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Regularity results for minimizers of a charged liquid drop model

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" L' Universo è più grande delle nostre idee. "

Introduction

This thesis is devoted to the study of qualitative and quantitative properties of minimizers of variational models describing the shape of charged liquid droplets. Charged droplets are supposed to minimize free energies which are composed by two terms: an aggregating one, due to surface tensions, and a disaggregating one due to the repulsion of charged particles. Several models proposed in the literature are based on this principle. One of the simplest and most used assumes that charged droplets are stationary points for the following free energy:

$$P(E) + \frac{Q^2}{\mathcal{C}(E)}. \tag{0.1}$$

Here, $E \subset \mathbb{R}^3$ corresponds to the volume occupied by the droplet, $P(E)$ is the De Giorgi perimeter, [23, Chapter 12], Q is the total charge and

$$\frac{1}{\mathcal{C}(E)} := \inf \left\{ \frac{1}{4\pi} \iint \frac{d\mu(x)d\mu(y)}{|x-y|} : \operatorname{spt}\mu \subset E, \mu(E) = 1 \right\}, \tag{0.2}$$

takes into account the repulsive forces between charged particles. Note that the probability measure μ in (0.2) can be thought of a (normalized) density of charges and that $\mathcal{C}(E)$ is the Newtonian capacity of the set E. In particular one assumes that the optimal shapes are given by the following variational problem with a prescribed volume $V > 0$:

$$\min_{|E|=V} \left\{ P(E) + \frac{Q^2}{\mathcal{C}(E)} \right\}, \tag{0.3}$$

where $|E| = \mathcal{L}^n(E)$ is the n dimensional Lebesgue measure of E .

Heuristically, one expects the perimeter term to dominate for small values of the charge Q forcing the droplet to have a spherical or almost spherical shape. While the repulsion term should become dominant for large values of Q leading to the formation of singularities and/or to the ill-posedness of (0.3). This heuristics is confirmed by the perturbative analysis of (0.1) around a spherical shape. This computation, performed for the first time by Lord Rayleigh in 1882, [27], shows that the spherical droplet is linearly stable only for Q smaller than a critical threshold.

The transition from a stable to an unstable behavior of spherical droplets has also been verified experimentally, starting from the work of Zeleny at the beginning of 1900 [34] (in a slightly different context). More precisely, it has been observed that a spherical droplet exposed to an electric field, remains stable until the total charge is below a critical value $Q_c > 0$, while, as soon as Q exceeds Q_c the droplet changes its appearance and the surface start to develop singularities, the so called Taylor's cones, [31]. Whenever $Q \geq Q_c$ a very thin steady jet composed by small but highly charged little balls is formed, [11, 12, 28, 33].

In spite of the interest in (0.3) for applications, a rigorous mathematical study of this model has been only performed in the last years, mostly thanks to the work of Goldman, Muratov, Novaga and Ruffini, [19, 20, 21, 25, 26].

The starting point of their analysis is the following remarkable and somehow disappointing observation: *Problem* (0.3) *is always ill-posed.* More precisely, in [19], it is shown that

$$\inf_{|E|=V}\left\{P(E)+\frac{Q^2}{\mathcal{C}(E)}\right\}=P(B^V),$$

where B^V is the ball of volume V. Since B^V is a competitor for the variational problem , this clearly implies that there are no minimizers of (0.3).

The above equality is obtained by constructing a minimizing sequence consisting of a ball of roughly volume V together with several balls with vanishing perimeter and volumes and very high charge escaping at infinity. Hence, on the mathematical side, the phenomena observed by Zeleny appears for *every value of the charge*. Let us also remark that ill-posedness of (0.3) is shown also if one assumes that all the sets involved in the minimization problem are a-priori bounded, [19, Theorem 1.3]. We refer also to Chapter 1 for a more detailed discussion.

It then becomes natural to investigate the local minimality of the ball, at least for "small" perturbations and small values of Q. In [19, Theorems 1.4 & 1.7] the linear stability of the ball in the small charge regime is upgraded to local minimality in a sufficiently strong topology. On the other hand, Muratov and Novaga showed that the ball is *never* a local minimizer of (0.3) under (smooth) perturbation which are small in L^∞, [25, Theorem 2]. We also refer to [21] where well-posedness is recovered under suitable geometric restrictions and to [26] for the case of "flat" droplets.

The main phenomena driving to the ill-posedness of (0.3) is the possibility of concentrating a high charge on small volumes. In order to avoid this situation, in [25], Muratov and Novaga proposed as a possible regularization mechanism the finite screening length in the conducting liquid, by introducing the entropic effects associated with the presence of free ions in the liquid, (see also [10, 32] for a related model). They suggested to consider the following *Debye-Hückel-type free energy* [1]

$$\mathcal{F}(E,u,\rho):=P(E)+Q^2\left\{\int_{\mathbb{R}^n}a_E|\nabla u|^2\,dx+K\int_E\rho^2\,dx\right\}. \tag{0.4}$$

Here

$$a_E(x):=\mathbf{1}_{E^c}+\beta\mathbf{1}_E,$$

where $\mathbf{1}_F$ is the characteristic function of a set $F\subset\mathbb{R}^n$ and $\beta>1$ is the permittivity of the droplet. The (normalized) density of charge $\rho\in L^2(\mathbb{R}^n)$ satisfies

$$\rho\mathbf{1}_{E^c}=0 \qquad \text{and} \qquad \int\rho=1, \tag{0.5}$$

and the electrostatic potential u is such that $\nabla u\in L^2(\mathbb{R}^n)$ and

$$-\operatorname{div}\big(a_E\nabla u\big)=\rho \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{0.6}$$

$K>0$ is a physical constant related to the model.

[1] Precisely they introduced the energy functional in Remark 1.2.3.

The variational model proposed in [25], where one assumes *a-priori* that all the sets involved are contained in a fixed (large) ball B_R , is the following

$$\min\big\{\mathcal{F}(E,u,\rho) : |E| = V, E \subset B_R, (u,\rho) \in \mathcal{A}(E)\big\}, \tag{0.7}$$

where we have set

$$\mathcal{A}(E) := \big\{(u,\rho) \in D^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \colon u \text{ and } \rho \text{ satisfy (0.6) and (0.5)}\big\}, \tag{0.8}$$

and

$$D^1(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathring{W}^{1,2}(\mathbb{R}^n)} \qquad \|\varphi\|_{\mathring{W}^{1,2}(\mathbb{R}^n)} = \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}.$$

Note that the class of admissible couples $\mathcal{A}(E)$ is non-empty only if $n \geq 3$, for this reason *this assumption will be in force throughout the whole thesis*, see also Remark 2.1.2.

Thanks to the a-priori boundedness assumption $E \subset B_R$, existence of a minimizer in the class of sets of finite perimeter can be easily shown, see [25, Theorem 3] and Chapter 1.

Note that the presence of the L^2 norm of ρ in the energy is exactly what prevents the concentration of charges. Indeed, if one assumes that $\beta = 1$ so that (0.6) reduces to

$$-\Delta u = \rho,$$

then the minimization problem (0.7) can be written, in dimension $n = 3$ as

$$\min_{|E|=V, E\subset B_R} P(E) + Q^2 \min\left\{\frac{1}{4\pi}\iint \frac{\rho(x)\rho(y)dxdy}{|x-y|} + K\int \rho^2 \text{ s.t. } \rho\mathbf{1}_{E^c} = 0, \int \rho = 1\right\},$$

which should be compared with (0.2) and (0.3). In view of this we also note that, on the mathematical ground, the variational problem (0.7) can also be considered as an "interpolation" between the classical Ohta-Kawasaki problem, and the free-interface problems arising in optimal design studied for instance in [3, 7, 15, 22].

Main results

Once existence of minimizers of (0.7) is obtained it is natural to investigate their qualitative and quantitative properties, also to understand to which extent the predictions of the model agree with the observed phenomenology. In particular the following questions arise, compare with [25]:

(i) Is every minimizer smooth, at least outside a small singular set?

(ii) Can one show that for small values of the total charge the minimizers of (0.7) are balls in agreement with experimental observations?

(iii) Is it possible to show existence/non-existence of minimizers removing the a-priori confinement assumption?

(iv) Which is the structure of (possible) singularities of minimizers? Do they agree with Taylor's cones[2]?

[2]Note that this is possible only if β is large compared to 1, see the discussion at the end of this introduction and Remark 2.3.6

In this thesis we investigate the regularity of minimizers. Precisely, we address question (i). We give also a sketch of (ii) and (partially) (iii) which will be treat in the forthcoming paper [24]. All the original results presented in this thesis are contained in the following works:

- **Title:** *Regularity of minimizers for a model of charged droplets*,
  Joint work with: G. De Philippis, J. Hirsch, [8].

- **Title:** *Minimality of the ball for a model of charged droplets*,
  Joint work with: E. Mukoseeva, [24].

The first main result is the following partial regularity theorem:

Theorem 1. Let $n \geq 3$ and $B \geq 1$. Then there exists $\eta = \eta(n, B) > 0$ with the following property: if E is a minimizer of (0.7) with $1 \leq \beta \leq B$ then there exists a closed set $\Sigma_E \subset \partial E$ such that $\mathcal{H}^{n-1-\eta}(\Sigma_E) = 0$ and $\partial E \setminus \Sigma_E$ is a $C^{1,\vartheta}$ manifold for all $\vartheta \in (0, 1/2)$.

As it is customary in Geometric Measure Theory, the proof Theorem 1 is based on an ε-regularity result.

By scaling (see Section 1.2.2), for $R \geq 1$ we will consider the following variational problem

$$(\mathcal{P}_{\beta,K,Q,R}) \qquad \min\left\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|,\ E \subset B_R\right\},$$

where,

$$\mathcal{F}_{\beta,K,Q}(E) := P(E) + Q^2\mathcal{G}_{\beta,K}(E), \tag{0.9}$$

and

$$\mathcal{G}_{\beta,K}(E) := \inf_{(u,\rho)\in\mathcal{A}(E)} \left\{ \int_{\mathbb{R}^n} a_E|\nabla u|^2 + K\int_E \rho^2 \right\}. \tag{0.10}$$

Furthermore, we fix now some notation which will be useful throughout the thesis.

Notation. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, $\nu \in \mathbb{S}^{n-1}$ and $r > 0$.

- Let $x_0 \in \partial E$. We define

  $$\mathbf{e}_E(x_0, r) := \inf_{\nu\in\mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{\partial^* E\cap B_r(x_0)} \frac{|\nu_E(y) - \nu|^2}{2}\, d\mathcal{H}^{n-1}(y),$$

  where $\partial^* E$ is the *reduced boundary* of E and ν_E is the *measure-theoretic unit normal* to ∂E, [23]. We call $\mathbf{e}_E(x_0, r)$ the *spherical excess* at a point x_0 and at a scale r.

- Let $(u_E, \rho_E) \in \mathcal{A}(E)$ be the minimizer of $\mathcal{G}_{\beta,K}(E)$ whose existence and uniqueness can be easily proved, see Proposition 2.1.3. We define the *normalized Dirichlet energy* at $x_0 \in \mathbb{R}^n$ as

  $$D_E(x_0, r) := \frac{1}{r^{n-1}} \int_{B_r(x_0)} |\nabla u_E|^2\, dx.$$

With these conventions, the ε-regularity result can be stated as follows, see also Theorem 3.2.1 for a slightly more precise statement.

Theorem 2. Given $n \geq 3$, $A > 0$ and $\vartheta \in (0, 1/2)$, there exits $\varepsilon_{\rm reg} = \varepsilon_{\rm reg}(n, A, \vartheta) > 0$ such that if E is minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) with $Q + \beta + K + \frac{1}{K} \leq A$, $x \in \partial E$ and

$$r + \mathbf{e}_E(x, r) + Q^2\, D_E(x, r) \leq \varepsilon_{\rm reg},$$

then $E \cap \mathbf{C}(x, r/2)$ coincides with the epi-graph of a $C^{1,\vartheta}$ function, where $\mathbf{C}(x, r/2)$ is defined in Notation 3.1.2. In particular, $\partial E \cap \mathbf{C}(x, r/2)$ is a $C^{1,\vartheta}$ $(n-1)$-dimensional manifold.

By exploiting a bootstrap argument we upgrade the $C^{1,\vartheta}$ regularity of ∂E to C^∞ smoothness. Precisely, we report here the ε-smoothness regularity version of this result.

Theorem 3. Given $n \geq 3$ and $A > 0$ there exits $\varepsilon_{\rm reg} = \varepsilon_{\rm reg}(n, A) > 0$ such that if E is minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) with $Q + \beta + K + \frac{1}{K} \leq A$,

$$x \in \partial E \quad \text{and} \quad r + \mathbf{e}_E(x, r) + Q^2\, D_E(x, r) \leq \varepsilon_{\rm reg},$$

then $E \cap \mathbf{C}(x, r/2)$ coincides with the epi-graph of a C^∞ function f. In particular, we have that $\partial E \cap \mathbf{C}(x, r/2)$ is a C^∞ $(n-1)$-dimensional manifold.

As mentioned before, thanks to the particular structure of the functional $\mathcal{F}$, one can expect that for the small total charge Q the attractive term, represented by the perimeter, dominates the non-local repulsive energy $\mathcal{G}$, forcing the droplet to have the spherical shape. By combining the above partial regularity results with the analysis of the linearized energy around a ball, we want to confirm this intuition by proving that the ball is the unique minimizer of the functional $\mathcal{F}$ for small values of the total charge Q. This is consistent with the experiments conducted by Zeleny. To be more precise, we want to prove the following theorem.

Theorem 4. Fix $K > 0$, $\beta > 1$. Then there exists $Q_0 > 0$ such that for all $Q < Q_0$ and any $R \geq 1$ the only minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) are the balls of radius 1.

The proof of Theorem 4 will be addressed in [24] and partially in Chapter 5.

Organization of the thesis

We shall now present an overview of the content of this thesis.

- **Chapter 1: On well-posedness of the classical charged liquid drops model.** In this chapter we review some of the existing literature. More precisely, we recall some recent results on the ill-posedness of the classical charged liquid drops model. Subsequently, we introduce the *Debye-Hückel type free energy* which will we study for the rest of the thesis, and we prove the existence of its minimizers in a reasonable class of competitors. We conclude the chapter by presenting some (open) questions concerning the model.
- **Chapter 2: Properties of minimizers.** Though the energy we are considering has a certain similarity with those studied in optimal design problems, the fact that the minimization problem in (1.16) is performed only among admissible pairs $(u, \rho) \in \mathcal{A}(E)$ makes very difficult to make local perturbations. In particular, problem ($\mathcal{P}_{\beta,K,Q,R}$) has (a priori) no local scaling invariance. For this reason in Section 2.1 we study carefully the energy $\mathcal{G}(E)$ and its minimizers (u_E, ρ_E). Moreover we establish boundedness of u_E and ρ_E.

In order to study the regularity of minimizers one needs to perform local variations and hope that these give localized (or almost localized) changes of the energy. This is not completely obvious due to the presence of a volume constraint and to the non-local character of $\mathcal{G}$. As it is well known the volume constraint can be relaxed into a "perturbed" minimality property of minimizers. In order to have estimates uniform in the structural parameters it will be important to have this "perturbed" minimality property uniform in the class of minimizers. In Section 2.2 we start studying how the energy varies according to a flow of diffeomorphism, which will be important in performing small volume adjustments and we establish the Euler Lagrange equations for minimizers. In Section 2.3 we prove the perturbed minimality property and we study the behavior of the energy under local perturbations. In Section 2.4 we prove the compactness of the class of minimizers in the L^1 topology.
The next step consists in establishing local perimeter and volume estimates for the minimizers of ($\mathcal{P}_{\beta,K,Q,R}$). Usually these estimates are easily obtained by combining minimality with local isoperimetric inequalities. Here, due to the non-local character of the energy term $\mathcal{G}(E)$ and the absence of a natural scaling invariance of the problem, more refined arguments are required. In particular we will first show that the energy $\mathcal{G}$ is monotone by set inclusion. This implies that E is an outer minimizer for the perimeter and leads to upper perimeter bounds and lower density estimates for E^c. Estimating the density of E is instead more complicated and requires to perform an inductive argument showing that if E has small relative measure in a ball $B_r(x)$, then the Dirichlet energy of u_E decays enough to preserve this information at smaller scales, leading to a contradiction. In doing this, higher integrability of the gradient of minimizers of $\mathcal{G}$ plays a key role. Local density estimates are obtained in Section 2.5 together with the boundedness of $D_E(x,r)$. This fact combined with the local density and perimeter estimates allows somehow to recover the scaling invariance of the problem.

- **Chapter 3: $C^{1,\vartheta}$-regularity.** The main step of the proof of Theorems 1 and 2 is the decay of the excess established in Section 3.1. Once the local scaling invariance of the problem is recovered, the proof of Theorem 2 follows the classical De Giorgi's idea of harmonic approximation. Namely we will show that in the regime of small excess and small normalized Dirichlet energy, ∂E can be well approximated by the graph of a function with "small" Laplacian. This leads to the decay of the excess which, thanks to the higher integrability of ∇u_E, in turn also implies the decay of the normalized Dirichlet energy and eventually allows to conclude the proof.
In Section 3.2 we prove Theorems 1 and 2. Theorem 2 will be an immediate consequence of Theorem 3.1.1 (see also Theorem 3.2.1 for a more quantitative version). Theorem 1 is proved by following the strategy of [15] where one combines the the ε-regularity result with the higher integrability of the ∇u_E and the classical regularity theory for minimal surfaces.
Let us remark that most of the above described difficulties arises only in the case when β is relatively large compared to 1. Indeed in the regime $\beta - 1 \ll 1$, Cordes estimates, see [5], imply that ∇u_E belongs to L^p with p large. In this case Hölder inequality immediately gives that the energy term $\mathcal{G}$ is lower order with respect to the perimeter at small scales. E will then be an *ω-minimizer* of the perimeter and the regularity theory follows for instance from [30], see Remark 2.3.6. In particular in this case one obtains full regularity in $n = 3$, thus excluding the formation of Taylor's cone singularities. This

phenomena was already observed in [29] for a different model of charged droplets.

- **Chapter 4: Higher regularity.** This chapter is devoted to enhance the partial regularity result obtained in Chapter 3 by proving the higher regularity of minimizers. By exploiting the Euler-Lagrange equation and the C^1-regularity of u up to the boundary, ∂E obtained in Chapter 3, we deduce the partial C^2-regularity of minimizers.
  Finally, by a bootstrap argument we are able to obtain the smooth partial regularity of minimizers.

- **Chapter 5: Minimality of the ball in the small charge regime.** The higher regularity result obtained in Chapter 4 is intereasting in itself but it is also the starting point to prove Theorem 4.

  In this chapter, precisely in Section 5.2 we show how to reduce the problem to the so-called *nearly spherical sets* in the small charge regime, by exploiting Theorem 3. The advantage is that for this particular class of sets we are able to deduce a Taylor expansion of the energy $\mathcal{G}$ near the ball B_1, Theorem 5.1.4. The proof of this theorem will be addressed in the forthcoming work [24].

  In Section 5.1 we deduce Theorem 4 for nearly-spherical sets in the small charge regime, by using Theorem 5.1.4.

1

On well-posedness of the classical charged liquid drops model

Contents

1.1 The classical charged liquid drops model

In this chapter we review some of the classical and well-known variational models which describe the equilibrium shape of charged liquid droplets.

Given a prescribed volume $V > 0$ we consider all $E \subset \mathbb{R}^n$ $(n > 2)$ occupied by a charged liquid droplet with volume V and total charge $Q > 0$. The energy functional which describe this phenomena is composed by the surface tension, represented by the perimeter $P(E)$, and the *Coulombic interaction energy* $\mathcal{I}(E)$:

$$\mathcal{E}_Q(E) := P(E) + Q^2\,\mathcal{I}(E), \tag{1.1}$$

where

$$\mathcal{I}(E) := \inf\left\{\frac{1}{4\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-2}}\,d\mu(x)d\mu(y)\;:\;\mathrm{spt}\mu\subset E, \mu(E)=1\right\}. \tag{1.2}$$

The probability measure μ can be though as a (normalized) density of charges. If $\mathcal{C}_2(E)$ is the *Newtonian capacity* of the set E i.e.

$$\mathcal{C}(E) := \min\left\{\int_{\mathbb{R}^n}|\nabla u|^2\,dx\;:\;u\in H^1_0(\mathbb{R}^n),\,u\geq 1\,\text{on}\;E\right\},$$

then

$$\mathcal{I}(E) = \frac{1}{\mathcal{C}(E)}.$$

The optimal equilibrium shapes of charge droplets are given by minimizing the functional $\mathcal{E}_Q$:

$$\min \{\mathcal{E}_Q(E) \,:\, E \subset \mathbb{R}^n \, \text{compact}, |E| = V\} . \tag{1.3}$$

Notice that

$$\mathcal{I}(\lambda E) = \lambda^{2-n}\, \mathcal{I}(E), \quad \forall \lambda > 0. \tag{1.4}$$

This implies that

$$\mathcal{E}_Q(E) = \lambda^{1-n}\, \mathcal{E}_{Q\lambda^{n-\frac{3}{2}}}(\lambda E), \quad \forall \lambda > 0.$$

In particular, by replacing Q with $Q\left(\frac{V}{\omega_n}\right)^{1-\frac{3}{2n}}$, we can assume $V = |B_1| := \omega_n$. So we will consider the following minimization problem:

$$\min \{\mathcal{E}_Q(E) \,:\, E \subset \mathbb{R}^n \, \text{compact}, |E| = |B_1|\} . \tag{1.5}$$

From a mathematical point of view the nature of this variational model is interesting because two different types of energies coexist in $\mathcal{E}_Q$: the shape of a charged liquid droplet is indeed determined by the competition between the local and attractive perimeter $P(E)$ and the non-local and repulsive electrostatic potential $\mathcal{I}$ which are respectively minimized and maximized by the unitary ball.

Thanks to a rigorous mathematical analysis of the model due to the recent contributions of Goldman, Muratov, Novaga and Ruffini the following unexpected observation arise: *the problem* (1.5) *is always ill-posed.* More precisely, in [19], they construct a minimizing sequence of competitors composed by a big uncharged ball of roughly volume $|B_1| = \omega_n$ and several balls, far from each other, highly charged with vanishing perimeter and volumes. Roughly speaking, in order to minimize the energy, it is convenient to disperse the charge at infinity. This is the content of the next theorem, proved in [19, Theorem 1.1].

Theorem 1.1.1. *Suppose that $n > 2$, then*

$$\inf_{|E|=\omega_n} \mathcal{E}_Q(E) = P(B_1),$$

where B_1 is the ball of volume ω_1.

Remark 1.1.2. Since B_1 is a competitor for the variational problem and it minimizes the perimeter we have

$$P(B_1) \leq P(E) < P(E) + \frac{Q^2}{\mathcal{C}(E)} = \mathcal{E}_Q(E), \tag{1.6}$$

for every E with $|E| = |B_1| = \omega_n$. Theorem 1.1.1 and inequality (1.6) clearly imply that there are no minimizers of (0.3).

Proof. Fix $d \in \mathbb{N}$. Consider d pairwise disjoint balls, $\{B_{r_d}(x_i)\}_{i=1}^d$, each of radius $r_d := \left(\frac{1}{d}\right)^\alpha$ with $\alpha > 0$ to be chosen later on. Define

$$E_d := \bigcup_{i=1}^d B_{r_d}(x_i) \cup B_R, \quad \text{where} \quad R = (1 - d\, r_d^n)^{\frac{1}{n}}.$$

Put on each ball $B_{r_d}(x_i)$ a charge $\frac{Q}{d}$ and assume that the big ball B_R is non-charged. Hence E_d has volume ω_n and it is a competitor for the problem. Moreover, recalling (1.4), we deduce

$$\begin{aligned} Q^2\, \mathcal{I}(E_d) = Q^2\, \mathcal{I}\left(\bigcup_{i=1}^d B_{r_d}(x_i)\right) &\leq d \left(\frac{Q}{d}\right)^2 \mathcal{I}(B_{r_d}) + \left(\frac{Q}{d}\right)^2 \varepsilon_{i,j}(d) \\ &\leq \frac{Q^2}{d} \left(\frac{1}{r_d}\right)^{n-2} \mathcal{I}(B_1) + \left(\frac{Q}{d}\right)^2 \varepsilon_{i.j}(d), \end{aligned}$$

where we have bounded the cross-interaction by

$$\varepsilon_{i,j}(d) := \sum_{i=1}^d \frac{1}{(|x_i - x_j| - 2r_d)^{n-2}}.$$

We assume also that the balls $\{B_{r_d}(x_i)\}_{i=1}^d$ are far away from each other in the sense that

$$\lim_{d\to\infty} \varepsilon_{i,j}(d) = 0.$$

Then, thanks to properties of the perimeter we have

$$\begin{aligned} \mathcal{E}_Q(E_d) &\leq P(B_R) + d\, P(B_{r_d}) + \frac{Q^2}{d} \left(\frac{1}{r_d}\right)^{n-2} \mathcal{I}(B_1) + \left(\frac{Q}{d}\right)^2 \varepsilon_{i.j}(d) \\ &\leq (1 - d\, r_d^n)^{\frac{n-1}{n}}\, P(B_1) + d\,(n-1)\omega_n\, r_d^{n-1} + \frac{Q^2}{d}\left(\frac{1}{r_d}\right)^{n-2} \mathcal{I}(B_1) \\ &+ \left(\frac{Q}{d}\right)^2 \varepsilon_{i.j}(d). \end{aligned} \tag{1.7}$$

On the other hand since B_1 minimizes the perimeter and $|E_d| = |B_1|$ then

$$P(B_1) \leq P(E_d) \leq \mathcal{E}_Q(E_d).$$

Choose $\alpha > 0$ such that

$$\lim_{d\to+\infty} d\, r_d^{n-1} = 0, \quad \text{and} \quad \lim_{d\to+\infty} \frac{1}{d}\left(\frac{1}{r_d}\right)^{n-2} = 0.$$

Therefore by letting $d \to +\infty$ in (1.7) we obtain the result. □

On the mathematical side, Theorem 1.1.1 shows that the instability of the ball, observed in the Zeleny's experiments whenever the voltage increase enough, appears here for *every value of the total charge*. In [19, Theorem 3.4] it was also proved that the problem is ill-posed even if all the set involved in the minimization problem are a-priori bounded:

Theorem 1.1.3. *Let $\Omega \subset \mathbb{R}^n$ be compact with $\partial\Omega$ smooth and let $0 < V < |\Omega|$. If $E_0 \subset \mathbb{R}^n$ is the solution of the isoperimetric problem*

$$\min\{P(E) \,:\, E \subset \Omega,\, |E| = V\},$$

then for every $Q > 0$ we have

$$\inf\{\mathcal{E}(E) \,:\, E \subset \Omega,\, |E| = V\} = P(E_0) + Q^2\, \mathcal{I}(\Omega).$$

By [19, Lemma 2.15, (i)] we have that for all $E \subset \mathbb{R}^n$ open and bounded if μ minimizes (1.2) then

$$\mathcal{I}(E) = \mathcal{I}(\partial E) \quad \text{and} \quad \text{spt}\mu \subset \partial E. \tag{1.8}$$

Thanks to (1.8) if $\Omega \subset \mathbb{R}^n$ is open with a smooth boundary then

$$\mathcal{I}(E) > \mathcal{I}(\Omega) \quad \forall E \subset\subset \Omega. \tag{1.9}$$

By combining Theorem 1.1.3 and (1.9) one can prove that the functional $\mathcal{E}_Q$ does not admit local volume constraint minimizers, [19, Theorem 3.5].

Since the set-construction in the above theorems, which generate the instability of the ball, consists of disconnected sets , it is reasonable to investigate the minimality of the ball under (smooth) perturbations which are small in L^∞. In [25, Theorem 2] Muratov and Novaga attest the ball instability also in this case.

For completeness we report here this result.

Theorem 1.1.4. *Let $n = 3$. Then for every $\delta > 0$ there exists a smooth function*

$$\varphi_\delta : \partial B_1 \to (-\delta, \delta)$$

such that the following set

$$\Omega_\delta := \left\{ x \in \mathbb{R}^n \, : \, |x| \leq 1 + \varphi_\delta \left(\frac{x}{|x|} \right) \right\}$$

is a competitor for the energy i.e. $|\Omega_\delta| = |B_1|$ and

$$\mathcal{E}_Q(\Omega_\delta) < \mathcal{E}_Q(B_1).$$

1.2 Well-Posedness

In order to restore well-posedness of the problem (1.5) there are two possible ways to follow:

(i) one can restrict the class of admissible sets.

(ii) one can modify the energy functional by some regularizing physical mechanisms.

1.2.1 Restriction of the admissible class

In order to overcome the issues which arise in Section 1.1 one should minimize the energy $\mathcal{E}_Q$ among all sets with a strong constraint on the curvature. These are the sets which satisfy the following geometric δ-ball condition.

Definition 1.2.1 (δ-ball condition)**.** Let $\delta > 0$. A set $E \subset \mathbb{R}^n$ satisfy the *internal* (resp. *external*) *δ-ball condition* if for every $x \in \partial E$ there exists a ball of radius δ contained in E (resp. contained in E^c) which is tangent to ∂E at x. If E satisfy both the external and internal δ-ball conditions then we say that E satisfy the *δ-ball condition*. Denote by $\mathcal{K}_\delta$ the class of all sets which satisfy the δ-ball condition.

The main advantages to restrict the admissible sets to $\mathcal{K}_\delta$ are that this regular sets satisfy some density estimates and the minimizing measure of $\mathcal{I}(E)$ is a uniformly bounded measure on ∂E. By exploiting this key properties in [19, Theorem 4.3] and [19, Theorem 5.6] Goldman, Novaga and Ruffini prove the following existence and regularity result.

Theorem 1.2.2. *Let $n > 2$. Then there exists $Q_0 = Q_0(n, 2) > 0$ such that the following problem*

$$\min \{\mathcal{E}_Q(E) \, : \, E \in \mathcal{K}_\delta \, , |E| = \omega_n\} \, , \tag{1.10}$$

admits a solution for every $Q \geq Q_0 \, \delta^n$. Moreover, for every $\delta > 0$ such that $\omega_n \, \delta^n \leq 1$ then there exists $Q_1 = Q_1(\delta) > 0$ such that the ball is the unique minimizer of the problem whenever $Q \leq Q_1$.

1.2.2 The Debye-Hückel type free energy

The main difficulty with the variational problem (1.5) comes from the tendency of charges to concentrate at the liquid interface: mathematically this phenomena is reflected in the freedom choice of the charge distribution i.e. the measure μ, in the variational model (1.5). Notice that, as mentioned before, the validity of Theorem 1.2.2 strongly depends on the geometric curvature assumption.

Instead to proceed as in the previous paragraph, in order to restore the well-posedness one should consider a physical regularizing mechanism in the functional. With this purpose in [25] Muratov and Novaga integrate the entropic effects associated with the presence of free ions in the liquid. The advantage of this model is that the charges now are distributed inside of the droplet E.

Precisely, they introduce the following *Debye-Hückel-type free energy* in every dimension $n \in \mathbb{N}$:

$$\mathcal{F}(E, u, \rho) := P(E) + Q^2 \left\{ \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_E \rho^2 \, dx \right\}. \tag{1.11}$$

Here

$$a_E(x) := \mathbf{1}_{E^c} + \beta \mathbf{1}_E,$$

where $\mathbf{1}_F$ is the characteristic function of a set F and $\beta > 1$ is the permittivity of the droplet. The (normalized) density of charge $\rho \in L^2(\mathbb{R}^n)$ satisfies

$$\rho \mathbf{1}_{E^c} = 0 \qquad \text{and} \qquad \int \rho = 1, \tag{1.12}$$

and the electrostatic potential u is such that $\nabla u \in L^2(\mathbb{R}^n)$ and

$$-\operatorname{div}\big(a_E \nabla u\big) = \rho \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{1.13}$$

$K > 0$ is a physical constant related to the model.

