means that the limit measure $\mathcal{H}^{n-1} \llcorner \mathrm{Tan}(x,\Gamma)$ has support disjoint from the open set $C^+(\eta^+(x), L)$. Thus, since

$$\eta^+(x) - (\eta^+(x) \cdot \nu(x))\nu(x) \in \mathrm{Tan}(x,\Gamma),$$

we have the uniform bound on the scalar product

$$\nu(x) \cdot \eta^+(x) > \sqrt{1-L}, \text{ for every } x \in \Gamma. \tag{2.57}$$

Now consider $\epsilon > 0$ small enough such that $(7+L)/8 - \epsilon > L$ (which is possible since $L < 1$) and $\epsilon < \frac{\sqrt{1-L}}{4}$. By compactness we can find a finite covering of $\mathbb{S}^{n-1}$, made of closed balls of radius $\epsilon/2$, say $(B_i)_{i=1}^{N(\epsilon)}$. Define for each $i = 1, \dots, N(\epsilon)$, $\Gamma_i := {\eta^+}^{-1}(B_i)$, then $\Gamma \subset \bigcup_i \Gamma_i$. For every $\Gamma_i \neq \emptyset$ $(i = 1, \dots, N(\epsilon))$ choose $x_i \in \Gamma_i$ and define $\eta_i^+ := \eta^+(x_i)$. We claim that:

$$\bigl\{x + C^+(\eta_i^+, (7+L)/8)\bigr\} \cap B_r(x) \cap \Gamma = \emptyset, \text{ for every } x \in \Gamma_i. \tag{2.58}$$

In order to show (2.58) it is enough to notice that if $y \in x + C^+(\eta_i^+,\ (7+L)/8)$ then by using $(7+L)/8 - \epsilon > L$, we have:

$$\begin{aligned} |\eta^+(x) \cdot (y-x)| = |\eta_i^+ \cdot (y-x) + (\eta^+(x) - \eta_i^+) \cdot (y-x)| &\geq |\eta_i^+ \cdot (y-x)| - \epsilon|y-x| \\ &\geq \biggl((7+L)/8 - \epsilon\biggr)|y-x| \\ &> L|y-x|, \end{aligned} \tag{2.59}$$

[1] By [53, Theorem 5.4.5] $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ admits an approximate tangent space.
[2] See [53, Definition 5.4.4] for the definition of approximate tangent space.

which implies $y \in x + C^+(\eta^+(x), L) \cap B_r(x)$ and proves the claim.
Now we work on Γ_i. Consider a basis of $\mathbb{R}^n$, say $\Xi_i := \{\xi_1, \dots, \xi_n\}$, such that:

$$\xi_j \in C^+\big(\eta_i^+, (7+L)/8\big), \text{ for every } j = 1, \dots, n. \tag{2.60}$$

Notice that by the fact $\epsilon < \frac{\sqrt{1-L}}{4}$ we have:

$$\begin{aligned}
\nu(x) \cdot \xi_j &= \nu(x) \cdot (\xi_j - \eta_i^+) + \nu(x) \cdot (\eta_i^+ - \eta^+(x)) + \nu(x) \cdot \eta^+(x) \\
&\geq -\sqrt{2(1 - \xi_j \cdot \eta_i^+)} - \epsilon + \sqrt{1-L} \\
&\geq -\frac{\sqrt{1-L}}{2} - \epsilon + \sqrt{1-L} \\
&\geq \frac{\sqrt{1-L}}{4}
\end{aligned}$$

Now proceeding exactly as in the proof of Theorem 2.1.13 we have for every $\xi_j \in \Xi_i$

$$\begin{aligned}
\int_{\pi^{\xi_j}(\Gamma_i^\epsilon \cap \{\nu \cdot \xi_j > 0\})} \sum_{t_k} |(\hat{\underline{u}}_y^{\xi_j})^+(t_k)| \theta^{\xi_j^+}(y + t_k \xi_j) \, dy \leq \int_{\pi^{\xi_j}(\Gamma_i^\epsilon \cap \{\nu \cdot \xi_j > 0\})} |D\hat{\underline{u}}_y^{\xi_j}|(\mathbb{R} \setminus (J_{\underline{u}})_y^{\xi_j}) \, dy \\
+ \int_{\Gamma_i \times \mathbb{R}} |\underline{u}(x)| \, dx.
\end{aligned} \tag{2.61}$$

Using (2.58) we have that $\theta^{\xi_j^+}(x) \geq r$ for every $\xi_j \in \Xi_i$ and every $x \in \Gamma_i$. In fact, since $\xi_j \in C^+\big(\eta_i^+, (7+L)/8\big)$, if for some $x = y + t\xi_j \in \Gamma_i$ it holds $\theta^{\xi_j^+}(x) < r$, by the definition of $\theta^{\xi_j}(\cdot)$ it means that there exists a point $\tilde{x} = y + \tilde{t}\xi_j \in \Gamma$ such that $0 < \tilde{t} - t < r$, hence

$$\tilde{x} \in \big\{x + C^+(\eta_i^+, (7+L)/8)\big\} \cap B_r(x) \cap \Gamma,$$

which is a contradiction with (2.58).

So by means of Coarea Formula applied to π^{ξ_j} on the set Γ_i, we can write:

$$\begin{aligned}
&r\frac{\sqrt{1-L}}{4} \int_{\Gamma_i} |\underline{u}^+(x) \cdot \xi_j| \, d\mathcal{H}^{n-1}(x) \leq \\
&\int_{\pi^{\xi_j}(\Gamma_i \cap \{\nu \cdot \xi_j > 0\})} |D\hat{\underline{u}}_y^{\xi_j}|(\mathbb{R} \setminus (J_{\underline{u}})_y^{\xi_j}) \, dy + \int_{\Gamma_i \times \mathbb{R}} |\underline{u}(x)| \, dx.
\end{aligned} \tag{2.62}$$

Summing the inequalities (2.62) for every $\xi_j \in \Xi_i$ we get

$$\int_{\Gamma_i} \sum_{\xi_j \in \Xi_i} |\underline{u}^+(x) \cdot \xi_j| \, d\mathcal{H}^{n-1}(x) \leq C'(n, L, r)\bigg(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \setminus J_{\underline{u}}) + \int_{\Gamma_i \times \mathbb{R}} |\underline{u}(x)| dx\bigg). \tag{2.63}$$

Now call $A_i \in \mathbb{M}^{n \times n}(\mathbb{R}^n)$, the matrix whose j-th columns is composed by the vector $\xi_j \in \Xi_i$. Then we have:

$$\sum_{\xi_j \in \Xi_i} |\underline{u}^+(x) \cdot \xi_j| \geq \bigg(\sum_{\xi_j \in \Xi_i} |\underline{u}^+(x) \cdot \xi_j|^2\bigg)^{\frac{1}{2}} \geq \frac{|\underline{u}^+(x)|}{\|A_i^{-T}\|_{\mathbb{M}^{n \times n}}}. \tag{2.64}$$

So finally we can write:

$$\int_{\Gamma_i} |\underline{u}^+(x)| \, d\mathcal{H}^{n-1}(x) \leq C''(n, L, r)\bigg(\hat{\mu}_{\underline{u}}(\mathbb{R}^n \setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}(x)| \, dx\bigg), \tag{2.65}$$

where $C''(n,L,r)$ is a constant which depends only on n,L,r. Analogously we have the same inequality for $|\underline{u}^-|$, so that by summing on $i=1,\dots,N(n)$ we obtain:

$$\int_\Gamma |\underline{u}^\pm(x)|\, d\mathcal{H}^{n-1}(x) \le C(n,L,r)\Big(\hat\mu_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n} |\underline{u}(x)|dx\Big), \tag{2.66}$$

which concludes the proof. □

Remark 2.1.18. *A particular case of Theorem 2.1.14 and of point (b) of Proposition 2.1.17 is when $u \in GSBV^p_p(\Omega;\Gamma)^n$ $(p\ge 1)$. By definition every $u\in GSBV^p_p(\Omega)^n$ is a vector field in $L^p(\Omega,\mathbb{R}^n)$ whose approximate gradient ∇u belongs to $\in L^p(\Omega,\mathbb{M}^{n\times n})$. Therefore $GSBV^p_p(\Omega;\Gamma)^n \subset GSBD^p_p(\Omega;\Gamma)$. In particular Theorem 2.1.14 and point (b) of Corollary 2.1.17 apply to $GSBV^p_p(\Omega;\Gamma)^n$ with $\mathcal{E}u$ replaced by ∇u.* ■

2.1.5 Trace inequalities with weighted volume measure

An alternative way to obtain a trace estimate without weight on $\partial^*\Omega\cup\Gamma$ is to consider a suitable weight Ψ defined on Ω as explained in the next theorem:

Theorem 2.1.19. *Let Ω and Γ be as in Theorem 2.1.13. Then there exists an $\mathcal{L}^n$-measurable function $\Psi:\Omega\to\mathbb{R}^+$ depending only on the geometry of Γ, its orientation ν, and on Ω, such that denoting with $u^\pm$ the traces of u according to Definition 2.1.11 ,we have:*

(a) The function $\Psi\in L^1_{loc}(\Omega)$. In particular:

$$\int_B \Psi\, dx \le C(n)\Big(\mathcal{H}^{n-1}(\Gamma\cap B) + 2\mathcal{H}^{n-1}(\partial B) + \mathcal{L}^n(B)\Big), \tag{2.67}$$

for every ball $B\subset\Omega$.

(b) The following inclusions hold true:

(i) $GBD(\Omega;\Gamma)\cap L^1(\Omega,\Psi\mathcal{L}^n)\subset BD(\Omega;\Gamma)$ and

$$\int_{\Gamma\cup\partial^*\Omega} |u^\pm|\, d\mathcal{H}^{n-1} \le C(n)\Big(\hat\mu_u(\Omega\setminus J_u) + \int_\Omega |u|\,\Psi dx\Big); \tag{2.68}$$

(ii) $GSBD^p(\Omega;\Gamma)\cap L^p(\Omega,\Psi\mathcal{L}^n)\subset SBD^p_p(\Omega;\Gamma)$ $(p\ge 1)$ and

$$\int_{\Gamma\cup\partial^*\Omega} |u^\pm|^p\, d\mathcal{H}^{n-1} \le C(n,p)\Big(\int_\Omega |\mathcal{E}u|^p\, dx + \int_\Omega |u|^p\,\Psi dx\Big). \tag{2.69}$$

(c) Given $p<n$ let $p^*=np/(n-p)$ be the usual critical Sobolev exponent, and consider $u\in GSBD^p(\Omega;\Gamma)\cap L^{p^*}(\Omega)\cap L^{\frac{p(n-1)}{n-p}}(\Omega,\Psi\mathcal{L}^n)$. Then:

$$\int_{\Gamma\cup\partial^*\Omega} |u^\pm|^{\frac{p(n-1)}{n-p}}\, d\mathcal{H}^{n-1} \le C(n,p)\Big(\int_\Omega |\mathcal{E}u|^p\, dx + \int_\Omega |u|^{\frac{p(n-1)}{n-p}}\,\Psi dx + \int_\Omega |u|^{p^\star}\, dx\Big). \tag{2.70}$$

(d) If $\Gamma\cup\partial^*\Omega$ satisfies the cone condition (see Definition 2.1.15) , then Ψ can be chosen equal to 1.

(e) If Γ is such that:

$$\int_\Omega \Psi^\gamma dx < \infty, \text{ for some } \gamma > 1, \tag{2.71}$$

then we have $GBD(\Omega;\Gamma) \cap L^{\frac{\gamma}{\gamma-1}}(\Omega) \subset BD(\Omega;\Gamma)$ and $GSBD^p(\Omega;\Gamma) \cap L^{\frac{\gamma p}{\gamma-1}}(\Omega) \subset SBD^p_p(\Omega;\Gamma)$.

Remark 2.1.20. *If $\Omega \subset \mathbb{R}^n$ is a Lipschitz-regular bounded domain, and $\Gamma = \emptyset$, then clearly $\partial^*\Omega (= \partial\Omega)$ satisfies the cone condition. Thanks to point (d) of the previous theorem,* $\operatorname{ess\,sup}_{x\in\Omega} \Psi < \infty$, *therefore* (2.70) *becomes:*

$$\int_{\partial\Omega} |Tr(u)|^{\frac{p(n-1)}{n-p}} \, d\mathcal{H}^{n-1} \le C(n,p,\Gamma,\Omega)\Big(\int_\Omega |\mathcal{E}u|^p \, dx + \int_\Omega |u|^{p^*} \, dx\Big). \tag{2.72}$$

Moreover one can prove that on the open set Ω holds true a Sobolev-like inequality of the form:

$$\|u\|_{p^*} \le C(n,p,\Gamma,\Omega)(\|\mathcal{E}u\|_p + \|u\|_p). \tag{2.73}$$

This last inequality, together with (2.72)*, proves the $L^{\frac{p(n-1)}{n-p}}(\partial\Omega, \mathcal{H}^{n-1})$- integrability of the trace of u, which is the usual critical exponent for the trace of Sobolev functions in $W^{1,p}(\Omega)$.* ■

Proof. (Theorem 2.1.19) Let $\underline{u} \in GBD(\mathbb{R}^n; \Gamma \cup \partial^*\Omega)$ be the function extended to 0 outside of Ω as in proposition 2.1.9. In order to simplify the notation, we write Γ to denote $\Gamma \cup \partial^*\Omega$, and ν to denote the orientation that coincides with the given orientation ν on Γ, and with ν_Ω on $\partial^*\Omega$.

By following the proofs of Theorem 2.1.13 and 2.1.14, thanks to our definitions of $u^\pm$ and by Proposition 2.1.9, we can prove the analogous of inequalities (2.68), (2.69), and (2.70) for the function $\underline{u}$.

We first prove (a) and (b): consider Λ the covering of $\mathbb{S}^{n-1}$ as in Remark 2.1.7 and by compactness we consider a subcovering $(C(\Xi_i,\delta))_{i=1}^N$. If we define for any $i = 1,\dots,N$, $\Gamma_i := \nu^{-1}\big(C(\Xi_i,\delta)\big)$ then $(\Gamma_i)_{i=1}^N$ is a finite measurable cover of Γ. Note that by definition of the covering Λ, for any $\xi \in \Xi_i$ we have $|\xi \cdot \nu(x)| > 1/\sqrt{n} - \delta$ for every $x \in \Gamma_i$.

Now we fix i and $\xi \in \Xi_i$. We write the generic point $x \in \mathbb{R}^n$ as $y + t\xi$ where $(y,t) \in \xi^\perp \times \mathbb{R}$, and from now on we will work on the set of points $y \in \pi^\xi(\Gamma_i)$ such that $\hat{\underline{u}}^\xi_y \in BV_{loc}(\mathbb{R})$ and $\mathcal{H}^0(\Gamma^\xi_y) < \infty$; from the Definition 1.4.18 of GBD and Remark 2.1.1 we already know that $\mathcal{H}^{n-1}$ almost all of y have these properties.

We call $(t_k)_{k=1}^{\mathcal{H}^0(\Gamma^\xi_y)}$ the point of the slicing Γ^ξ_y ordered such that $t_k < t_{k+1}$ for any k.

Since ξ has been fixed, in order to simplify the notation, we omit the dependence on ξ and write $\Gamma_i^+ := \Gamma_i \cap \{\nu\cdot\xi > 0\}$ and $\Gamma_i^- := \Gamma_i \cap \{\nu\cdot\xi < 0\}$. Let's focus for example on the set Γ_i^+. Proceeding exactly as in Theorem 2.1.13, we have for $\mathcal{H}^{n-1}$-a.e. $y \in \pi^\xi(\Gamma_i^+)$

$$(t_{k+1} - t_k)|(\hat{\underline{u}}^\xi_y)^+(t_k)| \le (t_{k+1} - t_k)\int_{t_k}^{t_{k+1}} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{t_{k+1}} |\hat{\underline{u}}^\xi_y(t)| \, dt,$$
$$\text{for } t_k \in \big(\Gamma_i^+\big)^\xi_y, \text{ and } t_{k+1} - t_k \le 1,$$

and

$$|(\hat{\underline{u}}^\xi_y)^+(t_k)| \le \int_{t_k}^{1+t_k} d|D\hat{\underline{u}}^\xi_y| + \int_{t_k}^{1+t_k} |\hat{\underline{u}}^\xi_y(t)| \, dt, \tag{2.74}$$
$$\text{for } t_k \in (\Gamma_i^+)^\xi_y, \text{ and } t_{k+1} - t_k > 1.$$

Since θ^ξ coincides with $t_{k+1}-t_k$ or 1 on the set $\{y+t\xi \mid t_k<t<t_{k+1}\}$, we can divide both sides of the previous inequality by θ^ξ and then we sum on $t_k \in \left(\Gamma_i^+\right)_y^\xi$ to get

$$\begin{aligned}\sum_{t_k} |(\hat{\underline{u}}_y^\xi)^+(t_k)| &\leq \sum_{t_k}\left(\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} d|D\hat{\underline{u}}_y^\xi| + \int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} \frac{|\hat{\underline{u}}_y^\xi(t)|}{\theta^\xi(y+t\xi)}dt\right) \\ &\leq |D\hat{\underline{u}}_y^\xi|(\mathbb{R}\setminus J_{\hat{\underline{u}}_y^\xi}) + \sum_{t_k}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} \frac{|\hat{\underline{u}}_y^\xi(t)|}{\theta^\xi(y+t\xi)}dt.\end{aligned} \tag{2.75}$$

The term in the left hand-side, and the last two addends on the right hand-side of (2.75) are measurable functions of y (as explained in the proof of Theorem 2.1.13). By Theorem 1.4.23 $J_{\hat{\underline{u}}_y^\xi} = (J_{\underline{u}}^\xi)_y^\xi$ for a.e. y. Since $(J_{\underline{u}}^\xi)_y^\xi \cap \Gamma_i = (J_{\underline{u}})_y^\xi \cap \Gamma_i$, by integrating over $\pi^\xi(\Gamma_i^+)$ we get:

$$\begin{aligned}\int_{\pi^\xi(\Gamma_i^+)} \sum_{t_k} |(\hat{\underline{u}}_y^\xi)^+(t_k)|\,dy &\leq \int_{\pi^\xi(\Gamma_i^+)} |D\hat{\underline{u}}_y^\xi|(\mathbb{R}\setminus (J_{\underline{u}})_y^\xi)\,dy \\ &+ \int_{\pi^\xi(\Gamma_i^+)}\left(\sum_{t_k\in(\Gamma_i^+)_y^\xi}\int_{t_k}^{t_k+\theta^{\xi^+}(y+t_k\xi)} \frac{|\underline{u}(y+t\xi)|}{\theta^\xi(y+t\xi)}\,dt\right)dy.\end{aligned}$$

Again by arguing as in the proof of Theorem 2.1.13, we find the same inequality on the set Γ_i^-, then by means of the Coarea formula on the rectifiable set Γ_i applied to the projection $\xi^\perp$, and by summing on every directions in Ξ_i, we get:

$$\int_{\Gamma_i} |\underline{u}^+(x)|\,d\mathcal{H}^{n-1}(x) \leq C(n)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}\sum_{\xi_j\in\Xi_i}\frac{|\underline{u}(x)|}{\theta^{\xi_j}}(x)\,dx\right). \tag{2.76}$$

Now define $\Psi^i, \Psi : \mathbb{R}^n \to \mathbb{R}^+$ as:

$$\Psi^i(x) := \sum_{\xi_j\in\Xi_i}\frac{1}{\theta^{\xi_j}(x)} \text{ and } \Psi(x) := \sum_{i=1}^{N(n)}\Psi^i(x). \tag{2.77}$$

To prove (a) it is enough to notice that for each $\xi \in \mathbb{S}^{n-1}$ and for each ball $B \subset \mathbb{R}^n$ we have:

$$\begin{aligned}\int_B \frac{1}{\theta^\xi}\,dx &= \int_{\pi^\xi(B)}\left(\int_{B_y^\xi}\frac{1}{\theta^\xi(y+t\xi)}\,dt\right)dy \\ &= \int_{\pi^\xi(B)}\left(\sum_{t_k\in((\Gamma\cap B)\cup\partial B)_y^\xi}\frac{t_{k+1}-t_k}{\theta^\xi(y+t\xi)}\right)dy \\ &\leq \int_{\pi^\xi(B)}\left(\mathcal{H}^0((\Gamma\cap B)_y^\xi) + 2 + \mathcal{H}^1(B_y^\xi)\right)dy \\ &\leq \mathcal{H}^{n-1}(\Gamma\cap B) + 2\mathcal{H}^{n-1}(\partial B) + \mathcal{L}^n(B).\end{aligned} \tag{2.78}$$

Hence $\Psi \in L^1_{loc}(\Omega)$. By summing on $i = 1,\ldots,N(n)$ inequality (2.76) becomes:

$$\int_\Gamma |\underline{u}^+(x)|\,d\mathcal{H}^{n-1}(x) \leq C'(n)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\,\Psi(x)dx\right). \tag{2.79}$$

Analogously we can prove the same inequality for the trace from below:

$$\int_\Gamma |\underline{u}^-(x)|\,d\mathcal{H}^{n-1}(x) \leq C'(n)\left(\hat{\mu}_{\underline{u}}(\mathbb{R}^n\setminus J_{\underline{u}}) + \int_{\mathbb{R}^n}|\underline{u}(x)|\,\Psi(x)dx\right). \tag{2.80}$$

Thanks to Proposition 2.1.9, (2.79) and (2.80) are exactly (2.68). In particular this means that the jump function $[u](x) = u^+(x) - u^-(x)$ belongs to $L^1(J_u, \mathcal{H}^{n-1})$, and as a consequence that $u \in BD(\Omega)$ (see Remark 1.4.24).

In order to pass to the L^p-norm in (2.69), we can proceed as in the proof of Theorem 2.1.14. Then by arguing as in the previous proof of inequality (2.68), we get also (ii) of point (b).

To prove (c) fix i and $\xi \in \Xi_i$. Notice that for $\mathcal{H}^{n-1}$-a.e. $y \in \pi^\xi(\Gamma_i)$ we have all the properties mentioned in the first lines of this proof and moreover $\underline{\hat{u}}^\xi_y \in L^{p*}(\mathbb{R})$, $\nabla \underline{\hat{u}}^\xi_y \in L^p(\mathbb{R})$. So we elevate the one dimensional sections $\underline{\hat{u}}^\xi_y$ to the power $p(n-1)/(n-p)$ and we notice that for $\mathcal{H}^{n-1}$-a.e. y we have $\underline{\hat{u}}^\xi_y \in W^{1,p}\big((t_k, t_{k+1})\big)$. Thus by means of the *chain rule formula* we get:

$$\begin{aligned}(t_{k+1} - t_k)|(\underline{\hat{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} \leq (t_{k+1} - t_k)\frac{p(n-1)}{n-p}\int_{t_k}^{t_{k+1}} |\underline{\hat{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \underline{\hat{u}}^\xi_y(r)|\, dr \\ + \int_{t_k}^{t_{k+1}} |\underline{\hat{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\ \text{for } t_k \in \big(\Gamma_i^+\big)^\xi_y, \text{ and } t_{k+1} - t_k \leq 1,\end{aligned} \tag{2.81}$$

and:

$$\begin{aligned}|(\underline{\hat{u}}^\xi_y)^+(t_k)|^{\frac{p(n-1)}{n-p}} \leq \frac{p(n-1)}{n-p}\int_{t_k}^{1+t_k} |\underline{\hat{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \underline{\hat{u}}^\xi_y(r)|\, dr \\ + \int_{t_k}^{1+t_k} |\underline{\hat{u}}^\xi_y(t)|^{\frac{p(n-1)}{n-p}}\, dt, \\ \text{for } t_k \in \big(\Gamma_i^+\big)^\xi_y, \text{ and } t_{k+1} - t_k > 1,\end{aligned} \tag{2.82}$$

Hölder's inequality with exponents $p/(p-1)$ and p, and then Young's inequality with the same exponents yelds to:

$$\begin{aligned}\int_{t_k}^{t_{k+1}} |\underline{\hat{u}}^\xi_y(r)|^{\frac{p(n-1)}{n-p}-1}|\nabla \underline{\hat{u}}^\xi_y(r)|\, dr \leq \left(\int_{t_k}^{t_{k+1}} |\underline{\hat{u}}^\xi_y(r)|^{p^*}\, dr\right)^{\frac{p-1}{p}}\left(\int_{t_k}^{t_{k+1}} |\nabla \underline{\hat{u}}^\xi_y(r)|^p\, dr\right)^{\frac{1}{p}} \\ \leq \frac{p-1}{p}\int_{t_k}^{t_{k+1}} |\underline{\hat{u}}^\xi_y(r)|^{p^*}\, dr + \frac{1}{p}\int_{t_k}^{t_{k+1}} |\nabla \underline{\hat{u}}^\xi_y(r)|^p\, dr,\end{aligned} \tag{2.83}$$

First we can use inequality (2.83) to estimate the first term in the right hand side of (2.81) and of (2.82), then we can argue in the same way as in the proof of (b) in order to get (c).

The proof of (d) is similar to the one of Theorem 2.1.17 always starting from inequalities (2.27) and (2.74).

Finally we prove (e). It is enough to apply Hölder inequality with the two conjugate exponents γ and $\gamma/(\gamma-1)$ to the integrals on the right hand side of (2.68) and of (2.69), respectively. □

Remark 2.1.21. *Under hypothesis of Theorem* (2.1.19) *inequalities* (2.69), (2.70), *statements* ((d)) *and* ((e)), *hold true in* $GSBV^p_p(\Omega;\Gamma)^n$ *with the full approximate gradient instead of the symmetric one as specified in Remark 2.1.18.* ■

2.2 Convergence in measure

This section is devoted to prove a fundamental result about the continuity of the trace operator. We will show that this operator acting on the space $GSBD^p_p(\Omega;\Gamma)$, is continuous in measure with respect to the notion of convergence (0.7). This result, together with our previous trace inequalities, allow us to deduce the continuity properties of the trace cited so far in the introduction.

2.2.1 Two auxiliary results

For convenience of the reader, we remind the notion of convergence in measure.

Definition 2.2.1. *(Convergence in measure). Let $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ a bounded positive Radon measure. Consider $(v_i)_{i\in\mathbb{N}}$ $v_i\colon \mathbb{R}^n \to \mathbb{R}$ a sequence of μ-measurable functions and let $v\colon \mathbb{R}^n \to \mathbb{R}$ be a μ-measurable function. Then (v_i) converges to v in μ-measure, if for any $\epsilon > 0$ and $\delta > 0$ there exists an index $\bar{i} \in \mathbb{N}$ such that:*

$$\mu\big(\{x \in \mathbb{R}^n \mid |v_i(x) - v(x)| > \epsilon\}\big) \leq \delta, \quad \forall i \geq \bar{i}. \tag{2.84}$$

Equivalently (v_i) converges to v in μ-measure if for every $\epsilon > 0$ we have

$$\lim_{i\to\infty} \mu\big(\{x \in \mathbb{R}^n \mid |v_i(x) - v(x)| > \epsilon\}\big) = 0. \tag{2.85}$$

Remark 2.2.2. *If (v_i) converges to v in measure, then there exists a subsequence v_{i_j} that converges to v pointwise μ-a.e. Moreover if $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ is concentrated on A, and $(A_j)_{j=1}^\infty$ is a μ-measurable covering of A, in order to check the convergence in measure, it is enough to check the convergence in measure of $(v_i \llcorner A_j)_{i=1}^\infty$ to $v \llcorner A_j$, for each $j = 1, 2, \ldots$.* ■

Now we introduce the notation for the truncated functions that we adopt in this section.

Definition 2.2.3. *(Truncation function). Let a, b be two real numbers. Define $\sigma_a, \sigma^b\colon \mathbb{R} \to \mathbb{R}$ to be the truncation function from below at level a anf from above at level b, respectively, as*

$$\sigma_a(t) := \begin{cases} a \text{ if } & t < a \\ t \text{ if } & t \geq a, \end{cases} \qquad \sigma^b(t) := \begin{cases} t \text{ if } & t < b \\ b \text{ if } & t \geq b. \end{cases} \tag{2.86}$$

Define $\sigma_a^b\colon \mathbb{R} \to \mathbb{R}$ to be the truncation function from below and above at level a and b $(a < b)$, as:

$$\sigma_a^b(t) := \begin{cases} a \text{ if } & t < a \\ t \text{ if } & a \leq t < b, \\ b \text{ if } & t \geq b. \end{cases} \tag{2.87}$$

Proposition 2.2.4. *Let $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ be a bounded positive Radon measure. Let $v_j\colon \mathbb{R}^n \to \mathbb{R}$ be a sequence of μ-measurable functions and let $v\colon \mathbb{R}^n \to \mathbb{R}$ be a μ-measurable function. Suppose that for any $a < b$ holds:*

$$\sigma_a^b(v_j) \rightharpoonup \sigma_a^b(v), \text{ weakly* in } L^\infty(\mathbb{R}^n, \mu), \text{ as } j \to \infty. \tag{2.88}$$

Then the sequence (v_j) converges to v in μ-measure.

Proof. First of all fix two positive parameters ϵ and δ as in Definition 2.2.1. Then find $M > 0$ big enough such that $\mu\big(\mathbb{R}^n \setminus \{-M \leq v < M\}\big) \leq \delta/2$ (this is possible because μ is a finite measure). To simplify the notation we write $V_M := \{-M \leq v < M\}$.

Let $\gamma := \min\{\frac{c\epsilon\delta}{2}, \frac{\epsilon}{2}\}$, where $c = \frac{1}{4\mu(\mathbb{R}^n)}$, and consider a partition of $[-M, M)$ made of intervals of the form $[t_i, t_{i+1})$, such that $t_{i+1} - t_i = \gamma$, for any $i = 1, \ldots, 2M/\gamma$ (we may suppose that $M = \gamma \cdot N$ where N is a sufficiently large natural number).

Define for any i the set $A_i := v^{-1}([t_i, t_{i+1}))$ and $\bar{t}_i := (t_{i+1} + t_i)/2$ the middle point between t_i and t_{i+1}. Notice that by triangular inequality and by recalling that $\gamma \leq \epsilon$:

$$\{|v_j - v| > \epsilon\} \cap A_i \subseteq \{|v_j - \bar{t}_i| > \frac{\epsilon}{2}\} \cap A_i \text{ for } i = 1, \dots, \frac{2M}{\gamma}, \text{ and } j \in \mathbb{N}. \tag{2.89}$$

This means that:

$$\begin{aligned} \mu\big(\{|v_j - v| > \epsilon\} \cap V_M\big) &\leq \sum_{i=1}^{2M/\gamma} \mu(\{|v_j - \bar{t}_i| > \frac{\epsilon}{2}\} \cap A_i) \\ = \sum_{i=1}^{2M/\gamma} \Big[\mu\big(\{v_j - \bar{t}_i > \frac{\epsilon}{2}\} &\cap A_i\big) + \mu\big(\{v_j - \bar{t}_i < -\frac{\epsilon}{2}\} \cap A_i\big)\Big]. \end{aligned} \tag{2.90}$$

For every i, let us introduce the function:

$$\frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j) - \bar{t}_i]}{\epsilon} = \begin{cases} 0 & \text{if } v_j - \bar{t}_i < 0 \\ 1 & \text{if } v_j - \bar{t}_i \geq \frac{\epsilon}{2} \\ \frac{2}{\epsilon}(v_j - \bar{t}_i) & \text{if } 0 \leq v_j - \bar{t}_i < \frac{\epsilon}{2}. \end{cases} \tag{2.91}$$

For every i and j we have:

$$\mathbb{1}_{\{v_j - \bar{t}_i > \epsilon/2\}}(x) \leq \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \bar{t}_i]}{\epsilon}, \ \forall x \in \mathbb{R}^n. \tag{2.92}$$

We claim that for every $i = 1, \dots, 2M/\gamma$, there exists a $j(i) \in \mathbb{N}$ (depending on i) such that for any $j > j(i)$:

$$\int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) \leq c\delta\mu(A_i), \tag{2.93}$$

where $c = \frac{1}{4\mu(A)}$. In fact using the hypothesis of weak convergence at any level of truncation we can write:

$$\begin{aligned} &\limsup_{j\to\infty} \int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) = \\ &= \limsup_{j\to\infty} \Bigg(\int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v_j(x)) - \sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v(x))]}{\epsilon}\, d\mu(x) \\ &\qquad + \int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) \Bigg) \\ &\leq \int_{A_i} \frac{2[\sigma_{\bar{t}_i}^{\bar{t}_i+\epsilon/2}(v(x)) - \bar{t}_i]}{\epsilon}\, d\mu(x) \leq \frac{2\gamma}{\epsilon}\mu(A_i) < c\delta\mu(A_i). \end{aligned} \tag{2.94}$$

Now define:

$$\tilde{j} := \max\{j(i) \mid i = 1, \dots, 2M/\gamma\}, \tag{2.95}$$

hence the following estimate holds true:

$$\sum_{i=1}^{2M/\gamma} \mu(\{v_j - \bar{t}_i > \frac{\epsilon}{2}\} \cap A_i) < \sum_{i=1}^{2M/\gamma} c\delta\mu(A_i) \leq c\delta\mu(\mathbb{R}^n) = \frac{\delta}{4}, \ \ \forall j \geq \tilde{j}. \tag{2.96}$$

Analogously we repeat the same argument for the other addends of (2.90). So finally we have:

$$\mu(\{|v_j - v| > \epsilon\} \cap V_M) \leq \delta/2. \tag{2.97}$$

By using (2.97) and by the definition of M, for any $j > \tilde{j}$ we get:

$$\begin{aligned} \mu(\{|v_j - v| > \epsilon\}) &= \mu\big(\{|v_j - v| > \epsilon\} \cap V_M\big) + \mu\big(\{|v_j - v| > \epsilon\} \setminus V_M\big) \\ &\leq \frac{\delta}{2} + \mu(\mathbb{R}^n \setminus V_M) \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned} \tag{2.98}$$

obtaining the desired estimate. □

The following proposition is a characterisation of rectifiable sets by means of sets with finite perimeter.

Proposition 2.2.5. *$\Gamma \subset \mathbb{R}^n$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set if and only if there exists a sequence of bounded open sets of finite perimeter $(U_i)_{i=1}^{\infty}$ such that*

$$\mathcal{H}^{n-1}(\Gamma \setminus \bigcup_{i=1}^{\infty} \partial^* U_j) = 0, \tag{2.99}$$

where $\partial^* U_j$ denotes the reduced boundary of U_i (see [2, Definition 3.54]).

Proof. Using [36, Theorem 3.2.29] we know that $\mathcal{H}^{n-1}$ almost all of Γ is contained in a countably union of $(n-1)$-dimensional manifolds of $\mathbb{R}^n$ of class C^1. So we can reduce ourselves to prove the statement for a single $(n-1)$-dimensional manifold M of class C^1; moreover by basic fact about differential geometry we have that M can be covered by countably many graphs of maps from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ of class C^1. So for our purpose it is enough to prove the proposition for a $(n-1)$-dimensional manifold of the form $M \subseteq \text{graph}(f)$ where $f \in C^1(\mathbb{R}^{n-1})$.

To prove this last assertion we can consider a countable measurable partition of $\mathbb{R}^{n-1}$ made for example by open cubes $(Q_i)_{i=0}^{\infty}$. For every $i \in \mathbb{N}$, up to a translation on M, we may assume that $\inf_{Q_i} f > 0$. Finally we define:

$$U_i := \{(y,t) \mid y \in Q_i,\, 0 < t < f(y)\}.$$

Clearly each U_i is an open set of finite perimeter such that:

$$\text{graph}(f \llcorner Q_i) \subset \partial^* U_j,$$

and

$$\mathcal{H}^{n-1}(M \setminus \bigcup_i \partial^* U_j) \leq \mathcal{H}^{n-1}\big(M \setminus \bigcup_i \text{graph}(f \llcorner Q_i)\big) = 0.$$

□

2.2.2 Convergence in measure of the traces

As we mentioned in the introduction, the notion of convergence 0.7 is useful in order to ensure the compactness for suitable minimizing sequences in several minimisation problems that come from the variational models of fracture mechanics. When we will speak about continuity of the traces, we will always refer to that notion of convergence.

Definition 2.2.6. *If Γ is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with orientation ν, for every $\xi \in \mathbb{S}^{n-1}$ we define the set $\Gamma^\xi := \{x \in \Gamma \mid \nu(x) \cdot \xi \neq 0\}$.*

We are now in position to prove our result about the convergence of traces in measure.

Theorem 2.2.7. *(Convergence in measure). Let Ω be an open set of $\mathbb{R}^n$ of finite perimeter, and let Γ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set in Ω with $\mathcal{H}^{n-1}(\Gamma) < \infty$ oriented by ν. Let $(u_i)_{i=1}^\infty$ be a sequence converging to $u \in GSBD_p^p(\Omega;\Gamma)$ $(p \geq 1)$ with respect to the convergence (0.7), then $(u_i)^\pm_{\Gamma\cup\partial^*\Omega}$ converges in measure (with respect to $\mathcal{H}^{n-1} \llcorner \Gamma \cup \partial^*\Omega$) to $u^\pm_{\Gamma\cup\partial^*\Omega}$.*

Remark 2.2.8. *Just to simplify the notation we prefer to give the proof of the previous theorem when Ω is the entire space $\mathbb{R}^n$. Using the extension argument given by proposition 2.1.9 the same argument works for the general case.* ■

Proof. Thanks to Proposition 2.2.5 there exists a countable family of bounded open sets of finite perimeter, say $\{U_j\}_{j=1}^\infty$, such that

$$\Gamma \subset \bigcup_{j=1}^\infty \partial^* U_j \tag{2.100}$$

up to a $\mathcal{H}^{n-1}$-negligible set. Hence, by Remark 2.2.2, in order to prove our statement we can reduce ourselves to prove that for every $j \in \mathbb{N}$, $(u_i)^\pm_\Gamma$ converges in $\mathcal{H}^{n-1} \llcorner (\Gamma \cap \partial^* U_j)$-measure to $u^\pm_\Gamma$. Because of the fact that:

$$(u_i)^+_\Gamma(x) = \begin{cases} (u_i)^+_{\partial^* U_j}(x) & \text{if } \nu_\Gamma(x) = \nu_{U_j}(x) \\ (u_i)^-_{\partial^* U_j}(x) & \text{if } \nu_\Gamma(x) = -\nu_{U_j}(x), \end{cases}$$

up to a measurable change of sign of ν_Γ, it is equivalent to prove that $(u_i)^\pm_{\partial^* U_j}$ converges to $u^\pm_{\partial^* U_j}$ in $\mathcal{H}^{n-1} \llcorner \partial^* U_j$-measure.
Now we fix $j \in \mathbb{N}$ and we prove that for any $\xi \in \mathbb{S}^{n-1}$, $(u_i)^+_{\partial^* U_j} \cdot \xi$ converges to $u^+_{\partial^* U_j} \cdot \xi$ in $\mathcal{H}^{n-1} \llcorner \partial^* U_j^\xi$-measure, and to simplify the notation, we denote $u_i^+ := (u_i)^+_{\partial^* U_j}$ and $u^+ := u^+_{\partial^* U_j}$.

By Proposition 2.2.4 it is enough to show that given any pair $a, b \in \mathbb{R}$ with $a < b$ we have:

$$\sigma_a^b(u_i^+ \cdot \xi) \rightharpoonup \sigma_a^b(u^+ \cdot \xi), \text{ weakly* in } L^\infty(\partial^* U_j^\xi, \mathcal{H}^{n-1}). \tag{2.101}$$

For each $i \in \mathbb{N}$ let $\underline{u}_i$ and $\underline{u}$ be the functions extended to zero outside of U_j (see Proposition 2.1.9). We know that $\underline{u}_i^+ = u_i^+$ and $\underline{u}^+ = u^+$ on $\partial^* U_j$, so we can prove our assertion for the sequence $(\underline{u}_i)_{i=1}^\infty$.

By hypothesis we have that $\underline{u}_i \to \underline{u}$ strongly in $L^1(\mathbb{R}^n, \mathbb{R}^n)$. As a consequence also $\sigma_a^b(\underline{u}_i \cdot \xi) \to \sigma_a^b(\underline{u} \cdot \xi)$ strongly in $L^1(\mathbb{R}^n)$ and in particular this means that:

$$D_\xi \sigma_a^b(\underline{u}_i \cdot \xi) \rightharpoonup D_\xi \sigma_a^b(\underline{u} \cdot \xi), \tag{2.102}$$

in the sense of distributions. Moreover we have the bound on the total variations along the direction ξ:

$$\begin{aligned}
\sup_{i\in\mathbb{N}} |D_\xi \sigma_a^b(\underline{u}_i\cdot\xi)|(\mathbb{R}^n) &\le \sup_{i\in\mathbb{N}} \int_{\mathbb{R}^n} |(\mathcal{E}(\underline{u}_i)\xi,\ \xi)|\ dx + |b-a|\mathcal{H}^{n-1}\big((\Gamma^\xi\cap U_j)\cup\partial^* U_j^\xi\big)\\
&= \int_{U_j} |(\mathcal{E}(u_i)\xi,\ \xi)|\ dx + |b-a|\mathcal{H}^{n-1}\big((\Gamma^\xi\cap U_j)\cup\partial^* U_j^\xi\big)\\
&\le \sup_{i\in\mathbb{N}} \mathcal{L}^n(U_j)^{1-\frac{1}{p}}\left(\int_{U_j} |\mathcal{E}u_i|^p\ dx\right)^{\frac{1}{p}} + |b-a|\mathcal{H}^{n-1}\big((\Gamma^\xi\cap U_j)\cup\partial^* U_j^\xi\big)\\
&< +\infty.
\end{aligned} \tag{2.103}$$

Hence the convergence in (2.102) still holds true weakly in $\mathcal{M}_b(\mathbb{R}^n)$. Since by hypothesis $\mathcal{E}(\underline{u}_i) \rightharpoonup \mathcal{E}(\underline{u})$ weakly in $L^1(\mathbb{R}^n;\mathbb{M}^{n\times n}_{sym})$, we can write:

$$D_\xi\sigma_a^b(\underline{u}_i\cdot\xi) - \mathcal{E}(\underline{u}_i)\xi\cdot\xi\mathcal{L}^n \rightharpoonup D_\xi\sigma_a^b(\underline{u}\cdot\xi) - \mathcal{E}(\underline{u})\xi\cdot\xi\mathcal{L}^n \text{ weakly in } \mathcal{M}_b(\mathbb{R}^n), \tag{2.104}$$

and it follows:

$$[\sigma_a^b(\underline{u}_i\cdot\xi)]\xi\cdot\nu\mathcal{H}^{n-1} \rightharpoonup [\sigma_a^b(\underline{u}\cdot\xi)]\xi\cdot\nu\mathcal{H}^{n-1} \text{ weakly in } \mathcal{M}_b(\mathbb{R}^n). \tag{2.105}$$

On the other hand, thanks to the truncation between a and b, the sequence $([\sigma_a^b(\underline{u}_i\cdot\xi)])_{i\in\mathbb{N}}$ is relatively sequentially compact in the weak* topology of L^∞, and call for example α one of its limits. Given any $\phi\in L^1\big(\mathbb{R}^n, \mathcal{H}^{n-1}\llcorner[(\Gamma\cap U_j)\cup\partial^* U_j]\big)$ we can use $\phi\xi\cdot\nu$ as test function in the weak* convergence:

$$\lim_{i_k\to\infty}\int_{(\Gamma\cap U_j)\cup\partial^* U_j} [\sigma_a^b(\underline{u}_{i_k}\cdot\xi)]\phi\xi\cdot\nu\ d\mathcal{H}^{n-1}(x) = \int_{(\Gamma\cap U_j)\cup\partial^* U_j} \alpha\phi\xi\cdot\nu\ d\mathcal{H}^{n-1}(x), \tag{2.106}$$

this together with (2.105) means that every weak* limits α is equal to $[\sigma_a^b(\underline{u}\cdot\xi)]$ on the set $(\Gamma^\xi\cap U_j)\cup\partial^* U_j^\xi$.

Recall that by Remark 2.1.10 $\underline{u}_i^- = 0$ a.e. on $\partial^* U_j$, and by Proposition 2.1.9 $\underline{u}_i^+ = u_i^+$ a.e. on $\partial^* U_j$, hence for every $i\in\mathbb{N}$:

$$[\sigma_a^b(\underline{u}_i\cdot\xi)] = \sigma_a^b(u_i^+\cdot\xi),\ \mathcal{H}^{n-1}\text{-a.e. on } \partial^* U_j,$$

and also:

$$[\sigma_a^b(\underline{u}\cdot\xi)] = \sigma_a^b(u^+\cdot\xi),\ \mathcal{H}^{n-1}\text{-a.e. on } \partial^* U_j.$$

Therefore:

$$\sigma_a^b(u_i^+\cdot\xi) \rightharpoonup \sigma_a^b(u^+\cdot\xi) \text{ weakly* in } L^\infty(\partial^* U_j^\xi, \mathcal{H}^{n-1}). \tag{2.107}$$

Using $\mathbb{R}^n\setminus U_j$ instead of U_j we can prove in the very same way that:

$$\sigma_a^b(u_i^-\cdot\xi) \rightharpoonup \sigma_a^b(u^-\cdot\xi) \text{ weakly* in } L^\infty(\partial^* U_j^\xi, \mathcal{H}^{n-1}).$$

Thanks to the arbitrariness of $\xi\in\mathbb{S}^{n-1}$, we can use the argument of Remark 2.1.7 to deduce:

$$\sigma_a^b(u_i^\pm) \rightharpoonup \sigma_a^b(u^\pm) \text{ weakly* in } L^\infty(\partial^* U_j, \mathcal{H}^{n-1}),$$

and thanks to the arbitrariness of $a, b\in\mathbb{R}$, by Proposition 2.2.4 we have:

$$u_i^\pm \to u^\pm \text{ in } \mathcal{H}^{n-1}\llcorner\partial^* U_j\text{-measure },$$

which is our desired result. □

Remark 2.2.9. *Let Ω and Γ be as in Theorem 2.2.7. As explained in Remark 2.1.18 we have the following inclusion $GSBV_p^p(\Omega;\Gamma)^n \subset GSBD_p^p(\Omega;\Gamma)$, hence thanks to Theorem 2.2.7, if $(u_i)_{i=1}^\infty \subset GSBV_p^p(\Omega;\Gamma)$ converges to $u \in GSBV_p^p(\Omega;\Gamma)$ with respect to the following notion of convergence:*

$$\begin{cases} \sup_i \left(\|u_i\|_p + \|\nabla u_i\|_p + \mathcal{H}^{n-1}(J_{u_i})\right) \leq C \\ u_i \to u, \ \ \textit{in } L^1(\Omega) \\ \nabla u_i \rightharpoonup \nabla u, \ \ \textit{weakly in } L^1(\Omega), \end{cases} \tag{2.108}$$

then $(u_i)^\pm_{\Gamma\cup\partial^*\Omega}$ converges in $\mathcal{H}^{n-1} \llcorner (\Gamma \cup \partial^*\Omega)$-measure to $u^\pm_{\Gamma\cup\partial^*\Omega}$. ■

2.3 Continuity of the trace and an application

2.3.1 The main result

Now we summarize our previous results, Theorems 2.1.14, 2.1.19, and 2.2.7, into the following theorem.

Theorem 2.3.1. *Let Ω be an open set of $\mathbb{R}^n$ of finite perimeter, and let Γ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set in Ω, with $\mathcal{H}^{n-1}(\Gamma) < \infty$ and oriented by ν. Consider the space $GSBD_p^p(\Omega;\Gamma)$ with $p > 1$ endowed with the notion of convergence (0.7), the functions $\Theta^\pm$ defined in Theorem 2.1.14 and Ψ given in Theorem 2.2.7. Then:*

(a) the trace operators from above and from below:

$$(\cdot)^\pm_{\Gamma\cup\partial^*\Omega} \colon GSBD_p^p(\Omega;\Gamma) \to L^q(\Gamma \cup \partial^*\Omega, \Theta^\pm\mathcal{H}^{n-1}) \, (p > 1), \tag{2.109}$$

are strongly continuous for every $q \in [1,p)$ and weakly continuous for every $q \in [1,p]$.

(b) if we add the uniform bound on the $\|\cdot\|_{L^p(\Omega,\Psi\mathcal{L}^n)}$-norm along the sequence in the notion of convergence (0.7), the trace operators from above and from below:

$$(\cdot)^\pm_{\Gamma\cup\partial^*\Omega} \colon GSBD^p(\Omega;\Gamma) \cap L^p(\Omega,\Psi\mathcal{L}^n) \to L^q(\Gamma\cup\partial^*\Omega, \mathcal{H}^{n-1}) \, (p > 1), \tag{2.110}$$

are strongly continuous for every $q \in [1,p)$ and weakly continuous for every $q \in [1,p]$.

Remark 2.3.2. *By Remarks 2.1.18, 2.1.21 and 2.2.9, the previous theorem applies also to the space $GSBV_p^p(\Omega;\Gamma)$. Moreover the continuity properties of the trace operators mentioned so far in the introduction, are simply a consequence of this previous theorem, when we restrict our attention on $\partial\Omega$. In fact when Ω is Lipschitz-regular the reduced boundary $\partial^*\Omega$ coincides $\mathcal{H}^{n-1}$-a.e. with the topological boundary $\partial\Omega$.* ■

Proof of Theorem 2.3.1. It is a consequence of the convergence in measure of the traces given by Theorem 2.2.7 together with estimate (2.39) to prove (a), and together with estimate (2.69) to prove (b). □

Remark 2.3.3. *In the case $p = 1$ Theorem 2.3.1 does not hold due to the non reflexivity of the spaces $L^1(\Gamma\cup\partial^*\Omega, \Theta^\pm\mathcal{H}^{n-1})$ and $L^1(\Gamma\cup\partial^*\Omega, \mathcal{H}^{n-1})$. This is coherent with the continuity property of the trace operator defined for example in $W^{1,1}(\Omega)$.* ■

2.3.2 A counterexample

Now we give a counterexample to the strong continuity of the trace operator in (2.109) when $q = p$:

Example 2.3.4. *Consider in $\mathbb{R}^2$ the set*

$$E := \bigcup_{n=1}^{\infty} \left(\left[-\frac{1}{2n^2}, \frac{1}{2n^2} \right]^2 + (n,0) \right),$$

made of infinitely many squares E_n of side $1/n^2$ and centered at $(n,0) \in \mathbb{R}^2$. Clearly E is a set of finite perimeter so we can choose as Γ its reduced boundary $\partial^* E$ oriented with respect to its inner measure theoretical normal ν_E. Define the sequence of functions $(u_n)_{n=1}^{\infty} \subset GSBD^2_2(\Omega;\Gamma)$ as

$$u_n(x) := \frac{1}{\sqrt{\mathcal{L}^2(E_n)}} \mathbb{1}_{E_n}(x) \text{ for every } n,$$

and notice that $\|u_n\|_2 = 1$ for any n.
Clearly the trace functions u_n^+ converges pointwise $(\Theta^+ \mathcal{H}^1 \llcorner \Gamma)$-a.e. to 0 for any choice of Θ^+ i.e. for any choice of an orthonormal basis $\{\xi_1, \xi_2\}$ of $\mathbb{R}^2$ as in (2.46). This means that any strong $L^2(\Gamma, \Theta^+\mathcal{H}^1)$ limit of u_n^+ must be the zero function. But we claim that for each choice of Θ^+ as in (2.46) we have that

$$\int_\Gamma |u_n^+|^2 \Theta^+ \, d\mathcal{H}^1 \geq C(\Theta^+) \|u_n\|_2^2 > 0,$$

where $C(\Theta^+)$ is a strictly positive constant which depends only on Θ^+, which is a contradiction. Remember that in order to construct Θ^+, we divide Γ in finitely many parts $(\Gamma_i)_{i=1}^N$, and we associate to each Γ_i an orthonormal basis $\{\xi_1^i, \xi_2^i\}$ such that $|\xi_1^i \cdot \nu_E(x)|, |\xi_2^i \cdot \nu_E(x)| > \frac{1}{2\sqrt{2}}$ for $x \in \Gamma_i$. This means that for each n there exists $i(n) \in \{1, \dots, N\}$, such that:

$$\mathcal{H}^1(\partial^* E_n \cap \Gamma_{i(n)}) \geq \frac{\mathcal{H}^1(\partial^* E_n)}{N}. \tag{2.111}$$

Possibly passing through a sub-sequence we may suppose for example that for every n:

$$\mathcal{H}^1\big(\{x \in \partial^* E_n \cap \Gamma_{i(n)} \mid \theta^{\xi_1^{i(n)}} \leq \theta^{\xi_2^{i(n)}}\}\big) \geq \frac{1}{2}\mathcal{H}^1(\partial^* E_n \cap \Gamma_{i(n)}). \tag{2.112}$$

To simplify the notation we omit the dependence on n and we write Γ_i, ξ_1^i, ξ_2^i to denote respectively $\Gamma_{i(n)}$, $\xi_1^{i(n)}$, $\xi_2^{i(n)}$; so we have:

$$\begin{aligned}
\int_\Gamma |u_n^+|^2 \Theta^+ \, d\mathcal{H}^1 &\geq \int_{\partial^* E_n \cap \Gamma_i} |u_n^+(x)|^2 \Theta^+(x) \, d\mathcal{H}^1(x) \\
&\geq \frac{1}{\mathcal{L}^2(E_n)} \int_{\partial^* E_n \cap \Gamma_i \cap \{\theta^{\xi_1^i} \leq \theta^{\xi_2^i}\}} \theta^{\xi_1^i}(x) \, d\mathcal{H}^1(x) \\
&\geq \frac{1}{\mathcal{L}^2(E_n)} \int_{\pi^{\xi_1^i}(E_n^{\xi_1^i})} \frac{1}{2\sqrt{2}} \mathcal{H}^1\big((E_n)_y^{\xi_1^i}\big) \, d\mathcal{L}^1(y) \\
&= \frac{\mathcal{L}^2(E_n^{\xi_1^i})}{\mathcal{L}^2(E_n)},
\end{aligned}$$

where for each n, $E_n^{\xi_1^i} = \big\{(y,t) \in E_n \mid y \in \pi^{\xi_1^i}\big(\mathcal{F}E_n \cap \Gamma_i \cap \{\theta^{\xi_1^i} \leq \theta^{\xi_2^i}\}\big)\big\}$. Finally from (2.111), (2.112), and $|\xi_1^i \cdot \nu_E(x)| > \frac{1}{2\sqrt{2}}$ (for every i), it is a geometric fact that for every n the quotient $\frac{\mathcal{L}^2(E_n^{\xi_1^i})}{\mathcal{L}^2(E_n)}$ is greater then a strictly positive real number which depends only on the $\mathcal{H}^1$-measure of the projection $\pi^{\xi_1}(E_n^{\xi_1^i})$ and on the scalar product $|\xi_1^i \cdot \nu_E|$. Hence we have showed the claim. △

2.3.3 An application to the theory of elasticity with cracks

Finally, we show an application of our results in the theory of elasticity with cracks. Before doing this, we want to make a remark between the classical way to formulate the Dirichlet condition and the way that we present via the theory of trace.

Remark 2.3.5. *Our formulation of the Dirichlet boundary condition is slightly more general than the standard one. In fact, a classical way to formulate the Dirichlet problem in a domain Ω is to consider a larger open set Ω' such that $\Omega \subset \Omega'$, $\partial_D\Omega = \Omega' \cap \partial\Omega$, and to prescribe the value of all admissible functions in $\Omega' \setminus \overline{\Omega}$. Precisely, given $u' \in W^{1,2}(\Omega' \setminus \overline{\Omega}; \mathbb{R}^n)$, one minimizes among all $u \in GSBD^2_2(\Omega')$ such that $u = u'$ on $\Omega' \setminus \overline{\Omega}$. In our case, since the jump sets are confined in $\Gamma \subset \Omega$, the minimisation problem is*

$$\min_{\substack{u \in GSBD^2_2(\Omega';\Gamma) \\ u=u' \text{ on } \Omega'\setminus\overline{\Omega}}} E(u), \tag{2.113}$$

where $E(u)$ could be for example the functional (2.115).

Given $w \in GSBD^2_2(\Omega;\Gamma)$, instead of (2.113), our formulation of the Dirichlet problem is of the form (see example 2.3.6)

$$\min_{\substack{u \in GSBD^2_2(\Omega;\Gamma) \\ u=w \text{ on } \partial_D\Omega}} E(u). \tag{2.114}$$

Since any admissible function in (2.113) has jump set J_u contained in Γ, and $\Gamma \subset \Omega$, we have that any minimiser of (2.113) is also a minimiser of (2.114). On the other hand, it is clear that if a minimiser of (2.114) is also a minimiser of (2.113) then there must exist $u' \in W^{1,2}(\Omega' \setminus \overline{\Omega}; \mathbb{R}^n)$ whose trace coincides with the trace of w on $\partial_D\Omega$. We show with an example that this is not always the case.

Consider $\Omega, \Omega' \subset \mathbb{R}^2$ to be $\Omega := (0,1) \times (0,1)$ and $\Omega' := (0,1) \times (-1,1)$, respectively. Define the Dirichlet part of the boundary to be $\partial_D\Omega := (0,1) \times \{0\}$. Let $\Gamma \subset \Omega$ be the graph of the function $f\colon (0,1) \to \mathbb{R}$ defined as $f(x) := x^2$. Now, since Γ forms a cusp with $\partial_D\Omega$, it is clear that there exists a function $u \in GSBD^2_2(\Omega;\Gamma)$ whose trace is not square integrable on $\partial_D\Omega$. But on the other hand, since $\Omega' \setminus \overline{\Omega}$ is a Lipschitz-regular domain, any function $u \in W^{1,2}(\Omega' \setminus \overline{\Omega}; \mathbb{R}^n)$ has trace on $\partial_D\Omega$ which is square integrable.

This shows that the approach (2.114) allows to treat more general Dirichlet boundary conditions. ■

Example 2.3.6. *Let $\Omega \subset \mathbb{R}^n$ be a regular domain which represents the reference configuration of an elastic body, and let $\Gamma \subset \mathbb{R}^n$ be a crack described by a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with finite $\mathcal{H}^{n-1}$-measure; we consider two disjoint measurable subsets of $\partial\Omega$, $\partial_D\Omega$ and $\partial_N\Omega$, which are the Dirichlet part and the Neumann part of the boundary, respectively. On the set $\partial\Omega \setminus (\partial_D\Omega \cup \partial_N\Omega)$ and on the crack we impose the homogeneous Neumann condition.*

We consider the following minimisation problem:

$$\min_{\substack{u \in GSBD^2_2(\Omega;\Gamma) \\ u=w \text{ on } \partial_D\Omega}} E(u) := \int_\Omega |\mathcal{E}u|^2 \, dx + \int_\Omega |u-g|^2 \, dx + \mathcal{H}^{n-1}(J_u) - \int_{\partial_N\Omega} F \cdot Tr(u) \, d\mathcal{H}^{n-1}, \tag{2.115}$$

where w is some function in $GSBD^2_2(\Omega;\Gamma)$, F is a vector field representing the traction force, and g is some square integrable vector field. Minimisation problems like (2.115), arise for example in [22] or in [70], in order to solve the wave equation or the equations of elastodynamics in a prescribed arbitrary growing cracks domain, respectively.

In order to prove the existence of a minimum, we need to specify the space of all admissible Neumann terms: let Θ^+ be the weight function given in Theorem 2.3.1, then we consider all the measurable vector fields F such that $\int_{\partial_N\Omega} \frac{F^2}{\Theta^+}\, d\mathcal{H}^{n-1} < \infty$, or equivalently such that $F = G\sqrt{\Theta^+}$ for some vector field $G \in L^2(\Omega;\mathbb{R}^n)$.

Roughly speaking the function Θ^+ measures, somehow, how much Γ is close to the boundary. From a physical point of view, this might be interpreted as the fact that, when the elastic material between the Neumann boundary and the crack is infinitesimally small, then the elastic reaction to the traction force should be extremely large; hence, in order to reach the equilibrium, the traction forces need to decrease their intensity (proportionally to Θ^+).

First of all we show the coercivity of $E(\cdot)$. By Theorem (2.3.1) we can bound the Neumann term from above as:

$$\begin{aligned}\int_{\partial_N\Omega} F\cdot Tr(u)\, d\mathcal{H}^{n-1} &\le \left(\int_{\partial_N\Omega} |G|^2\, d\mathcal{H}^{n-1}\right)^{1/2}\left(\int_{\partial_N\Omega} |Tr(u)|^2\Theta^+\, d\mathcal{H}^{n-1}\right)^{1/2}\\ &\le C\Big(\|\mathcal{E}u\|_{L^2} + \|u\|_{L^2}\Big),\end{aligned}$$

where $C > 0$ is a constant which depends only on the dimension n and on F. As a consequence we immediately deduce the coercivity:

$$E(u) \ge \|\mathcal{E}u\|^2_{L^2} + 2\|u\|^2_{L^2} - 2\|g\|^2_{L^2} - \mathcal{H}^{n-1}(\Gamma) - C(\|\mathcal{E}u\|_{L^2} + \|u\|_{L^2}).$$

Hence every minimizing sequence satisfies the uniform bound

$$\sup_k(\|\mathcal{E}u_k\|_{L^2} + \|u_k\|_{L^2} + \mathcal{H}^{n-1}(J_{u_k})) < \infty,$$

and we are in position to use the compactness result in [20, Theorem 11.3] to deduce that there exists $u \in GSBD^2_2(\Omega;\Gamma)$ such that (up to subsequences):

$$\begin{cases} u_k \to u, \ \ \text{in } L^1(\Omega)\\ \mathcal{E}u_k \rightharpoonup \mathcal{E}u, \ \ \text{weakly in } L^1(\Omega).\end{cases} \tag{2.116}$$

This means that (u_k) converges to u with respect to the notion of convergence (0.7), and by Theorem 2.2.7 u still satisfies $Tr(u) = Tr(w)$ on $\partial_D\Omega$.

The first two terms of $E(\cdot)$ are clearly lower semicontinuous with respect to the convergence (0.7), while $\liminf_{k\to\infty}\mathcal{H}^{n-1}(J_{u_k}) \ge \mathcal{H}^{n-1}(J_u)$ is ensured by Theorem 1.4.8. The Neumann term is even continuous: this is a simple consequence of the fact that by Theorem 2.3.1 the trace operator is weakly continuous in $L^2(\Omega, \Theta^+\mathcal{H}^{n-1})$, thus we can write:

$$\begin{aligned}\lim_k \int_{\partial_N\Omega} F\cdot Tr(u_k)\, d\mathcal{H}^{n-1} &= \lim_k \int_{\partial_N\Omega} \frac{G}{\sqrt{\Theta^+}}\cdot Tr(u_k)\ \Theta^+ d\mathcal{H}^{n-1}\\ &= \int_{\partial_N\Omega} \frac{G}{\sqrt{\Theta^+}}\cdot Tr(u)\ \Theta^+ d\mathcal{H}^{n-1}\\ &= \int_{\partial_N\Omega} F\cdot Tr(u)\, d\mathcal{H}^{n-1}.\end{aligned}$$

Hence our functional is coercive and lower semicontinuous, so we are in position to apply the standard direct method in the calculus of variation to deduce the existence of a minimiser. △

Chapter 3

On the blow-up of $GSBV$ functions

Contents

3.1 Weak Poincaré's inequality for indecomposable sets

This first section is devoted to the proof of a weak version of the Poincaré's inequality for indecomposable sets. We recall that given a connected Lipschitz-regular open set Ω, the Poincarè's inequality allows to control the L^p-distance of a function u from its average in terms of the L^p-norm of its gradient. Namely, for every $u \in W^{1,p}(\Omega)$ it holds

$$\left(\int_\Omega \left|u - \fint u\right|^p dx\right)^{\frac{1}{p}} \leq C(\Omega,p)\left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}. \tag{3.1}$$

In our case we want to derive a similar inequality in the context of $GSBV^p$-functions, when Ω is replaced by a generic indecomposable set. Since in general a function $u \in GSBV^p(\Omega)$ is not even integrable, the L^p-distance on the left hand side of (3.1) will be substituted by the L^0-distance, which is the one that induces the convergence in measure (see Definition 3.4.4).

Precisely, we prove the following inequality

$$\left(\int_F |u-m|^p \wedge 1\, dx\right)^{\frac{1}{p}} \leq C(F,p,\lambda)\left(\int_F |\nabla u|^p\, dx\right)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}}, \tag{3.2}$$

for every indecomposable set $F \subset \Omega$ with $|F| < \infty$ and for every $u \in GSBV^p(\Omega)$ such that $\mathcal{H}^{n-1}(J_u \cap F^{(1)}) = 0$. The real number m is the median of u on F (see Definition 3.1.8), ∇u is the approximate gradient of u (see Section 1.4.1), and λ is any positive real number in $(0, 1/2]$. The integral on the left hand side of (3.2) is equivalent to the L^0-distance on F (see (3.88)) between u and m. The function $C(F,p,\cdot)$ is decreasing and in general may blow-up as $\lambda \to 0^+$. Inequality (3.2) tells us that if $\int_F |\nabla u|^p\, dx$ is sufficiently small, then u is close to a single constant on F. This information will play a crucial role in order to derive the first main result of this chapter, namely Theorem 1.

3.1.1 The upper isoperimetric profile

Given $F \subset \Omega$ an indecomposable set we finite measure, we want to introduce an isoperimetric quantity h_F, which is a function $h_F\colon (0, \frac{1}{2}] \to (0,\infty)$, and which plays a similar role to the so called *Cheeger's constant*. We recall that when Ω is a bounded open set of $\mathbb{R}^n$, $(n \geq 2)$ the Cheeger's constant is defined as (see [58],[59])

$$h(\Omega) := \inf\left\{\frac{P(E)}{|E|} \mid E \subset \Omega,\, |E| > 0\right\}. \tag{3.3}$$

Let us remind that the Cheeger's constant was introduced in [15] to study lower bounds for the smallest eigenvalue of the Laplace operator on compact Riemannian manifold without boundary. As a consequence, one obtains the validity of a Poincaré's inequality with optimal constant uniformly bounded from below by a geometric constant. Precisely, for the case of Ω bounded open set of $\mathbb{R}^n$, let $\lambda_p(\Omega)$ be the smallest "eigenvalue" of the p-laplacian with Dirichlet boundary condition $(1 \leq p < \infty)$, i.e.

$$\lambda_p(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^p}^p}.$$

Then arguing as in [15] (see [57] [51]) one can easily show that

$$\lambda_p(\Omega) \geq \frac{h(\Omega)^p}{p^p}.$$

In our case, since we are interested in a weaker version of Poincaré's inequality for indecomposable sets without the assumption of Dirichlet boundary conditions, we need to work with a different notion of Cheeger's constant. Before starting with the definition, we need to prove a lower-semicontinuity property, which can be seen as a generalisation of the well known result of lower semicontinuity of the perimeter: given a sequence of sets (E_k) such that $\lim_{k\to\infty} |E_k \Delta E| = 0$, then for every open set Ω

$$\liminf_{k\to\infty} P(E_k;\Omega) \geq P(E;\Omega).$$

Proposition 3.1.1 (Lower semicontinuity)**.** *Let Ω be an open set of $\mathbb{R}^n$. Let $(E_k)_{k=1}^\infty$,$(E'_k)_{k=1}^\infty$ and E, E' be subsets of Ω with finite perimeter in Ω such that $E'_k \subset E_k$ and*

1. $\lim_{k\to\infty} |E_k \Delta E| = 0$;

2. $\lim_{k\to\infty} P(E_k;\Omega) = P(E;\Omega)$*;*

3. $\lim_{k\to\infty} |E'_k \Delta E'| = 0$*.*

Then it holds the following lower semicontinuity property

$$\liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) \geq \mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}). \tag{3.4}$$

Proof. Using the Leibniz's formula (1.2) we can write

$$\begin{aligned} P(E'_k;\Omega) = P(E'_k \cap E_k;\Omega) &= \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^* E_k \cap E_k'^{(1)}) \\ &\quad + \mathcal{H}^{n-1}(\{\nu_{E'_k} = \nu_{E_k}\}). \end{aligned} \tag{3.5}$$

Since $E'_k \subset E_k$ then $E_k'^{(1)} \subset E_k^{(1)}$, hence $E_k'^{(1)} \cap E_k^{(1/2)} = \emptyset$. This implies $\mathcal{H}^{n-1}(\partial^* E_k \cap E_k'^{(1)}) = 0$. Moreover, since $E'_k \subset E_k$ then $\mathcal{H}^{n-1}(\{\nu_{E'_k} \neq \nu_{E_k}\}) = 0$. Therefore (3.5) can be rewritten as

$$P(E'_k;\Omega) = \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^* E'_k \cap \partial^* E_k). \tag{3.6}$$

Analogously we have

$$P(E_k \setminus E'_k;\Omega) = \mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap \partial^* E_k). \tag{3.7}$$

Since $\mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap E_k^{(1)}) = \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)})$, then we can rewrite the previous equality as

$$P(E_k \setminus E'_k;\Omega) = \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap \partial^* E_k). \tag{3.8}$$

We claim that

$$\mathcal{H}^{n-1}(\partial^* E_k \setminus (\partial^* E'_k \cup \partial^*(E_k \setminus E'_k))) = 0 \tag{3.9}$$

and

$$\mathcal{H}^{n-1}((\partial^* E_k \cap \partial^* E'_k) \cap (\partial^* E_k \cap \partial^*(E_k \setminus E'_k)) = 0. \tag{3.10}$$

To show this, notice that by Theorem 1.3.2, for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$, if $x \in E_k^{(1/2)}$ then

$$\left\{x \in (E'_k)^{(0)} \text{ or } x \in (E'_k)^{(1/2)}\right\} \text{ and } \left\{x \in (E_k \setminus E'_k)^{(0)} \text{ or } x \in (E_k \setminus E'_k)^{(1/2)}\right\}.$$

But if $x \in E_k^{(1/2)}$ it cannot happen $x \in (E'_k)^{(0)}$ and $x \in (E_k \setminus E'_k)^{(0)}$, otherwise $x \in E_k^{(0)}$ which is a contradiction. This proves (3.9). Also if $x \in E_k^{(1/2)}$ then it cannot happen $x \in (E'_k)^{(1/2)}$ and $x \in (E_k \setminus E'_k)^{(1/2)}$, otherwise $x \in E_k^{(1)}$ which is again a contradiction. This proves (3.10).

By (3.9) and (3.10), summing (3.6) with (3.8) we obtain for every $k \in \mathbb{N}$

$$P(E'_k;\Omega) + P(E_k \setminus E'_k;\Omega) = 2\mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + P(E_k;\Omega). \tag{3.11}$$

Since $E' \subset E$, repeating the same argument we have also in this case

$$P(E';\Omega) + P(E \setminus E';\Omega) = 2\mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}) + P(E;\Omega). \tag{3.12}$$

Finally if we call $l := \liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)})$ (without loss of generality we can assume $l \in \mathbb{R}$), using (3.11) and the lower semicontinuity of the perimeter on Ω, we can write

$$\begin{aligned} 2\mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}) + P(E;\Omega) &= P(E';\Omega) + P(E \setminus E';\Omega) \\ &\leq \liminf_{k\to\infty} P(E'_k;\Omega) + P(E_k \setminus E'_k;\Omega) \\ &= \liminf_{k\to\infty} 2\mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + P(E_k;\Omega) \\ &= 2l + \lim_{k\to\infty} P(E_k;\Omega) \\ &= 2l + P(E;\Omega), \end{aligned}$$

which is our desired result. □

Remark 3.1.2. *If the* (E_k) *of the previous proposition are open sets, say for example* (U_k), *such that* $\mathcal{H}^{n-1}(U_k^{(1)}\Delta U_k)=0$ *then for every* k $P(E_k';U_k)=\mathcal{H}^{n-1}(\partial^* E_k'\cap U_k^{(1)})$, *and by the previous lower semi-continuity result we have*

$$\liminf_{k\to\infty} P(E_k';U_k)\geq P(E';U),$$

where we have also used $\mathcal{H}^{n-1}(\partial^* E'\cap U^{(1)})\geq P(E';U)$. ■

With the next definition we introduce the *upper isoperimetric profile.*

Definition 3.1.3 (Upper isoperimetric profile)**.** *Let* Ω *be an open set of* $\mathbb{R}^n$ $(n\geq 2)$ *and let* F *be* an indecomposable set *of* Ω *with* $|F|<\infty$. *For every* $\lambda\in(0,1/2]$ *we define*

$$h_F(\lambda):=\inf\left\{\frac{\mathcal{H}^{n-1}(\partial^* E\cap F^{(1)})}{|E|}\;\middle|\; E\subset F,\ \lambda|F|\leq|E|\leq|F|/2,\ P(E;\Omega)<\infty\right\}. \tag{3.13}$$

We call the function $h_F\colon(0,1/2]\to\mathbb{R}^+$ *the* upper isoperimetric profile *of* F.

Remark 3.1.4. *The upper isoperimetric profile is a non decreasing function. Moreover, if we take an indecomposable open set* $U\subset\Omega$ *such that* $|U|<\infty$, *and* $\mathcal{H}^{n-1}(U^{(1)}\Delta U)=0$, *then* (3.13) *reduces to*

$$h_U(\lambda):=\inf\left\{\frac{P(E;U)}{|E|}\;\middle|\; E\subset U,\ \lambda|U|\leq|E|\leq|U|/2,\ P(E;\Omega)<\infty\right\}.$$

Notice that $\inf_{\lambda>0}h_U(\lambda)$ *is not the Cheeger's constant in* (3.3), *since we look only at the relative perimeter of* E *inside* U, *while in* (3.3) *one is interested in the whole perimeter of* E.

Notice also that in literature (in particular in the context of Riemannian manifolds) the isoperimetric profile at λ *is defined by considering the infimum among all sets* E *with fixed volume* $|E|=\lambda|F|$. *Since we ask for* $|E|\geq\lambda|F|$ *we decide to call it upper isoperimetric profile.* ■

Finally, the next proposition is the core result of this subsection.

Proposition 3.1.5. *Let* Ω *be an open set of* $\mathbb{R}^n$ $(n\geq 2)$ *and let* F *be an indecomposable set of* Ω *with* $|F|<\infty$. *Then* $h_F(\lambda)>0$ *for every* $\lambda\in(0,1/2]$.

In particular, it holds the following relative isoperimetric inequality

$$|E|\leq\frac{1}{h_F(\lambda)}\,\mathcal{H}^{n-1}(\partial^* E\cap F^{(1)}), \tag{3.14}$$

for every $E\subset F$ *with* $\lambda|F|\leq|E|\leq|F|/2$ *and* $P(E;\Omega)<\infty$.

Proof. Let $\lambda\in(0,1/2]$ and consider

$$h_F(\lambda)=\inf_{\substack{\lambda|F|\leq|E|\leq|F|/2\\ E\subset F}}\frac{\mathcal{H}^{n-1}(\partial^* E\cap F^{(1)})}{|E|}. \tag{3.15}$$

Clearly $h_F(\lambda)$ is finite. We want to show that it is strictly positive. Consider a minimizing sequence $(E_k)_{k\in\mathbb{N}}$ i.e.

$$h_F(\lambda)=\lim_{k\to\infty}\frac{\mathcal{H}^{n-1}(\partial^* E_k\cap F^{(1)})}{|E_k|};$$

since

$$\begin{aligned} P(E_k;\Omega) &= \mathcal{H}^{n-1}(\partial^* E_k \cap F^{(1)}) + \mathcal{H}^{n-1}(\{\nu_F = \nu_{E_k}\}) \\ &\leq P(F;\Omega) + (h_F(\lambda)+\epsilon)|E_k| \\ &\leq P(F;\Omega) + (h_F(\lambda)+\epsilon)(|F|/2), \end{aligned}$$

then by using [2, Theorem 3.39], up to subsequence there exists a set $E_\infty \subset F$ having finite perimeter with $\lambda|F| \leq |E_\infty| \leq |F|/2$ and such that $\lim_{k\to\infty} |E_k \Delta E_\infty| = 0$. Moreover thanks to Proposition 3.1.5 we have

$$h_F(\lambda) = \lim_{k\to\infty} \frac{\mathcal{H}^{n-1}(\partial^* E_k \cap F^{(1)})}{|E_k|} \geq \frac{\mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)})}{|E_\infty|},$$

which means

$$h_F(\lambda) = \frac{\mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)})}{|E_\infty|}.$$

Finally, since $\lambda|F| \leq |E_\infty| \leq |F|/2$ and $F = E_\infty \cup (F \setminus E_\infty)$, by Proposition (1.3.8), the indecomposability of F forces $\mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)}) > 0$. This concludes the proof. □

Remark 3.1.6. *Notice that* $\inf_{\lambda>0} h_F(\lambda)$ *might be equal to zero. Indeed consider two sequences of positive real numbers* $(l_n)_{n=1}^\infty$ *and* $(\delta_n)_{n=1}^\infty$ *such that* $\sum_{n=1}^\infty l_n^2 < \infty$ *and* $\lim_{n\to\infty} \delta_n / l_n^2 = 0$. *Define an open set* $U \subset \mathbb{R}^2$ *made of an union of disjoint open squares* Q_n *of side* l_n, *each connected to an open big rectangle through small bridges of size* δ_n *as in figure* (3.1).

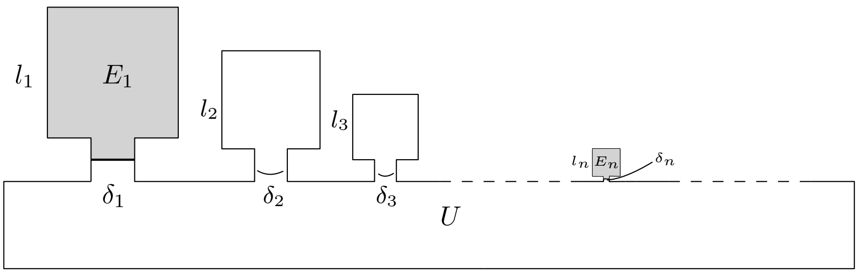

Figure 3.1: *Indecomposable set* U *with* $\inf_{\lambda>0} h_U(\lambda) = 0$.

By our choice of l_n, U *is a connected open set with finite perimeter, hence by Remark 1.3.6 it is indecomposable.*

For every $n \in \mathbb{N}$ *we define* $E_n \subset U$ *to be the square of side* l_n *union half of the* n*-th bridge as in figure* (3.1). *By our choice of* l_n *and* δ_n *we have*

$$\inf_{\lambda>0} h_U(\lambda) \leq \inf_{n\in\mathbb{N}} \frac{\mathcal{H}^1(\partial^* E_n \cap U^{(1)})}{|E_n|} = 0.$$

However, Proposition 3.1.5 tells us that this can happen only for sequences (E_n) *such that* $|E_n| \to 0$.

Moreover, by using the Coarea Formula, it can be proved that $\inf_{\lambda>0} h_F(\lambda) > 0$ *if and only if for every* $u \in BV(\Omega)$ *the following Poincaré's inequality holds true*

$$\int_F |u-m|\, dx \leq c\,|Du|(F^{(1)}),$$

where m *is the median of* u *on* F *(see Definition 3.1.8). In this case the best constant* c *which satisfies the previous inequality is exactly* $\inf_{\lambda>0} h_F(\lambda)$. ■

Remark 3.1.7. *Given F an indecomposable set of $\mathbb{R}^n$, then simply by definition, we have the following scaling property of the relative upper isoperimetric profile:*

$$h_F(\cdot) = r\, h_{\frac{F-x}{r}}(\cdot),$$

for every $r > 0$, $x \in \mathbb{R}^n$. ■

3.1.2 Weak Poincaré's inequality

We are now in position to prove the weak version of Poincaré's inequality (3.2). Before we need the following definition.

Definition 3.1.8. *Let $u\colon \Omega \to \overline{\mathbb{R}}$ be a measurable function. Given a measurable set $F \subset \Omega$ we define the* median *of u in F as*

$$m(u, F) := \inf\left\{ t \in \mathbb{R} \mid |\{u > t\} \cap F| \leq \frac{|F|}{2} \right\}.$$

Remark 3.1.9. *It holds*

$$|\{u > t\} \cap F| \leq \frac{|F|}{2} \text{ for } t \geq m(u, F), \qquad |\{u > t\} \cap F| > \frac{|F|}{2} \text{ for } t < m(u, F). \tag{3.16}$$

■

Theorem 3.1.10. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Given an indecomposable set $F \subset \Omega$ with $|F| < \infty$ and $\mathcal{H}^{n-1}(\Gamma \cap F^{(1)}) = 0$, then for every $u \in GSBV^p(\Omega; \Gamma)$ $(p \geq 1)$ and for every $\lambda \in (0, 1/2]$, there exists a measurable set $F^\lambda \subset F$ such that*

$$|F \setminus F^\lambda| \leq 2\lambda |F|, \tag{3.17}$$

and the following inequality holds true

$$\left(\int_{F^\lambda} |u - m|^p \, dx \right)^{\frac{1}{p}} \leq \frac{p}{h_F(\lambda)} \left(\int_{F^\lambda} |\nabla u|^p \, dx \right)^{\frac{1}{p}}, \tag{3.18}$$

where $m := m(u, F)$.

Proof. Let $v \in GSBV^p(\Omega; \Gamma)$ be a positive function such that

$$|\{v > t\} \cap F| \leq \frac{|F|}{2} \text{ for every } t > 0. \tag{3.19}$$

Define

$$s := \inf\{t : |\{v > t\} \cap F| \leq \lambda |F|\}, \tag{3.20}$$

and notice that

$$|\{v > t\} \cap F| \leq \lambda |F| \text{ for } t \geq s, \qquad |\{v > t\} \cap F| > \lambda |F| \text{ for } t < s. \tag{3.21}$$

If we set $v^s := v \wedge s$ we can write

$$\int_{F \cap \{v \leq s\}} v \, dx \leq \int_F v^s \, dx = \int_0^s |F \cap \{v^s > t\}| \, dt. \tag{3.22}$$

Since $\{v > t\} = \{v^s > t\}$ for every $t \in (0, s)$ and $v^s \in SBV(\Omega)$, then by (3.19), (3.21), and the definition of $h_F(\cdot)$ we have

$$\mathcal{H}^{n-1}(\partial^*\{v^s > t\} \cap F^{(1)}) = \mathcal{H}^{n-1}(\partial^*\{v > t\} \cap F^{(1)}) \geq h_F(\lambda)|F \cap \{v > t\}| = h_F(\lambda)|F \cap \{v^s > t\}|.$$

Then by (3.22) we can use the Coarea Formula for BV functions (see [2, Theorem 3.40]) to obtain

$$\begin{aligned} \int_{F^{(1)}\cap\{v\leq s\}} v\,dx &\leq \frac{1}{h_F(\lambda)}\int_0^s \mathcal{H}^{n-1}(\partial^*\{v^s > t\} \cap F^{(1)})\,dt \\ &= \frac{1}{h_F(\lambda)}|Dv^s|\big(F^{(1)}\big) \\ &= \frac{1}{h_F(\lambda)}\int_{F^{(1)}\cap\{v\leq s\}} |\nabla v|\;dx, \end{aligned} \tag{3.23}$$

where for the last equality we have used $\mathcal{H}^{n-1}(\Gamma \cap F^{(1)}) = 0$ together with the decomposition of the variation measure in BV.

Now define $(u-m)_+^p := [(u-m) \vee 0]^p$. Since by (3.16)

$$|\{(u-m)_+^p > t\} \cap F| \leq \frac{|F|}{2} \text{ for } t > 0,$$

we can apply (3.23) to the function $(u-m)_+^p$ instead of v to deduce that there exists $s^+ \geq 0$ satisfying (3.21) (where v is replaced by $(u-m)_+^p$ and s by s^+) and such that, thanks to the chain rule in BV (see [2, Theorem 3.99]), we can write

$$\int_{F\cap\{0<(u-m)_+^p\leq s^+\}} (u-m)_+^p\,dx \leq \frac{p}{h_F(\lambda)}\int_{F\cap\{0<(u-m)_+^p\leq s^+\}} (u-m)_+^{p-1}|\nabla u|\,dx \tag{3.24}$$

where we used that both integrals vanish on the set $\{(u-m)_+^p = 0\}$ and that $|F\Delta F^{(1)}| = 0$. Analogously, if we set $(u-m)_-^p := |(u-m) \wedge 0|^p$ by (3.16)

$$|\{(u-m)_-^p > t\} \cap F| \leq \frac{|F|}{2}, \text{ for } t > 0.$$

Arguing as before there exists $s^- > 0$ such that

$$\int_{F\cap\{0<(u-m)_-^p\leq s^-\}} (u-m)_-^p\,dx \leq \frac{p}{h_F(\lambda)}\int_{F\cap\{0<(u-m)_-^p\leq s^-\}} (u-m)_-^{p-1}|\nabla u|\,dx. \tag{3.25}$$

If we set $F^\lambda := \{m - (s^-)^{1/p} \leq u \leq m + (s^+)^{1/p}\} \cap F$ by (3.21) we have $|F \setminus F^\lambda| \leq 2\lambda|F|$. By summing the previous two inequalities and by using Hölder inequality we deduce

$$\left(\int_{F^\lambda} |u-m|^p\,dx\right)^{\frac{1}{p}} \leq \frac{p}{h_F(\lambda)}\left(\int_{F^\lambda} |\nabla u|^p\,dx\right)^{\frac{1}{p}}, \tag{3.26}$$

which immediately implies (3.18).

□

Corollary 3.1.11 (Weak Poincaré's inequality)**.** *Under the same hypothesis of Theorem 3.1.10 we have for every $\lambda \in (0, 1/2]$ and for every $u \in GSBV^p(\Omega;\Gamma)$ with $\mathcal{H}^{n-1}(\Gamma \cap F^{(1)}) = 0$*

$$\left(\int_F |u-m|^p \wedge 1\,dx\right)^{\frac{1}{p}} \leq \frac{p}{h_F(\lambda)}\left(\int_F |\nabla u|^p\;dx\right)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}}, \tag{3.27}$$

where $m := m(u, F)$.

Proof. Given $u \in GSBV^p(\Omega;\Gamma)$, we can consider F^λ and m as in Theorem 3.1.10. Then we can write

$$\begin{aligned}\left(\int_F |u-m|^p \wedge 1\, dx\right)^{\frac{1}{p}} &\leq \left(\int_{F^\lambda} |u-m|^p\, dx\right)^{\frac{1}{p}} + |F\setminus F^\lambda|^{\frac{1}{p}} \\ &\leq \frac{p}{h_F(\lambda)}\left(\int_F |\nabla u|^p\, dx\right)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}},\end{aligned} \tag{3.28}$$

which is exactly (3.27). □

3.2 The class $\mathcal{J}_p$

In this chapter we define the class of admissible jump sets $\mathcal{J}_p$ for which Theorems 1 and 2 hold true. Before starting with the definition we need two results concerning the decomposabilty property of sets with finite perimeter.

3.2.1 Indecomposable components of sets with finite perimeter

The following result is a well known fact about the decomposability property of sets of finite perimeter. Precisely, every set with finite perimeter E can be decomposed into a countable family of indecomposable sets (F_i) such that

$$\mathbb{1}_E = \sum_{i=1}^{\infty} \mathbb{1}_{F_i}, \quad \text{and} \quad P(E) = \sum_{i=1}^{\infty} P(F_i).$$

This result has been first announced (with a skecth of the proof) in [35, Subsection 4.2.25] in the more general setting of integral currents of $\mathbb{R}^n$. A complete proof in the context of sets of finite perimeter in $\mathbb{R}^n$ can be found in [52]. We are interested in the same result when $\mathbb{R}^n$ is replaced by a generic Lipschitz-regular open set Ω. Namely, whenever $E \subset \Omega$ is such that $P(E;\Omega) < \infty$, then there exists a countable family of indecomposable subset of Ω, say (F_i), such that

$$\mathbb{1}_E = \sum_{i=1}^{\infty} \mathbb{1}_{F_i}, \quad \text{and} \quad P(E;\Omega) = \sum_{i=1}^{\infty} P(F_i;\Omega).$$

This fact can be deduced by using [52, Lemma 3.1, Corollary 3.2, Proposition 3.3]. We decide to give a complete proof in the next proposition.

Proposition 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz-regular domain, and let $E \subset \Omega$ be such that $P(E;\Omega) < \infty$. Then there exists a Caccioppoli's indecomposable partition of E, which means a countable family $(F_i)_{i=1}^\infty$ of indecomposable sets such that*

1. $E \cap F_i^{(1)} = F_i$, *for every* $i \in \mathbb{N}$;
2. $\mathcal{H}^{n-1}(E \cap E^{(1)} \setminus \bigcup_{i=1}^\infty F_i) = 0$;
3. $|F_i \cap F_j| = 0$, *for* $i \neq j$;
4. $\sum_{i=1}^\infty P(F_i;\Omega) = P(E;\Omega)$;
5. $\mathcal{H}^{n-1}((\Omega \cap \partial^* E) \setminus \bigcup_{i=1}^\infty \partial^* F_i) = 0$;
6. $\mathcal{H}^{n-1}((\Omega \cap \partial^* F_i) \setminus \partial^* E) = 0$ *for every* $i \in \mathbb{N}$;

7. $\mathcal{H}^{n-1}(\Omega \cap \partial^* F_i \cap \partial^* F_j) = 0$ *for* $i \neq j$*.*

Moreover the family $(F_i)_{i=1}^\infty$ *is unique up to permutation of indices in the sense that given any family of indecomposable sets* $(F_i')_{i=1}^\infty$ *satisfying 1-4 then there exists a bijection* $\pi\colon \mathbb{N} \to \mathbb{N}$ *such that*

$$|F_i \Delta F'_{\pi(i)}| = 0 \text{ for every } i \in \mathbb{N}.$$

Remark 3.2.2. *Conditions 2 and 3 say in a more precise way that* $\mathbb{1}_E = \sum_{i=1}^\infty \mathbb{1}_{F_i}$*.* ■

Proof. We first prove that conditions 1-3 hold true. Since Ω is Lipschitz-regular, we know that E is a set of finite perimeter in $\mathbb{R}^n$, i.e. $P(E) < \infty$ (see [64, Subsection 6.5.1 Lemma 1]). By applying [52, Proposition 3.3] to the set $E \subset \mathbb{R}^n$, we deduce that there exists a countable family $(\tilde{F}_i)_{i\in I}$ satisfying 1 and 2 with the additional property $\tilde{F}_i \neq \emptyset$ for every $i \in I$, and

$$\sum_{i\in I} P(\tilde{F}_i) = P(E). \tag{3.29}$$

Now if the cardinality of I is a natural number N, we define $F_i := \tilde{F}_{\pi(i)}$ where π is any bijection of $\{1, \dots, N\}$ onto I, and $F_i = \emptyset$ for every $i > N$. While if the cardinality of I is equal to the cardinality of $\mathbb{N}$, we define $F_i := \tilde{F}_{\pi(i)}$ where π is any bijection of $\mathbb{N}$ onto I. Clearly the family $(F_i)_{i=1}^\infty$ satisfies 1 and 2. We show that it satisfies also 3. Indeed, we can use [52, Lemma 3.1] which says that

$$\mathcal{H}^{n-1}\left([\Omega \cap (\partial^* E \cup E^{(1)})] \setminus \bigcup_{i=1}^\infty \partial^* F_i \cup F_i^{(1)}\right) = 0,$$

where now $\partial^* E$ has to be intended as the reduced boundary of E as a subset of $\mathbb{R}^n$. Since $F_i^{(1)} \cap \partial^* E = \emptyset$ for every i, the only possibility is that

$$\mathcal{H}^{n-1}\left(\partial^* E \setminus \bigcup_{i=1}^\infty \partial^* F_i\right) = 0, \tag{3.30}$$

which by (3.29) implies

$$\bigcup_{i=1}^\infty \partial^* F_i = \partial^* E \ \text{ and } \ \mathcal{H}^{n-1}(\partial^* F_i \cap \partial^* F_j) = 0, \text{ for } i \neq j,$$

and in particular that

$$\mathcal{H}^{n-1}(\partial^* F_i \setminus \partial^* E) = 0, \text{ for every } i \in \mathbb{N}. \tag{3.31}$$

If $|F_i \cap F_j| > 0$ for some $i \neq j$ then by using Proposition 1.3.8, we deduce that $\mathcal{H}^{n-1}(\partial^* F_i \cap F_j^{(1)}) > 0$ which is in contradiction with (3.31) and shows condition 3.

We claim that

$$\sum_{i=1}^\infty P(F_i; \Omega) = P(E; \Omega).$$

To show this, notice that by applying the Leibniz's formula (1.2) to the couple of set Ω and E (both seen as set with finite perimeter in $\mathbb{R}^n$), since $E \subset \Omega$ we can write

$$\mathcal{H}^{n-1} \llcorner \partial^* E = \mathcal{H}^{n-1} \llcorner (\partial^*(E \cap \Omega)) = \mathcal{H}^{n-1} \llcorner (\partial^* E \cap \Omega^{(1)}) + \mathcal{H}^{n-1} \llcorner (\{\nu_E = \nu_\Omega\}), \tag{3.32}$$

and since by 1 we have $F_i \subset \Omega$ for every $i \in \mathbb{N}$, then we have also

$$\mathcal{H}^{n-1} \llcorner \partial^* F_i = \mathcal{H}^{n-1} \llcorner (\partial^*(F_i \cap \Omega)) = \mathcal{H}^{n-1} \llcorner (\partial^* F_i \cap \Omega^{(1)}) + \mathcal{H}^{n-1} \llcorner (\{\nu_{F_i} = \nu_\Omega\}). \tag{3.33}$$

By using [52, Corollary 3.2] together with (3.32) and (3.33) we deduce that

$$\mathcal{H}^{n-1}\Big(\{\nu_E = \nu_\Omega\} \setminus \bigcup_{i=1}^{\infty} \{\nu_{F_i} = \nu_\Omega\}\Big) = 0. \tag{3.34}$$

We can write

$$P(E) = \mathcal{H}^{n-1}(\partial^* E) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega^{(1)}) + \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\}) = P(E;\Omega) + \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\})$$

By using the lower semicontinuity of the perimeter and (3.34) we can continue the previous inequality

$$\begin{aligned} P(E) = P(E;\Omega) + \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\}) &\leq \sum_{i=1}^{\infty} P(F_i;\Omega) + \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) \\ &= \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\partial^* F_i \cap \Omega^{(1)}) + \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) \\ &= \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\partial^* F_i) = \sum_{i=1}^{\infty} P(F_i) = P(E), \end{aligned}$$

where we have also used that, since Ω is Lipschitz-regular, then $\mathcal{H}^{n-1}(\partial^* F_i \cap \Omega^{(1)}) = P(F_i;\Omega)$. By using again the lower semicontinuity of the perimeter and (3.34), we deduce that the only possibility for which (3.2.1) is actually an equality is that

$$P(E;\Omega) = \sum_{i=1}^{\infty} P(F_i;\Omega) \quad \text{and} \quad \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\}) = \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}),$$

which in particular implies our claim.

Properties 5-7 simply follows by [52, Corollary 3.2].

It remains to prove that a family of indecomposable sets $(F_i)_{i=1}^{\infty}$ satisfying 1-4 is unique. As before, since Ω is Lipschitz-regular, then the (F_i) are actually sets having finite perimeter in $\mathbb{R}^n$. Then, in view of [52, Proposition 3.3] it is enough to prove that

$$\sum_{i=1}^{\infty} P(F_i) = P(E). \tag{3.35}$$

In this case, suppose that (F_i) and (F_i') are two sequences of sets satisfying 1-4. By removing the sets in (F_i) and in (F_i') equal to the emptyset we end up we two families $(F_i)_{i\in I}$ and $(F_i')_{i\in I'}$ both satisfying 1-4 we the additional condition $F_i, F_i' \neq \emptyset$ for every $i \in I$ and $i \in I'$. Therefore, we are in position to apply the uniqueness of [52, Proposition 3.3] which says that there exists a bijection $\pi\colon I \to I'$ such that

$$|F_i \Delta F'_{\pi(i)}| = 0, \text{ for every } i \in I,$$

and this would be enough to obtain the uniqueness.

Now we show (3.35). Since by hypothesis

$$\sum_{i=1}^{\infty} P(F_i;\Omega) = P(E;\Omega),$$

by (3.32) and by (3.33) together with the fact $\mathcal{H}^{n-1}(\partial^* F_i \cap \Omega^{(1)}) = P(F_i;\Omega)$, it is enough to show

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) = \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\})$$

First of all since $|F_i \cap F_j| = 0$ it is easy to see that

$$\mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\} \cap \{\nu_{F_j} = \nu_\Omega\}) = 0 \text{ for } i \neq j. \tag{3.36}$$

In particular this implies that

$$\sum_{i=1}^{\infty} P(F_i) = \sum_{i=1}^{\infty} P(F_i;\Omega) + \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) < \infty. \tag{3.37}$$

We claim that

$$\mathcal{H}^{n-1}\left(\{\nu_E = \nu_\Omega\} \Delta \bigcup_{i=1}^{\infty} \{\nu_{F_i} = \nu_\Omega\}\right) = 0.$$

By using (3.37) we can apply [52, Lemma 3.1] and arguing as before this implies (3.34). To prove

$$\mathcal{H}^{n-1}\left(\bigcup_{i=1}^{\infty} \{\nu_{F_i} = \nu_\Omega\} \setminus \{\nu_E = \nu_\Omega\}\right) = 0,$$

we show that $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_i} = \nu_\Omega\}$ belongs to $\partial^* E$ for every $i \in \mathbb{N}$. For this purpose, define $\Gamma := \bigcup_{i=1}^{\infty} \partial^* F_i$. Thanks to (3.37) Γ is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. By properties 1-4, we can apply [52, Corollary 3.2], to deduce that

$$\mathcal{H}^{n-1}(\Omega \cap \partial^* F_i \cap \partial^* F_j) = 0 \text{ for } i \neq j.$$

This together with (3.36) tells us that

$$\mathcal{H}^{n-1}(\partial^* F_i \cap \partial^* F_j) = 0 \text{ for } i \neq j. \tag{3.38}$$

This last condition allows us to define an orientation of Γ, namely a measurable map $\nu\colon \Gamma \to \mathbb{S}^{n-1}$, in the following way

$$\nu(x) := \nu_{F_i}(x), \ \text{ for } x \in \partial^* F_i. \tag{3.39}$$

Therefore, if we set $u_i := \mathbb{1}_{\cup_{j=1}^{i} F_j}$, then we have $u_i \in SBV_1^1(\Omega;\Gamma)$ and $u_i \rightharpoonup u$ with respect to the notion of convergence (2.108) as $i \to \infty$. By applying Remark 2.2.9 we deduce

$$u_i^{\pm} \to u^{\pm} \text{ in } \mathcal{H}^{n-1}\text{-measure on } \Gamma. \tag{3.40}$$

Now fix $i_0 \in \mathbb{N}$. By (3.38) and the definition of $u^{\pm}$ (see Definition 1.4.1), for $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$ $u_i^+(x) = 1$ and $u_i^-(x) = 0$ for every i. Hence, by (3.40) this means also that $u^+(x) = 1$ and $u^-(x) = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$. By definition of $u^{\pm}$ (see Definition 1.4.3) we deduce that $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$ is a point of density $1/2$ for E, and by Theorem 1.3.2 also that $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$ belongs to $\partial^* E$. This concludes the proof.

□

Definition 3.2.3 (Indecomposable components)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \Omega$ with $P(E;\Omega) < \infty$. Let $(F_i)_{i=1}^{\infty}$ be the unique (up to permutation of indices) indecomposable partition of E given by Proposition 3.2.1. Then, for every $i \in \mathbb{N}$ we say that F_i is an* indecomposable components *of E.*

The following proposition says that whenever $|E_r \Delta E| \to 0$, and $P(E_r;\Omega) \to P(E;\Omega)$ then for every r it is possible to select an indecomposable components F_r of E_r, such that $|F_r \Delta E| \to 0$ and $P(F_r;\Omega) \to P(E;\Omega)$.

Proposition 3.2.4. *Let Ω be a bounded Lipschitz regular domain of $\mathbb{R}^n$, and let $(E_r)_{r\in(0,1)}$ be a family of sets contained in Ω with $P(E_r;\Omega) < \infty$. For each $r \in (0,1)$ let $(F_{r,i})_{i=1}^{\infty}$ be the Caccioppoli's indecomposable partition of E_r given by Proposition 3.2.1. Let $E_0 \subset \Omega$ be an indecomposable set. Suppose that*

1. $\lim_{r\to 0^+} |E_r \Delta E_0| = 0$
2. $\lim_{r\to 0^+} P(E_r;\Omega) = P(E_0;\Omega)$.

Then, for each $r \in (0,1)$ there exists $\sigma_r \in \mathbb{N}$ such that

$$\lim_{r\to 0^+} |F_{r,\sigma_r} \Delta E_0| = 0, \tag{3.41}$$

and

$$\lim_{r\to 0^+} P(F_{r,\sigma_r};\Omega) = P(E_0;\Omega). \tag{3.42}$$

Proof. Suppose that our proposition does not hold. Then there exists a $\delta > 0$ such that

$$\limsup_{r\to 0^+} \left(\inf_{i\in\mathbb{N}} |F_{r,i} \Delta E_0| \right) \geq \delta.$$

This implies the existence of a subsequence $(r_m)_{m=1}^{\infty}$ such that

$$|F_{r_m,i} \Delta E_0| > \delta, \tag{3.43}$$

for every $m \in \mathbb{N}$ and for every $i \in \mathbb{N}$.

Consider the Caccioppoli's partitions of Ω made of $(F_{r_m,i})_{i=1}^{\infty} \cup \Omega \setminus E_{r_m}$. Since Ω has finite Lebesgue measure, these partitions can be ordered. Thus we can apply the compactness theorem for Caccioppoli's ordered partition (see [2, Theorem, 4.19] and [2, Remark, 4.20]), to find a Caccioppoli's (ordered) partition of Ω, say $(F_{0,i})_{i=1}^{\infty}$ where one of the $F_{0,i}$ must be equal to $\Omega \setminus E_0$, such that up to subsequences we have

$$\lim_{m\to\infty} |F_{r_m,i} \Delta F_{0,i}| = 0 \text{ for every } i \in \mathbb{N}. \tag{3.44}$$

By removing the set $(\Omega \setminus E_0)$ from the partition, we obtain a ordered measurable partition of E_0, which we still call $(F_{0,i})_{i=1}^{\infty}$.

By (3.43), there exists a family $I \subset \mathbb{N}$ with cardinality strictly greater than 1, such that $F_{0,i} \neq \emptyset$ for every $i \in I$.

Using the lower semicontinuity of the perimeter and (3) of Proposition 3.2.1, we can write

$$\begin{aligned} \sum_{i\in I} P(F_{0,i};\Omega) \leq \sum_{i=0}^{\infty} \liminf_{m\to\infty} P(F_{k_m,i};\Omega) &\leq \liminf_{m\to\infty} \sum_{i=1}^{\infty} P(F_{k_m,i};\Omega) \\ &\leq \liminf_{m\to\infty} P(E_{k_m};\Omega) \\ &= P(E_0;\Omega), \end{aligned} \tag{3.45}$$

since $(F_{0,i})_{i\in I}$ is a (measurable) partition of E_0, (3.45) implies

$$\sum_{i\in I} P(F_{0,i};\Omega) = P(E_0;\Omega), \tag{3.46}$$

and by Remark 1.3.7 this is a contradiction with the indecomposability of E_0, hence this proves (3.41).

Finally we notice that

$$\begin{aligned} P(E_0;\Omega) &\le \liminf_{r\to 0^+} P(F_{r,\sigma_r};\Omega) \le \limsup_{r\to 0^+} P(F_{r,\sigma_r};\Omega) \\ &\le \limsup_{r\to 0^+} \sum_{i=0}^{\infty} P(F_{r,i};\Omega) \\ &= \limsup_{r\to 0^+} P(E_r;\Omega) \\ &= P(E_0;\Omega), \end{aligned}$$

and this gives (3.42). □

3.2.2 The property of non vanishing upper isoperimetric profile

In this section we present the notion of *non vanishing upper isoperimetric profile*. Before we need the following definitions.

Definition 3.2.5. *We say that a set $A \subset B_1(0)$ is* conical*, if*

$$|(A\cap \lambda A)\Delta \lambda A| = 0 \text{ for every } \lambda \in (0,1).$$

Definition 3.2.6. *Given $(A_r)_{r>0}$ a family of subsets of Ω, we say that it is left or right continuous, if for every $r_0 > 0$ we have*

$$\lim_{r\to r_0^-} |A_r \Delta A_{r_0}| = 0 \qquad or \qquad \lim_{r\to r_0^+} |A_r \Delta A_{r_0}| = 0,$$

respectively.

Moreover, given an open set $\Omega \subset \mathbb{R}^n$, given a set $A \subset \Omega$, and given a ball $B_r(x) \subset \Omega$ we will use the following notation

$$A_{r,x} := \frac{A-x}{r} \cap B_1(0),$$

and we will always make use of the following identity

$$A_{\lambda r,x} = \frac{A_{r,x}\cap B_\lambda(0)}{\lambda},\ \lambda \in (0,1].$$

For a given function $u\colon \Omega \to \mathbb{R}$, we define $u_{r,x}\colon B_1(0) \to \mathbb{R}$ as

$$u_{r,x}(y) := u(x+ry),\ y \in B_1(0).$$

Definition 3.2.7 (Non vanishing upper isoperimetric profile)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$. Given $x \in \Omega$ we say that Γ has a non vanishing upper isoperimetric profile at x if there exists $N_x \in \mathbb{N}$ such that*

(1) for every $1 \le j \le N_x$ there exists $(F_{r,j})_{0<r\le r_x}$ ($r_x > 0$) a left-continuous family of indecomposable subsets of $B_1(0)$, with the following properties

(1.1) $\mathcal{H}^{n-1}(\Gamma_{r,x} \cap F^{(1)}_{r,j}) = 0,\ r \in (0,r_x)$;

(1.2) $\liminf_{r\to 0^+} h_{F_{r,j}}(\lambda) > 0, \quad \lambda \in (0, 1/2]$;

(2) *there exists a measurable partition of* $B_1(0)$ *made of (nonempty) conical sets* $(E_{0,j})_{j=1}^{N_x}$ *with the following property*

(2.1) $\lim_{r\to 0^+} |F_{r,j} \Delta E_{0,j}| = 0$.

Remark 3.2.8. *In order to prevent misunderstandings, since* $F_{r,j}$ *are subsets of the unitary ball, in the definition of* $h_{F_{r,j}}$ *the infimum in (3.13) has to be taken among all sets with finite perimeter in* B_1. *Moreover, for a given* Γ *nor the family* $(F_{r,j})$ *neither* $(E_{0,j})$ *are unique. Nevertheless, if* Γ *has a non vanishing upper isoperimetric profile at* x, *then there exists the minimum number* $N_x (\geq 1)$ *for which (1) and (2) hold, and this number clearly depends on the geometry of* Γ. ■

Remark 3.2.9. *The property of non vanishing upper isoperimetric profile is stable under inclusion, in the sense that whenever* $\Gamma' \subset \Gamma$ *and* Γ *has a non vanishing upper isoperimetric profile at* x, *then also* Γ' *has the same property at* x. ■

We give a basic example which clarifies the concept of non vanishing upper isoperimetric profile.

Example 3.2.10. *Let* $M \subset \Omega$ *be an* $(n-1)$*-dimensional manifold of class* C^1. *Then* M *has a non vanishing upper isoperimetric profile for every* $x \in \Omega$. *To show this, let us first suppose* $x \in M$. *Then if we call* $\nu(x)$ *a unit normal to* M *at* x, *we know that there exists a sufficiently small value* $r_x > 0$ *and a* C^1 *function* $f \colon \nu(x)^\perp \to \mathbb{R}$ *such that*

$$B_r(x) \cap M = B_r(x) \cap graph(f), \;\; r \in (0, r_x).$$

By writing the generic $y \in \mathbb{R}^n$ *as* $y = (z, t)$ *where* $y \in \nu(x)^\perp$ *and* $t \in \mathbb{R}$, *we define*

$$F_1 := \{y \in B_{r_x}(x) \mid t < f(z)\} \qquad F_2 := \{y \in B_{r_x}(x) \mid t > f(z)\},$$

and

$$N_x = 2, \qquad F_{r,1} := (F_1)_{r,x}, \qquad F_{r,2} := (F_2)_{r,x}, \qquad r \in (0, r_x);$$

$$E_{0,1} := \{y \in B_1(0) \mid \nu(x) \cdot y < 0\}, \qquad E_{0,2} := \{y \in B_1(0) \mid \nu(x) \cdot y > 0\}.$$

To prove that the families $(F_{r,j})_{0<r<r_x}$ *are left-continuous* $(j = 1, 2)$, *it is equivalent to prove that for every* $r \in (0, r_x)$ *it holds*

$$\lim_{\lambda \to 1^-} \|\mathbb{1}_{F_{\lambda r,j}} - \mathbb{1}_{F_{r,j}}\|_{L^1(B_1)} = 0. \tag{3.47}$$

Since we have

$$\mathbb{1}_{F_{\lambda r,j}}(x) = \mathbb{1}_{F_{r,j}}(\lambda x), \quad x \in B_1(0), \quad j = 1, 2,$$

this means that the convergence (3.47) can be rewritten as

$$\lim_{\lambda \to 1^-} \int_{B_1} |\mathbb{1}_{F_{r,j}}(\lambda x) - \mathbb{1}_{F_{r,j}}(x)|\, dx = 0, \quad j = 1, 2,$$

and this last convergence follows by the continuity of the dilations in L^1. *Conditions (1.1),(2) and (2.1) follow easily by construction.*

To prove condition (1.2), one can use the C^1 regularity of f and an argument similar to the one in Example 3.5.5, to deduce that the open sets $F_{r,j}$ $(j = 1, 2)$ admit a Poincaré's inequality of the form

$$\int_{F_{r,j}} \left| u - \fint_{F_{r,j}} u \right| dx \leq c\,|Du|(F_{r,j}), \ u \in BV(B_1) \tag{3.48}$$

where $c > 0$ is a constant independent on $r \in (0, r_x)$. So given $E \subset F_{r,j}$ a set of finite perimeter in B_1, we can use $\mathbb{1}_E$ instead of u in (3.48) to deduce that

$$\min\{|E|, |F_{r,j} \setminus E|\} \leq c\,|D\mathbb{1}_E|(F_{r,j}) = c\,\mathcal{H}^{n-1}(\partial^* E \cap F_{r,j}) = c\,\mathcal{H}^{n-1}(\partial^* E \cap F_{r,j}^{(1)}),$$

where the right-most equality follows from the fact $F_{r,j} = F_{r,j}^{(1)}$. This implies

$$\liminf_{r \to 0^+} h_{F_{r,j}}(\lambda) \geq \frac{1}{c},$$

for every $\lambda \in (0, 1/2]$.

Another possibility to prove that M has a non-vanishing upper isoperimetric profile at $x \in M$, is to notice that, since M is an $(n-1)$-manifold of class C^1, we can always find a set of finite perimeter $E \subset \Omega$ such that $M \subset \partial^* E$. In this case we can make use of Proposition 3.2.12, which says that $\partial^* E$ admits a non-vanishing upper isoperimetric profile at every point $x \in \partial^* E$. Since the property of non-vanishing upper isoperimetric profile is stable under inclusion (Remark 3.2.9), this means that also M has this property for every $x \in M$.

Finally, the case $x \in \Omega \setminus M$ is much easier. Indeed, by the closeness of M there exists $r_x > 0$ small enough such that $B_r(x) \cap M = \emptyset$ for every $r \in (0, r_x)$. Then it is enough to set

$$N_x = 1, \qquad F_{r_1} := B_1(0), \qquad r \in (0, r_x);$$
$$E_{0,1} := B_1(0).$$

△

Now we are in position to introduce the space of all the admissible jump sets Γ.

Definition 3.2.11 (Admissible jump sets)**.** *Let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$ and let $1 < p \leq$. We say that Γ belongs to $\mathcal{J}_p$ if for every $x \in \Omega \setminus S_\Gamma$, where S_Γ is a set of Hausdorff dimension at most $n - p$, Γ has a non vanishing upper isoperimetric profile at x.*

We will use the next two propositions to construct example of sets living in $\mathcal{J}_p$ (see Chapter 3.5).

Proposition 3.2.12. *Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \Omega$ with $P(E; \Omega) < \infty$. Then the reduced boundary $\partial^* E$ has a non vanishing upper isoperimetric profile for every point x in the following set*

$$\{x \in \Omega \mid \Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = 0\} \cup \partial^* E.$$

Proof. First we deal with the case $x \in \partial^* E$.

We denote as H the half space given by Theorem 1.3.1 such that

$$\lim_{r \to 0^+} |(E_{r,x} \Delta H) \cap B_1(0)| = 0, \qquad \lim_{r \to 0^+} P(E_{r,x}; B_1(0)) = P(H; B_1(0)). \tag{3.49}$$

Clearly $H \cap B_1(0)$ and $B_1(0) \setminus H$ are conical and indecomposable sets. Thus we can apply Proposition 3.2.4, to find two familes $F_{r,1}$ and $F_{r,2}$ made of indecomposable components of $E_{r,x}$ and $B_1(0) \setminus E_{r,x}$, respectively, such that

$$\lim_{r\to 0^+} |(F_{r,1}\Delta H) \cap B_1(0)| = 0, \quad \lim_{r\to 0^+} P(F_{r,1}; B_1(0)) = P(E_{0,1}; B_1(0)), \tag{3.50}$$

$$\lim_{r\to 0^+} |F_{r,2}\Delta(B_1(0) \setminus H)| = 0, \quad \lim_{r\to 0^+} P(F_{r,2}; B_1(0)) = P(E_{0,2}; B_1(0)). \tag{3.51}$$

Given $r_x > 0$ such that $B_{r_x}(x) \subset \Omega$, we set

$$E_1 := E \cap B_{r_x}(x), \quad E_2 := B_{r_x}(x) \setminus E,$$

and

$$E_{0,1} = H \cap B_1(0), \quad E_{0,2} = B_1(0) \setminus H.$$

Clearly $E_{0,1}, E_{0,2}$ are conical and indecomposable sets. This choice guarantees also (1.1) and (2.1) of Definition 3.2.7. Moreover, using that for every r the sets $F_{r,j}$ are indecomposable components of $(E_j)_{r,x}$ $(j = 1, 2)$, respectively, together with (3.49), we can apply Proposition 3.2.13 to deduce that the family $r \to F_{r,j}$ are left-continuous.

Finally, in order to show (1.2) of Definition 3.2.7, we claim that

$$\liminf_{r\to 0^+} h_{F_{r,1}}(\lambda) \geq h_H(\lambda), \;\; \lambda \in (0, 1/2], \tag{3.52}$$

and

$$\liminf_{r\to 0^+} h_{F_{r,2}}(\lambda) \geq h_{B_1\setminus H}(\lambda), \;\; \lambda \in (0, 1/2]. \tag{3.53}$$

We prove for example (3.52). To this purpose fix $\lambda \in \big(0, \frac{1}{2}\big]$ and for every $r \in (0, r_x)$ consider $E_r \subset F_{r,1}$ with $P(E_r; B_1) < \infty$, such that

$$\frac{\mathcal{H}^{n-1}(\partial^* E_r \cap F_{r,1}^{(1)})}{|E_r|} \leq h_{F_{r,1}}(\lambda) + r, \qquad \lambda |F_{r,1}| \leq |E_r| \leq |F_{r,1}|/2. \tag{3.54}$$

We show that for every subsequence (r_m) such that $r_m \to 0^+$ as $m \to \infty$ then

$$\liminf_{m\to\infty} h_{F_{r_m,1}}(\lambda) \geq h_H(\lambda).$$

Without loss of generality we assume

$$\liminf_{m\to\infty} h_{F_{r_m,1}}(\lambda) = \lim_{m\to\infty} h_{F_{r_m,1}}(\lambda) = l < \infty.$$

Since $E_{r_m} \subset F_{r_m,1}$, by using Leibniz's formula 1.3.3 the inequalities (3.54) say to us

$$\sup_m P(E_{r_m}; B_1) \leq \sup_m [\, |E_{r_m}|\, h_{F_{r_m,1}}(\lambda) + P(F_{r_m,1}; B_1)] < \infty.$$

This means that, thanks to the compactness result [2, Theorem 3.39], eventually passing through another subsequence we have $\lim_{m\to\infty} |E_{r_m} \Delta E_0| = 0$, for some sets $E_0 \subset H$ with finite perimeter in $B_1(0)$ and with $\lambda |H| \leq |E_0| \leq |H|/2$. Hence thanks to (3.50), we are in position to apply the lower semicontinuity result of Proposition 3.1.1 to obtain

$$\liminf_{m\to\infty} h_{F_{r_m,1}}(\lambda) \geq \liminf_{m\to\infty} \frac{\mathcal{H}^{n-1}(E_{r_m} \cap F_{r_m,1}^{(1)})}{|E_{r_m}|} - r_m \geq \frac{\mathcal{H}^{n-1}(E_0 \cap H^{(1)})}{|E_0|} \geq h_H(\lambda).$$

The same argument shows the validity of (3.53). Since $h_H(\lambda) > 0$, this says that $\partial^* E$ admits a non-vanishing upper isoperimetric profile at x with $N_x = 2$.

In the case $x \in \Omega$ is such that $\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = 0$, we claim that we have two different sub-cases:

$$\lim_{r \to 0^+} |B_1(0) \setminus E_{r,x}| = 0, \tag{3.55}$$

or

$$\lim_{r \to 0^+} |E_{r,x}| = 0. \tag{3.56}$$

Indeed by a simple application of the relative isoperimetric inequality in the unitary ball we can write

$$\min\{|E_{x,r}|, |B_1(0) \setminus E_{x,r}|\}^{\frac{n-1}{n}} \leq C(n) \frac{P(E; B_1(0))}{r^{n-1}} = C(n) \frac{\mathcal{H}^{n-1}(\partial^* E \cap B_1(0))}{r^{n-1}},$$

and by the fact that $r \mapsto |E_{x,r}|$ is a continuous map on $(0, r_x)$ we deduce that one between (3.55) and (3.56) must occur. Suppose for example (3.55) holds. Given $r_x > 0$ such that $B_{r_x}(x) \subset \Omega$, we set

$$E_1 := E \cap B_{r_x}(x), \qquad E_{0,1} = B_1(0).$$

Arguing in the very same way as before, we can make use of Proposition 3.2.4 to find for every $r \in (0, r_x)$ an indecomposable component of $(E_1)_{r,x}$, say $F_{r,1}$, which form a left-continuous family and such that

$$\lim_{r \to 0^+} |F_{r,1} \Delta B_1(0)| = 0, \qquad \lim_{r \to 0^+} P(F_{r,1}; B_1) = 0.$$

Finally, by using again Proposition 3.1.1 we can prove in the very same way as before that

$$\liminf_{r \to 0^+} h_{F_{r,1}}(\lambda) \geq h_{B_1}(\lambda), \;\; \lambda \in (0, 1/2].$$

Case (3.56) can be treated in the same way. □

The following proposition gives a sufficient condition for which a family of indecomposable sets $(F_r)_{r>0}$ is left-continuous in the sense of Definition 3.2.6.

Proposition 3.2.13. *Let Ω be an open set of $\mathbb{R}^n$, let $E \subset \Omega$ with $P(E; \Omega) < \infty$, and let $r_x > 0$ be such that $B_{r_x}(x) \subset \Omega$. Suppose that there exists $E_0 \subset B_1(0)$, $|E_0| > 0$ and such that*

$$\lim_{r \to 0^+} |E_{r,x} \Delta E_0| = 0.$$

If F_r is an indecomposable component of $E_{r,x}$ $(0 < r < r_x)$ satisfying

$$\lim_{r \to 0^+} |F_r \Delta E_0| = 0,$$

then there exists $0 < r'_x \leq r_x$, such that the family $(F_r)_{r \in (0, r'_x)}$ is continuous from the left.

Proof. Fix $r \in (0, r_x)$ and define

$$\mathcal{F}_{\lambda r} := \frac{F_r \cap B_\lambda(0)}{\lambda}, \;\; \lambda \in (0, 1]. \tag{3.57}$$

Notice that arguing as in Example 3.2.10 it holds $\lim_{\lambda \to 1^-} |\mathcal{F}_{\lambda r} \Delta F_r| = 0$. In order to simplify the notation we will write $E_r := E_{r,x}$.

By applying Proposition 1.3.9 to the triple $E_{\lambda r}, \mathcal{F}_{\lambda r}, F_{\lambda r}$, since $\partial^*\mathcal{F}_{\lambda r} \subset \partial^* E_{\lambda r}$, we deduce that for every λ one only one of the following can hold

$$F_{\lambda r} \subseteq \mathcal{F}_{\lambda r} \text{ or } F_{\lambda r} \subseteq E_{\lambda r} \setminus \mathcal{F}_{\lambda r}, \quad (\mathcal{L}^n\text{-a.e.}). \tag{3.58}$$

We claim that there exists $0 < r''_x < r_x$ such that for every $r < r''_x$ there exists $0 < \delta''_r < 1$ such that

$$F_{\lambda r} \subset \mathcal{F}_{\lambda r} \quad \lambda \in (1-\delta''_r, 1], \quad (\mathcal{L}^n\text{-a.e.}). \tag{3.59}$$

Indeed let r''_x be small enough in such a way that $|F_r \Delta E_0| \leq |E_0|/8$ and $|E_r \setminus E_0| \leq |E_0|/8$. We define for every $0 < r < r''_x$ a real number $\delta''_r \in (0,1)$ such that $|\mathcal{F}_{\lambda r} \Delta F_r| \leq |E_0|/8$ for $\lambda \in (1-\delta''_r, 1]$. If with this choice of r''_x and δ''_r inclusion (3.59) does not hold, then there exists a positive r with $r < r''_x$ and a $\lambda \in (1-\delta''_r, 1]$ such that $F_{\lambda r} \subset E_{\lambda r} \setminus \mathcal{F}_{\lambda r}$, therefore

$$\begin{aligned}
\frac{7|E_0|}{8} \leq |E_0| - |F_{\lambda r}\Delta E_0| \leq |F_{\lambda r}| \leq |E_{\lambda r} \setminus \mathcal{F}_{\lambda r}| &\leq |E_{\lambda r}\Delta E_0 \setminus \mathcal{F}_{\lambda r}| + |E_0 \setminus \mathcal{F}_{\lambda r}| \\
&\leq |E_{\lambda r}\Delta E_0| + |E_0 \setminus F_r| + |F_r \setminus \mathcal{F}_{\lambda r}| \\
&\leq \frac{3|E_0|}{8},
\end{aligned}$$

which is a contradiction and proves our claim.