Remark 1.2.3. Actually in [25], the energy (1.11) is written as

$$\sigma P(E) + Q^2 \left\{ \frac{\beta_0}{2} \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_E \rho^2 \, dx \right\},$$

for suitable parameters σ and β_0 and the relation (1.13) is replaced by $-\beta_0 \operatorname{div}\big(a_E \nabla u\big) = \rho$. However it is easy to see that the parameters σ and β_0 can be absorbed in Q and K, see also the discussion below.

The variational model proposed in [25], where one assumes *a-priori* that all the sets are contained in a fixed (large) ball B_R, is the following

$$\min\big\{\mathcal{F}(E,u,\rho) : |E| = V, E \subset B_R, (u,\rho) \in \mathcal{A}(E)\big\}, \tag{1.14}$$

where we have set

$$\mathcal{A}(E) := \big\{(u,\rho) \in D^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\text{: } u \text{ and } \rho \text{ satisfy (1.13) and (1.12)}\big\}, \tag{1.15}$$

and

$$D^1(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathring{W}^{1,2}(\mathbb{R}^n)} \qquad \|\varphi\|_{\mathring{W}^{1,2}(\mathbb{R}^n)} = \|\nabla\varphi\|_{L^2(\mathbb{R}^n)}.$$

Note that the class of admissible couples $\mathcal{A}(E)$ is non-empty only if $n \geq 3$, for this reason *this assumption will be in force throughout the whole thesis*, see also Remark 2.1.2.

In order to keep track of the various dependence on the parameters let us first fix some notations, which will be useful also in the sequel. For $E \subset \mathbb{R}^n$ we define

$$\mathcal{G}_{\beta,K}(E) := \inf_{(u,\rho)\in\mathcal{A}(E)} \left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2 + K\int_E \rho^2\right\}, \tag{1.16}$$

where the set of admissible pairs $\mathcal{A}(E)$ is defined in (1.15) (if the dependence on the parameter is not relevant we will simply write $\mathcal{G}$). Since

$$(u,\rho) \in \mathcal{A}(E) \quad \Longrightarrow \quad \Big(\lambda^{2-n}u\Big(\frac{\cdot}{\lambda}\Big), \lambda^{-n}\rho\Big(\frac{\cdot}{\lambda}\Big)\Big) \in \mathcal{A}(\lambda E),$$

one has

$$\mathcal{G}_{\beta,\lambda^2K}(\lambda E) = \lambda^{2-n}\mathcal{G}_{\beta,K}(E).$$

Setting

$$\mathcal{F}_{\beta,K,Q}(E) := P(E) + Q^2\mathcal{G}_{\beta,K}(E),$$

one gets

$$\mathcal{F}_{\beta,K,Q}(E) = \lambda^{1-n}\mathcal{F}_{\beta,K\lambda^2,Q\lambda^{\frac{2n-3}{2}}}(\lambda E).$$

In particular, by replacing K and Q with $K(\omega_n/V)^{\frac{2}{n}}$ and $Q(\omega_n/V)^{1-\frac{3}{2n}}$ we can assume that $V = |B_1| =: \omega_n$. Namely, for $R \geq 1$ we will consider the following problem

$$\min\big\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|,\ E \subset B_R\big\}. \tag{$\mathcal{P}_{\beta,K,Q,R}$}$$

Thanks to the a-priori boundedness assumption $E \subset B_R$, in the following theorem we easily prove the existence of a minimizer in the class of admissible sets of finite perimeter.

Theorem 1.2.4 (Existence)**.** *Given $0 < \beta \leq 1$, $Q > 0$ and $V > 0$. Then the problem* (1.14) *admits a solution.*

Proof. We follow the approach of [25, Theorem 3] by applying the direct method of calculus of variations. Consider a minimizing sequence (E_h, u_h, ρ_h) for the energy functional $\mathcal{F}$ such that $|E_h| = V$, $E_h \subset B_R$ and $(u_h,\rho_h) \in \mathcal{A}(E_h)$ is admissible. Then, up to subsequence, there exists (E,u,ρ) such that

- $\sup_h P(E_h) < \infty$ and $E_h \subset B_R \implies E_h \to E$ in $L^1(\mathbb{R}^n)$ when $h \to \infty$,
- $\|u_h\|_{L^{2^*}(B_R)} \leq C\,\|\nabla u_h\|_{L^2(\mathbb{R}^n)} < \infty \implies u_h \rightharpoonup u$ in $W^{1,2}(B_R)$ when $h \to \infty$,

- $\sup_h \|\rho_h\|_{L^2(\mathbb{R}^n)} < \infty \implies \rho_h \rightharpoonup \rho$ in $L^2(\mathbb{R}^n)$ when $h \to \infty$.

We show now that the triple (E, u, ρ) is admissible for the problem. Indeed clearly $|E| = V$ and $E \subset B_R$. Moreover,

$$Q = \int_{\mathbb{R}^n} \rho_h \, dx = \int_{\mathbb{R}^n} \rho_h \, \mathbf{1}_{B_R} \, dx \to \int_{\mathbb{R}^n} \rho \, \mathbf{1}_{B_R} \, dx.$$

It is simply to prove also that $\int_{\mathbb{R}^n} \rho \, (1 - \mathbf{1}_E) \, \varphi \, dx = 0$ for every test function $\varphi \in C^1_c(\mathbb{R}^n)$. Then $\rho = 0$ a.e. on E^c. Now we want to prove that u solves $-\operatorname{div}(a_E \nabla u) = \rho$. To this purpose notice that

$$\left| \int_{\mathbb{R}^n} a_E \nabla u \cdot \nabla \varphi \, dx - \int_{\mathbb{R}^n} a_{E_h} \nabla u_h \cdot \varphi \, dx \right| \leq \left| \int_{\mathbb{R}^n} a_E \, (\nabla u - \nabla u_h) \cdot \nabla \varphi \, dx \right| + \left| \int_{\mathbb{R}^n} (a_E - a_{E_h}) \, \nabla u_h \cdot \nabla \varphi \, dx \right|. \tag{1.17}$$

The first term in the right hand side of the above equation converges to 0 as $h \to \infty$ since $\nabla u_h \rightharpoonup \nabla u$ in $L^2(\mathbb{R}^n)$. The second term also converges to 0 by using the L^1 convergence of E_h to E together with the following estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} (a_E - a_{E_h}) \, \nabla u_h \cdot \nabla \varphi \, dx \right| &\leq \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \, \|a_E - a_{E_h}\|_{L^2(\mathbb{R}^n)} \, \|\nabla u_h\|_{L^2(\mathbb{R}^n)} \\ &\leq \beta \, \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \, |E \Delta E_h|^{1/2} \, \|\nabla u_h\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Thus we have obtained that (E, u, ρ) is a competitor for the problem. It remains to prove the lower semicontinuity of $\mathcal{F}$. Notice that, by adding and subtract the term $u_h \rho_h$, we have

$$\int_{\mathbb{R}^n} \rho_h u \, dx \leq \|\rho_h\|_{L^2(\mathbb{R}^n)} \, \|u - u_h\|_{L^2(B_R)} + \int_{\mathbb{R}^n} \rho_h u_h \, dx.$$

The above inequality yields

$$\int_{\mathbb{R}^n} \rho_h u \, dx \leq \liminf_{h \to \infty} \int_{\mathbb{R}^n} \rho_h u_h \, dx.$$

Therefore, we conclude that

$$\begin{aligned} \int_{\mathbb{R}^n} a_E \, |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} \rho \, u \, dx &= \lim_{h \to \infty} \int_{\mathbb{R}^n} \rho_h u \, dx \\ &\leq \liminf_{h \to \infty} \int_{\mathbb{R}^n} \rho_h u_h \, dx = \liminf_{h \to \infty} \int_{\mathbb{R}^n} a_{E_h} \, |\nabla u_h|^2 \, dx. \end{aligned} \tag{1.18}$$

By exploiting (1.18), the lower semicontinuity of the perimeter and the L^2-norm of ρ we get

$$\mathcal{F}(E, u, \rho) \leq \liminf_{h \to \infty} \mathcal{F}(E_h, u_h, \rho_h).$$

So we conclude by the direct method of calculus of variations. □

Once we get the existence of the minimizers it is interesting to investigate some regularity properties concerning the minimizers. The following questions arise:

Questions:

(1) *Regularity.*
Are the minimizers smooth? Precisely, can we locally describe the topological boundary by the graph of a smooth function?

(2) *Shape of minimizers for small values of the charge.*
Is the ball the unique minimizer of the functional $\mathcal{F}$, for small values of the total charge Q? A positive answer is reasonable, indeed accordingly to the particular structure of the energy functional, one can expect that for small values of the total charge $Q > 0$ the perimeter term -which is minimized by the ball- becomes dominant in the energy.

(3) *Topological structure of minimizers for large values of the charge.*
As conjectured in [25] whenever $Q > Q(V)$ and the radius of the confinement R is sufficiently large, one should be able to show that minimizers are composed by a number of connected components (balls) whose size is comparable to the so called *Debye radius* r_D. The radius r_D depends only on the permittivity constants and others physical constants related to the model (for a precise definition of r_D see [25]). The reason of this phenomenon relies on the fact that spherical droplets whose size is much greater than r_D could be splitted into many little ones; then by redistributing the charge one should be able to decrease the total energy.

(4) *Existence issue.*
Can existence of minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) be shown also without assuming the a priori boundedness constraint? In analogy with the Ohta-Kawasaki model, this seems to be reasonable for small values of Q while for large values one expect the ill-posedness of the problem.

(5) *Structure of singularities.*
Finally, we want to mention another open problem which involves the structure of (possible) singularities of minimizers. For large values of the droplet permittivity $\beta >> 1$ (recall that 1 is the permittivity of the external liquid in the container), in line with physical experiments, one should conjecture the presence of cone-like singularities: the so called *Taylor's cones*. In [15] N. Fusco and V. Julin thanks to a monotonicity formula proved that conical *stationary* critical points (the Taylor's cones) for a similar free-interface problem, can occur only if the opening angle is neither too small nor to close to $\frac{\pi}{2}$ and the ratio of the two permittivities is sufficiently large.

In the next chapters we address points (1) and (2).

2

Properties of minimizers

Contents

In this chapter we provide some properties concerning the minimizers of the variational problem (1.14) introduced in Chapter 1.

2.1 Properties of minimizers of $\mathcal{G}$

In this section we start establishing some basic properties of minimizers of $\mathcal{G}$. We start with the following lemma. Recall that $2^* := 2n/(n-2)$ and $n \geq 3$.

Lemma 2.1.1. *Let $n \geq 3$, $\beta > 1$ and let $A : \mathbb{R}^n \to \mathrm{Sym}_n(\mathbb{R}^n)$ be a symmetric matrix valued function such that*

$$\mathrm{Id} \leq A(x) \leq \beta\, \mathrm{Id} \qquad \textit{for all } x \in \mathbb{R}^n.$$

Then for every $\rho \in L^{(2^*)'}$ (i.e. the dual of L^{2^*}) there exists a unique $u \in D^1(\mathbb{R}^n)$ such that

$$-\operatorname{div}(A\nabla u) = \rho. \tag{2.1}$$

Proof. Recall that for $u \in D^1(\mathbb{R}^n)$ one has the following Sobolev inequality

$$\|u\|_{2^*} \leq S(n)\|\nabla u\|_{L^2}.$$

In particular, by the assumptions on ρ and A the energy

$$\mathcal{E}(v) := \frac{1}{2}\int_{\mathbb{R}^n} A\nabla v \cdot \nabla v \, dx - \int_{\mathbb{R}^n} \rho v \, dx,$$

is finite. By Young's inequality $\mathcal{E}(v)$ is bounded from below by

$$\|\nabla v\|_{L^2}^2 - C(n)\|\rho\|_{(2^*)'}^2.$$

Hence, direct methods of the calculus of variations imply the existence of a unique minimizer which is the desired solution. Furthermore for the solution we have

$$\min_{v\in D^1(\mathbb{R}^n)} \mathcal{E}(v) = \mathcal{E}(u) = -\frac{1}{2}\int_{\mathbb{R}^n} A\nabla u \nabla u \, dx = -\frac{1}{2}\int_{\mathbb{R}^n} \rho u \, dx. \tag{2.2}$$

Notice that the minimum u of (2.2) is the unique solution of (2.1). Indeed let u_1 and u_2 be two minimizers and define $z := u_1 - u_2$, hence $-\operatorname{div}(A\nabla z) = 0$. By ellipticity of A we have

$$\int_{\mathbb{R}^n} |\nabla z|^2 \, dx = 0.$$

The Sobolev inequality implies $z = 0$. □

Remark 2.1.2. In dimension $n = 2$ the above lemma is easily seen to be false, indeed even for a smooth and compactly supported ρ, the solution of

$$-\Delta u = \rho,$$

does not in general satisfy $\nabla u \in L^2$.

By the above lemma , if $|E| < \infty$ the couple (u, ρ) defined by

$$\rho = \frac{\mathbf{1}_E}{|E|}, \qquad -\operatorname{div}(a_E \nabla u) = \rho,$$

is admissible, $(u, \rho) \in \mathcal{A}(E)$. By testing the equation with u and using the Sobolev embedding, we then get

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \le \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx = \fint_E u \, dx \le \left(\fint_E u^{2^*} \, dx\right)^{\frac{1}{2^*}} \le \frac{S(n)}{|E|^{\frac{1}{2^*}}}\|\nabla u\|_2.$$

In particular (recall $\beta > 1$)

$$\mathcal{G}(E) \le \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K\int_{\mathbb{R}^n} \rho^2 \, dx \le C(n, \beta, K, 1/|E|). \tag{2.3}$$

Proposition 2.1.3. *Let $E \subset \mathbb{R}^n$ be a set of finite measure. Then there exists a unique pair $(u_E, \rho_E) \in \mathcal{A}(E)$ minimizing $\mathcal{G}_{\beta,K}(E)$. Moreover,*

$$u_E + K\rho_E = \mathcal{G}_{\beta,K}(E) \qquad \textit{in } E, \tag{2.4}$$

and

$$0 \le u_E \le \mathcal{G}_{\beta,K}(E) \qquad \textit{and} \qquad 0 \le K\rho_E \le \mathcal{G}_{\beta,K}(E)\mathbf{1}_E. \tag{2.5}$$

In particular, $\rho_E \in L^p$ for all $p \in [1, \infty]$ and

$$\|\rho_E\|_p \le C(n, \beta, K, 1/|E|). \tag{2.6}$$

Proof. Existence of a minimizer is an immediate application of the direct methods in the calculus of variations. Uniqueness follows from the convexity of the admissible set $\mathcal{A}(E)$ and the strict convexity of the energy

$$(u,\rho) \mapsto \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho^2 \, dx.$$

Let now $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that

$$\psi \mathbf{1}_{E^c} = 0, \qquad \int_{\mathbb{R}^n} \psi \, dx = 0. \tag{2.7}$$

Let $v \in D^1(\mathbb{R}^n)$ be the solution of

$$-\operatorname{div}(a_E \nabla v) = \psi. \tag{2.8}$$

If (u_E, ρ_E) is the minimizing pair then $(v_\varepsilon, \rho_\varepsilon) = (u_E + \varepsilon v, \rho_E + \varepsilon \psi) \in \mathcal{A}(E)$ is admissible. Hence, by taking the derivative with respect to ε of its energy we get

$$0 = \int_{\mathbb{R}^n} a_E \nabla u_E \nabla v \, dx + K \int_{\mathbb{R}^n} \rho_E \psi \, dx \overset{(2.8)}{=} \int (u_E + K\rho_E)\psi \, dx.$$

Since this holds for all ψ satisfying (2.7) we get that $u_E + K\rho_E = \text{const}$ in E. By multiplying this equation by ρ_E and integrating over E we infer that the constant shall be equal to $\mathcal{G}(E)$, and this proves (2.4). In particular u_E solves

$$-\operatorname{div}(a_E \nabla u_E) = \frac{\mathcal{G}(E) - u_E}{K} \mathbf{1}_E. \tag{2.9}$$

By testing the above with $(\mathcal{G}(E) - u_E)_- = -\min\{0, \mathcal{G}(E) - u_E\}$ we obtain

$$0 = \int_{\{\mathcal{G}(E) < u_E\}} a_E \, |\nabla u_E|^2 \, dx + \int_{\{\mathcal{G}(E) < u\}} \frac{(\mathcal{G}(E) - u_E)^2}{K} \, dx,$$

which implies the second half of the first inequality in (2.5). Testing (2.9) with $u_- = -\min\{0, u\}$ we obtain the first half. The second inequality in (2.5) follows now from the first and (2.4). Inequality (2.6) follows from (2.3). □

We establish now the monotonicity of $\mathcal{G}$ with respect to set inclusion. We start from the following lemma.

Lemma 2.1.4. *Let A, $B : \mathbb{R}^n \to \mathrm{Sym}_n(\mathbb{R}^n)$ two symmetric matrix valued functions such that $\mathrm{Id} \le A(x) \le B(x)$ for all $x \in \mathbb{R}^n$. If $\rho \in L^{(2^*)'}(\mathbb{R}^n)$ and $u, v \in D^1(\mathbb{R}^n)$ are the unique solutions of*

$$-\operatorname{div}(A\nabla u) = \rho \qquad \text{and} \qquad -\operatorname{div}(B\nabla v) = \rho, \qquad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{2.10}$$

then

$$2\int_{\mathbb{R}^n} (B - A)\, \nabla v \cdot \nabla v \, dx + \int_{\mathbb{R}^n} B\, \nabla v \cdot \nabla v \, dx \le \int_{\mathbb{R}^n} A\, \nabla u \cdot \nabla u \, dx. \tag{2.11}$$

In particular,

$$\int_{\mathbb{R}^n} B\, \nabla v \cdot \nabla v \, dx \le \int_{\mathbb{R}^n} A\, \nabla u \cdot \nabla u \, dx.$$

Proof. Let $\mathcal{E}_A$ and $\mathcal{E}_B$ be the following functionals defined on $D^1(\mathbb{R}^n)$:

$$\mathcal{E}_A(w) := \frac{1}{2}\int_{\mathbb{R}^n} A\nabla w\cdot\nabla w\,dx - \int_{\mathbb{R}^n}\rho w\,dx,$$
$$\mathcal{E}_B(w) := \frac{1}{2}\int_{\mathbb{R}^n} B\nabla w\cdot\nabla w\,dx - \int_{\mathbb{R}^n}\rho w\,dx.$$

Hence $\mathcal{E}_A(w)\le\mathcal{E}_B(w)$ for every $w\in D^1(\mathbb{R}^n)$. Since the solutions of (2.10) are minimizers of these energies, compare with Lemma 2.1.1, we have

$$\mathcal{E}_A(u) = \min_{D^1(\mathbb{R}^n)}\mathcal{E}_A \le \min_{D^1(\mathbb{R}^n)}\mathcal{E}_B = \mathcal{E}_B(v).$$

Then

$$-\frac{1}{2}\int_{\mathbb{R}^n} A\,\nabla u\cdot\nabla u\,dx = \mathcal{E}_A(u)\le\mathcal{E}_B(v) = -\frac{1}{2}\int_{\mathbb{R}^n} B\,\nabla v\cdot\nabla v\,dx,$$

and

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}^n} B\nabla v\cdot\nabla v\,dx &= \int_{\mathbb{R}^n} B\nabla v\cdot\nabla v\,dx - \int_{\mathbb{R}^n}\rho v\,dx\\ &= \int_{\mathbb{R}^n}(B-A)\,\nabla v\cdot\nabla v\,dx + \int_{\mathbb{R}^n} A\,\nabla v\cdot\nabla v\,dx - \int_{\mathbb{R}^n}\rho v\,dx\\ &\ge \int_{\mathbb{R}^n}(B-A)\,\nabla v\cdot\nabla v\,dx - \frac{1}{2}\int_{\mathbb{R}^n} A\,\nabla u\cdot\nabla u\,dx,\end{aligned}$$

concluding the proof. □

The following corollary is an immediate consequence of the above lemma.

Corollary 2.1.5. *Let $E\subset F\subset\mathbb{R}^n$ be two sets of finite measure. Then*

$$\mathcal{G}_{\beta,K}(E)\ge\mathcal{G}_{\beta,K}(F).$$

Proof. Let (u_E,ρ_E) be the optimal pair for E and let v be a solution of

$$-\operatorname{div}(a_F\nabla v) = \rho_E.$$

Then (v,ρ_E) is admissible in the minimization problem defining $\mathcal{G}_{\beta,K}(F)$, hence

$$\mathcal{G}_{\beta,K}(F)\le\int_{\mathbb{R}^n} a_F|\nabla v|^2\,dx + K\int_{\mathbb{R}^n}\rho_E^2\,dx \le \int_{\mathbb{R}^n} a_E|\nabla u_E|^2\,dx + K\int_{\mathbb{R}^n}\rho_E^2\,dx = \mathcal{G}_{\beta,K}(E),$$

where the last inequality follows from Lemma 2.1.4. □

We conclude this section by proving the continuity of $\mathcal{G}$ under L^1 convergence. Recall that given two sets E and F, $E\Delta F := (E\cup F)\setminus(E\cap F)$ is their symmetric difference.

Proposition 2.1.6. *Let $\{E_h\}$ be a sequence of sets with $|E_h| =: V_h\to V>0$ and let E be such that $\lim_{h\to\infty}|E_h\Delta E| = 0$, so that in particular $|E| = V$. Assume that $\beta_h\to\beta$ and that $K_h\to K$ when $h\to\infty$. Then*

$$\lim_{h\to\infty}\mathcal{G}_{\beta_h,K_h}(E_h) = \mathcal{G}_{\beta,K}(E).$$

Moreover, ∇u_{E_h} and ρ_{E_h} converge in L^2 to ∇u_E and ρ_E respectively.

Proof. Note that by (2.3)

$$\sup_h \mathcal{G}_{\beta_h,K_h}(E_h) < +\infty. \tag{2.12}$$

Thus

$$\sup_h \int_{\mathbb{R}^n} |\nabla v_{E_h}|^2\,dx + \int_{\mathbb{R}^n} \rho_{E_h}^2\,dx < \infty.$$

Moreover,

$$a_h := a_{E_h} \xrightarrow{L^2} a_E = \mathbf{1}_{E^c} + \beta \mathbf{1}_E \qquad \text{as } h \to \infty.$$

In particular, if $(u_h, \rho_h) = (u_{E_h}, \rho_{E_h})$ is the minimizing pair for $\mathcal{G}_{\beta_h,K_h}(E_h)$, then up to subsequence there exist (u, ρ) such that

$$\nabla u_h \overset{L^2}{\rightharpoonup} \nabla u, \qquad a_h \nabla u_h \overset{L^2}{\rightharpoonup} a_E \nabla u, \qquad \rho_h \overset{L^2}{\rightharpoonup} \rho,$$

as $h \to \infty$. Since (u_h, ρ_h) are in $\mathcal{A}(E_h)$, one immediately deduces that $(u, \rho) \in \mathcal{A}(E)$ and thus, by lower semicontinuity,

$$\mathcal{G}_{\beta,K}(E) \le \int_{\mathbb{R}^n} a_E |\nabla u|^2\,dx + K \int_{\mathbb{R}^n} \rho^2\,dx \le \liminf_{h\to\infty} \left(\int a_h |\nabla u_h|^2\,dx + K_h \int \rho_h^2\,dx \right).$$

To prove the opposite inequality we take (u_E, ρ_E) to be the minimizing pair for $\mathcal{G}_{\beta,K}(E)$ and we define $(w_h, \widetilde{\rho}_h) \in \mathcal{A}(E_h)$ as

$$\widetilde{\rho}_h = \sigma_h\, \rho_E \mathbf{1}_{E_h}, \qquad -\operatorname{div}(a_h \nabla w_h) = \widetilde{\rho}_h,$$

where $\sigma_h \to 1$ is such that $\int_{\mathbb{R}^n} \widetilde{\rho}_h\,dx = 1$. Since

$$-\operatorname{div}(a_h \nabla(u_E - w_h)) = -\operatorname{div}((a_h - a_E)\nabla u_E) + \rho_E(\mathbf{1}_E - \sigma_h \mathbf{1}_{E_h}),$$

by testing with $u_E - w_h$ and by exploiting the Sobolev embedding we obtain

$$\begin{aligned}
\|\nabla(u_E - w_h)\|_2^2 &\le \int_{\mathbb{R}^n} a_h(\nabla u_E - \nabla w_h)\cdot(\nabla u_E - \nabla w_h)\,dx \\
&= \int_{\mathbb{R}^n} (a_h - a_E)\nabla u_E \cdot \nabla(u_E - w_h)\,dx + \int_{\mathbb{R}^n} \rho_E(\mathbf{1}_E - \sigma_h \mathbf{1}_{E_h})\rho_E(u_E - w_h)\,dx \\
&\le \|(a_h - a_E)\nabla u_E\|_2 \|\nabla(u_E - w_h)\|_2 + S(n)\,\|\rho_E(\mathbf{1}_E - \sigma_h \mathbf{1}_{E_h})\|_2 \|\nabla(u_E - w_h)\|_2.
\end{aligned}$$

Then Young's inequality implies that $\|\nabla(u_E - w_h)\|_2 \to 0$. Since also $\|\widetilde{\rho}_h - \rho_E\|_2 \to 0$ and $(w_h, \widetilde{\rho}_h) \in \mathcal{A}(E_h)$, we get that

$$\begin{aligned}
\limsup_{h\to\infty} \mathcal{G}_{\beta_h,K_h}(E_h) &\le \lim_{h\to\infty} \left(\int_{\mathbb{R}^n} a_h |\nabla w_h|^2\,dx + K_h \int_{\mathbb{R}^n} \widetilde{\rho}_h^2\,dx \right) \\
&= \int_{\mathbb{R}^n} a_E |\nabla u_E|^2\,dx + K \int_{\mathbb{R}^n} \rho^2\,dx = \mathcal{G}_{\beta,K}(E).
\end{aligned}$$

Strong convergence of ∇u_{E_h} and ρ_{E_h} is now a simple consequence of the convergence of energies. □

2.2 Small volume adjustments and Euler Lagrange equations

In this section we show how to adjust the volume of a given set without increasing too much its energy which will be instrumental both to prove compactness of the class of minimizers in Section 2.4 and to get rid of the volume constraint in studying regularity of solutions of ($\mathcal{P}_{\beta,K,Q,R}$), see Section 2.3. The "adjustment" lemma will be proved with the aid of a deformation via a family of diffeomorphism close to the identity. Finally we also establish the Euler Lagrange equations associated to ($\mathcal{P}_{\beta,K,Q,R}$). We start with the following lemma.

Lemma 2.2.1. *For every $\eta \in C_c^\infty(B_R;\mathbb{R}^n)$ there exists $t_0 = t_0(\operatorname{dist}(\operatorname{spt}\eta, \partial B_R) > 0$ such that $\{\varphi_t\}_{|t|\le t_0}$ defined by $\varphi_t(x) := x + t\,\eta(x)$ is a family of diffeomorphisms of B_R into itself. Moreover for some set $E \subset B_R$ let (u,ρ) be a solution of*

$$-\operatorname{div}(a_E \nabla u) = \rho.$$

Then setting

$$u_t := u \circ \varphi_t^{-1} \qquad \text{and} \qquad \widetilde{\rho}_t := \det(\nabla\varphi_t^{-1})\,\rho \circ \varphi_t^{-1},$$

we have

$$-\operatorname{div}\left(a_{E_t} A_t \nabla u_t\right) = \widetilde{\rho}_t, \tag{2.13}$$

where $\|A_t - \operatorname{Id}\|_\infty = O(t)$ and the implicit constant depends only on $\|\nabla\eta\|_\infty$.

Proof. The proof of the first part of the Lemma is straightforward. For the second we see that for $\psi \in C_c^\infty$, by change of variables $x = \varphi_t(y)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \widetilde{\rho}_t(x)\,\psi(x)\,dx &= \int_{\mathbb{R}^n} \rho(y)\psi(\varphi_t(y))\det(\nabla\varphi_t^{-1})(\varphi_t(y))\det(\nabla\varphi_t(y))\,dy \\
&= \int_{\mathbb{R}^n} a_E(y)\nabla u(y)\cdot\nabla\left(\psi\circ\varphi_t\right)(y)\,dy \\
&= \int_{\mathbb{R}^n} a_E(y)\nabla u(y)\cdot\left(\nabla\varphi_t(y)\right)^T\nabla\psi(\varphi_t(y))\,dy \\
&= \int_{\mathbb{R}^n} a_{E_t}\nabla u\circ\varphi_t^{-1}\left(\nabla\varphi_t\circ\varphi_t^{-1}\right)^T\nabla\psi\det\nabla\varphi_t^{-1}\,dx \\
&= \int_{\mathbb{R}^n} a_{E_t}\left(\nabla\varphi_t^{-1}\right)^{-T}\nabla u_t\cdot(\nabla\varphi_t{}^{-1})^{-T}\nabla\psi\det\nabla\varphi_t^{-1}\,dx \\
&= \int_{\mathbb{R}^n} a_{E_t}\,A_t\,\nabla u_t\cdot\nabla\psi\,dx.
\end{aligned}$$

Where we have used the equality $\nabla\varphi\circ\varphi_t^{-1} = (\nabla\varphi_t^{-1})^{-1}$ and for a matrix N we denoted by N^T its transpose and by N^{-T} for $(N^{-1})^T$. Hence u_t is a solution of 2.13 with

$$A_t = \det\nabla\varphi_t^{-1}\left(\nabla\varphi_t^{-1}\right)^{-T}\left(\nabla\varphi_t^{-1}\right)^{-1}.$$

By explicit computation we see that A_t satisfies the desired bound. □

We now show how the energy $\mathcal{G}$ changes by the effect of a family of diffeomorphism.

Lemma 2.2.2. *Let $E \subseteq B_R$ be a measurable set and let $\{\varphi_t\}_{|t|\le t_0}$ be a family of diffeomorphisms as in Lemma 2.2.1. Then*

$$\mathcal{G}_{\beta,K}(E_t) \le (1 + O(t))\,\mathcal{G}_{\beta,K}(E), \tag{2.14}$$

where $E_t := \varphi_t(E)$ and the implicit constant depends only on $\|\nabla\eta\|_\infty$. Moreover,

$$\begin{aligned}(2.15)\quad \mathcal{G}_{\beta,K}(E_t) \le{}& \mathcal{G}_{\beta,K}(E)\\ &+t\left(\int_{\mathbb{R}^n} a_E\Big(|\nabla u_E|^2\operatorname{div}\eta - 2\nabla u_E\cdot\nabla\eta\,\nabla u_E\Big)\,dx - K\int_{\mathbb{R}^n}\rho_E^2\operatorname{div}\eta\,dx\right)+O(t^2).\end{aligned}$$

Proof. Let $(u_E,\rho_E)\in\mathcal{A}(E)$ be a the optimal pair for $\mathcal{G}_{\beta,K}(E)$. By Lemma 2.2.1 $u_t = u_E\circ\varphi_t^{-1}$ solves (2.13) with $\widetilde{\rho}_t = \rho_E\circ\varphi_t^{-1}\det(\nabla\varphi_t^{-1})$. Let v_t be the solution of

$$(2.16)\qquad -\operatorname{div}\left(a_{E_t}\nabla v_t\right) = \widetilde{\rho}_t \qquad \text{in}\quad \mathcal{D}'(\mathbb{R}^n).$$

Step 1: We start by proving the following estimate

$$(2.17)\qquad \int_{\mathbb{R}^n} a_{E_t}\left(|\nabla v_t|^2 - |\nabla u_t|^2\right)\,dx \le O(t)\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx,$$

where the implicit constant depends only on $\|\nabla\eta\|_\infty$. In order to prove (2.17) we claim that

$$(2.18)\qquad \left(\int_{\mathbb{R}^n} a_{E_t}|\nabla(u_t-v_t)|^2\,dx\right)^{1/2} \le O(t)\left(\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx\right)^{1/2}.$$

Indeed assuming that (2.18) holds true and using that $|a|^2-|b|^2 = 2b\cdot(a-b)+|a-b|^2$ for every $a,b\in\mathbb{R}^n$, we have

$$(2.19)\qquad \begin{aligned}\int_{\mathbb{R}^n} a_{E_t}\left(|\nabla v_t|^2 - |\nabla u_t|^2\right)\,dx ={}& 2\int_{\mathbb{R}^n} a_{E_t}\nabla u_t\cdot\nabla(v_t-u_t)\,dx\\ &+\int_{\mathbb{R}^n} a_{E_t}|\nabla(u_t-v_t)|^2\,dx.\end{aligned}$$

We estimate the first term in the right hand side of (2.19). By (2.18), we find that

$$(2.20)\qquad \begin{aligned}\int_{\mathbb{R}^n} a_{E_t}\,\nabla u_t\cdot\nabla(v_t-u_t)\,dx &\le \left(\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^n} a_{E_t}|\nabla(u_t-v_t)|^2\,dx\right)^{1/2}\\ &\le O(t)\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx.\end{aligned}$$

By (2.19) and (2.20), we have:

$$\int_{\mathbb{R}^n} a_{E_t}\left(|\nabla v_t|^2 - |\nabla u_t|^2\right)\,dx \le O(t)\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx + O(t^2)\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx,$$

which proves (2.17).