Moreover, notice that

$$\lim_{\lambda \to 1^-} P(\mathcal{F}_{\lambda r}, B_1(0)) = \lim_{\lambda \to 1^-} \frac{1}{\lambda^{n-1}} P(F_r, B_\lambda(0)) = P(F_r, B_1(0)),$$

where for the last equality we used the continuity of the Radon measure $P(F_r, \cdot)$ with respect to monotone sequences. Therefore, since

1. $\lim_{\lambda\to 1^-} |\mathcal{F}_{\lambda r}\Delta F_r| = 0$;
2. $\lim_{\lambda\to 1^-} P(\mathcal{F}_{\lambda r}, B_1(0)) = P(F_r, B_1(0))$;
3. F_r is indecomposable,

we can apply Theorem 3.2.4 to deduce the existence of a sequence $(\tilde{F}_{\lambda r})_{\lambda\in(0,1)}$ such that for each λ $\tilde{F}_{\lambda r}$ is an element of the Caccioppoli's indecomposable partition of $\mathcal{F}_{\lambda r}$ and

$$\lim_{\lambda\to 1^-} |\tilde{F}_{\lambda r}\Delta F_r| = 0.$$

We claim that there exists $0 < r'_x < r_x$ such that for every $r < r'_x$ there exists $0 < \delta'_r < 1$ such that

$$\tilde{F}_{\lambda r} = F_{\lambda r} \quad \lambda \in (1-\delta'_r, 1], \quad (\mathcal{L}^n\text{-a.e.}).$$

Indeed, since $F_{\lambda r} \subset \mathcal{F}_{\lambda r}$ for every $r < r''_x$ and every $\lambda \in (1-\delta''_x, 1]$, since $F_{\lambda r}$ is indecomposable, and since $\partial^*\tilde{F}_{\lambda r} \subset \partial^*\mathcal{F}_{\lambda r}$ $\mathcal{H}^{n-1}$-a.e. (here we use property (5) for indecomposable components Proposition 3.2.1), we can apply Proposition 1.3.9 to the triple $\tilde{F}_{\lambda r}, F_{\lambda r}, \mathcal{F}_{\lambda r}$, to obtain that one and only one of the following holds

$$F_{\lambda r} \subset \tilde{F}_{\lambda r} \text{ or } F_{\lambda r} \subset \mathcal{F}_{\lambda r} \setminus \tilde{F}_{\lambda r}, \quad (\mathcal{L}^n\text{-a.e.}).$$

Since for every $r < r_x$ $|\mathcal{F}_{\lambda r}\Delta\tilde{F}_{\lambda r}| \to 0$ as $\lambda \to 1^-$ and $|F_r\Delta E_0| \to 0$ as $r \to 0^+$, arguing in the very same way as before, we can find $0 < r'_x < r''_x$ such that for every $r < r'_x$ there exists $0 < \delta'_r < \delta''_r$ such that

$$F_{\lambda r} \subset \tilde{F}_{\lambda r} \quad \lambda \in (1-\delta'_x, 1] \quad (\mathcal{L}^n\text{-a.e.}).$$

We want to prove that $F_{\lambda r} = \tilde{F}_{\lambda r}$ for every $r < r'_x$ and every $\lambda \in (1 - \delta'_r, 1]$. It is enough to show

$$|\tilde{F}_{\lambda r} \setminus F_{\lambda r}| = 0. \tag{3.60}$$

But since

$$\mathcal{F}^{(1)}_{\lambda r} = \frac{F^{(1)}_r \cap B_\lambda(0)}{\lambda} \subset \frac{E^{(1)}_r \cap B_\lambda(0)}{\lambda} = E^{(1)}_{\lambda r},$$

and since $F_{\lambda r}$ is an indecomposable component of $E_{\lambda r}$, by property (5) of Proposition 3.2.1 we must have $\mathcal{H}^{n-1}(\partial^* F_{\lambda r} \cap \mathcal{F}^{(1)}_{\lambda r}) = 0$. As a consequence, since $\tilde{F}^{(1)}_{\lambda r} \subset \mathcal{F}^{(1)}_{\lambda r}$, then also

$$\mathcal{H}^{n-1}(\partial^* F_{\lambda r} \cap \tilde{F}^{(1)}_{\lambda r}) = 0. \tag{3.61}$$

If (3.60) does not hold, since $\tilde{F}_{\lambda r} = F_{\lambda r} \cup \tilde{F}_{\lambda r} \setminus F_{\lambda r}$, by using Proposition 1.3.8 and the indecomposability of $\tilde{F}_{\lambda r}$, immediately follows

$$\mathcal{H}^{n-1}(\partial^* F_{\lambda r} \cap \tilde{F}^{(1)}_{\lambda r}) > 0,$$

which is in contradiction with (3.61) and proves the theorem. □

Remark 3.2.14. *By Proposition 3.2.12, the reduced boundary $\partial^* E$ of a set $E \subset \Omega$ with finite perimeter such that* $\dim_H(\Omega \setminus \partial^* E \cup \{x \mid \Theta(\mathcal{H}^{n-1} \llcorner \partial^* E; x) = 0\}) = n - p$*, belongs to $\mathcal{J}_p$.* ■

3.3 Properties of the blow-up in $GSBV^p(\Omega)$

This chapter contains the proof of Theorem 1. We proceed following four main steps: first we show that whenever Γ admits a non vanishing upper isoperimetric profile at x, then the medians $m_j(u, r, x)$ (see Definition 3.3.1) are left-countinuous functions of r for every $u \in GSBV^p(\Omega; \Gamma)$ $(1 < p \leq n)$; then we show that there exist suitable subsequences of radii (r_i) with $r_i \to 0^+$ as $i \to \infty$, such that if $\Gamma \in \mathcal{J}_p$ then for every $x \in \Omega$, up to a set of Hausdorff dimension $n - p$, the limit $\lim_{i \to \infty} m_j(u, r_i, x)$ exists and it is finite; combining this result with the left-continuity of $r \mapsto m_j(u, r, x)$ we are able to show that actually *the full limit*, i.e. $\lim_{r \to 0^+} m_j(u, r, x)$, exists and is finite; finally, this last information together with the weak Poincarè's inequality on balls (see Theorem 3.3.3) enables us to deduce the main theorem.

A significant effort is required to prove the convergence of the medians $m_j(r, u, x)$ as $r \to 0^+$ (Section 3.3.2). This requires a careful analysis of the behavior of the function u on points close to a single point x. In the classical theory, i.e. in the context of Sobolev functions, this issue is overcome by introducing the functional capacity (see [36]). This allows the authors to bypass the problem, and only as a consequence of their main result [36, Theorem 9], which is the analogous of Theorem 1, they can (eventually) deduce the convergence of the medians. However this approach rely on the fact that smooth functions are dense in $W^{1,p}(\Omega)$ while, at the best of our knowledge, it is not known whether a similar density result holds in $GSBV^p(\Omega; \Gamma)$ (see Remark 3.4.23). For this reason we adopt a different strategy which requires the study of the medians $m_j(u, r, x)$ as functions of r.

3.3.1 Weak Poincaré's inequality on balls

We start this section by proving a weak version of Poincare's inequality on balls. First, we need the following definition.

Definition 3.3.1. *Let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$ and let $x \in \Omega \setminus S_\Gamma$. Let $r_x > 0$ and $N_x \in \mathbb{N}$ be given by Definition 3.2.7. We define for every $r \in (0, r_x)$, $\overline{u}_{r,x} \colon B_{r_x}(x) \to \mathbb{R}$ as*

$$\overline{u}_{r,x}(y) := \begin{cases} m_j(u,r,x) & \text{on } x + rF_{r,j} \\ 0 & \text{otherwise.} \end{cases}$$

where $m_j(u,r,x) := m(u, x + rF_{r,j})$ (see Definition 3.1.8), and $(F_{r,j})_{j=1}^{N_x}$ are the indecomposable sets given by Definition 3.2.7.

Remark 3.3.2. *The median of u in F is invariant under rescaling and translation in the sense that*

$$m(u, F) = m(u_{r,x}, (F - x)/r), \;\; x \in \Omega, \;\; 0 < r < r_x.$$

This means that

$$m_j(u, r, x) = m(u_{r,x}, F_{r,j}),$$

where $m_j(u,r,x)$ is the one introduced in the previous definition. ■

Theorem 3.3.3 (Weak Poincare's inequality on balls)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Suppose that $\Gamma \subset \Omega$ has a non vanishing upper isoperimetric profile at x, then for every $\lambda \in (0, 1/2]$ there exists $r_\lambda > 0$ (depending also on x) such that*

$$\left(\int_{B_r(x)} |u - \overline{u}_{r,x}|^p \wedge 1 \, dy \right)^{\frac{1}{p}} \leq C(p,n) \left[\mathscr{H}_x(\lambda) r \left(\int_{B_r(x)} |\nabla u|^p \, dy \right)^{\frac{1}{p}} + (r^n \lambda)^{\frac{1}{p}} \right], \tag{3.62}$$

for every $r \leq r_\lambda$ and for every $u \in GSBV^p(\Omega; \Gamma)$, where

$$\mathscr{H}_x(\lambda) := \limsup_{r \to 0^+} \left[\max_{j=1,\dots,N_x} \left\{ \frac{2}{h_{F_{r,j}}(\lambda)} \right\} \right] < \infty. \tag{3.63}$$

Remark 3.3.4. *Poincaré-Wirtinger type inequality on balls for functions in SBV can be found in the seminal paper [29], and then generalised also to the $GSBV$-case in [10] (see also [17, 44] for Poincarè's-type inequalities in SBD). The inequality in [29] analyse the behavior of an SBV function u in balls B when the measure $\mathcal{H}^{n-1}(J_u \cap B)$ is small compared to the size of the ball. In particular it allows to control the oscillation of u (truncated at suitable levels depending on the size of $\mathcal{H}^{n-1}(J_u \cap B)$) in an L^q sense, only with the L^p-norm of ∇u neglecting the jump part of the distributional gradient. In this way in the limit case $\mathcal{H}^{n-1}(J_u \cap B) = 0$ it becomes the usual Poincaré's inequality for Sobolev functions. Precisely, if we denote with γ the constant of the relative isoperimetric inequality on balls, then whenever $u \in SBV(B)$ is such that*

$$(2\gamma \mathcal{H}^{n-1}(J_u \cap B))^{\frac{n}{n-1}} < \frac{|B|}{2}, \tag{3.64}$$

the following Poincaré-Wirtinger inequality holds true ($p < n$)

$$\left(\int_B |\overline{u} - m|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq C(n,p,\gamma) \left(\int_B |\nabla u|^p \, dx \right)^{\frac{1}{p}}, \tag{3.65}$$

where $\overline{u}$ is the function u truncated from above and from below at suitable levels (see [2, Subsection 4.3]) and m is the median of u on B.

The main difference between inequality (3.65) and (3.62) is that even if in the right hand side of (3.65) only the approximate gradient of u is present, the jump part of the derivative is implicitly involved through inequality (3.64). On the contrary, in (3.62) we can get rid of the bound (3.64). For instance, in Example 3.5.5 it is showed that there are functions $u \in SBV(\mathbb{R}^2)$ and balls $B_r(x)$ such that $\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner J_u, x) = +\infty$ while inequality (3.62) still holds true on $B_r(x)$ at any scale $r > 0$.

Moreover inequality (3.65) describe the fact that an SBV functions u, on points x for which $\Theta^{*(n-1)}(\mathcal{H}^{n-1}, \llcorner J_u, x) = 0$, behaves asymptotically (as $r \to 0^+$) as a Sobolev functions, i.e. its values on $B_r(x)$ are close to a single constant (the median on $B_r(x)$) in an L^q sense. In our case, we are interested to capture also the behavior of u on $B_r(x)$ when at the limit it behaves like a piecewise constant function. For this reason we need inequality (3.62). ■

Proof. Fix $\lambda > 0$ and let $x \in \Omega \setminus S_\Gamma$. By property (2.1) of Definition 3.2.7 we know that there exists $0 < r'_\lambda < r_x$ such that for every $r < r'_\lambda$

$$\sup_{r<r'_\lambda} |B_1(0) \setminus \bigcup_{j=1}^{N_x} F_{r,j}| \leq \lambda,$$

which means

$$\sup_{r<r'_\lambda} |B_r(x) \setminus \bigcup_{j=1}^{N_x} (x + rF_{r,j})| \leq r^n \lambda. \tag{3.66}$$

Moreover by the definition of lim sup we can consider r''_λ small enough such that

$$\sup_{r<r''_\lambda} \left[\max_{j=1,\dots,N_x} \left\{ \frac{1}{h_{F_{r,j}}(\lambda)} \right\} \right] \leq \mathscr{H}_x(\lambda) < \infty.$$

Since $u_{r,x} \in GSBV^p(B_1(0); \Gamma_{r,x})$ $(r < r_x)$ and thanks to the fact $\mathcal{H}^{n-1}(\Gamma_{r,x} \cap F_{r,j}^{(1)}) = 0$ for every $1 \leq j \leq N_x$, by applying Theorem 3.1.10 we know that there exist $F_{r,j}^\lambda \subset F_{r,j}$ with

$$|F_{r,j} \setminus F_{r,j}^\lambda| \leq \lambda |F_{r,j}|, \tag{3.67}$$

such that

$$\int_{F_{r,j}^\lambda} |u(x + ry) - m_{r,j}|^p \, dy \leq \left(\frac{pr}{h_{F_{r,j}}(\lambda)} \right)^p \int_{F_{r,j}^\lambda} |\nabla u(x + ry)|^p \, dy,$$

where $m_{r,j} := m_j(u, r, x)$.

If we define $F_r^\lambda := \bigcup_{j=1}^{N_x} F_{r,j}^\lambda$, then by summing on $j = 1, \dots, N_x$ both sides of the previous inequality if $r \leq \min\{r'_\lambda, r''_\lambda\}$ we obtain

$$\int_{F_r^\lambda} |u(x + ry) - \overline{u}_{r,x}(x + ry)|^p \, dy \leq (pr\mathscr{H}_x(\lambda))^p \int_{F_r^\lambda} |\nabla u(x + ry)|^p \, dy,$$

or equivalently

$$\int_{x+rF_r^\lambda} |u(y) - \overline{u}_{r,x}|^p \, dy \leq (pr\mathscr{H}_x(\lambda))^p \int_{x+rF_r^\lambda} |\nabla u(y)|^p \, dy.$$

By defining $F_r := \bigcup_{j=1}^{N_x} F_{r,j}$ and by using also (3.66) and (3.67) we can write

$$\begin{aligned}\int_{B_r(x)} |u(y)-\overline{u}_{r,x}|^p \wedge 1\,dy &\le \int_{x+rF_r^\lambda} |u(y)-\overline{u}_{r,x}|^p\,dy + |B_r(x)\setminus(x+rF_r^j)| \\ &\le (pr\mathscr{H}_x(\lambda))^p \int_{x+rF_r^\lambda} |\nabla u(y)|^p\ dy + r^n|F_r\setminus F_r^\lambda| + |B_r(x)\setminus(x+rF_r)| \\ &\le (pr\mathscr{H}_x(\lambda))^p \int_{B_r(x)} |\nabla u(y)|^p\ dy + (1+\omega_n)r^n\lambda \\ &\le C(n,p)\left[(\mathscr{H}_x(\lambda)r)^p \int_{B_r(x)} |\nabla u(y)|^p\ dy + r^n\lambda\right].\end{aligned}$$

which is exactly (3.62). □

Remark 3.3.5. *Under the hypothesis of the previous theorem, in the case the set* Γ *satisfies the stronger condition at* x

- $\bigcup_{j=1}^{N_x} F_{r,j} = B_1(0), \quad r \le r_x;$
- $\liminf_{r\to 0^+} \inf_{\lambda>0} h_{F_{r,j}}(\lambda) > 0, \quad j=1,\dots,N_x.$

then it is not difficult to show that inequality (3.62) *can be improved to*

$$\left(\int_{B_r(x)} |u-\overline{u}_{r,x}|^p\,dx\right)^{\frac{1}{p}} \le C(p,n)r\left(\int_{B_r(x)} |\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

■

3.3.2 Convergence of the medians

We start with a proposition which shows that for points x outside of the singular set S_Γ, the medians $m_j(u,r,x)$ are left-continuous functions of r.

Proposition 3.3.6. *Let* $\Gamma \in \mathscr{J}_p$ $(1 < p \le)$, $x \in \Omega \setminus S_\Gamma$ *and* $u \in GSBV^p(\Omega;\Gamma)$. *Then the maps*

$$r \mapsto m_j(u,r,x), \quad j=1,\dots,N_x,$$

are left-continuous for $r \in (0, r_x)$.

Proof. Let us consider for example $j=1$. Fix $r \in (0,r_x)$, then we want to show

$$\lim_{\lambda\to 1^-} m_1(u,\lambda r,x) = m_1(u,r,x). \tag{3.68}$$

Hence by definition of median, we have to prove that

$$\inf\{t\in\mathbb{R} \mid |\{u_{\lambda r,x} > t\}\cap F_{\lambda r,1}| \le |F_{\lambda r,1}|/2\},$$

converges to

$$\inf\{t\in\mathbb{R} \mid |\{u_{r,x} > t\}\cap F_{r,1}| \le |F_{r,1}|/2\},$$

as $\lambda \to 1^-$. First of all notice that

$$u_{\lambda r,x} \to u_{r,x} \text{ in } \mathcal{L}^n \llcorner B_1(0)\text{- measure.}$$

This implies that for every t except a countable set A, we have

$$|\{u_{\lambda r,x} > t\}\Delta\{u_{r,x} > t\} \cap B_1(0)| \to 0, \quad (\lambda \to 1^-).$$

Since $(F_{r,1})_r$ is a left-continuous family, then $|F_{\lambda r,1}\Delta F_{r,1}| \to 0$ as $\lambda \to 1^-$, we have also that for $t \in \mathbb{R} \setminus A$

$$|\{u_{\lambda r,x} > t\} \cap F_{\lambda r,1}| \to |\{u_{r,x} > t\} \cap F_{r,1}|, \quad (\lambda \to 1^-). \tag{3.69}$$

The convergence (3.69) implies that if $t \in \mathbb{R} \setminus A$ is such that $|\{u_{r,x} > t\} \cap F_{r,1}| < \frac{|F_{r,1}|}{2}$, then for λ close enough to 1^- we have

$$|\{u_{\lambda r,x} > t\} \cap F_{\lambda r,1}| < \frac{|F_{\lambda r,1}|}{2},$$

Analogously if $t \in \mathbb{R} \setminus A$ is such that $|\{u_{r,x} > t\} \cap F_{r,1}| > \frac{|F_{r,1}|}{2}$ then for every λ close enough to 1^- we have

$$|\{u_{\lambda r,x} > t\} \cap F_{\lambda r,1}| > \frac{|F_{\lambda r,1}|}{2}.$$

Therefore, the convergence (3.68) is established once we prove that for every $t > m_1(u_{r,x}, 1, 0)$ it holds

$$|\{u_{r,x} > t\} \cap F_{r,1}| < \frac{|F_{r,1}|}{2} \text{ for } t > m_1(u_{r,x}, 1, 0),$$

and

$$|\{u_{r,x} > t\} \cap F_{r,1}| > \frac{|F_{r,1}|}{2} \text{ for } t < m_1(u_{r,x}, 1, 0).$$

The last condition follows by the definition of median.

To prove the first condition we argue by contradiction. If it does not hold, then there exist $t_1 < t_2$ such that

1. $\{u_{r,x} > t\} \cap F_{r,1} = \{u_{r,x} > s\} \cap F_{r,1}$ for a.e. every $t, s \in (t_1, t_2)$,
2. $|\{u_{r,x} > t\} \cap F_{r,1}| = \frac{|F_{r,1}|}{2}$ for a.e. every $t \in (t_1, t_2)$.

By Coarea Formula there exists $t \in (t_1, t_2)$ such that $\{u_{r,x} > t\}$ is a set of finite perimeter in $B_1(0)$ such that $|\{u_{r,x} > t\} \cap F_{r,1}| = \frac{|F_{r,1}|}{2}$. Since $F_{r,1}$ is indecomposable, Proposition 1.3.8 says to us that $\mathcal{H}^{n-1}(\partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)}) > 0$. We claim that

$$\partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)} \subset S_{u_{r,x}}. \tag{3.70}$$

Indeed, by using Theorem 1.3.2, we know that $\mathcal{H}^{n-1}$-a.e. point $y \in \partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)}$ is a point of density $1/2$ for $\{u_{r,x} > t\}$. By (1) we deduce that every point of density $1/2$ for $\{u_{r,x} > t\}$ is also a point of density $1/2$ for $\{u_{r,x} > s\}$ for every $s \in (t_1, t_2)$. By definition of upper and lower approximate limit (Definition 1.4.1) we deduce that for every $\mathcal{H}^{n-1}$-a.e. $y \in \partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)}$ we have

$$u_{r,x}^-(y) \le t_1 < t_2 \le u_{r,x}^+(y),$$

which proves (3.70).

Finally, since by Theorem 1.4.7 $S_{u_{r,x}} \subset J_{u_{rx}} \subset \Gamma_{r,x}$ ($\mathcal{H}^{n-1}$-a.e.), condition (3.70) is in contradiction with property (1.1) of Definition 3.2.7 and this proves our desired result. □

The next theorem, which is the core result of this section, tells us that when the integrals $\int_{B_r(x)} |\nabla u|^p$ decays properly as $r \to 0^+$, then the medians $m_j(u, r, x)$ are convergent for suitable subsequences of radii $r_i \to 0^+$. In the proof we will use the following inequality which is true for each quadruple of measurable sets $A, B, C, D \subset \Omega$

$$|A\Delta B| \leq |C\Delta D| + |A\Delta C| + |B\Delta D|. \tag{3.71}$$

Theorem 3.3.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$ and let $x \in \Omega \setminus S_\Gamma$. Suppose that there exists some $\delta \in (0, p]$ with the following property*

$$\limsup_{r\to 0^+} \frac{1}{r^{n-p+\delta}} \int_{B_r(x)} |\nabla u|^p \, dx = 0. \tag{3.72}$$

Then for every sequence of radii $(r_i)_{i=1}^\infty$ such that

1. $\left(\frac{1}{2}\right)^{\frac{1}{2n}} < \frac{r_{i+1}}{r_i} \leq 1, \quad i \in \mathbb{N}$;
2. $\sum_{i=1}^\infty (r_i)^{\frac{\delta}{p}} < \infty$;

the sequence of medians $\big(m_j(u, r_i, x)\big)_{i=1}^\infty$ is Cauchy for every $j = 1, \dots, N_x$.

Proof. Choose $j \in \{1, \dots, N_x\}$. In order to simplify the notation we write

$$t_{r_i} := m_j(u, r_i, x) \qquad F_r := F_{r,j} \qquad E_0 := E_{0,j} \qquad a_i := \frac{r_i}{r_{i-1}}.$$

Fix $\epsilon > 0$ such that for every $n \in \mathbb{N}$ it holds $\epsilon \leq \frac{2a_i^n - 1}{4a_i^n + 4}$ (this is possible since by condition (1) it is enough to chose $0 < \epsilon \leq \frac{\sqrt{2}-1}{2\sqrt{2}+4}$), and consider $\bar{i} \in \mathbb{N}$ so big that for every $i \geq \bar{i}$

$$h_{F_{r_i}}(\epsilon) \geq \frac{1}{c(\epsilon)} := \frac{1}{2} \liminf_{i\to\infty} h_{F_{r_i}}(\epsilon) > 0.$$

This is possible by the definition of lim inf.

By using Theorem 3.1.10 with the function $u(x + r(\cdot)) \in GSBV^p(B_1(0); \Gamma_{r,x})$ and the indecomposable set F_{r_i}, we deduce that for every $i \geq \bar{i}$ and for every $\epsilon > 0$, there exists $F^\epsilon_{r_i} \subset F_{r_i} \subset B_1(0)$ such that

$$|F^\epsilon_{r_i}| \geq (1 - 2\epsilon)|F_{r_i}|, \tag{3.73}$$

and

$$\left(\int_{F^\epsilon_{r_i}} |u_{r_i,x} - t_{r_i}|^p \, dy\right)^{\frac{1}{p}} \leq c(\epsilon)p\left(\int_{F^\epsilon_{r_i}} |\nabla u_{r_i,x}|^p \, dy\right)^{\frac{1}{p}}. \tag{3.74}$$

Now for each $i \geq \bar{i}$ define

$$\mathcal{F}_i := a_i F^\epsilon_{r_i} \cap F^\epsilon_{r_{i-1}} \subset B_{a_i}(0).$$

Since $\mathcal{F}_i \subset a_i F_{r_i}$, we can give the following estimate

$$\begin{aligned} |\mathcal{F}_i| = |a_i F^\epsilon_{r_i} \cap F^\epsilon_{r_{i-1}}| &= |a_i F_{r_i} \setminus (a_i F_{r_i} \setminus a_i F^\epsilon_{r_i} \cup a_i F_{r_i} \setminus F^\epsilon_{r_{i-1}})| \\ &\geq |a_i F_{r_i}| - |a_i F_{r_i} \setminus a_i F^\epsilon_{r_i}| - |a_i F_{r_i} \setminus F^\epsilon_{r_{i-1}}|. \end{aligned}$$

By (3.73) and the fact $|F_{r_i} \Delta E_0| \to 0$ we can write

$$|a_i F_{r_i} \setminus a_i F^\epsilon_{r_i}| = a_i^n |F_{r_i} \setminus F^\epsilon_{r_i}| \leq a_i^n 2\epsilon |F_{r_i}| = a_i^n 2\epsilon[|E_0| + o(1)],$$

and by using also inequality (3.71) with $A = F_{r_{i-1}} \cap a_i F_{r_i}$, $B = a_i F_{r_i}$, $C = E_0 \cap a_i E_0$ and $D = a_i E_0$, we can write

$$\begin{aligned}|a_i F_{r_i} \setminus F^{\epsilon}_{r_{i-1}}| &\le |(F_{r_{i-1}} \cap a_i F_{r_i}) \setminus F^{\epsilon}_{r_{i-1}}| + |[(F_{r_{i-1}} \cap a_i F_{r_i}) \Delta a_i F_{r_i}] \setminus F^{\epsilon}_{r_{i-1}}| \\ &\le |F_{r_{i-1}} \setminus F^{\epsilon}_{r_{i-1}}| + |(F_{r_{i-1}} \cap a_i F_{r_i}) \Delta a_i F_{r_i}| \\ &= 2\epsilon |E_0| + |(E_0 \cap a_i E_0) \Delta a_i E_0| + o(1),\end{aligned}$$

and since E_0 is conical, then $|(E_0 \cap a_i E_0) \Delta a_i E_0| = 0$ for every $i \in \mathbb{N}$; as a consequence we can write

$$|a_i F_{r_i} \setminus F^{\epsilon}_{r_{i-1}}| \le 2\epsilon |E_0| + o(1).$$

Putting together our previous estimates we obtain

$$\begin{aligned}|\mathcal{F}_i| &\ge a_i^n |E_0| - a_i^n 2\epsilon |E_0| - 2\epsilon |E_0| + o(1) \\ &= |E_0|(a_i^n - a_i^n \epsilon - 2\epsilon) + o(1).\end{aligned}$$

By our choice of ϵ, we have $a_i^n - \epsilon(2a_i^n + 2) \ge \frac{1}{2}$, hence

$$|\mathcal{F}_i| \ge \frac{1}{2}|E_0| + o(1), \ i \in \mathbb{N}. \tag{3.75}$$

Therefore, for every $i \ge \bar{i}$, we can make use of (3.74), (3.75), and $a_i \le 1$, to deduce the following estimates

$$\begin{aligned}|t_{r_i} - t_{r_{i-1}}|^p &= \fint_{\mathcal{F}_i} |t_{r_i} - t_{r_{i-1}}|^p \, dy \le 2^{p-1} \fint_{\mathcal{F}_i} |u_{r_{i-1},x} - t_{r_i}|^p \, dy + 2^{p-1} \fint_{\mathcal{F}_i} |u_{r_{i-1},x} - t_{r_{i-1}}|^p \, dy \\ &= \frac{2^{p-1} a_i^n}{|\mathcal{F}_i|} \int_{F^{\epsilon}_{r_i}} |u_{r_i,x} - t_{r_i}|^p \, dy + \frac{2^{p-1}}{|\mathcal{F}_i|} \int_{F^{\epsilon}_{r_{i-1}}} |u_{r_{i-1},x} - t_{r_{i-1}}|^p \, dy,\end{aligned}$$

hence by using (3.74) and $a_i \le 1$ there exists $C = C(p, n, \epsilon) > 0$ such that

$$\begin{aligned}|t_{r_i} - t_{r_{i-1}}|^p &\le \frac{C}{|\mathcal{F}_i|} \left[\int_{F^{\epsilon}_{r_i}} |\nabla u_{r_i,x}|^p \, dy + \int_{F^{\epsilon}_{r_{i-1}}} |\nabla u_{r_{i-1},x}|^p \, dy \right] \\ &= \frac{C}{|\mathcal{F}_i|} \left[r_i^p \int_{F^{\epsilon}_{r_i}} |\nabla u(x + r_i y)|^p \, dy + r_{i-1}^p \int_{F^{\epsilon}_{r_{i-1}}} |\nabla u(x + r_{i-1} y)|^p \, dy \right],\end{aligned}$$

and finally by using (3.75) we have

$$\begin{aligned}|t_{r_i} - t_{r_{i-1}}|^p &= \frac{C}{1/2|E_0| + o(1)} \left[\frac{1}{r_i^{n-p}} \int_{x + r_i F^{\epsilon}_{r_i}} |\nabla u|^p \, dx + \frac{1}{r_{i-1}^{n-p}} \int_{x + r_{i-1} F^{\epsilon}_{r_{i-1}}} |\nabla u|^p \, dx \right] \\ &\le \frac{C r_i^{\delta}}{1/2|E_0| + o(1)} \left[\frac{1}{r_i^{n-p+\delta}} \int_{B_{r_i}(x)} |\nabla u|^p \, dx + \frac{(\frac{1}{2})^{-\delta/n}}{r_{i-1}^{n-p+\delta}} \int_{B_{r_{i-1}}(x)} |\nabla u|^p \, dx \right] \\ &\le C' r_i^{\delta},\end{aligned}$$

where, thanks also to (3.72), $C' > 0$ is a constant which depends only on x, j, p, n, ϵ.

These last inequality means

$$\sum_{i \ge \bar{i}}^{\infty} |t_{r_i} - t_{r_{i-1}}| \le C'^{\frac{1}{p}} \sum_{i=1}^{\infty} (r_i)^{\frac{\delta}{p}},$$

and this last series is convergent thanks to our choice of r_i. This implies that the sequence $(t_{r_i})_{i=1}^{\infty}$ is Cauchy. Since $1 \le j \le N_x$ was arbitrary, we prove the theorem.

□

The next proposition shows that for every sequence of radii r_i statisfying (1) and (2) of Theorem 3.3.7, then $\lim_{i\to\infty} m_j(u, r_i, x)$ is actually unique.

Proposition 3.3.8. *Under the hypothesis of Theorem 3.3.7 we have that there exists $l \in \mathbb{R}$ such that*

$$\lim_{i\to\infty} m_j(u, 1/2^{\alpha i}, x) = l,$$

for every $\alpha \in (0, \frac{1}{2n})$.

Proof. The fact that it exists for every $\alpha \in (0, 1/2n)$ is simply a consequence of the fact that the sequences $(1/2^{\alpha i})_{i=1}^{\infty}$ satisfy (1) and (2) of Proposition 3.3.7, hence $\big(m_j(u, 1/2^{\alpha i}, x)\big)_{i=1}^{\infty}$ is a Cauchy sequence.

To show that the limits do not depend on α, pick $0 < \alpha_1 < \alpha_2 < \frac{1}{2n}$, and consider for every $i \in \mathbb{N}$

$$r_i' := \frac{1}{2^{\alpha_1 i}} \quad \text{and} \quad r_i'' := \frac{1}{2^{\alpha_2 i}}.$$

Define a new sequence $(r_i)_{i=1}^{\infty}$ by riordering the $(r_i'), (r_i'')$: set $r_1 := \max\{\frac{1}{2^{\alpha_1}}, \frac{1}{2^{\alpha_2}}\}$, and then inductively

$$r_i := \max\{r \mid r \in (r_k')_{k=1}^{\infty} \cup (r_k'')_{k=1}^{\infty} \setminus (r_j)_{j=1}^{i-1}\}.$$

We want to prove that $(r_i)_{i=1}^{\infty}$ satisfies conditions (1) and (2) of Proposition 3.3.7. Once we prove this, we can apply Proposition 3.3.7 to deduce that $(m_j(u, r_i, x))_{i=1}^{\infty}$ is a Cauchy sequence for every $j = 1, \dots, N_x$ which implies

$$\lim_{i\to\infty} m_j(u, r_i', x) = \lim_{i\to\infty} m_j(u, r_i'', x).$$

To prove (1), notice that for every $i \in \mathbb{N}$

$$\left(\frac{1}{2}\right)^{\frac{1}{2n}} \le \min\left\{\frac{1}{2^{\alpha_1}}, \frac{1}{2^{\alpha_2}}\right\} \le r_i/r_{i-1} \le 1.$$

Indeed $r_i/r_{i-1} \le 1$ simply follows because by construction $r_i \to 0^+$ as $i \to \infty$. To prove the other inequality, suppose for example $r_{i-1} = r_j'$ and $r_i = r_k''$ for some j and k, then this means $r_k'' \ge r_{j+1}'$, and therefore

$$\min\left\{\frac{1}{2^{\alpha_1}}, \frac{1}{2^{\alpha_2}}\right\} \le \frac{r_{j+1}'}{r_j'} \le \frac{r_k''}{r_j'} = \frac{r_i}{r_{i-1}}.$$

The same argument works in the case $r_{i-1} = r_k''$ and $r_i = r_j'$ for some j and k.

To prove (2) it is enough to estimate

$$\lim_{N\to\infty} \sum_{i=1}^{N} (r_i)^{\frac{\delta}{p}} \le \lim_{N\to\infty} \left(\sum_{i=1}^{N} \frac{1}{2^{i\delta\alpha_1/p}} + \sum_{i=1}^{N} \frac{1}{2^{i\delta\alpha_2/p}}\right) = \frac{2^{\delta\alpha_1/p}}{2^{\delta\alpha_1/p} - 1} + \frac{2^{\delta\alpha_2/p}}{2^{\delta\alpha_2/p} - 1} - 2,$$

and this concludes the proof. □

The next proposition is the link between Proposition 3.3.6 and Proposition 3.3.8. It says that in order to compute the limit at ∞ of a right or left continuous function of one variable, *it is enough to compute it on arithmetic progressions.*

Proposition 3.3.9. *Let $f\colon [0,+\infty) \to \mathbb{R}$ be a right (or left) continuous function. Suppose that there exists $l \in \mathbb{R}$ and $0 < a < b < \infty$ such that for every $\alpha \in [a,b]$ it holds*

$$\lim_{\substack{i\to\infty \\ i\in\mathbb{N}}} f(i\alpha) = l. \tag{3.76}$$

Then

$$\lim_{t\to+\infty} f(t) = l.$$

Proof. Suppose not. Then there exists a sequence $t_k \to +\infty$ as $k \to \infty$ and an $\epsilon > 0$ such that

$$|f(t_k) - l| \geq \epsilon, \ \ k \in \mathbb{N}. \tag{3.77}$$

By the right continuity of f for every k there exists $\delta_k > 0$ with

$$|f(t) - f(t_k)| \leq \frac{1}{k}, \ \ t \in (t_k, t_k + \delta_k). \tag{3.78}$$

In order to simplify the notation write $I_k := (t_k, t_k + \delta_k)$.

We claim that there exists an $\alpha \in [a,b]$ such that for infinitely many k there exists an element of the sequence $(i\alpha)_{i=1}^{\infty}$ which belongs to I_k.

To prove the claim, for each $j \in \mathbb{N}$ define $A_j := \{\alpha \in [a,b] \mid \exists i \in \mathbb{N}, \ i\alpha \in I_j\}$, and then set $B_k := \bigcup_{j\geq k} A_j$. Notice that for every $k \in \mathbb{N}$, B_k is an open and dense subset of $[a,b]$. Indeed for every couple $\alpha_1, \alpha_2 \in [a,b]$ with $\alpha_1 < \alpha_2$, there exists $\bar{i} \in \mathbb{N}$ such that

$$(i+1)\alpha_1 < i\alpha_2, \ \ i \geq \bar{i}.$$

This means that the intervals $[i\alpha_1, i\alpha_2]$ overlap each others for $i \geq \bar{i}$, hence

$$(\bar{i}\alpha_1, \infty) \subset \bigcup_{i\geq\bar{i}} [i\alpha_1, i\alpha_2]. \tag{3.79}$$

To see that B_k are dense in $[a,b]$, it is enough to notice that for every open interval $(\alpha_1, \alpha_2) \subset [a,b]$, by (3.79), there must exists a $\alpha \in (\alpha_1, \alpha_2)$, a $j \geq k$ and a $i \geq \bar{i}$ such that $i\alpha \in I_j$ which implies $\alpha \in B_k$.

The fact that B_k are open is even easier. If $i\alpha_0 \in I_k$ for some $i, k \in \mathbb{N}$ and some $\alpha_0 \in [a,b]$, since I_k is open then by continuity of the map $t \mapsto i\,t$ there exists a relative open neighborhood U of α_0 in $[a,b]$, such that for every $\alpha \in U$ $i\alpha \in I_k$.

Using Baire's Lemma on the complete metric space $[a,b]$ we deduce that

$$\bigcap_{k\in\mathbb{N}} B_k \neq \emptyset.$$

So pick $\alpha \in \bigcap_{k\in\mathbb{N}} B_k$, and notice that by definition of B_k, for every k there exists $i_k \in \mathbb{N}$ and $j_k \geq k$ such that $i_k\alpha \in I_{j_k}$. By (3.77) and (3.78) we can write

$$|f(i_k\alpha) - l| \geq |f(t_{j_k}) - l| - |f(t_{j_k}) - f(i_k\alpha)| \geq \epsilon - \frac{1}{j_k}, \ \ (k \in \mathbb{N}).$$

Since by construction $j_k \to \infty$ as $k \to \infty$, the previous inequality contradicts (3.76) and proves the proposition. □

Finally, we are in position to prove the convergence of the medians cited so far at the beginning of this section.

Theorem 3.3.10. *Let Ω be an open set of $\mathbb{R}^n$, let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$ and let $u \in GSBV^p(\Omega;\Gamma)$. Then for every $x \in \Omega$, except on a set of Hausdorff dimension at most $n-p$, $\lim_{r\to 0^+} m_j(u,r,x)$ exists finite for every $1 \leq j \leq N_x$.*

Proof. For every $\delta > 0$, consider $A_\delta \subset \Omega \setminus S_\Gamma$ the set of point x such that

$$\limsup_{r\to 0^+} \frac{1}{r^{n-p+\delta}} \int_{B_r(x)} |\nabla u|^p \, dx > 0. \tag{3.80}$$

By applying for example [34, Theorem 3, Section 2.4.3] we have $\mathcal{H}^{n-p+\delta}(A_\delta) = 0$. Moreover since $A_{\delta_1} \subset A_{\delta_2}$, for $\delta_1 \leq \delta_2$, we have that if we fix $\delta_0 > 0$ then

$$\mathcal{H}^{n-p+\delta_0}\left(\bigcap_{\delta>0} A_\delta\right) = 0.$$

Since $\delta_0 > 0$ is arbitrary we deduce

$$\dim_{\mathcal{H}}\left(\bigcap_{\delta>0} A_\delta\right) = n-p. \tag{3.81}$$

Now pick $x \in \Omega \setminus S_\Gamma$ and $x \notin \bigcap_{\delta>0} A_\delta$. If we define for every $1 \leq j \leq N_x$

$$f(t) := m_j(u, 1/2^t, x), \quad \lambda \geq 0,$$

then by Proposition 3.3.6, $f(t)$ is continuous from the right, and thanks to Proposition 3.3.8, there exists $l \in \mathbb{R}$ such that for every $0 < \alpha \leq \frac{1}{2n}$, we have

$$\lim_{i\to\infty} f(\alpha i) = l.$$

Therefore we are in position to apply Proposition 3.3.9 and to deduce that

$$\lim_{t\to+\infty} f(t) = l.$$

As a consequence, since $\lim_{r\to 0^+} \log_{\frac{1}{2}}(r) = +\infty$ we finally deduce that

$$\lim_{r\to 0^+} m_j(u,r,x) = \lim_{r\to 0^+} f(\log_{\frac{1}{2}}(r)) = \lim_{t\to+\infty} f(t) = l.$$

□

3.3.3 Convergence of the blow-up

Theorem 3.3.11. *Let Ω be an open set of $\mathbb{R}^n$, let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$ and let $u \in GSBV^p(\Omega;\Gamma)$. Then for every $x \in \Omega$ except a set of Hausdorff dimension at most $n-p$, there exists a piecewise constant function $u_x(\cdot)\colon B_1(0) \to \mathbb{R}$ such that*

$$\lim_{r\to 0^+} \int_{B_1(0)} |u_{r,x} - u_x| \wedge 1 \, dy = 0. \tag{3.82}$$

Moreover using the notation of Definitions 3.2.7 and 3.1.8 we have that

$$u_x(y) = m_j(u,x) \text{ if } y \in E_{0,j}, \tag{3.83}$$

where $m_j(u,x) := \lim_{r\to 0^+} m_j(u,r,x)$ for $1 \leq j \leq N_x$.

Proof. Let A_δ $(\delta > 0)$ be the sets defined in the proof of Theorem 3.3.10 and define $A := \bigcap_{\delta>0} A_\delta$. By (3.81) $\dim_{\mathcal{H}}(A) = n - p$, hence also $\dim_{\mathcal{H}}(A \cup S_\Gamma) = n - p$.

We claim that every $x \in \Omega \setminus (S_\Gamma \cup A)$ satisfies (3.82) and (3.83). By Theorem 3.3.10 we know that $\lim_{r\to 0^+} m_j(u, r, x)$ exists for every $1 \le j \le N_x$. Therefore, by defining

$$m_j(u, x) = \lim_{r\to 0^+} m_j(u, r, x), \ 1 \le j \le N_x,$$

and recalling condition (2.1) of Definition 3.2.7 we immediately deduce

$$\lim_{r\to 0^+} \int_{B_1(0)} |\overline{u}_{r,x}(x + ry) - u_x(y)| \wedge 1 \, dy = 0. \tag{3.84}$$

Now in order to prove (3.82), it is enough to use the weak Poincaré's inequality on balls (3.3.3) together with (3.84), to deduce that for every $\lambda \in (0, 1/2]$

$$\begin{aligned} \limsup_{r\to 0^+} \int_{B_1(0)} |u(x + ry) - u_x(y)| \wedge 1 \, dy &\le \limsup_{r\to 0^+} \int_{B_1(0)} |u(x + ry) - \overline{u}_{r,x}(x + ry)| \wedge 1 \, dy \\ = \limsup_{r\to 0^+} \fint_{B_r(x)} |u - \overline{u}_{r,x}| \wedge 1 \, dy &\le C(p, n) \limsup_{r\to 0^+} \left[\mathscr{H}_x(\lambda) \left(\frac{1}{r^{n-p}} \int_{B_r(x)} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + \lambda^{\frac{1}{p}} \right] \\ &\le C(p, n)\lambda^{\frac{1}{p}}. \end{aligned}$$

Finally, by letting $\lambda \to 0^+$ we deduce (3.82). □

Remark 3.3.12. *Since the result of Theorem 3.3.11 is local, then it still holds for the space* $GSBV^p_{loc}(\Omega; \Gamma)$. ■

Remark 3.3.13. *It is not difficult to show that if we substitute condition (1.2) in the definition of non vanishing upper isoperimetric profile with the stronger condition*

$$\bigcup_{j=1}^{N_x} F_{r,j} = B_1(0), \quad (r \le r_x) \quad \textit{and} \quad \liminf_{r\to 0^+} \inf_{\lambda>0} h_{F_{r,j}}(\lambda) > 0, \quad (j = 1, \dots, N_x), \tag{3.85}$$

then, by using Remark 3.3.5, it is possible to show that the convergence (3.82) *actually holds with respect to the* L^p*-convergence (in Example 3.5.5 we construct a non trivial admissible jump set, such that it admits a non vanishing upper isoperimetric profile with the stronger condition* (3.85) *at every point* x *).* ■

Remark 3.3.14. *When we deal with Sobolev spaces, namely* $\Gamma = \emptyset$*, Theorem 1 implies the well known result that given* $u \in W^{1,p}_{loc}(\Omega)$*, then every point* x*, up to a set of Hausdorff dimension at most* $n - p$*, is a Lebesgue point of* u*. The function*

$$u(x) = \log\log |x|^{-1},$$

which belongs to $W^{1,2}(B_1(0))$*,* $B_1(0) \subset \mathbb{R}^2$ *shows that the dimension* $n - p$ *is optimal in Theorem 1.* ■

3.4 A notion of capacity for functions with prescribed jump

This chapter is devoted to the proof of Theorem 2. For this purpose we need to introduce a suitable notion of capacity for functions in $GSBV^p(\Omega; \Gamma)$. Let us recall that given $A \subset \mathbb{R}^n$, the classical p-capacity in the context of Sobolev functions is defined as (see for example [36] or [34])

$$\mathrm{Cap}_p(A) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\, dx \mid u \in K^p,\ u \geq 1 \text{ a.e. in an open neighborhood of } A\right\}, \quad (3.86)$$

where $K^p := \{u\colon \mathbb{R}^n \to \mathbb{R} \mid u \geq 0,\ u \in L^{p*}(\mathbb{R}^n),\ \nabla u \in L^p(\mathbb{R}^n)\}$. Moreover, the following result can be interpreted as a capacitary version of Chebyshev's inequality (see for example [36, Section 7] or [34, Lemma 1, Section 4.8]).

Proposition 3.4.1. *Assume $u \in K^p$ and $\epsilon > 0$. Let*

$$A := \{x \in \mathbb{R}^n \mid m(u,r,x) > \epsilon \textit{ for some } r > 0\},$$

where $m(u,r,x)$ denotes the median of u on $B_r(x)$ (see Definition 3.1.8). Then

$$\mathrm{Cap}_p(A) \leq \frac{c}{\epsilon^p}\int_{\mathbb{R}^n} |\nabla u|^p\, dx,$$

where $c = c(n,p)$.

The previous proposition suggest us that given $A \subset \mathbb{R}^n$, if we define a variant of the p-capacity in the following way

$$\mathrm{Cap}'_p(A) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\, dx \mid u \in K^p,\ \limsup_{r\to 0^+} m(u,r,x) > 1 \text{ for every } x \in A\right\}, \quad (3.87)$$

then

$$\mathrm{Cap}'_p(A) \leq \mathrm{Cap}_p(A) \leq c2^p\, \mathrm{Cap}'_p(A),$$

for some constant $c > 0$ and for every $A \subset \mathbb{R}^n$. Indeed, if $u \in K^p$ is such that $u \geq 1$ a.e. in an open neighborhood of A, clearly u satisfies $\limsup_{r\to 0^+} m(u,r,x) \geq 1$ for every $x \in A$ and we obtain

$$\mathrm{Cap}'_p(A) \leq \mathrm{Cap}_p(A).$$

On the other hand, given $\delta > 0$, let $u \in K^p$ be such that $\limsup_{r\to 0^+} m(u,r,x) \geq 1$ for every $x \in A$ and

$$\int_{\mathbb{R}^n} |\nabla u|^p\, dx < \mathrm{Cap}'_p(A) - \delta.$$

By definition of lim sup for every $x \in A$ there exists r_x such that $m(u,r_x,x) > 1/2$. Therefore

$$A \subset \{x \in \mathbb{R}^n \mid m(u,r,x) > 1/2 \text{ for some } r > 0\},$$

and by the capacitary Chebyshev's inequality the previous inclusion together with the monotonicity of the p-capacity immediately imply

$$\mathrm{Cap}_p(A) \leq c2^p \int_{\mathbb{R}^n} |\nabla u|^p\, dx \leq c2^p(\mathrm{Cap}'_p(A) - \delta).$$

Thanks to the arbitrariness of $\delta > 0$, we deduce

$$\mathrm{Cap}_p(A) \leq c\, \mathrm{Cap}'_p(A).$$

Hence, it is possible to define an equivalent notion of capacity by looking at the medians of u for *every* $x \in A$. Since for technical reason we prefer to define a notion of capacity where the infimum (3.86) does not depend on a *a.e.*-condition, the variant introduced in (3.87) seems to fit better our purpose. However, if we want to mimic definition in (3.87), we should take into account different medians, i.e. $(m_j(u,r,x))_{j=1}^{N_x}$, depending on Γ and x (see Definition 3.3.1). Since we prefer to define a capacity which is a priori independent on Γ, we decide to give a slightly different definition which is based on the notion of approximate limit (see Definition (3.93)).

3.4.1 Convergence with respect to an outer measure

In this subsection we want to fix the notion of convergence with respect to an outer measure and to define a suitable function space which will be useful in view of Theorem 2.

For convenience of the reader we recall the definition of outer measure.

Definition 3.4.2 (Outer measure)**.** *An outer measure on Ω is any set function $\mu\colon \mathcal{P}(\Omega) \to [0,+\infty]$ satisfying the following properties*

(a) $\mu(\emptyset) = 0$*;*

(b) $\mu(A_1) \le \mu(A_2)$*, whenever* $A_1 \subset A_2$ *(monotonicity);*

(c) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i)$ *(countably sub-additivity).*

Definition 3.4.3. *Let $\mu\colon \mathcal{P}(\Omega) \to [0,\infty]$ be an outer measure. Given $A \subset \Omega$, we say that a property $\mathcal{P}(x)$, defined for $x \in A$, holds μ-quasi everywhere, and we use the abbreviation* μ-q.e.*, if there exists a set $N \subset A$, with $\mu(N) = 0$, such that $\mathcal{P}(x)$ holds for every $x \in A \setminus N$.*

We recall that the convergence in measure (see 2.2.1) can be metrized.

Definition 3.4.4. *We denote by $L^0(B_1)$ (see [50]) the Fréchet space of all (equivalence classes of) Lebesgue measurable real-functions on B_1 equipped with the topology of convergence in measure. This topology can be defined for example by the Lévy-metric*

$$\|u - v\|_{L^0(B_1)} := \int_{B_1} |u - v| \wedge 1 \, dx, \;\; u, v \in L^0(B_1). \tag{3.88}$$

Remark 3.4.5. *Notice that, with abuse of notation, we use the symbol $\|u - v\|_{L^0(B_1)}$ to denote the Lévy-metric even if it is not a norm.* ■

By means of Theorem 1, whenever $\Gamma \in \mathcal{J}_p$ to each function $u \in GSBV^p(\Omega;\Gamma)$ we can associate a map $u_{(\cdot)}\colon \Omega \to L^0(B_1)$ defined everywhere except on a set of Hausdorff dimension $n - p$. Given an outer measure μ on Ω, we want to define a space which contains functions defined μ-q.e. from Ω to $L^0(B_1)$, and to endow such a space with a notion of convergence in μ-measure.

Definition 3.4.6. *Let μ be an outer measure on Ω. Let X be the real vector space of all functions $u\colon \Omega \to L^0(B_1)$, and consider the equivalence relation*

$$u \sim v \quad \textit{iff} \quad \mu(\{x \in \Omega \mid u(x) \neq v(x)\}) = 0. \tag{3.89}$$

We define

$$U_\mu(\Omega; L^0(B_1)) := X/\sim,$$

i.e. the space consisting of all equivalence classes obtained as the quotient of X with respect to $\sim$.

Remark 3.4.7. *Notice that, since μ is an outer measure, (3.89) makes sense even without any measurability conditions on the functions u and v.* ■

Definition 3.4.8. *Let μ be an outer measure on Ω, let $(u_k)_{k=1}^{\infty}$ and u be functions in $U_\mu(\Omega; L^0(B_1))$. We say that (u_k) converges to u in* μ-measure *if*

$$\lim_{k\to\infty} \mu(\{x \in \Omega \mid \|u_k - u\|_{L^0(B_1)} > \epsilon\}) = 0, \tag{3.90}$$

for every $\epsilon > 0$.

Convergence in μ-measure implies up to subsequence pointwise convergence μ-q.e.. This is the content of the next proposition.

Proposition 3.4.9. *Let μ be an outer measure on Ω, let $(u_k)_{k=1}^\infty$ and u be functions in $U_\mu(\Omega; L^0(B_1))$. Suppose $u_k \to u$ in μ-measure, then there exists a subsequence (k_j) such that for μ-q.e. $x \in \Omega$*

$$\lim_{j\to\infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0,$$

Proof. For every $j \in \mathbb{N}$ choose $k_j \in \mathbb{N}$ such that

$$\mu\left(\left\{x \in \Omega \mid \|u_{k_j} - u\|_{L^0(B_1)} > \frac{1}{j}\right\}\right) \leq \frac{1}{2^j}.$$

Set $A_j := \left\{x \in \Omega \mid \|u_{k_j} - u\|_{L^0(B_1)} \leq \frac{1}{j}\right\}$, define $B_i := \bigcap_{j\geq i} A_j$ and finally $B := \bigcup_{i=1}^\infty B_i$. Suppose $x \in B$, then $x \in B_i$ for some i and hence $x \in A_j$ for every $j \geq i$. Therefore

$$\|u_{k_j}(x) - u(x)\|_{L^0(B_1)} \leq \frac{1}{j}, \text{ for } j \geq i,$$

which means

$$\lim_{j\to\infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0.$$

Finally we can use the monotonicity and the countable sub-additivity of μ to estimate

$$\mu(\Omega \setminus B) \leq \mu(\Omega \setminus B_i) \leq \sum_{j\geq i} \mu(A_j) \leq \frac{1}{2^{i-1}},$$

and by the arbitrariness of i we deduce $\mu(\Omega \setminus B) = 0$. □

The convergence in μ-measure can be metrized in the following way.

Proposition 3.4.10. *Let μ be an outer measure on Ω with $\mu(\Omega) < \infty$, and let $u, v \in U_\mu(\Omega; L^0(B_1))$. The metric $d(u, v)$ defined by*

$$d(u, v) := \inf_{\delta>0} \mu(\{\|u - v\|_{L^0(B_1)} > \delta\}) + \delta,$$

induces the convergence in measure (3.90), and it gives to $U_\mu(\Omega; L^0(B_1))$ the structure of a complete metric space.

Proof. We start by proving that $d(\cdot, \cdot)$ is a metric.

First of all suppose that $d(u, v) = 0$ then we want to prove that $\mu(\{\|u - v\|_{L^0(B_1)} > 0\}) = 0$. Indeed, if $d(u, v) = 0$, then for every $\delta > 0$, $\mu(\{\|u - v\|_{L^0(B_1)} > \delta\}) = 0$. Since $\{\|u - v\|_{L^0(B_1)} > 0\} = \bigcup_{k=1}^\infty \{\|u - v\|_{L^0(B_1)} > 1/k\}$, by using the monotonicity and countable sub-additivity of μ we can write

$$\begin{aligned} 0 \leq \mu(\{\|u - v\|_{L^0(B_1)} > 0\}) - \mu\Bigg(\bigcup_{k=1}^{\bar{k}} \{\|u - v\|_{L^0(B_1)} > 1/k\}\Bigg) &\leq \mu\Bigg(\bigcup_{k=\bar{k}+1}^\infty \{\|u - v\|_{L^0(B_1)} > 1/k\}\Bigg) \\ &\leq \sum_{k=\bar{k}+1}^\infty \mu(\{\|u - v\|_{L^0(B_1)} > 1/k\}) \\ &= 0, \end{aligned}$$

which immediately implies $\mu(\{\|u-v\|_{L^0(B_1)}>0\})=0$.

The equality $d(u,v)=d(v,u)$ is obvious.

Finally we need to prove the triangular inequality. For this purpose notice that for every triple of functions $u,v,g\colon\Omega\to L^0(B_1)$ it holds

$$\{\|u-v\|_{L^0(B_1)}>\delta_1+\delta_2\}\subset\{\|u-g\|_{L^0(B_1)}>\delta_1\}\cup\{\|g-v\|_{L^0(B_1)}>\delta_2\}.$$

Given $\epsilon>0$, let δ_1 and δ_2 be positive real numbers such that

$$d(u,g)+\epsilon\geq\mu(\{\|u-g\|_{L^0(B_1)}>\delta_1\})+\delta_1,\quad d(g,v)+\epsilon\geq\mu(\{\|g-v\|_{L^0(B_1)}>\delta_2\})+\delta_2.$$

Then

$$\begin{aligned}d(u,v)&=\inf_{\delta>0}\mu(\{\|u-v\|_{L^0(B_1)}>\delta\})+\delta\\&\leq\mu(\{\|u-v\|_{L^0(B_1)}>\delta_1+\delta_2\})+\delta_1+\delta_2\\&\leq[\mu(\{\|u-g\|_{L^0(B_1)}>\delta_1\})+\delta_1]+[\mu(\{\|g-v\|_{L^0(B_1)}>\delta_2\})+\delta_2]\\&\leq d(u,g)+d(g,v)+2\epsilon,\end{aligned}$$

and letting $\epsilon\to0+$ this implies the triangular inequality.

Given $(u_k)_{k=1}^\infty\subset U_\mu(\Omega;L^0(B_1))$ and $u\in U_\mu(\Omega;L^0(B_1))$, we claim that $\lim_{k\to\infty}d(u_k,u)=0$ if and only if u_k converge to u in μ-measure. Let us first suppose $\lim_{k\to\infty}d(u_k,u)=0$. Then by definition of $d(\cdot,\cdot)$, it turns out that for every k there exist $\delta_k\searrow0$ such that

$$\lim_{k\to\infty}\mu(\{\|u_k-u\|_{L^0(B_1)}>\delta_k\})=0.$$

Hence, given $\epsilon>0$ we can find $\overline{k}$ big enough such that for every $k\geq\overline{k}$ $\{\|u_k-u\|_{L^0(B_1)}>\epsilon\}\subset\{\|u_k-u\|_{L^0(B_1)}>\delta_k\}$, which implies

$$\lim_{k\to\infty}\mu(\{\|u_k-u\|_{L^0(B_1)}>\epsilon\})\leq\lim_{k\to\infty}\mu(\{\|u_k-u\|_{L^0(B_1)}>\delta_k\})=0.$$

This gives the convergence in μ-measure.

Now suppose that u_k converge to u in μ-measure. Then we can write for every $\epsilon>0$

$$\lim_{k\to\infty}\inf_{\delta>0}\mu(\{\|u_k-u\|_{L^0(B_1)}>\delta\})+\delta\leq\lim_{k\to\infty}\mu(\{\|u_k-u\|_{L^0(B_1)}>\epsilon\})+\epsilon=\epsilon,$$

which immediately implies $\lim_{k\to\infty}d(u_k,u)=0$.

Finally, we have to prove that $U_\mu(\Omega;L^0(B_1))$ endowed with the metric $d(\cdot,\cdot)$ is complete. For this purpose, suppose that the sequence $(u_k)_{k=1}^\infty$ is Cauchy. Given a sequence $(\lambda_j)_j$ of positive real numbers such that $\sum_{j=1}^\infty\lambda_j<\infty$, there exists a subsequence $(k_j)_j$ such that

$$d(u_{k_{j_1}},u_{k_{j_2}})\leq\lambda_j,\ \text{for every } j_1,j_2\geq j,$$

which means that for every j there exists $0<\delta_j\leq\lambda_j$ such that (without loss of generality we may also suppose $\delta_j\searrow0$)

$$\mu(\{\|u_{k_j}-u_{k_{j+1}}\|_{L^0(B_1)}>\delta_j\})+\delta_j\leq\lambda_j.\tag{3.91}$$

Define $A_j:=\{\|u_{k_j}-u_{k_{j+1}}\|_{L^0(B_1)}>\delta_j\}$ and set $B_j:=\bigcup_{m\geq j+1}A_m$. We claim that u_{k_j} converges point wise for every $x\in\Omega\setminus\bigcap_{j=1}^\infty B_j$. Indeed, if $x\in\Omega\setminus\bigcap_{j=1}^\infty B_j$ means that there exists $\overline{j}$ such that $x \notin B_{\overline{j}}$, which, by definition of $B_{\overline{j}}$, implies $x \notin A_j$ for every $j \geq \overline{j}+1$. For this reason we have

$$\|u_{k_j}(x) - u_{k_{j+1}}(x)\|_{L^0(B_1)} \leq \lambda_j, \text{ for every } j \geq \overline{j}+1,$$

and this immediately implies that $(u_{k_j}(x))_j$ is a Cauchy sequence in $L^0(B_1)$. By the completeness of $L^0(B_1)$ we deduce that there exists a function $u\colon \Omega \setminus \bigcap_{j=1}^{\infty} B_j \to L^0(B_1)$ such that

$$\lim_{j\to\infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0. \tag{3.92}$$

Since by the monotonicity of μ we have

$$\mu\bigg(\bigcap_{j=1}^{\infty} B_j\bigg) \leq \lim_{j\to\infty} \sum_{m\geq j} \mu(A_m) \leq \lim_{j\to\infty} \sum_{m\geq j} \lambda_m = 0,$$

we deduce that the function u is a well defined element of $U_\mu(\Omega; L^0(B_1))$.

We claim that the subsequence $(u_{k_j})_j$ converges in μ-measure to u. Indeed given any $\epsilon > 0$ we have

$$\{\|u_{k_j} - u\|_{L^0(B_1)} > \epsilon\} \subset \{\|u_{k_j} - u\|_{L^0(B_1)} > \delta_j\}$$

for every j big enough such that $\delta_j \leq \epsilon$. This means that

$$\lim_{j\to\infty} \mu(\{\|u_{k_j} - u\|_{L^0(B_1)} > \epsilon\}) \leq \lim_{j\to\infty} \mu(\{\|u_{k_j} - u\|_{L^0(B_1)} > \delta_j\}).$$

By using (3.91) we can deduce

$$\begin{aligned}
\mu(\{\|u_{k_j} - u\|_{L^0(B_1)} > \delta_j\}) &\leq \sum_{m=j}^{\infty} \mu(\{\|u_{k_m} - u_{k_{m+1}}\|_{L^0(B_1)} > \delta_j\}) \\
&\leq \sum_{m=j}^{\infty} \mu(\{\|u_{k_m} - u_{k_{m+1}}\|_{L^0(B_1)} > \delta_m\}) \\
&\leq \sum_{m=j}^{\infty} \lambda_m,
\end{aligned}$$

which by the fact $\sum_{j=1}^{\infty} \lambda_j < \infty$ implies our claim. Since we already know that the convergence in μ-measure implies the convergence in the metric $d(\cdot,\cdot)$, we can write

$$\lim_{j\to\infty} d(u_{k_j}, u) = 0.$$

This together with the fact that the sequence $(u_k)_k$ is Cauchy in the metric $d(\cdot,\cdot)$, easily implies that the full sequence satisfies

$$\lim_{k\to\infty} d(u_k, u) = 0,$$

and we are done. □

Remark 3.4.11. *The space $U_\mu(\Omega; L^0(B_1))$ equipped with the distance defined in the previous proposition is actually a Fréchet space.* ■

3.4.2 The outer measure C_p

Let us start with the definition of capacity.

Definition 3.4.12 (p-Capacity)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. We define the p-Capacity $(1 < p \leq n)$ of a set $A \subset \Omega$ as*

$$C_p(A) := \inf \left\{ \int_\Omega (|\nabla u|^p + |u|^p)\, dx \mid u \in GSBV(\Omega;\Gamma),\, u^+(x) \geq 1 \text{ on } A \right\}, \tag{3.93}$$

where $u^+(x)$ is the upper approximate limit defined in 1.4.1.

Remark 3.4.13. *In (3.93) we consider also the L^p-norm of the function, while in (3.86) only the L^p-norm of the gradient is present. This is simply because we want to avoid that functions u belonging to the kernel of ∇ could trivialise the infimum in (3.93). We remember that the kernel of the approximate gradient of $GSBV(\Omega;\Gamma)$ functions is made up of piecewise constant functions whose jump sets are contained in a Caccioppoli's partition subordinated to Γ. This result can be found for example in [14] for SBV functions; the case $GSBV$ can be easily recoverd by a truncation argument. For example, with this choice the scaling property $C_p(\lambda A) = \lambda^{n-p} C_p(A)$ (see [34, Section 4.7.1]) is lost. Anyway, we do not need this property to develop our theory.* ■

Proposition 3.4.14. *For every set $A \subset \Omega$ we have*

$$C_p(A) = \inf \left\{ \int_\Omega (|\nabla u|^p + |u|^p)\, dx \mid u \in GSBV(\Omega;\Gamma),\, u^+(x) \geq 1 \text{ on } A,\, 0 \leq u \leq 1 \right\}.$$

Proof. Let $u_0^1 := (u \wedge 1) \vee 0$. Since $u^+(x) \geq 1$ if and only $u_0^{1+}(x) \geq 1$, it is enough to notice that if $u \in GSBV^p(\Omega;\Gamma)$ then

$$\int_\Omega |\nabla u_0^1|^p + |u_0^1|^p\, dx \leq \int_\Omega |\nabla u|^p + |u|^p\, dx,$$

and this concludes the proof. □

Proposition 3.4.15. *$C_p(\cdot)$ is an outer measure on Ω.*

Proof. Clearly $C_p(\cdot)$ is monotone and $C_p(\emptyset) = 0$. Hence we need only to prove the countable sub-additivity.

Let $(A_k)_{k=1}^\infty$ be a countable family of subsets of Ω and define $A := \bigcup_{k=1}^\infty A_k$. Without loss of generality we can assume $\sum_k C_p(A_k) < \infty$. For each k we can find $u_k \in GSBV^p(\Omega;\Gamma)$, $0 \leq u_k \leq 1$, and $u_k^+(x) \geq 1$ on A_k such that

$$\int_\Omega |\nabla u_k|^p + |u_k|^p\, dx \leq C_p(A_k) + \frac{\epsilon}{2^k}.$$

We define $u := \sup_{k \in \mathbb{N}} u_k$, and we claim that $u \in GSBV^p(\Omega;\Gamma)$ and $u^+(x) \geq 1$ on A. Indeed, since the u_k are bounded functions in $GSBV_p^p(\Omega;\Gamma)$, we have $u_k \in SBV(\Omega)$. Therefore by using the chain rule in BV [2, Theorem 3.99], if we set $u_m := \sup_{1 \leq k \leq m} u_k$, we have

$$\int_\Omega |\nabla u_m|^p\, dx \leq \sum_{k=1}^m \int_\Omega |\nabla u_k|^p\, dx,$$

hence

$$\sup_m \int_\Omega |\nabla u_m|^p + |u_m|^p \, dx \leq \sum_{k=1}^{\infty} C_p(A_k) + \frac{\epsilon}{2^k}. \tag{3.94}$$

Thanks to (3.94) we can use the compactness result [2, Theorem 4.36] for $GSBV(\Omega)$ together with [21, Remark 2.9] to deduce that $u \in GSBV^p_p(\Omega; \Gamma)$ and moreover

$$u_m \to u \text{ strongly in } L^1(\Omega) \qquad \nabla u_m \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega). \tag{3.95}$$

Moreover if $x \in A$ then $x \in A_k$ for some k, therefore $u_k^+(x) \geq 1$, and since $u \geq u_k$ for every k we deduce $u^+(x) \geq u_k^+(x)$, hence

$$A \subset \{x \in \Omega \mid u^+(x) \geq 1\}.$$

Therefore by using the lower semicontinuity of the L^p-norm with respect to the convergence (3.95), we have

$$C_p(A) \leq \int_\Omega |\nabla u|^p + |u|^p \, dx \leq \liminf_{m\to\infty} \int_\Omega |\nabla u_m|^p + |u_m|^p \, dx \leq \sum_{k=1}^{\infty} C_p(A_k) + \epsilon,$$

which implies the countably sub-additivity of $C_p(\cdot)$ thanks to the arbitrariness of ϵ. □

3.4.3 Relations between C_p and $\mathcal{H}^{n-p}$

In this subsection we derive the relation between C_p and $\mathcal{H}^{n-p}$. Let us notice that Proposition 3.4.16 and property 2 of Theorem 3.4.17 are obtained mainly as in the Sobolev case, and do not depend on the fact $\Gamma \in \mathcal{J}_p$, while property 1 of Theorem 3.4.17 strongly rely on the validity of Theorem 1, i.e. on $\Gamma \in \mathcal{J}_p$.

Proposition 3.4.16. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. For every $1 < p \leq n$ there exists a constant $c = c(n,p) > 0$ such that for every $A \subset \Omega$*

$$C_p(A) \leq c\,\mathcal{H}^{n-p}(A).$$

Proof. First of all if $B_r(x) \subset \Omega$, then $C_p(B_r(x))$ can be rewritten as

$$\inf \left\{ \int_\Omega (|\nabla v|^p + |v|^p) \, dy \mid v(y) = u\left(\frac{x-y}{r}\right),\ u \in GSBV(\Omega'; \Gamma'),\ u^+(x) \geq 1 \text{ on } B_1(0) \right\},$$

where $\Omega' = (\Omega - x)/r$ and $\Gamma' = (\Gamma - x)/r$.

Notice that for $r \leq 1$ we have

$$\int_\Omega (|\nabla v|^p + |v|^p) \, dy = r^n \int_{\Omega'} (r^{-p} |\nabla u|^p + |u|^p) \, dy \leq r^{n-p} \int_{\Omega'} (|\nabla u|^p + |u|^p) \, dy.$$

Hence, by choosing $u(x) := \mathrm{dist}(x, \mathbb{R}^n \setminus B_2(0)) \wedge 1$ whenever $x \in \Omega'$, it follows

$$C_p(B_r(x)) \leq 2^{n+1} \omega_n r^{n-p} \quad (r \leq 1).$$

Let $(C_i)_{i=1}^\infty$ be a family of sets contained in Ω which is a cover of A and $\mathrm{diam} C_i \leq 1$. For each i there exists a ball $B_{r_i}(x_i)$ such that $C_i \subset B_{r_i}(x_i)$ and $r_i = \mathrm{diam}(C_i)$. Therefore

$$C_p(A) \leq \sum_{i=1}^\infty C_p(C_i) \leq \sum_{i=1}^\infty C_p(B_{r_i}(x_i)) \leq 2^{n+1} \omega_n \sum_{i=1}^\infty r_i^{n-p} \leq 2^{2n+1-p} \omega_n \sum_{i=1}^\infty \left(\frac{\mathrm{diam}\, C_i}{2} \right)^{n-p}.$$

Hence if we set $c := 2^{2n+1-p}\omega_n$, then

$$C_p(A) \le c\,\mathcal{H}^{n-p}(A).$$

$\square$

Whenever $u\colon \Omega \to \mathbb{R}$ is such that u_x is a piecewise constant function of the form of Theorem 3.3.11, then by definition of upper approximate limit (Definition 1.4.1), it is easy to see that

$$u^+(x) = \max_{1\le j\le N_x} m_j(u,x). \tag{3.96}$$

We shall use this simple observation to deduce more precise relations between the p-capacity and the Hausdorff measure.

Theorem 3.4.17. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then for every $A \subset \Omega$ and for every $1 < p \le n$ we have*

1. *$C_p(A) = 0$ and $\Gamma \in \mathcal{J}_p$ imply $dim_{\mathcal{H}}(A) \le n-p$;*
2. *$\mathcal{H}^{n-p}(A) < \infty$ implies $C_p(A) = 0$.*

Proof. Suppose $C_p(A) = 0$ and $\Gamma \in \mathcal{J}_p$. By hypothesis we can find a sequence $(u_k)_{k=1}^{\infty} \subset GSBV^p(\Omega;\Gamma)$, $0 \le u_k \le 1$, such that

(i) $u_k^+(x) \ge 1$, for every $x \in A$;

(ii) $\int_\Omega (|\nabla u_k|^p + |u_k|^p)\,dx \le \frac{1}{k^2}$, for every $k \in \mathbb{N}$.

Define $u := \sum_{k=1}^{\infty} u_k$. Since by Proposition 1.4.10 $GSBV^p_p(\Omega;\Gamma)$ is a Banach space, by (ii) we deduce that $u \in GSBV^p_p(\Omega;\Gamma)$. Thanks to Theorem 3.3.11, if we call S_k the set of $x \in \Omega$ where the blow-up of u_k does not exist, then $\dim_{\mathcal{H}}(S_k) \le n-p$. By setting $S := \bigcup_{k=1}^{\infty} S_k$ clearly

$$\dim_{\mathcal{H}}(S) \le n-p. \tag{3.97}$$

Property (i) above together with (3.96) imply that for every k and for every $x \in A \setminus S$ the blow up of u_k at x is of the form

$$(u_k)_x = \sum_{j=1}^{N_x} m_j(u_k,x)\mathbb{1}_{E_{0,j}}, \quad \text{and} \quad \max_{1\le j\le N_x} m_j(u_k,x) \ge 1. \tag{3.98}$$

Since $u_k \ge 0$ for every k, we have $u(y) \ge \sum_{k=1}^{M} u_k(y)$ for every $M \in \mathbb{N}$ and every $y \in B_1(0)$. For this reason, by using the linearity of the blow-up and again (3.96), we have

$$u^+(x) \ge \left(\sum_{k=1}^{M} u_k(x)\right)^+ = \max_{1\le j\le N_x}\left[\sum_{k=1}^{M} m_j(u_k,x)\right].$$

By letting $M \to \infty$, thanks to (3.98) we deduce that $A \setminus S$ is contained in the set of point x where $u^+(x) = +\infty$. By Theorem 3.3.11 together with observation (3.96) we deduce that

$$\dim_{\mathcal{H}}(A \setminus S) \le n-p,$$

which together with (3.97) is exactly (1).

To prove 2 we follow the proof given in [36, Section 3]. Suppose $\mathcal{H}^{n-p}(A) < 2^{p-n}\gamma < \infty$ for some $\gamma > 0$. By denoting as $\mathcal{S}^{n-p}$ the $(n-p)$-dimensional spherical measure (see [35, Paragraph 2.10.2]), we have

$$\mathcal{S}^{n-p}(A) \leq 2^{n-p}\mathcal{H}^{n-p}(A) < \gamma.$$

We claim that for every $m \in \mathbb{N}$ we can find an open set V_m and a function $u_m \in W^{1,p}(\Omega)$ such that

(a) $A \subset V_m = \bigcup_{i=1}^{\infty} B_{r_i}(x_i)$, $\sup_i r_i^p \leq (m+1)^{-p}\big(\sum_{k=1}^{m+1} k^{-1}\big)^{-p}$;

(b) $B_{2r_i}(x_i) \subset V_{m-1}$ $(V_m \subset V_{m-1})$;

(c) $u_m^+(x) = 1$ on V_m, $\operatorname{spt}(Du_m) \subset V_{m-1} \setminus V_m$, $\int_\Omega |Du_m|^p\,dx \leq c\,\gamma$,

where $c := c(n,p) > 0$.

We start by setting $V_0 := \Omega$ and $u_0 := 1$. Set $\delta_{m+1} := (m+1)^{-1}\big(\sum_{k=1}^{m+1} k^{-1}\big)^{-1}$. To define V_m and u_m, by using

$$\sum_{i=1}^{\infty} \mathcal{S}^{n-p}(A \cap \{x \mid 2^i < \operatorname{dist}(x, V_{m-1}) \leq 2^{i-1}\}) < \gamma,$$

we can find a sequence of balls $(B_{r_i}(x_i))_{i=1}^{\infty}$ such that $B_{2r_i}(x_i) \subset V_{m-1}$, $\sup_i r_i \leq \delta_m$, and

$$A \subset V_m := \bigcup_{i=1}^{\infty} B_{r_i}(x) \quad \text{and} \quad \sum_{i=1}^{\infty} \omega_{n-p} r_i^{n-p} \leq \gamma.$$

Define $h_i \in W^{1,p}(\Omega)$ as

$$h_i(x) = 1 \text{ if } |x - x_i| \leq r_i, \quad h_i(x) = 0, \quad \text{if} \quad |x - x_i| \geq 2r_i,$$

$$h_i(x) = 2 - |x - x_i|/r_i \quad \text{if} \quad r_i < |x - x_i| < 2r_i.$$

Since $\int_\Omega |Dh_i|^p\,dx = r_i^{-p}\omega_n[(2r_i)^n - r_i^n] = c\,\omega_{n-p} r_i^{n-p}$, if we define $u_m := \sup_{i=1}^{\infty} h_i$, then

$$\int_\Omega |Du_m|^p\,dx \leq \int_\Omega \sum_{i=1}^{\infty} |Dh_i|^p\,dx \leq c\,\gamma.$$

In this way (a),(b) and (c) are satisfied.

Define $u := \sum_{k=1}^{\infty} k^{-1} u_k$. Since by construction $\operatorname{spt}(Du_m) \subset V_{m-1} \setminus V_m$ we have $|\operatorname{spt}(Du_m) \cap \operatorname{spt}(Du_{m+1})| = 0$ for every $m \in \mathbb{N}$. Therefore we can write

$$\int_\Omega |Du|^p\,dx = \int_\Omega \sum_{k=1}^{\infty} k^{-p}|Du_k|^p\,dx \leq c\,\gamma\,\lambda,$$

where $\lambda := \sum_{k=1}^{\infty} k^{-p}$. By using

$$u(x) \leq \sum_{k=1}^{m} k^{-1} \text{ if } x \in V_{m-1} \setminus V_m,$$

we can estimate

$$\int_\Omega |u|^p\,dx = \int_\Omega \left|\sum_{k=1}^{\infty} k^{-1}u_k\right|^p dx \leq \sum_{m=1}^{\infty} \int_{V_{m-1}\setminus V_m} \left(\sum_{k=1}^{m} k^{-1}\right)^p dx \leq \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} k^{-1}\right)^p |V_{m-1}|. \tag{3.99}$$

If we call $(B_{r_i}(x))$ the balls relative to V_{m-1}, i.e. $V_{m-1} = \bigcup_{i=1}^{\infty} B_{r_i}(x)$, then

$$\left(\sum_{k=1}^{m} k^{-1}\right)^p |V_{m-1}| \leq \left(\sum_{k=1}^{m} k^{-1}\right)^p \sum_{i=1}^{\infty} \omega_n r_i^n \leq \frac{\omega_n}{\omega_{n-p}} \sum_{i=1}^{\infty} m^{-p} \omega_{n-p} r_i^{n-p}.$$

Therefor we can continue inequality (3.99) in the following way

$$\int_\Omega |u|^p\, dx \leq \frac{\omega_n}{\omega_{n-p}} \sum_{m,i=1}^{\infty} m^{-p}\, \omega_{n-p} r_i^{n-p} \leq \frac{\omega_n}{\omega_{n-p}} \sum_{m=1}^{\infty} m^{-p}\gamma = \frac{\omega_n}{\omega_{n-p}} \gamma\, \lambda.$$

We claim that

$$u^+(x) \geq \sum_{k=1}^{m} k^{-1}, \quad x \in V_m. \tag{3.100}$$

To prove (3.100) it is sufficient to show that for every $t < \sum_{k=1}^{m} k^{-1}$ the superlevel $\{u > t\}$ has strictly positive density at every $x \in V_m$.

Using the fact that $V_m \subset V_{m-1}$ together with property (c), we have that

$$u_k^+(x) \geq 1, \quad 1 \leq k \leq m, \quad x \in V_m. \tag{3.101}$$

Hence, by choosing any $t < \sum_{k=1}^{m} k^{-1}$, since

$$\bigcap_{k=1}^{m} \{u_k > (1-\delta)\} \subset \{u > t\},$$

for any $0 < \delta < 1$ such that $\sum_{k=1}^{m} (1-\delta) k^{-1} = t$, and since by (3.101) each set $\{u_k > (1-\delta)\}$ has strictly positive density at $x \in V_m$, we deduce that $\{u > t\}$ has strictly positive density at every $x \in V_m$.

For this reason by definition of p-capacity it immediately follows

$$C_p(V_m) \leq \left(\sum_{k=1}^{m} k^{-1}\right)^{-p} \int_\Omega (|\nabla u|^p + |u|^p)\, dx \leq \left(\sum_{k=1}^{m} k^{-1}\right)^{-p} c'\,(\gamma\,\lambda).$$

Sending $m \to \infty$ in the previous inequality we deduce $C_p(A) = 0$.

□

3.4.4 The main result

Let $1 < p \leq n$ and $\Gamma \in \mathcal{J}_p$. Given $u \in GSBV^p(\Omega; \Gamma)$, by Theorem 3.3.11 we know that $u_x \in L^0(B_1)$ is well defined for every $x \in \Omega$ up to a singular set of Hausdorff dimension at most $n-p$. If we call S such a singular set, this means that for every $1 < q < p$ we have $\mathcal{H}^{n-q}(S) = 0$, and by Proposition 3.4.16 also $C_q(S) = 0$. Therefore, for every $1 < q < p$, u_x is a well defined element in the Fréchet space $U_{C_q}(\Omega; L^0(B_1))$ (see Definition 3.4.6). Unfortunately, we can not conclude the same for $q = p$. For this reason we need to introduce a further outer measure.

Definition 3.4.18 (Lower p-capacity)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, and let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$. Given any set $A \subset \Omega$ we define the* lower p-capacity *as*

$$C_p^-(A) := \sup_{1<q<p} C_q(A). \tag{3.102}$$

Proposition 3.4.19. *$C_p^-(\cdot)$ is an outer measure. In addition,*

$$C_p^-(A) = 0 \quad \textit{iff} \quad C_q(A) = 0 \textit{ for every } 1 < q < p. \tag{3.103}$$

Proof. $C_p^-(\cdot)$ is an outer measure simply because it is obtained as supremum of a family of outer measures. Property (3.103) follows by construction. □

Proposition 3.4.20. *Let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$, then for every $u \in GSBV^p(\Omega;\Gamma)$ we have that u_x is a well defined element in $U_{C_p^-}(\Omega; L^0(B_1))$.*

Proof. By Theorem 3.3.11 we know that u_x exists everywhere except on a singular set S of Hausdorff dimension at most $n-p$. This means that if we call S the set of points where the blow-up $u_{r,x}$ does not converge, then for every $\delta > 0$ $\mathcal{H}^{n-p+\delta}(S) = 0$. As a consequence by Proposition 3.4.16 this means also $C_{p-\delta}(S) = 0$. Finally, relation (3.103) gives the conclusion of the theorem. □

Proposition 3.4.21 (Capacitary Chebyshev's inequality)**.** *Let Ω be a bounded open set of $\mathbb{R}^n$ and let $\Gamma \in \mathcal{J}_p$ with $1 < p \leq n$. Then for every $\epsilon > 0$ and for every $1 < q < p$ it holds*

$$C_q\big(\{x \in \Omega \mid \|u_x\|_{L^0(B_1)} > \omega_n \epsilon\}\big) \leq \frac{1}{\epsilon^q} \int_\Omega (|\nabla u|^q + |u|^q)\, dx, \tag{3.104}$$

for every $u \in GSBV^p(\Omega;\Gamma)$.

Proof. Renormalizing by ϵ, in order to prove (3.104), it is enough to show that up to a C_q-negligible set the following inclusion holds true

$$\{x \in \Omega \mid \|u_x\|_{L^0(B_1)} > \omega_n\} \subset \{x \in \Omega \mid |u|^+(x) \geq 1\}. \tag{3.105}$$

By Theorem 3.3.11 together with Theorem 3.4.16 we know that except a C_q-negligible set, we have

$$|u|_x(y) = \sum_{j=1}^{N_x} m_j(|u|, x) \mathbb{1}_{E_{0,j}}(y), \;\; y \in B_1(0).$$

Using (3.96) we know that $|u|^+(x) \geq 1$ if and only if at least one of the $(m_j(|u|,x))_{j=1}^{N_x}$ is greater or equal than one.

Now suppose by contradiction that $\max_{1\leq j\leq N_x} m_j(x) < 1$ and $\|u_x\|_{L^0(B_1)} > \omega_n$. Then

$$\begin{aligned}\|u_x\|_{L^0(B_1)} &= \left\| \sum_{j=1}^{N_x} m_j(|u|,x) \mathbb{1}_{F_{r,j}} \right\|_{L^0(B_1)} \\ &= \sum_{j=1}^{N_x} \int_{F_{r,j}} |m_j(|u|,x)| \wedge 1\, dy \\ &\leq \omega_n,\end{aligned}$$

which immediately implies (3.105) and the proposition. □

Theorem 3.4.22. *Let Ω be a bounded open set of $\mathbb{R}^n$ and let $\Gamma \in \mathcal{J}_p$ with $1 < p \leq n$. Suppose that $(u_k)_{k=1}^\infty \subset GSBV_p^p(\Omega;\Gamma)$ is such that*

$$\|u_k - u\|_{L^p} + \|\nabla u_k - \nabla u\|_{L^p} \to 0, \;\; \textit{as } k \to \infty.$$

Then $(u_k)_x$ converge to u_x in the Fréchet space $U_{C_p^-}(\Omega; L^0(B_1))$.

Proof. We shall prove that given $\epsilon, \delta > 0$, then there exists $\overline{k}$ such that for every $k \geq \overline{k}$

$$C_p^-\big(\{x \in \Omega \mid \|(u_k)_x - u_x\|_{L^0(B_1)} > \omega_n \epsilon\}\big) \leq \delta.$$

Thanks to Theorem 3.3.11 there exists a set S with $\dim_H(S) \leq n - p$ such that $(u_k)_x$ and u_x exist for every $x \in \Omega \setminus S$ and for every $k \in \mathbb{N}$. Moreover by Theorem 3.4.17 we know that $C_q(S) = 0$ for every $1 < q < p$, which by Proposition 3.4.19 implies $C_p^-(S) = 0$. Therefore, since by linearity we have for every $x \in \Omega \setminus S$ the relation $(u_k)_x - u_x = (u_k - u)_x$, by using the capacitary Chebyshev's inequality for every $1 < q < p$ we get

$$C_q\big(\{x \in \Omega \mid \|(u_k)_x - u_x\|_{L^0(B_1)} > \omega_n \epsilon\}\big) \leq \frac{1}{\epsilon^q} \int_\Omega (|\nabla u_k - \nabla u|^q + |u_k - u|^q)\, dx.$$

Finally, by the definition of C_p^- it is enough to choose $\overline{k}$ big enough such that for every $k \geq \overline{k}$

$$\sup_{1<q<p} \frac{1}{\epsilon^q} \int_\Omega (|\nabla u_k - \nabla u|^q + |u_k - u|^q)\, dx \leq \delta,$$

which is possible by using Hölder inequality to pass from the exponent q to p. □

Putting together Theorems 3.4.17, 3.4.22 and (3.103) we are able to prove the second second main result of this chapter.

Proof of Theorem 2. Let us first suppose Ω bounded. Putting together the previous result with Theorem 3.4.9 we have that there exists a subsequence k_j such that

$$\lim_{j\to\infty} \|(u_{k_j})_x - u_x\|_{L^0(B_1)} = 0,$$

for every $x \in \Omega$ except a C_p^--negligible set S. Putting together Theorem 3.4.17 with (3.103) it easily follows $\dim_{\mathcal{H}}(S) \leq n - p$.

For general Ω, we set $\Omega_i := \Omega \cap B_i(0)$ $(i \in \mathbb{N})$. For every i we can apply the previous result on the bounded open set Ω_i to obtain a sequence $(k_j^i)_{j=1}^\infty$ and a set $S_i \subset \Omega_i$ with $\dim_{\mathcal{H}}(S_i) \leq n - p$, such that

$$\lim_{j\to\infty} \|(u_{k_j^i})_x - u_x\|_{L^0(B_1)} = 0, \text{ for every } x \in \Omega_i \setminus S_i.$$

We can also suppose that $(k_j^{i+1})_{j=1}^\infty$ is a subsequence of $(k_j^i)_{j=1}^\infty$ for every i. By a diagonal argument we define for every $j \in \mathbb{N}$ $k_j := k_j^j$, and we obtain that

$$\lim_{j\to\infty} \|(u_{k_j})_x - u_x\|_{L^0(B_1)} = 0, \text{ for every } x \in \Omega \setminus \bigcup_{i=1}^\infty S_i.$$

Finally, since every S_i has Hausdorff dimension which does not exceed $n-p$, then also $\bigcup_{i=1}^\infty S_i$ has Hausdorff dimension which does not exceed $n - p$. This proves the theorem. □

Remark 3.4.23. *In [31] the authors are able to prove a density result for the space $SBV^p(\Omega)$. More precisely, if Ω is an open set with Lipschitz boundary and $u \in SBV^p(\Omega)$, then there exists a sequence of functions $u_j \in SBV^p(\Omega)$ and of compact C^1 manifolds with C^1 boundary $M_j \subset\subset \Omega$ and such that $J_{u_j} \subseteq M_j$ but $\mathcal{H}^{n-1}(M_j \setminus J_{u_j}) = 0$ and*

$$u_j \in C^\infty(\Omega \setminus \overline{J_{u_j}}), \qquad \|u_j - u\|_{L^1} \to 0, \qquad \|\nabla u_j - \nabla u\|_{L^p} \to 0, \qquad \mathcal{H}^{n-1}(J_{u_j} \Delta J_u) \to 0.$$

It is natural to ask whether the hypothesis $\mathcal{H}^{n-1}(J_{u_j}\Delta J_u) \to 0$ can be improved to

$$J_{u_j} \subset J_u \text{ for every } j \in \mathbb{N}.$$

In other words we can rephrase this question in the following way: given $\Gamma \subset \Omega$ a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set, then is it true that the closure in SBV^p with respect to the norm given by $\|\nabla u\|_{L^p} + \|u\|_{L^1}$ of all functions v such that

$$v \in C^\infty(\Omega \setminus \overline{J_u}), \qquad J_u \subset M \cap \Gamma, \qquad M \text{ is any } C^1 \text{ manifolds with } C^1 \text{ boundary}, \tag{3.106}$$

is the whole of $SBV^p(\Omega;\Gamma) \cap L^1(\Omega)$?

The answer is in general no. Consider $\Gamma_0 \subset \mathbb{R}^2$ the union of three half lines starting from the origin. Let $\Gamma \subset \mathbb{R}^3$ be defined by $\Gamma_0 \times \mathbb{R}$ and let l be the straight line $\{(0,0,t) \mid t \in \mathbb{R}\}$. The set Γ disconnects $\mathbb{R}^3 \setminus \Gamma$ into three connected components $\Omega_1, \Omega_2, \Omega_3$. Let $v\colon \mathbb{R}^3 \to \mathbb{R}$ be the function which assumes three different constant values on each of the connected components, say $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$. Clearly $v \in SBV^p(\Omega;\Gamma)$ for every $p \in [1,3)$. We claim that for $p \in (2,3)$, the function v cannot be approximated in SBV^p by functions satisfying (3.106). Indeed, any function $u \in SBV^p(\Omega)$ satisfying (3.106) has the property that v_x is defined everywhere, except on a $(3-p)$-dimensional Hausdorff set, and it is a function taking at most two values. By using a slightly modified version of Theorem 2 (where we have to substitute the L^p convergence of the functions with the L^1 convergence), we deduce that any limit u in $SBV^p(\Omega;\Gamma)$ of functions satisfying (3.106), inherits the property that its blow-up converges to a function u_x which takes at most two values for every x except a set of Hausdorff dimension $3-p$. However for every point $x \in l$, v_x assumes three different values, namely $\alpha_1, \alpha_2, \alpha_3$. Since $\dim_{\mathcal{H}}(l) = 1$, this implies that for every $p \in (2,3)$, v cannot be approximated by functions satisfying (3.106). ■

3.5 More on the class $\mathcal{J}_p$

We dedicate this last part of the chapter to construct sets living in $\mathcal{J}_p$. In the first part of this section we show that finite unions of graph of certain Sobolev functions belong to $\mathcal{J}_p$. In particular we deduce that an admissible jump set does not need to be essentially closed (see Remark 3.5.2). In the second part of this section we present a counterexample to Theorem 1.

3.5.1 Some examples

Let $n \geq 3$ and $1 < p \leq n-1$. We write the generic point $x \in \mathbb{R}^n$ as $x = (y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We define $\mathcal{W}^{1,p}(\mathbb{R}^{n-1})$ as the space of all Sobolev functions $f \in W^{1,p}(\mathbb{R}^{n-1})$ such that for every $y \in \mathbb{R}^{n-1}$ except a set of Hausdorff dimension $n-1-p$, y is a Lebesgue point for the distributional gradient Df [1].

Now let $f \in \mathcal{W}^{1,p}(\mathbb{R}^{n-1})$ and consider its sub-graph

$$S_f^- := \left\{x \in \mathbb{R}^n \mid t < f(y),\, y \in \mathbb{R}^{n-1}\right\}.$$

It is well known that S_f^- is a set having locally finite perimeter in $\mathbb{R}^n$.

[1] By using the theory of capacity it is easy to see that $W^{2,p}(\mathbb{R}^{n-1}) \subset \mathcal{W}^{1,p}(\mathbb{R}^{n-1})$

Consider the following sets

$$A := \left\{ y \in \mathbb{R}^{n-1} \mid \exists\, \tilde{f}(y) \in \mathbb{R},\ \lim_{r\to 0} \fint_{B_r^{n-1}(y)} |f(z) - \tilde{f}(y)|\, dz \to 0 \right\},$$

and

$$B := \left\{ y \in \mathbb{R}^{n-1} \mid \exists\, \tilde{D}f(y) \in \mathbb{R}^n,\ \lim_{r\to 0} \fint_{B_r^{n-1}(y)} |Df(z) - \tilde{Df}(y)|\, dz \to 0 \right\},$$

where $B_r^{n-1}(y)$ is the $(n-1)$-dimensional ball of radius r centered at y. To be precise we will call the graph of f the set of points of the form

$$\mathrm{graph}(f) := \{(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y \in A \cap B,\ t = \tilde{f}(y)\}.$$

Proposition 3.5.1. *Let $f \in \mathcal{W}^{1,p}(\mathbb{R}^{n-1})$ with $1 < p \leq n-1$ and $n \geq 3$. Then* graph(f) *belongs to $\mathcal{J}_p$.*

Proof. By using the theory of capacity developed in [36] (see also [34, Section 4.7]), and the definition of $\mathcal{W}^{1,p}$, we know that

$$\dim_{\mathcal{H}}(\mathbb{R}^{n-1} \setminus A \cap B) \leq n - 1 - p.$$

Therefore, it follows for example by [63, Corollary 8.11] that

$$\dim_{\mathcal{H}}([\mathbb{R}^{n-1} \setminus (A \cap B)] \times \mathbb{R}) \leq n - p.$$

We claim that for every $x = (y,t) \in \mathbb{R}^n$ such that $y \in A \cup B$ one and only one of the following conditions occur

- $x \in \partial^* S_f^-$;
- $\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* S_f^-, x) = 0.$

By Proposition 3.2.12, this would imply that $\partial^* S_f^-$ has a non vanishing upper isoperimetric profile at x.

We first prove that for every $x = (y,t) \in (A \cup B) \times \mathbb{R}$ such that $t < \tilde{f}(y)$, it holds

$$\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* S_f^-, x) = 0, \tag{3.107}$$

or equivalently

$$\lim_{r\to 0^+} \mathcal{H}^{n-1}\big((\partial^* S_f^-)_{r,x}\big) = 0. \tag{3.108}$$

Now, since $\lim_{r\to 0} \fint_{B_r^{n-1}(y)} |f(z) - \tilde{f}(y)|\, dz = 0$, then by a change of variable in the integral we have

$$\lim_{r\to 0^+} \|f(y + r(\cdot)) \to \tilde{f}(y)\|_{L^1(B_1^{n-1}(0))} = 0.$$

In particular, this means that for every $\epsilon > 0$

$$\lim_{r\to 0^+} |\{|f(y + r(\cdot)) - \tilde{f}(y)| > \epsilon\} \cap B_1^{n-1}(0)| = 0. \tag{3.109}$$

For every $z \in B_1^{n-1}(0)$ such that $|f(y + rz) - \tilde{f}(y)| \leq \epsilon$ we have

$$\frac{|f(y + rz) - t|}{r} \geq \frac{|\tilde{f}(y) - t|}{r} - \frac{|f(y + rz) - \tilde{f}(y)|}{r} \geq \frac{\tilde{f}(y) - t}{r} - \frac{\epsilon}{r},$$

and if $\epsilon < \frac{\tilde{f}(y)-t}{2}$, by the previous inequalities we deduce

$$\frac{|f(y+rz)-t|}{r} > \frac{\tilde{f}(y)-t}{2r}.$$

Hence, for sufficiently small value of r, we have

$$\frac{|f(y+rz)-t|}{r} > 1.$$

Therefore, for sufficiently small value of r we have

$$\{|f(y+r(\cdot))-t|/r \leq 1\} \cap B_1^{n-1}(0) \subset \{|f(y+r(\cdot))-\tilde{f}(y)| > \epsilon\} \cap B_1^{n-1}(0).$$

Notice that

$$(\partial^* S_f^-)_{r,x} \subset \left\{(z,s) \in B_1^{n-1}(0) \times (-1,1) \mid s = \frac{f(y+rz)-t}{r}\right\}.$$

As a consequence, for sufficiently small value of r we have the following inequalities

$$\mathcal{H}^{n-1}\big((\partial^* S_f^-)_{r,x}\big) \leq \int_{\{|f(y+r(\cdot))-t|/r\leq 1\}\cap B_1^{n-1}(0)} \sqrt{1+|Df(z)|^2}\,dz \tag{3.110}$$

$$\leq \int_{\{|f(y+r(\cdot))-\tilde{f}(y)|>\epsilon\}\cap B_1^{n-1}(0)} \sqrt{1+|Df(z)|^2}\,dz. \tag{3.111}$$

Therefore, by using (3.109) and the definition of B we deduce

$$\lim_{r\to 0^+} \int_{\{|f(y+r(\cdot))-\tilde{f}(y)|>\epsilon\}\cap B_1^{n-1}(0)} \sqrt{1+|Df(z)|^2}\,dz = 0,$$

which proves claim (3.107).

Analogously one can prove that if $x = (y,t) \in (A \cup B) \times \mathbb{R}$ is such that $\tilde{f}(y) < t$ then (3.107) holds.

Finally it remains to prove that if $x \in \mathrm{graph}(u)$ then $x \in \partial^* S_f^-$. First of all, since y is a Lebesgue point for u and a Lebesgue point for Du, by using [2, Theorem 3.83], u is approximately differentiable at y, i.e.

$$\lim_{r\to 0^+} \int_{B_1^{n-1}(0)} \frac{|f(y+rz)-f(y)-\tilde{D}u(y)\cdot rz|}{r}\,dz = 0.$$

Therefore if we set $L_y(z) := \tilde{D}u(y)\cdot z$, then

$$\frac{f(y+r(\cdot))-\tilde{f}(y)}{r} \to L_y(\cdot), \quad \text{in } L^1(B_1^{n-1}(0)), \text{ as } r \to 0^+. \tag{3.112}$$

This means that if we define $C_1(0)$ to be the cylinder given by $B_1^{n-1}(0) \times (-1,1)$, we can write

$$\lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) = \lim_{r\to 0^+} \int_{B_1^{n-1}(0)} \sqrt{1+|Df(y+rz)|^2}. \tag{3.113}$$

Moreover, if we call H_x^- the lower half space relative to the unit vector $\frac{(-\tilde{D}u(y),1)}{\sqrt{1+|\tilde{D}u(y)|^2}}$, we can continue equality (3.113) in the following way

$$\begin{aligned}\lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) &= \lim_{r\to 0^+} \int_{B_1^{n-1}(0)} \sqrt{1+|Df(y+rz)|^2}\,dz \\ &= P(H_x^-; C_1(0)),\end{aligned} \tag{3.114}$$

where we used that y is a Lebesgue point for Du. Putting together (3.112) with (3.114) we deduce

(i) $(S_f^- - x)/r \to H_x^-$ in measure in $C_1(0)$ as $r \to 0^+$;

(ii) $\lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) = P(H_x^-; C_1(0))$.

Since $B_1(0) \subset C_1(0)$, condition (i) implies in particular

$$(S_f^- - x)/r \to H_x^-, \text{ in measure in } B_1(0), \text{ as } r \to 0^+. \tag{3.115}$$

Moreover, since $P(H_x^-; \partial B_1(0)) = 0$ we have

$$\begin{aligned} P(H_x^-; C_1(0)) &= \lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) \\ &\geq \limsup_{r\to 0^+} [P((S_f^- - x)/r; B_1(0)) + P((S_f^- - x)/r; C_1(0) \setminus \overline{B}_1(0))] \\ &\geq \limsup_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) + \liminf_{r\to 0^+} P((S_f^- - x)/r; C_1(0) \setminus \overline{B}_1(0)) \\ &\geq \liminf_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) + \liminf_{r\to 0^+} P((S_f^- - x)/r; C_1(0) \setminus \overline{B}_1(0)) \\ &\geq P(H_x^-; B_1(0)) + P(H_x^-; C_1(0) \setminus \overline{B}_1(0)) \\ &= P(H_x^-; C_1(0)), \end{aligned}$$

which implies

$$\limsup_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) = \liminf_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) = P(H_x^-; B_1(0)). \tag{3.116}$$

Putting together (3.115) and (3.116) we can use [2, Proposition 1.62] for the measures $D\mathbb{1}_{(S_f^- - x)/r}$ $(0 < r < 1)$, to deduce that

$$D\mathbb{1}_{(S_f^- - x)/r}(B_1(0)) \to D\mathbb{1}_{H_x^-}(B_1(0)) = \omega_{n-1}\, \nu_{H_x}(x), \quad \text{as } r \to 0^+, \tag{3.117}$$

where ν_{H_x} is the inner unit vector relative to H_x^-. Finally, by (3.116) we deduce

$$\lim_{r\to 0^+} \frac{P(S_f^-; B_r(x))}{r^{n-1}} = \omega_{n-1},$$

which together with (3.117) implies

$$\lim_{r\to 0+} \frac{D\mathbb{1}_{S_f^-}(B_r(x))}{|D\mathbb{1}_{S_f^-}|(B_r(x))} = \lim_{r\to 0+} \frac{r^{n-1}\, \mu_r(B_1(0))}{P(S_f^-; B_r(x))} = \lim_{r\to 0+} \frac{\mu_r(B_1(0))}{\omega_{n-1}} = \nu_{H_x}(x),$$

and this is exactly (1.1), hence we can conclude $x \in \partial^* S_f^-$. □

Remark 3.5.2. *Since for $n-1-2p > 0$ it is possible to construct functions $u \in W^{2,p}(\mathbb{R}^{n-1})$ such that the topological closure of their graphs have arbitrarily large n-dimensional Lebesgue measure, with the previous example we have shown that a generic set in $\mathcal{J}_p$ is not essentially closed.* ■

Proposition 3.5.3. *Let Ω be an open set of $\mathbb{R}^n$ $(n \geq 3)$, and let $(\Gamma_i)_{i=1}^M$ $(M \in \mathbb{N})$, be sets such that for every i there exists $\xi_i \in \mathbb{S}^{n-1}$ and $f \in \mathcal{W}^{1,p}(\xi_i^\perp)$ $(1 < p \leq n-1)$ with $\Gamma_i := graph(f_i) \cap \Omega$. Then $\Gamma := \bigcup_{i=1}^M \Gamma_i$ belongs to $\mathcal{J}_p$*

Proof. Proposition 3.5.1 shows that for every $x \in \Omega$ and for every $1 \leq i \leq M$, except an $(n-p)$-dimensional Hausdorff set, one and only one of the following conditions occurs

- $x \in \partial^* S^-_{f_i}$;
- $\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* S_{f_i}, x) = 0$.

Now fix such an $x \in \Omega$. By reordering the index i we may suppose for example that there exists $k \in \mathbb{N}$ such that for every $1 \le i \le k$ $x \in \partial^* S^-_{f_i}$ and for every $k < i \le M$ $\Theta^{*(n-1)}(S_{f_i}, x) = 0$. Without loss of generality we may also suppose that if $1 \le i_1 < i_2 \le k$ and Γ_{i_1} Γ_{i_2} have the same tangent space at x, then the theoretic normals of $S^-_{f_{i_1}}$ and $S^-_{f_{i_2}}$ are the same at x. For the same reason, without loss of generality, we may suppose that for every $k < i \le M$ than x is a point of density 1 for $S^-_{f_i}$.

Given $r > 0$ such that $B_r(x) \subset \Omega$, we set for $1 \le i \le k$

$$E_i^- := S^-_{f_i} \cap B_r(x) \text{ and } E_i^+ := S^+_{f_i} \cap B_r(x).$$

and

$$E^-_{0,i} := \{y \in B_1(0) \mid \nu_{\Gamma_i}(x) \cdot y < 0\} \text{ and } E^+_{0,i} := \{y \in B_1(0) \mid \nu_{\Gamma_i}(x) \cdot y > 0\},$$

For $k < i \le M$ we set

$$E_{i,1} := S^-_{f_i} \cap B_r(x),$$

and

$$E_{0,i} := B_1(0).$$

By eventually reordering again the first k index, we may assume that there exist $k_1, k_2, \dots, k_m$ $(m \le k)$ such that

$$\nu_{\Gamma_{i_1}} = \nu_{\Gamma_{i_2}} \text{ if and only if } k_j \le i_1, i_2 < k_{j+1}.$$

Now we want to define the sets $F_{r,j}$ and $E_{0,j}$ of Definition 3.2.7. For this purpose let us denote as Σ_2^M the family of maps from $\{1, \dots, M\}$ into $\{-, +\}$. Given $\sigma \in \Sigma_2^M$ we define

$$E_\sigma = \bigcap_{i=1}^M E_i^{\sigma(i)},$$

and

$$E_{0,\sigma} = \bigcap_{i=1}^M E_{0,i}^{\sigma(i)},$$

whenever $E_{0,\sigma} \ne \emptyset$.

We have $1 \le N_x \le 2^M$. Instead of the index $j = 1, \dots, N_x$ appearing in Definition 3.2.7, we have indexed our sets by $\sigma \in \Sigma_2^M$. Notice that

$$\lim_{r \to 0^+} |(E_\sigma)_{r,x} \Delta E_{0,\sigma}| = 0.$$

Moreover $E_{0,\sigma}$ are conical and indecomposable sets, since they are intersection of half spaces.

Notice that by our choice of $x \in \Omega$ we have that

$$\lim_{r \to 0^+} P((E_i^\pm)_{r,x}; B_1(0)) = P(E^\pm_{0,i}; B_1(0)),\ i = 1, \dots, M.$$

By construction we have also that, since $E_{0,\sigma} \ne \emptyset$, then $\sigma(i_1) = \sigma(i_2)$ for every $k_j \le i_1, i_2 < k_{j+1}$ and for every $j = 1, \dots, m$. This means that the family $(E_{0,i}^{\sigma(i)})_{i=1}^M$ satisfies also point (3) of Lemma 3.5.4. Therefore we can deduce that

$$\lim_{r \to 0^+} P((E_\sigma)_{r,x}; B_1(0)) = P(E_{0,\sigma}; B_1(0)).$$

Hence, we are in position to apply Proposition 3.2.4 and to deduce that for every $\sigma \in \Sigma_2^M$ such that $E_{0,\sigma} \neq \emptyset$, there exist indecomposable components of $(E_\sigma)_{r,x}$, say $F_{r,\sigma}$ such that

$$\lim_{r\to 0^+} |F_{r,\sigma} \Delta E_{0,\sigma}| = 0, \tag{3.118}$$

and

$$\lim_{r\to 0^+} P(F_{r,\sigma}; B_1(0)) = P(E_{0,\sigma}; B_1(0)). \tag{3.119}$$

This gives immediately condition (1.1) and (2.1) of Definition 3.2.7, and using Proposition 3.2.13 we deduce also that for every $1 \le j \le M$ the families $(F_{r,j})_{0<r<r_x}$ are left-continuous.

Finally, by (3.119) we can use the same argument as in the proof of Proposition 3.2.12 to deduce

$$\liminf_{r\to 0^+} h_{F_{r,\sigma}}(\lambda) \ge h_{E_{0,\sigma}}(\lambda), \ \ \lambda \in (0, 1/2],$$

which implies condition (1.2) of Definition 3.2.7 since $h_{E_{0,\sigma}}(\lambda) > 0$ for every $\lambda \in (0, 1/2]$. □

Lemma 3.5.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $(E_{r,i})_{i=1}^M$ ($M \in \mathbb{N}$) be sets having finite perimeter in Ω. Suppose that there exist sets $(E_{0,i})_{i=1}^M$ having finite perimeter in Ω such that*

1. $\lim_{r\to 0^+} |E_{r,i} \Delta E_{0,i}| = 0, \ 1 \le i \le M$;
2. $\lim_{r\to 0^+} P(E_{r,i}; \Omega) = P(E_{0,i}; \Omega), \ 1 \le i \le M$;
3. $\mathcal{H}^{n-1}(\partial^* E_{0,i_1} \cap \partial^* E_{0,i_2} \cap \{\nu_{E_{0,i_1}} \neq \nu_{E_{0,i_2}}\}) = 0, \ 1 \le i_1 < i_2 \le M$.

Then we have

$$\lim_{r\to 0^+} P\Big(\bigcap_{i=1}^M E_{r,i}; B_1(0)\Big) = P\Big(\bigcap_{i=1}^M E_{0,i}; B_1(0)\Big).$$

Proof. We proceed by induction on M. For $M = 1$ there is nothing to prove. By induction suppose that our statement holds for $M-1$, then we want to show that it still holds for M. For this purpose, suppose to have $(E_{r,i})_{i=1}^M$ satisying (1)-(3). If we consider the first $M-1$ sets $(E_{r,i})_{i=1}^{M-1}$, then they clearly still satisfy (1)-(3), hence by inductive hypothesis we have

$$\lim_{r\to 0^+} P\Big(\bigcap_{i=1}^{M-1} E_{r,i}; B_1(0)\Big) = P\Big(\bigcap_{i=1}^{M-1} E_{0,i}; B_1(0)\Big). \tag{3.120}$$

If we define $E'_r := \bigcap_{i=1}^{M-1} E_{r,i}$, $E'_0 := \bigcap_{i=1}^{M-1} E_{0,i}$ and $E_r := E_{r,M}$, $E_0 := E_{0,M}$, then we have that the couple E_r, E'_r still satisfies (1)-(3): the first is clearly satisfied; the second follows from (3.120); for the third just notice that if $x \in \partial^* E'_0 \cap \partial^* E_0$ then there must exists $1 \le i \le M-1$ such that $x \in \partial^* E_{0,i} \cap \partial^* E_{0,M}$, therefore if $\nu_{E'_0}(x) = -\nu_{E_0}(x)$ also $\nu_{E_{0,i}}(x) = -\nu_{E_{0,M}}(x)$. This immediately implies $\mathcal{H}^{n-1}(\partial^* E'_0 \cap \partial^* E_0 \cap \{\nu_{E'_0} \neq \nu_{E_0}(x)\}) = 0$. Hence we are reduced to prove our statement for $M = 2$.

In order to do that, we notice that by Theorem 1.3.2 the following identities hold

$$\begin{aligned} P(E'_r; B_1(0)) &= \mathcal{H}^{n-1}(\partial^* E'_r \cap E_r^{(1)}) + \mathcal{H}^{n-1}(\partial^* E'_r \cap E_r^{(0)}) \\ &\quad + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} = \nu_{E_r}\}) + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}), \end{aligned} \tag{3.121}$$

and analogously

$$\begin{aligned} P(E_r; B_1(0)) &= \mathcal{H}^{n-1}(\partial^* E_r \cap E_r'^{(1)}) + \mathcal{H}^{n-1}(\partial^* E_r \cap E_r'^{(0)}) \\ &\quad + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} = \nu_{E_r}\}) + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}). \end{aligned} \tag{3.122}$$

Summing both sides of (3.121) and (3.122), and using Leibniz formulas (1.2) for the reduced boundary of an intersection of sets with finite perimeter we get

$$\begin{aligned} P(E'_r;B_1(0)) + P(E_r;B_1(0)) = P(E'_r \cap E_r;B_1(0)) + P(E'^c_r \cap E^c_r;B_1(0)) \\ + 2\mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}). \end{aligned} \tag{3.123}$$

Taking the lim sup on both sides we get

$$\begin{aligned} P(E'_0;B_1(0)) + P(E_0;B_1(0)) &= \limsup_{r\to 0^+}[P(E'_r \cap E_r;B_1(0)) + P(E'^c_r \cap E^c_r;B_1(0)) \\ &+ 2\mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\})] \\ &\geq \liminf_{r\to 0^+}[P(E'_r \cap E_r;B_1(0)) + P(E'^c_r \cap E^c_r;B_1(0)) \\ &+ 2\mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\})] \\ &\geq P(E'_0 \cap E_0;B_1(0)) + P(E'^c_0 \cap E^c_0;B_1(0)) \\ &+ 2\liminf_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}) \\ &= P(E'_0;B_1(0)) + P(E_0;B_1(0)) \\ &+ 2\liminf_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}), \end{aligned} \tag{3.124}$$

where in the last equality we use again identity (3.123) for E'_0, E_0 and the fact $\mathcal{H}^{n-1}(\partial^* E'_0 \cap \partial^* E_0 \cap \{\nu_{E'_0} \neq \nu_{E_0}(x)\}) = 0$.

By (3.124) we immediately deduce

$$\liminf_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}) = 0.$$

Moreover since (3.124) is true for every subsequence $r_j \to 0^+$ we can choose r_j such that

$$\limsup_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}) = \lim_{j\to\infty} \mathcal{H}^{n-1}(\partial^* E'_{r_j} \cap \partial^* E_{r_j} \cap \{\nu_{E'_{r_j}} \neq \nu_{E_{r_j}}\}),$$

and we immediately deduce that

$$\lim_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}) = 0.$$

Using this last information again in (3.124), we obtain

$$\lim_{r\to 0^+}[P(E'_r \cap E_r;B_1(0)) + P(E'^c_r \cap E^c_r;B_1(0))] = P(E'_0 \cap E_0;B_1(0)) + P(E'^c_0 \cap E^c_0;B_1(0)),$$

which by the lower semicontinuity of the perimeter implies separately

$$\lim_{r\to 0^+} P(E'_r \cap E_r;B_1(0)) = P(E'_0 \cap E_0;B_1(0)),$$

and

$$\lim_{r\to 0^+} P(E'^c_r \cap E^c_r;B_1(0)) = P(E'^c_0 \cap E^c_0;B_1(0)).$$

This is exactly our desired result. □

The purpose of the previous propositions is to show that the class $\mathcal{J}_p$ is much richer than the class of C^1-manifolds. Nevertheless, we were able to cover condition (1.2) of Definition 3.2.7, by using the convergence of the perimeter

$$\lim_{r\to 0^+} P(F_{r,j};B_1(0)) = P(E_{0,j};B_1(0)), \ 1 \leq j \leq N_x. \tag{3.125}$$

However, we want to show that (3.125) is not necessary in order to have a non-vanishing upper isoperimetric profile at x. In the next example we exhibit a rectifiable set Γ in $\mathbb{R}^2$ such that there exists a set of Hausdorff dimension α $(0 < \alpha < 1)$ on which Γ admits an asymptotic upper isoperimetric profile but the limit (3.125) diverges to $+\infty$.

Example 3.5.5 (Cantor's home)**.** *We work in* $\mathbb{R}^2$*. We define a sequence of closed set, say* $(J_n)_{n=1}^{\infty}$*, following the usual way to construct the* Cantor's middle third set *(see [63, Subsection 4.10]): let* $J_1 := [0,1]$ *and define* $J_n := \frac{J_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{J_{n-1}}{3}\right)$*. Set* $C_n := \bigcap_{k=1}^{n} J_n$*.*

Now fix $2 < s < 3$ *and consider by induction the following sets:*

$$\mathscr{C}_1 := J_1 \times \left[0, \frac{1}{s-1}\right],$$

and

$$\mathscr{C}_n := \mathscr{C}_{n-1} \setminus (C_{n-1} \setminus C_n \times (-\infty, s_n)), \quad (n \geq 2)$$

where

$$s_n := \frac{s^{1-n}}{(s-1)} = \sum_{i=n}^{\infty} \frac{1}{s^i} \quad \textit{and} \quad \frac{1}{s-1} = \sum_{i=1}^{\infty} \frac{1}{s^i}.$$

We define the Cantor's home $\mathscr{C} \subset \mathbb{R}^2$ *as*

$$\mathscr{C} := \bigcap_{n=1}^{\infty} \mathscr{C}_n.$$

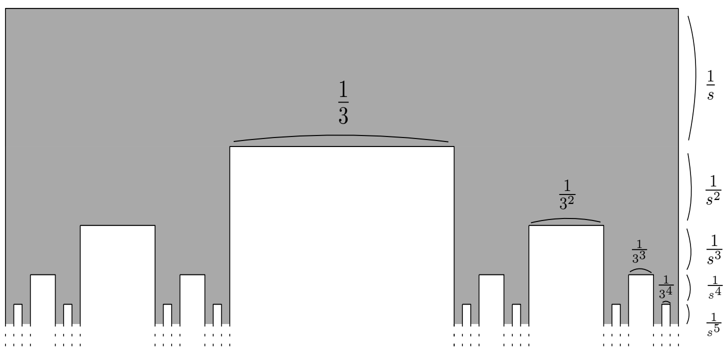

Figure 3.2: *Cantor's home*

By construction $\mathscr{C}$ *is a closed set and* $P(\mathscr{C}) < \infty$*. Indeed it can be easily verified that*

$$P(\mathscr{C}_{n+1}) := P(\mathscr{C}_n) + \frac{2^n}{(s-1)s^n}, \; n = 1, 2, \ldots,$$

which means

$$P(\mathscr{C}) \leq \liminf_{n\to\infty} P(\mathscr{C}_n) = P(\mathscr{C}_1) + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2^n}{(s-1)s^n} < \infty,$$

where in the last inequality we have used $s > 2$*.*

We claim that $\partial^*\mathscr{C}$ *admits a non vanishing upper isoperimetric profile at every* $x \in \mathbb{R}^2$*. As a consequence* $\partial^*\mathscr{C} \in \mathcal{J}_p$ *for every* $p > 1$*.*

To prove our claim, notice that if we call $C \subset [0,1]$ the Cantor's set, i.e.

$$C = \bigcap_{n=1}^{\infty} J_n, \tag{3.126}$$

then it is easy to see that for every $x \in \mathbb{R}^2 \setminus (C \times \{0\})$ our claim is satisfied. Therefore, we need only to prove that for $x \in C \times \{0\}$ our claim holds.

If $x \in C \times \{0\}$, by using the fact that the number of connected components of $J_n \cap (x_1 - \frac{r}{2}, x_1 + \frac{r}{2})$ can be asymptotically estimated by $2^{n - \log_{1/3} r}$ as $n \to \infty$, together with the fact that $s < 3$, it is possible to check that

$$\Theta^n(\mathcal{L}^2 \llcorner \mathscr{C}, x) = 0, \tag{3.127}$$

while

$$\Theta^{n-1}(\mathcal{H}^1 \llcorner \partial^* \mathscr{C}, x) = +\infty. \tag{3.128}$$

Now we denote the generic point $x \in \mathbb{R}^2$ as $x = (x_1, x_2)$ where $x_1, x_2 \in \mathbb{R}$ and we prove that conditions (1) and (2) of Definition 3.2.7 are satisfied with $N_x = 1$. Instead of the balls $B_r(x)$ we prefer to work with the squares $Q_r(x)$. It is clear that everything will be true also for balls.

Pick $x \in \mathscr{C} \times \{0\}$ and consider $r_x > 0$ such that $Q_{r_x}(x) \subset \Omega$. Set

- $E_{0,1} := Q_1(0)$;
- $F_{r,1} := Q_1(0) \setminus \frac{\mathscr{C} - x}{r}$ *for every* $r \leq r_x$.

First of all, since for each r the sets $F_{r,1}$ are connected open sets with finite perimeter, then they are indecomposable (see Remark 1.3.6). Thanks to (3.127), we can apply Proposition 3.2.13 to deduce that there exists an $0 < r'_x < r_x$ such that for $r \in (0, r'_x)$ the family $(F_{r,1})_r$ is left-continuous. Moreover, conditions (1.1) and (2.1) immediately follow from construction and from (3.127), respectively.