Let us now prove (2.18). By testing (2.13) and (2.16) with v_t-u_t we get

$$\begin{aligned}&\int_{\mathbb{R}^n} a_{E_t}\,\nabla v_t\cdot\nabla(v_t-u_t)\,dx = \int_{\mathbb{R}^n}\widetilde{\rho}_t\,(v_t-u_t)\,dx = \int_{\mathbb{R}^n} a_{E_t}\,A_t\,\nabla u_t\cdot\nabla(v_t-u_t)\,dx\\ &= \int_{\mathbb{R}^n} a_{E_t}\,(A_t-\mathrm{Id})\,\nabla u_t\cdot\nabla(v_t-u_t)\,dx + \int_{\mathbb{R}^n} a_{E_t}\,\nabla u_t\cdot\nabla(v_t-u_t)\,dx.\end{aligned}$$

Rearranging terms and recalling that $|A_t-\mathrm{Id}\,| = O(t)$, this gives

$$\int_{\mathbb{R}^n} a_{E_t}|\nabla(v_t-u_t)|^2\,dx \le O(t)\int_{\mathbb{R}^n}|\nabla v_t-\nabla u_t||\nabla u_t|\,dx,$$

which, by Young's inequality, implies (2.18).

Step 2: By changing of variables

$$\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx = \int_{\mathbb{R}^n} |(\nabla\varphi_t)^{-T}\nabla u|^2 \det\nabla\varphi_t\,dx.$$

Moreover

$$\nabla\phi_t = \mathrm{Id} + t\nabla\eta + o(t) \qquad \text{and} \qquad \det\nabla\phi_t = 1 + t\,\mathrm{div}\,\eta + o(t),$$

which gives

$$\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx = (1+O(t))\int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx. \tag{2.21}$$

and the more precise equality

$$\begin{aligned}\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx &= \int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx \\ &\quad + t\int_{\mathbb{R}^n} a_E\Big(\mathrm{div}\,\eta\,|\nabla u_E|^2 - 2\nabla u_E\cdot\nabla\eta\,\nabla u_E\Big)\,dx + o(t).\end{aligned} \tag{2.22}$$

In the same way we get

$$\int_{E_t}\widetilde{\rho}_t^2\,dx = \int_E \frac{\rho^2}{\det\nabla\varphi_t}\,dx = (1+O(t))\int_E \rho^2\,dx. \tag{2.23}$$

Furthermore, since $\det\nabla\varphi_t = 1 + t\,\mathrm{div}\,\eta + o(t)$, we also get

$$\int_{E_t}\widetilde{\rho}_t^2\,dx = \int_E \rho^2\,dx - t\int_E \rho_E^2\,\mathrm{div}\,\eta\,dx + o(t). \tag{2.24}$$

Step 4: Since, by its definition

$$\int_{\mathbb{R}^n}\widetilde{\rho}_t\,dx = 1, \qquad \widetilde{\rho}_t\mathbf{1}_{E_t^c} = 0,$$

and v_t solves (2.16), we see that $(v_t,\widetilde{\rho}_t)\in\mathcal{A}(E_t)$. Hence, by combining (2.17), (2.21) and (2.23) we obtain

$$\begin{aligned}\mathcal{G}_{\beta,K}(E_t) &\le \int_{\mathbb{R}^n} a_{E_t}|\nabla v_t|^2\,dx + K\int_{\mathbb{R}^n}\widetilde{\rho}_t^2\,dx \\ &\le (1+O(t))\int_{\mathbb{R}^n} a_{E_t}|\nabla u_t|^2\,dx + K\int_{\mathbb{R}^n}\widetilde{\rho}_t^2\,dx \\ &\le (1+O(t))\left(\int_{\mathbb{R}^n} a_E|\nabla u_E|^2\,dx + K\int_E \rho_E^2\,dx\right) = (1+O(t))\mathcal{G}_{\beta,K}(E),\end{aligned}$$

which proves (2.14). The proof of (2.15) is obtained by combining the above argument with (2.22) and (2.24). □

In the next corollary we deduce also the Euler Lagrange equation for minimizers of $(\mathcal{P}_{\beta,K,Q,R})$.

Corollary 2.2.3. *Let E be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then*

$$\int_{\partial^* E}\mathrm{div}_E\,\eta\,d\mathcal{H}^{n-1} + Q^2\left\{\int_{\mathbb{R}^n} a_E\Big(|\nabla u_E|^2\,\mathrm{div}\,\eta - 2\nabla u_E\cdot\nabla\eta\,\nabla u_E\Big)\,dx - K\int_{\mathbb{R}^n}\rho_E^2\,\mathrm{div}\,\eta\,dx\right\} = 0, \tag{2.25}$$

for all $\eta\in C_c^1(B_R;\mathbb{R}^n)$ with $\int_E \mathrm{div}\,\eta\,dx = 0$.

Proof. Let E be a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) and let $\{\varphi_t\}_{|t|\le t_0}$ be a family of diffeomorphisms as in Lemma 2.2.1 with $\int_E \operatorname{div}\eta\, dx = 0$. Since $|E_t| = V$ then E_t is a competitor of the problem. Hence, by the minimality of E we have

$$0 \le \frac{\mathcal{F}_{\beta,K,Q}(E_t) - \mathcal{F}_{\beta,K,Q}(E)}{t} = \frac{P(E_t) - P(E)}{t} + Q^2 \left\{ \frac{\mathcal{G}_{\beta,K}(E_t) - \mathcal{G}_{\beta,K}(E)}{t} \right\}, \ \forall t \in (0, |t_0|)\,.$$

By combining the Taylor expansion of the perimeter, [23, Theorem 17.8],

$$P(\varphi_t(E)) = P(E) + t \int_{\partial^* E} \operatorname{div}_E \eta\, d\mathcal{H}^{n-1} + o(t), \quad \text{where} \quad \operatorname{div}_E \eta = \operatorname{div}\eta - \nu_E \cdot \nabla\eta\, \nu_E,$$

with (2.15) we obtain

$$\begin{aligned} 0 \le \liminf_{t\to 0^+} \frac{\mathcal{F}_{\beta,K,Q}(E_t) - \mathcal{F}_{\beta,K,Q}(E)}{t} &\le \int_{\partial^* E} \operatorname{div}_E \eta\, d\mathcal{H}^{n-1} \\ &+ Q^2 \left\{ \int_{\mathbb{R}^n} a_E \Big(|\nabla u_E|^2 \operatorname{div}\eta - 2\nabla u_E \cdot \nabla\eta\, \nabla u_E \Big)\, dx - K \int_{\mathbb{R}^n} \rho_E^2 \operatorname{div}\eta\, dx \right\}. \end{aligned}$$

By replacing η with $-\eta$ in the above inequality we obtain (2.2.3). □

The next series of results are modeled after [2] and allow to do small volume adjustments without increasing too much the perimeter, see also [23, Chapter 17]. The first lemma is elementary.

Lemma 2.2.4. *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter and let U be an open set such that $P(E,U) > 0$. Then there exist $\gamma = \gamma(E) > 0$ and a vector field $\eta_E \in C^1_c(U;\mathbb{R}^n)$ with $\|\eta\|_{C^1} \le 1$ such that*

$$\int_E \operatorname{div}\eta_E\, dx \ge \gamma(E) > 0.$$

Proof. Since

$$P(E,U) = \sup\left\{ \int_E \operatorname{div}\eta\, dx \ : \ \eta \in C^1_c(U;\mathbb{R}^n),\ \|\eta\|_\infty \le 1 \right\},$$

we find a vector field $\widetilde{\eta} \in C^1_c(U;\mathbb{R}^n)$ with $\|\widetilde{\eta}\|_\infty \le 1$ such that

$$\int_E \operatorname{div}\widetilde{\eta}\, dx \ge P(E,U)/2.$$

Taking $\eta = \widetilde{\eta}/\|\widetilde{\eta}\|_{C^1}$ we obtain the desired conclusion. □

In order to have uniform controls on the constants involved in our regularity theory, we need to upgrade the above lemma in the following one. Note that this time the constants depend only on the upper bound on the perimeter, in particular they do not depend on R.

Lemma 2.2.5. *For every $P > 0$ there exist two constants $\bar\gamma = \bar\gamma(n,P) > 0$ and $\bar\delta = \bar\delta(n,P) > 0$ such that if $R \in (1,\infty)$ and $E \subset B_R$ satisfies*

$$\frac{|B_1|}{2} \le |E| \le \frac{3|B_1|}{2}, \qquad P(E) \le P\,, \tag{2.26}$$

then there exists a vector field $\eta \in C^1_c(B_{R-\bar\delta};\mathbb{R}^n)$ with $\|\eta\|_{C^1} \le 1$ such that

$$\int_E \operatorname{div}\eta\, dx \ge \bar\gamma.$$

Proof. Let us argue by contradiction: assume that there exist a sequence of radii R_k and a sequence of sets E_k satisfying (2.26) such that

$$\int_{E_k} \operatorname{div} \eta \, dx \to 0 \qquad \text{for all } \eta \in C^1_c(B_{R_k-\bar\delta};\mathbb{R}^n), \text{ with } \|\eta\|_{C^1} \le 1, \tag{2.27}$$

where $\bar\delta = \bar\delta(n,P)$ is a small constant to be fixed later only in dependence of n and P. By [23, Remark 29.11] there exist points $y_k \in \mathbb{R}^n$ and a constant $\delta_1 = \delta_1(n,P)$ such that

$$|E_k \cap B_1(y_k)| \ge 2\delta_1.$$

Then by taking $z_k \in E_k \cap B_1(y_k) \subset B_{R_k} \cap B_1(y_k)$ we get

$$|E_k \cap B_2(z_k)| \ge 2\delta_1 \qquad z_k \in B_{R_k}.$$

Let us now detail the proof in the case in which, up to subsequences, $R_k \to \infty$ and $\partial B_{R_k-1} \cap B_2(z_k) \ne \emptyset$. The other cases are actually simpler and we explain how to modify the argument at the end of the proof. We first note that since $\partial B_{R_k-1} \cap B_2(z_k) \ne \emptyset$, we can take $x_k \in \partial B_{R_k}$ such that

$$|E_k \cap B_4(x_k)| \ge |E_k \cap B_2(y_k)| \ge 2\delta_1 \qquad x_k \in \partial B_{R_k}.$$

Now a simple geometric argument ensures that

$$\lim_{\delta\to 0} \sup_k |B_4(x_k) \cap (B_{R_k} \setminus B_{R_k-\delta})| \to 0.$$

In particular we can choose $\delta_2 = \delta_2(n,P)$ such that

$$|E_k \cap B_4(x_k) \cap B_{R_k-\delta_2}| \ge \delta_1. \tag{2.28}$$

Let us now assume that, up to subsequences and a possible rotation of coordinates

$$F_k := \big(E_k \cap B_4(x_k) \cap B_{R_k-\delta_2}\big) - x_k \to F, \quad \frac{x_k}{R_k} \to e_1,$$

where the first limit exits due to our assumption on the perimeters. In particular

$$B_4 \cap B_{R_k-\delta_2}(-x_k) \to \widehat{B} = B_4(0) \cap \{x_1 < -\delta_2\},$$

and $F \subset \widehat{B}$. Note that by (2.28), $F \ne \emptyset$ and, since $|F_k| \le 3|B_1|/2$, $|\widehat{B} \setminus F| > 0$. In particular, $P(F,\widehat{B}) > 0$. By Lemma 2.2.4, we can find a constant $\gamma = \gamma(F) > 0$ and vector field $\eta_F \in C^1_c(\hat{B};\mathbb{R}^n)$ with $\|\eta\|_{C^1} \le 1$ such that

$$\gamma \le \int_F \operatorname{div} \eta_F \, dx.$$

For k large the vector field $\eta_k(\cdot) = \eta_F(\cdot + x_k)$ satisfies $\eta_k \in C^1_c(B_{R_k-\delta_2/2};\mathbb{R}^n)$, $\|\eta\|_{C^1} \le 1$ and contradicts (2.27) with $\bar\delta = \delta_2/2$.

Let us conclude by explaining how to modify the proof in the case in which either $B_2(z_k) \cap \partial B_{R_k-1} = \emptyset$ or $R_k \to \bar R < \infty$. In the first case one argue as above by considering the sets $F_k := \big(E_k \cap B_2(z_k)\big) - z_k$ and by noticing that the vector fields $\eta_k(\cdot) = \eta_F(\cdot + y_k)$ with $F := \lim F_k$ are compactly supported in $B_{R_k-1/2}$. In the second case one can simply reproduce the above argument. □

The next proposition will be crucial in removing the volume constraint and in making comparison estimates for minimizers of ($\mathcal{P}_{\beta,K,Q,R}$).

Proposition 2.2.6. *For every $P > 0$ there exist constants $\bar\sigma = \bar\sigma(n, P) > 0$ and $C = C(n)$ such that if $R \in (1, \infty)$ and $E \subset B_R$ satisfies*

$$\frac{|B_1|}{2} \le |E| \le \frac{3|B_1|}{2}, \qquad P(E) \le P\,,$$

then for all $\sigma \in (-\bar\sigma, \bar\sigma)$ there exists $F_\sigma \subset B_R$ such that

$$|F_\sigma| = |E| + \sigma \qquad \text{and} \qquad |\mathcal{F}_{\beta,K,Q}(F_\sigma) - \mathcal{F}_{\beta,K,Q}(E)| \le C|\sigma|\,\mathcal{F}_{\beta,K,Q}(E).$$

Proof. By Lemma 2.2.5 we can find $\bar\gamma = \bar\gamma(n, P) > 0$, $\bar\delta = \bar\delta(n, P)$ and a vector field $\eta \in C^1_c\big(B_{R-\bar\delta}; \mathbb{R}^n\big)$ with $\|\eta\|_{C^1} \le 1$ such that

$$\bar\gamma \le \int_E \operatorname{div}\eta\, dx. \tag{2.29}$$

Define a family of diffeomorphisms $\varphi_t := \mathrm{Id} + t\,\eta$ and note that, since $\operatorname{dist}(\operatorname{spt}(\eta, \partial B_R) \ge \bar\delta(n, P)$, they send B_R into itself for $|t| \le t_0(n, P)$. By Taylor expansion

$$|E_t| = |E| + t\int_E \operatorname{div}\eta\, dx + O(t^2)|E|. \tag{2.30}$$

and

$$P(E_t) = P(E) + t\int_{\partial^* G} \operatorname{div}_E \eta\, d\mathcal{H}^{n-1} + O(t^2)P(E),$$

where the implicit constants depends only on $\|\nabla\eta\|_\infty \le 1$. Moreover

$$\mathcal{G}_{\beta,K}(E_t) \le (1 + C|t|)\mathcal{G}_{\beta,K}(E), \tag{2.31}$$

where $E_t = \varphi_t(E)$ and the constant in (2.31) depends only on $\|\nabla\eta\|_\infty \le 1$. Hence we can find $t_1 = t_1(n, P) > 0$ such that

$$\big||E_t| - |E|\big| \ge |t|\,\frac{\bar\gamma}{2} \qquad \text{(by (2.29))}, \tag{2.32a}$$

and

$$|\mathcal{F}_{\beta,K,Q}(E_t) - \mathcal{F}_{\beta,K,Q}(E)| \le C|t|\mathcal{F}_{\beta,K,Q}(E). \tag{2.32b}$$

for every $|t| \le t_1$. By equations (2.32a) and (2.32b) we get

$$|\mathcal{F}_{\beta,K,Q}(E_t) - \mathcal{F}_{\beta,K,Q}(E)| \le C\mathcal{F}_{\beta,K,Q}(E)\big||E_t| - |E|\big|.$$

Let $g(t) := |E_t|$ and note that thanks to (2.30) and (2.29), g is increasing in a neighborhood of 0. Take $\bar\sigma > 0$ such that $(|E| - \bar\sigma, |E| + \bar\sigma) \subseteq g\big((-t_1, t_1)\big)$. Then for every $|\sigma| \le \bar\sigma$ there exists $t_\sigma > 0$ such that $|E_{t_\sigma}| = |E| + \sigma$. Setting $F_\sigma = E_{t_\sigma}$ we obtain the desired conclusion. □

2.3 Λ-minimality and local variations

In order to study the regularity of minimizers it will be convenient to understand what is the behavior under small perturbations in balls. In this section we start by removing the volume constraint by showing that minimizers are Λ-minimizer of $\mathcal{F}$ under small perturbations. In order to keep track of the dependence of the parameters in Theorem 2, it will be important that this "almost"-minimality depends only on the structural parameter of the problem. We thus start by fixing the following convention, which will be in force throughout all the rest of the thesis.

Convention 2.3.1 (Universal constants)**.** Given $A > 0$, we say that

- the parameters β, K, Q with $\beta \geq 1$ are *controlled by* A if
$$\beta + K + \frac{1}{K} + Q \leq A.$$
- A constant is *universal* if it depends only on the dimension n and on A.
- For two positive quantities X and Y, we will write $X \lesssim Y$ if there exists a universal constant C such that $X \leq CY$ and we write $X \gtrsim Y$ if $Y \lesssim X$.

Note in particular that universal constants *do not depend* on the size of the container where the minimization problem is solved. Moreover we also remark here the following elementary fact: since B_1 is always a competitor for ($\mathcal{P}_{\beta,K,Q,R}$), if E is a minimizer then

$$P(E) \leq \mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(B_1) \leq C(n, A), \tag{2.33}$$

whenever β, K, Q are controlled by A.

Let us now introduce the following perturbed minimality condition.

Definition 2.3.2 (($\Lambda, \bar{r}$)-minimizer)**.** We say that E is a $(\Lambda, \bar{r})$-minimizer of the energy $\mathcal{F}$ if there exist constants $\Lambda > 0$ and $\bar{r} > 0$ such that for every ball $B_r(x) \subseteq \mathbb{R}^n$ with $r \leq \bar{r}$ we have

$$\mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(F) + \Lambda\,|E\Delta F| \qquad \text{whenever } E\Delta F \subset B_{\bar{r}}(x). \tag{2.34}$$

Remark 2.3.3. Note that if E is $(\bar{\Lambda}, \bar{r})$-minimizer than it is also a $(\bar{\Lambda}_1, \bar{r}_1)$-minimizer whenever $\bar{\Lambda}_1 \geq \bar{\Lambda}$ and $\bar{r}_1 \leq \bar{r}$. Hence there is no loss of generality in assuming that $\bar{r} \leq 1$.

We can now establish the desired Λ-minimality property for minimizers of ($\mathcal{P}_{\beta,K,Q,R}$).

Proposition 2.3.4. *Let $A > 0$ and let β, K, Q with $\beta \geq 1$ be controlled by A and let $R \geq 1$. Then there exist $\Lambda_1, \bar{r}_1 > 0$ universal such that all minimizers ($\mathcal{P}_{\beta,K,Q,R}$) satisfy*

$$\mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(F) + \Lambda_1\,|E\Delta F|,$$

whenever $F \subset B_R$ and $E\Delta F \subset B_r(x_0)$, $r \leq \bar{r}_1$.

Proof. Clearly we can suppose that

$$\mathcal{F}_{\beta,K,Q}(F) \leq \mathcal{F}_{\beta,K,Q}(E) \lesssim 1\,,$$

since otherwise the result is trivial. In particular $P(F)$ is bounded by a universal constant P. Let $\bar{\sigma}$ and C be the parameters in Proposition 2.2.6 associated to P. If $\bar{r}_1$ is chosen small enough we have

$$|E\Delta F| \leq \omega_n \bar{r}_1^n < \bar{\sigma}.$$

Moreover, since $|E| = |B_1|$, $|F| \in (|B_1|/2, 3|B_1|/2)$. Hence we can apply Proposition 2.2.6 to F to obtain a set $\widetilde{F} \subset B_R$ such that $|\widetilde{F}| = |B_1|$ and

$$\mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(\widetilde{F}) \leq \big(1 + C\big||\widetilde{F}| - |F|\big|\big)\, \mathcal{F}_{\beta,K,Q}(F), \tag{2.35}$$

where the first inequality is due to the minimality of E. Since $\mathcal{F}_{\beta,K,Q}(F) \lesssim 1$ and

$$\big||\widetilde{F}| - |F|\big| = ||E| - |F|| \leq |F\Delta E|,$$

we obtain the conclusion for a suitable universal constant Λ_1.

□

We conclude this section by establishing the following "local" minimality properties of minimizers ($\mathcal{P}_{\beta,K,Q,R}$). Note that in (ii) below we are not requiring F to be contained in B_R.

Proposition 2.3.5. *Let $A > 0$, and let β, K, Q be controlled by A and $R \geq 1$. Then there exist universal constants Λ_2 and $\bar{r}_2$ such that given a minimizer E of ($\mathcal{P}_{\beta,K,Q,R}$) we have that it satisfies the following two properties:*

(i) *for every set of finite perimeter $F \subseteq E$ with $E \setminus F \subset B_r(x)$ and $r \leq \bar{r}_2$ it holds:*

$$P(E) \leq P(F) + \Lambda_2|E \setminus F| + \Lambda_2 Q^2 \int_{E\setminus F} |\nabla u_E|^2\, dx, \tag{2.36}$$

where u_E the minimizer in (1.16).

(ii) *For every set of finite perimeter $F \supseteq E$ with $F \setminus E \subset B_r(x)$ and $r \leq \bar{r}_2$ it holds:*

$$P(E) \leq P(F) + \Lambda_2|F \setminus E|. \tag{2.37}$$

In particular, if u_E is the minimizer in (1.16) it holds:

$$P(E) \leq P(F) + \Lambda_2|E\Delta F| + \Lambda_2 Q^2 \int_{E\Delta F} |\nabla u_E|^2\, dx, \tag{2.38}$$

whenever $F\Delta E \subset B_r(x)$ with $r \leq \bar{r}_2$.

Proof. We start proving (i). Let E be a minimizer and (u_E, ρ_E) be the minimizing pair for $\mathcal{G}(E)$. Let $F \subseteq E$ be such that $E \setminus F \subset B_r(x)$ with $r \leq \bar{r}_1$ where $\bar{r}_1$ is the constant defined in Proposition 2.3.4, by possibly choosing r_1 smaller, we can assume that

$$|F| \geq \frac{|E|}{2} = \frac{|B_1|}{2}. \tag{2.39}$$

Let us set

$$\rho = (\rho_E + \lambda_F)\, \mathbf{1}_F \qquad \text{where} \qquad \lambda_F = \frac{\int_{E\setminus F} \rho_E\, dx}{|F|},$$

and let u be the solution of

$$-\operatorname{div}(a_F \nabla u) = \rho.$$

Note that $(u, \rho) \in \mathcal{A}(F)$ and thus, by using the Λ-minimality of E established in Proposition 2.3.4,

$$P(E) + Q^2\Big(\int_{\mathbb{R}^n} a_E|\nabla u_E|^2\, dx + K\int \rho_E^2\, dx\Big) \leq P(F) + Q^2\Big(\int_{\mathbb{R}^n} a_F|\nabla u|^2\, dx + K\int \rho^2\, dx\Big) + \Lambda|E \setminus F|.$$

Item (i) will then follow if we can prove

$$\int_{\mathbb{R}^n} \left(\rho^2 - \rho_E^2\right)\, dx \lesssim |E \setminus F|, \tag{2.40}$$

and

$$\int_{\mathbb{R}^n} \left(a_F\, |\nabla u_F|^2 - a_E |\nabla u|^2\right)\, dx \lesssim |E \setminus F| + \int_{E\setminus F} |\nabla u_E|^2\, dx. \tag{2.41}$$

To prove (2.40) we estimate

$$\begin{aligned} \int_{\mathbb{R}^n} (\rho^2 - \rho_E^2)\, dx &= - \int_{E\setminus F} \rho_E^2\, dx + \int_F (\lambda_F^2 + 2\rho_E\, \lambda_F)\, dx \\ &\le - \int_{E\setminus F} \rho_E^2\, dx + \frac{|E\setminus F|}{|F|} \int_{E\setminus F} \rho_E^2\, dx + 2\|\rho_E\|_\infty \int_{E\setminus F} \rho_E\, dx \\ &\le 2\|\rho_E\|_\infty^2 |E\setminus F|, \end{aligned}$$

where in the first inequality we have used (2.39) and the definition of λ_F. By (2.6), $\|\rho_E\|_\infty \lesssim 1$ and this concludes the proof of (2.40).

Let us now prove (2.41). First note that

$$\begin{aligned} \int_{\mathbb{R}^n} \left(a_F|\nabla u|^2 - a_E|\nabla u_E|^2\right)\, dx &= \int_{\mathbb{R}^n} a_F(\, |\nabla u|^2 - |\nabla u_E|^2)\, dx \\ &\quad + \int_{\mathbb{R}^n} (a_F - a_E)\, |\nabla u_E|^2\, dx. \end{aligned} \tag{2.42}$$

Testing the equations satisfied by u_E and u with u_E and u respectively and subtracting the result we obtain also

$$\int_{\mathbb{R}^n} \left(a_F|\nabla u|^2 - a_E|\nabla u_E|^2\right)\, dx = \int_{\mathbb{R}^n} u\rho\, dx - \int_{\mathbb{R}^n} u_E\rho_E\, dx. \tag{2.43}$$

Subtracting (2.42) from two times (2.43) we get

$$\begin{aligned} \int_{\mathbb{R}^n} \left(a_F|\nabla u|^2 - a_E|\nabla u_E|^2\right)\, dx &= \int_{\mathbb{R}^n} a_F\left(|\nabla u_E|^2 - |\nabla u|^2\right)\, dx \\ &\quad + \int_{\mathbb{R}^n} (a_E - a_F)\, |\nabla u_E|^2\, dx \\ &\quad + 2\int_{\mathbb{R}^n} u\rho\, dx - 2\int_{\mathbb{R}^n} u_E\rho_E\, dx. \end{aligned} \tag{2.44}$$

Moreover,

$$\begin{aligned} \int_{\mathbb{R}^n} a_F\left(|\nabla u_E|^2 - |\nabla u|^2\right)\, dx &= 2\int_{\mathbb{R}^n} a_F\, \nabla u\cdot(\nabla u_E - \nabla u)\, dx \\ &\quad + \int_{\mathbb{R}^n} a_F\, |\nabla u_E - \nabla u|^2\, dx \\ &= 2\int_{\mathbb{R}^n} \rho(u_E - u)\, dx \\ &\quad + \int_{\mathbb{R}^n} a_F|\nabla u_F - \nabla u|^2\, dx. \end{aligned} \tag{2.45}$$

Combining (2.44) and (2.45) we then obtain:

$$\begin{aligned} \int_{\mathbb{R}^n} \left(a_F|\nabla u|^2 - a_E|\nabla u_E|^2\right)\, dx &= 2\int_{\mathbb{R}^n} (\rho - \rho_E)\, u_E\, dx + \int_{\mathbb{R}^n} a_F\, |\nabla u - \nabla u_E|^2\, dx \\ &\quad + \int_{\mathbb{R}^n} (a_E - a_F)\, |\nabla u_E|^2\, dx. \end{aligned} \tag{2.46}$$

We start to estimate the first term in the right hand side of (2.46). By using Proposition 2.1.3 and by arguing as in the proof of (2.40) the first term can be easily estimated as

$$\int_{\mathbb{R}^n} (\rho - \rho_E)\, u_E \, dx \lesssim |E \setminus F|.$$

To estimate the second term in the right hand side of (2.46), we write

$$-\operatorname{div}\big(a_F(\nabla u - \nabla u_E)\big) = \rho - \rho_E + \operatorname{div}\big((a_F - a_E)\,\nabla u_E\big).$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} a_F\, |\nabla u - \nabla u_E|^2 \, dx &= \int_{\mathbb{R}^n} (\rho - \rho_E)\,(u - u_E)\, dx \\ &\quad + \int_{\mathbb{R}^n} (a_E - a_F)\,\nabla u_E \cdot (\nabla u - \nabla u_E)\, dx \\ &\le \|\rho - \rho_E\|_{(2^*)'}\, \|u - u_E\|_{2^*} \\ &\quad + \left(\int_{\mathbb{R}^n} (a_F - a_E)^2\, |\nabla u_E|^2\right)^{\frac{1}{2}} \|\nabla u - \nabla u_E\|_2.\end{aligned}$$

By the Sobolev embedding and Young inequality (and recalling that $1 \le a_F \le \beta$), the above inequality immediately implies

$$\int_{\mathbb{R}^n} a_F\, |\nabla u - \nabla u_E|^2 \, dx \lesssim \int_{\mathbb{R}^n} (a_F - a_E)^2\, |\nabla u_E|^2 \, dx + \|\rho - \rho_E\|^2_{(2^*)'}.$$

By the definition of ρ, the second term is $\lesssim |E \setminus F|$ (note that $2/(2^*)' \ge 1$) while the first one is less than

$$\beta^2 \int_{E\setminus F} |\nabla u_E|^2 \, dx.$$

Since also the third term in (2.46) can be estimated by the above integral, this concludes the proof of (2.41).

Let us now prove (ii). Let $F \supseteq E$, note that $P(F \cap B_R) \le P(F)$ and that $(F \cap B_R) \setminus E \subset F \setminus E$. Hence if we can prove (i) for subsets of B_R we will get it for all sets. Let us then assume that $E \subseteq F \subseteq B_R$. By Λ-minimality of E

$$\mathcal{F}_{\beta,K,Q}(E) \le \mathcal{F}_{\beta,K,Q}(F) + \Lambda |F \setminus E|.$$

Since, by Lemma 2.1.4, $\mathcal{G}_{\beta,K}(E) \ge \mathcal{G}_{\beta,K}(F)$ the conclusion follows. □

Remark 2.3.6. We record here the following simple consequence of (2.38). Assume that $|\nabla u_E|^2 \in L^p$, then (2.38) and Hölder inequality imply that for F such that $F \Delta E \subset B_r(x)$ with $r \le \bar{r}_2$,

$$\begin{aligned}P(E) &\le P(F) + \Lambda_2 |E\Delta F| + \Lambda_2 Q^2 \int_{E\Delta F} |\nabla u|^2 \, dx \\ &\le P(F) + \Lambda_2 |B_r| + \Lambda_2 Q^2 |B_r|^{1-\frac{1}{p}} \|\nabla u_E\|^2_{2p} \le P(F) + C r^{n-\frac{n}{p}}.\end{aligned}$$

In particular if $p > n$, then $n - \frac{n}{p} > n - 1$ and thus E is a ω minimizers of the perimeter in the sense of [30]. Hence ∂E is a C^1 manifold outside a singular closed set Σ of dimension at most $(n-8)$. Note that by Cordes estimate, [5], the assumption $|\nabla u_E|^2 \in L^p$ with $p > n$ is satisfied wherever $\beta - 1 \ll 1$. In particular, in this regime, Taylor cones singularities are excluded in $\mathbb{R}^3$.

2.4 Compactness of minimizers

In this section we prove that the class of minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) is a compact subset of L^1.