In order to show that also condition (1.2) is satisfied, first of all notice that for each $r < r_x$ the sets $F_{r,1}$ are open connected and of finite perimeter, hence in particular they are indecomposable. We claim that for every $r \in (0, r_x)$ and every $\lambda \in (0, 1/2]$ we have

$$h_{F_{r,1}}(\lambda) \geq \frac{1}{3}. \tag{3.129}$$

In order to show (3.129) we shall prove that for every $r \in (0,1)$

$$\min\{|E|, |F_{r,1} \setminus E|\} \leq 3\mathcal{H}^1(\partial^* E \cap F_{r,1}^{(1)}),\ E \subset F_{r,1}. \tag{3.130}$$

This can be achieved by proving that for every $r \in (0,1)$ it holds the following Poincaré's inequality

$$\int_{F_{r,1}} |u - \overline{u}|\, dx \leq 3|Du|(F_{r,1}),\ u \in BV(Q_1(0)), \tag{3.131}$$

where $\overline{u} := \fint_{F_{r,1}} u$. Then (3.130) follows by choosing $u = \mathbb{1}_E$ in (3.131), since in this case

$$\int_{F_{r,1}} |u - \overline{u}|\, dx \geq \min\{|E|, |F_{r,1} \setminus E|\},$$

and

$$\mathcal{H}^{n-1}(\partial^* E \cap F_{r,1}^{(1)}) \geq \mathcal{H}^{n-1}(\partial^* E \cap F_{r,1}) = |Du|(F_{r,1}).$$

Given $t \in \mathbb{R}$ we write

$$F_t := \{x_2 \in \mathbb{R} \mid (t, x_2) \in F\}, \text{ and } F^t := \{x_1 \in \mathbb{R} \mid (x_1, t) \in F\}.$$

Notice that for each $r \in (0, r_x)$ the sets $F_{r,1}$ have the following two properties

1. $(x_1, x_2) \in F_{r,1}$ *and* $(x_1, y_2) \in F_{r,1}$ *implies* $(x_1, \lambda x_2 + (1-\lambda)y_2) \in F_{r,1}$ *for every* $\lambda \in [0, 1]$*;*
2. $(x_1, x_2) \in F_{r,1}$*,* $(y_1, x_2) \in F_{r,1}$ *and* $x_2 \in (-1/2, 0)$ *implies* $(\lambda x_1 + (1-\lambda)y_1, x_2) \in F_{r,1}$ *for every* $\lambda \in [0, 1]$*.*

We show that any set $F \subset Q_1(0)$ satisfying (1) and (2) admits a Poincaré's inequality like (3.131)*.*

Indeed we have

$$\begin{aligned}\int_F |u - \overline{u}|\, dx &= 2\int_{-1/2}^{0} \left[\fint_F \left| u(x_1, x_2) - \left(\fint_F u(y_1, y_2)\, dy_1 d\, y_2 \right) \right| dx_1\, dx_2 \right] dt \\ &\le 2\int_{-1/2}^{0} \left[\int_F \left(\fint_F |u(x_1, x_2) - u(y_1, y_2)|\, dy_1 d\, y_2 \right) dx_1\, dx_2 \right] dt.\end{aligned}$$

If $t \in (-1/2, 0)$, using the triangle inequality we can write

$$|u(x_1, x_2) - u(y_1, y_2)| \le |u(x_1, x_2) - u(x_1, t)| + |u(x_1, t) - u(y_1, t)| + |u(y_1, t) - u(y_1, y_2)|,$$

hence by the Fundamental Theorem of Calculus we have

$$\begin{aligned}\int_F |u - \overline{u}|\, dx \le\ & 2\int_{-1/2}^{0} \left[\int_F \left(\fint_F |D_2 u|(F_{x_1})\, dy_1 d\, y_2 \right) dx_1\, dx_2 \right] dt \\ &+ 2\int_{-1/2}^{0} \left[\int_F \left(\fint_F |D_1 u|(F^t)|\, dy_1 d\, y_2 \right) dx_1\, dx_2 \right] dt \\ &+ 2\int_{-1/2}^{0} \left[\int_F \left(\fint_F |D_2 u|(F_{y_1})\, dy_1 d\, y_2 \right) dx_1\, dx_2 \right] dt,\end{aligned}$$

and finally by using Fubini's Theorem

$$\begin{aligned}\int_F |u - \overline{u}|\, dx \le\ & \int_{-1/2}^{1/2} |D_2 u|(F_{x_1}) |\mathcal{L}^1(F_{x_1})\, dx_1 + 2|F| \int_{-1/2}^{0} |D_1 u|(F^t)\, dt \\ &+ \int_{-1/2}^{1/2} |D_2 u|(F_{y_1}) |\mathcal{L}^1(F_{y_1})\, dy_1 \\ &\le |D_2 u|(F) + 2|F||D_1 u|(F) + |D_2 u|(F) \\ &\le (1 + 2|F|)|Du|(F).\end{aligned}$$

Since $|F| \le 1$, this shows exactly (3.131)*.* △

3.5.2 A counterexample

Now we want to exploit the idea of the previous example to show that, the indecomposability condition together with condition (1.2) of Definition 3.2.7 are crucial in order to get the validity of Theorem 1.

Example 3.5.6 (Optimality for the class $\mathcal{J}_p$). *We start by showing that the indecomposability assumption on the sets $(F_{r,j})$ in Definition 3.2.7, cannot be removed in order to get the validity of Theorem 1. For this purpose we shall construct a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set Γ in $\mathbb{R}^2$ with $\mathcal{H}^{n-1}(\Gamma) < \infty$, such that there exists $\Gamma_0 \subset \mathbb{R}^2$ with $dim_{\mathcal{H}}(\Gamma_0) = \log_3(2)$, with the following properties*

(a) for every $x \in \Gamma_0$ there exists a left-continuous family of sets $F_r \subset B_1(0)$ $(r > 0)$ satisfying $\lim_{r\to 0^+} |F_r \Delta B_1(0)| = 0$;

(b) there exists a function $u \in SBV^2(\mathbb{R}^2; \Gamma)$ such that for every $x \in \Gamma_0$ the blow-up $u_{r,x}$ does not converge.

To construct such a Γ, we start by considering $\tilde{\mathscr{C}} \subset \mathbb{R}^2$ the reflection of the Cantor's home given in Example 3.5.5 with respect to the x_1 axis, i.e.

$$\tilde{\mathscr{C}} := \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, -x_2) \in \mathscr{C}\}.$$

We define $E := \mathscr{C} \cup \tilde{\mathscr{C}}$. The set E can be seen also as the limit in measure of $\mathscr{C}_n \cup \overline{\mathscr{C}}_n$ when $n \to \infty$, where $\mathscr{C}_n$ is the approximated Cantor's home at the n-th step (see Example 3.5.5) and $\tilde{\mathscr{C}}_n$ is its reflection with respect to the x_1-axis (see Figure 3.3). Clearly E has a non vanishing upper isoperimetric profile for every $x \in \mathbb{R}^2 \setminus (C \times \{0\})$, where C denotes the Cantor's set (see (3.126)). By arguing as in Example 3.5.5 we know that E is a set of finite perimeter and

$$\Theta^n(\mathcal{L}^n \llcorner E, x) = 0, \quad x \in C \times \{0\}.$$

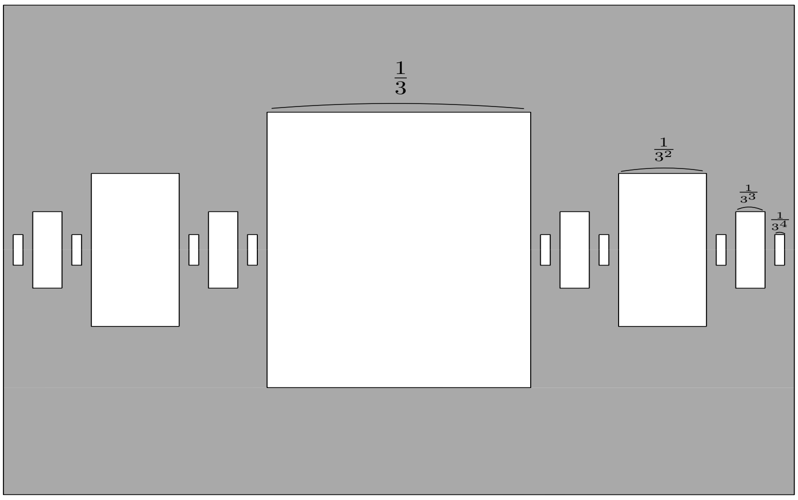

Figure 3.3: *Approximation of $\mathscr{C} \cup \tilde{\mathscr{C}}$ at the fifth step.*

Now let (C_n) and (s_n) be the sequence of sets and the sequence of numbers defined in Example 3.5.5, respectively. Define the following function

$$u(x) := \begin{cases} 1 & \text{if } x \in C_{2n-1} \setminus C_{2n} \times (-s_{2n}, s_{2n}) \\ -1 & \text{if } x \in C_{2n} \setminus C_{2n+1} \times (-s_{2n+1}, s_{2n+1}) \\ 0 & \text{otherwise.} \end{cases} \tag{3.132}$$

By Proposition 1.4.10 we know $u \in SBV^2(\mathbb{R}^2; \partial^* E)$. Call $f\colon \mathbb{R} \to \mathbb{R}$ the restriction of u to the x_1 axis, i.e.

$$f(\cdot) := u(\cdot, 0).$$

Notice that given $x \in C \times \{0\}$, then if the blow-up of u at $x = (x_1, x_2)$ converges as $r \to 0^+$ then also the blow-up of f at x_1 must converge. To see this, by using the fact that the parameter s of Example 3.5.5 has been chosen strictly less than 3, it is not difficult to show that

$$\lim_{r \to 0^+} \mathcal{L}^1(\{y_1 \in (-1, +1) \mid u_{r,x}(y_1, y_2) = f_{r,x_1}(y_1) \text{ for every } y_2\}) = 2.$$

This means that any limit $u_{0,x}$ in $L^1(B_1)$ of $(u_{r,x})$ must be constant along the segments orthogonal to the x_1-axis and contained in B_1, and moreover it must satisfy

$$\lim_{r \to 0^+} \|f_{r,x_1}(\cdot) - u_{0,x}(\cdot, 0)\|_{L^1} = 0.$$

Therefore, given $x = (x_1, x_2)$, if we want to prove that $(u_{r,x})$ does not converge, we can reduce ourselves to prove that f_{r,x_1} does not converge.

In view of the previous observation, we claim that for every $x_1 \in C \times \{0\}$, except on a countable set A, f_{r,x_1} does not converge as $r \to 0^+$. For this purpose, we show that given $x_1 \in C \setminus A$, then for every $\epsilon > 0$ there exists a couple of radii $r, r' \leq \epsilon$ such that

$$\|f_{r,x_1} - f_{r',x_1}\|_{L^0(B_1)} \geq \frac{1}{2}. \tag{3.133}$$

To see this it is convenient to write every point in the Cantor's set in base 3. This means that given $x \in \Omega$, then there exists a map σ defined on every positive integer number with values in $\{0, 2\}$, such that

$$x = \sum_{i=1}^{\infty} \frac{\sigma(i)}{3^i}.$$

Define the set A to be the set of points x_1 in the Cantor's set such that there exists $i_0 \in \mathbb{N}$ (depending on x_1) such that for every $i \geq i_0$ the function σ alternates consecutively the values 0 and 2. Clearly the set A is countable. We want to prove our claim on every point $x_1 \in C \setminus A$.

Let $x_1 \in C \setminus A$, then by definition of A there exists a sufficiently large value of n such that $1/3^n \leq \epsilon/2$ and $\sigma(n) = \sigma(n+1) = 0$ or $\sigma(n) = \sigma(n+1) = 2$. Let us suppose to be in the case $\sigma(n) = 2$ (the case $\sigma(n) = 0$ can be treated in the same way). Since $x_1 \in C$, then x_1 belongs to a connected component (an interval) of J_{n-1}, say I (J_n are those defined in Example 3.5.5). We can consider a partion of I made of three closed intervals I_1, I_2, I_3 (with overlapping end-points) each of length $\frac{|I|}{3}$ where I_1 is the most-left one, I_3 is the most right one, and I_2 is in between. By construction of the sets (J_n), we know that $J_n \cap I = I_1 \cup I_3$, and since $\sigma(n) = 2$ then $x_1 \in I_3$. By (3.132), we know that f takes value 1 or -1 on I_2. Let us suppose for example 1. As before we can consider a partion of I_3 made of three closed intervals $I_{3,1}, I_{3,2}, I_{3,3}$ (with overlapping end-points) each of length $|I_3|/3$ where $I_{3,1}$ is the most-left one, $I_{3,3}$ is the most right one, and $I_{3,2}$ is in between. In addition, by (3.132), we know that f assumes the value -1 on $I_{3,2}$. As before, since $\sigma(n+1) = 2$, then $x_1 \in I_{3,3}$.

Now call a the left end-point of I_2 and b the left hand-point of $I_{3,2}$. Clearly we have the following estimates

$$(x_1 - a) \leq \frac{2}{3^n} \quad \text{and} \quad (x_1 - b) \leq \frac{2}{3^{n+1}}. \tag{3.134}$$

Moreover, if we set $r = x_1 - a$ and $r' = x_1 - b$, then we can write

$$\begin{aligned}\|f_{r,x_1} - f_{r',x_1}\|_{L^0(B_1)} &= \fint_{B_1} |f(x_1 + (x_1 - a)y_1) - f(x_1 + (x_1 - b)y_1)| \wedge 1\, dy_1 \\ &= \fint_{B_{(x_1-a)}} |f(x_1 + y_1) - f(x_1 + (x_1 - b)/(x_1 - a)y_1)| \wedge 1\, dy_1.\end{aligned}$$

Using the fact that, by construction, the dilated interval $\frac{(x_1-a)}{(x_1-b)}(I_{3,2} - x_1)$ has left hand-point coincident with the left hand point of the interval $I_2 - x_1$, and that $|f(x_1+y_1) - f(x_1+(x_1 - b)/(x_1 - a)y_1)| = 2$ on $(I_2 - x_1) \cap \frac{(x_1-a)}{(x_1-b)}(I_{3,2} - x_1)$, we can continue the previous inequality by writing

$$\begin{aligned}\|f_{x_1,(x_1-a)} - f_{x_1,(x_1-b)}\|_{L^0(B_1)} &\geq \frac{1}{(x_1-a)}\left|(I_2 - x_1) \cap \frac{(x_1-a)}{(x_1-b)}(I_{3,2} - x_1)\right| \\ &\geq \frac{1}{(x_1-a)} \min\left\{|I_2|, \frac{(x_1-a)}{(x_1-b)}|I_{3,2}|\right\} \\ &= \min\left\{\frac{3^{-n}}{(x_1-a)}, \frac{3^{-(n+1)}}{(x_1-b)}\right\} \\ &\geq \frac{1}{2},\end{aligned}$$

where for the last inequality we use (3.134). This proves our claim and shows that at every $x \in (C \setminus A) \times \{0\}$ the blow-up $u_{r,x}$ does not converge as $r \to 0^+$.

Finally, by setting $\Gamma := \partial^* E$, $\Gamma_0 := (C \setminus A) \times \{0\}$, and $F_r := B_1(0) \setminus E_{r,x}$ we obtain (a) and (b) (the left continuity of the family $(F_{r,x})_{r>0}$ can be easily obtained by exploiting the continuity with respect to the L^1-convergence of the dilations). As a consequence we deduce that $\Gamma \notin \mathcal{J}_p$ for every $p \in (2-\log_3(2), 2]$. In this case it is clear that what fails in the definition of non vanishing upper isoperimetric profile is the indecomposability of the sets $(F_{r,x})_{r>0}$ for every $x \in C \times \{0\}$.

Exploiting the previous idea, it is possible to construct sets Γ, Γ_0 and a function u satisfying (a) and (b) with the additional property that the sets F_r are indecomposable. This together with Theorem 1, immediately implies that on every point of Γ_0 the set Γ satisfies all the properties of Definition 3.2.7 except (1.2). This shows that condition (1.2) is crucial in view of Theorem 1.

The idea is to connect each white rectangle of E (see Figure 3.3) by small bridges without altering the local behavior of the set E on points of the Cantor's set $C \times \{0\}$. To do this, define for each $n \geq 1$, $\delta_n := 1/7^n$. We start by connecting the two white rectangles whose horizontal sides have length $1/3^2$ with the white rectangle whose horizontal sides have length $1/3$ (see figure 3.3) by subtracting to the set E a thin horizontal bridge in the following way

$$E_1 := E \setminus (1/3^2, 1 - 1/3^2) \times (s_3 - \delta_1, s_3).$$

By induction, if we call n-th thin bridge $R_n := (1/3^{n+1}, 1 - 1/3^{n+1}) \times (s_{n+2} - \delta_n, s_{n+2})$ (s_n are defined in Example 3.5.5), then we define for general n (see Figure 3.4 for $n = 3$)

$$E_n := E_{n-1} \setminus R_n.$$

Since by the choice of (δ_n) the rectangles (R_n) are disjoint, by subtracting to E_{n-1} the rectangle R_n, one adds an amount of perimeter which is exactly $2(2^{n+1} - 2)/3^{n+1}$, i.e.

$$P(E_n) = P(E_{n-1}) + 2(2^{n+1} - 2)/3^{n+1}.$$

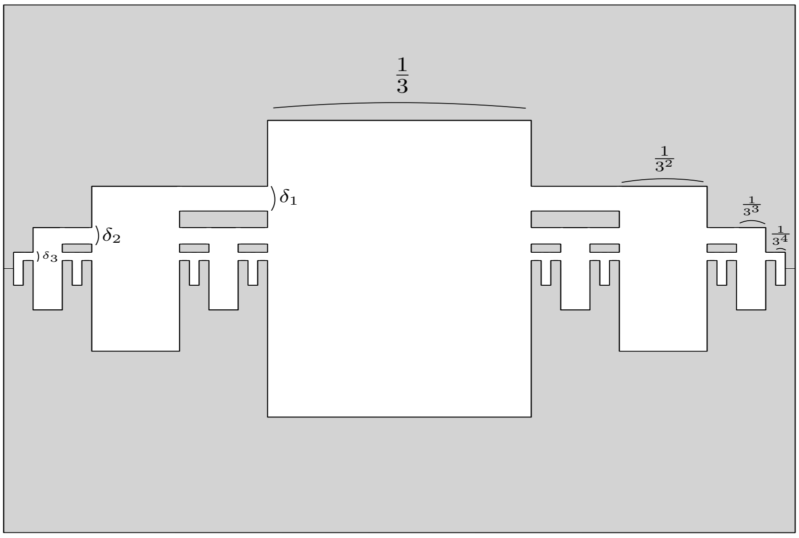

Figure 3.4: E_3

This means that by defining

$$E' := \bigcap_{n=1}^{\infty} E_n,$$

then E' is a closed set of finite perimeter in $\mathbb{R}^2$.

Since $E' \subset E$, this means that we still have the property that for every $x \in C \times \{0\}$ it holds

$$\Theta^n(\mathcal{L}^n \llcorner E', x) = 0,$$

but with the additional property that for every $r > 0$ the open sets $B_1(0) \setminus E'_{r,x}$ are connected and with finite perimeter and hence indecomposable (see Remark 1.3.6). The connectedness comes from the fact that if Q_1 is a connected components (white rectangle) of the set $[C_{n-1} \setminus C_n] \times (-s_n, s_n)$ for some n (where (C_n) and (s_n) are defined in Example 3.5.5), and Q_2 is a connected components (white rectangle) of the set $[C_{m-1} \setminus C_m] \times (-s_m, s_m)$ for some m, both Q_1 Q_2 with non empty intersection with $B_r(x)$ $(x \in C \times \{0\})$, then there must exist a sufficiently large $M \geq \max\{n, m\}$ for which the bridge R_M connects $Q_1 \cap B_r(x)$ with $Q_2 \cap B_r(x)$.

Now we define the function $v \in SBV^2(\mathbb{R}^2; \partial^* E')$ in the following way. If $x \notin \bigcup_{n=1}^{\infty} R_n \cap E$ we define $v(x) := u(x)$ where u is the function defined in (3.132). If $x \in R_n \cap E$ for some n, then by construction there exists a connected components of $J_{n+1} \subset [0,1]$ (see Example 3.5.5), say I, such that $x \in I \times (s_{n+2} - \delta_n, s_{n+2})$. We have two cases: suppose that $I \times (s_{n+2} - \delta_n, s_{n+2})$ connects two rectangles where v has the same value, i.e. -1 or $+1$, then we simply define $v(x)$ to be exactly -1 or $+1$, respectively; otherwise suppose that v changes value (for example from -1 to $+1$), then if we call $p\colon I \to \mathbb{R}$ the linear interpolation between -1 and $+1$ we define

$$v(x) := p(\pi_1(x)),$$

where $\pi_1\colon \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first component (we proceed analogously if v changes value from $+1$ to -1).

Clearly, since v differs from u on a set which is contained in E, and $\Theta^n(\mathcal{L}^n \llcorner E, x) = 0$ for every $x \in C \times \{0\}$, then the blow-up of v has the same behavior of the blow-up of u at *each point in $C \times \{0\}$. It remains to prove that $\nabla v \in L^2(\mathbb{R}^2)$. But by our choice of δ_n, an easy computation shows that we can estimate from above*

$$\int_{\mathbb{R}^2} |\nabla v|^2 \, dx \leq 2 \sum_{n=1}^{\infty} \frac{6^{n+1}}{7^n} < \infty.$$

△

3.6 Non convergence of the blow-up

In this last part we construct a set in $E \subset \mathbb{R}^2$ with the property that its blow-up $(E - x)/r$ does not converge in measure on every point of a set having Hausdorff dimension equal to 1. To show this, we need the following theorem which can be deduced from the results obtained in [74] (see also [45] for a simpler proof). Anyway we decide to present this result with an alternative proof which is more convenient for our purpose.

Theorem 3.6.1. *Let $N \subset (0,1)$. Then N has zero Lebesgue measure if and only if there exists Lipschitz function $u\colon (0,1) \to \mathbb{R}$ such that u is not differentiable from the right at every point of N.*

Proof. If f is lipschitz then the set of point where it is not right differentiable has Lebesgue measure zero from Rademacher's Theorem.

Now let $N \subset \mathbb{R}$ be such that $|N| = 0$. We claim that there exists a Borel set $F \subset \mathbb{R}$ such that for every $x \in N$ we have

$$0 = \liminf_{r \to 0^+} \frac{|F \cap (x, x+r)|}{r} < \limsup_{r \to 0^+} \frac{|F \cap (x, x+r)|}{r} = 1. \tag{3.135}$$

To prove this, notice that for every $0 < \epsilon \leq 1/6$, since $|N| = 0$, we can find a cover of N made of open and disjoint subintervals of $(0,1)$, say $(I_i)_{i=1}^{\infty}$, such that

1. $\sum_{i=1}^{\infty} |I_i| \leq \epsilon$;
2. $N \cap I_i \subset \bigcup_{j=1}^{\infty} \{x \in I_i \mid 1/2^{s(j+1)} < \operatorname{dist}(x, \mathbb{R} \setminus I_i) < 1/2^{sj}\}$, for some $s \in [1/2, 1]$ depending on I_i.

Indeed (1) simply follows by the fact that $|N| = 0$. Moreover, let $a < b$ be the end points of the intervals I_i; then since $|N| = 0$ we have that for every j

$$|\{s \in [1/2, 1] \mid a \in N - 1/2^{sj}\}| = 0,$$

therefore also

$$\left| \bigcup_{j=1}^{\infty} \{s \in [1/2, 1] \mid a \in N - 1/2^{sj}\} \right| = 0.$$

For this reason by choosing $s \in \bigcup_{j=1}^{\infty} \{s \in [1/2, 1] \mid a \in N - 1/2^{sj}\}$ we obtain that

$$a + 1/2^{sj} \notin N, \quad j = 1, 2, 3, \ldots .$$

By repeating the same argument for the right end point we can find $s \in [1/2, 1]$ satisfying (2).

Define $\mathcal{I}^1 \subset (0,1)$ to be a cover of N made of open intervals satisfying (1) and (2) with $\epsilon = 1/6$. By induction we define $\mathcal{I}^i$ in the following way. For every $I \in \mathcal{I}^{i-1}$ we consider the set

$$N_j := N \cap \{x \in I \mid 1/2^{s(j+1)} < \mathrm{dist}(x, \mathbb{R} \setminus I) < 1/2^{sj}\},$$

where $s \in [1/2, 1]$ is relative to I. Since $|N_j| = 0$ we can use the claim to find a cover of N_j made of open and disjoint subintervals of $\{x \in I \mid 1/2^{s(j+1)} < \mathrm{dist}(x, \mathbb{R} \setminus I) < 1/2^{sj}\}$, say $(I_i)_{i=1}^{\infty}$, satisfying (1) and (2) with $\epsilon = 1/6^j$. Finally, we call $\mathcal{I}^i$ the family made of all open intervals obtained as in the previous procedure, by letting I varies in $\mathcal{I}^{i-1}$ and j varies in $\mathbb{N}$.

We set

$$F := \bigcup_{i=1}^{\infty} \Big(\bigcup_{I \in \mathcal{I}^{2i-1}} I \setminus \bigcup_{I \in \mathcal{I}^{2i}} I \Big), \tag{3.136}$$

and we claim that F does the job. Clearly F is Borel since every $\mathcal{I}^i$ is a family made of open intervals. Moreover, whenever $x \in N$, since for every i the family $\mathcal{I}_i$ covers N, then for every i there exists $\overline{I} \in \mathcal{I}^i$ such that $x \in \overline{I}$. Moreover, by property (2) there exists j such that $x \in \{x \in \overline{I} \mid 1/2^{s(j+1)} < \mathrm{dist}(x, \mathbb{R} \setminus \overline{I}) < 1/2^{sj}\}$ for some $s \in [1/2, 1]$. Therefore

$$\Big(x, x + \frac{1}{2^{s(j+2)}}\Big) \subset \bigcup_{k=j-1}^{j+1} \{x \in \overline{I} \mid 1/2^{s(k+1)} < \mathrm{dist}(x, \mathbb{R} \setminus \overline{I}) < 1/2^{sk}\},$$

and since by construction the intervals $I \in \mathcal{I}_{i+1}$ which are contained in the set $\{x \in I_i \mid 1/2^{s(k+1)} < \mathrm{dist}(x, \mathbb{R} \setminus I_i) < 1/2^{sk}\}$ are such that $\sum |I| \leq \epsilon$ with $\epsilon = 1/6^k$, we can write

$$\Big| \bigcup_{I \in \mathcal{I}^{i+1}} I \cap \Big(x, x + \frac{1}{2^{s(j+2)}}\Big) \Big| \leq \sum_{k=j-1}^{j+1} \frac{1}{6^k}$$

If i is odd, by (3.136) we have that $\overline{I} \setminus \bigcup_{I \in \mathcal{I}_{i+1}} I \subset F$, and then

$$\begin{aligned} \frac{|F \cap (x, x + 1/2^{s(j+2)})|}{1/2^{s(j+2)}} &\geq 2^{s(j+2)} \Big(\frac{1}{2^{s(j+2)}} - \sum_{k=j-1}^{j+1} \frac{1}{6^k} \Big) = 1 - \frac{2^{s(j+2)}}{6^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &= 1 - \frac{2^{(s-1)j+3s}}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &\geq 1 - \frac{2^3}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k}, \end{aligned} \tag{3.137}$$

where for the last inequalty we use $s \in [1/2, 1]$. Therefore for each i odds there exists a corresponding j_i, with $j_i \to \infty$ as $i \to \infty$, satisfying (3.137), hence by letting $i \to \infty$ among all odds numbers, this proves

$$\limsup_{r \to 0^+} \frac{|F \cap (x, x + r)|}{r} = 1.$$

If i is even, by (3.136) we have that $F \cap \overline{I} \subset \bigcup_{I \in \mathcal{I}_{i+1}} I$, and then

$$\begin{aligned} \frac{|F \cap (x, x + 1/2^{s(j+2)})|}{1/2^{s(j+2)}} &\leq 2^{s(j+2)} \sum_{k=j-1}^{j+1} \frac{1}{6^k} = \frac{2^{s(j+2)}}{6^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &= \frac{2^{(s-1)j+3s}}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &\leq \frac{2^3}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k}, \end{aligned} \tag{3.138}$$

where for the last inequalty we use $s \in [1/2, 1]$. Arguing as before, this proves

$$\liminf_{r \to 0^+} \frac{|F \cap (x, x + r)|}{r} = 0.$$

Finally, define

$$u(t) := |F \cap (0, t)|, \ t \in (0, 1).$$

Clearly u is 1-Lipschitz and moreover $(u(x + r) - u(x))/r = |(F \cap (x, x + r))|/r$ for every $0 < r < 1 - x$. By (3.135) we immediately deduce that u is not right differentiable at every $x \in N$. □

Theorem 3.6.2. *There exists a set $E \subset \mathbb{R}^2$ of finite perimeter, such that the set of point where its blow-up $(E - x)/r$ does not converge locally in measure has Hausdorff dimension* 1.

Proof. Let $N \subset (0, 1)$ be a set of Hausdorff dimension equal to 1. It can be easily constructed as a countable union of sets N_k with positive $\mathcal{H}^{1-1/k}$-measure. Clearly N has zero Lebesgue measure.

Let $u \colon (0, 1) \to \mathbb{R}$ be the 1-lipschitz function given by the previous proposition, which is not right differentiable at every point of N. Define $E := \{x \in \mathbb{R}^2 \mid 0 < x_1 < 1, \ 0 < x_2 < u(x_1)\}$. We claim that at every point x of the form $x_1 \in N$ and $x_2 = u(x_1)$ the blow-up of E at x does not converge in measure.

Indeed, since u is 1-Lipschitz then in the cylinder $C_1(0)$ the set $(E - x)/r$ can be described as the subgraph of the function $y_1 \to (u_{r,x_1}(y_1) - u(x_1))/r$ for $y_1 \in (-1, 1)$ and for every $r < \min x_1, 1 - x_1$. Hence the the convergence of $(E - x)/r$ locally in measure implies in particular the convergence in $L^1_{loc}((0, 1))$ of the sequence $((u_{r,x_1}(y_1) - u(x_1))/r)$ to some function $v(y_1)$. Moreover, since for every $\lambda > 0$ we have

$$v(\lambda y_1) = \lim_{r \to 0^+} (u_{r,x_1}(\lambda y_1) - u(x_1))/r = \lambda \lim_{r \to 0^+} (u_{\lambda r,x_1}(y_1) - u(x_1))/\lambda r = \lambda v(y_1),$$

we have that v is positively one-homogeneous. But since u is 1-Lipschitz, then the L^1_{loc} convergence can be improved to a uniform convergence on the closed interval $[0, 1]$, i.e.

$$\lim_{r \to 0^+} \sup_{y_1 \in [0,1]} \left| \frac{u(x_1 + r y_1) - u(x_1)}{r} - v(y_1) \right| = 0,$$

and thanks to the positively one homogeneity of v this immediately implies the right differentiability of u at x_1 with $u'(x_1) = v(1)$ which is a contradiction. This proves the theorem. □

Chapter 4

Elastodynamics in domains with growing cracks

Contents

4.1 Preliminary results

In this first section, after recalling some notation, we provide the fundamental tools to prescribe the boundary conditions for the equations of elastodynamics in a domain with arbitrary growing cracks (0.13), which we recall to be

$$\ddot{u}(t) - \operatorname{div}[\mathbb{C}\,\mathcal{E}u(t)] - \gamma\operatorname{div}[\mathbb{B}\,\mathcal{E}\dot{u}(t)] = f(t), \quad \text{in } \Omega\setminus\Gamma(t), \tag{4.1}$$

where $\Omega\subset\mathbb{R}^n$ is a sufficiently regular open set representing the reference configuration and the family $(\Gamma(t))$ represents the crack sets at each time t which are increasing in time with respect to sets inclusion, i.e. $\Gamma(s)\subseteq\Gamma(t)$ for $s<t$. More precisely, we split $\partial\Omega$ into two disjoint Borel subsets $\partial_N\Omega$ and $\partial_D\Omega$, the Neumann and the Dirichlet part of the boundary, respectively. By using the results obtained in Chapter 2, we first introduce the space of all admissible Neumann force, for which we will be able to prescribe the Neumann boundary conditions on $\partial_N\Omega$(see Definition 4.1.5). As it will be clear from Definition 4.1.5, this space depends on the geometry of the crack sets, through the weight function Θ introduced in Chapter 2. In order to prescribe the Dirichlet boundary conditions, we need to introduce the space of all displacements with zero trace on $\partial_D\Omega$. It is crucial that such space is sequentially closed with respect to the weak notion of convergence (0.7). Since in our context the crack sets are in general only $(n-1)$-rectifiable and with finite $\mathcal{H}^{n-1}$-measure, they might have

non trivial interaction with $\partial_D\Omega$ (they might be dense subset of Ω), hence this property does not follow by standard arguments (as in the context of Sobolev spaces) which would ensure the closeness of the trace operator. In this case the closure property is a direct consequence of Theorem 2.2.7 (see Proposition 4.1.9 and Remark 4.1.10). In the last part of this section, precisely in Lemma 4.1.13, we give a precise meaning to the second derivative in time $\ddot{u}(t)$ appearing in (4.1). Precisely, given a function $u(t)$ assuming values at each time t in a different separable Hilbert space V_t, such that $V_s \subset V_t$ whenever $s < t$, we specify the value of $\ddot{u}(t)$ as an element of the dual space of V_t.

4.1.1 Notation

We denote the space of $n\times n$ matrices with real entries as $\mathbb{M}^{n\times n}$ endowed with the euclidean scalar product

$$\xi\cdot\eta := \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\xi_{ij}\eta_{ij}\right),\quad \xi,\eta\in\mathbb{M}^{n\times n},$$

and we denote by $|\cdot|$ the associated norm. With $\mathcal{L}(\mathbb{M}^{n\times n})$ we denote the space of continuous linear maps of $\mathbb{M}^{n\times n}$ into itself. The subspace of symmetric $n\times n$ matrices is denoted by $\mathbb{M}^{n\times n}_{sym}$. Moreover, when Ω is an open set of $\mathbb{R}^n$, in order to simplify the notation, we denote the space $L^2(\Omega,\mathbb{R}^n)$ by H, with scalar products $\langle\cdot,\cdot\rangle_H$ and with associated norm $\|\cdot\|_H$. Analogously we denote the space $L^2(\Omega,\mathbb{M}^{n\times n})$ by H_n, with scalar product $\langle\cdot,\cdot\rangle_{H_n}$ and with associated norm $\|\cdot\|_{H_n}$

Definition 4.1.1. *We say that $\mathbb{C}\colon\Omega\to\mathcal{L}(\mathbb{M}^{n\times n})$ is a bounded symmetric and positive definite tensor field, if it is $\mathcal{L}^n$-measurable and*

- $\|\mathbb{C}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{M}^{n\times n}))}<\infty$,
- $\mathbb{C}(x)\xi\in\mathbb{M}^{n\times n}_{sym},\ \forall\xi\in\mathbb{M}^{n\times n},$ *for a.e.* $x\in\Omega$,
- $\mathbb{C}(x)\xi\cdot\eta=\xi\cdot\mathbb{C}(x)\eta,\ \forall\xi,\eta\in\mathbb{M}^{n\times n},$ *for a.e.* $x\in\Omega$ *(symmetry),*
- $\mathbb{C}(x)\xi\cdot\xi\geq\gamma_0|\xi|^2,\ \forall\xi\in\mathbb{M}^{n\times n}_{sym},$ *for a.e.* $x\in\Omega$ $(\gamma_0>0)$ *(positiveness),*

(which are the usual assumptions in linear elasticity). The striclty positive number γ_0 is called ellipticity constant of $\mathbb{C}$. Under the previous assumption on $\mathbb{C}$, given any $\mathcal{L}^n$-measurable function $\xi\colon\Omega\to\mathbb{M}^{n\times n}_{sym}$, we write

$$\|\xi\|_{H_n^{\mathbb{C}}} := \int_\Omega \mathbb{C}(x)\xi(x)\cdot\xi(x)\,dx. \tag{4.2}$$

Remark 4.1.2. *Thanks to the symmetry and positiveness properties of $\mathbb{C}$, it follows that the function $\|\cdot\|_{H_n^{\mathbb{C}}}$ defined on the real vector space of all measurable functions $\xi\colon\Omega\to\mathbb{M}^{n\times n}_{sym}$ is a norm. Moreover, by using also the L^∞-bound, the norm $\|\cdot\|_{H_n^{\mathbb{C}}}$ is equivalent to the norm $\|\cdot\|_{H_n}$.* ■

4.1.2 Boundary conditions

We recall that $GSBD^2_2(\Omega)$ is the space of all functions $u\in GSBD(\Omega)$ such that $u\in L^2(\Omega,\mathbb{R}^n)$, and their symmetric approximate gradients $\mathcal{E}u$ belong to $L^2(\Omega,\mathbb{M}^{n\times n}_{sym})$ (see Definition 1.4.28).

Proposition 4.1.3. *Let Ω be an open set of $\mathbb{R}^n$, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then the space $GSBD^2_2(\Omega;\Gamma) := \{u \in GSBD^2_2(\Omega) \mid J_u \subset \Gamma\}$ endowed with the scalar product*

$$\langle u, v\rangle_2 = \langle u, v\rangle_H + \langle \mathcal{E}u, \mathcal{E}v\rangle_{H_n},$$

is a separable Hilbert space.

Proof. By using Proposition 1.4.29 together with Remark 1.4.30 we already know that it is an Hilbert space. To prove the separability consider the embedding $j\colon GSBD^2_2(\Omega;\Gamma) \to L^2(\mathbb{R}^n \times \mathbb{R}^{n^2})$ defined by $j(u) := (u, \mathcal{E}u)$. By the well known fact that subspace of separable metric space are separable, since j is an embedding, we deduce that also $GSBD^2_2(\Omega;\Gamma)$ is separable. □

The dual $GSBD^2_2(\Omega;\Gamma)^*$ will not be identified with $GSBD^2_2(\Omega;\Gamma)$, but instead will be endowed with a pairing consistent with the L^2 inner product, as is usually done for the duals of Sobolev spaces. Since

$$GSBD^2_2(\Omega;\Gamma) \subset L^2(\Omega, \mathbb{R}^n)$$

is a dense embedding, we have

$$L^2(\Omega, \mathbb{R}^n) = L^2(\Omega, \mathbb{R}^n)^* \subset GSBD^2_2(\Omega;\Gamma)^*,$$

and $L^2(\Omega, \mathbb{R}^n)$ is densely embedded in $GSBD^2_2(\Omega;\Gamma)^*$.

In the case Ω has also finite perimeter, the trace operator $Tr(\cdot)$ can be extended to the space $GSBD(\Omega;\Gamma)$, using the notion of *approximate limit* on the point of the reduced boundary $\partial^*\Omega$ (see Definition 2.1.11). Moreover the following theorem holds true.

Theorem 4.1.4. *Let Ω be an open set of $\mathbb{R}^n$ of finite perimeter, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then there exists a measurable function $\Theta\colon \partial^*\Omega \to \mathbb{R}^+$ such that*

(a) $\mathcal{H}^{n-1}(\{\Theta = 0\}) = 0$ and $\Theta \in L^\infty(\partial^*\Omega, \mathcal{H}^{n-1})$ (in particular $\|\Theta\|_\infty \le 1$);

(b) For every $u \in GSBD^2_2(\Omega;\Gamma)$ we have

$$\left(\int_{\partial^*\Omega} |Tr(u)|^2 \Theta \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \le C(\|u\|_H + \|\mathcal{E}u\|_{H_n}), \tag{4.3}$$

where $C = C(n) > 0$ is a constant depending only on the dimension.

Proof. Let Θ^+ be the weight functions given by Theorem 2.1.14. If we define

$$\Theta(x) := \Theta^+(x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^*\Omega,$$

then (a) and (b) are a direct consequence of Theorem 2.1.14 with $p = 2$. □

Now let $\partial_N\Omega$ be a Borel subset of $\partial^*\Omega$ which we call the Neumann part of the boundary. In order to impose to Equation (4.1) a prescribed Neumann boundary condition on $\partial_N\Omega$, we are led to study the continuity property of the following linear form (see Remark 4.2.6)

$$u \mapsto \int_{\partial_N\Omega} F \cdot Tr(u)\, d\mathcal{H}^{n-1}, \quad u \in GSBD^2_2(\Omega;\Gamma), \tag{4.4}$$

where $F\colon \partial_N\Omega \to \mathbb{R}^n$ is some measurable vector field representing the Neumann force. In view of inequality (4.3) we introduce the following space.

Definition 4.1.5 (Admissible Neumann term)**.** *Let Ω, Γ, and Θ be as in Theorem 4.1.4, and let $\partial_N\Omega$ be a Borel subset of $\partial^*\Omega$. We define $N_\Theta := L^2(\partial_N\Omega, \Theta\mathcal{H}^{n-1})$, and we denote by N_Θ^* its dual. We identify N_Θ^* with the space of measurable vector fields $F\colon \partial_N\Omega \to \mathbb{R}^n$ such that*

$$\int_{\partial_N\Omega} \frac{|F|^2}{\Theta}\, d\mathcal{H}^{n-1} < \infty.$$

We consider the corresponding duality pairing between N_Θ^* and N_Θ given by

$$\langle F, g\rangle_\Theta := \int_{\partial_N\Omega} F\cdot g\; d\mathcal{H}^{n-1},$$

whenever $F \in N_\Theta^*$ and $g \in N_\Theta$. The induced norm is denoted by $\|\cdot\|_{N_\Theta}^*$.

Putting together the definition of N_Θ and Theorem 4.1.4 we have the following result.

Proposition 4.1.6. *Let Ω, Γ, and Θ be as in Theorem 4.1.4, and let $\partial_N\Omega$ be a Borel subset of $\partial^*\Omega$. If $F \in N_\Theta^*$ then the linear form defined in (4.4) belongs to $GSBD_2^2(\Omega;\Gamma)^*$.*

Proof. It is enough to use inequality (4.3) to have

$$\int_{\partial_N\Omega} F\cdot Tr(u)\; d\mathcal{H}^{n-1} \le \|F\|_{N_\Theta}^* \|Tr(u)\|_{N_\Theta} \le C\|F\|_{N_\Theta}^*\big(\|u\|_H + \|\mathcal{E}u\|_{H_n}\big).$$

□

Remark 4.1.7. *Our choice of Neumann forces, in some sense, is natural. Indeed looking at the construction of Θ made in Section 2.1.1, roughly speaking, the function Θ measures the "closeness" of Γ to the boundary. From a physical point of view, this might be interpreted as the fact that, when the elastic material between the Neumann boundary and the crack is infinitesimally small, then the elastic reaction to a constant (in modulus) traction force should be extremely large; hence, in order to reach the equilibrium, the traction force has to decrease their intensity (proportionally to Θ) on the points where the cracks reach the Neumann part of the boundary.* ■

Definition 4.1.8. *Let $\partial_D\Omega \subseteq \partial^*\Omega$ be a Borel set. Given a measurable function $g\colon \partial_D\Omega \to \mathbb{R}^n$, we define*

$$GSBD_{2,g}^2(\Omega;\Gamma) := \{u \in GSBD_2^2(\Omega;\Gamma) \mid Tr(u) = g,\ \text{on } \partial\Omega_D\}. \tag{4.5}$$

The following Proposition will allow us to prescribe the Dirichlet boundary condition on the Dirichlet of the boundary $\partial_D\Omega \subseteq \partial^*\Omega$.

Proposition 4.1.9. *Let Ω, Γ be as in Theorem 4.1.4, let $\partial_D\Omega$ be a Borel subset of $\partial^*\Omega$, and let $g : \partial_D\Omega \to \mathbb{R}^n$ be a measurable function. Then the set $GSBD_{2,g}^2$ of Definition 4.1.8 is an affine closed subspace of $GSBD_2^2(\Omega;\Gamma)$*

Proof. Suppose $GSBD_{2,g}^2 \neq \emptyset$ otherwise the theorem immediately follows.

The only non trivial fact is to show that it is closed. Given a sequence $(u_k)_{k=1}^\infty \subset GSBD_{2,g}^2(\Omega;\Gamma)$ such that $\lim_{k\to\infty}\|u_k - u\|_2 = 0$, we can apply the compactness Theorem 1.4.27 with $p = 2$, to deduce that $u_k \rightharpoonup u$ as $k \to \infty$ with respect to the notion of convergence (0.7). Therefore, as a direct consequence of Theorem 2.2.7, we deduce $Tr(u) = g$ and we are done. □

Remark 4.1.10. *Thanks to our previous proposition, $GSBD_{2,0}^2(\Omega;\Gamma)$ is actually an Hilbert space with scalar product inherited as a subspace of $GSBD_2^2(\Omega;\Gamma)$.* ■

4.1.3 Generalised second derivative in time

Now fix $T > 0$, and fix $\Gamma \subset \Omega$ a countable $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Consider for $t \in [0,T]$ an increasing family of cracks $t \mapsto \Gamma(t)$, namely

$$\Gamma(s) \subseteq \Gamma(t) \subseteq \Gamma \text{ if } s \leq t.$$

For simplicity of notation, we denote $GSBD^2_2(\Omega;\Gamma)$ by V and $GSBD^2_{2,0}(\Omega;\Gamma(t))$ by V_t. The norm in V is denoted by $\|\cdot\|$, the norm in V_t with $\|\cdot\|_t$. Note that for $s < t$ we have $V_s \subset V_t \subset V$, and as we already mentioned, since $V \subset H$ is densely embedded in H, we have the embedding $H \subset V^*$ and the density of H in V^*. Similarly H is a dense subspace of V_t^* for every $t \in [0,T]$. To simplify the notation we denote the pairing $\langle \cdot,\cdot\rangle_{V^*}$ between V^* and V by $\langle \cdot,\cdot\rangle$, and the associated dual norm by $\|\cdot\|^*$. Analogously, we denote the pairing $\langle \cdot,\cdot\rangle_{V_t^*}$ between V_t^* and V_t by $\langle \cdot,\cdot\rangle_t$, and the associated dual norm by $\|\cdot\|_t^*$. We note that these pairings are the unique continuous bilinear maps on $V^* \times V$ and $V_t^* \times V_t$ such that $\langle f, v\rangle = \langle f, v\rangle_H$ and $\langle f, v_t\rangle_t = \langle f, v_t\rangle_H$ whenever $f \in H, v \in V, v_t \in V_t$.
If $s < t$ then V_s is not dense in V_t and so V_t^* is not embedded in V_s^*. Anyway we can introduce the projection operators from V_t^* to V_s^* in the following way.

Definition 4.1.11. *Let $s < t$ and let $i : V_s \to V_t$ denote the embedding $V_s \subset V_t$. Let f be an element of V_t^*. Then we define the projection map P_{st} of V_t^* onto V_s^* as*

$$\langle P_{st} f, v_s\rangle_s := \langle f, i(v_s)\rangle_t \quad \textit{for any } v_s \in V_s. \tag{4.6}$$

Remark 4.1.12. *Note that the projection maps defined above are continuous and in particular $\|P_{st} f\|_s^* \leq \|f\|_t^*$. When there is no misunderstanding, we omit the notation $P_{st} f$, since the action of $f \in V_t^*$ on elements of $V_s \subset V_t$ is clear from the context.* ■

As already mentioned, since our displacement $u(t)$ lives at each time in different spaces V_t, in order to give a meaning to the second derivative in time $\ddot{u}(t)$ in equation (4.1), we need the following lemma.

Lemma 4.1.13. *Let $u \in W^{1,\infty}(0,T;H)$. Assume that there exists a positive function $g \in L^2(0,T)$, such that for every $s, t \in [0,T]$ with $s < t$, we have*

$$u \in W^{2,2}(t,T;V_s^*), \text{ and } \|\ddot{u}(r)\|_s^* \leq g(r) \text{ for a.e. } r \in (t,T). \tag{4.7}$$

Then there exists a set $E \subset [0,T]$ of full measure, such that for every $t \in E$ there exists $w(t) \in V_t^*$ with the following properties

$$\|w(t)\|_t^* \leq \tilde{g}(t), \tag{4.8}$$

where $\tilde{g}(t) = \limsup_{h\to 0^+} \frac{1}{h}\int_t^{t+h} g(r)\, dr$, and

$$\lim_{h\to 0^+} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = w(t), \text{ weakly in } V_t^*, \tag{4.9}$$

and

$$\lim_{h\to 0} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = w(t), \text{ strongly in } V_s^* \text{ for every } s < t. \tag{4.10}$$

In particular for every $s \in [0,T]$ the functions $t \mapsto u(t)$ and $t \mapsto P_{st} w(t)$, considered as functions from (s,T) to V_s^*, belong to $W^{2,2}(s,T;V_s^*)$ and $L^2(s,T;V_s^*)$, respectively. Moreover $\ddot{u}(t) = P_{st} w(t)$ in V_s^* for a.e. $t \in (s,T)$.

Remark 4.1.14. *Under the previous hypothesis on u, for every $t \in [0,T]$ $\dot{u}(t)$ is a well defined element of H. More precisely the functions $\dot{u} : [0,T] \to H$ is weakly continuous, i.e. for every $t_k \to t \in [0,T]$ we have*

$$\dot{u}(t_k) \rightharpoonup \dot{u}(t) \text{ weakly in } H, \text{ as } k \to \infty. \tag{4.11}$$

Indeed thanks to the fact $u \in W^{2,2}(0,T;V_0^*)$ we have that $\dot{u}(t)$ is a well defined element in V_0^* for every $t \in [0,T]$ and

$$\dot{u}(t_k) \rightharpoonup \dot{u}(t), \quad \text{in } V_0^*. \tag{4.12}$$

Moreover, since $u \in W^{1,\infty}(0,T;H)$, given $t \in [0,T]$ then there exists a sequence (t_i) such that $t_i \to t$ and

$$\sup_i \|\dot{u}(t_i)\|_H^2 \le \|\dot{u}\|_{L^\infty(0,T;H)}. \tag{4.13}$$

By (4.12), together with the fact that H is embedded in V_0^*, we deduce that any weak limit in H of $(\dot{u}(t_i))$ must be equal to $\dot{u}(t)$. This means that (4.13) implies $\dot{u}(t) \in H$ and more precisely

$$\liminf_{i\to\infty} \|\dot{u}(t_i)\|_H \ge \|\dot{u}(t)\|_H.$$

As a consequence we have that $\dot{u}(t)$ is a well defined element of H for every $t \in [0,T]$.

In order to prove (4.11), since $(\dot{u}(t_k))_{k\in\mathbb{N}}$ is bounded in H uniformly in k, by arguing exactly as before we deduce that any weak limit in H of $(\dot{u}(t_k))$ must be equal to $\dot{u}(t)$ which is exactly our (4.11). ■

Remark 4.1.15. *In the proof of Lemma 4.1.13, we are able to show that the convergence in (4.9) holds only for positive h.* ■

In the proof of Lemma 4.1.13 we shall use the following result on increasing sequences of subspaces of separable Hilbert spaces proved in [20, Lemma 2.3].

Lemma 4.1.16. *Let $\{V_t \mid t \in [0,T]\}$ be an increasing family of closed linear subspaces of a separable Hilbert space V. Then, there exists a countable set $S \subset [0,T]$ such that for all $t \in [0,T] \setminus S$, we have*

$$V_t = \overline{\bigcup_{s<t} V_s}.$$

Proof. (Lemma 4.1.13) Let $D \subset [0,T]$ be a countable dense set. Given $s \in D$, thanks to (4.7) for a.e. $t > s$ there exists $\ddot{u}(t)$ as an element of V_s^* and $\|\ddot{u}(t)\|_s^* \le \tilde{g}(t)$ a.e.. By the fact that D is countable we have a set $E' \subset (0,T)$ of full measure, such that if $t \in E'$ then $\ddot{u}(t)$ exists as an element of V_s^* and $\|\ddot{u}(t)\|_s^* \le \tilde{g}(t)$ for every $s \in (0,t) \cap D$. Moreover, by density, for any $s_1 < t$ there exists $s_2 \in D$ with $s_1 < s_2 < t$, such that thanks to the continuity of the projection map $P_{s_1 s_2}$, we have the relation between $\ddot{u}(t)$ computed in $V_{s_2}^*$ and in $V_{s_1}^*$, given by

$$\lim_{h\to 0} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = \ddot{u}(t) \text{ in } V_{s_2}^* \Rightarrow \lim_{h\to 0} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = P_{s_1 s_2}\ddot{u}(t) \text{ in } V_{s_1}^*, \tag{4.14}$$

This means that for every $t \in E'$ and for every $s < t$

$$\ddot{u}(t) \text{ exists in } V_s^*, \quad \text{and} \quad \|\ddot{u}(t)\|_s^* \le \tilde{g}(t), \tag{4.15}$$

and more precisely, two derivatives computed on different V_s^* are related by (4.14). Hence for every $t \in E'$, $\ddot{u}(t)$ is a well defined element of V_s^* for every $s < t$ and moreover the bound on the second derivative in (4.15) implies

$$u \in W^{2,2}(s,T;V_s^*) \text{ for every } s \in (0,T). \tag{4.16}$$

Now define

$$E := E' \cup \{t \in [0,T] \mid \overline{\bigcup_{s<t} V_s} = V_t \text{ and } \tilde{g}(t) < \infty\}, \tag{4.17}$$

By Lemma 4.1.16, Lebesgue's Differentiation Theorem, and the definition of E', it holds that E has still full measure in $[0,T]$.

Now let $t \in E$. In order to prove (4.8), we notice that for every $\phi \in V_t$ there exists an increasing sequence $(s_n)_{n=1}^\infty$ converging to t and a sequence of functions $(\phi_{s_n})_{n=1}^\infty$ with $\phi_{s_n} \in V_{s_n}$ strongly converging to ϕ in V_t. Therefore we can define $w(t)\colon V_t \to \mathbb{R}$ as

$$\langle w(t), \phi\rangle_t := \lim_{n\to\infty} \langle \ddot{u}(t), \phi_{s_n}\rangle_{s_n} \text{ for every } \phi \in V_t. \tag{4.18}$$

Notice that if $n > m$ then

$$\begin{aligned} \langle \ddot{u}(t), \phi_{s_n}\rangle_{s_n} - \langle \ddot{u}(t), \phi_{s_m}\rangle_{s_m} &= \langle \ddot{u}(t), \phi_{s_n} - \phi_{s_m}\rangle_{s_n} \\ &\le \|\ddot{u}(t)\|_{s_n}^* \|\phi_{s_n} - \phi_{s_m}\|_{s_n} \\ &\le \tilde{g}(t)\|\phi_{s_n} - \phi_{s_m}\|_t, \end{aligned} \tag{4.19}$$

and this shows that the limit (4.18) exists and does not depend on the approximating sequence $(\phi_{s_n})_{n=1}^\infty$. Clearly this defines a continuous linear functional on V_t such that $\|w(t)\|_t^* \le \tilde{g}(t)$. This is exactly (4.8).

To prove (4.9), we fix $\epsilon > 0$ and $\phi \in V_t$, then we can find $s < t$ and $\phi_s \in V_s$ such that $\|\phi_s - \phi\|_t \le \epsilon$. Hence

$$\begin{aligned} &\lim_{h\to 0^+} \left\langle \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - w(t), \phi \right\rangle_t = \lim_{h\to 0^+} \left\langle \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - w(t), \phi_s + (\phi - \phi_s) \right\rangle_t \\ &= \lim_{h\to 0^+} \left\langle \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - w(t), \phi_s \right\rangle_t + \lim_{h\to 0^+} \left\langle \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - w(t), \phi - \phi_s \right\rangle_t . \end{aligned} \tag{4.20}$$

By using the definition of $w(t)$ we immediately deduce that $\lim_{h\to 0^+} \langle \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - w(t), \phi_s\rangle_t = 0$, therefore we can continue the previous estimate with

$$\begin{aligned} \lim_{h\to 0^+} \left\langle \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - w(t), \phi \right\rangle_t &\le \lim_{h\to 0^+} \left\langle \frac{\dot{u}(t+h) - \dot{u}(t)}{h} - w(t), \phi_s \right\rangle_t \\ &= \lim_{h\to 0^+} \frac{1}{h} \int_t^{t+h} \langle \ddot{u}(r) - w(t), \phi - \phi_s\rangle_t \, dr \\ &\le \lim_{h\to 0^+} \left(\frac{1}{h} \int_t^{t+h} g(r)\, dr + \tilde{g}(t) \right) \|\phi - \phi_s\|_t \\ &\le 2\tilde{g}(t)\epsilon, \end{aligned} \tag{4.21}$$

where we used the fact that $u \in W^{2,2}(t,T;V_t^*)$ together with the fundamental theorem of calculus. The arbitrariness of ϵ gives assertion (4.9) and concludes the proof. □

Definition 4.1.17. *Under the assumption of Lemma 4.1.13, the element $w(t)$ of V_t^* defined in (4.9) for a.e. $t \in [0,T]$ is denoted by $\ddot{u}(t)$.*

The second derivative in time defined in Lemma 4.1.13 satisfies the following integration by parts formula.

Lemma 4.1.18. *Let u be as in Lemma 4.1.13 and consider $\varphi \in L^2(0,T;V) \cap W^{1,2}(0,T;H)$ such that $\varphi(t) \in V_t$ for every $t \in [0,T]$. Then the map $t \mapsto \langle \dot u(t), \varphi(t)\rangle_H$ is absolutely continuous on $[0,T]$ and more precisely*

$$\langle \dot u(t_2), \varphi(t_2)\rangle_H - \langle \dot u(t_1), \varphi(t_1)\rangle_H = \int_{t_1}^{t_2} \langle \ddot u(\tau), \varphi(\tau)\rangle_\tau + \langle \dot u(\tau), \dot\varphi(\tau)\rangle_\tau \, d\tau, \tag{4.22}$$

for every $0 \le t_1 < t_2 \le T$.

Proof. By Remark 4.1.14 we know that $t \mapsto \dot u(t)$ is weakly continuous in H. Therefore, since $t \mapsto \varphi(t)$ is strongly continuous in H, we deduce that $t \mapsto \langle \dot u(t), \varphi(t)\rangle_H$ is a continuous real valued map.

First of all we prove our assertion for $\varphi(\cdot)_h := \varphi(\cdot - h)$ instead of $\varphi(\cdot)$. Fix any $t \in [h, T-h]$. Since $\varphi(\cdot - h) \in V_t$ on the time interval $[t, t+h]$ and $u \in W^{2,2}(t, t+h; V_t^*)$, we easily deduce that

$$\langle \dot u(t_2), \varphi_h(t_2)\rangle_H - \langle \dot u(t_1), \varphi_h(t_1)\rangle_H = \int_{t_1}^{t_2} \langle \ddot u(\tau), \varphi_h(\tau)\rangle_\tau + \langle \dot u(\tau), \dot\varphi_h(\tau)\rangle_\tau \, d\tau, \tag{4.23}$$

for every $t_1, t_2 \in (t, t+h)$ $(t_1 < t_2)$. Since t was arbitrary and $\langle \dot u(\cdot), \varphi_h(\cdot)\rangle_H$ is continuous, we can actually obtain (4.23) for every $t_1, t_2 \in [h, T-h]$ $(t_1 < t_2)$.

Finally thanks to the fact $\varphi \in W^{1,2}(0,T;H)$ the left hand side of (4.23) converges to $\langle \dot u(t_2), \varphi(t_2)\rangle_H - \langle \dot u(t_1), \varphi(t_1)\rangle_H$ as $h \to 0^+$, while using also $\varphi \in L^2(0,T;V)$ (in particular the continuity of the translations in L^2), the right hand side of (4.23) converges to $\int_{t_1}^{t_2} \langle \ddot u(\tau), \varphi(\tau)\rangle_\tau + \langle \dot u(\tau), \dot\varphi(\tau)\rangle_\tau \, d\tau$, and we are done. □

4.2 The damped system of elastodynamics

In this section we deal with the damped system of elastodynamics:

$$\begin{cases} \ddot u(t) - \operatorname{div} \mathbb{C}\mathcal{E}u(t) - \gamma \operatorname{div} \mathbb{B}\mathcal{E}\dot u(t) = f(t), & \text{on } \Omega \setminus \Gamma(t) \\ u(t) = w(t), & \text{on } \partial_D\Omega \\ (\mathbb{C}\mathcal{E}u(t) + \mathbb{B}\mathcal{E}\dot u(t))\nu = F(t), & \text{on } \partial_N\Omega \\ (\mathbb{C}\mathcal{E}u(t) + \mathbb{B}\mathcal{E}\dot u(t))\nu = 0, & \text{on } \Gamma^\pm(t) \\ u(0) = u^0, & \text{in } V_0 \\ \dot u(0) = u^1, & \text{in } H \end{cases} \tag{4.24}$$

where $w(t)$ and $F(t)$ are the Dirichlet and Neumann boundary conditions, u^0 and u^1 are the initial conditions; here ν denotes the outer normal to $\partial_N\Omega$ and is an orientation on $\Gamma(t)$. Both boundary conditions and initial conditions have to be intended in a suitable weak sense which will be specified in the sequel (see Definition 4.2.2 and Remark 4.2.3).

4.2.1 Definition of solution

From now on we consider the following standing assumptions:

(a) $\Omega \subset \mathbb{R}^n$ is an open set of finite perimeter;

(b) $(\Gamma(t))_{t\in[0,T]}$ is an increasing family of crack sets:

$$\Gamma(s) \subseteq \Gamma(t) \subseteq \Gamma \quad \text{for } s < t,$$

where $\Gamma \subseteq \Omega$ is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$;

(c) $\partial_D\Omega$, $\partial_N\Omega$ are two disjoint Borel subsets of $\partial^*\Omega$, the Dirichlet and the Neumann part of the reduced boundary, respectively, such that $\partial_D\Omega \cup \partial_N\Omega = \partial^*\Omega$.

(d) $\mathbb{C}$ and $\mathbb{B}$ are bounded symmetric and positive definite tensor fields with ellipticity constant γ_0 and γ_1, respectively (see Definition 4.1.1).

Remark 4.2.1. *In the case of homogeneous Neumann condition everywhere, the existence result proved in the next subsection holds true under the only hypothesis on Ω of being open (as it has already been done for the scalar case in [22]). However, since we deal also with mixed non-homogeneous Neumann and Dirichlet boundary conditions we need to add the assumption on Ω of having finite perimeter. In this situation it is possible to prescribe the boundary conditions only on the reduced boundary $\partial^*\Omega$. If in addition Ω is also Lipschitz-regular, then $\mathcal{H}^{n-1}(\partial^*\Omega\Delta\partial\Omega) = 0$, and therefore the boundary conditions can be prescribed on the entire topological boundary $\partial\Omega$.* ■

Here we give the precise definition of solution to (4.24).

Definition 4.2.2. *Assume (a), (b), (c) and (d). With the notation introduced in Section 4.1.1, let $f \in L^2(0,T;V^*)$, let $w \in W^{2,2}(0,T;H) \cap W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n))$ and let $F \in L^2(0,T;N_\Theta)$ where Θ is the function relative to the crack set Γ given by Theorem 4.1.4. We say that u is a solution to (4.24) on the time-dependent domain $t \mapsto \Omega \setminus \Gamma(t)$ with Dirichlet boundary condition $w(t)$ on $\partial_D\Omega$, Neumann boundary condition $F(t)$ on $\partial_N\Omega$, and homogeneous Neumann boundary conditions on $\Gamma(t)$, if*

$$u \in W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;V). \tag{4.25}$$

$$\text{For every } t \in [0,T]\ u(t) - w(t) \in V_t. \tag{4.26}$$

$$\text{For every } s \in [0,T)\ u \in W^{2,2}(s,T;V_s^*), \text{ and} \tag{4.27}$$

$$\|P_{st}\ddot{u}(t)\|_s^* \le g(t) \text{ for a.e. } t \in (s,T), \text{ for some } g \in L^2(0,T). \tag{4.28}$$

$$\lim_{h\to 0^+} \int_h^T \frac{\|\dot{u}(t) - \dot{u}(t-h)\|_H^2}{h}\, dt = 0. \tag{4.29}$$

For a.e. $t \in [0,T]$

$$\langle \ddot{u}(t), \phi\rangle_t + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\phi\rangle_{H_n} - \langle F(t), Tr(\phi)\rangle_\Theta =$$
$$= \langle f(t), \phi\rangle_t, \text{ for every } \phi \in V_t \tag{4.30}$$

where $\ddot{u}(t)$ is the one given by Definition 4.1.17.

Given $u^0 \in V$ such that $u^0 - w(0) \in V_0$ and $u^1 \in H$, since $t \mapsto u(t)$ is strongly continuous in V the initial value for u is well defined as an element of V. Moreover we are able to prescribe the initial conditions for $\dot{u}(0)$ asking

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \|\dot{u}(t) - u^1\|_H^2\, dt = 0. \tag{4.31}$$

The work of the external forces on the solution u over a time interval $[t_1, t_2] \subset [0, T]$ is given by

$$\mathcal{W}_{load}(u; t_1, t_2) := \int_{t_1}^{t_2} \langle f(t), \dot{u}(t) \rangle_t \, dt \tag{4.32}$$

which is well defined by (4.25) and the fact $f \in L^2(0, T; V^*)$. One would expect that the work on the solution u due to the varying Dirichlet boundary conditions w and Neumann boundary conditions F over a time interval $[t_1, t_2] \subset [0, T]$ is given by

$$\begin{aligned} \mathcal{W}_{bdry}(u; t_1, t_2) := & \int_{t_1}^{t_2} \left(\int_{\partial_D \Omega} (\mathbb{C}\mathcal{E}u(t) + \mathbb{B}\mathcal{E}\dot{u}(t))\nu \cdot \dot{w}(t) \, d\mathcal{H}^{n-1} \right) dt \\ & + \int_{t_1}^{t_2} \langle F(t), \dot{u}(t) \rangle_\Theta \, dt. \end{aligned} \tag{4.33}$$

Unfortunately, under the assumptions (4.25)-(4.28) the trace of the normal derivative cannot be defined, not even in a weaker sense, because both $\mathcal{E}u(t)$ and $\mathcal{E}\dot{u}(t)$ in general belong only to H_n. We decide to solve this problem following [23, Proposition 3.1], by using the weak formulation of the work due to the Dirichlet boundary conditions:

$$\begin{aligned} \mathcal{W}^D_{bdry}(u; t_1, t_2) := & \langle \dot{u}(t_2), \dot{w}(t_2) \rangle_H - \langle \dot{u}(t_1), \dot{w}(t_1) \rangle_H - \int_{t_1}^{t_2} \langle \ddot{w}(t), \dot{u}(t) \rangle_H \, dt \\ - \int_{t_1}^{t_2} \langle F(t), \dot{w}(t) \rangle_\Theta \, dt - & \int_{t_1}^{t_2} \langle f(t), \dot{w}(t) \rangle_t \, dt + \int_{t_1}^{t_2} \langle \mathbb{C}\,\mathcal{E}u(t) + \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\dot{w}(t) \rangle_{H_n} \, dt. \end{aligned} \tag{4.34}$$

With these notations, the energy balance that we are able to prove for the solution u to (4.24) has the following form:

$$\begin{aligned} & \frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|^2_{H_n^{\mathbb{C}}} + \int_0^t \|\mathcal{E}\dot{u}(\tau)\|^2_{H_n^{\mathbb{B}}} \, d\tau = \\ & = \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u; 0, t) + \mathcal{W}_{bdry}(u; 0, t). \end{aligned} \tag{4.35}$$

Remark 4.2.3. *Since $\mathcal{E}u(t)$ and $\mathcal{E}\dot{u}(t)$ are in general only elements of H_n, it does not make sense to talk about their traces. For this reason the Neumann boundary conditions, $F(t)$ on $\partial_N\Omega$ and homogeneous on both sides of $\Gamma(t)$, have to be intended in a weak sense by means of integration by parts in equation* (4.30). ■

Remark 4.2.4. *Condition* (4.29) *is technical and is related to the presence of the damping term. Moreover it plays a crucial role in view of the energy balance* (4.35) *(see Proposition 4.2.7).* ■

4.2.2 Existence and uniqueness results

We start by proving this partial existence result which is the core of this section.

Theorem 4.2.5 (Partial existence)**.** *Assume (a), (b), (c) and (d). Let f, w and F be as in Definition 4.2.2. Given two initial conditions $u^0 \in V$ such that $u^0 - w(0) \in V_0$ and $u^1 \in H$, then there exists a function u satisfying* (4.25)-(4.28) *and* (4.30) *of Definition 4.3.1, with initial conditions $u(0) = u^0$ and* (4.31)*. Moreover u satisfies the energy inequality*

$$\begin{aligned} & \frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|^2_{H_n^{\mathbb{C}}} + \int_0^t \|\mathcal{E}\dot{u}(\tau)\|^2_{H_n^{\mathbb{B}}} \, d\tau \leq \\ & \leq \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u; 0, t) + \mathcal{W}_{bdry}(u; 0, t) \end{aligned} \tag{4.36}$$

for a.e. $t \in [0,T]$.

Remark 4.2.6. *Since* $F \in L^2(0,T;N_\Theta)$*, by Proposition 4.1.6 we have that* $\langle F(t), Tr(\phi)\rangle_\Theta$ *is actually a duality pairing between* V_t^* *and* V_t*. Therefore we can absorb the Neumann term into the forcing term defining*

$$\langle \tilde{f}(t), \phi\rangle_t := \langle f(t), \phi\rangle_t + \langle F(t), Tr(\phi)\rangle_\Theta,$$

and we can reduce ourselves to prove the Theorem when (4.30) *has the simplest form*

$$\langle \ddot{u}(t), \phi\rangle_t + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\phi\rangle_{H_n} = \langle f(t), \phi\rangle_t.$$

■

Proof. For $k \in \mathbb{N}$, we set $\tau_k := T/k$ and $t_k^j := j\tau_k$. For $j = 0, 1, 2, \ldots k$ we define $f_k^j \in V_T^*$ by

$$f_k^j := \frac{1}{\tau_k}\int_{t_k^j}^{t_k^{j+1}} f(\tau)\, d\tau, \tag{4.37}$$

and

$$w_k^j := w(t_k^j), \tag{4.38}$$

(we use $w \in W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n))$ implies that for every $t \in [0,T]$ $w(t)$ is well defined in $H^1(\Omega;\mathbb{R}^n)$).

Inductively we define u_k^j for $j = -1, 0, \ldots, k$ by the following:

$$u_k^0 := u^0,\ \ u_k^{-1} := u^0 - \tau_k u^1; \tag{4.39}$$

then, for $j = 0, 1, \ldots, k-1$, the function u_k^{j+1} is the minimiser in $V_{t_k^{j+1}} + w_k^{j+1}$ of

$$u \mapsto \left\| \frac{u - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k} \right\|_H^2 + \|\mathcal{E}u\|^2_{H_n^{\mathbb{C}}} + \frac{1}{\tau_k}\|(\mathcal{E}u - \mathcal{E}u_k^j)\|^2_{H_n^{\mathbb{B}}} - 2\langle f_k^j, u\rangle_{t_k^{j+1}}. \tag{4.40}$$

Thanks to the ellipticity hypothesis on $\mathbb{C}$ and $\mathbb{B}$, at each step the above functional is coercive in $V_{t_k^{j+1}} + w_k^{j+1}$ because it is greater than

$$c_k[\|u\|_H^2 + (\gamma_0+\gamma_1)\|\mathcal{E}u\|^2_{H_n}] - 2\|f_k^j\|^*_{t_k^{j+1}}(\|u\|_H + \|\mathcal{E}u\|_{H_n}) - a_k^{j+1} \tag{4.41}$$

where $c_k := \min\{1, 1/\tau_k^2\}$, a_k^{j+1} is a constant depending only on k j, and $\gamma_0, \gamma_1 > 0$ are the ellipticity constants of $\mathbb{C}$ and $\mathbb{B}$, respectively. By using also that the first three terms in (4.40) are lower-semicontinuous (here we use the symmetry and positivity of $\mathbb{C}$ and $\mathbb{B}$) while the term $\langle f_k^j, u\rangle_{t_k^{j+1}}$ is even continuous with respect to the weak convergence in $V_{t_k^{j+1}}$, we deduce that the functional in (4.40) admits a minimiser u_k^{j+1} in $V_{t_k^{j+1}}$. The Euler equation for u_k^{j+1} is

$$\left\langle \frac{u_k^{j+1} - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k}, \frac{\phi}{\tau_k}\right\rangle_H + \langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}\phi\rangle_{H_n} + \frac{1}{\tau_k}\langle \mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}\phi\rangle_{H_n} = \langle f_k^j, \phi\rangle_{t_k^{j+1}} \tag{4.42}$$

for every $\phi \in V_{t_k^{j+1}}$. Then using $u_k^{j+1} - u_k^j - (w_k^{j+1} - w_k^j) \in V_{t_k^{j+1}}$ as ϕ we can write

$$\left\| \frac{u_k^{j+1} - u_k^j}{\tau_k} \right\|_H^2 - \left\langle \frac{u_k^{j+1} - u_k^j}{\tau_k}, \frac{u_k^j - u_k^{j-1}}{\tau_k} \right\rangle_H - \left\langle \frac{u_k^{j+1} - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k}, \frac{w_k^{j+1} - w_k^j}{\tau_k} \right\rangle_H +$$
$$+\|\mathcal{E}u_k^{j+1}\|^2_{H_n^{\mathbb{C}}} - \langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}u_k^j\rangle_{H_n} - \langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} + \frac{1}{\tau_k}\|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|^2_{H_n^{\mathbb{B}}} +$$
$$-\frac{1}{\tau_k}\langle \mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} = \langle f_k^j, u_k^{j+1} - u_k^j\rangle_{t_k^{j+1}} - \langle f_k^j, w_k^{j+1} - w_k^j\rangle_{t_k^{j+1}}.$$

Now using the identity $\|a\|^2 - \langle a,b\rangle = \frac{1}{2}\|a\|^2 + \frac{1}{2}\|a-b\|^2 - \frac{1}{2}\|b\|^2$, multiplying by 2, and rearranging, we get

$$\begin{aligned}
&\left\|\frac{u_k^{j+1}-u_k^j}{\tau_k}\right\|_H^2 + \left\|\frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}\right\|_H^2 - 2\left\langle \frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}, \frac{w_k^{j+1}-w_k^j}{\tau_k}\right\rangle_H + \\
&\quad + \|\mathcal{E}u_k^{j+1}\|_{H_n^{\mathbb{C}}}^2 + \|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|_{H_n^{\mathbb{C}}}^2 + 2\langle \mathbb{C}\,\mathcal{E}u_k^{j+1}, \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} + \\
&\quad + 2\frac{1}{\tau_k}\|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|_{H_n^{\mathbb{B}}}^2 - 2\frac{1}{\tau_k}\langle \mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} = \\
&\quad = \left\|\frac{u_k^j-u_k^{j-1}}{\tau_k}\right\|_H^2 + \|\mathcal{E}u_k^j\|_{H_n^{\mathbb{C}}} + 2\langle f_k^j,\ u_k^{j+1}-u_k^j\rangle_{t_k^{j+1}} - 2\langle f_k^j, w_k^{j+1}-w_k^j\rangle_{t_k^{j+1}}.
\end{aligned}$$

Summing from $j=0$ to $i \in \{1,\dots,k\}$ and using (4.39), we get

$$\begin{aligned}
&\left\|\frac{u_k^{i+1}-u_k^i}{\tau_k}\right\|_H^2 + \|\mathcal{E}u_k^{i+1}\|_{H_n^{\mathbb{C}}}^2 + \sum_{j=0}^{i}\left\|\frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}\right\|_H^2 + \sum_{j=0}^{i}\|\mathcal{E}u_n^{j+1} - \mathcal{E}u_k^j\|_{H_n^{\mathbb{C}}}^2 + \\
&+ 2\frac{1}{\tau_k}\sum_{j=0}^{i}\|\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j\|_{H_n^{\mathbb{B}}}^2 = \|u^1\|_H^2 + \|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}}^2 + 2\sum_{j=0}^{i}(\langle f_k^j,\ u_k^{j+1}-u_k^j\rangle_{t_k^{j+1}} + \\
&+ 2\sum_{j=0}^{i}\langle f_k^j, w_k^{j+1}-w_k^j\rangle_{t_k^{j+1}} + 2\sum_{j=0}^{i}\langle \mathbb{C}\,\mathcal{E}u_k^{j+1} + \frac{1}{\tau_k}\mathbb{B}(\mathcal{E}u_k^{j+1} - \mathcal{E}u_k^j), \mathcal{E}w_k^{j+1} - \mathcal{E}w_k^j\rangle_{H_n} + \\
&- 2\sum_{j=0}^{i}\left\langle \frac{u_k^j-u_k^{j-1}}{\tau_k}, \frac{w_k^{j+1}-w_k^j}{\tau_k} - \frac{w_k^j-w_k^{j-1}}{\tau_k}\right\rangle_H + 2\left\langle \frac{u_k^{i+1}-u_k^i}{\tau_k}, \frac{w_k^{i+1}-w_k^i}{\tau_k}\right\rangle_H \\
&- 2\left\langle \frac{u_k^1-u_k^0}{\tau_k}, \frac{w_k^1-w_k^0}{\tau_k}\right\rangle_H
\end{aligned}$$

We define the piece-wise affine discrete approximations $u_k,\ v_k,\ w_k,\ z_k\colon [0,T] \to V$ for $t \in (t_k^j, t_k^{j+1}]$ by

$$u_k(t) := u_k^j + \frac{t-t_k^j}{\tau_k}(u_k^{j+1}-u_k^j), \tag{4.43}$$

$$w_k(t) := w_k^j + \frac{t-t_k^j}{\tau_k}(w_k^{j+1}-w_k^j), \tag{4.44}$$

$$v_k(t) := \frac{u_k^j-u_k^{j-1}}{\tau_k} + \frac{t-t_k^j}{\tau_k}\left(\frac{u_k^{j+1}-u_k^j}{\tau_k} - \frac{u_k^j-u_k^{j-1}}{\tau_k}\right) \tag{4.45}$$

$$z_n(t) := \frac{w_k^j-w_k^{j-1}}{\tau_k} + \frac{t-t_k^j}{\tau_k}\left(\frac{w_k^{j+1}-w_k^j}{\tau_k} - \frac{w_k^j-w_k^{j-1}}{\tau_k}\right), \tag{4.46}$$

and the piece-wise constant discrete approximations $\tilde{u}_k,\ \tilde{w}_k,\ f_k\colon [0,T] \to V$ for $t \in (t_k^j, t_k^{j+1}]$ by

$$\tilde{u}_k(t) := u_k^{j+1}, \tilde{w}_k(t) := w_k^{j+1}, f_k(t) := f_k^j. \tag{4.47}$$

(4.48)

Rewriting the previous equality in terms of $u_k,\ w_k,\ \tilde{u}_k,\ v_k,\ z_k$ we get the discrete energy balance for every $t \in (t_k^j, t_k^{j+1})$:

$$\begin{aligned} &\|\dot u_k(t)\|_H^2 + \|\mathcal{E} u_k(t_k^{j+1})\|^2_{H_n^{\mathbb{C}}} + \tau_k \int_0^{t_k^{j+1}} \|\dot v_k(\tau)\|_H^2 \, d\tau + \tau_k \int_0^{t_k^{j+1}} \|\mathcal{E}\dot u_k(\tau)\|^2_{H_n^{\mathbb{C}}}\, d\tau + \\ &+ \int_0^{t_k^{j+1}} \|\mathcal{E}\dot u_k(\tau)\|^2_{H_n^{\mathbb{B}}}\, d\tau = \|u^1\|_H^2 + \|\mathcal{E} u^0\|^2_{H_n^{\mathbb{C}}} + 2\int_0^{t_k^{j+1}} \langle f_k(\tau), \dot u_k(\tau)\rangle_{t_k^{j+1}}\, d\tau + \\ &+ 2\int_0^{t_k^{j+1}} \langle f_k(\tau), \dot w_k(\tau)\rangle_{t_k^{j+1}} + \langle \mathbb{C}\,\mathcal{E}\tilde u_k(\tau), \mathcal{E}\dot w_k(\tau)\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot u_k(\tau), \mathcal{E}\dot w_k(\tau)\rangle_{H_n}\, d\tau + \\ &- 2\int_0^{t_k^{j+1}} \langle \dot u_k(\tau), \dot z_k(\tau)\rangle_H\, d\tau + 2\langle \dot u_k(t), \dot w_k(t)\rangle_H - 2\langle u^1, \dot w_k(0)\rangle_H \end{aligned} \tag{4.49}$$

Let $M_k := \sup_{t\in(0,T)} \|\dot u_k(t)\|_H$, $L_k := \sup_{t\in(0,T)} \|\mathcal{E}\tilde u_k(t)\|_{H_n^{\mathbb{C}}}$. By (4.49) we can give the estimate

$$M_k^2 + L_k^2 + \|\mathcal{E}\dot u_k\|^2_{L^2(0,T;H_n^{\mathbb{B}})} \le a(M_k + L_k + \|\mathcal{E}\dot u_k\|_{L^2(0,T;H_n^{\mathbb{B}})}) + b, \tag{4.50}$$

where a and b are constants that depends only on $\|f\|_{L^2(0,T;V^*)}$, $\|w\|_{W^{1,2}(0,T;V)}$, $\|w\|_{W^{2,2}(0,T;H)}$, $\|u_1\|_H$ and on T. As a consequence we can deduce the following

$$\mathcal{E} u_k(t) \text{ and } \mathcal{E}\tilde u_k(t) \text{ are bounded in } H_n \text{ uniformly in } t \text{ and } k, \tag{4.51}$$

$$\dot u_k(t) \text{ and } v_k(t) \text{ are bounded in } H \text{ uniformly in } t \text{ and } k \tag{4.52}$$

$$\mathcal{E}\dot u_k \text{ is bounded in } L^2(0,T;H_n) \text{ uniformly in } k. \tag{4.53}$$

Notice also that the fact $u^0 \in H$ implies that u_k is bounded in H uniformly in t and k. This together with (4.51) gives

$$u_k(t) \text{ is bounded in } V \text{ uniformly in } t \text{ and } k. \tag{4.54}$$

Furthermore, using (4.43)-(4.45) and (4.47), we can rewrite (4.42) for all $t \in (t_k^j, t_k^{j+1})$ as

$$\langle \dot v_k(t), \phi\rangle_H + \langle \mathbb{C}\,\mathcal{E}\tilde u_k(t), \mathcal{E}\phi\rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot u_k(t), \mathcal{E}\phi\rangle_{H_n} = \langle f_k(t), \phi\rangle_{t_k^{j+1}} \tag{4.55}$$

for every $\phi \in V_{t_k^{j+1}}$. The last equation leads us to write for all $t \in (t_k^j, t_k^{j+1})$

$$\|\dot v_k(t)\|^*_{t_k^{j+1}} \le \|\mathcal{E}\tilde u_k(t)\|_{H_n^{\mathbb{C}}} + \|\mathcal{E}\dot u_k(t)\|_{H_n^{\mathbb{B}}} + \|f_k(t)\|^*_{t_k^{j+1}}. \tag{4.56}$$

In particular fix $s \in [0,T)$, then for every $t_1, t_2 \in [s,T]$ with $t_1 < t_2$, we have

$$\int_{t_1}^{t_2} \|P_{st}\dot v_k(t)\|_s^*\, dt \le \int_{t_1}^{t_2} (\|\mathcal{E}\tilde u_k(t)\|_{H_n^{\mathbb{C}}} + \|\mathcal{E}\dot u_k(t)\|_{H_n^{\mathbb{B}}} + \|f_k(t)\|_s^*)\, dt. \tag{4.57}$$

Now fix a dense set $D \subset [0,T]$. Using (4.51)-(4.53), (4.56), and a diagonal argument, we obtain a subsequence, not relabeled, such that

$$u_k \rightharpoonup u \text{ weakly in } W^{1,2}(0,T;V) \tag{4.58}$$

$$v_k \rightharpoonup v \text{ weakly in } L^2(0,T;H), \tag{4.59}$$

$$v_k \rightharpoonup v \text{ weakly in } W^{1,2}(s,T;V_s^*) \text{ for every } s \in D, \tag{4.60}$$

and if we call $\tilde g$ a weak limit of $t \mapsto \|\mathcal{E}\dot u_k(t)\|_{H_n^{\mathbb{B}}}$ in $L^2(0,T)$, we have

$$\int_{t_1}^{t_2} \|P_{st}\dot v(t)\|_s^*\, dt \le M|t_2 - t_1| + \int_{t_1}^{t_2} (\tilde g(t) + \|f(t)\|_s^*)\, dt, \tag{4.61}$$

for every $t_1, t_2 \in [s, T]$ with $t_1 < t_2$.

Moreover by using the continuity of the projection maps P_{st} and the density of D, an argument similar to that in the proof of Lemma 4.1.13 shows that (4.60) and (4.61) become

$$v_k \rightharpoonup v \text{ weakly in } W^{1,2}(s, T; V_s^*) \text{ for every } s \in [0, T), \tag{4.62}$$

and

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}(t)\|_s^* \, dt \leq M|t_2 - t_1| + \int_{t_1}^{t_2} (\tilde{g}(t) + \|f(t)\|_s^*) \, dt, \tag{4.63}$$

for every $s \in [0, T)$ and every $t_1, t_2 \in [s, T]$ with $t_1 < t_2$. In particular

$$\|P_{st}\dot{v}(t)\|_s^* \leq M + \tilde{g}(t) + \|f(t)\|_s^*, \tag{4.64}$$

for every $s \in [0, T)$ and a.e. $t > s$.

By (4.52) it is easy to see that

$$u \in W^{1,\infty}(0, T; H), \tag{4.65}$$

and this together with convergence (4.58) gives (4.25).

Now we want to show

$$\dot{u}(t) = v(t) \text{ in } H \text{ for a.e. } t \in [0, T]. \tag{4.66}$$

First of all for $t \in (t_k^j, t_k^{j+1})$ we have $\dot{u}_k(t) = v_k(t_k^{j+1})$ and so

$$\|\dot{u}_k(t) - v_k(t)\|_{t_k^{j+1}}^* = \|v_k(t_k^{j+1}) - v_k(t)\|_{t_k^{j+1}}^* \leq \int_{t_k^j}^{t_k^{j+1}} \|\dot{v}_k(\tau)\|_{t_k^{j+1}}^* \, d\tau \leq \tau_k^{\frac{1}{2}} C, \tag{4.67}$$

where C is a uniform bound on the L^2 norm of the right-hand side of (4.56). Then for all $s < t$ we have $\|\dot{u}_k(t) - v_k(t)\|_s^* \leq \tau_k^{\frac{1}{2}} C$, and this together with (4.59) implies $\dot{u}_k \rightharpoonup v$ weakly in $L^2(s, T; V_s^*)$ for any $s \in [0, T)$. But also $\dot{u}_k \rightharpoonup \dot{u}$ weakly in $L^2(0, T; H)$ by (4.58). So $v(t) = \dot{u}(t)$ in V_s^*, for every $s \in [0, T)$ and for a.e. $t \in (s, T)$. Since $v(t)$ and $\dot{u}(t)$ belong to H, and H is embedded in V_s^* for every $s \in [0, T]$ we finally get that $v(t) = \dot{u}(t)$ as elements of H for a.e. $t \in [0, T]$. This together with (4.58) and (4.64) allows to conclude that

$$u \in W^{2,2}(t, T; V_s^*) \text{ for every } s, t \in [0, T] \text{ with } s < t. \tag{4.68}$$

Let $g(t) := M + \tilde{g}(t) + \|f(t)\|_t^*$ then by (4.64) we have

$$\|P_{st}\ddot{u}(t)\|_s^* \leq g(t), \tag{4.69}$$

for every $s \in [0, T)$ and for a.e. $t > s$, hence we obtain (4.28).

Now we investigate the convergence of the constant piecewise interpolated $\tilde{u}_k$. Since by (4.58) u_k are Lipschitz with values in H uniformly in k, as before we get that

$$\tilde{u}_k \rightharpoonup u \text{ weakly in } L^2(0, T; H), \tag{4.70}$$

and since by (4.51) $\mathcal{E}\tilde{u}_k$ is bounded in $L^2(0, T; H_n)$, we also obtain that up to subsequences

$$\tilde{u}_k \rightharpoonup u \text{ weakly in } L^2(0, T; V) \tag{4.71}$$

Furthermore, note that $\tilde{u}_k(t-\tau_k) - \tilde{w}_k(t-\tau_k) \in V_t$ for every $t \in [0,T]$, and that by (4.58) we can write for every for every $t \in (t_k^j, t_k^{j+1}]$

$$\|u_k(t) - \tilde{u}_k(t-\tau_k)\|_V = \|u_k(t) - u_k(t_k^j)\|_V \leq \int_{t_k^j}^{t_k^{j+1}} \|\dot{u}_k(\tau)\|_V \, d\tau \leq \tau_k^{1/2} C,$$

where C is a constant independent on k. This means that, by using also $w \in W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n))$, we can write

$$\tilde{u}_k(\cdot - \tau_k) - \tilde{w}_k(\cdot - \tau_k) \rightharpoonup u - w \text{ weakly in } L^2(0,T;V). \tag{4.72}$$

Since the linear subspace $\{v \in L^2(0,T;V) \mid v(t) \in V_t \text{ for a.e. } t \in [0,T]\}$ is strongly closed, it is also weakly closed in $L^2(0,T;V)$. Therefore $u(t) \in V_t + w(t)$ for a.e. $t \in [0,T]$. Moreover for every $t \in (0,T]$ there exist an increasing sequence $t_i \in [0,T]$ converging to t such that $u(t_i) - w(t_i) \in V_{t_i}$ for every i. Thanks to (4.58) we know that $t \mapsto u(t) - w(t)$ is a strongly continuous map with values in V, and we obtain $u(t)-w(t) \in V_t$ for every $t \in (0,T]$. Together with the initial condition $u(0) = u^0 \in V_0$ we obtain (4.26). Moreover thanks to (4.68) and (4.69) we are in position to apply Lemma 4.1.13 to the function $u - w$ and hence to deduce that for a.e. $t \in [0,T]$

$$\frac{\dot{u}(t+h) - \dot{u}(t)}{h} \rightharpoonup \ddot{u}(t) \text{ weakly in } V_t^* \text{ as } h \to 0^+, \tag{4.73}$$

and this argument also show (4.27).

Now we want to show that (4.30) holds for a.e. $t \in [0,T]$ for every $\phi \in V_t$. We claim that there exists a negligible set $W \subset [0,T]$ such that for $s \in D$ and for all $\phi \in V_s$, we have

$$\langle \ddot{u}(t), \phi \rangle_s + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\phi \rangle_{H_n} = \langle f(t), \phi \rangle_s, \tag{4.74}$$

for every $t \in (s,T] \setminus W$.

To prove the claim, first we fix $s \in D$ and $\phi \in V_s$. Using (4.55) we have for a.e. $t > s$

$$\langle \dot{v}_k(t), \phi \rangle_s + \langle \mathbb{C}\,\mathcal{E}\tilde{u}_k(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}_k(t), \mathcal{E}\phi \rangle_{H_n} = \langle f_k(t), \phi \rangle_s. \tag{4.75}$$

Hence we have also

$$\int_s^T (\langle \dot{v}_k(t), \phi \rangle_s + \langle \mathbb{C}\,\mathcal{E}\tilde{u}_k(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}_k(t), \mathcal{E}\phi \rangle_{H_n} - \langle f_k(t), \phi \rangle_s) \, dt = 0. \tag{4.76}$$

By construction $f_k \to f$ strongly in $L^2(0,T;V^*)$. We know that $\dot{v}_k \rightharpoonup \dot{v}$ weakly in $L^2(s,T;V_s^*)$ by (4.119). Since $\dot{u} = v$ in $W^{1,2}(s,T;V_s^*)$, we also have that $\ddot{u} = \dot{v}$ in $L^2(s,T;V_s^*)$. Using also (4.58) and (4.71), we can pass to the limit in (4.76) to have

$$\int_s^T (\langle \ddot{u}(t), \phi \rangle_s + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi \rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\phi \rangle_{H_n} - \langle f(t), \phi \rangle_s) \, dt = 0. \tag{4.77}$$

By (4.75) we deduce that the integrand in (4.77) is zero for a.e. $t > s$. Since V_s is separable, the set N_s of $t > s$ for which (4.75) does not hold can be taken independent of ϕ. We set W to be the union over $s \in D$ of the sets N_s, so that W also has measure zero. It follows that for every $s \in D$ and for every $t \in (s,T] \setminus W$ (4.74) holds, and this shows the claim.

Using Lemma 4.1.16 it follows that for a.e. t and for every $\phi \in V_t$, there exist $s_i \nearrow t$ with $s_i \in D$ and $\phi_i \in V_{s_i}$, such that $\phi_i \to \phi$ strongly in V_t. Now note that if t belongs also to $(0,T] \setminus W$, by our previous claim we have

$$\begin{aligned} &\langle \ddot{u}(t), \phi_i \rangle_t + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi_i \rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}(t), \mathcal{E}\phi_i \rangle_{H_n} - \langle f(t), \phi_i \rangle_t \\ &\quad = \langle \ddot{u}(t), \phi_i \rangle_{s_i} + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi_i \rangle_{H_n} + \langle \mathbb{B}\,\mathcal{E}\dot{u}, \mathcal{E}\phi_i \rangle_{H_n} - \langle f(t), \phi_i \rangle_{s_i} = 0. \end{aligned} \tag{4.78}$$

The convergence of the ϕ_i to ϕ gives (4.30).

Since by construction

$$f_k \to f \text{ strongly in } L^2(0,T;V^*), \tag{4.79}$$
$$w_k \to w \text{ strongly in } W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n)), \tag{4.80}$$
$$\dot w_k \to \dot w \text{ strongly in } H \text{ for every } t\in[0,T], \tag{4.81}$$
$$\dot z_k \to L^2(0,T;H) \text{ strongly in } L^2(0,T;H), \tag{4.82}$$

using also (4.58) and (4.71), passing to the limit as $k\to\infty$ in (4.49), we obtain (4.36) by lower semicontinuity.
To prove (4.31), it is equivalent to show that there exists a set $N\subset[0,T]$ of measure zero such that for every $t_i\in[0,T]\setminus N$ with $t_i\to 0$, we have

$$\dot u(t_i)\to u^1 \text{ strongly in } H \tag{4.83}$$

(by Remark 4.1.14 $\dot u(t)$ is a well defined element in H for every $t\in[0,T]$). By (4.25) and (4.36) we have

$$\|\dot u(t)\|_H^2+\|\mathcal{E}u(t)\|^2_{H_n^{\mathbb{C}}}\le\|u^1\|_H^2+\|\mathcal{E}u^0\|^2_{H_n^{\mathbb{C}}}+o\,(1),\quad \text{as } t\to 0^+ \tag{4.84}$$

for every $t\in[0,T]\setminus N$ where $N\subset[0,T]$ is a set of measure zero. Now let (t_i) be such that $t_i\in[0,T]\setminus N$ and $t_i\to 0$. By Remark 4.1.14, we already know that

$$\dot u(t)\rightharpoonup\dot u(0)\quad\text{weakly in } H,\text{ as } t\to 0^+. \tag{4.85}$$

Moreover by (4.62) together with (4.66) we can write for a.e. $t\in[0,T]$

$$\dot u(t)-\dot u(0)=\int_0^t\ddot u(\tau)\,d\tau=\lim_{k\to\infty}\int_0^t\dot v_k(\tau)\;d\tau=\lim_{k\to\infty}v_k(t)-u^1\;\text{ in } V_0^*, \tag{4.86}$$

hence by choosing t such that $v_k(t)\to\dot u(t)$ in V_0^* (which is possible by (4.62)), we deduce that $\dot u(0)=u^1$ in V_0^*. Since both $\dot u(0)$ and u^1 are element of H, and H is embedded in V_0^*, we deduce $\dot u(0)=u^1$ in H. Therefore the convergence (4.85) becomes

$$\dot u(t_i)\rightharpoonup u^1\quad\text{weakly in } H.$$

This means that (4.83) is equivalent to

$$\limsup_{i\to\infty}\|\dot u(t_i)\|_H^2\le\|u^1\|_H^2. \tag{4.87}$$

Since by (4.25) $t\mapsto\mathcal{E}u(t)$ is strongly continuous in H_n we clearly have $\mathcal{E}u(t_i)\to\mathcal{E}u(0)$ strongly in H_n. By using the convergence (4.58) and a similar argument to (4.86) we deduce that $u(0)=u^0$, hence that $\mathcal{E}u(t_i)\to\mathcal{E}u^0$ strongly in H_n. Therefore, by using t_i in inequality (4.84) and passing to the limit as $i\to\infty$, we deduce exactly (4.87) and hence also (4.83). □

Proposition 4.2.7. *Let u be the function given by Theorem 4.3.2, then u satisfies condition (4.29) and the energy balance (4.35) for every t Lebesgue point of $\|\dot u(\cdot)\|_H^2$.*

Proof. First of all we claim that if $t\in(0,T)$ is a Lebesgue point of $\|\dot u(\cdot)\|_H^2$ then t is also a Lebesgue point of $\dot u(\cdot)$, i.e.

$$\lim_{h\to 0}⨍_{t-h}^{t+h}\|\dot u(\tau)-\dot u(t)\|_H^2\,d\tau=0. \tag{4.88}$$

Since by Remark 4.1.14 we know that $t \mapsto \dot{u}(t)$ is weakly continuous in H, then it follows that

$$\dot{u}(t+h\tau) \rightharpoonup \dot{u}(t), \text{ weakly in } L^2(-1,1;H), \text{ as } h \to 0^+.$$

This means that in order to prove (4.88) it is enough to show that $\lim_{h\to 0} \fint_{-1}^{1} \|\dot{u}(t+h\tau)\|_H^2\, d\tau = \|\dot{u}(t)\|_H^2$. But this is guaranteed by the fact that t is a Lebesgue point of $\|\dot{u}(\cdot)\|_H^2$.

Now let w be the Dirichlet boundary condition considered in Definition 4.2.2. We note that for every $h \in (0,T)$ and for a.e. $t \in [0,T]$ the functions $u(t)-u(t-h)-(w(t)-w(t-h)) \in V_t$. Hence if we define $z(t) := u(t)-w(t)$ we can test equation (4.30) with $\frac{z(t)-z(t-h)}{h}$ and integrate on (h,T) to get

$$\begin{aligned}
&\int_h^t \left\langle \ddot{u}(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau + \int_h^t \left\langle \mathbb{C}\,\mathcal{E}u(\tau), \frac{\mathcal{E}z(\tau)-\mathcal{E}z(\tau-h)}{h} \right\rangle_{H_n} d\tau \\
&\quad + \int_h^t \left\langle \mathbb{B}\,\mathcal{E}\dot{u}(\tau), \frac{\mathcal{E}z(\tau)-\mathcal{E}z(\tau-h)}{h} \right\rangle_{H_n} d\tau \\
&\quad = \int_h^t \left\langle f(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau + \int_h^t \left\langle F(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\Theta d\tau.
\end{aligned} \tag{4.89}$$

Since $u \in W^{1,2}(0,T;V)$ and $w \in W^{2,2}(0,T;H) \cap W^{1,2}(0,T;V)$, we take the limit as $h \to 0^+$ on both side of the previous equality

$$\begin{aligned}
&\lim_{h\to 0^+} \int_h^t \left\langle \ddot{u}(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau + \int_0^t \langle \mathbb{C}\,\mathcal{E}u(\tau), \mathcal{E}\dot{z}(\tau)\rangle_{H_n} d\tau \\
&\quad + \int_0^t \langle \mathbb{B}\,\mathcal{E}\dot{u}(\tau), \mathcal{E}\dot{z}(\tau)\rangle_{H_n} d\tau = \int_0^t \langle f(\tau), \dot{z}(t)\rangle_\tau d\tau + \int_0^t \langle F(\tau), \dot{z}(\tau)\rangle_\Theta d\tau
\end{aligned} \tag{4.90}$$

In order to compute the limit of the first term in the left hand-side of (4.90), we use Lemma 4.1.18 to write

$$\begin{aligned}
\int_h^t \left\langle \ddot{u}(\tau), \frac{z(\tau)-z(\tau-h)}{h} \right\rangle_\tau d\tau = &\left\langle \dot{u}(t), \frac{z(t)-z(t-h)}{h} \right\rangle_H - \left\langle \dot{u}(h), \frac{z(h)-z(0)}{h} \right\rangle_H \\
&- \int_h^t \left\langle \dot{u}(\tau), \frac{\dot{z}(\tau)-\dot{z}(\tau-h)}{h} \right\rangle_H d\tau
\end{aligned} \tag{4.91}$$

Since t satisfies (4.88), u satisfies the initial condition (4.31), and $\dot{u}(\cdot)$ is weakly continuous in H, we have

$$\lim_{h\to 0^+} \left\langle \dot{u}(t), \frac{u(t)-u(t-h)}{h} \right\rangle_H = \|\dot{u}(t)\|_H^2 \text{ and } \lim_{h\to 0^+} \left\langle \dot{u}(h), \frac{u(h)-u(0)}{h} \right\rangle_H = \|u^1\|_H^2.$$

Moreover by using the identity $\langle \dot{u}(\tau), \dot{u}(\tau)-\dot{u}(\tau-h)\rangle_H = -\frac{1}{2}\|\dot{u}(\tau-h)\|_H^2 + \frac{1}{2}\|\dot{u}(\tau)\|_H^2 + \frac{1}{2}\|\dot{u}(\tau)-\dot{u}(\tau-h)\|_H^2$, we can write

$$\begin{aligned}
\int_h^t \left\langle \dot{u}(\tau), \frac{\dot{u}(\tau)-\dot{u}(\tau-h)}{h} \right\rangle_H d\tau = &\frac{1}{2h}\int_{t-h}^t \|\dot{u}(\tau)\|_H^2\, d\tau - \frac{1}{2h}\int_0^h \|\dot{u}(\tau)\|_H^2\, d\tau \\
&+ \frac{1}{2h}\int_h^t \|\dot{u}(\tau)-\dot{u}(\tau-h)\|_H^2\, d\tau.
\end{aligned}$$

Again using condition (4.31) and the fact that t is a Lebesgue point of $\|\dot{u}(\cdot)\|_H^2$, we can write

$$\begin{aligned}
\lim_{h\to 0^+} \int_h^t \left\langle \dot{u}(\tau), \frac{\dot{u}(\tau)-\dot{u}(\tau-h)}{h} \right\rangle_H d\tau = &\frac{1}{2}\|\dot{u}(t)\|_H^2 - \frac{1}{2}\|u^1\|_H^2 \\
&+ \lim_{h\to 0^+} \frac{1}{2}\int_h^t \frac{\|\dot{u}(\tau)-\dot{u}(\tau-h)\|_H^2}{h}\, d\tau,
\end{aligned} \tag{4.92}$$

By using the regularity assumption $w \in W^{2,2}(0,T;H)$ a simple calculation leads to

$$\lim_{h\to 0^+}\int_h^t \left\langle \ddot u(\tau), \frac{w(\tau)-w(\tau-h)}{h}\right\rangle_\tau d\tau = \langle \dot u(t), \dot w(t)\rangle_H - \langle u^1, \dot w(0)\rangle_H - \int_0^t \langle \dot u(\tau), \ddot w(\tau)\rangle_H \, d\tau. \tag{4.93}$$

Putting together (4.92) with (4.93), we can take the limit on both side of(4.91) to get

$$\begin{aligned}\lim_{h\to 0^+}\int_h^t \left\langle \ddot u(\tau), \frac{z(\tau)-z(\tau-h)}{h}\right\rangle_\tau d\tau &= \frac12\|\dot u(t)\|_H^2 - \frac12\|u^1\|_H^2\\ &+ \lim_{h\to 0^+}\frac12\int_h^t \frac{\|\dot u(\tau)-\dot u(\tau-h)\|_H^2}{h}\,d\tau - \langle \dot u(t), \dot w(t)\rangle_H\\ &+\langle u^1, \dot w(0)\rangle_H + \int_0^t \langle \dot u(\tau), \ddot w(\tau)\rangle_H \, d\tau.\end{aligned} \tag{4.94}$$

Putting together (4.90) with (4.94) by using the definition of $\mathcal{W}_{load}(u;0,t)$ and $\mathcal{W}_{bdry}(u;0,t)$ (see (4.32),(4.33),(4.34)) we obtain for every Lebesgue point t of $\|\dot u(\cdot)\|_H^2$

$$\begin{aligned}&\frac12\|\dot u(t)\|_H^2 + \frac12\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 + \int_0^t \|\mathcal{E}\dot u(\tau)\|_{H_n^{\mathbb{B}}}^2 d\tau - \lim_{h\to 0^+}\frac12\int_h^t \frac{\|\dot u(\tau)-\dot u(\tau-h)\|_H^2}{h}d\tau\\ &= \frac12\|u^1\|_H^2 + \frac12\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t).\end{aligned} \tag{4.95}$$

But since we already know u satisfies the energy inequality (4.36), we immediately conclude that both condition (4.29) and the energy balance (4.35) hold. □

Remark 4.2.8. *Since $\dot u(t) - \dot w(t) \in V_t$ for a.e. $t \in [0,T]$, we can use it as test function in (4.30) and integrate on $(0,t)$ to obtain*

$$\begin{aligned}&\int_0^t \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau \; d\tau + \frac12\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 + \int_0^t \|\mathcal{E}\dot u(\tau)\|_{H_n^{\mathbb{B}}}^2 \; d\tau\\ &\qquad= \frac12\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}}^2 + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t).\end{aligned}$$

Comparing this last identity with the energy balance (4.35), we have for a.e. $t \in (0,T)$ that

$$\frac12\|\dot u(t)\|_H^2 - \frac12\|u^1\|_H^2 = \int_0^t \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau \; d\tau,$$

and since $\tau \mapsto \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau \in L^1(0,T)$ we deduce that $\|\dot u(\cdot)\|_H^2 \in W^{1,1}(0,T)$. ■

Putting together Theorem 4.2.5 and Proposition 4.2.7, we deduce the existence of a solution u to the damped system of elastodynamic. Moreover using (4.95) we can also obtain the uniqueness of solutions considered in Definition 4.2.2. This is the content of the next theorem.

Theorem 4.2.9 (Existence and uniqueness)**.** *Under hypothesis of Theorem 4.2.5 there exists a unique solution u considered in Definition 4.2.2. Moreover u satisfies the energy balance*

$$\begin{aligned}&\frac12\|\dot u(t)\|_H^2 + \frac12\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 + \int_0^t \|\mathcal{E}\dot u(\tau)\|_{H_n^{\mathbb{B}}}^2 \; d\tau =\\ &\quad= \frac12\|u^1\|_H^2 + \frac12\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t)\end{aligned} \tag{4.96}$$

for a.e. $t \in [0,T]$.

Proof. The existence of a solution satisfying (4.96) is simply a consequence of Theorem 4.2.5 and Proposition 4.2.7.

To show uniqueness, we notice that (4.25)-(4.30) are all preserved under linear combinations. Therefore, the difference v between two solutions is a solution with Dirichlet and Neumann homogeneous conditions, with forcing term $f = 0$, and satisfying $v(0) = 0$ and

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot v(t)\|_h^2 \, dt = 0.$$

Moreover using the same argument as in Proposition 4.2.7, since (4.29) holds for v, we have

$$\frac{1}{2}\|\dot v(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}v(t)\|^2_{H_n^{\mathbb{C}}} + \int_0^t \|\mathcal{E}\dot v(\tau)\|^2_{H_n^{\mathbb{B}}} \, d\tau = 0,$$

for a.e. $t \in [0,T]$. Therefore $\dot v(t) = 0$ a.e. on $[0,T]$. Since $v \in W^{1,\infty}(0,T;H)$ and $v(0) = 0$, we conclude $v(t) = 0$ a.e. on $[0,T]$. □

Finally, one can also prove that the energy balance (4.35) holds for every $t \in [0,T]$ and that the map $t \mapsto \dot u(t)$ is strongly continuous in H. For the proof of this result we refer to [22, Lemma 3.10].

Proposition 4.2.10. *Under the assumptions of Theorem 4.2.5 , let u be the solution of the damped wave equation considered in Definition (4.2.2), with initial conditions $u(0) = u^0$ and (4.31) . Then $t \mapsto \dot u(t)$ is continuous from $[0,T]$ to H and the energy balance (4.35) holds for every $t \in [0,T]$.*

4.3 The undamped system of elastodynamics

In this section we study the undamped system of elastodynamics

$$\begin{cases} \ddot u(t) - \operatorname{div}\mathbb{C}\mathcal{E}u(t) = f(t), & \text{on } \Omega\setminus\Gamma(t) \\ u(t) = w(t), & \text{on } \partial_D\Omega \\ \mathbb{C}\mathcal{E}u(t)\nu = F(t), & \text{on } \partial_N\Omega \\ \mathbb{C}\mathcal{E}u(t)\nu = 0, & \text{on } \Gamma^{\pm}(t) \\ u(0) = u^0, & \text{in } V_0 \\ \dot u(0) = u^1, & \text{in } H \end{cases} \tag{4.97}$$

As before both boundary conditions and initial conditions have to be intended in a suitable weak sense.

4.3.1 Definition of solution

Definition 4.3.1. *Assume (a), (b), (c) and (d). With the notation introduced in Section 4.1.1, let $f \in W^{1,2}(0,T;V^*)$, let $w \in W^{2,2}(0,T;H) \cap W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n))$ and let $F \in W^{1,2}(0,T;N_\Theta)$ where Θ is the function relative to the crack set Γ given by Theorem 4.1.4 . We say that u is a solution of (0.13) on the time-dependent domain $t \mapsto \Omega\setminus\Gamma(t)$ with Dirichlet boundary condition $w(t)$ on $\partial_D\Omega$, Neumann boundary condition $F(t)$ on $\partial_N\Omega$, and* *homogeneous Neumann boundary conditions on* $\Gamma(t)$*, if*

$$u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H). \tag{4.98}$$

$$\text{For every } t \in [0,T]\; u(t) - w(t) \in V_t. \tag{4.99}$$

$$\text{For every } s \in [0,T)\; u \in W^{2,2}(s,T;V_s^*) \text{ and} \tag{4.100}$$

$$\|P_{st}\ddot{u}(t)\|_s^* \le g(t) \text{ for a.e. } t \in (s,T), \text{ for some } g \in L^2(0,T). \tag{4.101}$$

For a.e. $t \in [0,T]$

$$\langle \ddot{u}(t), \phi\rangle_t + \langle \mathbb{C}\,\mathcal{E}u(t), \mathcal{E}\phi\rangle_{H_n} - \langle F(t), Tr(\phi)\rangle_\Theta = \langle f(t), \phi\rangle_t, \text{ for every } \phi \in V_t \tag{4.102}$$

where $\ddot{u}(t)$ *is the one given by Definition 4.1.17.*

Given $u^0 \in V$ *such that* $u^0 - w(0) \in V_0$ *and* $u^1 \in H$*, since* $t \mapsto u(t)$ *is strongly continuous in* H *the initial value for* u^0 *is well defined as element of* H*. Moreover we are able to prescribe the initial conditions respectively for* $\mathcal{E}u(0)$ *and* $\dot{u}(0)$ *asking*

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\mathcal{E}u(t) - \mathcal{E}u^0\|_{H_n}^2\, dt = 0, \tag{4.103}$$

and

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot{u}(t) - u^1\|_H^2\, dt = 0. \tag{4.104}$$

Since in this case, $\dot{u}(t)$ is in general only an element in H, we need to consider also a weakened formulation of the work due to the Neumann boundary conditions. More precisely, the term appearing in the work due to the boundary forces $\mathcal{W}_{bdry}(u;t_1,t_2)$, which in the damped case read as

$$\int_{t_1}^{t_2} \langle F(t), \dot{u}(t)\rangle_\Theta\, dt,$$

becomes

$$\langle F(t_2), u(t_2)\rangle_\Theta - \langle F(t_1), u(t_1)\rangle_\Theta - \int_{t_1}^{t_2} \langle \dot{F}(t), u(t)\rangle_\Theta\, dt. \tag{4.105}$$

for every time interval $[t_1,t_2] \subset [0,T]$.

4.3.2 Existence result

Theorem 4.3.2. *Assume (a), (b), (c) and (d). Let* f*,* w *and* F *be as in Definition 4.3.1. Given two initial conditions* $u^0 \in V$ *such that* $u^0 - w(0) \in V_0$ *and* $u^1 \in H$*, then there exists a solution* u *of* (0.13) *with initial conditions* (4.103) *and* (4.104)*. Moreover* u *satisfies the energy inequality*

$$\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}}^2 \le \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + \mathcal{W}_{load}(u;0,t) + \mathcal{W}_{bdry}(u;0,t) \tag{4.106}$$

for a.e. $t \in [0,T]$.

Proof. Since the argument is similar to the one given for Theorem 4.3.2, we simply give a sketch of the proof.

For $k \in \mathbb{N}$, we set $\tau_k := T/k$ and $t_k^j := j\tau_k$. For $j = 1,2,\dots,k$, by using that $f \in W^{1,2}(0,T;V^*)$ and $w \in W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n))$, so that for every t $f(t)$ and $w(t)$ are well defined elements of V^* and $H^1(\Omega;\mathbb{R}^n)$, respectively, we define $f_k^j \in V^*$ and $w_k^j \in H^1(\Omega;\mathbb{R}^n)$ by

$$f_k^j := f(t_k^j),\ w_k^j := w(t_k^j). \tag{4.107}$$

Inductively we define u_k^j for $j = -1, 0, \dots, k$ by the following:

$$u_k^0 := u^0, \; u_k^{-1} := u^0 - \tau_k u^1; \tag{4.108}$$

then, for $j = 0, 1, \dots, k-1$, the function u_k^{j+1} is the minimiser in $V_{t_k^{j+1}} + w_k^{j+1}$ of

$$u \mapsto \left\| \frac{u - u_k^j}{\tau_k} - \frac{u_k^j - u_k^{j-1}}{\tau_k} \right\|_H^2 + \|\mathcal{E}u\|^2_{H_n^{\mathbb{C}}} - 2\langle f_k^j, u\rangle_{t_k^{j+1}}. \tag{4.109}$$

Then if we define u_k, $\tilde{u}_k$, w_k, $\tilde{w}_k$, v_k, and z_k as in (4.43)-(4.47), then proceeding exactly as in 4.2.5, we deduce the following bounds

$$\mathcal{E}u_k(t) \text{ and } \mathcal{E}\tilde{u}_k(t) \text{ are bounded in } H_n \text{ uniformly in } t \text{ and } k, \tag{4.110}$$

$$\dot{u}_k(t) \text{ and } v_k(t) \text{ are bounded in } H \text{ uniformly in } t \text{ and } k \tag{4.111}$$

$$u_k(t) \text{ is bounded in } V \text{ uniformly in } t \text{ and } k. \tag{4.112}$$

Furthermore, using the Euler equation for u_k^{j+1} we can write for all $t \in (t_k^j, t_k^{j+1})$

$$\langle \dot{v}_k(t), \phi\rangle_H + \langle \mathbb{C}\,\mathcal{E}\tilde{u}_k(t), \mathcal{E}\phi\rangle_{H_n} = \langle f_k(t), \phi\rangle_{t_k^{j+1}} \tag{4.113}$$

for every $\phi \in V_{t_k^{j+1}}$. The last equation leads us to write for all $t \in (t_k^j, t_k^{j+1})$

$$\|\dot{v}_k(t)\|^*_{t_k^{j+1}} \le \|\mathcal{E}\tilde{u}_k(t)\|_{H_n^{\mathbb{C}}} + \|f_k(t)\|^*_{t_k^{j+1}}. \tag{4.114}$$

In particular fix $s \in [0, T)$, then for every $t_1, t_2 \in [s, T]$ with $t_1 < t_2$, we have

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}_k(t)\|^*_s \, dt \le \int_{t_1}^{t_2} (\|\mathcal{E}\tilde{u}_k(t)\|_{H_n^{\mathbb{C}}} + \|f_k(t)\|^*_s) \, dt. \tag{4.115}$$

By using (4.110) and the fact that by construction $f_k \to f$ strongly in $L^2(0, T; V^*)$, we deduce that, eventually passing through a subsequence, if we call v a weak limit of (v_k) in $W^{1,2}(s, T; V_s^*)$, then there exists a constant $M > 0$ such that passing to the limit as $k \to \infty$ in both sides of (4.115) we obtain

$$\int_{t_1}^{t_2} \|P_{st}\dot{v}(t)\|^*_s \, dt \le |t_2 - t_1| M + \int_{t_1}^{t_2} \|f_k(t)\|^*_s) \, dt,$$

for every $t_1, t_2 \in [s, T]$ with $t_1 < t_2$. Again following the steps after inequality (4.57) in the proof of 4.2.5, we deduce that up to a subsequence, not relabeled, we have

$$u_k \rightharpoonup u, \text{ weakly in } W^{1,2}(0, T; H), \tag{4.116}$$

$$\tilde{u}_k \rightharpoonup u, \text{ weakly in } L^2(0, T; V) \tag{4.117}$$

$$v_k \rightharpoonup v \text{ weakly in } L^2(0, T; H), \tag{4.118}$$

$$v_k \rightharpoonup v \text{ weakly in } W^{1,2}(s, T; V_s^*), \; s \in [0, T). \tag{4.119}$$

Moreover $u \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; H)$, $\dot{u}(t) = v(t)$ a.e. on $[0, T]$, and

$$\|P_{st}\ddot{u}(t)\|^*_t \le M + \|f(t)\|^*_s, \tag{4.120}$$

for every $s \in [0, T)$ and for a.e. $t > s$.

In order to prove that $u(t) - w(t) \in V_t$ for every $t \in [0,T]$ we need to show that

$$\tilde{u}_k(\cdot - \tau_k) - \tilde{w}_k(\cdot - \tau_k) \rightharpoonup h \text{ weakly in } L^2(0,T;V), \tag{4.121}$$

then the proof follows exactly the one of 4.2.5. To prove (4.121), notice that by (4.116) we have for every $t \in (t_k^j, t_k^{j+1}]$

$$\|u_k(t) - \tilde{u}_k(t-\tau_k)\|_H = \|u_k(t) - u_k(t_k^j)\|_H \le \int_{t_k^j}^{t_k^{j+1}} \|\dot{u}_k(\tau)\|_H \, d\tau \le \tau_k^{1/2} C,$$

where C is a constant independent on k. This means that, by using also $w \in W^{1,2}(0,T;H^1(\Omega;\mathbb{R}^n))$, we can write

$$\tilde{u}_k(\cdot - \tau_k) - \tilde{w}_k(\cdot - \tau_k) \rightharpoonup u - w \text{ weakly in } L^2(0,T;H). \tag{4.122}$$

By (4.117) we have also that, up to subsequences,

$$\tilde{u}_k(\cdot - \tau_k) - \tilde{w}_k(\cdot - \tau_k) \rightharpoonup h \text{ weakly in } L^2(0,T;V).$$

But V is embedded in H, therefore by (4.122) every weak limit h must coincide with $u - w$ and we are done. The proof that u is a solution, and that satisfies energy inequality (4.106) proceeds as in the damped case.

Finally, it remains to prove that u satisfies the initial conditions (4.103) (4.104). We start by showing that $u(t)$ is a well defined element of V for every $t \in [0,T]$. By (4.98) we deduce that $u(t)$ is a well defined element of H for every $t \in [0,T]$ and that

$$u(t_i) \to u(t) \text{ strongly in } H \text{ for every } t \in [0,T]. \tag{4.123}$$

Moreover by using $u \in L^\infty(0,T;V)$ we know that given $t \in [0,T]$ there exists a sequence $t_i \to t$ such that

$$\sup_i \|u(t_i)\|_V \le \|u\|_{L^\infty(0,T;V)}.$$

Therefore, by the fact that V is embedded in H together with (4.123) we deduce that any weak limit in V of $(u(t_i))$ must be equal to $u(t)$. This means that

$$u(t_i) \rightharpoonup u(t) \quad \text{weakly in } V \text{ as } t_i \to t$$

and

$$\|u\|_{L^\infty(0,T;V)} \ge \liminf_{i\to\infty} \|u(t_i)\|_V \ge \|u(t)\|_V.$$

Hence, for every $t \in [0,T]$, $u(t)$ is a well defined element of V.

Moreover given any $t \in [0,T]$ and given any sequence $t_i \to t$, by the previous inequality we deduce that

$$\sup_i \|u(t_i)\|_V \le \|u\|_{L^\infty(0,T;V)},$$

therefore arguing exactly as before we deduce

$$u(t_i) \rightharpoonup u(t) \text{ weakly in } V. \tag{4.124}$$

This shows that the map $t \mapsto u(t)$ is weakly continuous in V for every $t \in [0,T]$.

Now in order to prove (4.104) and (4.104), it is equivalent to show that there exists a set $N \subset [0,T]$ of measure zero, such that for every sequence t_i with $t_i \in [0,T] \setminus N$ and $t_i \to 0$, then

$$\dot{u}(t_i) \to u^1 \text{ strongly in } H, \tag{4.125}$$

and

$$\mathcal{E}u(t_i) \to \mathcal{E}u^0 \text{ strongly in } H_n. \tag{4.126}$$

To prove this notice that inequality (4.106) together with (4.98) and the regularity assumptions on w tells us that

$$\|\dot{u}(t)\|_H^2 + \|\mathcal{E}u(t)\|_{H_n^{\mathbb{C}}} \leq \|u^1\|_H^2 + \|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} + o(1), \quad \text{as } t \to 0^+ \tag{4.127}$$

for every $t \in [0,T]\backslash N$ where N is a set of measure zero. Now let (t_i) be such that $t_i \in [0,T]\backslash N$ and $t_i \to 0$.