Proposition 2.4.1. *Let $K_h, Q_h \in \mathbb{R}$, $\beta_h \geq 1$ and $R_h \geq 1$ be such that*

$$K_h \to K > 0\,, \quad \beta_h \to \beta \geq 1\,, \quad R_h \to R \geq 1\,, \quad Q_h \to Q \geq 0,$$

when $h \to \infty$. For every $h \in \mathbb{N}$ let E_h be a minimizer of ($\mathcal{P}_{\beta_h,K_h,Q_h,R_h}$).
Then, up to a non relabelled subsequence, there exists a set of finite perimeter E such that

$$\lim_{h\to\infty} |E \Delta E_h| = 0. \tag{2.47}$$

Moreover, E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) and

$$\mathcal{F}_{\beta,K,Q}(E) = \lim_{h\to\infty} \mathcal{F}_{\beta_h,K_h,Q_h}(E_h), \qquad \lim_{h\to\infty} P(E_h) = P(E).$$

Proof. Since if $R_h = 1$ for all h the problem is trivial (recall that $|E_h| = |B_1|$) we can assume that R_h and R are strictly bigger than one. Moreover B_1 is always an admissible competitor and thus,

$$\limsup_{h\to\infty} \mathcal{F}_{\beta_h,K_h,Q_h}(E_h) \leq \limsup_{h\to\infty} \mathcal{F}_{\beta_h,K_h,Q_h}(B_1) = C(n, K, Q, \beta).$$

In particular the perimeters of E_h are uniformly bounded and since all the sets are included in, say, B_{2R} there exists a non relabelled subsequence and set $E \subset B_R$ such that (2.47) hold true. Since the perimeter is lower-semicontinuous and, by Proposition 2.1.6, $\mathcal{G}$ is continuous we also get that

$$\mathcal{F}_{\beta,K,Q}(E) \leq \liminf_{h\to\infty} \mathcal{F}_{\beta_h,K_h,Q_h}(E_h). \tag{2.48}$$

We now show that E is a minimizer. For let $F \subset B_R$ with $|F| = |B_1|$. Since $R_h \to R$, we can find $\lambda_h \to 1$ such that $F_h := \lambda_h F \subset B_{R_h}$. Clearly, $F_h \to F$, $|F_h| = |F| + o(1)$ and $P(F_h) = P(F) + o(1)$. Thus

$$\mathcal{F}_{\beta,K,Q}(F) = \mathcal{F}_{\beta_h,K_h,Q_h}(F_h) + o(1). \tag{2.49}$$

By Proposition 2.2.6 applied to F_h we can find sets $\widetilde{F}_h \subset B_{R_h}$ such that $|\widetilde{F}_h| = |B_1|$ and

$$\mathcal{F}_{\beta_h,K_h,Q_h}(\widetilde{F}_h) = \mathcal{F}_{\beta_h,K_h,Q_h}(F_h) + o(1) = \mathcal{F}_{\beta,K,Q}(F) + o(1).$$

where in the last equality we have used (2.49). By minimality of E_h we get

$$\mathcal{F}_{\beta_h,K_h,Q_h}(E_h) \leq \mathcal{F}_{\beta_h,K_h,Q_h}(\widetilde{F}_h) = \mathcal{F}_{\beta,K,Q}(F) + o(1),$$

which combined with (2.48) implies the minimality of E. By choosing $E = F$ we also deduce the convergence of the energies and, by Proposition 2.1.6, this implies the convergence of the perimeters. □

2.5 Decay of the Dirichlet energy and density estimates

The goal of this section is to deduce some density estimates for the volume and the perimeter of minimizers which are crucial in the proofs of the regularity results.

2.5.1 Decay of the Dirichlet energy

Following [15], in this subsection we establish an almost Lipschitz decay for the Dirichlet energy of u_E in certain regimes. Namely when the set or the complement almost fill a ball or when the set is very close to an half space.

We start by recalling the following higher integrability theorem. The proof can be found for instance in [18].

Theorem 2.5.1. *Let $f \in L^1(B_r)$ and $g \in L^q(B_r)$ for some $q > 1$. Assume that whenever $B_s \subset\subset B_{2s} \subset\subset B_r$ we have*

$$\fint_{B_s} f\,dx \leq B\left\{\left(\fint_{B_{2s}} f^m\,dx\right)^{\frac{1}{m}} + \fint_{B_{2s}} g\,dx\right\}, \tag{2.50}$$

for some $B > 0$ and $0 < m < 1$. Then there exists $p > 1$ such that $f \in L^p(B_{r/2})$ and

$$\fint_{B_{\frac{r}{2}}} f^p\,dx \leq B\left\{\left(\fint_{B_r} f\,dx\right)^p + \fint_{B_r} g^p\,dx\right\}. \tag{2.51}$$

Thanks to Theorem 2.5.1 we obtain higher integrability for ∇u_E.

Lemma 2.5.2. *Let E be a set of finite measure and let $(u,\rho) \in \mathcal{A}(E)$. Then there exists $C = C(n,\beta)$ and $p = p(n,\beta) > 1$ such that for all balls $B_r(x) \subset \mathbb{R}^n$*

$$\fint_{B_r(x)} |\nabla u|^{2p}\,dx \leq C\left\{\left(\fint_{B_{2r}(x)} |\nabla u|^2\,dx\right)^p + r^{2p}\fint_{B_{2r}(x)} \rho^{2p}\,dx\right\}. \tag{2.52}$$

Furthermore, the constants C and p depend only on an upper bound for β.

Proof. Without loss of generality assume $x = 0$ and by scaling $r = 1$. Let $B_s \subset\subset B_{2s} \subset\subset B_1$. Take a cutoff function $\zeta \in C_c^\infty(B_{2s})$ such that $\zeta = 1$ on B_s, $|\nabla\zeta| \leq \frac{2}{s}$ and $|\zeta| \leq 1$. Denote by $u_{2s} := \fint_{B_{2s}} u\,dx$. By testing (1.13) with $\varphi := (u - u_{2s})\zeta^2$ we get

$$\int_{\mathbb{R}^n} a_E\,\zeta^2|\nabla u|^2\,dx + 2\int_{\mathbb{R}^n} a_E\,(u - u_{2s})\zeta\,\nabla\zeta\cdot\nabla u\,dx = \int_{\mathbb{R}^n} \rho\,(u - u_{2s})\zeta^2\,dx. \tag{2.53}$$

Thus (2.53) yields

$$\int_{B_{2s}} a_E\,\zeta^2|\nabla u|^2\,dx \leq \int_{B_{2s}} |\rho|\,|u - u_{2s}|\,|\zeta|^2\,dx + 2\beta\int_{B_{2s}} |u - u_{2s}|\,|\zeta|\,|\nabla\zeta|\,|\nabla u|\,dx. \tag{2.54}$$

By applying Young in (2.54) we obtain

$$\begin{aligned}\int_{B_{2s}} \zeta^2|\nabla u|^2\,dx \leq{}& \left(\frac{1}{2} + \frac{4\beta}{\vartheta}\right)\frac{1}{s^2}\int_{B_{2s}} |u - u_{2s}|^2\,dx + \frac{s^2}{2}\int_{B_{2s}} \rho^2\,dx\\ &+ \beta\,\vartheta\int_{B_{2s}} \zeta^2|\nabla u|^2\,dx,\end{aligned} \tag{2.55}$$

for every $\vartheta > 0$. Hence by choosing $\vartheta < \frac{1}{\beta}$ we get

$$\int_{B_{2s}} \zeta^2|\nabla u|^2\,dx \lesssim \frac{1}{s^2}\int_{B_{2s}} |u - u_{2s}|^2\,dx + s^2\int_{B_{2s}} \rho^2\,dx, \tag{2.56}$$

By applying the Sobolev-Poincaré inequality in (2.56) we have

$$\int_{B_s} |\nabla u|^2\,dx \lesssim \frac{1}{s^2}\left(\int_{B_{2s}} |\nabla u|^{2m}\,dx\right)^{\frac{1}{m}} + \int_{B_{2s}} \rho^2\,dx, \tag{2.57}$$

with $m = n\,(n+2)^{-1} < 1$. Divide (2.57) by $\mathcal{L}^n(B_s) = \omega_n\, s^n$:

$$\fint_{B_s} |\nabla u|^2\, dx \lesssim \left(\fint_{B_{2s}} |\nabla u|^{2m}\, dx \right)^{\frac{1}{m}} + \fint_{B_{2s}} \rho^2\, dx. \tag{2.58}$$

Therefore the hypothesis of Theorem 2.5.1 are satisfied with $f = |\nabla u|^2$ and $g = \rho^2$. Hence the higher integrability follows from Theorem 2.5.1. □

We start with the following elementary lemma where the optimal decay is obtained in some limit situations.

Lemma 2.5.3. *Let $\beta \geq 1$ and $\rho \in L^\infty(\mathbb{R}^n)$. Then there exists a dimensional constant $C = C(n) > 0$ such that:*

(i) *if $v \in W^{1,2}(B_r(x))$ is a solution of*

$$-\Delta v = \rho,$$

then for all $\lambda \in (0,1)$

$$\fint_{B_{\lambda r}(x)} |\nabla v|^2\, dx \leq C \fint_{B_r(x)} |\nabla v|^2\, dx + \frac{C}{\lambda^n} r^2\, \|\rho\|_\infty^2. \tag{2.59}$$

(ii) *If $v \in W^{1,2}(B_r(x))$ is a solution of*

$$-\operatorname{div}(a_H \nabla v) = \rho, \qquad a_H = \beta \mathbf{1}_H + \mathbf{1}_{H^c},$$

where $H := \{y \in \mathbb{R}^n : (y-x)\cdot e \leq 0\}$ for some $e \in \mathbb{S}^{n-1}$. Then for all $\lambda \in (0,1)$

$$\fint_{B_{\lambda r}(x)} a_H |\nabla v|^2\, dx \leq C \fint_{B_r(x)} a_H |\nabla v|^2\, dx + \frac{C}{\lambda^n} r^2\, \|\rho\|_\infty^2. \tag{2.60}$$

Proof. We just prove point (ii) since (i) is a particular case (and well known). By scaling and translating, we can assume without loss of generality that $x = 0$ and $r = 1$. Let w be the solution of

$$\begin{cases} -\operatorname{div}(a_H \nabla w) = 0 & \text{in } B_1 \\ w = v & \text{on } \partial B_1, \end{cases}$$

so that $u = v - w$ solves

$$\begin{cases} -\operatorname{div}(a_H \nabla u) = \rho & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

By multiplying the last equation by u, applying Poincaré inequality we obtain

$$\int_{B_1} a_H |\nabla u|^2\, dx \leq \|\rho\|_2 \|u\|_2 \leq C(n)\|\rho\|_\infty \|\nabla u\|_2 \leq C(n)\|\rho\|_\infty \left(\int_{B_1} a_H |\nabla u|^2\, dx \right)^{\frac{1}{2}}.$$

where we have used that $a_H \geq 1$. Hence

$$\int_{B_1} a_H |\nabla v - \nabla w|^2\, dx = \int_{B_1} a_H |\nabla u|^2\, dx \leq C\|\rho\|_\infty^2. \tag{2.61}$$

Moreover, by [15, Lemma 2.3],

$$\fint_{B_\lambda} a_H |\nabla w|^2\, dx \leq \fint_{B_1} a_H |\nabla w|^2\, dx.$$

Hence,

(2.62)

$$
\begin{aligned}
\fint_{B_\lambda} a_H|\nabla v|^2\,dx &\le 2\fint_{B_\lambda} a_H|\nabla w|^2\,dx + 2\fint_{B_\lambda} a_H|\nabla v-\nabla w|^2\,dx\\
&\le 2\fint_{B_1} a_H|\nabla w|^2\,dx + 2\fint_{B_\lambda} a_H|\nabla v-\nabla w|^2\,dx\\
&\le 4\fint_{B_1} a_H|\nabla v|^2\,dx + 2\fint_{B_\lambda} a_H|\nabla v-\nabla w|^2\,dx + 4\fint_{B_1} a_H|\nabla v-\nabla w|^2\,dx.
\end{aligned}
$$

which together with (2.61) concludes the proof. □

As in [15], we now exploit the higher integrability Lemma 2.5.2 to obtain an "almost version" of the above decay.

Proposition 2.5.4 (Decay of Dirichlet energy)**.** *Let $\beta \ge 1$ then there exists a constant $C = C(n,\beta)$ with the following property: if $E \subset \mathbb{R}^n$, u and ρ satisfy*

$$
-\operatorname{div}(a_E\nabla u) = \rho, \qquad a_E = \beta\mathbf{1}_E + \mathbf{1}_{E^c},
$$

then for all $\lambda \in \left(0,\frac{1}{2}\right)$ there exists $\varepsilon_0 = \varepsilon_0(\lambda,\beta) > 0$ such that

(i) *if*

$$
\text{either} \quad \frac{|E\cap B_r(x)|}{|B_r(x)|} \le \varepsilon_0 \quad \text{or} \quad \frac{|B_r(x)\setminus E|}{|B_r(x)|} \le \varepsilon_0,
$$

then

$$
\fint_{B_{\lambda r}(x)} |\nabla u|^2\,dx \le C\fint_{B_r(x)} |\nabla u|^2\,dx + \frac{Cr^2}{\lambda^n}\|\rho\|_\infty^2.
$$

(ii) *If*

$$
\frac{|(E\Delta H)\cap B_r(x)|}{|B_r(x)|} \le \varepsilon_0,
$$

where $H := \left\{y \in \mathbb{R}^n : (y-x)\cdot e \le 0\right\}$ for some $e \in \mathbb{S}^{n-1}$, then

$$
\fint_{B_{\lambda r}(x)} |\nabla u|^2\,dx \le C\fint_{B_r(x)} |\nabla u|^2\,dx + \frac{Cr^2}{\lambda^n}\|\rho\|_\infty^2.
$$

Moreover, the constants C and ε_0 can be chosen to depend only on un upper bound on β.

Proof. We detail the proof of item (ii). Item (i) can be obtained in a similar way and we sketch the argument at the end of the proof. Without loss of generality, by translating, we can assume $x = 0$. Let $\lambda \in (0,1/2)$ be given and let v the solution of

$$
\begin{cases}
-\operatorname{div}(a_H\nabla v) = \rho & \text{in } B_{r/2}\\
v = u & \text{on } \partial B_{r/2}.
\end{cases}
$$

where $a_H = \beta\mathbf{1}_H + \mathbf{1}_{H^c}$. In particular, $w = (u-v) \in W_0^{1,2}(B_{r/2})$ and

$$
-\operatorname{div}(a_H\nabla w) = -\operatorname{div}((a_E - a_H)\nabla u).
$$

By testing the above equation with w and using Young inequality we get

$$
\int_{B_{r/2}} |\nabla u - \nabla v|^2\,dx \le \int_{B_{r/2}} (a_E - a_H)^2|\nabla u|^2\,dx \le \beta^2\int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2\,dx.
$$

By the higher integrability lemma there exists $p > 1$ such that

$$\left(\fint_{B_{r/2}} |\nabla u|^{2p}\, dx\right)^{\frac{1}{p}} \le C \fint_{B_r} |\nabla u|^2\, dx + C\, r^{n+2}\, \|\rho\|_\infty^2. \tag{2.63}$$

Hence by exploiting Hölder inequality with exponent p we have

$$\begin{aligned}\int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2\, dx &\le \left|(E\Delta H)\cap B_{r/2}\right|^{1-\frac{1}{p}} \left(\int_{B_{r/2}} |\nabla u|^{2p}\, dx\right)^{\frac{1}{p}}\\ &\le c(2,p)\,|B_r| \left(\frac{|(E\Delta H)\cap B_r|}{|B_r|}\right)^{1-\frac{1}{p}} \left(\fint_{B_{r/2}} |\nabla u|^{2p}\, dx\right)^{\frac{1}{p}}\\ &\le C\,\varepsilon_0^{1-\frac{1}{p}} \left\{\int_{B_r} |\nabla u|^2\, dx + r^{n+2}\, \|\rho\|_\infty^2\right\}.\end{aligned} \tag{2.64}$$

Therefore, (recall $r < 1$) the above estimates yield

$$\int_{B_{r/2}} |\nabla w|^2\, dx \le C\left\{(\beta-1)^2\,\varepsilon_0^{1-\frac{1}{p}} \int_{B_r} |\nabla u|^2\, dx + r^{n+2}\|\rho\|_\infty^2\right\}. \tag{2.65}$$

Since the decay estimate (2.60) apply to v, we can argue as in the proof of (2.62) to obtain

$$\fint_{B_{\lambda r}} |\nabla u|^2\, dx \le C \fint_{B_r} |\nabla u|^2\, dx + \frac{C\varepsilon_0^{1-\frac{1}{p}}}{\lambda^n} \int_{B_r} |\nabla u|^2\, dx + \frac{C\|\rho\|_\infty^2}{\lambda^n}.$$

Choosing $\varepsilon_0 = \varepsilon_0(n,\lambda) \ll \lambda$ sufficiently small we conclude the proof of (ii). The proof of (i) can be obtained in the same way by comparing u to a solution of $-\Delta u = \rho$ (or $-\beta\Delta u = \rho$) and by using (2.59). □

2.5.2 Density estimates

In this section we establish scaling invariant upper and lower bounds for the perimeter and for the measure of a minimizer in balls. We also establish a universal upper bound for the normalized Dirchlet energy of the minimizer of u_E. We start with the following proposition which is a simple consequence of the outward minimizing property of E established in Proposition 2.3.5 (ii).

Proposition 2.5.5. *Let $A > 0$, and let β, K, Q be controlled by A and $R \ge 1$. Then there exist universal constants C_o and r_o such that, if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, $r \in (0, r_o)$, then*[1]

$$P(E, B_r(x)) \le C_o r^{n-1} \qquad \text{for all } x \in \partial E \text{ and } r \in (0, r_o), \tag{2.66}$$

and

$$\frac{|B_r(x)\setminus E|}{|B_r(x)|} \ge \frac{1}{C_o} \qquad \text{for all } x \in E^c \text{ and } r \in (0, r_o), \tag{2.67}$$

[1]Here and in the sequel we will always work with the representative of E such that

$$\partial E = \left\{x : \frac{|B_r(x)\setminus E|}{|B_r(x)|} \cdot \frac{|B_r(x)\cap E|}{|B_r(x)|} > 0 \quad \text{for all } r > 0\right\},$$

see [23, Proposition 12.19].

Proof. We let Λ_2 and $\bar{r}_2$ be the constants appearing in Proposition 2.3.5 we take $r_o \le \bar{r}_2$. For $r \le r_o$, we plug $F = E \cup B_r(x)$ in (2.37) and we obtain, after simple manipulations,

$$P(E, B_r(x)) \le \mathcal{H}^{n-1}(\partial B_r(x) \setminus E) + \Lambda_2 |E \setminus B_r(x)| \le n\omega_n r^{n-1} + \Lambda_2 \omega_n r^n.$$

Hence, assuming that $\Lambda_2 r_o \le 1$, we immediately get $P(E, B_r(x)) \lesssim r^{n-1}$. To obtain the lower density bound for E^c we set $m(r) := |B_r(x) \setminus E|$ and we use the isoperimetric inequality to deduce

$$\begin{aligned} m(r)^{\frac{n-1}{n}} &= |B_r(x) \setminus E|^{\frac{n-1}{n}} \lesssim P(E \setminus B_r(x)) \\ &= P(E, B_r(x)) + \mathcal{H}^{n-1}(\partial B_r(x) \setminus E) \\ &\lesssim \mathcal{H}^{n-1}(\partial B_r(x) \setminus E) + |E \setminus B_r(x)| \\ &\lesssim m'(r) + m(r), \end{aligned}$$

where we have used that, by co-area formula $m'(r) = \mathcal{H}^{n-1}(\partial B_r(x) \setminus E)$. If we choose r_o such that $Cm(r)^{\frac{1}{n}} \le C(n\omega_n)^{\frac{1}{n}} r_o \le 1/2$ where C is the implied universal constant in the above estimate, we obtain

$$m(r)^{\frac{n-1}{n}} \lesssim m'(r).$$

Since $x \in \partial E$, $m(r) > 0$ for all $r > 0$ then the above inequality implies that

$$\frac{\mathrm{d}}{\mathrm{d}r} m(r)^{\frac{1}{n}} \gtrsim 1 \qquad \text{for all } r \in (0, r_o).$$

Hence $m(r) \gtrsim r^n$ and this concludes the proof. ☐

The next lemma establishes a universal bound on the normalized Dirichlet integral.

Lemma 2.5.6. *Let $A > 0$, and let β, K, Q be controlled by A and $R \ge 1$. Then there exists a universal constant C_e such that, if E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), then for all $x \in \overline{B}_R$,*

$$Q^2 D_E(x,r) = \frac{Q^2}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2 \, dy \le C_e. \tag{2.68}$$

Proof. The estimates is clearly true if $r \ge r_0$ where $r_0 = r_0(n, A)$ (recall that $Q^2 \int_{\mathbb{R}^n} |\nabla u_E|^2 \, dx \le \mathcal{F}_{\beta,K,Q}(E) \lesssim 1$). Hence we can assume that $r \le r_0 \ll 1$. We claim the following: there exist constants $\lambda = \lambda(n, A) \in (0, 1/2)$, $C = C(n, A)$ and $r_0 = r_0(n, A)$ such that

(a) If $x \in \partial B_R$ and $r \le r_0$, then

$$Q^2 D_E(x, \lambda r) \le \frac{1}{2} Q^2 D_E(x, r) + C. \tag{2.69}$$

(b) If $x \in B_R$ and $r \le \min\{\mathrm{dist}(x, \partial B_R), r_0/2\}$, then

$$Q^2 D_E(x, \lambda r) \le \frac{1}{2} Q^2 D_E(x, r) + C. \tag{2.70}$$

Let $\varepsilon \ll 1$ to be fixed and let $r_0 = r_0(\varepsilon) \ll \bar{r}_1$ where $\bar{r}_1$ is the constant in Proposition 2.3.4 and such that the following holds true

$$x \in \partial B_R \quad \text{and } r \le r_0 \quad \Longrightarrow \quad \frac{|(B_R \cap B_r(x)) \Delta H_x|}{|B_r(x)|} \le \varepsilon, \tag{2.71}$$

where $H_x := \{y : (y - x) \cdot x \le 0\}$ is the supporting half space of B_R at x. Note that since the curvatures of ∂B_R are universally bounded (recall that $R \ge 1$), this can be achieved by choosing r_0 small only in dependence of ε.

Let now $x \in \overline{B}_R$ and $r \leq r_0$ be a radius satisfying either condition (a) (if $x \in \partial B_R$) or condition (b) (if $x \in B_R$) above. Let (u_E, ρ_E) be the minimizers for $\mathcal{G}(E)$ and consider

$$F = (E \cup B_r(x)) \cap B_R.$$

We define u to be the solution of

$$-\operatorname{div}(a_F \nabla u) = \rho_E. \tag{2.72}$$

Note that $(u, \rho_E) \in \mathcal{A}(F)$ since $F \supset E$. Hence, by Proposition 2.3.4,

$$\begin{aligned}
P(E) + Q^2\Big(&\int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx\Big) \\
&\leq P(F) + Q^2\Big(\int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx\Big) + \Lambda_1 |F \setminus E| \\
&\leq P(E \cup B_r(x)) + Q^2\Big(\int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx\Big) + \Lambda_1 |B_r(x)|,
\end{aligned}$$

where we have used that $F \setminus E \subset B_r(x)$ and that $P(F) \leq P(E \cup B_r(x))$, by the convexity of B_R. Rearranging terms we get

$$Q^2\Big(\int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx - \int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx\Big) \leq P(E \cup B_r(x)) - P(E) + \Lambda_1 |B_r(x)| \lesssim r^{n-1}.$$

Recall now that u_E solves

$$-\operatorname{div}(a_E \nabla u_E) = \rho_E,$$

and we use (2.11) in Lemma 2.1.4 to infer that

$$\int_{\mathbb{R}^n} (a_F - a_E) |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx - \int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx \lesssim \frac{r^{n-1}}{Q^2}. \tag{2.73}$$

Since

$$-\operatorname{div}(a_E \nabla (u_E - u)) = -\operatorname{div}((a_F - a_E) \nabla u),$$

by testing with $u_E - u$ and by Young inequality we get

$$Q^2 \int_{\mathbb{R}^n} |\nabla u_E - \nabla u|^2 \, dx \leq Q^2 \int_{\mathbb{R}^n} (a_F - a_E)^2 |\nabla u|^2 \, dx \lesssim r^{n-1},$$

where the last inequality follows from (2.73).

We want now apply Proposition 2.5.4 to u. Note that since

$$F \cap B_r(x) = B_r(x) \cap B_R,$$

then the assumption are satisfied both in case (a) (thanks to (2.71)) and in case (b) (since $B_r(x) \subset B_R$). Hence, given $\lambda \in (0, 1/2)$, we have:

$$\begin{aligned}
\frac{1}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u_E|^2 \, dy &\leq \frac{2}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u - \nabla u_E|^2 \, dy + \frac{2}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u|^2 \, dy \\
&\leq \frac{2}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u - \nabla u_E|^2 \, dy + \frac{C\lambda}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2 \, dy + \frac{C r^2 \|\rho_E\|_\infty}{\lambda^{n-1}} \\
&\leq \frac{C}{\lambda^{n-1}} \frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla u - \nabla u_E|^2 \, dy + \frac{C\lambda}{r^{n-1}} \int_{B_r(x)} |\nabla u_E|^2 \, dy + \frac{C r^2 \|\rho_E\|_\infty}{\lambda^{n-1}},
\end{aligned} \tag{2.74}$$

for a constant $C = C(n, A)$ provided ε (and thus r_0) is chosen sufficiently small. Since by (2.6) $\|\rho_E\|_\infty \lesssim 1$, we deduce from (2.74) that

$$Q^2 D_E(x, \lambda r) \le C\lambda\, Q^2 D_E(x, r) + \frac{C(n, A)}{\lambda^{n-1}}. \tag{2.75}$$

Now choosing $\lambda = \lambda(n, A)$ such that $C\lambda = 1/2$ we conclude the proof of the claim. Note that this fixes ε and thus r_0 as functions depending only on n and A.

To conclude the proof we have to show that (a) and (b) above implies that

$$S := \sup_{y \in \overline{B}_R} \sup_{0<s\le r_0} Q^2 D_E(y, s) \le C(n, A).$$

We first assume that $S < +\infty$ and show that we can bound it by a universal constant. Let $\bar{y} \in \overline{B}_R$ and $\bar{s} \in 0 < s \le r_0$ be such that

$$\frac{3S}{4} \le Q^2 D_E(\bar{y}, \bar{s}).$$

Let us distinguish a few cases:

- *Case 1:* $\bar{y} \in \partial B_R$. If $\bar{s} \le \lambda r_0$, (2.69) implies that

$$\frac{3S}{4} \le Q^2 D_E(\bar{y}, \bar{s}) \le \frac{1}{2} Q^2 D_E\Big(\bar{y}, \frac{\bar{s}}{\lambda}\Big) + C \le \frac{1}{2} S + C,$$

and we are done. On the other end if $\bar{s} \ge \lambda r_0$, then

$$\begin{aligned} \frac{3S}{4} \le Q^2 D_E(\bar{y}, \bar{s}) &\le \frac{Q^2}{(\lambda r_0)^{n-1}} \int_{\mathbb{R}^n} |\nabla u_E|^2\, dx \\ &\le \frac{1}{(\lambda r_0)^{n-1}} \mathcal{F}_{\beta,K,Q}(E) \le C(n, A). \end{aligned}$$

- *Case 2:* $\bar{y} \in B_R$. If $\bar{s} \le \lambda \min\{\operatorname{dist}(\bar{y}, \partial B_R), r_0/2\}$, we can use (2.70) and we argue as in the first part of Case 1. If $\bar{s} \ge \lambda r_0/2$ we argue instead as in the second part of Case 1 to conclude. We are thus left to consider the case

$$\lambda \operatorname{dist}(\bar{x}, \partial B_R) \le \bar{s} \le \lambda r_0/2.$$

In this case $B_{\bar{s}}(\bar{y}) \subset B_{r_0}(\bar{y})$, $\bar{y} \in \partial B_R$ and

$$\frac{3S}{4} \le Q^2 D_E(\bar{y}, \bar{s}) \le \frac{1}{2} S + C.$$

Thus we are done.

To show that one can actually assume that $S < +\infty$ one can consider

$$S_\delta = \sup_{y \in \overline{B}_R} \sup_{\delta \le s \le r_0} Q^2 D_E(y, s) \le C(n, A)\delta^{1-n},$$

and argue as above to show that $S_\delta \le C(n, A)$. Letting $\delta \to 0$ we conclude the proof. □

We are now ready to complete the proof of density and perimeter estimates.

Proposition 2.5.7. *Let $A > 0$, and let β, K, Q be controlled by A and $R \ge 1$. Then there exist universal constants $C_{\rm i}$ and $\bar{r}_{\rm i}$ such that, if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then*

$$P(E, B_r(x)) \ge \frac{r^{n-1}}{C_{\rm i}} \qquad \text{for all } x \in \partial E \text{ and } r \in (0, \bar{r}_{\rm i}), \tag{2.76}$$

and

$$\frac{|B_r(x) \cap E|}{|B_r(x)|} \ge \frac{1}{C_{\rm i}} \qquad \text{for all } x \in E \text{ and } r \in (0, \bar{r}_{\rm i}), \tag{2.77}$$

Proof. We start showing the validity of (2.76) and we divide the proof in few steps.

• *Step 1*: We claim that for every $\lambda \in (0, 1/4)$, there exist $\varepsilon_1 = \varepsilon_1(\lambda, , A)$, $C_1 = C_1(n, A)$ and and $\bar{r} = \bar{r}(n, A, \lambda)$ such that if

$$P(E, B_r(x)) \leq \varepsilon r^{n-1} \qquad \varepsilon \leq \varepsilon_1 \quad r \leq \bar{r}\,,$$

then,

(2.78)

$$P(E, B_{\lambda r}(x)) + Q^2 \int_{B_{\lambda r}(x)} |\nabla u_E|^2\, dy \leq C_1 \lambda^n \bigg(P(E, B_r(x)) + Q^2 \int_{B_r(x)} |\nabla u_E|^2\, dy + r^n \bigg).$$

For the ease of notation let us assume that $x = 0$. Let $\lambda \in (0, 1/4)$ be fixed. By the relative isoperimetric inequality

$$\left(\min \left\{ \frac{|E \cap B_r|}{|B_r|}, \frac{|B_r \setminus E|}{|B_r|} \right\} \right)^{\frac{n-1}{n}} \leq C(n) \frac{P(E, B_r)}{r^{n-1}} \lesssim \varepsilon.$$

By (2.67) and by choosing $\varepsilon_1, \bar{r} \ll 1$ we get

$$\frac{|E \cap B_r|}{|B_r|} \leq C(n) \left(\frac{P(E, B_r)}{r^{n-1}} \right)^{\frac{1}{n-1}} \frac{P(E, B_r)}{r^{n-1}} \lesssim \varepsilon^{\frac{1}{n-1}} \frac{P(E, B_r)}{r^{n-1}}. \tag{2.79}$$

Let us choose $t \in (\lambda r, 2\lambda r)$ such that

$$\begin{aligned} \mathcal{H}^{n-1}(E \cap \partial B_t) &\leq \fint_{\lambda r}^{2\lambda r} \mathcal{H}^{n-1}(E \cap \partial B_s)\, ds \\ &\leq \frac{|E \cap B_{2\lambda r}|}{\lambda r} \leq C(n, \lambda) \varepsilon^{\frac{1}{n-1}} P(E, B_r). \end{aligned} \tag{2.80}$$

By testing (2.37) with $F = E \setminus B_t$ we obtain

$$P(E, B_t) \leq \mathcal{H}^{n-1}(E \cap \partial B_t) + \Lambda_2 |E \cap B_t| + \Lambda_2 Q^2 \int_{E \cap B_t} |\nabla u_E|^2\, dx, \tag{2.81}$$

which together with (2.80) and recalling that $t \in (\lambda r, 2\lambda r)$, implies that

$$\begin{aligned} &P(E, B_{\lambda r}) + Q^2 \int_{B_{\lambda r}} |\nabla u_E|^2\, dx \\ &\qquad \leq C(n, \lambda) \varepsilon^{\frac{1}{n-1}} P(E, B_r) + (\Lambda_2 + 1) Q^2 \int_{B_{2\lambda r}} |\nabla u_E|^2\, dx + \Lambda_2 |B_{2\lambda r}|. \end{aligned} \tag{2.82}$$

If we now choose $\varepsilon_1 = \varepsilon_1(\lambda) \ll 1$, (2.79) allow to apply Proposition 2.5.4 (i). Hence by also choosing $\bar{r} \ll \lambda$ we deduce that

$$\begin{aligned} \int_{B_{2\lambda r}} |\nabla u_E|^2\, dx &\leq C(n, A) \lambda^n \left(\int_{B_r} |\nabla u_E|^2\, dx + \frac{\bar{r}^2}{\lambda^n} r^n \right) \\ &\leq C(n, A) \lambda^n \left(\int_{B_r} |\nabla u_E|^2\, dx + r^n \right), \end{aligned} \tag{2.83}$$

where we have used that by (2.6), $\|\rho_E\|_\infty \lesssim 1$. By gathering equations (2.82) and (2.83) we then get

$$P(E,B_{\lambda r}) + Q^2\int_{B_{\lambda r}} |\nabla u_E|^2\,dx \le C(n,\lambda)\varepsilon^{\frac{1}{n-1}} P(E,B_r) + C(n,A)\lambda^n\left(Q^2\int_{B_r}|\nabla u_E|^2\,dx + r^n\right).$$

If we choose $\varepsilon_1 = \varepsilon_1(n,A,\lambda) \ll 1$ such that $C(n,\lambda)\varepsilon^{\frac{1}{n-1}} \le \lambda^n$ the above inequality implies (2.78).

• *Step 2*: We now prove the validity of (2.76). By density it is enough to prove it at all $x \in \partial^* E$. Again we set coordinates so that $x = 0$. Let us choose $\lambda = \lambda(n,A) \in (0,1/4)$ such that $C_1\lambda \le 1/2$ where C_1 is the constant appearing in (2.78) and let $\bar r$ and ε_1 be the corresponding constants (which now depend only on A and n). We claim that

$$P(E,B_r) + Q^2\int_{B_r}|\nabla u_E|^2\,dx \ge \frac{\varepsilon_1}{2}\,r^{n-1} \qquad \text{for all } r \le \min\{r_1,\varepsilon_1/2\}. \tag{2.84}$$

Indeed otherwise, by (2.78) and the choice of λ

$$P(E,B_{\lambda r}) + Q^2\int_{B_{\lambda r}}|\nabla u_E|^2\,dx \le \frac{\lambda^{n-1}}{2}\left(P(E,B_r) + Q^2\int_{B_r}|\nabla u_E|^2\,dx + \frac{\varepsilon_1}{2}\,r^{n-1}\right) \le \frac{\varepsilon_1}{2}(\lambda r)^{n-1}.$$

We can thus iterate the above estimate and deduce that

$$\liminf_{r\to 0}\frac{P(E,B_r)}{r^{n-1}} = 0\,,$$

in contradiction with the assumption that $0 \in \partial^* E$. Let now $\bar\lambda \ll \varepsilon_1$ to be chosen where ε_1 is the constant obtained above. Let ε_2 and r_2 be the constants corresponding to $\bar\lambda$ in Step 1. We claim that if we choose $\bar\lambda$ small enough depending only on n and A then

$$P(E,B_r) \ge \varepsilon_2 r^{n-1} \qquad \text{for all } r \le r_3, \tag{2.85}$$

where $r_3 \ll \min\{r_2,r_1\}$ will depend only on n and A. Indeed otherwise we can apply Step 1, (2.66), and Lemma 2.5.6 to get

$$\begin{aligned}P(E,B_{\bar\lambda r}) + Q^2\int_{B_{\bar\lambda r}}|\nabla u_E|^2\,dx &\le C(n,A)\bar\lambda^n\left(P(E,B_r(x)) + Q^2\int_{B_r}|\nabla u_E|^2\,dx + r^n\right)\\ &\le \bar C(n,A)\bar\lambda(\bar\lambda r)^{n-1},\end{aligned}$$

where $\varepsilon_2 \ll \varepsilon_1$ and $r_2 \ll r_1$ are universal constants. If $\bar\lambda$ is chosen so that $\bar C(n,A)\bar\lambda \le \varepsilon_1/4$ this contradicts (2.84) and thus proves (2.76) with $c_i \le \varepsilon_2$.

• *Step 3*: We now prove the validity of (2.77). Assume indeed that

$$\frac{|E\cap B_r|}{|B_r|} \le \varepsilon_4 \qquad \text{for } r \le r_4,$$

with $\varepsilon_4, r_4 \ll 1$ to be fixed only in term of n and A. Then, by exploiting Lemma 2.5.2 and Lemma 2.5.6, for all $s \in (r/4, r/2)$ we have

$$\begin{aligned} Q^2 \int_{B_s} |\nabla u_E|^2\, dx &\le Q^2 |E \cap B_s|^{1-\frac{1}{p}} \left(\int_{B_s} |\nabla u_E|^{2p}\, dx \right)^{\frac{1}{p}} \\ &\lesssim Q^2 \left(\frac{|E \cap B_r|}{|B_r|} \right)^{1-\frac{1}{p}} \int_{B_{2s}} |\nabla u_E|^2\, dx \lesssim \varepsilon_4^{1-\frac{1}{p}} r^{n-1} \lesssim \varepsilon_4^{1-\frac{1}{p}} s^{n-1}. \end{aligned} \tag{2.86}$$

Moreover, by the co-area formula, there exists $s \in (r/4, r/2)$ such that

$$\mathcal{H}^{n-1}(E \cap B_s) \le \fint_{r/4}^{r/2} \mathcal{H}^{n-1}(E \cap B_t) dt \le \frac{4|E \cap B_r|}{r} \lesssim \varepsilon_4 r^{n-1} \lesssim \varepsilon_4 s^{n-1}. \tag{2.87}$$

By testing (2.36) with $E \setminus B_s$ we get

$$P(E, B_s) \le \mathcal{H}^{n-1}(E \cap B_s) + \Lambda_2 |B_s| + Q^2 \Lambda_2 \int_{B_s} |\nabla u_E|^2\, dx,$$

which together with (2.86) and (2.87) and provided $r_4 \ll \varepsilon_4 \ll 1$ implies

$$P(E, B_s) \le C \varepsilon_4^{1-\frac{1}{p}} s^{n-1},$$

for a suitable universal constant C. Choosing ε_4 small with respect to ε_2 we get

$$P(E, B_s) \le \varepsilon_2 s^{n-1},$$

in contradiction with (2.85).

□

3

$C^{1,\vartheta}$-regularity

Contents

This chapter is devoted to the ε-regularity result for minimizers, Theorem 2 and the partial regularity result, Theorem 1.

3.1 Decay of the excess

Since the seminal works of De Giorgi and Almgren, [1, 6] the proof of Theorem 2 is based on an excess decay theorem, namely

Theorem 3.1.1 (Excess improvement)**.** *Let $A > 0$, and let β, K, Q be controlled by A and $R \geq 1$. There exists a universal constant $C_{\text{dec}} > 0$ such that for all $\lambda \in (0, 1/4)$ there exists $\varepsilon_{\text{dec}} = \varepsilon_{\text{dec}}(n, A, \lambda) > 0$ satisfying the following: if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and*

$$x \in \partial E, \quad r + Q^2 D_E(x, r) + \mathbf{e}_E(x, r) \leq \varepsilon_{\text{dec}},$$

then

$$Q^2 D_E(x, \lambda r) + \mathbf{e}_E(x, \lambda r) \leq C_{\text{dec}} \lambda \Big(\mathbf{e}_E(x, r) + Q^2 D_E(x, r) + r \Big). \tag{3.1}$$

As it is customary, the proof of the above theorem is based on an "harmonic approximation" technique. More precisely we will go through the following steps:

(i) In the small excess regime, the boundary of E can be well approximated by the graph of a Lipschitz function f with Dirichlet energy bounded by the excess.

(ii) If the excess and the normalized Dirichlet of u_E are small, then f is almost harmonic.

(iii) Almost harmonicity of f implies closeness to an harmonic function g in the L^2 topology. By classical estimates for harmonic functions then an L^2 type of excess of g decays. This in turn implies the decay of the *flatness* $\mathbf{f}$ of E, see (3.35) below for the definition.

(iv) Via a Caccioppoli type inequality, the decay of the flatness can be transferred to the decay of the excess.

(v) Via Proposition 2.5.4 (ii), the decay of the excess implies the decay of the normalized Dirchlet energy.

Usually, Step (i) is obtained by reproducing at most points and at all scale an height type bound for ∂E in the small excess regime. This relies on the scaling invariance of the problem studied. Step (ii) and (iv) are obtained by simple comparison arguments and Step (iii) is based on a compactness argument together with the classical regularity theory for harmonic functions.

In our situation the problem does not enjoy of a nice scaling behaviour, due to the global constraint $\int_{\mathbb{R}^n} \rho_E \, dx = 1$. However, the local estimates obtained in the previous section are exactly what we need to carry on the proof of Step (i), see Lemma 3.1.3 below.

3.1.1 Lipschitz approximation

In this subsection we prove the Lipschitz approximation lemma. Let us first recall a few notations that will be useful in the remaining part of the thesis.

Notation 3.1.2. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{n-1}$, $\nu \in \mathbb{S}^{n-1}$ and $r > 0$.

- We call $\mathbf{p}^\nu(x) := x - (x \cdot \nu)\,\nu$ and $\mathbf{q}^\nu(x) := (x \cdot \nu)\,\nu$, respectively, the *orthogonal projection* onto the plane $\nu^\perp$ and the *projection* on ν. For simplicity we denote $\mathbf{p}(x) := \mathbf{p}^{e_n}(x)$ and $\mathbf{q}(x) := \mathbf{q}^{e_n}(x) = x_n$.
- We define the *cylinder* with center at x and radius r with respect to the direction ν as

$$\mathbf{C}(x, r, \nu) := \big\{ y \in \mathbb{R}^n \,:\, |\mathbf{p}^\nu(y - x)| < r\,, |\mathbf{q}^\nu(y - x)| < r \big\}.$$

  We write $\mathbf{C}_r(x) := \mathbf{C}(x, r, e_n)$, $\mathbf{C}_r := \mathbf{C}(0, r, e_n)$ and $\mathbf{C} := \mathbf{C}_1$.
- We denote the $(n-1)$-dimensional *disk* centered at z and of radius r by

$$\mathbf{D}(z, r) := \big\{ y \in \mathbb{R}^{n-1} : |y - z| < r \big\}.$$

  For simplicity we write $\mathbf{D}_r(z) := \mathbf{D}(z, r)$, $\mathbf{D}_r := \mathbf{D}(0, r)$ and $\mathbf{D} := \mathbf{D}(0, 1)$.
- The *cylindrical excess* in a direction $\nu \in \mathbb{S}^{n-1}$ is defined as

$$\mathbf{e}_E(x, r, \nu) = \frac{1}{r^{n-1}} \int_{\mathbf{C}(x,r,\nu) \cap \partial^* E} \frac{|\nu_E(y) - \nu|^2}{2} \, d\mathcal{H}^{n-1}(y)\,,$$

  so that

$$\mathbf{e}_E(x, r) \leq \inf_{\nu \in S^{n-1}} \mathbf{e}_E(x, r, \nu). \tag{3.2}$$

The following height bound is crucial in the sequel. Note that it does not require any minimality property on E, only the validity of inequality (3.3) at all scales.

Lemma 3.1.3. *Let $C > 0$. Then there exists an increasing function $\omega_C : (0,1) \to \mathbb{R}$ with $\omega_C(0^+) = 0$ depending only on C such that if $E \subseteq \mathbb{R}^n$ is a set of finite perimeter in $\mathbf{C}(x, 2r)$ satisfying the following properties:*

(i) $x \in \partial E$,

(ii) *for all $y \in \partial E$ and s such that $B_s(y) \subset \mathbf{C}(x, 2r)$*

$$\frac{1}{C} \leq \frac{|E \cap B_s(y))|}{|B_s(y)|} \leq \Big(1 - \frac{1}{C}\Big), \qquad P(E, B_s(y)) \leq C s^{n-1}, \tag{3.3}$$

then

$$\mathbf{e}_E(x, 2r, e_n) < t \qquad \Longrightarrow \sup_{y \in \mathbf{C}(x,r) \cap \partial E} |\mathbf{q}(y - x)| \leq \omega_C(t) r, \tag{3.4}$$

$$\big|\{y \in \mathbf{C}(x,r) \cap E \,:\, \mathbf{q}(y-x) > \omega_C(t) r\}\big| = 0, \tag{3.5}$$

$$\big|\{y \in \mathbf{C}(x,r) \setminus E \,:\mathbf{q}(y-x) < -\omega_C(t) r\}\big| = 0. \tag{3.6}$$

Proof. Note that the assumptions are scaling and translation invariant, hence we can assume that $x = 0$ and $r = 1$. For every $t \in (0,1)$ let

$$\mathcal{M}_t := \big\{\text{sets of finite perimeter satisfying } \mathbf{e}_E(0, 2, e_n) < t, \text{ (i) and (ii)}\big\}.$$

For every $E \subseteq \mathbb{R}^n$ let us call

$$\begin{aligned} h_E &:= \sup_{x \in \mathbf{C} \cap \partial E} |\mathbf{q} x|, \\ g_E &:= \inf \big\{s \in [0,1] \,:\, |\{x \in \mathbf{C} \cap E \,:\, \mathbf{q} x > s\}| = 0\big\} \quad \text{and} \\ f_E &:= \inf \big\{s \in [0,1] \,:\, |\{x \in \mathbf{C} \setminus E \,:\, \mathbf{q} x < -s\}| = 0\big\}. \end{aligned} \tag{3.7}$$

Define the functions $\omega_1, \omega_2, \omega_3 : (0,1) \to \mathbb{R}$ as

$$\omega_1(t) := \sup_{E \in \mathcal{M}_t} h_E, \;\; \omega_2(t) := \sup_{E \in \mathcal{M}_t} g_E \;\; \text{and} \;\;\; \omega_3(t) := \sup_{E \in \mathcal{M}_t} f_E. \tag{3.8}$$

Let $\omega_C := \max\{\omega_1, \omega_2, \omega_3\}$. Notice that ω_C is increasing since it is the maximum of increasing functions and by definition it satisfies (3.4), (3.5), and (3.6). Let us prove that $\omega_C(0^+) = 0$. Assume by contradiction that $\lim_{t \to 0^+} \omega_C(t) > 0$ then there exist a sequence $t_k \searrow 0$ and $L > 0$ such that $\omega_C(t_k) > L$ for all k. We now distinguish three cases.

Case 1: Up to subsequences $\omega_C(t_k) = \omega_1(t_k)$ for every $k \in \mathbb{N}$. For every k there exists $E_k \in \mathcal{M}_{t_k}$ such that $h_{E_k} \geq L$. By (3.3) up to subsequences there exists a set of finite perimeter $E \subseteq \mathbb{R}^n$ such that $E_k \cap \mathbf{C}_r \to E \cap \mathbf{C}_r$ whenever $r < 2$ and

$$\lim_{k \to +\infty} \mathbf{e}_{E_k}(0, 2, e_n) = 0. \tag{3.9}$$

Now take $\mathbf{C}_s \subset \mathbf{C}_r \subset \mathbf{C}_2$ with $s > 1$. By the lower semicontinuity of the excess we obtain that $\mathbf{e}_E(0, s, e_n) = 0$. Moreover let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence such that $x_k \in \partial E_k \cap \mathbf{C}$ and let us assume that $x_k \to x$. By (ii) one easily deduce that

$$\min\big\{|E \cap B_s(x))|, |B_s(x) \setminus E|\big\} \geq \frac{|B_s(x)|}{C},$$

which implies that $x \in \partial E$ (recall that we are working with the representative of E such that $\partial E = \operatorname{spt} D\mathbf{1}_E$). This in particular implies that $0 \in \partial E$. Since $\mathbf{e}_E(0, s, e_n) = 0$ we get

$H := \{x : \mathbf{q}x < 0\} = E$. However, if $x_k \in \partial E_k$ is such that $|\mathbf{q}x_k| \geq L$, up to a subsequence, we can assume that $x_k \to \bar{x} \in \partial E = \{x : \mathbf{q}x = 0\}$, a contradiction.

Case 2: Up to subsequences $\omega_C(t_k) = \omega_2(t_k)$ for every $k \in \mathbb{N}$. Hence for every k there exists $E_k \in \mathcal{M}_{t_k}$ such that $g_{E_k} \geq L$. Note that if $\ell \in (0, L)$ then

$$\left|\{x \in \mathbf{C} \cap E_k \,:\, \mathbf{q}x > \ell\}\right| > 0 \qquad \text{for all } k \in \mathbb{N}. \tag{3.10}$$

Hence (3.10) implies that, up to extracting a subsequence,

either

$$\text{there exists } \ell \in (0, L) \text{ such that for } k \text{ there exists } x_k \in \mathbf{C} \cap \partial E_k \cap \{\mathbf{q}x > \ell\}, \tag{3.11}$$

or

$$\mathbf{1}_{E_k \cap \{\mathbf{q}x > 0\}} \longrightarrow \mathbf{1}_{\{\mathbf{q}x>0\}} \text{ in } L^1(\mathbf{C}). \tag{3.12}$$

Indeed if by contradiction (3.11) does not hold then for every $j \gg 1$ there exists $k_j \in \mathbb{N}$ such that $\mathbf{q}x \leq \frac{1}{j}$ for every $x \in \mathbf{C} \cap \partial E_{k_j}$. By (3.10), since $\{\mathbf{q}x > \frac{1}{j}\}$ is connected, then necessarily $\mathbf{C} \cap E_{k_j} \supseteq \{\mathbf{q}x > \frac{1}{j}\}$. By letting $j \to +\infty$ we get (3.12).

By arguing as in Case 1, we have that $E_k \to \{\mathbf{q}x \leq 0\}$, hence (3.12) cannot hold. Hence (3.11) holds, which is again in contradiction with Case 1.

Case 3: Up to subsequences $\omega_C(t_k) = \omega_3(t_k)$ for every $k \in \mathbb{N}$. This case can be ruled out by arguing as in Case 2 (or by working with E^c which satisfies the same assumption of E).

Therefore ω_C is the required function. □

First of all, thanks to the following lemma, [23, Lemma 22.11] we define the *excess measure*.

Lemma 3.1.4 (Excess measure)**.** *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter such that $0 \in \partial E$. Denote by $M := \mathbf{C} \cap \partial E$. If there exists $h \in (0, 1)$ such that*

$$\sup_{x \in \mathbf{C} \cap \partial E} |\mathbf{q}x| \leq h, \tag{3.13}$$

$$\left|\{x \in \mathbf{C} \cap E \,:\, \mathbf{q}x > h\}\right| = 0, \tag{3.14}$$

$$\left|\{x \in \mathbf{C} \setminus E \,:\, \mathbf{q}x < -h\}\right| = 0, \tag{3.15}$$

then for every Borel set $F \subseteq \mathbf{D}$ we have $\mathcal{H}^{n-1}(F) \leq \mathcal{H}^{n-1}\left(M \cap \mathbf{p}^{-1}(F)\right)$. Moreover,

$$\mu(F) := \mathcal{H}^{n-1}\left(M \cap \mathbf{p}^{-1}(F)\right) - \mathcal{H}^{n-1}(F),$$

is a Radon measure on $\mathcal{B}(\mathbb{R}^{n-1})$, where $\mathcal{B}(\mathbb{R}^{n-1})$ is the σ-algebra of Borel sets. The measure μ is called the excess measure.

We are now ready to prove the following Lipschitz approximation lemma.

Lemma 3.1.5 (Lipschitz approximation I)**.** *Let $C > 0$. Then there exist $\varepsilon_{\mathrm{L}} = \varepsilon_{\mathrm{L}}(n, C) > 0$ and $C_{\mathrm{L}} = C_{\mathrm{L}}(n, C) > 0$ with the following property: if E is a set of finite perimeter in $\mathbf{C}(x, 4r)$ satisfying*

(i) *$x \in \partial E$,*

(ii) *for all* $y \in \partial E \cap \mathbf{C}(x, 2r)$ *such that* $B_s(y) \subset \mathbf{C}(x, 2r)$

$$\frac{1}{C} \le \frac{|E \cap B_s(y))|}{|B_s(y)|} \le \Big(1 - \frac{1}{C}\Big), \qquad P(E, B_s(y)) \le C s^{n-1},$$

(iii)

$$\mathbf{e}_E(x, 2r, e_n) \le \varepsilon_{\mathrm{L}},$$

then there exists a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ *with*

$$\text{(3.16)} \qquad \mathrm{Lip}(f) \le 1, \quad \frac{1}{r^{n-1}} \int_{\mathbf{D}_r} |\nabla f|^2 \le C_{\mathrm{L}} \mathbf{e}_E(x, 2r, e_n), \quad \frac{\|f\|_\infty}{r} \le \omega_C\big(\mathbf{e}_E(x, 2r, e_n)\big),$$

such that, defining $\Gamma_f := x + \{(z, f(z)) \,:\, z \in \mathbf{D}_r\}$,

$$\text{(3.17)} \qquad \frac{\mathcal{H}^{n-1}\big((\partial E \cap \mathbf{C}(x, r, e_n))\Delta\Gamma_f\big)}{r^{n-1}} \le C_{\mathrm{L}} \mathbf{e}_E(x, 2r, e_n),$$

where ω_C *is the function in Lemma 3.1.3.*

Proof. Without loss of generality we assume $x = 0$ and $r = 1$. Denote by

$$\|z\| := \max\{|\mathbf{p}z|, |\mathbf{q}z|\} \quad \text{for every } z \in \mathbb{R}^n.$$

Let $\varepsilon_L \in (0, 1)$ such that $\omega_C(\varepsilon_L) < 1$, where ω_C is the non decreasing function of Lemma 3.1.3. Define

$$G := \left\{ y \in \partial E \cap \mathbf{C} \,:\, \sup_{0<s<\frac{3}{4}} \mathbf{e}_E(y, s, e_n) \le \varepsilon_L \right\}.$$

Fix $y \in G$ and $x \in \partial E \cap \mathbf{C}$ with $\|x - y\| < \frac{3}{8}$. Let $F := \frac{E-y}{\|x-y\|}$. Notice that F is a set of finite perimeter in $\mathbf{C}_{\frac{2}{\|x-y\|}}$ with $0 \in \partial F$ which satisfies the density estimates of the volume and the perimeter in $\mathbf{C}_{\frac{2}{\|x-y\|}}$. Moreover by definition of the excess and G we obtain

$$\mathbf{e}_F\,(0, 2, e_n) = \mathbf{e}_E\,(y, 2\|x - y\|, e_n) \le \varepsilon_L.$$

Hence, by choosing $t := \mathbf{e}_F\,(0, 2, e_n) \le \varepsilon_L$ in Lemma 3.1.3 we deduce
(3.18)

$$\sup_{\mathbf{C}\cap\partial E} |\mathbf{q}x| \le \omega_C(t) \implies \frac{|\mathbf{q}(x - y)|}{\|x - y\|} \le \omega_C(t) \quad \text{and} \quad \|x-y\| = |\mathbf{p}(x-y)| \;\forall x \in \partial E\cap\mathbf{C}, y \in G.$$

Thus (3.18) implies that $\mathbf{p}$ is invertible on G. Hence there exists $f : \mathbf{p}(G) \to \mathbb{R}$ such that $f(\mathbf{p}z) = \mathbf{q}z$ for every $z \in G$. Therefore,

$$\text{(3.19)} \qquad |f(\mathbf{p}x) - f(\mathbf{p}y))| \le \omega_C(t)\,\|\mathbf{p}x - \mathbf{q}y\| \;\; \forall x, y \in G.$$

By McShane's lemma we can extend f to all $\mathbb{R}^{n-1}$ with the same Lipschitz constant (we continue to call f also the extension). Thus we have found a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\text{(3.20)} \qquad \mathrm{Lip}(f) \le \omega_C(\varepsilon_L) < 1 \quad \text{and} \quad G \subseteq \Gamma_f = \{(z, f(z)) \,:\, z \in \mathbf{D}\}.$$

Thus we obtain the first and the third inequality of (3.16). Let us prove (3.17). In this proof $C_L = C_L(n) \ge 0$ is a constant which changes line by line. By definition of G for every $y \in (\partial E \cap \mathbf{C}) \setminus G$ there exists a radius $s \in (0, 3/4)$ such that

$$\text{(3.21)} \qquad \varepsilon_L\, s^{n-1} < s^{n-1}\mathbf{e}_E(y, s, e_n) = \int_{\partial E\cap\mathbf{C}(y,s)} \frac{|\nu_E(x) - e_n|^2}{2}\, d\mathcal{H}^{n-1}(x).$$

By applying Besicovitch's theorem we can find a contant $\varsigma(n)>0$ depending only on n and a countable disjoint family of balls $\left\{B_{\sqrt{2}s_h}(y_h)\right\}_{h\in\mathbb{N}}$ with $\mathbf{C}(y_h,s_h)\subseteq B_{\sqrt{2}s_h}(y_h)\subseteq \mathbf{C}_2$ which satisfy (3.21) such that

$$\begin{aligned}\mathcal{H}^{n-1}(\partial E\cap\mathbf{C}\setminus G)&\leq\varsigma(n)\sum_{h\in\mathbb{N}}\mathcal{H}^{n-1}\left(\partial E\cap\mathbf{C}\setminus G\cap B_{\sqrt{2}s_h}(y_h)\right)\\&\leq\varsigma(n)\sum_{h\in\mathbb{N}}\mathcal{H}^{n-1}\left(\partial E\cap\mathbf{C}\cap B_{\sqrt{2}s_h}(y_h)\right)\\&\leq\varsigma(n)2^{\frac{n-1}{2}}c_2\sum_{h\in\mathbb{N}}s_h^{n-1}.\end{aligned}\tag{3.22}$$

Hence by (3.21) and (3.22) we have

$$\begin{aligned}\mathcal{H}^{n-1}(\partial E\cap\mathbf{C}\setminus G)&\leq\varsigma(n)2^{\frac{n-1}{2}}\frac{c_2}{\varepsilon_L}\sum_{h\in\mathbb{N}}\varepsilon_L s_h^{n-1}\\&\leq\varsigma(n)2^{\frac{n-1}{2}}\frac{c_2}{\varepsilon_L}\sum_{h\in\mathbb{N}}\int_{\partial E\cap\mathbf{C}(y_h,s_h)}\frac{|\nu_E(x)-e_n|^2}{2}\,d\mathcal{H}^{n-1}(x)\\&\leq\varsigma(n)2^{\frac{n-1}{2}}\frac{c_2}{\varepsilon_L}\mathbf{e}_E(0,2,e_n).\end{aligned}\tag{3.23}$$

Since $\partial E\cap\mathbf{C}\setminus\Gamma_f\subseteq\partial E\cap\mathbf{C}\setminus G$ then

$$\mathcal{H}^{n-1}(\partial E\cap\mathbf{C}\setminus\Gamma_f)\leq C_L\,\mathbf{e}_E(0,2,e_n),\tag{3.24}$$

for some constant $C_L>0$. Area formula for Lipschitz functions and Lemma 3.1.4 yield

(3.25)

$$\begin{aligned}\mathcal{H}^{n-1}(\Gamma_f\setminus\partial E\cap\mathbf{C})&=\int_{\mathbf{p}(\Gamma_f\setminus\partial E\cap\mathbf{C})}\sqrt{1+|\nabla f(z)|^2}\,dz\leq\sqrt{1+\mathrm{Lip}(f)^2}\,\mathcal{H}^{n-1}(\mathbf{p}(\Gamma_f\setminus\partial E\cap\mathbf{C}))\\&\leq\sqrt{2}\,\mathcal{H}^{n-1}\left(\partial E\cap\mathbf{C}\cap\mathbf{p}^{-1}\left(\mathbf{p}(\Gamma_f\setminus\partial E\cap\mathbf{C})\right)\right).\end{aligned}$$

It is simply to prove that $\partial E\cap\mathbf{C}\setminus\Gamma_f\subseteq\partial E\cap\mathbf{C}\cap\mathbf{p}^{-1}\left(\mathbf{p}(\Gamma_f\setminus\partial E\cap\mathbf{C})\right)$. Therefore by combining (3.24) and (3.25) we found

$$\mathcal{H}^{n-1}(\Gamma_f\setminus\partial E\cap\mathbf{C})\leq\varsigma(n)\,2^{\frac{n}{2}}\frac{c_2}{\varepsilon_L}\mathbf{e}_E(0,2,e_n)\leq C_L\,\mathbf{e}_E(0,2,e_n).\tag{3.26}$$

Thus we have proved (3.17). Let us prove the second estimate of (3.16). One can simply see that there exists $\alpha:\mathbb{R}^n\to\{-1,1\}$ such that

$$\nu_E(x)=\alpha(x)\frac{(\nabla f(\mathbf{p}x),1)}{\sqrt{1+|\nabla f(\mathbf{p}x)|^2}}\quad\mathcal{H}^{n-1}\text{a.e. in }\partial E\cap\mathbf{C}.\tag{3.27}$$

Notice that, thanks to our notations,

$$(\mathbf{p}\nu_E)^2+(\mathbf{q}\nu_E)^2=1\quad\text{and}\quad\mathbf{q}\nu_E=\nu_E\cdot e_n.$$

Hence

$$|\mathbf{p}\nu_E|^2=\frac{|\nabla f(\mathbf{p}x)|^2}{1+|\nabla f(\mathbf{p}x)|^2}\quad\mathcal{H}^{n-1}\text{a.e. in }\partial E\cap\mathbf{C}.$$

On the other hand

$$\begin{aligned}\int_{\partial E\cap\mathbf{C}\cap\Gamma_f}\frac{|\mathbf{p}\nu_E|^2}{2}\,d\mathcal{H}^{n-1}&\leq\int_{\partial E\cap\mathbf{C}}\frac{|\mathbf{p}\nu_E|^2}{2}\,d\mathcal{H}^{n-1}\\&\leq\int_{\partial E\cap\mathbf{C}}\frac{|\nu_E-e_n|^2}{2}\,d\mathcal{H}^{n-1}=\mathbf{e}_E(0,1,e_n),\end{aligned}\tag{3.28}$$

where in the second inequality we have used

$$\frac{|\mathbf{p}\nu_E|^2}{2} = \frac{1-(\nu_E\cdot e_n)^2}{2} \le 1-\nu_E\cdot e_n = \frac{|\nu_E - e_n|^2}{2}.$$

Therefore by putting together (3.27) and (3.28) we obtain

$$\begin{aligned}\mathbf{e}_E(0,1,e_n) \ge \int_{\partial E\cap\mathbf{C}\cap\Gamma_f} \frac{|\nabla f(\mathbf{p}x)|^2}{1+|\nabla f(\mathbf{p}x)|^2}\, d\mathcal{H}^{n-1} &= \int_{\mathbf{p}(\partial E\cap\mathbf{C}\cap\Gamma_f)} \frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}}\, dx' \\ &\ge \frac{1}{2\sqrt{2}}\int_{\mathbf{p}(\partial E\cap\mathbf{C}\cap\Gamma_f)} |\nabla f|^2\, dx',\end{aligned}$$

where in the last inequality we have used the first inequality of (3.16). By using the first and the third inequality of (3.16) we have

$$\begin{aligned}\int_{\mathbf{p}(\partial E\cap\mathbf{C}\cap\Gamma_f)} |\nabla f|^2\, dx \le \mathcal{H}^{n-1}\left(\mathbf{p}(\partial E\cap\mathbf{C}\Delta\Gamma_f)\right) &\le \mathcal{H}^{n-1}\left(\mathbf{p}(\partial E\cap\mathbf{C}\Delta\Gamma_f)\right) \\ \le C_L\mathbf{e}_E(0,2,e_n) &\le 2^{n-1}C_L\mathbf{e}_E(0,1,e_n).\end{aligned}$$

Thus, we get the result. □

Remark 3.1.6. If E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), the assumption of the Lipschitz approximation lemma are satisfied with some universal constant C by (2.66) and (2.76). Hence we can cover most of its boundary by the graph of a Lipschitz function f.