By arguing as in the proof of Theorem 4.2.5 we can make use of convergence (4.116) and convergence (4.119) to deduce that $u^0 = u(0)$ and $u^1 = \dot{u}(0)$, respectively. Moreover, by Remark 4.1.14 we have $\dot{u}(t_i) \to u^1$ weakly in H and by (4.124) we have $u(t_i) \to u^0$ weakly in V. Therefore, to prove (4.125) and (4.126) it is enough to show that

$$\limsup_{i\to\infty} \|\dot{u}(t_i)\|_H^2 \leq \|u^1\|_H^2 \quad \text{and} \quad \limsup_{i\to\infty} \|\mathcal{E}u(t_i)\|_{H_n}^2 \leq \|\mathcal{E}u^0\|_{H_n}^2. \tag{4.128}$$

By lower semi-continuity we can write

$$\begin{aligned} \|u^1\|_H^2 + \|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}} &\leq \liminf_{i\to\infty} \|\dot{u}(t_i)\|_H^2 + \liminf_{i\to\infty} \|\mathcal{E}u(t_i)\|_{H_n^{\mathbb{C}}}^2 \\ &\leq \liminf_{i\to\infty} (\|\dot{u}(t_i)\|_H^2 + \|\mathcal{E}u(t_i)\|_{H_n^{\mathbb{C}}}^2) \\ &\leq \limsup_{i\to\infty} (\|\dot{u}(t_i)\|_H^2 + \|\mathcal{E}u(t_i)\|_{H_n^{\mathbb{C}}}^2) \\ &\leq \|u^1\|_H^2 + \|\mathcal{E}u^0\|_{H_n^{\mathbb{C}}}, \end{aligned}$$

where for the last inequality we have used inequality (4.127) on the points t_i. This immediately implies (4.128) and shows that (4.125) and (4.126) hold true. □

Chapter 5

Energy-dissipation balance of a smooth growing crack

Contents

5.1 Preliminaries

5.1.1 Standing assumptions

We consider a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, we take a Borel subset $\partial_D\Omega$ of $\partial\Omega$ (possibly empty), and we denote by $\partial_N\Omega$ its complement. The two sets $\partial_D\Omega$ and $\partial_N\Omega$ are the Dirichlet and the Neumann part of the boundary, respectively. We fix a $C^{3,1}$ curve $\gamma : [0, \ell] \to \overline{\Omega}$ parametrized by arc-length, with end-points on $\partial\Omega$; namely, denoting by Γ the support of γ, we assume $\Gamma \cap \partial\Omega = \gamma(0) \cup \gamma(\ell)$. Let $T > 0$ and $s : [0, T] \to (0, \ell)$ be a non decreasing function of class $C^{3,1}$. We set

$$\Gamma(t) := \{\gamma(\sigma) \mid 0 \leq \sigma \leq s(t)\}\,.$$

Given a matrix field $A : \overline{\Omega} \to \mathbb{R}^{2\times 2}_{sym}$ with smooth ($C^{2,1}$ would be enough) coefficients satisfying the ellipticity condition

$$A(x)\xi \cdot \xi \geq c_0|\xi|^2 \quad \forall\, x \in \overline{\Omega}\,,\ \xi \in \mathbb{R}^2\,, \tag{5.1}$$

a function $f \in C^0([0, T]; H^1(\Omega)) \cap \mathrm{Lip}([0, T]; L^2(\Omega))$, and suitable initial data u^0 and u^1 (for the precise regularity, see Theorems 5.2.4 and 5.2.10), we consider the differential equation

$$\ddot{u}(t) - \mathrm{div}(A\nabla u(t)) = f(t) \quad \text{in } \Omega \setminus \Gamma(t)\,, \tag{5.2}$$

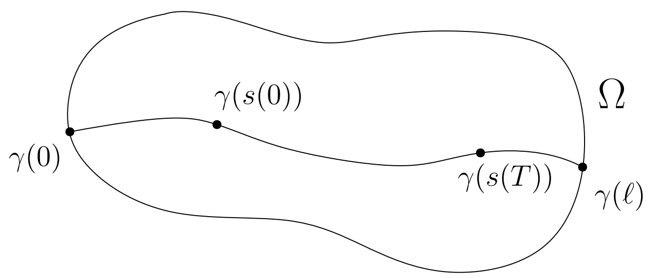

Figure 5.1: *The endpoints of* Γ *are* $\gamma(0)$ *and* $\gamma(\ell)$ *and belong to* $\partial\Omega$. *We study the evolution of the fracture along* Γ *from* $\gamma(s(0))$ *to* $\gamma(s(T))$.

with initial conditions

$$u(0) = u^0\,, \;\; \dot{u}(0) = u^1\,,$$

and boundary conditions

$$u(t) = 0 \quad \text{on } \partial_D\Omega\,, \quad A\nabla u(t)\cdot\nu = 0 \quad \text{on } \partial_N\Omega \cup \Gamma(t)\,, \tag{5.3}$$

where ν denotes the unit normal vector. The equation (5.2) has to be intended in the following weak sense (see also [25, Definition 2.4]): for every $t \in (0, T)$

$$\langle \ddot{u}(t), \varphi\rangle_{H^{-1}_D(\Omega\setminus\Gamma(t))} + \langle A\nabla u(t), \nabla\varphi\rangle_{L^2} = \langle f(t), \nabla\varphi\rangle_{L^2}, \tag{5.4}$$

for every test function φ in the space $H^1_D(\Omega\setminus\Gamma(t))$ of H^1 functions in the fractured domain $\Omega\setminus\Gamma(t)$ which vanish on $\partial_D\Omega$. We implicitly require $u(t)$ to be in $H^1_D(\Omega\setminus\Gamma(t))$ and $\ddot{u}(t)$ to be in its dual $H^{-1}_D(\Omega\setminus\Gamma(t))$, for every $t\in(0,T)$. This definition of solution coincides with 4.3.1 except for the fact that here we are in the scalar case, and due to the fact that $\Gamma(t)$ are closed we can define the gradient ∇u in the sense of distribution in $\Omega\setminus\Gamma(t)$.

Furthermore, we assume that the velocity of s is bounded by the constant c_0 as follows:

$$|\dot{s}(t)|^2 \leq c_0 - \delta \quad \forall t \in [0, T]\,, \tag{5.5}$$

for some constant $0 < \delta \leq c_0$. This bound is due to the fact that the relation with the ellipticity constant c_0 of A will guarantee the resolvability of the problem (see also (5.13) in Lemma 5.1.1). Moreover (5.5) can be interpreted by saying that the crack evolves more slowly than the speed of elastic waves in the medium.

5.1.2 The change of variables approach

We fix $t_0, t_1 \in [0, T]$ such that $0 < t_1 - t_0 < \rho$, with ρ sufficiently small. A comment on the value of ρ is postponed to Remark 5.1.3. In the following, we perform 4 changes of variables: first we act on the operator A, transforming it into the identity on the fracture set; then we straighten the crack in a neighborhood of $\gamma(s(t_0))$; then we recall the time-dependent change of variables introduced in [25], that brings $\Gamma(t)$ into $\Gamma(t_0)$ for every $t \in [t_0, t_1]$; finally, we perform a last change of variables in a neighborhood of the (fixed) crack tip, in order to make the principal part of the transformed equation equal to the minus Laplacian. For the sake of clarity, at each step, we use the superscript i, $i = 1, \ldots, 4$, to denote the new objects: the domain Ω^i, the fracture set Γ^i, and the time-dependent crack $\Gamma^i(t)$. We will also introduce the matrix fields A^i, which characterize the leading part (with respect to the spatial variables) $-\operatorname{div}(A^i\nabla v)$ of the PDE (5.2) transformed.

Step 1. Thanks to the standing assumptions on A, we may find a matrix field Q of class $C^{2,1}(\overline{\Omega};\mathbb{R}^{2\times 2})$ such that, for every $x \in \Omega$,

$$Q(x)A(x)Q^T(x) = I\,, \tag{5.6}$$

being I the identity matrix. In particular we can choose $Q(x)$ to be equal to the square root matrix of $A^{-1}(x)$, namely $Q(x) = Q^T(x)$ and $Q^2(x) = A^{-1}(x)$. It is easy to prove the existence of a smooth diffeomorphism χ (again, $C^{3,1}$ would be enough) of $\overline{\Omega}$ which is the identity in a neighborhood of $\partial\Omega$ and satisfies $\nabla\chi(x) = Q(x)$ on $\Gamma \cap V$, being V a suitable neighborhood of $\gamma(s(t_0))$. Notice that the constraint $D\chi = Q$ cannot be satisfied in the whole domain, since the lines of Q in general are not curl free. We set

$$\begin{aligned}&\Omega^1 := \Omega\,,\ \Gamma^1 := \chi(\Gamma)\,,\ \Gamma^1(t) := \chi(\Gamma(t))\,,\\ &A^1(x) := [\nabla\chi\, A\, \nabla\chi^T](\chi^{-1}(x))\,.\end{aligned}$$

Clearly, the matrix A^1 satisfies an ellipticity condition of type (5.1) for a suitable constant $C_1 > 0$ and it equals the identity matrix on Γ^1. Moreover, we may easily write the arc-length parametrization γ^1 of Γ^1 exploiting that of Γ, by setting

$$\gamma^1 := \chi\circ\gamma\circ\beta\,, \quad \text{with} \quad \beta^{-1}(\sigma) := \int_0^\sigma |(\chi\circ\gamma)\prime|\,\mathrm{d}\tau\,.$$

Accordingly, the time-dependent fracture $\Gamma^1(t)$ is parametrized by

$$\Gamma^1(t) = \gamma^1(s^1(t))\,, \quad \text{with} \quad s^1 := \beta^{-1}\circ s\,.$$

Note that the function s^1 is of class $C^{3,1}$ and, thanks to (5.6) and (5.5), satisfies the following bound:

$$|\dot{s}^1(t)|^2 = \left|\frac{\mathrm{d}\beta^{-1}}{\mathrm{d}s}(s(t))\right|^2 |\dot{s}(t)|^2 \le \max_{\|\xi\|=1\,,\,x\in\Gamma\cap V} |\nabla\chi(x)\xi|^2|\dot{s}(t)|^2 \le (1-c_1^2)\,, \tag{5.7}$$

where, for brevity, we have set $c_1^2 := \delta/c_0$. Moreover, for the sake of clarity, we also fix a notation for the maximal acceleration: we set c_2 as

$$c_2 := \max_{t\in[t_0,t_1]} |\ddot{s}^1(t)|\,. \tag{5.8}$$

A direct computation proves that c_2 is bounded and depends on c_0, δ, $\ddot{s}$, γ'', and $\nabla^2\chi$.

Step 2. Now we provide a change of variables Λ of class $C^{2,1}$ which straightens the crack in a neighborhood of $\gamma^1(s^1(t_0))$. First, up to further compose Λ with a rigid motion, we may assume that the crack-tip of $\Gamma^1(t_0)$ is at the origin, and the tangent vector to Γ^1 at the origin is horizontal, namely

$$\gamma^1(s^1(t_0)) = 0\,, \quad (\gamma^1)'(s^1(t_0)) = e_1 = (1,0)\,.$$

For brevity, we set $\sigma_0 := s^1(t_0)$. We begin by transforming a tubular neighborhood U of the fracture near 0 into a square: setting

$$U := \{\gamma^1(\sigma_0+\sigma) + \tau\nu^1(\sigma_0+\sigma) \mid \sigma\in(-\varepsilon,\varepsilon)\,,\tau\in(-\varepsilon,\varepsilon)\}$$

with $\nu^1 := (\gamma^1)'^{\perp}$ and $\varepsilon > 0$ such that $U \subset\subset \Omega$, we define $\Lambda\colon U \to (-\varepsilon,\varepsilon)^2$ as the inverse of the function $(\sigma,\tau) \mapsto \gamma^1(\sigma+\sigma_0) + \tau\nu^1(\sigma+\sigma_0)$. The global diffeomorphism is obtained by extending Λ to the whole Ω. Accordingly, we set

$$\begin{aligned}&\Omega^2 := \Lambda(\Omega^1)\,,\ \Gamma^2 := \Lambda(\Gamma^1)\,,\ \Gamma^2(t) := \Lambda(\Gamma^1(t))\,,\\ &A^2(x) := [\nabla\Lambda\, A^1\, \nabla\Lambda^T](\Lambda^{-1}(x))\,.\end{aligned}$$

The matrix field A^2 still satisfies an ellipticity condition of type (5.1), for a suitable constant.

For $x \in \Gamma^2$ in a neighborhood of the origin, setting $y := \Lambda^{-1}(x) \in \Gamma^1$, we have

$$A^2(x) = \nabla\Lambda(y)\, A^1(y)\, \nabla\Lambda^T(y) = \nabla\Lambda(y)\, \nabla\Lambda^T(y) = [(\nabla(\Lambda^{-1}))^T(x)\, \nabla(\Lambda^{-1})(x)]^{-1} = I\,.$$

The last equality follows from

$$\frac{\partial(\Lambda^{-1})}{\partial\sigma}(\sigma,\tau) = (\gamma^1)'(\sigma_0+\sigma) + \tau(\nu^1)'(\sigma_0+\sigma)\,, \quad \frac{\partial(\Lambda^{-1})}{\partial\tau}(\sigma,\tau) = \nu^1(\sigma_0+\sigma)\,, \tag{5.9}$$

and the fact that here we consider x of the form $x = (\sigma,0)$. In particular, we may be more precise on the ellipticity constant of A^2 restricted to a neighborhood of the origin: for every $0 < \epsilon < 1$, there exists $r > 0$ such that

$$A^2(x)\xi\cdot\xi \geq (1-\epsilon)|\xi|^2 \quad \forall \xi \in \mathbb{R}^2\,,\ \forall |x| \leq r\,. \tag{5.10}$$

Finally, we underline that if $\rho := t_1 - t_0$ is small enough (see also Remark 5.1.3), the whole set $\Gamma^1(t_1)\setminus\Gamma^1(t_0)$ is contained in U, so that the time-dependent fracture $\Gamma^2(t)$ satisfies

$$\Gamma^2(t) = \Gamma^2(t_0) \cup \{(\sigma,0) \mid 0 \leq \sigma \leq s^1(t) - s^1(t_0)\}\,,$$

for every $t \in [t_0,t_1]$.

Step 3. Here we introduce a family of 1-parameter C^2 diffeomorphisms $\Psi(t,\cdot)$, $t \in [t_0,t_1]$, which transform every $\Omega^2\setminus\Gamma^2(t)$ into $\Omega^2\setminus\Gamma^2(t_0)$. All in all, we map the non cylindrical domain $\{(x,t)\ :\ x \in \Omega^2\setminus\Gamma^2(t)\,,\ t \in [t_0,t_1]\}$ into the cylindrical one $(\Omega^2\setminus\Gamma^2(t_0)) \times [t_0,t_1]$. This construction can be found in [66] and [25, Example 1.14], thus we limit ourselves to recall the main properties: the diffeomorphism $\Psi : [t_0,t_1]\times\overline{\Omega} \to \overline{\Omega}$ satisfies

$$\Psi(t_0) = id\,, \quad \Psi(t)_{\lfloor\partial\Omega} = id\,, \quad \Psi(t)(\Gamma^2(t)) = \Gamma^2(t_0)\,,$$

being id the identity map. The corresponding matrix field is

$$A^3(t,x) := [D\Psi\, A^2 D\Psi^T - \dot\Psi\otimes\dot\Psi](\Psi^{-1}(t,x))\,.$$

Note that A^2 does not depend on time, while A^3 does.

In a neighborhood of the origin,

$$\Psi(t,x) = x - (s^1(t) - s^1(t_0))e_1 \quad \text{and} \quad \Psi^{-1}(t,x) = x + (s^1(t) - s^1(t_0))e_1\,, \tag{5.11}$$

so that $D\Psi = I$, $\dot\Psi = -\dot s^1 e_1$, and for $x = (x_1,0)$ with x_1 small enough in modulus,

$$A^3(t,x) = \begin{pmatrix} 1-|\dot s^1(t)|^2 & 0\\ 0 & 1\end{pmatrix}.$$

Step 4. In this last step we apply a change of variables P near the origin (namely the crack-tip of $\Gamma^2(t_0)$), in order to make the matrix field A^4, constructed as in the previous steps, satisfy

$A^4(t,0) = I$ for every $t \in [t_0, t_1]$. To this aim, we recall the construction introduced in [66, §4].

We define $\alpha : [t_0, t_1] \to \mathbb{R}^+$ and $d : [t_0, t_1] \times \Omega \to \mathbb{R}^+$ as

$$\alpha(t) := \sqrt{1 - |\dot{s}^1(t)|^2}\,,$$
$$d(t,x) := \alpha(t) k_\eta(|x|) + (1 - k_\eta(|x|)) c_1\,,$$

where k_η is the following cut-off function:

$$k_\eta(\tau) := \begin{cases} 1 & \text{if } 0 \le \tau < \eta/2 \\ \left(2\frac{\tau}{\eta} - 2\right)^2 \left(4\frac{\tau}{\eta} - 1\right) & \text{if } \eta/2 \le \tau < \eta \\ 0 & \text{if } \tau \ge \eta\,. \end{cases} \tag{5.12}$$

Here η is a positive parameter, whose precise value will be specified later, small enough such that $B_\eta(0) \subset \Omega$. Eventually, we set

$$P(t,x) := \left(\frac{x_1}{d(t,x)}, x_2 \right)\,.$$

For every $t \in [t_0, t_1]$, P defines a diffeomorphism of Ω into its dilation in the horizontal direction

$$\Omega^4 := \left\{ \left(\frac{x_1}{c_1}, x_2 \right) \mid x \in \Omega \right\}\,,$$

which maps 0 in 0 and $\Gamma^3(t_0) := \Gamma^2(t_0)$ into a fixed set $\Gamma^4(t_0)$, horizontal near the origin. Accordingly, the matrix field A^4 associated to this transformation reads

$$A^4(t,x) = \left[\nabla P\, A^3\, \nabla P^T - \dot{P} \otimes \dot{P} - \nabla P\, \dot{\Psi}(\Psi^{-1}) \otimes \dot{P} - \dot{P} \otimes \nabla P\, \dot{\Psi}(\Psi^{-1}) \right] (P^{-1}(t,x))\,.$$

The properties of A^4 are gathered in the following

Lemma 5.1.1. *There exists a constant $c_4 > 0$ such that, for every $t \in [t_0, t_1]$ and $x \in \Omega^4$,*

$$A^4(t,x)\xi \cdot \xi \ge c_4 |\xi|^2\,, \quad \forall \xi \in \mathbb{R}^2\,. \tag{5.13}$$

Moreover, for every $t \in [t_0, t_1]$, there holds

$$A^4(t,0) = I\,. \tag{5.14}$$

Finally, there exists a vector field $M : \partial_N \Omega^4 \cup \Gamma^4(t_0) \to \mathbb{R}^2$ such that, for every $t \in [t_0, t_1]$ and $x \in \partial_N \Omega^4 \cup \Gamma^4(t_0)$,

$$(A^4)^T(t,x) n(x) = M(x)\,, \tag{5.15}$$

and $M(x) = n(x) = e_2$ in a neighborhood of the tip of $\Gamma^4(t_0)$.

Proof. Let $t \in [t_0, t_1]$ and $x \in \Omega^4$ be fixed. Setting $y := P^{-1}(t,x) \in \Omega$, we distinguish the three cases $|y| < \eta/2$, $\eta/2 \le |y| \le \eta$, and $|y| > \eta$, where η is the constant introduced in (5.12).

Without loss of generality, up to take η smaller, recalling (5.11), we may assume that in $B_\eta(0)$

$$\nabla \Psi(t, \Psi^{-1}(t,y)) = I\,, \quad \dot{\Psi}(t, \Psi^{-1}(y)) = -\dot{s}^1(t) e_1\,,$$

so that

$$A^3(t, P^{-1}(t,x)) = A^3(t,y) = A^2(y) - |\dot{s}^1(t)|^2 e_1 \otimes e_1\,.$$

Moreover, we take $\eta < r$ with r associated to $\epsilon = c_1^2/2$ as in (5.10), so that the ellipticity constant of A^2 in $B_\eta(0)$ is $(1 - c_1^2/2)$.

If $|y| < \eta/2$ we have

$$\nabla P(t,y) = \begin{pmatrix} \frac{1}{\alpha(t)} & 0 \\ 0 & 1 \end{pmatrix}, \quad \dot{P}(t,y) = \begin{pmatrix} -y_1 \frac{\dot{\alpha}(t)}{\alpha^2(t)} \\ 0 \end{pmatrix},$$

thus

$$A^4(t,x) = \begin{pmatrix} \frac{1}{\alpha(t)} & 0 \\ 0 & 1 \end{pmatrix} A^2(y) \begin{pmatrix} \frac{1}{\alpha(t)} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{|\dot{s}^1(t)|^2}{\alpha(t)^2} + y_1 \frac{2\dot{s}^1(t)\dot{\alpha}(t)}{\alpha^3(t)} + y_1^2 \frac{\dot{\alpha}^2(t)}{\alpha^4(t)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $P^{-1}(t,0) = 0$ and $A^2(0) = I$, we immediately get (5.14). For ξ arbitrary vector of $\mathbb{R}^2$, we have

$$(A^4(t,x)\xi) \cdot \xi \geq \left[\frac{1 - c_1^2/2 - |\dot{s}^1(t)|^2}{\alpha^2(t)} - 2y_1 \frac{\dot{s}^1(t)\dot{\alpha}(t)}{\alpha^3(t)} - y_1^2 \frac{\dot{\alpha}^2(t)}{\alpha^4(t)} \right] \xi_1^2 + (1 - c_1^2/2)\xi_2^2 .$$

In view of the bounds (5.5), (5.8), and (5.7), we get

$$|\dot{\alpha}(t)| \leq \frac{c_2}{c_1}, \quad c_1 \leq |\alpha(t)| \leq 1 ,$$

in particular

$$A^4(t,x)\xi \cdot \xi \geq \left(\frac{c_1^2}{2} - 2\eta \frac{c_2}{c_1^4} - \eta^2 \frac{c_2^2}{c_1^6} \right) \xi_1^2 + \frac{\xi_2^2}{2} .$$

The coefficient of ξ_1 is bounded from below, provided that η is small enough. This gives the statement (5.13) for $y \in B_{\eta/2}(0)$.

Let now $\eta/2 < |y| < \eta$. In this case we have

$$DP(t,y) = \frac{1}{d^2} \begin{pmatrix} d - y_1\partial_1 d & -y_1 \partial_2 d \\ 0 & d^2 \end{pmatrix}, \quad \dot{P}(t,y) = \frac{1}{d^2} \begin{pmatrix} -y_1 \partial_t d \\ 0 \end{pmatrix}.$$

Again exploiting the ellipticity of A^2 with constant $(1 - c_1^2/2) \geq \frac{1}{2}$ and setting

$$m := y_1^2 (\partial_t d)^2 + 2y_1 \dot{s}^1(t)(\partial_t d)(d - y_1 \partial_1 d) , \quad p := (d - y_1 \partial_1 d) , \quad q := -y_1 \partial_2 d ,$$

we get

$$\begin{aligned} A^4(t,x)\xi \cdot \xi &\geq \frac{1}{2} \|DP^T(t,y)\xi\|^2 - \frac{m}{d^4}\xi_1^2 = \frac{1}{2d^4} \left[(p^2 + q^2 - 2m)\xi_1^2 + 2qd^2 \xi_1 \xi_2 + d^4 \xi_2^2 \right] \\ &\geq \frac{1}{2} \left[p^2 - \left(\frac{1}{\varepsilon} - 1 \right) q^2 - 2|m| \right] \xi_1^2 + \frac{1}{2}(1 - \varepsilon)\xi_2^2 , \end{aligned} \tag{5.16}$$

where in the last inequality we have have used $d \leq 1$ and the Young's inequality, with $0 < \varepsilon < 1$, whose precise value will be fixed later. Let us prove that, if η and ε are well chosen, the coercivity of A^4 is guaranteed. The identities

$$\nabla d(t,y) = (\alpha(t) - c_1) \frac{y}{|y|} k_\eta'(|y|) , \quad \partial_t d(t,y) = - \frac{\dot{s}^1(t)\ddot{s}^1(t) k_\eta(|y|)}{\alpha(t)} ,$$

together with the bounds

$$0 \leq k_\eta \leq 1 , \quad c_1 \leq d \leq \alpha \leq 1 , \quad - \frac{3}{\eta} \leq k_\eta' \leq 0 ,$$

give

$$\begin{aligned}
&\frac{1}{d^4} \geq 1\,, \\
&p = d + \frac{y_1^2}{|y|}(\alpha - c_1)|k_\eta'(|y|)| \geq d \geq c_1\,, \\
&q^2 = (\alpha - c_1)^2 \frac{y_1^2 y_2^2}{|y|^2}(k_\eta'(|y|))^2 \leq 9(1-c_1)^2\,, \\
&|m| \leq \frac{42 c_2 (1-c_1^2)}{c_1}\eta + \frac{c_2^2(1-c_1^2)}{c_1^2}\eta^2\,.
\end{aligned}$$

Inserting these estimates into (5.16), we infer that

$$A^4(t,x)\xi \cdot \xi \geq \left[\frac{c_1^2}{2} - \frac{9}{2}\left(\frac{1}{\varepsilon} - 1\right)(1-c_1)^2 - \frac{42 c_2(1-c_1^2)}{c_1}\eta - \frac{c_2^2(1-c_1^2)}{c_1^2}\eta^2\right]\xi_1^2 + \frac{1-\varepsilon}{2}\xi_2^2\,.$$

Taking

$$\varepsilon = \frac{9(1-c_1)^2}{c_1^2/2 + 9(1-c_1)^2} \in (0,1)$$

we have

$$\frac{c_1^2}{2} - \frac{9}{2}\left(\frac{1}{\varepsilon} - 1\right)(1-c_1)^2 = \frac{c_1^2}{4}\,.$$

Thus, taking η small enough, we obtain the desired coercivity of A^4.

Finally, if $|y| > \eta$ we have

$$\nabla P(t,y) = \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \dot{P}(t,y) = 0\,,$$

and condition (5.13) is readily satisfied in view of the ellipticity of A^3.

The assertion (5.15) is clearly verified for A^2: the matrix field does not depend on time and equals to the identity on the fracture, in a neighborhood of the origin. The last diffeomorphisms Ψ and P both act in a neighborhood of the origin modifying the set only in the horizontal component; in particular they don't modify the normal to the fracture in a neighborhood of the origin. As for the external boundary, Ψ is the identity and P acts as a constant dilation, so that

$$W(x) = \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & 1 \end{pmatrix} A^2(c_1 x_1, x_2) \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & 1 \end{pmatrix} n(x) \quad \text{on } \partial_N \Omega^4\,.$$

□

Remark 5.1.2. *The idea of the proof of Lemma 5.1.1 is taken from [66, Lemma 4.1]. Let us underline the main differences: in [66] the authors deal with the identity matrix as starting matrix field (here instead we have A^3) and consider only the dynamics for which the acceleration of the tip is bounded by a precise constant depending on c_1 (in place of our bound c_2, not fixed a priori). We also point out that in [66] the study of the ellipticity of the transformed matrix field, in the annulus $\eta/2 < |y| < \eta$, is carried out forgetting the coefficients out of the diagonal.* ■

Remark 5.1.3. *In our construction, a control on the maximal amplitude ρ of the time interval $[t_0, t_1]$ is needed only in Step 2: roughly speaking, in order to straighten the set $\Gamma^1(t_1)\setminus\Gamma^1(t_0)$ and to remain inside Ω, we need to have enough room. A sufficient condition is that the length of the set, which is at most $\rho\max_{t\in[0,T]}\dot{s}^1(t)$, has to be less than or equal to the distance of the crack-tip $\gamma^1(s^1(t))$ from the boundary $\partial\Omega$, which is, thanks to the assumption $\Gamma^1(T)\setminus\Gamma^1(0)\subset\subset\Omega$, bounded from below by a positive constant. Notice that if we considered also a further diffeomorphism which is the identity in a neighborhood of $\Gamma^1(T)\setminus\Gamma^1(0)$ and stretches Ω near the boundary, then our results could be stated for every time $t\in[0,T]$.* ■

5.2 Proof of the representation result

In this section we derive the decomposition result (0.17) locally in time, namely in a time interval $[t_0, t_1]$ small enough (see §5.1.2 and Remark 5.1.3). Finally, in §5.2.4, we give a global representation of u, valid in the whole time interval $[0, T]$.

5.2.1 Preliminaries on semigroup theory

Here we recall some classical facts of semigroup theory. Standard references on the subject are the books [67] and [49].

Let V be a Banach space and $\mathcal{A}(t) : \mathrm{D}(\mathcal{A}(t)) \subset V \to V$ a differential operator. Consider the evolution problem

$$\partial_t U(t) + \mathcal{A}(t)U(t) = G(t)\,, \tag{5.17}$$

with initial condition $U(0) = U_0$ (the boundary conditions are encoded in the function space X).

Definition 5.2.1. *A triplet $\{\mathcal{A}; V, W\}$ consisting of a family $\mathcal{A} = \{\mathcal{A}(t)\,,\ t \in [0, T]\}$ and a pair of real separable Banach spaces V and W is called a constant domain system if the following conditions hold:*

i) the space W is embedded continuously and densely in V;

ii) for every t the operator $\mathcal{A}(t)$ is linear and has constant domain $\mathrm{D}(\mathcal{A}(t)) \equiv W$;

iii) the family $\mathcal{A}$ is a stable family of (negative) generators of strongly continuous semigroups on V;

iv) the operator $\partial_t\mathcal{A}$ is essentially bounded from $[0, T]$ to the space of linear functionals from W to V.

Theorem 5.2.2. *Let $\{\mathcal{A}; V, W\}$ form a constant domain system. Let $U^0 \in W$ and $G \in \mathrm{Lip}([0, T]; V)$. Then there exists a unique solution $U \in C([0, T]; W) \cap C^1([0, T]; V)$ of (5.17) with $U(0) = U^0$.*

5.2.2 Local representation result in the cylindrical domain

The chain of transformations introduced in §5.1.2 defines the family of time-dependent diffeomorphisms

$$\Phi(t) := P(t) \circ \Psi(t) \circ \Lambda \circ \chi\,, \quad \Phi(t) : \overline{\Omega} \to \overline{\Omega}^4\,, \tag{5.18}$$

which map Γ into Γ^4, $\Gamma(t)$ into $\Gamma^4(t_0)$ for every $t \in [t_0, t_1]$, and $\partial\Omega$ into $\partial\Omega^4$. More precisely, the Dirchlet part $\partial_D\Omega$ is mapped into $\partial_D\Omega^4 := \{(\Lambda_1(x)/c_2, \Lambda_2(x)) \ : \ x \in \partial_D\Omega\}$, the Neumann one $\partial_N\Omega$ into $\partial_N\Omega^4 := \{(\Lambda_1(x)/c_2, \Lambda_2(x)) \ : \ x \in \partial_N\Omega\}$. For the sake of clarity, we denote by x the variables in Ω and by y the new variables in Ω^4.

Looking for a solution u to (5.2) is equivalent to look for $v := u \circ \Phi^{-1}$, solution to

$$\ddot{v}(t) - \mathrm{div}(A^4\nabla v(t)) + p(t) \cdot \nabla v(t) - 2q(t) \cdot \nabla\dot{v}(t) = g(t) \quad \text{in } \Omega^4 \setminus \Gamma^4(t_0)\,, \tag{5.19}$$

supplemented by the boundary conditions

$$v = 0 \quad \text{on } \partial_D\Omega^4\,, \quad \partial_M v = 0 \quad \text{on } \partial_N\Omega^4 \cup \Gamma^4(t_0)\,, \tag{5.20}$$

and by suitable initial conditions. Here M is the vector field introduced in (5.15) - Lemma 5.1.1, and (see also [25])

$$\begin{aligned}
&p(t,y) := -[A^4(t,y)\nabla(\det\nabla\Phi^{-1}(t,y)) + \partial_t(q(t,y)\det\nabla\Phi^{-1}(t,y))]\det\nabla\Phi(t,\Phi^{-1}(t,y))\,,\\
&q(t,y) := -\dot{\Phi}(t,\Phi^{-1}(t,y))\,,\\
&g(t,y) := f(t,\Phi^{-1}(t,y))\,.
\end{aligned}$$

The equation (5.19) has to be intended in the weak sense, namely valid for every $t \in (0,T)$ in duality with an arbitrary test function in the space $H^1_D(\Omega^4 \setminus \Gamma^4(t_0))$ of H^1 functions in the (fixed) fractured domain $\Omega^4 \setminus \Gamma^4(t_0)$ with zero trace on $\partial_D\Omega^4$. We implicitly require $v(t)$ and $\dot{v}(t)$ to be in $H^1_D(\Omega^4 \setminus \Gamma^4(t_0))$, and $\ddot{v}(t)$ to be in the dual $H^{-1}_D(\Omega^4 \setminus \Gamma^4(t_0))$, for every $t \in (0,T)$.

The characterisation of u will follow from that of v, slightly easier to be derived. As already pointed out in the Introduction, the advantages in dealing with problem (5.19) are essentially 3: first of all, the domain is cylindrical and constant in time; then, the fracture set is straight near the tip; finally, even if the coefficients depend on space and time, the principal part of the spatial differential operator is constant at the crack-tip.

Before stating the result, we define

$$\mathcal{H} := \{v \in H^2(\Omega^4 \setminus \Gamma^4(t_0)) \mid (5.20) \text{ hold true}\} \oplus \{k\zeta S \ : \in \mathbb{R}\}\,,$$

where ζ is a cut-off function whose support contains the origin and

$$S(y) := Im(\sqrt{y_1 + iy_2})\,. \tag{5.21}$$

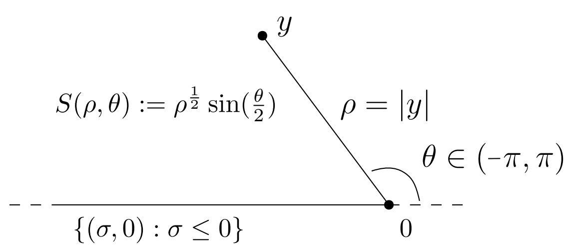

Figure 5.2: *In polar coordinates, the function S reads $S(r,\theta) = r^{1/2}\sin(\theta/2)$, where r is the distance from the origin and $\theta \in [-\pi,\pi]$ is the angle which has a discontinuity on the horizontal half line $\{x_1 \le 0\}$.*

Proposition 5.2.3. *Take $v^0 \in \mathcal{H}$, $v^1 \in H^1_D(\Omega^4 \setminus \Gamma^4(t_0))$, and $g \in \mathrm{Lip}([t_0,t_1];L^2(\Omega^4))$. Then there exists a unique solution v to (5.19)-(5.20) with $v(t_0)=v^0$, $\dot v(t_0)=v^1$ in the class*

$$v \in C([t_0,t_1];\mathcal{H}) \cap C^1([t_0,t_1];H^1_D(\Omega^4 \setminus \Gamma^4(t_0))) \cap C^2([t_0,t_1];L^2(\Omega^4))\,.$$

Proof. Once we show that the triplet $\{\mathcal{A};V;W\}$ defined by

$$\begin{aligned}
\mathcal{A}(t) &:= \begin{pmatrix} 0 & -1 \\ -\operatorname{div}(A^4(t)\nabla(\cdot)) + p(t)\cdot\nabla(\cdot) & -2q(t)\cdot\nabla(\cdot) \end{pmatrix},\\
V &:= H^1_D(\Omega^4 \setminus \Gamma^4(t_0)) \times L^2(\Omega^4)\,,\\
W &:= \mathcal{H} \times H^1_D(\Omega^4 \setminus \Gamma^4(t_0))\,,
\end{aligned}$$

is a constant domain system in $[t_0,t_1]$ (cf. Definition 5.2.1), we are done. Indeed, we are in a position to apply Theorem 5.2.2 with

$$G(t) := \begin{pmatrix} 0 \\ g(t) \end{pmatrix},$$

and the searched v is the second component of the solution U to (5.17).

The detailed proof of properties (i)-(iv) in Definition 5.2.1 can be found in [66, Theorem 4.7], with the appropriate modifications (see Remark 5.1.2). Here we limit ourselves to list the main ingredients.

First of all, the domain of $\operatorname{div}(A^4(t)\nabla(\cdot))$ is constant in time: in view (5.14), its principal part, evaluated at the crack tip, is the Laplace operator for every t, thus the domain of $\operatorname{div}(A^4(t)\nabla(\cdot))$ can be decomposed as the sum $\{v \in H^2(\Omega^4 \setminus \Gamma^4(t_0)) \;:\; (5.20) \text{ holds true}\} \oplus \{\zeta S\} =: \mathcal{H}$ (cf. [47, Theorem 5.2.7]); moreover, in view of (5.15), the boundary conditions (5.20) do not depend on time.

Other key points are the equi coercivity in time of the bilinear form

$$(w_0,w_1) \mapsto (A^4(t)\nabla w_0)\cdot\nabla w_1$$

in $H^1_D(\Omega^4 \setminus \Gamma^4(t_0))$, guaranteed by (5.13), and the property

$$\int_{\Omega^4\setminus\Gamma^4(t_0)} (q(t)\cdot\nabla\varphi)\,\varphi\,\mathrm{d}y = -\frac{1}{2}\int_{\Omega^4\setminus\Gamma^4(t_0)} \varphi^2 \operatorname{div} q(t)\,\mathrm{d}y\,,$$

valid for every $\varphi \in H^1_D(\Omega^4 \setminus \Gamma^4(t_0))$.

Finally, the needed continuity of the differential operator is ensured by the following regularity properties of the coefficients: for every $i,j,k \in \{1,2\}$,

$$\begin{aligned}
&A^4_{i,j}(t) \in C^0(\Omega^4) \quad \forall t \in [t_0,t_1]\\
&A^4_{i,j}\,,\ p_i\,,\ q_i \in \mathrm{Lip}([t_0,t_1];L^\infty(\Omega^4))\,,\\
&\|\partial_k A^4_{i,j}(t)\|_{L^\infty(\Omega^4)}\,,\ \|\operatorname{div} q(t)\|_{L^\infty(\Omega^4)} \le C\,,
\end{aligned}$$

for a suitable constant $C>0$ independent of t. □

5.2.3 Local representation result in the time-dependent domain

We are now in a position to prove the following representation result for u.

Theorem 5.2.4. *Let* $f \in C^0([t_0,t_1];H^1(\Omega)) \cap \mathrm{Lip}([t_0,t_1];L^2(\Omega))$. *Consider* u^0 *and* u^1 *of the form*

$$u^0 - k^0 \zeta S(\Phi(t_0)) \in H^2(\Omega \setminus \Gamma(t_0)), \tag{5.22}$$

$$u^1 - \nabla u^0 \cdot \Big(\nabla \Phi^{-1}(t_0, \Phi(t_0))\dot{\Phi}(t_0)\Big) \in H^1(\Omega \setminus \Gamma(t_0)), \tag{5.23}$$

with u^0 *satisfying the boundary conditions* (5.3), $u^1 = 0$ *on* $\partial_D \Omega$, ζ *cut-off function with support in a neighborhood of* $\gamma(s(t_0))$, *and* $k^0 \in \mathbb{R}$. *Then there exists a unique solution to* (5.2)-(5.3) *with initial conditions* $u(t_0) = u^0$, $\dot{u}(t_0) = u^1$ *of the form*

$$u(t,x) = u^R(t,x) + k(t)\zeta(t,x)S(\Phi(t,x)) \quad t \in [t_0,t_1],\, x \in \Omega \setminus \Gamma(t), \tag{5.24}$$

where $\zeta(t)$, $t \in [t_0,t_1]$, *is a* C^2 *(in time) family of cut-off functions with support in a neighborhood of* $\gamma(s(t))$, *and* k *is a* C^2 *function in* $[t_0,t_1]$ *such that* $k(t_0) = k^0$. *Moreover, for every* $t \in [t_0,t_1]$ *we have* $u^R(t) \in H^2(\Omega \setminus \Gamma(t))$, *and*

$$u^R \in C^2([t_0,t_1];L^2(\Omega)), \quad \nabla u^R \in C^1([t_0,t_1];L^2(\Omega;\mathbb{R}^2)), \quad \nabla^2 u^R \in C^0([t_0,t_1];L^2(\Omega;\mathbb{R}^{2\times 2})).$$

Remark 5.2.5. *Notice that the equality* $u(t,x) = v(t,\Phi(t,x))$ *implies that*

$$u^0 = v^0(\Phi(t_0)), \quad u^1 = v^1(\Phi(t_0)) + \nabla v^0(\Phi(t_0)) \cdot \dot{\Phi}(t_0).$$

The last term reads $\dot{\Phi}(t_0) = \dot{P}(t_0, \Psi\circ\Lambda\circ\chi) + \nabla P(t_0,\Psi\circ\Lambda\circ\chi)\cdot\dot{\Psi}(t_0,\Lambda\circ\chi)$. *A priori,* ∇v^0 *is just in* L^2 *in a neighborhood of the origin and its gradient behaves like* $|y|^{-3/2}$; *nevertheless, since* $\dot{P}(t,y) \sim (y_1, 0)$, *we recover the* L^2 *integrability of the gradient of* $\nabla v^0(\Phi(t_0)) \cdot \dot{P}(t_0, \Psi\circ\Lambda\circ\chi)$. *The same reasoning does not apply for the term* $\nabla v^0(\Phi(t_0)) \cdot \Big(\nabla P(t_0, \Psi\circ\Lambda\circ\chi)\cdot \dot{\Psi}(t_0,\Lambda\circ\chi)\Big)$, *since the singularity of* ∇v^0 *in a neighborhood of the orgin is not compensated by* $\nabla P \dot{\Psi}$. *Therefore we are not free to take* $u^1 \in H^1_D(\Omega \setminus \Gamma(t_0))$ *(as, on the contrary, is done in [66]).* ■

Remark 5.2.6. *Note that the solution* u *to* (5.2)-(5.3) *displays a singularity only at the crack-tip. Clearly, the fracture is responsible for this lack of regularity. On the other hand, the Dirichlet-Neumann boundary conditions do not produce any further singularity, due to the compatible initial data chosen.* ■

5.2.4 Global representation result in the time-dependent domain

We conclude the section by showing an alternative representation formula which can be expressed for every time. This is done providing another expression for the singular function, as in [56], whose computation does not require to straighten the crack. To simplify the notation we reduce ourselves to the case $A = I$, so that the diffeomorphism χ coincides with the identity.

The chosen singular part of the solution to problem (5.2)-(5.3) is a suitable riparametrization of the function S introduced in (5.21). More precisely, fixed $t_0, t_1 \in [0,T]$ with $0 < t_1 - t_0 < \rho$, for every $t \in [t_0,t_1]$ and x in a neighborhood of $r(t) := \gamma(s(t))$, the singular part reads

$$S\left(\frac{\Lambda_1(x) - (s(t) - s(t_0))}{\sqrt{1 - |\dot{s}(t)|^2}}, \Lambda_2(x)\right). \tag{5.25}$$

To compute (5.25) it is necessary to know the expression of Λ, which is explicit only for small time and locally in space. We hence provide a more explicit formula for the singular part, which has also the advantage of being defined for every time: for every $t \in [0,T]$ we set

$$\hat{S}(t,x) := Im\left(\sqrt{\frac{(x-r(t))\cdot\gamma'(s(t))}{\sqrt{1-|\dot{s}(t)|^2}} + i\,(x-r(t))\cdot\nu(s(t))}\right), \tag{5.26}$$

where $\nu(\sigma) \perp \gamma'(\sigma)$ and $\hat{S}(t)$ is given by the unique continuous determination of the complex square function such that in $x = r(t) + \sqrt{1-|\dot{s}(t)|^2}\gamma'(s(t))$ takes value 1 and its discontinuity set lies on $\Gamma(t)$. Roughly speaking, if we forget the term $\sqrt{1-|\dot{s}(t)|^2}$, the function (5.26) is the determination of $Im(\sqrt{y_1+iy_2})$ in the orthonormal system with center $\gamma(s(t))$ and axes $\gamma'(s(t))$ and $\nu(s(t))$.

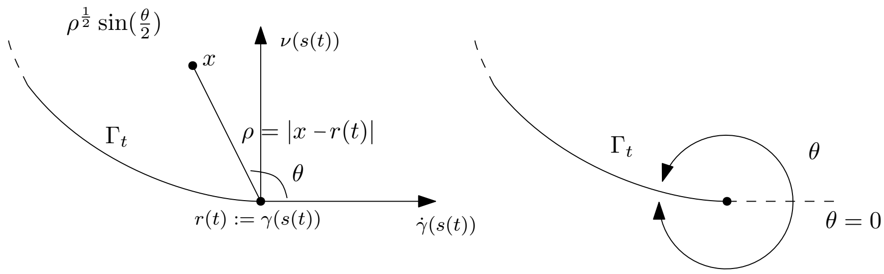

Figure 5.3: *A possible choice of determination of* $Im(\sqrt{y_1+iy_2})$*, with* $\Gamma(t)$ *as discontinuity set.*

For every $t \in [0,T]$ let $R(t) \in SO(2)^+$ be the matrix that rotates the orthonormal system with axes $\gamma'(s(t))$ and $\nu(s(t))$ in the one with axes e_1 and e_2. Thanks to our construction of Λ, and in particular to (5.9), the matrix $R(t)$ coincides with $\nabla\Lambda(r(t))$ in $[t_0,t_1]$. By setting

$$L(t) := \begin{pmatrix} \frac{1}{\sqrt{1-|s(t)|^2}} & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\Phi}(t,x) := L(t)R(t)(x-r(t)),$$
$$\tilde{\Omega}(t) := \tilde{\Phi}(t,\Omega), \quad \tilde{\Gamma}(t) := \tilde{\Phi}(t,\Gamma(t)),$$

we may also write $\hat{S}(t,x) = \tilde{S}(t,\tilde{\Phi}(t,x))$, where $\tilde{S}(t,\cdot)$ is given by the continuous determination of $Im(\sqrt{y_1+iy_2})$ in $\tilde{\Omega}(t) \setminus \tilde{\Gamma}(t)$ such that in $y = (1,0)$ takes the value 1.

Lemma 5.2.7. *Let* $\zeta(t)$*,* $t \in [t_0,t_1]$*, be a* C^2 *(in time) family of cut-off functions with support in a neighborhood of* $r(t)$*. Define the function*

$$w(t,x) := \zeta(t,x)S(\Phi(t,x)) - \hat{S}(t,x) \quad t \in [t_0,t_1],\ x \in \Omega \setminus \Gamma(t). \tag{5.27}$$

Then $w(t) \in H^2(\Omega \setminus \Gamma(t))$ *for every* $t \in [t_0,t_1]$.

Proof. Let us fix $t \in [t_0,t_1]$. The function $w(t)$ is of class C^2 in $\Omega \setminus \Gamma(t)$ and belongs to the space $H^1(\Omega \setminus \Gamma(t)) \cap H^2((\Omega \setminus \Gamma(t)) \setminus B_\epsilon(r(t)))$ for every $\epsilon > 0$. Hence it remains to prove the L^2-integrability of its second spatial derivatives in $B_\epsilon(r(t))$. Let us choose $\epsilon > 0$ so small that $\zeta(t) = 1$ on $B_\epsilon(r(t))$. For every $i, j = 1, 2$ in $B_\epsilon(r(t))$ we have

$$\begin{aligned}\partial^2_{ji} w(t) &= \sum_{h=1}^{d} [\partial_h S(\Phi(t)) \partial^2_{ji} \Phi_h(t) - \partial_h \tilde{S}(t, \tilde{\Phi}(t)) \partial^2_{ji} \tilde{\Phi}_h(t)] \\ &\quad + \sum_{h,k=1}^{d} [\partial^2_{hk} S(\Phi(t)) \partial_j \Phi_k(t) \partial_i \Phi_h(t) - \partial^2_{hk} \tilde{S}(t, \tilde{\Phi}(t)) \partial_j \tilde{\Phi}_k(t) \partial_i \tilde{\Phi}_h(t)] \\ &=: I_1(t) + I_2(t),\end{aligned}$$

where $\Phi_i(t)$ and $\tilde{\Phi}_i(t)$ are the i-th components of $\Phi(t)$ and $\tilde{\Phi}(t)$, respectively.

Notice that $\nabla S(\Phi(t)), \nabla \tilde{S}(t, \tilde{\Phi}(t)) \in L^2(B_\epsilon(r(t)); \mathbb{R}^2)$, while $D^2\Phi(t)$ and $D^2\tilde{\Phi}(t)$ are uniformly bounded in Ω. Therefore $I_1(t) \in L^2(B_\epsilon(r(t)))$ and there exists a positive constant C, independent of t, such that

$$|I_1(t, x)| \leq C|x - r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\epsilon(r(t)) \setminus \Gamma(t)\,,$$

provided that $\epsilon > 0$ is small enough.

As for $I_2(t)$, we estimate it from above as

$$\begin{aligned}|I_2(t)| \leq & \sum_{h,k=1}^{d} |\partial^2_{hk} S(\Phi(t)) - \partial^2_{hk} \tilde{S}(t, \tilde{\Phi}(t))| |\partial_j \tilde{\Phi}_k(t)| |\partial_i \tilde{\Phi}_h(t)| \\ & + \sum_{h,k=1}^{d} |\partial^2_{hk} S(\Phi(t))| |\partial_j \Phi_k(t) \partial_i \Phi_k(t) - \partial_j \tilde{\Phi}_k(t) \partial_i \tilde{\Phi}_h(t)|\,.\end{aligned} \tag{5.28}$$

Let us study the right-hand side of (5.28). By choosing ϵ small enough and by using the definitions of $\Phi(t)$ and $\tilde{\Phi}(t)$, we deduce that for every $x \in B_\epsilon(r(t))$

$$|\partial_j \Phi_k(t, x) \partial_i \Phi_h(t, x) - \partial_j \tilde{\Phi}_k(t, x) \partial_i \tilde{\Phi}_h(t, x)| \leq \frac{2}{c_1^2} \|\nabla \Lambda\|_\infty \|\nabla^2 \Lambda\|_\infty |x - r(t)|\,, \tag{5.29}$$

since $\|D\Phi(t)\|_\infty \leq \frac{1}{c_1}\|D\Lambda\|_\infty$, $\|D\tilde{\Phi}(t)\|_\infty \leq \frac{1}{c_1}\|D\Lambda\|_\infty$, and

$$|\nabla \Phi(t, x) - \nabla \tilde{\Phi}(t, x)| \leq \frac{1}{c_1} |\nabla \Lambda(x) - R(t)| \leq \frac{1}{c_1} \|\nabla^2 \Lambda\|_\infty |x - r(t)|\,.$$

Moreover, the function S satisfies $|\nabla^2 S(y)| \leq M|y|^{-\frac{3}{2}}$ in $\mathbb{R}^2 \setminus \{(\sigma, 0) : \sigma \leq 0\}$ for a positive constant M, while Λ is invertible and $|P(t, x)| \geq |x|$. This allows us to conclude that

$$|\partial^2_{hk} S(\Phi(t, x))| \leq M \|\nabla \Lambda^{-1}\|_\infty^{\frac{3}{2}} |x - r(t)|^{-\frac{3}{2}} \quad \text{for every } x \in B_\epsilon(r(t)) \setminus \Gamma(t)\,. \tag{5.30}$$

For the analysis of the second term in the right-hand side of (5.28), we fix $x \in B_\epsilon(r(t))$ and we consider the segment $[\Phi(t, x), \tilde{\Phi}(t, x)] := \{\lambda \Phi(t, x) + (1 - \lambda) \tilde{\Phi}(t, x) : \lambda \in [0, 1]\}$ and the function $d(t, x) := \operatorname{dist}([\Phi(t, x), \tilde{\Phi}(t, x)], 0)$. We claim that we can choose $\epsilon > 0$ so small that

$$d(t, x) \geq \frac{1}{2} |x - r(t)| \quad \text{for every } x \in B_\epsilon(r(t))\,. \tag{5.31}$$

Indeed, let $y \in [\Phi(t, x), \tilde{\Phi}(t, x)]$ be such that $|y| = d(t, x)$, then

$$|\tilde{\Phi}(t, x)| \leq |y| + |\tilde{\Phi}(t, x) - y| \leq |y| + |\tilde{\Phi}(t, x) - \Phi(t, x)|\,.$$

Since $|P(t,x)| \geq |x|$ and $R(t)$ is a rotation, for ϵ small we deduce that $|\tilde{\Phi}(t,x)| \geq |x - r(t)|$. On the other hand, by Lagrange's theorem there exists $z = z(t,x) \in B_\epsilon(r(t))$ such that

$$\begin{aligned}\Phi(t,x) &= \Phi(t,r(t)) + \nabla\Phi(t,r(t))(x-r(t)) + \nabla^2\Phi(t,z)(x-r(t))\cdot(x-r(t))\\ &= \tilde{\Phi}(t,x) + \nabla^2\Phi(t,z)(x-r(t))\cdot(x-r(t))\,.\end{aligned}$$

Hence we derive the estimate

$$|\Phi(t,x) - \tilde{\Phi}(t,x)| \leq \frac{1}{c_1}\|\nabla^2\Lambda\|_\infty |x-r(t)|^2 \quad \text{for every } x \in B_\epsilon(r(t))\,, \tag{5.32}$$

which implies

$$d(t,x) \geq |x-r(t)| - \frac{1}{c_1}\|\nabla^2\Lambda\|_\infty |x-r(t)|^2 \quad \text{for every } x \in B_\epsilon(r(t))\,.$$

In particular, we obtain (5.31) by choosing $\epsilon < c_1/(2\|\nabla^2\Lambda\|_\infty)$. Notice that ϵ does not depend on $t \in [t_0,t_1]$.

Let us now fix $x \in B_\epsilon(r(t)) \setminus \Gamma(t)$. Thanks to our construction of Φ and $\tilde{\Phi}$, it is possible to find two other determinations $S^\pm(t)$ of $Im(\sqrt{y_1 + iy_2})$ in $\mathbb{R}^2$ such that their discontinuity sets $\Gamma^\pm(t)$ do not intersect the segment $[\Phi(t,x), \tilde{\Phi}(t,x)]$, which is far way from 0. Moreover, we choose them in such a way that $S^+(t)$ is positive along $\{(\sigma,0) : \sigma \leq 0\}$, while $S^-(t)$ is negative, and $S(\Phi(t,x)) = S^\pm(t,\Phi(t,x))$ if and only if $\tilde{S}(t,\tilde{\Phi}(t,x)) = S^\pm(t,\tilde{\Phi}(t,x))$; notice that $|\nabla^3 S^\pm(t,y)| \leq M|y|^{-\frac{5}{2}}$ for a positive constant M and for every $y \in \mathbb{R}^2 \setminus \Gamma^\pm(t)$. By using Lagrange's theorem, (5.31), and (5.32), we deduce that

$$\begin{aligned}|\partial^2_{hk} S(\Phi(t,x)) - \partial^2_{hk}\tilde{S}(t,\tilde{\Phi}(t,x))| &= |\partial^2_{hk} S^\pm(t,\Phi(t,x)) - \partial^2_{hk} S^\pm(t,\tilde{\Phi}(t,x))|\\ &\leq |\nabla^3 S^\pm(t,z)||\Phi(t,x) - \tilde{\Phi}(t,x)|\\ &\leq \frac{M}{c_1}\|\nabla^2\Lambda\|_\infty |d(t,x)|^{-\frac{5}{2}}|x-r(t)|^2\\ &\leq \frac{4\sqrt{2}M}{c_1}\|\nabla^2\Lambda\|_\infty |x-r(t)|^{-\frac{1}{2}}\,,\end{aligned} \tag{5.33}$$

where $z = z(t,x) \in [\Phi(t,x), \tilde{\Phi}(t,x)]$. Hence, by combining (5.28) with (5.29), (5.30), and (5.33), we obtain the existence of a positive constant C such that

$$|I_2(t,x)| \leq C|x-r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\epsilon(r(t)) \setminus \Gamma(t)\,.$$

In particular we get the following bound for $\nabla^2 w$:

$$|\nabla^2 w(t,x)| \leq C|x-r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\epsilon(r(t)) \setminus \Gamma(t)\,, \tag{5.34}$$

and consequently $w(t) \in H^2(\Omega \setminus \Gamma(t))$ for every $t \in [t_0,t_1]$. □

In the following two lemmas, we investigate the regularity in time for w.

Lemma 5.2.8. *Under the same assumptions of Lemma 5.2.7, the function w introduced in (5.27) belongs to the space $C^0([t_0,t_1];L^2(\Omega))$. Moreover, we have $\nabla w \in C^0([t_0,t_1];L^2(\Omega;\mathbb{R}^2))$ and $\nabla^2 w \in C^0([t_0,t_1];L^2(\Omega;\mathbb{R}^{2\times 2}))$.*

Proof. The function $\zeta(S\circ\Phi)$ is in $C^0([t_0,t_1];L^2(\Omega))$, since $S \in C^2(\mathbb{R}^2 \setminus \{(\sigma,0) : \sigma \leq 0\}) \cap L^2_{loc}(\mathbb{R}^2)$ and Φ is continuous in $[t_0,t_1]\times\overline{\Omega}$. We also claim that $\hat{S} = \tilde{S}\circ\tilde{\Phi} \in C^0([t_0,t_1]\times(\Omega\setminus\Gamma)) \cap L^\infty((t_0,t_1)\times\Omega)$. Indeed, let $(t^*,x^*) \in [t_0,t_1]\times(\Omega\setminus\Gamma)$ and let $((t_j,x_j))_{j\in\mathbb{N}} \subset [t_0,t_1]\times(\Omega\setminus\Gamma)$

be a sequence of points converging to (t^*, x^*). By the convergence $\tilde\Phi(t_j, x_j) \to \tilde\Phi(t^*, x^*) \in \tilde\Omega(t^*) \setminus \tilde\Gamma(t^*)$ as $j \to \infty$, there exists $\bar{j} \in \mathbb{N}$ such that

$$\tilde S(t_j, \tilde\Phi(t_j, x_j)) = \tilde S(t^*, \tilde\Phi(t_j, x_j)) \quad \text{for every } j \geq \bar{j}\,.$$

This allows us to conclude that $\hat S(t_j, x_j) \to \hat S(t^*, x^*)$ as $j \to \infty$, since the function $\tilde S(t^*)$ is continuous in $\tilde\Omega(t^*)\setminus\tilde\Gamma(t^*)$. Furthermore, there exists $M > 0$ such that $|\hat S(t, x)| \leq M|\tilde\Phi(t, x)|^{\frac{1}{2}}$ for $x \in \Omega\setminus\Gamma$ and $t \in [t_0, t_1]$, which yields that $\hat S$ is uniformly bounded in $\Omega\setminus\Gamma$. We hence derive the claim, which implies $\hat S \in C^0([t_0, t_1]; L^2(\Omega))$, by the dominated convergence theorem.

By arguing as before, we can easily derive that $\nabla(\zeta(S\circ\Phi))$ belongs to $C^0([t_0, t_1]; L^2(\Omega; \mathbb{R}^2))$, while $\nabla\hat S = \nabla\tilde\Phi^T(\nabla\tilde S \circ \tilde\Phi) \in C^0([t_0, t_1] \times (\Omega \setminus \Gamma); \mathbb{R}^2)$. Therefore, thanks to the estimate $|\nabla\tilde S(t, \tilde\Phi(t, x))| \leq M|\tilde\Phi(t, x)|^{-\frac{1}{2}}$ for $x \in \Omega \setminus \Gamma$ and $t \in [t_0, t_1]$, and the dominated converge theorem, we conclude that $\nabla\hat S \in C^0([t_0, t_1]; L^2(\Omega; \mathbb{R}^2))$.

Finally, notice that the function $\nabla^2 w$ is continuous in $[t_0, t_1] \times (\Omega \setminus \Gamma)$. Let us now fix $t^* \in [t_0, t_1]$ and let $(t_j)_{j\in\mathbb{N}}$ be a sequence of points in $[t_0, t_1]$ such that $t_j \to t^*$ as $j \to \infty$. Thanks to the estimate (5.34), we can find $\bar{j} \in \mathbb{N}$ and $\epsilon > 0$ such that

$$|\nabla^2 w(t_j, x)| \leq C|x - r(t_j)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\epsilon(r(t_j)) \setminus \Gamma \text{ and } j \geq \bar{j}\,,$$

with C independent of j. Here we have used the fact that the constant in (5.34) can be chosen uniform in time. Furthermore, the functions $\nabla^2 w(t_j)$ are uniformly bounded with respect to j outside the ball $B_\epsilon(r(t_j))$. We can apply the generalised dominated convergence theorem to deduce that $\nabla^2 w(t_j)$ converges strongly to $\nabla^2 w(t^*)$ in $L^2(\Omega; \mathbb{R}^{2\times2})$, which implies $\nabla w^2 \in C^0([t_0, t_1]; L^2(\Omega; \mathbb{R}^{2\times2}))$. □

Lemma 5.2.9. *Under the same assumptions of Lemma 5.2.7, the function w introduced in (5.27) is an element of $C^2([0, T]; L^2(\Omega))$; moreover $\nabla w \in C^1([0, T]; L^2(\Omega; \mathbb{R}^2))$.*

Proof. For every $x \in \Omega \setminus \Gamma$ the function $t \mapsto w(t, x)$ is differentiable in $[t_0, t_1]$ and

$$\dot w(t, x) = \frac{\mathrm{d}}{\mathrm{d}t} w(t, x) = \dot\zeta(t, x) S(\Phi(t, x)) + \zeta(t, x)\nabla S(\Phi(t, x)) \cdot \dot\Phi(t, x) - \nabla\tilde S(t, \tilde\Phi(t, x)) \cdot \dot{\tilde\Phi}(t, x)\,.$$

Indeed, fixed $(t^*, x^*) \in [t_0, t_1] \times (\Omega \setminus \Gamma)$, we can find $\bar h > 0$ such that for every $|h| \leq \bar h$

$$\frac{\tilde S(t^* + h, \tilde\Phi(t^* + h, x^*)) - \tilde S(t^*, \tilde\Phi(t^*, x^*))}{h} = \frac{\tilde S(t^*, \tilde\Phi(t^* + h, x^*)) - \tilde S(t^*, \tilde\Phi(t^*, x^*))}{h}\,,$$

thanks to the fact that $\tilde\Phi(t^* + h, x^*) \to \tilde\Phi(t^*, x^*) \in \tilde\Omega(t^*) \setminus \tilde\Gamma(t^*)$ for every $x^* \in \Omega \setminus \Gamma$ as $h \to 0$. In particular, $\frac{1}{h}[\tilde S(t^* + h, \tilde\Phi(t^* + h, x^*)) - \tilde S(t^*, \tilde\Phi(t^*, x^*))] \to \nabla\tilde S(t^*, \tilde\Phi(t^*, x^*)) \cdot \dot{\tilde\Phi}(t^*, x^*)$, since $\tilde S(t^*) \in C^2(\tilde\Omega(t^*) \setminus \tilde\Gamma(t^*))$. Hence for every $(t, x) \in [t_0, t_1] \times (\Omega \setminus \Gamma)$ and $h \in \mathbb{R}$ such that $t + h \in [t_0, t_1]$ we may write

$$\frac{w(t + h, x) - w(t, x)}{h} = \frac{1}{h}\int_t^{t+h} \dot w(\tau, x)\,\mathrm{d}\tau\,.$$

By arguing as in the proof of the previous lemma we deduce that $\dot w \in C^0([t_0, t_1]; L^2(\Omega))$. Therefore we obtain that as $h \to 0$

$$\frac{1}{h}\int_t^{t+h} \dot w(\tau)\,\mathrm{d}\tau \to \dot w(t) \quad \text{in } L^2(\Omega) \text{ for every } t \in [t_0, t_1]\,,$$

and consequently $\frac{1}{h}[w(t + h) - w(t)] \to \dot w(t)$ in $L^2(\Omega)$.

Similarly, for every $x \in \Omega\backslash\Gamma$ the map $t \mapsto \dot{w}(t,x)$ is differentiable in $[t_0, t_1]$, with derivative

$$\begin{aligned}\ddot{w}(t,x) = \frac{\mathrm{d}}{\mathrm{d}t}\dot{w}(t,x) &= \ddot{\zeta}(t,x)S(\Phi(t,x)) + 2\dot{\zeta}(t,x)\nabla S(\Phi(t,x))\cdot\dot{\Phi}(t,x)\\ &+ \zeta(t,x)\nabla S(\Phi(t,x))\cdot\ddot{\Phi}(t,x) - \nabla\tilde{S}(t,\tilde{\Phi}(t,x))\cdot\ddot{\tilde{\Phi}}(t,x)\\ &+ \zeta(t,x)\nabla^2 S(\Phi(t,x))\cdot[\dot{\Phi}(t,x)\otimes\dot{\Phi}(t,x) - \dot{\tilde{\Phi}}(t,x)\otimes\dot{\tilde{\Phi}}(t,x)]\\ &+ [\zeta(t,x)\nabla^2 S(\Phi(t,x)) - \nabla^2\tilde{S}(t,\tilde{\Phi}(t,x))]\dot{\tilde{\Phi}}(t,x)\otimes\dot{\tilde{\Phi}}(t,x)\,.\end{aligned}$$

We may find $\epsilon > 0$ so small that $|\dot{\Phi}(t,x) - \dot{\tilde{\Phi}}(t,x)| \leq C|x - r(t)|$ in $B_\epsilon(r(t))$ for every $t \in [t_0, t_1]$ and for a positive constant C. Therefore, we can proceed as in the proof of Lemma 5.2.7 to obtain that $\ddot{w}(t) \in L^2(\Omega)$ for every $t \in [t_0, t_1]$, with

$$|\ddot{w}(t,x)| \leq C|x - r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\epsilon(r(t)) \setminus \Gamma(t)\,.$$

In particular, arguing as in Lemma 5.2.8, this uniform estimate implies that $\ddot{w} \in C^0([t_0, t_1]; L^2(\Omega))$. We can hence repeat the same procedure adopted before for $\dot{w}$ to conclude that as $h \to 0$

$$\frac{\dot{w}(t+h) - \dot{w}(t)}{h} \to \ddot{w}(t) \quad \text{in } L^2(\Omega) \text{ for every } t \in [t_0, t_1]\,,$$

which gives that $w \in C^2([t_0, t_1]; L^2(\Omega))$.

Finally, also the function $t \mapsto \nabla w(t,x)$ is differentiable in $[t_0, t_1]$ for every $x \in \Omega \setminus \Gamma$ and

$$\begin{aligned}\nabla\dot{w}(t,x) &= \frac{\mathrm{d}}{\mathrm{d}t}\nabla w(t,x)\\ &= \nabla\dot{\zeta}(t,x)S(\Phi(t,x)) + \nabla\zeta(t,x)\nabla S(\Phi(t,x))\cdot\dot{\Phi}(t,x) + \dot{\zeta}(t,x)\nabla\Phi(t,x)^T\nabla S(\Phi(t,x))\\ &+ \zeta(t,x)\nabla\dot{\Phi}(t,x)^T\nabla S(\Phi(t,x)) - \nabla\dot{\tilde{\Phi}}(t,x)^T\nabla\tilde{S}(t,\tilde{\Phi}(t,x))\\ &+ [\zeta(t,x)\nabla\Phi(t,x)^T - \nabla\tilde{\Phi}(t,x)^T]\nabla^2 S(\Phi(t,x))\dot{\Phi}(t,x)\\ &+ \zeta(t,x)\nabla\tilde{\Phi}(t,x)^T\nabla^2 S(\Phi(t,x))[\dot{\Phi}(t,x) - \dot{\tilde{\Phi}}(t,x)]\\ &+ \nabla\tilde{\Phi}(t,x)^T[\zeta(t,x)\nabla^2 S(\Phi(t,x)) - \nabla^2\tilde{S}(t,\tilde{\Phi}(t,x))]\dot{\tilde{\Phi}}(t,x)\,.\end{aligned}$$

Moreover there exists $\epsilon > 0$ so small that for every $t \in [t_0, t_1]$

$$|\nabla\dot{w}(t,x)| \leq C|x - r(t)|^{-\frac{1}{2}} \quad \text{for every } x \in B_\epsilon(r(t)) \setminus \Gamma(t)\,,$$

which implies the continuity of the map $t \mapsto \nabla\dot{w}(t)$ from $[t_0, t_1]$ to $L^2(\Omega;\mathbb{R}^2)$. Therefore, as $h \to 0$ we get that

$$\frac{\nabla w(t+h) - \nabla w(t)}{h} \to \nabla\dot{w}(t) \quad \text{in } L^2(\Omega;\mathbb{R}^2) \text{ for every } t \in [t_0, t_1]\,,$$

and in particular $\nabla w \in C^1([t_0, t_1]; L^2(\Omega;\mathbb{R}^2))$. □

Thanks to the previous lemmas we derive the following decomposition result.

Theorem 5.2.10. *Let $f \in C^0([0,T]; H^1(\Omega)) \cap \mathrm{Lip}([0,T]; L^2(\Omega))$. Consider u^0 and u^1 of the form*

$$u^0 - k^0\hat{S}(0) \in H^2(\Omega \setminus \Gamma(0))\,, \tag{5.35}$$

$$u^1 - k^0\dot{\hat{S}}(0) \in H^1(\Omega \setminus \Gamma(0))\,, \tag{5.36}$$

with u^0 satisfying the boundary conditions (5.3), $u^1 = 0$ on $\partial_D\Omega$, and $k^0 \in \mathbb{R}$. Then there exists a unique solution to (5.2)-(5.3) with initial condition $u(0) = u^0$ and $\dot{u}(0) = u^1$ of the form

$$u(t,x) = \hat{u}^R(t,x) + k(t)\hat{S}(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma(t)\,, \tag{5.37}$$

where $k \in C^2([0,T])$ and $\hat{u}^R(t) \in H^2(\Omega \setminus \Gamma(t))$ for every $t \in [0,T]$. Moreover

$$\hat{u}^R \in C^2([0,T];L^2(\Omega)), \quad \nabla\hat{u}^R \in C^1([0,T];L^2(\Omega;\mathbb{R}^2)), \quad \nabla^2\hat{u}^R \in C^0([0,T];L^2(\Omega;\mathbb{R}^{2\times 2}))\,. \tag{5.38}$$

In particular the function k does not depend on the choice of Φ, but only on Γ and s.

Proof. Thanks to our assumptions on f, u^0 and u^1 we can apply Theorem 5.2.4 with $t_0 = 0$. Indeed, in view of the computations done before, we have

$$\hat{S}(0) - \zeta(0)S(\Phi(0)) \in H^2(\Omega \setminus \Gamma(0))\,,$$

which gives (5.22). In particular

$$\nabla u^0 - k^0\zeta(0)\nabla\Phi(0)^T\nabla S(\Phi(0)) \in H^1(\Omega \setminus \Gamma(0))\,,$$

from which we derive

$$k^0\dot{\hat{S}}(0) - \nabla u^0 \cdot \Big(D\Phi^{-1}(0,\Phi(0))\dot{\Phi}(0)\Big) \in H^1(\Omega \setminus \Gamma(0))\,,$$

since $\dot{\hat{S}}(0) - \zeta(0)\nabla\Phi(0)^T\nabla S(\Phi(0)) \in H^1(\Omega \setminus \Gamma(0))$, by arguing as in the previous lemmas. Therefore, also condition (5.23) is satisfied. This implies the representation formula (5.24) in $[0,t_1]$, with $t_1 < \rho$. By combining (5.24) with Lemma 5.2.7, we deduce (5.37) in $[0,t_1]$. Indeed, we can write

$$u(t) = \hat{u}^R(t) + k(t)\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma(t),\, t \in [0,t_1]\,,$$

where $\hat{u}^R(t) := u^R(t) + k(t)[\zeta(t)S(\Phi(t)) - \hat{S}(t)] \in H^2(\Omega \setminus \Gamma(t))$.

We can repeat this construction starting from t_1 and we find a finite number of times $(t_i)_{i=0}^n$, with $0 =: t_0 < t_1 < \cdots < t_{n-1} < t_n := T$ such that the solution u to (5.2)-(5.3) with initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$ can be written for $i = 1,\dots,n$ as

$$u(t) = \hat{u}_i^R(t) + k_i(t)\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma(t),\, t \in [t_{i-1},t_i]\,.$$

Define $k\colon [0,T] \to \mathbb{R}$ and $\hat{u}^R\colon [0,T] \to H^2(\Omega \setminus \Gamma)$ as $k(t) := k_i(t)$ and $\hat{u}^R := \hat{u}_i^R$ in $[t_{i-1},t_i]$ for every $i = 1,\dots,n$, respectively. The functions k and $\hat{u}^R$ are well defined and do not depend on the particular choice of $(t_i)_{i=0}^n$. Indeed, if we have

$$u(t) = \hat{u}_1^R(t) + k_1(t)\hat{S}(t) = \hat{u}_2^R(t) + k_2(t)\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma(t)$$

for a time $t \in [0,T]$, then we derive

$$\hat{u}_1^R(t) - \hat{u}_2^R(t) = [k_2(t) - k_1(t)]\hat{S}(t) \quad \text{in } \Omega \setminus \Gamma(t)\,.$$

Since the left-hand side belongs to $H^2(\Omega \setminus \Gamma(t))$ while $\hat{S}(t)$ is in $H^1(\Omega \setminus \Gamma(t)) \setminus H^2(\Omega \setminus \Gamma(t))$, such identity can be true if and only if $k_1(t) = k_2(t)$ and $\hat{u}_1^R(t) = \hat{u}_2^R(t)$. Hence, we deduce that $k \in C^2([0,T])$ and that u satisfies the decomposition result (5.37) in $[0,T]$.

Finally, by combining the regularity in time of w, proved in Lemmas 5.2.8 and 5.2.9, with the definition of $\hat{u}^R$, we conclude that $\hat{u}^R$ satisfies (5.38) □

Remark 5.2.11. *When $A \neq I$ all the previous result are still true if we define*

$$\hat{S}(t,x) := Im\left(\sqrt{\frac{A^{-1}(r(t))\,(x-r(t))\cdot\gamma'(s(t))}{c_{A,\gamma'}(t)\sqrt{1-|c_{A,\gamma'}(t)|^2|\dot{s}(t)|^2}}+i\frac{(x-r(t))\cdot\nu(s(t))}{c_{A,n}(t)}}\right), \tag{5.39}$$

where $c_{A,\gamma'}(t) := |A^{-1/2}(r(t))\gamma'(s(t))|$, $c_{A,n}(t) := |A^{1/2}(r(t))\nu(s(t))|$, with $A^{1/2}$ and $A^{-1/2}$ the square root matrices of A and A^{-1}, respectively, and where $\hat{S}(t)$ is given by the unique continuous determination of the complex square function such that in $x = r(t)+\sqrt{1/|c_{A,\gamma'}(t)|^2-|\dot{s}(t)|^2}\gamma'(s(t))$ takes the value 1 and its discontinuity set lies on $\Gamma(t)$. Indeed, by exploiting the following identities in $[t_0,t_1]$

$$(\gamma^1)'(s^1(t)) = \frac{A^{-1/2}(r(t))\gamma'(s(t))}{|A^{-1/2}(r(t))\gamma'(s(t))|}, \quad \nu^1(s^1(t)) = \frac{A^{1/2}(r(t))\nu(s(t))}{|A^{1/2}(r(t))\nu(s(t))|}$$
$$\dot{s}^1(t) = |A^{-1/2}(r(t))\gamma'(s(t))|\dot{s}(t), \quad \nabla\chi(r(t)) = A^{-1/2}(r(t)),$$

where $(\gamma^1)'$ and ν^1 are, respectively, the tangent and the normal unit vectors to the curve Γ^1 in the point $\gamma^1(s^1(t))$, the function (5.39) can be rewritten as

$$Im\left(\sqrt{\frac{D\chi(r(t))\,(x-r(t))\cdot(\gamma^1)'(s^1(t))}{\sqrt{1-|\dot{s}^1(t)|^2}}+i\nabla\chi(r(t))\,(x-r(t))\cdot\nu^1(s^1(t))}\right).$$

In this case it is enough to set $\tilde{\Phi}(t,x) := L(t)R(t)\nabla\chi(r(t))(x-r(t))$, where L and R are constructed starting from γ^1 and s^1, and we can proceed again as in Lemmas (5.2.7), (5.2.8), and (5.2.9), thanks to the fact that for every $t \in [t_0,t_1]$ and $x \in B_\epsilon(r(t))$

$$|\Phi(t,x)-\tilde{\Phi}(t,x)| \le C|x-r(t)|^2, \quad |\nabla\Phi(t,x)-\nabla\tilde{\Phi}(t,x)| \le C|x-r(t)|,$$
$$|\dot{\Phi}(t,x)-\dot{\tilde{\Phi}}(t,x)| \le C|x-r(t)|.$$

We hence obtain the decomposition result (5.37) with singular part (5.39). As a byproduct, arguing as in Theorem 5.2.10, we derive that the values of k do not depend on the particular construction of Φ, but only on A, Γ, and s.

We point out that the condition $|\dot{s}(t)|^2 < 1/|c_{A,\gamma'}(t)|^2$, which we need in order to define $\hat{S}$, is implied by (5.5). Indeed

$$1 = \nabla\chi(r(t))A(r(t))\nabla\chi^T(r(t))\gamma'(s(t))\cdot\gamma'(s(t)) \ge c_0|A(r(t))^{-1/2}\gamma'(s(t))|^2 = c_0|c_{A,\gamma'}(t)|^2.$$

■

5.3 The energy-dissipation balance

In this section we derive formula (0.18) for the energy

$$\mathcal{E}(t) := \frac{1}{2}\|\dot{u}(t)\|^2_{L^2} + \frac{1}{2}\|A^{1/2}\nabla u(t)\|^2_{L^2},$$

associated to u, solution to (5.2)-(5.3) with initial conditions $u(0) = u^0$, $\dot{u}(0) = u^1$.

The computation is divided into three steps: first, in Proposition 5.3.5 we consider straight cracks when A is the identity matrix; then, in Theorem 5.3.7 we adapt the techniques to curved fractures; finally, in Remark 5.3.9 we generalise the former results to $A \neq I$. To this aim, some preliminaries are in order: first, in Remark 5.3.1 we compute the partial derivatives of u in a more convenient way, then in Lemmas 5.3.2 and 5.3.3 we provide two key results, based on Geometric Measure Theory. Once this is done, we deduce formula (0.16) in the time interval $[t_0, t_1]$ where the decomposition (5.24) holds.

For brevity of notation, in this section we consider $[t_0, t_1] = [0, 1]$. All the results can be easily extended to the general case. The global result in $[0, T]$ easily follows by iterating the procedure a finite number of steps, and using both the additivity of the integrals and the fact that k depends only on A, Γ, and s (see Theorem 5.2.10 and Remark 5.2.11).

Remark 5.3.1. *Let us focus our attention on a fracture which is straight in a neighborhood of the tip. Without loss of generality, we may fix the origin so that for every* $t \in [0, 1]$

$$\Gamma(t) \setminus \Gamma(0) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 \le s(t) - s(0),\ x_2 = 0\}\,.$$

The diffeomorphisms χ and Λ introduced in §5.1.2 can be both taken equal to the identity, so that, in a neighborhood of the origin, the diffeomorphisms $\Phi(t)$ defined in (5.18) simply read

$$\Phi(t, x) = \left(\frac{x_1 - (s(t) - s(0))}{\sqrt{1 - |\dot{s}(t)|}}, x_2 \right)\,.$$

Accordingly, the decomposition result in Theorem 5.2.4 states that the solution u to the wave equation (5.2)-(5.3) can be decomposed as

$$u(t, x) = u^R(t, x) + k(t)\zeta(t, x)\overline{S}(t, x)\,, \tag{5.40}$$

where, for brevity, we have set $\overline{S}(t, x) \coloneqq S(\Phi(t, x))$. We recall that $u^R \in C^2([0, 1]; L^2(\Omega))$, $\nabla u \in C^1([0, 1]; L^2(\Omega; \mathbb{R}^2))$, $\nabla^2 u \in C^0([0, 1]; L^2(\Omega; \mathbb{R}^{2\times 2}))$, $u^R(t) \in H^2(\Omega \setminus \Gamma(t))$ for every $t \in [0, 1]$, $k \in C^2([0, 1])$, $\zeta \in C^1([0, 1] \times \Omega)$, and $S(x) = \frac{x_2}{\sqrt{2}\sqrt{|x| + x_1}}$.

Let us now compute the partial derivatives of u. Since

$$\nabla S(x) = \frac{1}{2\sqrt{2}|x|}\left(\frac{-x_2}{\sqrt{|x| + x_1}}\,, \sqrt{|x| + x_1} \right),$$

$$\partial^2_{11} S(x) = \frac{2x_1x_2 + x_2|x|}{4\sqrt{2}|x|^3\sqrt{|x| + x_1}}\,, \quad \partial^2_{22} S(x) = -\frac{2x_1x_2 + x_2|x|}{4\sqrt{2}|x|^3\sqrt{|x| + x_1}}\,,$$

$$\partial^2_{12} S(x) = \partial^2_{21} S(x) = \frac{\sqrt{|x| + x_1}(|x| - 2x_1)}{4\sqrt{2}|x|^3}\,,$$

we get

$$\nabla u(t, x) = \nabla u^R(t, x) + k(t)\nabla\zeta(t, x)\overline{S}(t, x) + k(t)\zeta(t, x)\nabla\overline{S}(t, x)\,, \tag{5.41}$$

$$\dot{u}(t, x) = \dot{u}^R(t, x) + \dot{k}(t)\zeta(t, x)\overline{S}(t, x) + k(t)\dot{\zeta}(t, x)\overline{S}(t, x) + k(t)\zeta(t, x)\dot{\overline{S}}(t, x)\,. \tag{5.42}$$

We claim that

$$\dot{u}(t)\nabla u(t) - k^2(t)\zeta^2(t)\dot{\overline{S}}(t)\nabla\overline{S}(t) \in W^{1,1}(\Omega \setminus \Gamma(t); \mathbb{R}^2)\,,$$

for every $t \in [0, 1]$.

In fact $\nabla u^R(t, x)$, $\zeta(t, x)\overline{S}(t, x)$, $\dot{u}^R(t, x)$, $\zeta(t, x)\overline{S}(t, x)$, and $k(t)\dot{\zeta}(t, x)\overline{S}(t, x)$ are functions in $W^{1,2}(\Omega \setminus \Gamma(t))$ for every $t \in [0, 1]$; by the Sobolev embeddings theorem we deduce that each of the previous functions belongs to $L^p(\Omega)$ for every $p \ge 1$; using also the explicit form of $\overline{S}(t, x)$ and $\dot{\overline{S}}(t, x)$, one can also check that both of these functions are elements of $W^{1,4/3}(\Omega \setminus \Gamma(t))$. Having this in mind, we can easily conclude that the products of each term appearing in (5.41) with each term appearing in (5.42), except $k^2(t)\zeta^2(t)\dot{\overline{S}}(t)\nabla\overline{S}(t)$, are functions in $W^{1,1}(\Omega \setminus \Gamma(t); \mathbb{R}^2)$ for every $t \in [0, 1]$. ■

Lemma 5.3.2. *Let $a, b \in \mathbb{R}$ with $a < 0$ and $b > 0$ and define $H^+ := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 \geq 0\}$ to be the upper half plane in $\mathbb{R}^2$. Let $g\colon H^+ \to \mathbb{R}$ be bounded, continuous at the origin, and call ω a modulus of continuity for g at $x = 0$. Then*

$$\left|\frac{1}{\epsilon}\int_0^\epsilon \left(\int_a^b g(x_1,x_2)\frac{x_2}{x_1^2+x_2^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2 - \pi g(0,0)\right| \leq \|g\|_{L^\infty(H^+)}\big(2\varepsilon^{1/2}|b-a|+\theta(\epsilon)\big)+\pi\omega(\epsilon^{1/4})\,, \tag{5.43}$$

where

$$\theta(\epsilon) := \left|\pi - \int_0^1 \arctan\left(\frac{b}{\epsilon x_2}\right) - \arctan\left(\frac{a}{\epsilon x_2}\right)\mathrm{d}x_2\right|.$$

In particular, for every $g\colon H^+ \to \mathbb{R}$ bounded and continuous at the origin, we have

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_0^\epsilon\left(\int_a^b g(x_1,x_2)\frac{x_2}{x_1^2+x_2^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2 = \pi g(0,0)\,.$$

Proof. After a change of variable on the integral in (5.43), we can rewrite it as

$$\int_0^1\left(\int_a^b g(x_1,\epsilon x_2)\frac{\epsilon x_2}{x_1^2+(\epsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2\,.$$

Note that

$$\int_a^b \frac{\epsilon x_2}{x_1^2+(\epsilon x_2)^2}\,\mathrm{d}x_1 = \int_a^b \partial_1\arctan\left(\frac{x_1}{\epsilon x_2}\right)\mathrm{d}x_1 = \arctan\left(\frac{b}{\epsilon x_2}\right) - \arctan\left(\frac{a}{\epsilon x_2}\right),$$

therefore

$$\begin{aligned}
&\left|\int_0^1\left(\int_a^b g(x_1,\epsilon x_2)\frac{\epsilon x_2}{x_1^2+(\epsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2 - \pi g(0,0)\right| \\
&\leq \left|\int_0^1\left(\int_a^b [g(x_1,\epsilon x_2)-g(0,0)]\frac{\epsilon x_2}{x_1^2+(\epsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2\right| + g(0,0)\theta(\epsilon) \\
&\leq \left|\int_0^1\left(\int_{(a,b)\setminus(-\epsilon^{1/4},\epsilon^{1/4})} [g(x_1,\epsilon x_2)-g(0,0)]\frac{\epsilon x_2}{x_1^2+(\epsilon x_2)^2}\,\mathrm{d}x_1\right)\mathrm{d}x_2\right| + \pi\omega(\epsilon^{1/4}) + g(0,0)\theta(\epsilon)\,.
\end{aligned}$$

Using the estimate

$$\sup_{x\in[(a,b)\setminus(-\epsilon^{1/4},\epsilon^{1/4})]\times(0,1)}\frac{\epsilon x_2}{(x_1^2+(\epsilon x_2)^2)} \leq \frac{\epsilon^{1/2}}{1+\epsilon^{3/2}} \leq \epsilon^{1/2}\,,$$

valid for every $\epsilon \in (0,1)$, we can continue the above chain of inequalities with

$$\begin{aligned}
&\leq \epsilon^{1/2}\int_0^1\left(\int_{(a,b)\setminus(-\epsilon^{1/4},\epsilon^{1/4})}|g(x_1,\epsilon x_2)-g(0,0)|\,\mathrm{d}x_1\right)\mathrm{d}x_2 + \pi\omega(\epsilon^{1/4}) + g(0,0)\theta(\epsilon) \\
&\leq 2\epsilon^{1/2}\|g\|_{L^\infty(H^+)}|b-a| + \pi\omega(\epsilon^{1/4}) + g(0,0)\theta(\epsilon)\,,
\end{aligned}$$

which is (5.43), and the proof is concluded. □

Lemma 5.3.3. *Let $\Omega \subset \mathbb{R}^2$, let $\gamma\colon [0,\ell] \to \Omega$ be a Lipschitz curve, and set $\Gamma := \{\gamma(\sigma) \in \Omega \,:\, \sigma \in [0,\ell]\}$. For every $\epsilon > 0$ define $\varphi_\epsilon(x) := \frac{\mathrm{dist}(x,\Gamma)}{\epsilon} \wedge 1$. Then for each $u \in W^{1,1}(\Omega\setminus\Gamma)$ and for each $v\colon \Omega \to \mathbb{R}$ bounded and such that*

$$\lim_{x\to\overline{x}} v(x) = v(\overline{x}) \text{ for every } \overline{x} \in \Gamma\,,$$

we have

$$\lim_{\epsilon\to 0^+}\int_{\mathrm{dist}^+(x,\Gamma)<\epsilon} u(x)v(x)|\nabla\varphi_\epsilon(x)|\,\mathrm{d}x = \int_\Gamma u^+(y)v(y)\,\mathrm{d}\mathcal{H}^1(y)\,,$$

where u^+ *is the trace on* Γ *from above and*

$$\{\mathrm{dist}^+(x,\Gamma)<\epsilon\} := \bigcup_{\sigma\in[0,\ell]} B_\epsilon(\gamma(\sigma))\cap\{x\in\Omega \mid x\cdot(\gamma'(\sigma))^\perp>0\}\,.$$

Equivalently,

$$\lim_{\epsilon\to 0^+}\int_{\mathrm{dist}^-(x,\Gamma)<\epsilon} u(x)v(x)|\nabla\varphi_\epsilon(x)|\,\mathrm{d}x = \int_\Gamma u^-(y)v(y)\,\mathrm{d}\mathcal{H}^1(y)\,,$$

where u^- *is the trace on* Γ *from below and*

$$\{\mathrm{dist}^-(x,\Gamma)<\epsilon\} := \bigcup_{\sigma\in[0,\ell]} B_\epsilon(\gamma(\sigma))\cap\{x\in\Omega \mid x\cdot(\gamma'(\sigma))^\perp<0\}\,.$$

Proof. It is enough to apply the coarea formula to the Lipschitz maps φ_ϵ. □

Remark 5.3.4. *In what follows we compute the energy balance in the case of homogeneous Neumann conditions on the whole* $\partial\Omega$*. However, the same proof applies with no changes to the case of Dirichlet boundary conditions. For example, to treat the homogeneous Dirichlet condition on* $\partial_D\Omega\subseteq\partial\Omega$*, it is enough to check that the time derivative of the solution* $\dot u(t)$ *has still zero trace on* $\partial\Omega$*, in such a way that it still remains an admissible test function. But this is simply because the incremental quotient in time* $[u(t+h)-u(t)]/h$ *converges to* $\dot u(t)$ *as* $h\to 0$*, strongly in* H^1 *in a sufficiently small neighborhood of* $\partial_D\Omega$*, so that* $\dot u$ *has still zero trace on the Dirichlet part of the boundary.*

Analogously, if we prescribe a regular enough non-homogeneous Dirichlet boundary condition, we can rewrite the wave equation changing the forcing term f *appearing in its right-hand side, and turn the non-homogeneous Dirichlet condition into a homogeneous one. Also in this case, the computations follow unchanged.* ■

Proposition 5.3.5. *Let* $\Omega\subset\mathbb{R}^2$ *be a Lipschitz regular domain, and let* $\big(\Gamma(t)\big)_{t\in[0,1]}$ *be a family of rectilinear cracks inside* Ω*, of the form*

$$\Gamma(t)\setminus\Gamma(0) := \{(x_1,x_2)\in\mathbb{R}^2 \mid 0<x_1\le s(t)-s(0),\ x_2=0\}\,,$$

where $s\in C^2([0,1])$ *and* $\dot s(t)\ge 0$ *for every* $t\in[0,1]$*.*

Suppose that a function $u\colon[0,1]\times\Omega\to\mathbb{R}$ *can be decomposed as in (5.40) and satisfies the wave equation with homogeneous Neumann boundary conditions on the boundary and on the cracks:*

$$\ddot u(t)-\Delta u(t) = f(t) \text{ in } \Omega\setminus\Gamma(t)\,, \tag{5.44}$$

for a.e. $t\in[0,1]$*, with initial conditions* $u(0)=u^0$ *and* $\dot u(0)=u^1$*. Then for every* $t\in[0,1]$*,* u *satisfies the energy balance*

$$\mathcal{E}(t)-\mathcal{E}(0)+\mathcal{H}^1(\Gamma(t)\setminus\Gamma(0)) = \int_0^t\langle f(\tau),\dot u(\tau)\rangle_{L^2}\,\mathrm{d}\tau \tag{5.45}$$

if and only if the stress intensity factor k *is constantly equal to* $\frac{2}{\sqrt{\pi}}$ *in the set* $\{\dot s>0\}$*.*

Proof. By hypothesis the function u can be decomposed as $u(t,x) = u^R(t,x)+k(t)\zeta(t,x)\overline{S}(t,x)$, where $u^R(t) \in H^2(\Omega \setminus \Gamma(t))$, $\zeta(t)$ is a cut-off function supported in a neighborhood of the moving tip of $\Gamma(t)$, and

$$\overline{S}(t,x) = S\left(\frac{x_1 - (s(t)-s(0))}{\sqrt{1-|\dot{s}(t)|^2}}, x_2\right),$$

where $S(x_1,x_2) = \frac{x_2}{\sqrt{2}\sqrt{|x|+x_1}}$.

Fix $\bar{t} \in [0,1]$. For every $\epsilon > 0$ define $\varphi_\epsilon(x) = \frac{\mathrm{dist}(x,\Gamma(\bar{t})\setminus\Gamma(0))}{\epsilon} \wedge 1$. Since $\varphi_\epsilon \dot{u}(t) \in H^1(\Omega \setminus \Gamma(t))$, we can use it as test function in (5.44), and we get

$$\begin{aligned}\int_0^{\bar{t}} \langle \ddot{u}(t), \varphi_\epsilon \dot{u}(t)\rangle_{H^{-1}(\Omega\setminus\Gamma(t))}\,\mathrm{d}t + \int_0^{\bar{t}} \langle \nabla u(t), \nabla \dot{u}(t)\varphi_\epsilon\rangle_{L^2}\,\mathrm{d}t \\ + \int_0^{\bar{t}} \langle \nabla u(t), \nabla\varphi_\epsilon \dot{u}(t)\rangle_{L^2}\,\mathrm{d}t = \int_0^{\bar{t}} \langle f(t), \dot{u}(t)\varphi_\epsilon\rangle_{L^2}\,\mathrm{d}t\,.\end{aligned} \tag{5.46}$$

Using integration by parts with the fact that $t \mapsto \|\dot{u}(t)\|^2_{L^2(\Omega,\varphi_\epsilon\mathcal{L}^2)}$ is absolutely continuous, we obtain

$$\begin{aligned}\int_0^{\bar{t}} \langle \ddot{u}(t), \varphi_\epsilon \dot{u}(t)\rangle_{H^{-1}(\Omega\setminus\Gamma(t))}\,\mathrm{d}t &= \frac{1}{2}\int_0^{\bar{t}} \frac{\mathrm{d}}{\mathrm{d}t}\|\dot{u}(t)\|^2_{L^2(\Omega,\varphi_\epsilon\mathcal{L}^2)}\,\mathrm{d}t \\ &= \frac{1}{2}\|\dot{u}(\bar{t})\|^2_{L^2(\Omega,\varphi_\epsilon\mathcal{L}^2)} - \frac{1}{2}\|\dot{u}(0)\|^2_{L^2(\Omega,\varphi_\epsilon\mathcal{L}^2)}\,,\end{aligned}$$

and passing to the limit as $\epsilon \to 0^+$, by dominated convergence Theorem, we have

$$\lim_{\epsilon\to 0^+} \int_0^{\bar{t}} \langle \ddot{u}(t), \varphi_\epsilon \dot{u}(t)\rangle_{H^{-1}(\Omega\setminus\Gamma(t))}\,\mathrm{d}t = \frac{1}{2}\|\dot{u}(\bar{t})\|^2_{L^2} - \frac{1}{2}\|\dot{u}(0)\|^2_{L^2}\,.$$

Analogously, taking the limit as $\epsilon \to 0$ in the second term in the left-hand side and in the right-hand side of (5.46), we have, respectively,

$$\begin{aligned}&\lim_{\epsilon\to 0} \int_0^{\bar{t}} \langle \nabla u(t), \nabla\dot{u}(t)\varphi_\epsilon\rangle_{L^2}\,\mathrm{d}t = \frac{1}{2}\|\nabla u(\bar{t})\|^2_{L^2} - \frac{1}{2}\|\nabla u(0)\|^2_{L^2}\,, \\ &\lim_{\epsilon\to 0} \int_0^{\bar{t}} \langle f(t), \dot{u}(t)\varphi_\epsilon\rangle_{L^2}\,\mathrm{d}t = \int_0^{\bar{t}} \langle f(t), \dot{u}(t)\rangle_{L^2}\,\mathrm{d}t\,.\end{aligned}$$

The most delicate term is the third one in the left-hand side of (5.46). First of all, we write the partial derivatives explicitly:

$$\begin{aligned}&\nabla[k(t)\zeta(t,x)\overline{S}(t,x)] = k(t)\nabla\zeta(t,x)\overline{S}(t,x) + k(t)\zeta(t,x)\nabla\overline{S}(t,x)\,, \\ &\frac{\mathrm{d}}{\mathrm{d}t}[k(t)\zeta(t,x)\overline{S}(t,x)] = \dot{k}(t)\zeta(t,x)\overline{S}(t,x) + k(t)\dot{\zeta}(t,x)\overline{S}(t,x) + k(t)\zeta(t,x)\dot{\overline{S}}(t,x)\,.\end{aligned}$$

Moreover, if we set $\Phi_1(t,x) = \frac{x_1 - s(t)}{\sqrt{1-|\dot{s}(t)|^2}}$, we have

$$\nabla\overline{S}(t,x) = \left(\frac{1}{\sqrt{1-|\dot{s}(t)|^2}}\partial_1 S(\Phi_1(t,x),x_2), \partial_2 S(\Phi_1(t,x),x_2)\right)$$

and

$$\begin{aligned}\dot{\overline{S}}(t,x) &= \left[\frac{-\dot{s}(t)(1-|\dot{s}(t)|^2) + \dot{s}(t)\ddot{s}(t)(x_1-(s(t)-s(0)))}{(1-|\dot{s}(t)|^2)^{3/2}}\right]\partial_1 S(\Phi_1(t,x),x_2) \\ &= \dot{\Phi}_1(t,x)\sqrt{1-|\dot{s}(t)|^2}\,\partial_1\bar{S}(t,x)\,.\end{aligned}$$

Thanks to Remark 5.3.1, we know that the only contribution to the limit as $\epsilon \to 0$ is given by the following term:

$$\int_0^{\bar{t}} k^2(t)\langle \zeta^2(t,x)\nabla\overline{S}(t,x), \nabla\varphi_\epsilon(x)\dot{\overline{S}}(t,x)\rangle_{L^2(\Omega;\mathbb{R}^2)}\,\mathrm{d}t\,.$$

Therefore, we need to compute

$$\lim_{\epsilon\to 0^+}\int_0^{\bar{t}}\left(\int_{\{\mathrm{dist}(x,\Gamma(t)\setminus\Gamma(0))<\epsilon\}} k^2(t)\zeta^2(t,x)\nabla\overline{S}(t,x)\cdot\nabla\varphi_\epsilon(x)\dot{\overline{S}}(t,x)\,\mathrm{d}x\right)\mathrm{d}t\,. \tag{5.47}$$

To this aim, we set $I_\epsilon(t) := \int_{\{\mathrm{dist}(x,\Gamma(t)\setminus\Gamma(0))<\epsilon\}} k^2(t)\zeta^2(t,x)\nabla\overline{S}(t,x)\cdot\nabla\varphi_\epsilon(x)\dot{\overline{S}}(t,x)\,\mathrm{d}x$ and we decompose I_ϵ as $I_\epsilon^+ + I_\epsilon^-$, where I_ϵ^+ is the integral I_ϵ restricted to the upper half plane $\{x_2>0\}$ and I_ϵ^- is the integral I_ϵ restricted to the lower half plane $\{x_2<0\}$.

Let us focus on $I_\epsilon^+(t)$.

For brevity, we write $r(t) := (s(t)-s(0),0)$ for every $t\in[0,T]$. Then the gradient of φ_ϵ reads

$$\nabla\varphi_\epsilon = \begin{cases} \frac{e_2}{\epsilon} & \text{in } \{x\in\mathbb{R}^2 \mid 0\le x_1\le s(\bar{t})-s(0),\ 0\le x_2<\epsilon\} \\ \frac{x}{\epsilon|x|} & \text{in } \{x\in\mathbb{R}^2 \mid x\in B_\epsilon(0),\ x_1<0,\ x_2\ge 0\} \\ \frac{x-r(\bar{t})}{\epsilon|x-r(\bar{t})|} & \text{in } \{x\in\mathbb{R}^2 \mid x\in B_\epsilon(r(\bar{t})),\ x_1>s(\bar{t})-s(0),\ x_2\ge 0\} \\ 0 & \text{otherwise on } \{x_2\ge 0\}\,. \end{cases}$$

Thus we get

$$\begin{aligned} I_\epsilon^+(t) = &\ \frac{1}{\epsilon}\int_{[0,s(\bar{t})-s(0)]\times(0,\epsilon]} k^2(t)\sqrt{1-|\dot{s}(t)|^2}\zeta^2(t,x)\partial_2\overline{S}(t,x)\dot{\Phi}_1(t,x)\partial_1\overline{S}(t,x)\,\mathrm{d}x \\ &+\frac{1}{\epsilon}\int_{B_\epsilon(0)\cap\{x_1<0\}\times\{x_2\ge0\}} k^2(t)\sqrt{1-|\dot{s}(t)|^2}\zeta^2(t,x)\left(\nabla\overline{S}(t,x)\cdot\frac{x}{|x|}\right)\dot{\Phi}_1(t,x)\partial_1\overline{S}(t,x)\,\mathrm{d}x \\ &+\frac{1}{\epsilon}\int_{B_\epsilon(r(\bar{t}))\cap\{x_1>s(\bar{t})-s(0)\}\times\{x_2\ge0\}} k^2(t)\sqrt{1-|\dot{s}(t)|^2}\zeta^2(t,x)\left(\nabla\overline{S}(t,x)\cdot\frac{x-r(\bar{t})}{|x-r(\bar{t})|}\right)\dot{\Phi}_1(t,x)\partial_1\overline{S}(t,x)\,\mathrm{d}x\,. \end{aligned} \tag{5.48}$$

We notice that the last two terms in (5.48) have integrands which are bounded on the domains of integration, and so passing to the limit as ϵ goes to 0 they do not give any contribution. Thus we only have to analyze the first term of (5.48). Recalling that $\zeta(x,t) = \zeta\big(\Phi_1(t,x),x_2\big)$, $\overline{S}(t,x) = S\big(\Phi_1(t,x),x_2\big)$, $\Phi_1(t,x) = \frac{x_1-(s(t)-s(0))}{\sqrt{1-|\dot{s}(t)|^2}}$, and making the change of variable $x_1'\sqrt{1-|\dot{s}(t)|^2} = x_1-(s(t)-s(0))$, we rewrite the first term of (5.48) as

$$\begin{aligned} &-\frac{k^2(t)\dot{s}(t)}{\epsilon}\left(\int_0^\epsilon\int_{a_t}^{b_t}\zeta^2(x_1,x_2)\partial_1S(x_1,x_2)\partial_2S(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right) \\ &+\frac{k^2(t)\dot{s}(t)\ddot{s}(t)}{\epsilon\sqrt{1-|\dot{s}(t)|^2}}\left(\int_0^\epsilon\int_{a_t}^{b_t}x_1\zeta^2(x_1,x_2)\partial_1S(x_1,x_2)\partial_2S(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right), \end{aligned} \tag{5.49}$$

where the interval (a_t,b_t) denotes the segment $\frac{\{0<x_1<s(\bar{t})-s(0)\}-(s(t)-s(0))}{\sqrt{1-|\dot{s}(t)|^2}}$.

Notice that

$$\begin{aligned} &-\frac{k^2(t)\dot{s}(t)}{\epsilon}\left(\int_0^\epsilon\int_{a_t}^{b_t}\zeta^2(x_1,x_2)\partial_1S(x_1,x_2)\partial_2S(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\right) \\ &\qquad=\frac{k^2(t)\dot{s}(t)}{\epsilon}\left(\int_0^\epsilon\int_{a_t}^{b_t}\zeta^2(x_1,x_2)\frac{x_2}{8|x|^2}\,\mathrm{d}x_1\mathrm{d}x_2\right) \end{aligned}$$

and that the function $(x_1, x_2) \mapsto \zeta^2(x_1, x_2)$ is bounded and continuous in $(0, 0)$, therefore we are in a position to apply Lemma 5.3.2, which gives, in the limit as $\epsilon \to 0^+$,

$$\lim_{\epsilon \to 0^+} \frac{k^2(t)\dot{s}(t)}{\epsilon} \left(\int_0^\epsilon \int_{a_t}^{b_t} \zeta^2(x_1, x_2) \frac{x_2}{8|x|^2} \, \mathrm{d}x_1 \mathrm{d}x_2 \right) = \frac{\pi}{8} k^2(t)\dot{s}(t)\zeta^2(0, 0) .$$

Arguing in the very same way, we can show that the limit as $\epsilon \to 0^+$ of the second term of (5.49), thanks to the presence of x_1, is zero. This means that the limit of $I_\epsilon^+(t)$ is

$$\lim_{\epsilon \to 0^+} I_\epsilon^+(t) = \frac{\pi}{8} \dot{s}(t) k^2(t) ,$$

and, similarly,

$$\lim_{\epsilon \to 0^+} I_\epsilon^-(t) = \frac{\pi}{8} \dot{s}(t) k^2(t) .$$

All in all,

$$\lim_{\epsilon \to 0^+} I_\epsilon(t) = \lim_{\epsilon \to 0^+} [I_\epsilon^+(t) + I_\epsilon^-(t)] = \frac{\pi}{4} k^2(t)\dot{s}(t) .$$

Thanks to the estimate in (5.43), we infer that the family of functions $(I_\epsilon^+(t))_{\epsilon>0}$ are dominated on $[0, 1]$ by a bounded function, and the same holds for $(I_\epsilon^-(t))_{\epsilon>0}$; by the dominated convergence Theorem, we can pass the limit in (5.47) inside the integral in time, and we can write

$$\lim_{\epsilon \to 0^+} \int_0^{\bar{t}} I_\epsilon(t) \, \mathrm{d}t = \int_0^{\bar{t}} \frac{\pi}{4} k^2(t)\dot{s}(t) \, \mathrm{d}t .$$

So we deduce that the energy balance in (5.45) holds for every $\bar{t} \in [0, 1]$ if and only if the stress intensity factor $k(t)$ is equal to $\frac{2}{\sqrt{\pi}}$ whenever $\dot{s}(t) > 0$. □

Remark 5.3.6. *We underline that our approach is different to that of Dal Maso, Larsen, and Toader [23, §4]: in order to derive the energy balance associated to a horizontal crack opening with constant velocity c, they prove that the kinetic+elastic energy of $u(t)$ is constant in the moving ellipse $E_r(t) = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; (x_1 - ct)^2/(1 - c^2) + x_2^2 \leq r^2\}$ centered at the crack tip $(ct, 0)$, for some small $r > 0$, and they make the explicit computation of the energy in $\mathbb{R}^2 \setminus E_r(t)$.* ■

We now generalise the previous result to non straight cracks.

Theorem 5.3.7. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz regular domain, and let $\big(\Gamma(t)\big)_{t\in[0,1]}$ be a family of growing cracks inside Ω. Assume that there exists a bi-Lipschitz map $\Lambda \colon \Omega \to \Omega$ with the following properties:*

1. $\Lambda(\Gamma(t) \setminus \Gamma(0)) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 \leq s(t) - s(0), \, x_2 = 0\}$, *where* $s \in C^2([0, 1])$ *and* $\dot{s}(t) \geq 0$ *for every* $t \in [0, 1]$,
2. $\mathcal{H}^1\big(\Lambda(\Gamma(t) \setminus \Gamma(0))\big) = \mathcal{H}^1\big(\Gamma(t) \setminus \Gamma(0)\big)$ *for every* $t \in [0, 1]$,
3. $\lim_{x\to\bar{x}} \nabla\Lambda(x) = \nabla\Lambda(\bar{x}) \in SO(2)^+$, *for every* $\bar{x} \in \overline{\Gamma(1) \setminus \Gamma(0)}$.

Suppose that a function $u \colon [0, 1] \times \Omega \to \mathbb{R}$ can be decomposed as in (5.40) and satisfies the wave equation with homogeneous Neumann boundary conditions on the boundary and on the cracks:

$$\ddot{u}(t) - \Delta u(t) = f(t) \text{ in } \Omega \setminus \Gamma(t) , \tag{5.50}$$

for a.e. $t \in [0,1]$, *with initial conditions* $u(0) = u^0$ *and* $\dot u(0) = u^1$. *Then for every* $t \in [0,1]$, u *satisfies the following energy balance*

$$\mathcal{E}(t) - \mathcal{E}(0) + \mathcal{H}^1(\Gamma(t) \setminus \Gamma(0)) = \int_0^t \langle f(\tau), \dot u(\tau)\rangle_{L^2}\, \mathrm{d}\tau \tag{5.51}$$

if and only if the stress intensity factor k *is constantly equal to* $\frac{2}{\sqrt{\pi}}$ *in the set* $\{\dot s > 0\}$.

Proof. In view of (5.40), we have $u(t,x) = u^R(t,x) + k(t)\zeta(t,\Lambda(x))\overline{S}(t,\Lambda(x))$, with $u^R(t) \in H^2(\Omega \setminus \Gamma(t))$, $\zeta(t,\Lambda(\cdot))$ a cut-off function supported in a neighborhood of the moving tip of $\Gamma(t)$, and

$$\overline{S}(t,\Lambda(x)) = S\bigg(\frac{\Lambda_1(x) - (s(t) - s(0))}{\sqrt{1 - |\dot s(t)|^2}}, \Lambda_2(x)\bigg),$$

where $S(x_1,x_2) = \frac{x_2}{\sqrt{2}\sqrt{|x| + x_1}}$.

As in the proof of Proposition 5.3.5, we fix $\bar t \in [0,1]$ and, for every $\epsilon > 0$, we define $\varphi_\epsilon(x) = \frac{\mathrm{dist}(x, \Gamma(\bar t)\setminus\Gamma(0))}{\epsilon} \wedge 1$. Since $\varphi_\epsilon \dot u(t) \in H^1(\Omega \setminus \Gamma(t))$, we can use it as test function in (5.50), and we get

$$\begin{aligned}\int_0^{\bar t} \langle \ddot u(t), \varphi_\epsilon \dot u(t)\rangle_{H^{-1}(\Omega\setminus\Gamma(t))}\, \mathrm{d}t + \int_0^{\bar t} \langle \nabla u(t), \nabla \dot u(t)\varphi_\epsilon\rangle_{L^2}\, \mathrm{d}t& \\ + \int_0^{\bar t} \langle \nabla u(t), \nabla\varphi_\epsilon \dot u(t)\rangle_{L^2}\, \mathrm{d}t = \int_0^{\bar t} \langle f(t), \dot u(t)\varphi_\epsilon\rangle_{L^2}\, \mathrm{d}t\,.&\end{aligned} \tag{5.52}$$

Integrating by parts, we easily obtain

$$\lim_{\epsilon\to 0^+} \int_0^{\bar t} \langle \ddot u(t), \varphi_\epsilon \dot u(t)\rangle_{H^{-1}(\Omega\setminus\Gamma(t))}\, \mathrm{d}t = \frac12 \|\dot u(\bar t)\|^2_{L^2} - \frac12 \|\dot u(0)\|^2_{L^2}\,, \tag{5.53}$$

$$\lim_{\epsilon\to 0^+} \int_0^{\bar t} \langle \nabla u(t), \nabla \dot u(t)\varphi_\epsilon\rangle_{L^2}\, \mathrm{d}t = \frac12 \|\nabla u(\bar t)\|^2_{L^2} - \frac12 \|\nabla u(0)\|^2_{L^2}\,, \tag{5.54}$$

$$\lim_{\epsilon\to 0^+} \int_0^{\bar t} \langle f(t), \dot u(t)\varphi_\epsilon\rangle_{L^2}\, \mathrm{d}t = \int_0^{\bar t} \langle f(t), \dot u(t)\rangle_{L^2}\, \mathrm{d}t\,. \tag{5.55}$$

The asymptotics as $\epsilon \to 0$ of the third term in the left-hand side of (5.52) is more delicate to handle. To simplify the notation, we set

$$\overline{\zeta}(t,x) := \zeta(t,\Lambda(x)) \text{ and } \overline{\varphi}_\epsilon(x) := \varphi_\epsilon(\Lambda^{-1}(x))\,.$$

Using Lemma 5.3.3 and Remark 5.3.1, as in the proof of the previous proposition in the rectilinear case, we have that the only contribution to the limit as $\epsilon \to 0$ is given by the term

$$\begin{aligned}&\int_\Omega k^2(t)\overline{\zeta}^2(t,x)\big[\nabla\overline{S}(t,\Lambda(x)) \cdot \nabla\varphi_\epsilon(x)\big]\dot{\overline{S}}(t,\Lambda(x))\, \mathrm{d}x \\ &= \int_\Omega k^2(t)\alpha(t)\big[\nabla\Lambda^T(x)\nabla\overline{S}(t,\Lambda(x)) \cdot \nabla\varphi_\epsilon(x)\big]\overline{\zeta}^2(t,x)\dot\Phi_1(t,\Lambda(x))\partial_1\overline{S}(t,\Lambda(x))\, \mathrm{d}x \\ &= \int_\Omega k^2(t)\alpha(t)\big[\nabla\Lambda^T(\Lambda^{-1}(x))\nabla\overline{S}(t,x) \cdot \nabla\varphi_\epsilon(\Lambda^{-1}(x))\big]\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\overline{S}(t,x)\,|J\Lambda^{-1}(x)|\, \mathrm{d}x \\ &= \int_\Omega k^2(t)\alpha(t)\big[\nabla\overline{S}(t,x) \cdot B(\Lambda^{-1}(x))\nabla\overline{\varphi}_\epsilon(x)\big]\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\overline{S}(t,x)\,|J\Lambda^{-1}(x)|\, \mathrm{d}x\,,\end{aligned} \tag{5.56}$$

where $\Phi_1(t,x) := \frac{x_1-(s(t)-s(0))}{\sqrt{1-|\dot s(t)|^2}}$, $B(x) := \nabla\Lambda(x)\nabla\Lambda^T(x)$, and $\alpha(t) := \sqrt{1-|\dot s(t)|^2}$. In the last equality we used the coarea formula applied with the Lipschitz change of variables Λ^{-1}.

Thanks to our construction of Λ, for any x belonging to a suitable small neighborhood of $\{\Lambda(\Gamma(1))\}$ we have

$$B(\Lambda^{-1}(x)) = \begin{pmatrix} b_{11}(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where $b_{11}\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that $b_{11}(x_1,0)=1$. The last term in (5.56) can be split as

$$\int_\Omega k^2(t)\alpha(t)b_{11}(x)\partial_1\overline{\varphi}_\epsilon(x)\zeta^2(t,x)\dot\Phi_1(t,x)[\partial_1\overline{S}(t,x)]^2\,|J\Lambda^{-1}(x)|\,\mathrm{d}x \\ + \int_\Omega k^2(t)\alpha(t)\partial_2\overline{S}(t,x)\partial_2\overline{\varphi}_\epsilon(x)\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\overline{S}(t,x)\,|J\Lambda^{-1}(x)|\,\mathrm{d}x\,.$$

By construction of Λ, each line parallel to $\{x_2=0\}$ is mapped by Λ^{-1} into a level set of φ_ϵ; more precisely $\varphi_\epsilon(\Lambda^{-1}(\{x_2=s\})) = \frac{s}{\epsilon}\wedge 1$, and this means that on the set of points $\{\mathrm{dist}(x,\Lambda(\Gamma(1)))\le\epsilon\}$, we have

$$\nabla\overline{\varphi}_\epsilon(x) = \begin{cases} \frac{e_2}{\epsilon} & \text{in } \{x\in\mathbb{R}^2 \mid 0\le x_1\le s(\bar t)-s(0),\ 0\le x_2<\epsilon\} \\ \frac{x}{\epsilon|x|} & \text{in } \{x\in\mathbb{R}^2 \mid x\in B_\epsilon(0),\ x_1<0,\ x_2\ge 0\} \\ \frac{x-r(\bar t)}{\epsilon|x-r(\bar t)|} & \text{in } \{x\in\mathbb{R}^2 \mid x\in B_\epsilon(r(\bar t)),\ x_1>s(\bar t)-s(0),\ x_2\ge 0\} \\ 0 & \text{otherwise on } \{x_2\ge 0\}\,, \end{cases}$$

where, for brevity, we have set $r(t) := (s(t)-s(0),0)$ for every $t\in[0,1]$.

Since Λ is a bi-Lipschitz map, $|J\Lambda^{-1}|$ is bounded, thus by hypothesis (3) we have

$$\lim_{x\to(s(t)-s(0),0)} |J\Lambda^{-1}(x)| = 1\,,$$

for every $t\in[0,1]$.

Moreover, in view of assumption (3), we have that $|J\Lambda^{-1}|$ is continuous on the compact set $\overline{\Gamma(1)\setminus\Gamma(0)}$, hence uniformly continuous; therefore, proceeding exactly as in the proof of Proposition 5.3.5, we can write

$$\lim_{\epsilon\to0^+}\int_\Omega k^2(t)\alpha(t)\partial_2\overline{S}(t,x)\partial_2\overline{\varphi}_\epsilon(x)\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\overline{S}(t,x)\,|J\Lambda^{-1}(x)|\,\mathrm{d}x = \frac{\pi}{4}k^2(t)\dot s(t)\,. \tag{5.57}$$

Again by hypothesis (3), we can apply estimate (5.43) and deduce that the sequence of integrands in (5.57) is dominated in t, so that we can apply the Dominated Convergence Theorem to deduce

$$\lim_{\epsilon\to0^+}\int_0^{\bar t}\left(\int_\Omega k^2(t)\alpha(t)\partial_2\overline{S}(t,x)\partial_2\overline{\varphi}_\epsilon(x)\zeta^2(t,x)\dot\Phi_1(t,x)\partial_1\overline{S}(t,x)\,|J\Lambda^{-1}(x)|\,\mathrm{d}x\right)\mathrm{d}t \\ = \int_0^{\bar t}\frac{\pi}{4}k^2(t)\dot s(t)\,\mathrm{d}t\,. \tag{5.58}$$

By combining (5.52) with (5.53)-(5.55) and 5.58, we infer that

$$\mathcal{E}(\bar t)-\mathcal{E}(0)+\frac{\pi}{4}\int_0^{\bar t}k^2(t)\dot s(t)\,\mathrm{d}t = \int_0^{\bar t}\langle f(t),\dot u(t)\rangle_{L^2}\,\mathrm{d}t\,. \tag{5.59}$$

Hence, the energy-dissipation balance (5.51) is satisfied if and only if

$$\int_0^{\bar{t}} \frac{\pi}{4} k^2(t)\dot{s}(t)\,\mathrm{d}t = s(\bar{t}) = \mathcal{H}^1\big(\Lambda(\Gamma(\bar{t}) \setminus \Gamma(0))\big) = \mathcal{H}^1(\Gamma(\bar{t}) \setminus \Gamma(0)) \text{ for every } \bar{t} \in [0,1]\,,$$

which is true if and only if $k(t)$ is equal to $\frac{2}{\sqrt{\pi}}$ whenever $\dot{s}(t) > 0$. This concludes the proof. □

Remark 5.3.8. *Our approach is constructive and allows us to show the existence of pairs* $(\Gamma(t), u(t))$ *satisfying the energy-dissipation balance* (5.51)*. Under the standing assumptions on* $\Gamma(t)$*, it is enough to take* f *associated to* $2/\sqrt{\pi}\xi(\Phi(t,x))S(\Phi(t,x))$ *(which of course is* $u(t)$*), where* ξ *is a suitable cut-off function supported in a small neighborhood of the origin. In order to ensure the homogeneous Neumann condition on the fracture, we choose* ξ *satisfying* $\partial_2\xi(y_1, 0) = 0$ *for every* $y_1 \in \mathbb{R}$*. This can be achieved, e.g., by taking* $\xi(y_1, y_2) = \varphi(y_1)\varphi(y_2)$*, where* $\varphi \in C_c^\infty(\mathbb{R})$ *has compact support contained in* $(-\varepsilon, \varepsilon)$ *and satisfies* $\varphi \equiv 1$ *in* $(-\varepsilon/2, \varepsilon/2)$*, for some* $\varepsilon > 0$*.* ■

Remark 5.3.9. *When in equation* (0.14) *the matrix* A *is (possibly) not the identity, an energy balance similar to* (5.59) *is still valid: for every* $t \in [0,1]$*, there holds*

$$\mathcal{E}(t) - \mathcal{E}(0) + \frac{\pi}{4}\int_0^t k^2(\tau)a(\tau)\dot{s}(\tau)\,\mathrm{d}\tau = \int_0^t \langle f(\tau), \dot{u}(\tau)\rangle_{L^2}\,\mathrm{d}\tau\,, \tag{5.60}$$

where a *is a function depending only on* A*,* Γ*, and* s*, and it is given by*

$$a(t) := |A^{-1/2}(r(t))\gamma'(s(t))| \cdot |A^{1/2}(r(t))\nu(s(t))| \cdot \sqrt{\det A(r(t))}\,.$$

Here $A^{1/2}$ *and* $A^{-1/2}$ *denote the square root of the symmetric and positive definite matrices* A *and* A^{-1}*, respectively, and* $\gamma'(s(t))$ *and* $\nu(s(t))$ *are the tangent and normal unit vectors to* Γ *at the point* $r(t) := \gamma(s(t))$*, respectively. In this case, the energy-dissipation balance* (0.16) *holds true if and only if the stress intensity factor* $k(t)$ *satisfies*

$$k(t) = \frac{2}{\sqrt{\pi a(t)}}$$

during the crack opening, namely when $\dot{s}(t) > 0$*.*

In order to derive formula (5.60)*, we use the decomposition result* (5.24) *rewritten as*

$$u(t,x) = u^R(t,x) + k(t)\zeta(t,x)\overline{S}(t,\chi(x))\,,$$

where $\overline{S}(t,x)$ *is the singular part of the solution relative to the transformed curve* $\Gamma^1 = \chi(\Gamma)$*. Then we proceed as in the previous theorem and proposition: we test the PDE with* $\dot{u}(t)\varphi_\epsilon$ *(where* $\varphi_\epsilon(x) = \frac{dist(x,\Gamma(\bar{t})\setminus\Gamma(0))}{\epsilon} \wedge 1$*), and as before, we note that the only delicate term is the one that converges to the integral in the left hand-side of* (5.60)*:*

$$\lim_{\epsilon\to 0^+}\int_0^{\bar{t}} k^2(t)\bigg(\int_\Omega \zeta^2(t,x)[A(x)\nabla\overline{S}(t,\chi(x)) \cdot \nabla\varphi_\epsilon(x)]\,\dot{\overline{S}}(t,\chi(x))\,\mathrm{d}x\bigg)\,\mathrm{d}t\,.$$

By applying the change of variables χ^{-1}*, we can rewrite the space integral in the previous expression as follows:*

$$\int_\Omega \zeta^2(t,x)[\nabla\chi A\nabla\chi^T](\chi^{-1}(x))\nabla\overline{S}(t,x) \cdot \nabla\chi^{-T}(x)\nabla\varphi_\epsilon(\chi^{-1}(x))\,\dot{\overline{S}}(t,x)|J\chi^{-1}(x)|\,\mathrm{d}x\,.$$

Finally, we work on the transformed curve Γ^1*, exactly as in the previous theorem, using the property of the singular part* $\overline{S}(t,x)$ *together with the following facts: by construction,* $[\nabla\chi A\nabla\chi^T](\chi^{-1}(x))$ *is a continuous function which agrees with the identity on the points of* Γ^1*;* $\nabla\chi^{-T}(x)\nabla\varphi_\epsilon(\chi^{-1}(x))$ *is a continuous function equal to* $\frac{1}{\epsilon}|A^{1/2}(r(t))\nu(s(t))|\,\nu^1(s^1(t))$ *on the points of* Γ^1*, where* $\nu^1(s^1(t))$ *denotes the normal unit vector to* Γ^1 *at the point* $\gamma^1(s^1(t))$*; the velocity* $\dot{s}^1$ *of the curve* Γ^1 *satisfies* $\dot{s}^1(t) = |A^{-1/2}(r(t))\gamma'(s(t))|\dot{s}(t)$*; finally,* $|J\chi^{-1}(x)|$ *is a continuous function equal to* $\sqrt{\det A(r(t))}$ *on the points of* Γ^1*.* ■

Remark 5.3.10. *By combining the representation results 5.2.4 and 5.2.10 with Theorem 5.3.7 and Remarks 5.3.4, 5.3.9, we deduce that whenever* u^0*,* u^1 *satisfy the conditions of Theorem 5.2.10, then the unique solution* u *to* (5.2)-(5.3) *satisfies* (5.60)*. This formula gives an important quantitative information on* k *and* s *which satisfy the energy dissipation balance* (0.16)*: for every* $t \in [0,T]$

$$\left[\frac{2}{\sqrt{\pi a(t)}} - k(t)\right]\dot{s}(t) = 0\,.$$

In particular, in the set $\{t \mid \dot{s}(t) > 0\} \subset [0,T]$ *the stress intensity factor* k *coincides with the function* $2/\sqrt{\pi a}$*.* ■

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