A comparison argument implies that the Laplacian of f is small in a suitable negative norm. More precisely we have the following:

Proposition 3.1.7 (Lipschitz approximation II). *Let $A>0$, and let β, K, Q be controlled by A and $R\ge 1$. Then there exist universal constants ε_{lip}, C_{lip} and a "universal" increasing function (i.e. depending only on n and A) ω_{lip} with $\omega_{\text{lip}}(0+)=0$ such that if E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), $x\in\partial E$ and*

$$r+\mathbf{e}_E(x,2r,e_n)\le\varepsilon_{\text{lip}},$$

then there exists a function f satisfying (3.16) and (3.17) with C_L and ω_C replaced by C_{lip} and ω_{lip} respectively. Moreover,

$$\frac{1}{r^{n-1}}\left|\int_{\mathbf{D}_r}\nabla f\cdot\nabla\varphi\, dz\right| \le C_{\text{lip}}\|\nabla\varphi\|_\infty\Bigg(\mathbf{e}_E(x,2r,e_n)+r+Q^2D_E(x,2r)\Bigg), \tag{3.29}$$

for every $\varphi\in C^1_c(\mathbf{D}_r)$.

Proof. Upper and lower perimeter estimates established in (2.66) and (2.76) ensure that in every cylinder $\mathbf{C}(x,4r)$ centered at $x\in\partial E$, E satisfies the assumption of Lemma 3.1.5 with a universal constant $C=C(n,A)$ provided r is smaller than a universal radius $\bar r$. This proves the first part of the proposition.

To prove the second part we use properties of minimizers which are scaling invariant. So, without loss of generality we may assume $x=0$ and $r=1$. Let $\varphi\in C^1_c(\mathbf{D})$, we may also assume $\|\nabla\varphi\|_\infty=1$. We can construct a family of diffeomorphisms $\{\psi_t(x)\}_t$ of type $\psi_t(x)=x+t\varphi(\mathbf{p}x)e_n$ such that $\psi_t(E)\Delta E\subset\subset\mathbf{C}_2$. Let $E_t:=\psi_t(E)$. By plugging in (2.38)

$$F_t := (E_t\cap\mathbf{C}_2)\cup(E\setminus\mathbf{C}_2)\,,$$

we obtain

$$P(E)\le P(F_t)+\Lambda_2|E\Delta F_t|+\Lambda_2Q^2\int_{E\Delta F_t}|\nabla u|^2\, dx, \tag{3.30}$$

Therefore,

$$P(E,\mathbf{C}_2) \le P(E_t,\mathbf{C}_2) + \Lambda_2 |E\Delta E_t \cap \mathbf{C}_2| + \Lambda_2 Q^2 \int_{E\Delta E_t \cap \mathbf{C}_2} |\nabla u|^2\, dx. \tag{3.31}$$

By [23, Lemma 17.9] and upper density estimates of the perimeter we have

$$|E\Delta E_t \cap \mathbf{C}_2| \le C\,|t|\,P(E,C_2) \le C(n,A)\,|t|, \quad \forall |t| \text{ small enough.}$$

By exploiting the area formula, [23, Theorem 11.6] and [23, Lemma 23.10] one can prove the following inequality

$$P(E,\mathbf{C}_2) \le P(E_t,\mathbf{C}_2) - t\int_{\partial E\cap \mathbf{C}} (\nu_E\cdot\nabla f)(\nu_E\cdot e_n)\, d\mathcal{H}^{n-1} + C(n)\,t^2 \quad \forall |t| \text{ small enough.}$$

The above equations yield

$$|t|\,\left|\int_{\partial E\cap\mathbf{C}} (\nu_E\cdot\nabla f)(\nu_E\cdot e_n)\, d\mathcal{H}^{n-1}\right| \le C(n)|t|^2 + \Lambda_2\, C(n,A)\,|t| + \Lambda_2 Q^2 \int_{E\Delta E_t\cap\mathbf{C}_2} |\nabla u|^2\, dx. \tag{3.32}$$

On the other hand

$$\int_{\partial E\cap\mathbf{C}\setminus\Gamma_f} (\nu_E\cdot\nabla f)(\nu_E\cdot e_n)\, d\mathcal{H}^{n-1} \le \|\nabla\varphi\|_\infty\, \mathcal{H}^{n-1}(\partial E\cap\mathbf{C}\Delta\Gamma_f), \quad \text{and}$$
$$\int_{\partial E\cap\mathbf{C}\cap\Gamma_f} (\nu_E\cdot\nabla f)(\nu_E\cdot e_n)\, d\mathcal{H}^{n-1} = \int_{\mathbf{p}(\partial E\cap\mathbf{C}\cap\Gamma_f)} \frac{\nabla f\cdot\nabla\varphi}{\sqrt{1+|\nabla f|^2}}\, dz.$$

Furthermore,

$$\begin{aligned}\int_{\mathbf{p}(\partial E\cap\mathbf{C}\cap\Gamma_f)} \frac{\nabla f\cdot\nabla\varphi}{\sqrt{1+|\nabla f|^2}}\, dz &\le \|\nabla\varphi\|_\infty\, \mathcal{H}^{n-1}(\mathbf{p}(\partial E\cap\mathbf{C}\Delta\Gamma_f))\\ &\le \|\nabla\varphi\|_\infty\, \mathcal{H}^{n-1}(\partial E\cap\mathbf{C}\Delta\Gamma_f).\end{aligned}$$

Hence by choosing $|t|$ small enough in (3.32) and by the third inequality in (3.16) we obtain

$$\left|\int_{\mathbf{D}} \frac{\nabla f\cdot\nabla\varphi}{\sqrt{1+|\nabla f|^2}}\, dz\right| \le C_{\text{lip}}\,\|\nabla\varphi\|_\infty \left(\mathbf{e}_E\,(x,2,e_n) + 1 + Q^2\int_{\mathbf{C}_2} |\nabla u|^2\, dx\right), \tag{3.33}$$

for every $\varphi\in C^1_c(\mathbf{D})$. Thanks to (3.33) we prove (3.29) indeed

$$\begin{aligned}\left|\frac{\nabla f\cdot\nabla\varphi}{\sqrt{1+|\nabla f|^2}} - \nabla f\cdot\nabla\varphi\right| &= |\nabla f\cdot\nabla\varphi|\,\frac{\sqrt{1+|\nabla f|^2}-1}{\sqrt{1+|\nabla f|^2}}\\ &\le \frac{|\nabla\varphi|}{2}\,\frac{|\nabla f|^2}{\sqrt{1+|\nabla f|^2}} \le \frac{|\nabla\varphi|}{2}\,|\nabla f|^2,\end{aligned} \tag{3.34}$$

where in the first inequality we have used $\sqrt{1+x}-1 \le \frac{x}{2}$ for every $x>0$ and in the last inequality the first of (3.16). By integrating (3.34) over $\mathbf{D}$ and by using the second estimate in (3.16) and (3.33) we obtain (3.29).

□

3.1.2 The Caccioppoli inequality

By (3.29) one will deduce that under the assumption of Theorem 3.1.1, there exists an harmonic function $h : \mathbf{D}_r \to \mathbb{R}$ which is close to f in L^2. This closeness, together with the regularity theory for harmonic function will allow to deduce the decay of an L^2 type excess of f and thus for E. In order to pass from the L^2 excess to the classical one, one needs to esablish a Caccioppoli type inequality. To this end we introduce the following notion.

Definition 3.1.8 (Flatness)**.** Given a set $E \subset \mathbb{R}^n$ of finite perimeter we define the *flatness* of E at the point $x \in \mathbb{R}^n$, at the scale $r > 0$ with respect to the direction $\nu \in \mathbb{S}^{n-1}$ as

$$\mathbf{f}_E(x,r,\nu) := \frac{1}{r^{n-1}} \inf_{h\in\mathbb{R}} \int_{\mathbf{C}(x,r,\nu)\cap\partial^* E} \frac{|\nu\cdot(y-x)-h|^2}{r^2}\, d\mathcal{H}^{n-1}(y). \tag{3.35}$$

Proposition 3.1.9 (Caccioppoli inequality)**.** *Let $A > 0$, and let β, K, Q be controlled by A and $R \geq 1$. Then there exist universal constants ε_{cac}, and C_{cac} such that if E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), $x \in \partial E$, and*

$$r + \mathbf{e}_E(x, 4r, e_n) \leq \varepsilon_{\text{cac}},$$

then

$$\mathbf{e}_E(x,r,e_n) \leq C_{\text{cac}}\Big(\mathbf{f}_E(x,2r,e_n) + r + Q^2 D_E(x,2r)\Big). \tag{3.36}$$

In order to prove Proposition 3.1.9 we start with the following lemma which is a weak version of the Caccioppoli inequality.

Notation 3.1.10. Let $x \in \mathbb{R}^n$, $s, r \geq 0$. We denote

$$\mathbf{C}^r_s(x) := \{y \in \mathbb{R}^n \,:\, |\mathbf{p}(y-x)| < r\,, |\mathbf{q}(x-y)| < s\}\,.$$

Lemma 3.1.11 (Weak reverse Caccioppoli inequality)**.** *Let $E \subseteq \mathbb{R}^n$ be a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) in $\mathbf{C}_{4r}(x_0)$ and suppose $(u,\rho) \in \mathcal{A}(E)$ minimizes $\mathcal{G}_{\beta,K}(E)$. Assume that*

$$\begin{aligned} &|\mathbf{q}(x-x_0)| < \frac{r}{8} \qquad \forall x \in \mathbf{C}_{2r}(x_0)\cap\partial E,\\ &\left|\left\{x \in \mathbf{C}_{2r}(x_0)\setminus E \,:\, \mathbf{q}(x-x_0) < -\frac{r}{8}\right\}\right| = \left|\left\{x \in \mathbf{C}_{2r}(x_0)\cap E \,:\, \mathbf{q}(x-x_0) > \frac{r}{8}\right\}\right|. \end{aligned} \tag{3.37}$$

Then if

$$\mathbf{C}^r_s(y) \subseteq \boldsymbol{C}_{2r}(x_0) \quad \textit{and} \quad \mathcal{H}^{n-1}(\partial E \cap \partial\mathbf{C}^r_s(y)) = 0, \tag{3.38}$$

we have that for every $|h| < \frac{r}{4}$ the following ineqality holds

$$\begin{aligned} &P\left(E, \mathbf{C}^r_{\frac{s}{2}}(y)\right) - \mathcal{H}^{n-1}\left(\mathbf{D}_{\frac{s}{2}}(\mathbf{p}y)\right)\\ &\lesssim \Big(P(E,\mathbf{C}^r_s(y)) - \mathcal{H}^{n-1}(\mathbf{D}_s(\mathbf{p}y))\Big)^{\frac{1}{2}} \left\|\frac{(\mathbf{q}\cdot - h)^2}{s^2}\right\|_{L^2(\mathbf{C}^r_s(y),\, \mathcal{H}^{n-1}_{\llcorner \partial E})} +\\ &+ r\, s^{n-1} + Q^2 \int_{\mathbf{C}^r_{rs}(y)} |\nabla u|^2\, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.39}$$

Proof. Without loss of generality we may assume $x_0 = y = 0$. Since E is a set of finite perimeter we can construct a sequence $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ of open sets with ∂E_k of class C^∞ and $\varepsilon_k \to 0^+$ such that

$$\chi_{E_k} \to \chi_E \text{ locally in } L^1, \;\; \mathcal{H}^{n-1}_{\llcorner \partial E_k} \overset{*}{\rightharpoonup} \mathcal{H}^{n-1}_{\llcorner \partial E} \;\; \textit{and} \;\; \partial E_k \subseteq (\partial E)^{\varepsilon_k}, \tag{3.40}$$

where we have denoted by $(\partial E)^\varepsilon$ the ε-neighborhood of ∂E. Hence, for k sufficiently large we obtain

$$\begin{gathered} |\mathbf{q}x| < \frac{r}{4} \;\; \forall x \in \mathbf{C}_{2r} \cap \partial E_k, \\ \left\{x \in \mathbf{C}_{2r} \,:\, \mathbf{q}x < -\frac{r}{4}\right\} \subseteq \mathbf{C}_{2r} \cap E_k \subseteq \left\{x \in \mathbf{C}_{2r} \,:\, \mathbf{q}x < \frac{r}{4}\right\}. \end{gathered} \tag{3.41}$$

Notice that for every k the sets $E_k^r := \frac{E_k}{r}$ satisfy (3.41) with $r = 1$. By the coarea formula and (3.40) we deduce

$$\lim_{k\to\infty} \mathcal{H}^{n-1}\left(\partial \mathbf{C}_{r's} \cap (E_k^r)^{(1)} \Delta E_k^r\right) = 0. \tag{3.42}$$

for almost every $s \in \left(\frac{2}{3}, \frac{3}{4}\right)$. Now fix $\vartheta \in \left(0, \frac{1}{4}\right)$ and $|h| < \frac{1}{4}$. Thus we can apply [23, Lemma 24.8] to each E_k^r with respect to ϑ and h. Therefore, one can see that there exists $r' \in \left(\frac{2}{3}, \frac{3}{4}\right)$, a (not relabeled) subsequence of $\{E_k\}_{k\in\mathbb{N}}$ and a family of open sets $\{F_k\}_{k\in\mathbb{N}}$ such that

$$F_k \cap \partial \mathbf{C}_{r's} = E_k^r \cap \partial \mathbf{C}_{rs'}, \tag{3.43}$$

and satisfy the following property

$$\begin{aligned} P\left(F_k, \mathbf{C}_{r's}\right) - \mathcal{H}^{n-1}\left(\mathbf{D}_s\right) \leq C(n)\,\vartheta\,\left(P\left(E_k^r, \mathbf{C}_s\right) - \mathcal{H}^{n-1}\left(\mathbf{D}_s\right)\right) \\ + \frac{C(n)}{\vartheta}\left\|\frac{(\mathbf{q}\cdot - h)^2}{s^2}\right\|_{L^2(\mathbf{C}_s,\, \mathcal{H}^{n-1}_{\llcorner \partial^* E_k^r})}. \end{aligned} \tag{3.44}$$

For every $k \in \mathbb{N}$ let us define

$$G_k = \left(rF_k \cap \mathbf{C}^r_{rr's}\right) \cup \left(E \setminus \mathbf{C}^r_{rr's}\right). \tag{3.45}$$

Let $\sigma_k^r := \mathcal{H}^{n-1}\left(\partial \mathbf{C}^r_{rr's} \cap (E_k)^{(1)} \Delta E_k\right)$. Hence, by (3.42) we have $\lim_{k\to\infty} \sigma_k^r = 0$. Note that $G_k \Delta E \subset\subset \mathbf{C}^r_{rr's} \subset \mathbf{C}_{2r}$.Thanks to the properties of minimizers we can apply Proposition 2.3.5 for each k with $F = G_k$ in order to obtain

$$P\left(E, \mathbf{C}^r_{rr's}\right) \leq P\left(rF_k, \mathbf{C}^r_{rr's}\right) + \sigma_k^r + \Lambda_2 \left|(E\Delta rF_k) \cap \mathbf{C}^r_{rr's}\right| + \Lambda_2 Q^2 \int_{\mathbf{C}^r_{rr's}} |\nabla u|^2 \, dx. \tag{3.46}$$

Therefore, (3.46) yields

$$P(E^r, \mathbf{C}_{r's}) \leq P(F_k, \mathbf{C}_{r's}) + \sigma_k^r + \Lambda_2\, r\, \left|(E^r \Delta F_k) \cap \mathbf{C}_{r's}\right| + \Lambda_2 Q^2 \int_{\mathbf{C}_{r's}} |\nabla u^r|^2 \, dx, \tag{3.47}$$

where $u^r(x) := \frac{u(rx)}{\sqrt{r}}$ for every $x \in \mathbb{R}^n$. By combining (3.44) and (3.47) we get

$$\begin{aligned} & P(E^r, \mathbf{C}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(\mathbf{D}_{\frac{s}{2}}) \leq P(E^r, \mathbf{C}_{r's}) - \mathcal{H}^{n-1}\left(\mathbf{D}_s\right) \\ & \leq C(n)\,\vartheta\,\left(P\left(E_k^r, \mathbf{C}_s\right) - \mathcal{H}^{n-1}\left(\mathbf{D}_s\right)\right) + \frac{C(n)}{\vartheta}\left\|\frac{(\mathbf{q}\cdot - h)^2}{s^2}\right\|_{L^2\left(\mathbf{C}_s, \mathcal{H}^{n-1} \llcorner \partial^* E_k^r\right)} \\ & + \sigma_k^r + \Lambda_2\, r\, \left|(E^r \Delta F_k) \cap \mathbf{C}_{r's}\right| + \Lambda_2 Q^2 \int_{\mathbf{C}_{r's}} |\nabla u^r|^2 \, dx. \end{aligned} \tag{3.48}$$

Let $k \to \infty$ in (3.48), by (3.38) and $|(E^r \Delta F_k) \cap \mathbf{C}_{r's}| \leq |\mathbf{C}_{r's}| \leq C(n)\, s^{n-1}$ we obtain

$$\begin{aligned} & P(E^r, \mathbf{C}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(\mathbf{D}_{\frac{s}{2}}) \\ & \leq C(n)\,\vartheta\,\left(P\left(E^r, \mathbf{C}_s\right) - \mathcal{H}^{n-1}\left(\mathbf{D}_s\right)\right) + \frac{C(n)}{\vartheta}\left\|\frac{(\mathbf{q}\cdot - h)^2}{s^2}\right\|_{L^2\left(\mathbf{C}_s, \mathcal{H}^{n-1} \llcorner \partial^* E^r\right)} \\ & + \Lambda_2 C(n)\, r\, s^{n-1} + \Lambda_2 Q^2 \int_{\mathbf{C}_{r's}} |\nabla u^r|^2 \, dx. \end{aligned} \tag{3.49}$$

Hence for every $0 < \vartheta < \frac{1}{4}$ we have

$$\begin{aligned} &P(E, \mathbf{C}^r_{r\frac{s}{2}}) - \mathcal{H}^{n-1}\left(\mathbf{D}_{r\frac{s}{2}}\right) \\ &\leq C(n)\,\vartheta\,\left(P\left(E, \mathbf{C}^r_{rs}\right) - \mathcal{H}^{n-1}\left(\mathbf{D}_{rs}\right)\right) + \frac{C(n)}{\vartheta}\left\|\frac{(\mathbf{q}\cdot -rh)^2}{(rs)^2}\right\|_{L^2\left(\mathbf{C}_{rs}, \mathcal{H}^{n-1}\llcorner\partial^* E\right)} \\ &+ \Lambda_2 C(n)\, r\,(rs)^{n-1} + \Lambda_2 Q^2 \int_{\mathbf{C}^r_{rr's}} |\nabla u|^2\, dx. \end{aligned} \tag{3.50}$$

By Proposition 3.1.4 one can prove that (3.50) holds true for every $\vartheta > 0$. Then by minimizing in ϑ we get the result. □

Proof of Proposition 3.1.9. Without loss of generality we may assume $x = 0$ and $\nu = e_n$. First of all prove the following claim.
Claim: The following inequality holds true

$$\frac{P\left(E, \mathbf{C}_r\right) - \mathcal{H}^{n-1}\left(\mathbf{D}_r\right)}{r^{n-1}} \lesssim \left(\frac{1}{r^{n-1}}\int_{\mathbf{C}_{2r}\cap\partial E}\frac{|\mathbf{q}x - h|^2}{r^2}\, d\mathcal{H}^{n-1} + 2r + Q^2\, D_E(0, 2r)\right). \tag{3.51}$$

Proof of Claim: Since E is a minimizer then it satisfies density perimeter bounds on $\mathbf{C}_{4r}$ for some $r \in (0,1)$ sufficiently small. Hence $E^r := \frac{E}{r}$ satisfies hypothesis of Lemma 3.1.3. Let $\varepsilon_{\text{cac}} \in (0,1)$ be such that $\omega(\varepsilon_{\text{cac}}) < \frac{1}{8}$, where ω is the function as in Lemma 3.1.3. Thus since $\mathbf{e}(E^r, 0, 4, e_n) < \varepsilon_{\text{cac}}$, we have

$$\begin{aligned} &\sup_{x\in\mathbf{C}_2\cap\partial E^r} |\mathbf{q}x| < \frac{1}{8}, \\ &\left|\left\{x \in \mathbf{C}_2 \cap E^r \,:\, \mathbf{q}x > \frac{1}{8}\right\}\right| = 0, \quad \text{and} \quad \left|\left\{x \in \mathbf{C}_2 \setminus E^r \,:\, \mathbf{q}x < -\frac{1}{8}\right\}\right| = 0. \end{aligned} \tag{3.52}$$

Clearly by scaling we obtain

$$\begin{aligned} &\sup_{x\in\mathbf{C}_{2r}\cap\partial E} |\mathbf{q}x| < \frac{r}{8}, \\ &\left|\left\{x \in \mathbf{C}_{2r} \cap E \,:\, \mathbf{q}x > \frac{r}{8}\right\}\right| = 0, \quad \text{and} \quad \left|\left\{x \in \mathbf{C}_{2r} \setminus E \,:\, \mathbf{q}x < -\frac{r}{8}\right\}\right| = 0. \end{aligned} \tag{3.53}$$

By (3.53) and Lemma 3.1.4 we can define the excess measure μ as follows

$$\mu(G) := P\left(E, \mathbf{C}_{2r} \cap \mathbf{p}^{-1}(G)\right) - \mathcal{H}^{n-1}\left(G\right), \quad \forall\, G \subseteq \mathbf{D}_{2r} \ \text{ Borel.} \tag{3.54}$$

Moreover,

$$\mathcal{H}^{n-1}(G) = \int_{\mathbf{C}_{2r}\cap\partial^* E\cap\mathbf{p}^{-1}(G)} (\nu_E \cdot e_n)\, d\mathcal{H}^{n-1}, \quad \forall\, G \subseteq \mathbf{D}_{2r} \ \text{ Borel.} \tag{3.55}$$

Fix $|h| < \frac{r}{4}$. Let $y \in \mathbb{R}^n$ and $s > 0$ be such that

$$\mathbf{D}_{2s}(y) \subseteq \mathbf{D}_{2r} \quad \text{and} \quad \mathcal{H}^{n-1}\left(\partial E \cap \partial \mathbf{C}^r_{2s}(y)\right) = 0, \tag{3.56}$$

The weak Caccioppoli inequality (see Proposition (3.1.11)) yields

$$\begin{aligned} \mu(\mathbf{D}_s(y)) &\lesssim \left(\mu(\mathbf{D}_{2s}(y))\right)^{\frac{1}{2}}\left\|\frac{(\mathbf{q}\cdot -h)^2}{s^2}\right\|_{L^2\left(\mathbf{C}^r_s(y),\, \mathcal{H}^{n-1}\llcorner\partial E\right)} \\ &+ r\, s^{n-1} + Q^2 \int_{\mathbf{C}^r_{2s}(y)} |\nabla u|^2\, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.57}$$

Let us set

$$d := \inf_{|h|<\frac{r}{4}} \int_{\mathbf{C}_{2r}\cap\partial E} |\mathbf{q}x - h|^2 \, d\mathcal{H}^{n-1}. \tag{3.58}$$

Hence for every $\mathbf{D}_{2s}(y) \subseteq \mathbf{D}_{2r}$ (which implies $s \le r < 1$) we get

$$\begin{aligned} s^2\,\mu(\mathbf{D}_s(y)) &\lesssim \sqrt{s^2\,\mu(\mathbf{D}_{2s}(y))\,d} + s^2\,r^n + s^2\,Q^2 \int_{\mathbf{C}^r_{2s}(y)} |\nabla u|^2\,dx. \\ &\lesssim \sqrt{s^2\,\mu(\mathbf{D}_{2s}(y))\,d} + r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_{2s}(y)} |\nabla u|^2\,dx. \end{aligned} \tag{3.59}$$

Define

$$S := \sup\left\{s^2\mu(\mathbf{D}_s(y)) \,:\, \mathbf{D}_{2s}(y) \subseteq \mathbf{C}_{2r}\right\}. \tag{3.60}$$

Clearly $S < +\infty$, indeed by perimeter density bounds we have that for every $\mathbf{D}_s(y) \subseteq \mathbf{D}_{2r}$ we have $s^2\mu(\mathbf{D}_s(y)) \le \mu(\mathbf{D}_{2r}) \le P(E, \mathbf{C}_{2r}) < +\infty$. Let us fix $\mathbf{D}_{2s}(y) \subseteq \mathbf{D}_{2r}$ and cover $\mathbf{D}_s(y)$ by finitely many balls $\{\mathbf{D}_{\frac{s}{4}}(y_k)\}_{k\in\mathbb{N}}^N$ centered at $y_k \in \mathbf{D}_s(y)$. Therefore,

$$\begin{aligned} s^2\,\mu(\mathbf{D}_s(y)) &\lesssim \sum_{k=1}^N \left(\frac{s}{4}\right)^2 \mu(\mathbf{D}_{\frac{s}{4}}(y_k)) \\ &\lesssim \sum_{k=1}^N \sqrt{\left(\frac{s}{2}\right)^2 \mu(\mathbf{D}_{\frac{s}{2}}(y_k))\,d} + N\left(r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_r(y)} |\nabla u|^2\,dx\right) \\ &\lesssim \sqrt{S\,d} + r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_r(y)} |\nabla u|^2\,dx. \end{aligned} \tag{3.61}$$

By arbitrariness of $\mathbf{D}_s(y)$ we can pass to the sup on the left hand side of (3.61)

$$S \lesssim \sqrt{S\,d} + r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_r(y)} |\nabla u|^2\,dx. \tag{3.62}$$

Now we distinguish two cases:
Case 1: Suppose $\sqrt{S\,d} \le r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_r(y)} |\nabla u|^2\,dx$. Then

$$S \lesssim r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_r(y)} |\nabla u|^2\,dx. \tag{3.63}$$

Case 2: Suppose $\sqrt{S\,d} \ge r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_r(y)} |\nabla u|^2\,dx$. Then $S \lesssim \sqrt{S\,d}$, which is equivalent to

$$S \lesssim d. \tag{3.64}$$

Therefore, in both cases

$$S \lesssim d + r^{n+2} + r^2\,Q^2 \int_{\mathbf{C}^r_r(y)} |\nabla u|^2\,dx. \tag{3.65}$$

By applying (3.65) with $y = 0$, $s = r$ and by dividing (3.65) by r^{n+1} we prove the claim for every $|h| \le \frac{r}{4}$. Clearly for every $|h| \ge \frac{r}{4}$ we have

$$\int_{\mathbf{C}_{2r}\cap\partial E} \frac{(\mathbf{q}x - h)^2}{r^2}\,d\mathcal{H}^{n-1} \ge \frac{P(E, C_r)}{8^2}, \tag{3.66}$$

hence the claim is proved for every $|h| \ge \frac{r}{4}$. Thus the claim follows.

By (3.55) the claim implies the result. □

3.1.3 Dirichlet improvement

We now show that in the small excess regime there is fixed scale decay of the Dirichlet energy.

Proposition 3.1.12 (Decay of the Dirichlet energy)**.** *Let $A > 0$, and let β, K, Q be controlled by A and $R \geq 1$. There exists a universal constant $C_{\text{dir}} > 0$ such that for all $\lambda \in (0, 1/2)$ there exists $\varepsilon_{\text{dir}} = \varepsilon_{\text{dir}}(n, A, \lambda)$ satisfying the following: if E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), $x \in \partial E$ and*

$$r + \mathbf{e}_E(x, r, e_n) \leq \varepsilon_{\text{dir}}, \tag{3.67}$$

then

$$D_E(x, \lambda r) \leq C_{\text{dir}} \lambda \Big(D_E(x, r) + r \Big).$$

Proof. By (2.66) and (2.76) we have that if r is universally small we can apply Lemma 3.1.3 to E in $\mathbf{C}(x, r)$ to obtain a universal modulus of continuity ω such that for $H = \{y : \mathbf{q}(y - x) \leq 0\}$,

$$\frac{|(E \Delta H) \cap B_{r/2}(x)|}{|B_{r/2}|} \leq \omega(\varepsilon_{\text{dir}}).$$

By Lemma 2.5.4 (ii) (applied in $B_{r/2}(x)$) and the above inequality, for all $\lambda \in (0, 1/2)$ we can choose $\varepsilon_{\text{dir}} = \varepsilon_{\text{dir}}(n, A, \lambda)$ sufficiently small such that

$$\begin{aligned} D_E(x, \lambda r) &\leq C(n, A) \lambda \left(D_E(x, \frac{r}{2}) + \frac{r^3}{\lambda^n} \right) \\ &\leq C(n, A) \lambda \left(D_E(x, r) + \frac{\varepsilon_{\text{dir}}^2 r}{\lambda^n} \right) \leq C(n, A) \lambda \big(D_E(x, r) + r \big), \end{aligned}$$

where in the first inequality we have also exploited (2.6) and in the second the obvious inequality $D_E(x, r/2) \leq 2^{n-1} D_E(x, r)$. This concludes the proof. □

3.1.4 Excess improvement

In this section we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. We claim that there exists a universal constant C_{exc} such that for all $\lambda \in (0, 1/8)$ there exists $\varepsilon_{\text{exc}} = \varepsilon_{\text{exc}}(n, A, \lambda)$ satisfying the following: for all minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) with β, K, Q controlled by A and $R \geq 1$ if $x \in \partial E$ the following holds

$$\mathbf{e}_E(x, r) + Q^2 D_E(x, r) + r \leq \varepsilon_{\text{exc}} \quad \Longrightarrow \quad \mathbf{e}_E(x, \lambda r) \leq C_{\text{exc}} \lambda \Big(\mathbf{e}_E(x, r) + Q^2 D_E(x, r) + r \Big).$$

Note that the above claim, combined with Proposition 3.1.12 immediately implies the conclusion of the Theorem. Let us assume hence that there exists $\lambda \in (0, 1/8)$ a sequence of minimizers $E_k \subset B_{R_k}$ with parameters β_k, K_k, Q_k controlled by A, radii r_k and points $x_k \in \partial E_k$ such that

$$\varepsilon_k = \mathbf{e}_{E_k}(x_k, r_k) + Q^2 D_E(x_k, r_k) + r_k \to 0$$

but

$$\mathbf{e}_{E_k}(x_k, \lambda r_k) \geq C_{\text{exc}} \lambda \varepsilon_k \tag{3.68}$$

for a suitable universal constant C_{exc}. Note that up to translating and rotating we can assume that $x_k = 0$ and that

$$\mathbf{e}_{E_k}(0, r_k) = \mathbf{e}_{E_k}(0, r_k, e_n).$$

We apply Proposition 3.1.7 to each E_k. Hence, there exists a sequence of 1-Lipschitz functions $f_k : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\frac{\mathcal{H}^{n-1}\left(\mathbf{C}_{\frac{r_k}{2}} \cap \partial E_k \Delta \Gamma_{f_k}\right)}{r_k^{n-1}} \leq 2^{n-1} C_{\text{lip}} \varepsilon_k \,, \tag{3.69a}$$

$$\frac{1}{r_k^{n-1}} \int_{\mathbf{D}_{r_k/2}} |\nabla f_k|^2 \, dx \leq 2^{n-1} C_{\text{lip}} \varepsilon_k \,, \tag{3.69b}$$

$$\|f_k\|_\infty \leq \omega(\varepsilon_k) r_k, \tag{3.69c}$$

$$\left| \fint_{\mathbf{D}_{\frac{r_k}{2}}} \nabla f_k \cdot \nabla \varphi \, dx \right| \leq 2^{n-1} C_{\text{lip}} \|\nabla \varphi\|_\infty \varepsilon_k \qquad \text{for all } \varphi \in C^1_c(\mathbf{D}_{\frac{r_k}{2}}). \tag{3.69d}$$

Let us set

$$g_k := \frac{f_k^{r_k} - m_k}{\sqrt{\varepsilon_k}} \quad \text{where} \quad m_k := \fint_{\mathbf{D}_{\frac{r_k}{2}}} f_k^{r_k} \,, \quad \text{and} \quad f_k^{r_k}(z) := \frac{f_k(r_k z)}{r_k}.$$

By the Poincaré -Wirtinger inequality and (3.69b),

$$\sup_k \|g_k\|_{W^{1,2}(\mathbf{D}_{\frac{1}{2}})} \leq 2^{n-1} C_{\text{lip}} \,. \tag{3.70}$$

Hence there exists g in $W^{1,2}(\mathbf{D}_{\frac{1}{2}})$ such that $g_k \rightharpoonup g$ weakly in $W^{1,2}(\mathbf{D}_{\frac{1}{2}})$ to some g and strongly in $L^2(\mathbf{D}_{\frac{1}{2}})$. Moreover by (3.69d), for all $\varphi \in C^1_c(\mathbf{D}_{\frac{1}{2}})$

$$\begin{aligned} \left| \int_{\mathbf{D}_{\frac{1}{2}}} \nabla g \cdot \nabla \varphi \, dx \right| &= \lim_{k \to +\infty} \frac{1}{\sqrt{\varepsilon_k}} \left| \int_{\mathbf{D}_{\frac{1}{2}}} \nabla f_k^{r_k} \cdot \nabla \varphi \, dx \right| \\ &= \lim_{k \to +\infty} \frac{1}{\sqrt{\varepsilon_k}} \left| \fint_{\mathbf{D}_{\frac{r_k}{2}}} \nabla f_k \cdot \nabla \varphi_{r_k} \, dx \right| = 0, \end{aligned} \tag{3.71}$$

where $\varphi_{r_k}(z) = r_k \varphi(z/r_k) \in C^1_c(\mathbf{D}_{\frac{r_k}{2}})$ satisfies $\|\nabla \varphi_{r_k}\|_\infty = \|\nabla \varphi\|_\infty$. Hence g is harmonic. By the mean value property and (3.70)

$$\sup_{\mathbf{D}_{1/4}} |\nabla^2 g|^2 \leq C(n) \int_{\mathbf{D}_{\frac{1}{2}}} |\nabla g|^2 \, dx \leq C(n, A).$$

By Taylor expansion,

$$|g(z) - g(0) - \nabla g(0) \cdot z| \leq C(n, A)|z|^2 \qquad \text{for all } z \in \mathbf{D}_{\frac{1}{4}}. \tag{3.72}$$

If $2\lambda \in (0, 1/4)$ we can integrate the above inequality to get

$$\fint_{\mathbf{D}_{2\lambda}} |g(z) - g(0) - \nabla g(0) \cdot z|^2 \, dz \leq C(n, A) \, \lambda^4.$$

Recall that, by the mean value property of harmonic functions, for every $r \leq \frac{1}{2}$ we have

$$(g)_r := \fint_{\mathbf{D}_r} g \, dx = g(0) \quad \text{and} \quad (\nabla g)_r = \nabla g(0).$$

Hence,

$$\begin{aligned} \lim_{k \to +\infty} \fint_{\mathbf{D}_{2\lambda}} |g_k(z) - (g_k)_{2\lambda} - (\nabla g_k)_{2\lambda} \cdot z|^2 \, dz &= \fint_{\mathbf{D}_{2\lambda}} |g(z) - (g)_{2\lambda} - (\nabla g)_{2\lambda} \cdot z|^2 dz \\ &= \fint_{\mathbf{D}_{2\lambda}} |g(z) - g(0) - \nabla g(0) \cdot z|^2 \, dz \\ &\leq C(n, A) \lambda^4. \end{aligned}$$

which, by the definition of g_k and changing variables implies

$$\lim_{k\to+\infty}\frac{1}{\varepsilon_k(\lambda r_k)^{n+1}}\int_{\mathbf{D}_{2\lambda r_k}}\Big|f_k(z)-(f_k)_{2\lambda r_k}-(\nabla f_k)_{2\lambda r_k}\cdot z\Big|^2\,dz\le C(n,A)\,\lambda^2. \tag{3.73}$$

Let us define

$$\nu_k:=\frac{\Big(-(\nabla f_k)_{2\lambda r_k}\,,1\Big)}{\sqrt{1+\Big|(\nabla f_k)_{2\lambda r_k}\Big|^2}}\qquad h_k:=\frac{(f_k)_{2\lambda r_k}}{\sqrt{1+\Big|(\nabla f_k)_{2\lambda r_k}\Big|^2}},$$

and note that, by (3.69b), Jensen inequality and (3.69c)

$$|\nu_k-e_n|^2\le C\Big(\fint_{\mathbf{D}_{\lambda r_k}}|\nabla f_k|\,dx\Big)^2\le C(n,A,\lambda)\varepsilon_k\qquad\text{and}\qquad|h_k|\le C\omega(\varepsilon_k)r_k. \tag{3.74}$$

Since the f_k's are 1-Lipschitz, (3.73) implies

$$\begin{aligned}&\limsup_{k\to+\infty}\frac{1}{\varepsilon_k(\lambda r_k)^{n+1}}\int_{\Gamma_{f_k}\cap\mathbf{C}_{2\lambda r_k}}|\nu_k\cdot x-h_k|^2\,d\mathcal{H}^{n-1}(x)\\&\qquad\le\lim_{k\to+\infty}\frac{\sqrt{2}}{\varepsilon_k(\lambda r_k)^{n+1}}\int_{\mathbf{D}_{2\lambda r_k}}\Big|f_k(z)-(f_k)_{2\lambda r_k}-(\nabla f_k)_{2\lambda r_k}\cdot z\Big|^2\,dz\le C(n,A)\,\lambda^2.\end{aligned}$$

and thus

$$\limsup_{k\to+\infty}\frac{1}{\varepsilon_k(\lambda r_k)^{n+1}}\int_{\Gamma_{f_k}\cap\partial E_k\cap\mathbf{C}_{2\lambda r_k}}|\nu_k\cdot x-h_k|^2\,d\mathcal{H}^{n-1}(x)\le C(n,A)\,\lambda^2. \tag{3.75}$$

On the other hand, (3.69a), Lemma (3.1.3) and (3.74) imply

$$\begin{aligned}&\frac{1}{\varepsilon_k(\lambda r_k)^{n+1}}\int_{(\partial E_k\setminus\Gamma_{f_k})\cap\mathbf{C}_{2\lambda r_k}}|\nu_k\cdot x-h_k|^2\,d\mathcal{H}^{n-1}(x)\\&\qquad\le C(n,A,\lambda)\frac{\mathcal{H}^{n-1}\big((\partial E_k\Delta\Gamma_{f_k})\cap\mathbf{C}_{r_k}\big)}{\varepsilon_k r_k^{n-1}}\left(|\nu_k-e_n|^2+\sup_{x\in\partial E_k\cap\mathbf{C}_{r_k}}\frac{|\mathbf{q}x|}{r_k^2}+\frac{|h_k|^2}{r_k^2}\right)\\&\qquad\le C(n,A,\lambda)\frac{\mathcal{H}^{n-1}\big((\partial E_k\Delta\Gamma_{f_k})\cap\mathbf{C}_{r_k}\big)}{\varepsilon_k r_k^{n-1}}\big(\varepsilon_k+\omega(\varepsilon_k)\big)=o(1).\end{aligned} \tag{3.76}$$

Combining (3.75) and (3.76) we deduce that

$$\begin{aligned}&\limsup_{k\to\infty}\frac{\mathbf{f}_{E_k}(0,2\lambda r_k,\nu_k)}{\varepsilon_k}\\&\qquad\le\limsup_{k\to\infty}\frac{1}{\varepsilon_k(\lambda r_k)^{n+1}}\int_{\partial E_k\cap\mathbf{C}_{2\lambda r_k}}|\nu_k\cdot x-h_k|^2\,d\mathcal{H}^{n-1}(x)\le C(n,A)\,\lambda^2.\end{aligned} \tag{3.77}$$

On the other hand, by the perimeter density estimates (2.66) and (3.74)

$$\begin{aligned}\mathbf{e}_{E_k}(0,4\lambda r_k,\nu_k)&\le\frac{1}{(4\lambda r_k)^{n-1}}\int_{\partial E_k\cap\mathbf{C}_{4\lambda r_k}}\frac{|\nu_{E_k}-\nu_k|^2}{2}\,d\mathcal{H}^{n-1}\\&\le C(n,\lambda)\left(\mathbf{e}_{E_k}(0,r_k,e_n)+|e_n-\nu_k|^2\frac{P(E,B_{r_k})}{r_k^{n-1}}\right)=o(1).\end{aligned}$$

Hence we can apply Proposition 3.1.9 in $B_{4\lambda r_k}$ to get that

$$\begin{aligned}\mathbf{e}_{E_k}(0,\lambda r_k) &\le \mathbf{e}_{E_k}(0,\lambda r_k,\nu_k)\\ &\le C_{\mathrm{cac}}\Big(\mathbf{f}_{E_k}(0,2\lambda r_k,\nu_k)+Q^2D_{E_k}(0,2\lambda r_k)+\lambda r_k\Big),\end{aligned}\tag{3.78}$$

where in the first inequality we have used (3.2). Furthermore, by Proposition 3.1.12 applied in B_{r_k} we have

$$Q^2D_{E_k}(0,2\lambda r_k)\le C_{\mathrm{dir}}\lambda(Q^2D_{E_k}(0,r_k)+Q^2r_k)\le C(n,A)\lambda\varepsilon_k.\tag{3.79}$$

Combining (3.77), (3.78) and (3.79) we thus infer that

$$\limsup_{k\to\infty}\frac{\mathbf{e}(0,\lambda r_k)}{\varepsilon_k}\le C(n,A)\lambda\,,$$

in contradiction with (3.68) if C_{exc} is chosen big enough depending only on n and A. □

3.2 Proof of Theorems 1 and 2

In this section we prove Theorem 1 and 2. Theorem 2 is an immediate consequence of the following slightly more general theorem.

Theorem 3.2.1. *Let $A>0$ $\vartheta\in(0,1)$, and let β,K,Q be controlled by A and $R\ge 1$. There exist constants $C_{\mathrm{reg}}(n,A,\vartheta)>0$ and $\varepsilon_{\mathrm{reg}}=\varepsilon_{\mathrm{reg}}(n,A,\vartheta)>0$ if E is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), $x\in\partial E$, $r>0$ and $\nu\in S^{n-1}$ are such that*

$$r+Q^2D_E(x,2r)+\mathbf{e}_E(x,2r,\nu)\le\varepsilon_{\mathrm{reg}},$$

then there exists a $C^{1,\vartheta}$ function $f:\mathbb{R}^{n-1}\to\mathbb{R}$ with [1]

$$f(0)=0\,,\quad |\nabla f(0)-\nu|^2+r^{\vartheta}[\nabla f]^2_{\vartheta/2}\le C_{\mathrm{reg}}\big(r+Q^2D_E(x,2r)+\mathbf{e}_E(x,2r,\nu)\big)\,,$$

such that

$$E\cap B_r(x)=\Big\{y\in B_r(x):\nu\cdot(y-x)\le f(\mathbf{p}^{\nu}(y-x))\Big\}.$$

Proof. Given $\vartheta\in(0,1)$ we fix $\bar\lambda\in(0,1/8)$ be such that

$$C_{\mathrm{dec}}\bar\lambda+\bar\lambda\le\bar\lambda^{\vartheta},\tag{3.80}$$

and we let $\bar\varepsilon$ be the corresponding $\varepsilon_{\mathrm{dec}}$ in Theorem 3.1.1. Note that $\bar\varepsilon$ depends only on n, A and ϑ. We now choose $\varepsilon_{\mathrm{reg}}$ so that for all $y\in\partial E\cap B_r(x)$

$$\begin{aligned}r+Q^2D_E(y,r)+\mathbf{e}_E(y,r)&\le r+Q^2D_E(y,r,\nu)+\mathbf{e}_E(y,r,\nu)\\ &\le 2^{n-1}\big(r+Q^2D_E(x,2r,\nu)+\mathbf{e}_E(x,2r,\nu)\big)\le 2^{n-1}\varepsilon_{\mathrm{reg}}\le\bar\varepsilon.\end{aligned}$$

Hence we can apply Theorem 3.1.1 and (3.80) to deduce that for all $y\in\partial E\cap B_{r/2}(x)$,

$$\bar\lambda r+Q^2D_E(y,\bar\lambda r)+\mathbf{e}_E(y,\bar\lambda r)\le\bar\lambda^{\vartheta}\big(r+Q^2D_E(y,r)+\mathbf{e}_E(y,r)\big).$$

[1] Here

$$[\nabla f]_{\vartheta/2}:=\sup_{x\neq y}\frac{|\nabla f(x)-\nabla f(y)|}{|x-y|^{\frac{\vartheta}{2}}}.$$

Iterating we get
$$\mathbf{e}_E(y,\bar\lambda^k r)\le \bar\lambda^{k\vartheta}\big(r+Q^2D_E(y,r)+\mathbf{e}_E(y,r)\big),$$
which implies
$$\mathbf{e}_E(y,s)\le C(\vartheta)\Big(\frac{s}{r}\Big)^{\vartheta}\big(r+Q^2D_E(y,r)+\mathbf{e}_E(y,r)\big)\qquad \text{for all } s\le r.$$
By classical arguments this together with the density estimates (2.66) and (2.76), imply that for all $y\in B_r(x)\cap\partial E$ there exists ν_y such that
$$\mathbf{e}_E(y,s/2,\nu_y)=C(n,\vartheta,A)\Big(\frac{s}{r}\Big)^{\vartheta}\big(r+Q^2D_E(y,r)+\mathbf{e}_E(y,r)\big)\qquad \text{for all } s\le r.$$
and
$$|\nu_y-\nu|^2\le C(n,A)\big(r+Q^2D_E(y,2r,\nu)+\mathbf{e}_E(y,r)\big),$$
The last two display yield the desired conclusion, see for instance [23, Theorem 26.3] or [17, Theorem 4.8]. □

We can now prove Theorem 1 by following the arguments [15].

Proof of Theorem 1. By Theorem 2, if we set
$$\Sigma_E=\Big\{x\in\partial E:\limsup_{r\to0}\mathbf{e}_E(x,r)+D_E(x,r)>0\Big\}\,,$$
then $\partial E\setminus\Sigma_E$ is a $C^{1,\vartheta}$ manifold for all $\vartheta\in(0,1/2)$. Hence we will conclude the proof if we show that
$$\mathcal{H}^{n-1-\eta}(\Sigma_E)=0,$$
for some $\eta=\eta(n,B)>0$. Recall that by Lemma 2.5.2, $|\nabla u_E|^{2p}\in L^1_{\rm loc}$ for some $p=p(n,B)>1$, hence, by Hölder inequality
$$\Sigma_E^1=\Big\{x:\limsup_{r\to0}D_E(x,r)>0\Big\}\subset\Bigg\{x:\limsup_{r\to0}\frac{1}{r^{n-p}}\int_{B(x,r)}|\nabla u_E|^{2p}>0\Bigg\}.$$
Hence, by [13, Theorem 2.10], $\mathcal{H}^{n-p}(\Sigma_E^1)=0$. We now show that
$$\mathcal{H}^{\alpha}(\Sigma_E\setminus\Sigma_E^1)=0$$
for all $\alpha>n-8$ which clearly concludes the proof. Let us fix $\alpha>n-8$ and assume the contrary. By [17, Proposition 11.3], there will be a point
$$x\in\Sigma_E^2:=\Big\{x\in\partial E:\limsup_{r\to0}\mathbf{e}_E(x,r)>0\,,\lim_{r\to0}D_E(x,r)=0\Big\},$$
and a sequence $r_k\to0$ such that
$$\limsup_{k\to\infty}\frac{\mathcal{H}^{\alpha}_{\infty}(\Sigma_E^2\cap B(x,r_k))}{r_k^{\alpha}}\ge c(\alpha)>0.$$
where $\mathcal{H}^s_\infty$ is the infinity Hausdorff pre-measure. Let us set $E_k=(E-x)/r_k$ and note that by (2.66) and the above equation
$$P(E_k,B_s)\lesssim s^{n-1}\qquad,$$

for all $s > 0$ and

$$\limsup_{k\to\infty} \mathcal{H}^{\alpha}_{\infty}(\Sigma^2_{E_k} \cap B_1) \geq c(\alpha), \tag{3.81}$$

where $\Sigma^2_{E_k} = (\Sigma^2_E - x)/r_k$. Up to subsequences, $E_k \to F$. We claim that F is a local minimizer of the perimeter. Indeed if $G\Delta F \Subset B_s$, by averaging we choose $t \in (s, 2s)$ such that

$$\mathcal{H}^{n-1}((E_k\Delta G) \cap \partial B_t) = \mathcal{H}^{n-1}((E_k\Delta F) \cap \partial B_t) \leq \frac{|(E_k\Delta F) \cap B_{2s}|}{s} = \sigma_k \to 0.$$

With this choice, defining $G_k = (x + r_k G) \cap B_{tr_k}(x) \cup (E \setminus B_{tr_k}(x))$ and note that $E\Delta G_k \Subset B_{2sr_k}(x)$. Hence by (2.38) and classical computations

$$\begin{aligned} P(F, B_t) - P(G, B_t) &\leq \limsup_{k\to\infty} \frac{P(E_k, B_{tr_k}(x)) - P(G_k, B_{tr_k}(x))}{r_k^{n-1}} \\ &\lesssim \limsup_{k\to\infty} \sigma_k + s^n r_k + s^{n-1} D_E(x, sr_k) = 0, \end{aligned}$$

which implies the desired minimality property. Moreover, by using $G = F$ we also deduce that $P(E_k, B_s) \to P(F, B_s)$ for almost all $s > 0$.

Let now Σ_F be the singular set of F, and recall that, by the regularity theory for set of minimal perimeter [23, Part III], $\mathcal{H}^{\alpha}(\Sigma_F) = \mathcal{H}^{\alpha}_{\infty}(\Sigma_F) = 0$. Hence by the definition of Hausdorff measure, for all $\delta > 0$ there exists an open set U_δ such that

$$\Sigma_F \cap B_2 \subset U_\delta \qquad \text{and} \qquad \mathcal{H}^{\alpha}_{\infty}(U_\delta) \leq \delta.$$

We claim that there exists $k = k_\delta > 0$ such that $\Sigma^2_{E_k} \cap B_1 \subset U_\delta$ which will be in contradiction with (3.81) if δ is chosen small enough. Assume the claim is false, hence there is a sequence of points $\Sigma^2_{E_k} \cap B_1 \ni y_k \to \bar{y} \in \overline{B_1}$ with $\mathrm{dist}(\bar{y}, \Sigma_F) > 0$. It is easy to see that, by the lower perimeter estimates (2.76), $\bar{y} \in \partial F$. Hence by regularity, for all $\varepsilon > 0$ there exists $r > 0$ such that

$$\mathbf{e}_F(\bar{y}, r) \leq \varepsilon.$$

By perimeter convergence, this implies that, for k large

$$\mathbf{e}_E(x + r_k y_k, r r_k) = \mathbf{e}_{E_k}(y_k, r) \leq \mathbf{e}_F(\bar{y}, r) + \varepsilon \leq 2\varepsilon.$$

Choosing $\varepsilon \ll 1$ we can apply Theorem 2 to deduce that $x + r_k y_k \notin \Sigma^2_E$, i.e. $y_k \notin \Sigma^2_{E_k}$. This final contradiction concludes the proof.

□

4

Higher regularity

Contents

In this chapter we improve Theorem 2 obtained in Chapter 3. Precisely, we deduce the partial smooth regularity of minimizers.

4.1 $C^{2,\vartheta}$-regularity

The first step is to obtain more regularity for a couple $(u,\rho) \in \mathcal{A}(E)$, where $E \subset \mathbb{R}^n$ is a minimizer of the problem $(\mathcal{P}_{\beta,K,Q,R})$: we prove that u is $C^{1,\eta}$-regular up to the boundary of E. We start with some preliminary results.

Notation 4.1.1. Let $E \subset \mathbb{R}^n$ be such that $\partial E \cap \mathbf{C}(x_0,r)$ is described by the graph of a regular function f.

- If $x \in \mathbb{R}^n$, we write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.
- We denote ν_E the outer-unit normal to ∂E. Moreover, we extend ν_E at every point in the following way

$$\nu_E(x', x_n) = \nu_E(x', f(x')) \quad \forall x = (x', x_n) \in \mathbf{C}(x_0, r).$$

- Let u be a solution of

$$-\mathrm{div}(a_E \nabla u) = \rho_E \quad \text{in } \mathcal{D}'\left(B_r(x_0)\right),$$

where

$$\rho_E \in L^\infty\left(B_r(x_0)\right) \quad \text{and} \quad a_E = \beta \mathbf{1}_E + \mathbf{1}_{E^c}.$$

We will denote by

$$T_E u := \left(\partial_{\nu_E^\perp} u, (1 + (\beta - 1)\mathbf{1}_E)\partial_{\nu_E} u\right),$$

where

$$\partial_{\nu_E^\perp} u := \nabla u - (\nabla u \cdot \nu_E)\, \nu_E \quad \text{and} \quad \partial_{\nu_E} u := (\nabla u \cdot \nu_E)\, \nu_E.$$

- Let $g \in L^1(B_r(x))$. We write

$$[g]_{x,r} := \fint_{B_r(x)} g \, dx,$$

the *mean value* of g. For simplicity we denote $[g]_r := [g]_{0,r}$.

First of all we recall the following lemma, the proof can be found in [4, Theorem 7.53].

Lemma 4.1.2. *Let v be a solution of*

$$-\mathrm{div}(a_H \nabla v) = \rho_H \quad \text{in } \mathcal{D}'\left(B_1(x_0)\right),$$

where $\rho_H \in L^\infty\left(B_1(x_0)\right)$ and

$$H := \{y \in \mathbb{R}^n \,:\, (y - x_0) \cdot e_n \leq 0\}, \quad a_H = \beta \mathbf{1}_H + \mathbf{1}_{H^c}.$$

Then there exist $\gamma \in (0,1)$ and a constant $C_0 = C_0(n, \beta, \|\rho_H\|_\infty) > 0$ such that

$$\int_{B_{\lambda r}(x_0)} |T_H v - [T_H v]_{x_0,\lambda r}|^2 \, dx \leq C_0 \lambda^{n+2\gamma} \int_{B_r(x_0)} |T_H v - [T_H v]_{x_0,r}|^2 \, dx + C_0\, r^{n+1}, \tag{4.1}$$

for all $\lambda \in (0,1)$ small enough, where $T_H v := \big(\partial_1 v, \ldots, \partial_{n-1} v, (1 + (\beta - 1)\mathbf{1}_E)\partial_n v\big)$.

By arguing similarly to the proof of Lemma 4.1.2 one can show the following lemma.

Lemma 4.1.3. *Let $H \subset \mathbb{R}^n$ be the half space. Let $v \in W^{1,2}(B_r)$ be a solution of*

$$-\,\mathrm{div}(A \nabla v) = \mathrm{div}\, G, \quad \text{in } \mathcal{D}'\left(B_r\right),$$

where

$$G^+ := G\, \mathbf{1}_{H^c} \in C^{0,\alpha}(H^c), \quad G^- := G\, \mathbf{1}_H \in C^{0,\alpha}(H),$$

A is an elliptic matrix and $A^+ = A\, \mathbf{1}_{H^c}$, $A^- = A\, \mathbf{1}_H$ have coefficients respectively in $C^{0,\alpha}(B_r \cap \overline{H}^c)$ and $C^{0,\alpha}(B_r \cap \overline{H})$. Then

$$v^+ := v\, \mathbf{1}_{H^c} \in C^{1,\alpha}(B_{r/2} \cap \overline{H}^c), \quad v^- := v\, \mathbf{1}_H \in C^{1,\alpha}(B_{r/2} \cap \overline{H}).$$

Moreover, there exists a constant $C := C\left(r, \|G^+\|_{C^{0,\alpha}}, \|G^-\|_{C^{0,\alpha}}, \|A^+\|_{C^{0,\alpha}}, \|A^-\|_{C^{0,\alpha}}\right) > 0$ such that

$$[\nabla v^+]_{C^{1,\alpha}(\overline{H}^c \cap B_{r/2})} \leq C \quad \text{and} \quad [\nabla v^-]_{C^{1,\alpha}(\overline{H} \cap B_{r/2})} \leq C. \tag{4.2}$$

Lemma 4.1.4. *Given a minimizer E of ($\mathcal{P}_{\beta,K,Q,R}$) let $(u, \rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$. Assume that $\partial E \cap \mathbf{C}(x_0, r)$ is a $C^{1,\vartheta}$-manifold. Then for every $\gamma \in (0,1)$ there exists $0 < \bar{r} \leq r$ and $C > 0$ such that the following inequality holds true*

$$Q^2 \int_{B_s(x_0)} |\nabla u|^2 \, dx \leq C\, s^{n-\gamma}, \tag{4.3}$$

for every $s \leq \bar{r}$.

Proof. Fix $\gamma \in (0,1)$. Choose $\lambda \in (0,1/4)$ such that

$$C_{\text{dec}}\,\lambda \le \lambda^{1-\gamma},$$

where C_{dec} is as in Theorem 3.1.1. Let $s = s(\lambda) < 1$ be such that

$$C_{\text{dir}}(C_{\text{e}}+1)\,s(\lambda) \le \frac{\varepsilon_{\text{dec}}(\lambda)}{2}, \tag{4.4}$$

where ε_{dec} , C_{dir} and C_{e} are as in Theorem 3.1.1, Theorem 3.1.12 and Lemma 2.5.6. Define

$$\varepsilon(\lambda) := \min\left\{ s^{n-1}\frac{\varepsilon_{\text{dec}}(\lambda)}{2}, \varepsilon_{\text{dir}}(\lambda)\right\}.$$

Since $\partial E \cap \mathbf{C}(x_0, r)$ is regular we can take a radius $0 < \bar{r} < r$ such that

$$\bar{r} + \mathbf{e}_E(x_0,\bar{r}) \le \varepsilon(\lambda).$$

Then, thanks to the definition of $\varepsilon(\lambda)$, Theorem 3.1.12 and (4.4) we have

$$Q^2\, D_E(x_0, s\bar{r}) \le C_{\text{dir}} s\,(Q^2\, D_E(x_0,\bar{r}) + \bar{r}) \le C_{\text{dir}}(C_{\text{e}}+1)s \le \frac{\varepsilon_{\text{dec}}(\lambda)}{2}. \tag{4.5}$$

Furthermore, notice that

$$s\bar{r} + \mathbf{e}_E(x_0, s\bar{r}) \le \bar{r} + \frac{1}{s^{n-1}}\mathbf{e}_E(x_0,\bar{r}) \le \frac{\varepsilon_{\text{dec}}(\lambda)}{2}. \tag{4.6}$$

By combining (4.5) and (4.6) we have

$$s\bar{r} + Q^2\, D_E(x_0, s\bar{r}) + \mathbf{e}_E(x_0, s\bar{r}) \le \varepsilon_{\text{dec}}(\lambda). \tag{4.7}$$

The hypothesis of Theorem 3.1.12 are satisfied, hence (recall $\lambda s\bar{r} \le \varepsilon_{\text{dec}}(\lambda)$)

$$\begin{aligned} &Q^2\, D_E(x_0, \lambda s\bar{r}) + \mathbf{e}_E(x_0, \lambda s\bar{r}) + \lambda r \le \lambda^{1-\gamma}\,\left(\mathbf{e}_E(x_0, s\bar{r}) + Q^2\, D_E(x_0, s\bar{r}) + s\bar{r}\right) \\ &\le \lambda^{1-\gamma}\varepsilon_{\text{dec}}(\lambda) \le \varepsilon_{\text{dec}}(\lambda). \end{aligned} \tag{4.8}$$

By exploiting again Theorem 3.1.12 we obtain

(4.9)

$$\begin{aligned} Q^2\, D_E(x_0, \lambda^2 s\bar{r}) + \mathbf{e}_E(x_0, \lambda^2 s\bar{r}) + \lambda^2 s\bar{r} &\le \lambda^{(1-\gamma)}\,\left(\mathbf{e}_E(x_0, \lambda s\bar{r}) + Q^2\, D_E(x_0, \lambda s\bar{r}) + \lambda s\bar{r}\right) \\ &\le \lambda^{2(1-\gamma)}\,\left(\mathbf{e}_E(x_0, s\bar{r}) + Q^2\, D_E(x_0, s\bar{r}) + s\bar{r}\right) \\ &\le \lambda^{2(1-\gamma)}\varepsilon_{\text{dec}}(\lambda) \le \varepsilon_{\text{dec}}(\lambda). \end{aligned}$$

By iterating this argument k times we conclude that

$$Q^2\, D_E(x_0, \lambda^k s\bar{r}) + \mathbf{e}_E(x_0, \lambda^k s\bar{r}) + \lambda^k s\bar{r} \le \lambda^{k(1-\gamma)}\varepsilon_{\text{dec}}(\lambda), \quad \forall k \in \mathbb{N}. \tag{4.10}$$

In particular, the above equation yields

$$Q^2\, D_E(x_0, \lambda^k s\bar{r}) \le \lambda^{k(1-\gamma)}\varepsilon_{\text{dec}}(\lambda), \quad \forall k \in \mathbb{N}. \tag{4.11}$$

Therefore,

$$Q^2 \int_{B_{\lambda^k s\bar{r}}(x_0)} |\nabla u|^2\, dx \le C\,(\lambda^k s\bar{r})^{(n-\gamma)}, \quad \forall k \in \mathbb{N}, \tag{4.12}$$

for some constant $C > 0$. □

Proposition 4.1.5. *Let E be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u,\rho)\in\mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$ and $f\in C^{1,\vartheta}(\mathbf{D}(x_0',r))$. Suppose that $Q\le 1$ and*

$$E\cap\mathbf{C}(x_0,r)=\big\{x=(x',x_n)\in\mathbf{D}(x_0',r)\times\mathbb{R}\,:\,x_n<f(x')\big\}\cap\mathbf{C}(x_0,r),$$

for some $0<r\le\min\{\bar r,1\}$, where $\bar r$ is as in Lemma 4.1.4. Then there exist $\alpha=\alpha(\vartheta)\in(0,1)$ and a constant $C:=C(n,\beta,\vartheta,\|\rho\|_\infty)>0$ such that

$$Q^2\int_{B_{\lambda r}(x_0)}|T_Eu-[T_Eu]_{x_0,\lambda r}|^2\,dx\le C\,Q^2\lambda^{n+2\alpha}\int_{B_r(x_0)}|T_Eu-[T_Eu]_{x_0,r}|^2\,dx+C\,r^{n+\alpha},\tag{4.13}$$

Proof. Without loss of generality assume $0\in\partial E$, $x_0=0$. Note that during this proof the constant $C=C(n,\beta,\gamma,\|\rho\|_\infty)>0$ changes line by line. Let $\lambda\in(0,1/2)$ be given and let v be the solution of

$$\begin{cases}-\mathrm{div}(a_H\nabla v)=\rho & \text{in } B_{r/2},\\ v=u & \text{on } \partial B_{r/2}.\end{cases}$$

In particular, $w=v-u\in W_0^{1,2}(B_{r/2})$ and

$$-\mathrm{div}(a_H\nabla w)=-\mathrm{div}\left((a_E-a_H)\nabla u\right).\tag{4.14}$$

Since $[T_Eg]_s$ minimizes the functional $m\mapsto\int_{B_s}|T_Eg-m|^2\,dx$ we have

$$\begin{aligned}\int_{B_{\lambda r}}|T_Eu-[T_Eu]_{\lambda r}|^2\,dx&\le 2\left(\int_{B_{\lambda r}}|T_Hu-[T_Eu]_{\lambda r}|^2\,dx+\int_{B_{\lambda r}}|T_Eu-T_Hu|^2\,dx\right)\\&\le 2\left(\int_{B_{\lambda r}}|T_Hu-[T_Hu]_{\lambda r}|^2\,dx+\int_{B_{\lambda r}}|T_Eu-T_Hu|^2\,dx\right).\end{aligned}\tag{4.15}$$

We want now to estimate the first term in the right hand side of (4.15). Notice that, since $u=v-w$, by linearity of T_H we have

$$|T_Hu-[T_Hu]_{\lambda r}|^2\le 2\left(|T_Hv-[T_Hv]_{\lambda r}|^2+|T_Hw-[T_Hw]_{\lambda r}|^2\right).$$

Hence, integrating the above inequality on $B_{\lambda r}$ we obtain

$$\begin{aligned}\int_{B_{\lambda r}}|T_Hu-[T_Hu]_{\lambda r}|^2\,dx&\le 2\left(\int_{B_{\lambda r}}|T_Hv-[T_Hv]_{\lambda r}|^2\,dx+\int_{B_{\lambda r}}|T_Hw-[T_Hw]_{\lambda r}|^2\,dx\right)\\&\le 2\left(\int_{B_{\lambda r}}|T_Hv-[T_Hv]_{\lambda r}|^2\,dx+\int_{B_{\lambda r}}|T_Hw-[T_Hw]_{r/2}|^2\,dx\right).\end{aligned}$$

Therefore, (recall the definition of w)

$$\begin{aligned}\int_{B_{\lambda r}}|T_Hu-[T_Hu]_{\lambda r}|^2\,dx&\le 2\left(\int_{B_{\lambda r}}|T_Hw|^2\,dx+\int_{B_{\lambda r}}|T_Hv-[T_Hv]_{\lambda r}|^2\,dx\right)\\&\le C\left(\int_{B_{\lambda r}}|\nabla w|^2\,dx+\int_{B_{\lambda r}}|T_Hv-[T_Hv]_{\lambda r}|^2\,dx\right).\end{aligned}\tag{4.16}$$

To estimate the second term in the right hand side of (4.15) recall the Notation 4.1.1

$$\partial_{\nu_E^\perp}u=\nabla u-(\nabla u\cdot\nu_E)\,\nu_E\quad\text{and}\quad\partial_{e_n^\perp}u=\nabla u-(\nabla u\cdot e_n).$$

Hence

$$|T_Eu-T_Hu|=|(\nabla u\cdot\nu_E)\,\nu_E-(\nabla u\cdot e_n)\,e_n|\le 2|\nabla u|\,|\nu_E-e_n|.$$

Therefore,

$$\int_{B_{\lambda r}} |T_E u - T_H u|^2\, dx \le 4 \int_{B_{\lambda r}} |\nabla u|^2\, |\nu_E - e_n|^2\, dx. \tag{4.17}$$

Combining (4.15), (4.16) and (4.17) we obtain

$$\begin{aligned}\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\, dx &\le C \int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\, dx \\ &\quad + C \int_{B_{r/2}} |\nabla w|^2\, dx + C \int_{B_{\lambda r}} |\nabla u|^2\, |\nu_E - e_n|^2\, dx.\end{aligned} \tag{4.18}$$

By Lemma 4.1.2 we have

$$\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\, dx \le C\, \lambda^{n+2\gamma} \int_{B_{r/2}} |T_H v - [T_H v]_r|^2\, dx + C\, r^{n+1}. \tag{4.19}$$

By arguing as above one can easily see that

$$\begin{aligned}\int_{B_{r/2}} |T_H v - [T_H v]_{\lambda r}|^2\, dx &\le C \int_{B_{r/2}} |T_E u - [T_E u]_{\lambda r}|^2\, dx \\ &\quad + C \int_{B_{r/2}} |\nabla w|^2\, dx + C \int_{B_{r/2}} |\nabla u|^2\, |\nu_E - e_n|^2\, dx.\end{aligned} \tag{4.20}$$

Hence

$$\begin{aligned}\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\, dx &\le C\, \lambda^{n+2\gamma} \int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\, dx \\ &\quad + C \int_{B_{r/2}} |\nabla u|^2\, |\nu_E - e_n|^2\, dx + C \int_{B_{r/2}} |\nabla w|^2\, dx.\end{aligned} \tag{4.21}$$

We need to estimate the last two terms in the right hand side of the above inequality. Since $f \in C^{1,\vartheta}(\mathbf{D}_r)$, then there exists a constant $C > 0$ such that

$$\frac{|(E \Delta H) \cap B_r|}{|B_r|} \le C\, r^{\vartheta}. \tag{4.22}$$

By testing (4.14) with w we deduce

$$\int_{B_{r/2}} |\nabla w|^2\ dx \le \int_{B_{r/2}} a_H\ |\nabla w|^2\ dx = \int_{B_{r/2}} (a_E - a_H)\, \nabla u \cdot \nabla w\, dx. \tag{4.23}$$

By applying Hölder inequality in (4.23) we obtain

$$\int_{B_{r/2}} |\nabla w|^2\ dx \le \int_{B_{r/2}} (a_E - a_H)^2\ |\nabla u|^2\ dx \le C(\beta) \int_{(E \Delta H) \cap B_{r/2}} |\nabla u|^2\ dx. \tag{4.24}$$

By the higher integrability Lemma 2.5.2, there exists $p > 1$ such that

$$\left(\frac{1}{|B_{r/2}|} \int_{B_{r/2}} |\nabla u|^{2p} \right)^{\frac{1}{p}} dx \le C\, \frac{1}{|B_r|} \int_{B_r} |\nabla u|^2\, dx + C\, r^{n+2}\, \|\rho\|_\infty^2. \tag{4.25}$$

Hence by exploiting Hölder inequality with exponent p, (4.22) and (4.25) we have

(4.26)
$$\begin{aligned}\int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2\,dx &\le \left|(E\Delta H)\cap B_{r/2}\right|^{1-\frac{1}{p}} \left(\int_{B_{r/2}} |\nabla u|^{2p}\,dx\right)^{\frac{1}{p}} \\ &\le c(2,p)\,|B_r| \left(\frac{|(E\Delta H)\cap B_r|}{|B_r|}\right)^{1-\frac{1}{p}} \left(\frac{1}{|B_{r/2}|}\int_{B_{r/2}} |\nabla u|^{2p}\,dx\right)^{\frac{1}{p}} \\ &\le C\,r^{\vartheta\left(1-\frac{1}{p}\right)} \left\{\int_{B_r} |\nabla u|^2\,dx + r^{n+2}\,\|\rho\|_\infty^2\right\}.\end{aligned}$$

Therefore, (4.24) together with (4.26) (recall $r<1$) yield

$$\int_{B_{r/2}} |\nabla w|^2\,dx \le C\left\{\beta^2\,r^{\vartheta\left(1-\frac{1}{p}\right)}\int_{B_r}|\nabla u|^2 + r^{n+2}\|\rho\|_\infty^2\right\}. \tag{4.27}$$

On the other hand by Lemma 4.1.4 we have

$$Q^2\int_{B_s}|\nabla u|^2\,dx \le C\,s^{n-\gamma} \quad \forall\, s<\bar r. \tag{4.28}$$

Hence by combining (4.27) and (4.28) we obtain

$$\int_{B_r}|\nabla w|^2\,dx \le C\left\{\beta^2\,r^{\vartheta\left(1-\frac{1}{p}\right)+n-\gamma} + r^{n+2}\|\rho\|_\infty^2\right\}. \tag{4.29}$$

Finally, we estimate the second term in (4.21). Notice that

$$\begin{aligned}\int_{B_{r/2}}|\nabla u|^2\,|\nu_E-e_n|^2\,dx &= \int_{B_{r/2}}|\nabla u(x',x_n)|^2\,|\nu_E(x',x_n)-e_n|^2\,dx \\ &= \int_{B_{r/2}}|\nabla u|^2\,|\nu_E(x',f(x'))-e_n|^2\,dx.\end{aligned}$$

Since $\sqrt{1+t}\le 1+\frac{t}{2}$ for every $t>0$, then

$$\left|\nu_E(x',f(x'))-e_n\right|^2 = 2-\frac{2}{\sqrt{1+|\nabla f(x')|^2}} \le 2\left(\frac{\sqrt{1+|\nabla f(x')|^2}-1}{\sqrt{1+|\nabla f(x')|^2}}\right) \le |\nabla f(x')|^2. \tag{4.30}$$

Thanks to (4.28) and (4.30), since ∇f is ϑ-Hölder, we deduce

$$Q^2\int_{B_{r/2}}|\nabla u|^2\,|\nu_E-e_n|^2\,dx \le C\,r^{n+2\vartheta-\gamma}. \tag{4.31}$$

Let

$$\alpha := \min\left\{\gamma, \vartheta\,(1-1/p)-\gamma, 2\vartheta-\gamma\right\}.$$

Therefore, by multiplying (4.21) and (4.27) with Q^2 and by recalling that $Q<1$ we have that (4.31) imply (4.13). □

For completeness we recall the following integral characterization of Hölder continuous functions, [4, Theorem 7.51] and a simple iteration lemma, [4, Lemma 7.54].

Lemma 4.1.6 (Campanato's lemma)**.** *Let $p\ge 1$ and $g\in L^p(B_{2R}(x_0))$. Assume that there exist $\sigma\in(0,1)$ and $c>0$ such that for every $x\in B_R(x_0)$*

$$\frac{1}{|B_r|}\int_{B_r(x)}|g(y)-[g]_{x,r}|^p\,dy \le c^p\left(\frac{r}{R}\right)^{p\sigma}, \quad \forall B_r(x)\subset B_R(x_0). \tag{4.32}$$

Then g is σ-Hölder continuous in $B_R(x_0)$.

Lemma 4.1.7. *Let $0 < q < p$. Suppose that $h : (0, a) \to [0, +\infty)$ is an increasing function such that*

$$h(r) \le c_1 \left(\frac{r}{R}\right)^p h(R) + c_2 R^q, \quad \text{for every } 0 < r < R, \tag{4.33}$$

where c_1 and c_2 are positive constants. Then there exists $c = c(p, q) > 0$ such that

$$h(r) \le c \left\{ \left(\frac{r}{R}\right)^q h(R) + r^q \right\}, \quad \text{for every } 0 < r < R \text{ small enough.} \tag{4.34}$$

We are now ready to prove that u is regular up to the boundary.

Theorem 4.1.8. *Let E be a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), let $(u, \rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$ and $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$. Suppose $Q \le 1$ and*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} : x_n < f(x')\} \cap \mathbf{C}(x_0, r),$$

for some $0 < r \le \min\{\bar{r}, 1\}$, where $\bar{r}$ is as in Lemma 4.1.4. Then there exists $\eta = \eta(\vartheta) \in (0, 1)$ such that $Q\, u_\beta \in C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))$ and $Q\, u_1 \in C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))$, where $u_\beta := u\,\mathbf{1}_E$ and $u_1 := u\,\mathbf{1}_{E^c}$. Furthermore, let $A > 0$ and let β, K, Q be controlled by A and $R \ge 1$ then there exists a universal constant $C = C(n, A) > 0$ such that

$$\|Q\, u_\beta\|_{C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))} \le C \quad \text{and} \quad \|Q\, u_1\|_{C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))} \le C. \tag{4.35}$$

Proof. Let $u_Q := Q\, u$. By Proposition 4.13 there exists $C = C(n, \beta, \gamma, \|\rho\|_\infty) > 0$ such that

$$\int_{B_{\lambda r}(x_0)} |T_E u_Q - [T_E u_Q]_{x_0, \lambda r}|^2\, dx \le C \lambda^{n+2\alpha} \int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x_0, r}|^2\, dx + C\, r^{n+\alpha}, \tag{4.36}$$

where $\alpha \in (0, 1)$ is as in Proposition 4.13. Therefore, by arguing similarly to the proof of [4, Theorem 7.53], Lemma 4.1.7 implies that there exists a universal constant $C = C(n, A) > 0$ such that

$$\frac{1}{|B_r|} \int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x,r}|^2\, dy \le C \left(\frac{r}{R}\right)^{2\eta}, \quad \forall B_r(x_0) \subset B_R. \tag{4.37}$$

for some $\eta = \eta(\vartheta) \in (0, 1)$. Hence, by Lemma 4.1.6, recalling the definition of T_E, we get $u_Q \mathbf{1}_E \in C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))$ and $u_Q \mathbf{1}_{E^c} \in C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))$ and (4.35). □

In the next proposition we rewrite the Euler-Lagrange equation (see Corollary 2.2.3) in a more convenient form by exploiting the regularity of ∂E.

Proposition 4.1.9 (Euler-Lagrange equation)**.** *Let E be a minimizer for ($\mathcal{P}_{\beta,K,Q,R}$) and $(u, \rho) \in \mathcal{A}(E)$. Assume that $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r))$ and*

$$E \cap \mathbf{C}(x_0, r) = \{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} : x_n < f(x')\} \cap \mathbf{C}(x_0, r).$$

Then there exists a constant $C > 0$ such that

(4.38)

$$-\mathrm{div}\left(\frac{\nabla f(x')}{\sqrt{1 + |\nabla f(x')|^2}}\right) = Q^2 \left(\beta |\nabla u_\beta|^2 - |\nabla u_1|^2 - K\, \rho^2\right)(x', f(x')) + Q^2 \left(\beta \partial_n u_\beta\, \nabla u_\beta - \partial_n u_1\, \nabla u_1\right)(x', f(x')) \cdot (-\nabla f(x'), 1) + C,$$

for almost every point $x' \in \mathbf{D}(x_0', r)$, (where u_β and u_1 are defined as in Theorem 4.1.8).

Proof. Let $E \subset \mathbb{R}^n$ be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and let $(u,\rho) \in \mathcal{A}(E)$.

Notice that $E \cap \mathbf{C}(x_0,r)$ is an open set of $\mathbb{R}^n$. Moreover, by an approximation argument, we can integrate over $E \cap \mathbf{C}(x_0,r)$ the following identity,

$$\begin{aligned}|\nabla u_\beta|^2 \operatorname{div}\eta &= \operatorname{div}(|\nabla u_\beta|^2\eta) - \nabla|\nabla u_\beta|^2 \cdot \eta \\ &= \operatorname{div}(|\nabla u_\beta|^2\eta) + 2\operatorname{div}(\nabla u_\beta \nabla u_\beta \cdot \eta) - 2\Delta u_\beta\, \nabla u_\beta \cdot \eta + 2\nabla u_\beta \cdot \nabla\eta\nabla u_\beta,\end{aligned}$$

for every $\eta \in C_c^\infty(\mathbf{C}(x_0,r),\mathbb{R}^n)$. Therefore,

$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)} \left(|\nabla u_\beta|^2 \operatorname{div}\eta - 2\nabla u_\beta \cdot \nabla\eta\nabla u_\beta\right) dx &= \int_{E\cap\mathbf{C}(x_0,r)} \operatorname{div}(|\nabla u_\beta|^2\eta)\, dx \\ &\quad + \int_{E\cap\mathbf{C}(x_0,r)} 2\operatorname{div}(\nabla u_\beta\nabla u_\beta \cdot \eta)\, dx \\ &\quad - \int_{E\cap\mathbf{C}(x_0,r)} 2\Delta u_\beta\, \nabla u_\beta \cdot \eta\, dx.\end{aligned} \tag{4.39}$$

On the other hand since $(u,\rho) \in \mathcal{A}(E)$ then

$$-\beta\Delta u_\beta = \rho, \quad \text{in}\, \mathcal{D}'(E \cap \mathbf{C}(x_0,r)).$$

Moreover, by Proposition 2.1.3 we deduce

$$\nabla u_\beta = -K\,\nabla\rho, \quad \text{in } E \cap \mathbf{C}(x_0,r).$$

Then, by multiplying equation (4.39) by β, we have

$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)} \beta\left(|\nabla u_\beta|^2 \operatorname{div}\eta - 2\nabla u_\beta \cdot \nabla\eta\nabla u_\beta\right) dx &= \int_{E\cap\mathbf{C}(x_0,r)} \beta\operatorname{div}(|\nabla u_\beta|^2\eta)\, dx \\ &\quad + \int_{E\cap\mathbf{C}(x_0,r)} 2\beta\operatorname{div}(\nabla u_\beta\nabla u_\beta \cdot \eta)\, dx \\ &\quad - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho \cdot \eta\, dx.\end{aligned} \tag{4.40}$$

Integrating by parts the first and the second term in the right hand side of (4.40) we can write

$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)} \beta\left(|\nabla u_\beta|^2 \operatorname{div}\eta - 2\nabla u_\beta \cdot \nabla\eta\nabla u_\beta\right) dx &= \int_{\partial E\cap\mathbf{C}(x_0,r)} \beta|\nabla u_\beta|^2\eta \cdot \nu_E\, d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial E\cap\mathbf{C}(x_0,r)} 2\beta\,(\nabla u_\beta \cdot \eta)(\nabla u_\beta \cdot \nu_E)\, d\mathcal{H}^{n-1} \\ &\quad - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho \cdot \eta\, dx.\end{aligned} \tag{4.41}$$

By arguing similarly as above one can also prove

$$\begin{aligned}\int_{E^c\cap\mathbf{C}(x_0,r)} \left(|\nabla u_1|^2 \operatorname{div}\eta - 2\nabla u_1 \cdot \nabla\eta\nabla u_1\right) dx &= \int_{E^c\cap\mathbf{C}(x_0,r)} \operatorname{div}(|\nabla u_1|^2\eta)\, dx \\ &\quad - \int_{E^c\cap\mathbf{C}(x_0,r)} 2\operatorname{div}(\nabla u_1\nabla u_1 \cdot \eta)\, dx.\end{aligned} \tag{4.42}$$

Integrating by parts the right hand side of (4.42) we can write

$$\int_{E^c\cap \mathbf{C}(x_0,r)} \left(|\nabla u_1|^2 \operatorname{div}\eta - 2\nabla u_1\cdot\nabla\eta\nabla u_1\right)dx = -\int_{\partial E\cap\mathbf{C}(x_0,r)} |\nabla u_1|^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1} + \int_{\partial E\cap\mathbf{C}(x_0,r)} 2\,(\nabla u_1\cdot\eta)\,(\nabla u_1\cdot\nu_E)\,d\mathcal{H}^{n-1}. \tag{4.43}$$

Therefore, by combining (4.41) and (4.43) we get

$$\begin{aligned}\int_{\mathbb{R}^n} a_E\left(\operatorname{div}\eta\,|\nabla u|^2 - 2\nabla u\cdot\nabla\eta\nabla u\right)dx &= \int_{\partial E}\left(\beta|\nabla u_\beta|^2 - |\nabla u_1|^2\right)\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &+\int_{\partial E\cap\mathbf{C}(x_0,r)} 2\big(\beta\,(\nabla u_\beta\cdot\eta)(\nabla u_\beta\cdot\nu_E) - (\nabla u_1\cdot\eta)\,(\nabla u_1\cdot\nu_E)\big)\,d\mathcal{H}^{n-1}\\ &-K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\cdot\eta\,dx.\end{aligned}\tag{4.44}$$

Notice that the following identity hold true

$$\begin{aligned}K\int_{\mathbb{R}^n}\rho^2\operatorname{div}\eta &= K\int_{E\cap\mathbf{C}(x_0,r)}\operatorname{div}(\rho^2\eta)\,dx - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\,\cdot\eta dx\\ &= K\int_{\partial E\cap\mathbf{C}(x_0,r)}\rho^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1} - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\,\cdot\eta dx.\end{aligned}\tag{4.45}$$

By combining the Euler-Lagrange equation of Theorem 2.2.3, (4.44) and (4.45) we find

$$\begin{aligned}\int_{\partial E}\operatorname{div}_E\eta\,d\mathcal{H}^{n-1} &= Q^2\int_{\partial E}\left(\beta|\nabla u_\beta|^2 - |\nabla u_1|^2 - K\,\rho^2\right)\,\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &+Q^2\,2\int_{\partial E}\beta(\eta\cdot\nabla u_\beta)\,(\nabla u_\beta\cdot\nu_E) - (\eta\cdot\nabla u_1)\,(\nabla u_1\cdot\nu_E)\,d\mathcal{H}^{n-1},\end{aligned}\tag{4.46}$$

for every $\eta\in C^1_c(B_r(x_0),\mathbb{R}^n)$ with $\int_E \operatorname{div}\eta\,dx = 0$. We need to prove the following claim.
Claim: The function $f:\mathbf{C}(x_0',r)\to\mathbb{R}^n$ is a weak solution (precisely among all test functions with integral 0 on E) of the following equation in $\mathbf{C}(x_0,r)\cap\partial E$:

$$\begin{aligned}-\operatorname{div}\left(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\right) &= Q^2\left(\beta|\nabla u_\beta|^2 - |\nabla u_1|^2 - K\,\rho^2\right)(x',f(x'))\\ &+Q^2\,\left(\beta\partial_n u_\beta\,\nabla u_\beta - \partial_n u_1\,\nabla u_1\right)(x',f(x'))\cdot(-\nabla f(x'),1),\end{aligned}\tag{4.47}$$

for all $x'\in\mathbf{D}(x_0',r)$.
Proof of the claim. The tangential divergence of η on ∂E is

$$\operatorname{div}_E\eta := \operatorname{div}\eta - \sum_{i,j=1}^n(\nu_E)_i\,(\nu_E)_j\,\partial_j\eta_i \quad \text{on } \partial E,\tag{4.48}$$

where $\nu_E:\partial E\to\mathbb{S}^{n-1}$ is the normal vector to ∂E:

$$\nu_E := \frac{1}{\sqrt{1+|\nabla f|^2}}(-\nabla f,1).\tag{4.49}$$

Let $\eta := (0,\dots,0,\eta_n)$, then by (4.48) we have

$$\operatorname{div}_E\eta := \partial_n\eta_n + \frac{1}{1+|\nabla f|^2}\left\{\sum_{j=1}^n\partial_j\eta_n\,\partial_j f - \partial_n\eta_n\right\}\quad\text{on } \partial E.\tag{4.50}$$

Choose $\eta_n(x) := \varphi(\mathbf{p}x)\, s(x_n)$, where $\varphi \in C_c^1(\mathbf{D}(x_0,r))$ and $s : (-1,1) \to \mathbb{R}^n$ is such that $s(t) = 1$ for every $|t| \le \|f\|_\infty$. Since now η_n does not depend on the n-component on ∂E, we have

$$\eta \cdot \nu_E = \frac{\varphi(\mathbf{p}x)}{\sqrt{1+|\nabla f|^2}} \quad \text{on } \partial E \cap \mathbf{C}(x_0,r), \tag{4.51}$$

and the above equation (4.50) reads as

$$\mathrm{div}_E \eta := \frac{1}{1+|\nabla f|^2}\, \nabla\varphi \cdot \nabla f \quad \text{on } \partial E \cap \mathbf{C}(x_0,r). \tag{4.52}$$

Moreover,

$$\begin{aligned}
\int_E \mathrm{div}\eta\, dx &= \int_{\partial E} (\eta\cdot\nu_E)\, d\mathcal{H}^{n-1} = \int_{\partial E\cap \mathbf{C}(x_0,r)} \eta_n(\nu_E\cdot e_n)\, d\mathcal{H}^{n-1} \\
&= \int_{\partial E\cap \mathbf{C}(x_0,r)} \varphi(\mathbf{p}x) s(f(x))\, (\nu_E\cdot e_n)\, d\mathcal{H}^{n-1} \\
&= \int_{\partial E\cap \mathbf{C}(x_0,r)} \frac{\varphi(\mathbf{p}x)}{\sqrt{1+|\nabla f(\mathbf{p}x)|}}\, d\mathcal{H}^{n-1} = \int_{\mathbf{p}(\partial E\cap \mathbf{C}(x_0,r))} \varphi\, dx = 0.
\end{aligned}$$

This implies that η is admissible in (4.46). Hence by using η as a test function in (4.46), by combining (4.51) and (4.52) the claim follows.

The claim clearly implies (4.38). □

We prove now the partial $C^{2,\vartheta}$-regularity of minimizers.

Theorem 4.1.10 ($C^{2,\vartheta}$-regularity)**.** *Given $n \ge 3$, $A > 0$ and $\vartheta \in (0,1/2)$, there exists $\varepsilon_{\rm reg} = \varepsilon_{\rm reg}(n,A,\vartheta) > 0$ such that if E is minimizer of ($\mathcal{P}_{\beta,K,Q,R}$), $Q+\beta+K+\frac{1}{K} \le A$,*

$$x_0 \in \partial E \quad \text{and} \quad r + \mathbf{e}_E(x_0,r) + Q^2\, D_E(x_0,r) \le \varepsilon_{\rm reg},$$

then $E \cap \mathbf{C}(x_0,r/2)$ coincides with the epi-graph of a $C^{2,\vartheta}$-function f. In particular, we have that $\partial E \cap \mathbf{C}(x_0,r/2)$ is a $C^{2,\vartheta}$ $(n-1)$-dimensional manifold and

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0',r/2))} \le C\,(n,A,r,\vartheta)\,. \tag{4.53}$$

Proof. Choose $\varepsilon_{\rm reg}$ as in Theorem 2. Then there exists $f \in C^{1,\vartheta}(\mathbf{D}(x_0',r/2))$ such that

$$E \cap \mathbf{C}(x_0,r/2) = \left\{x = (x',x_n) \in \mathbf{D}(x_0',r/2)\times\mathbb{R} \,:\, x_n < f(x')\right\}.$$

Exploiting Proposition 4.1.9 we have

$$-\,\mathrm{div}\left(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\right) = G(x',f(x')) \quad \text{for a.e. } x' \in \mathbf{D}(x_0',r/2), \tag{4.54}$$

where,

$$\begin{aligned}
G(x',f(x')) &= Q^2\left(\beta|\nabla u_\beta|^2 - |\nabla u_1|^2 - K\rho^2\right)(x',f(x')) \\
&\quad + Q^2\,\left(\beta\partial_n u_\beta\, \nabla u_\beta - \partial_n u_1\, \nabla u_1\right)(x',f(x'))\cdot(-\nabla f(x'),1) + C, \quad x' \in \mathbf{D}(x_0',r/2).
\end{aligned}$$

Hence (4.54) is equivalent to

$$-\,\mathrm{div}\,(M(\nabla f)) = G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0,r/2), \tag{4.55}$$

where

$$M(\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}, \quad \forall \xi \in \mathbb{R}^n.$$

By [23, Theorem 27.1] we can take the derivatives of (4.55). Then,

$$-\operatorname{div}\left(\nabla M(\nabla f)\, \nabla \partial_i f\right) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2) \tag{4.56}$$

for every $i = 1, \dots, n$. Notice that

$$\nabla M(\xi) = \frac{1}{\sqrt{1+|\xi|^2}} \left(\mathrm{Id} - \frac{\xi \otimes \xi}{1+|\xi|^2} \right) \quad \forall \xi \in \mathbb{R}^n.$$

Hence the matrix $\nabla M(\nabla f)$ is uniformly elliptic, more precisely

$$|\eta|^2 \geq \nabla M(\nabla f)\eta \cdot \eta \geq (1 + \|\nabla f\|_\infty)^{-3/2}\, |\eta|^2 \quad \forall \eta \in \mathbb{R}^n.$$

By Theorem 4.1.8 it follows that G is Hölder continuous. By the definition of M and by the regularity of f we also have that $\nabla M(\nabla f)$ is Hölder continuous. Hence the following Schauder estimates hold in this case

$$[\nabla \partial_i f]_{C^{0,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C\left\{ \|\partial_i f\|_{L^2(\mathbf{D}(x_0', r/2))} + [G]_{C^{0,\eta}(\mathbf{D}(x_0', r/2))} \right\} \quad \forall i = 1, \dots, n,$$

for some constant C depending on r. In particular, f is $C^{2,\vartheta}$ and

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C\left\{ \|\nabla f\|_{L^2(\mathbf{D}(x_0', r/2))} + [G]_{C^{0,\eta}(\mathbf{C}(x_0, r/2))} \right\}.$$

By definition of G, recalling (4.35) and Proposition 2.1.3, by Poincaré inequality and since f is Lipschitz, one can easily see that there exists $C = C(n, A, \vartheta, r) > 0$ such that

$$[G]_{C^{0,\vartheta}(\mathbf{C}(x_0, r/2))} \leq C(n, A, \vartheta, r).$$

By the Lipschitz approximation theorem it follows that

$$\frac{1}{r^{n-1}} \int_{\mathbf{D}(x_0', r/2)} |\nabla f|^2 \, dz \leq C_L\, \mathbf{e}_E(x_0, r) \leq C_L\, \varepsilon_{\mathrm{reg}}, \tag{4.57}$$

which implies (4.53). □

4.2 Smooth regularity

In this section, by a bootstrap argument, we obtain the smooth partial regularity of minimizers.

Let us start with the following lemma.

Lemma 4.2.1. *Let $k \in \mathbb{N}$, $k \geq 2$ and f is $C^{k,\vartheta}(\mathbf{D})$. There exists $\varepsilon > 0$ such that if*

$$\|f\|_{C^{2,\vartheta}(\mathbf{D})} \leq \varepsilon \quad \text{and} \quad f(0) = 0,$$

then there exists a diffeomorphism $\Phi \in C^{k-1,\vartheta}$, $\Phi : \mathbf{C}_{1-\varepsilon} \to \mathbf{C}_{1-\varepsilon}$, such that

$$\Phi(\Gamma_f \cap \mathbf{C}_{1-\varepsilon}) = \{x = (x', x_n) \in \mathbf{D}_{1-\varepsilon} \times \mathbb{R} : x_n = 0\},$$

where Γ_f is the graph of f. Moreover,

$$\left(\nabla\Phi(\Phi^{-1}(x))\, (\nabla\Phi(\Phi^{-1}(x)))^T\right)_{jn} = 0 \ \ \forall j \neq n \quad \text{and} \quad \left(\nabla\Phi(\Phi^{-1}(x))\, (\nabla\Phi(\Phi^{-1}(x)))^T\right)_{nn} \neq 0. \tag{4.58}$$

Proof. By defining

$$\Psi(x', x_n) := (x', f(x')) + x_n \frac{(-\nabla f(x'), 1)}{\sqrt{1 + |\nabla f(x')|^2}} \quad \forall x = (x', x_n) \in \mathbf{C}_{1-\varepsilon},$$

one can see that $\Phi := \Psi^{-1}$. □

Theorem 4.2.2 (C^∞-regularity)**.** *Given $n \geq 3$ and $A > 0$, there exists $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A) > 0$ such that if E is minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) with $Q + \beta + K + \frac{1}{K} \leq A$,*

$$x_0 \in \partial E \quad \text{and} \quad r + \mathbf{e}_E(x_0, r) + Q^2 D_E(x_0, r) \leq \varepsilon_{\mathrm{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epi-graph of a C^∞-function f. In particular we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a C^∞ $(n-1)$-dimensional manifold. Moreover, for every $\vartheta \in (0, \frac{1}{2})$ there exists a constant $C(n, A, k, r, \vartheta) > 0$ such that

$$[f]_{C^{k,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C(n, A, k, r, \vartheta), \tag{4.59}$$

for every $k \in \mathbb{N}$.

Proof. If we choose $\varepsilon_{\mathrm{reg}}$ as in Theorem 4.1.10 then there exists $f \in C^{2,\vartheta}(\mathbf{D}(x_0', r/2))$ such that

$$E \cap \mathbf{C}(x_0, r/2) = \left\{ x = (x', x_n) \in \mathbf{D}(x_0', r/2) \times \mathbb{R} \, : \, x_n < f(x') \right\}.$$

Exploiting Proposition 4.1.9 we have

$$-\operatorname{div}\left(\frac{\nabla f(x')}{\sqrt{1 + |\nabla f(x')|^2}} \right) = G(x', f(x')), \quad \text{for a.e. } x' \in \mathbf{D}(x_0, r/2). \tag{4.60}$$

where,

$$\begin{aligned} G(x', f(x')) &= Q^2 \left(\beta |\nabla u_\beta|^2 - |\nabla u_1|^2 - K \rho^2\right)(x', f(x')) \\ &\quad + Q^2 \left(\beta \partial_n u_\beta \, \nabla u_\beta - \partial_n u_1 \, \nabla u_1\right)(x', f(x')) \cdot (-\nabla f(x'), 1) + C, \quad x' \in \mathbf{D}(x_0, r/2). \end{aligned}$$

Hence (4.60) is equivalent to

$$-\operatorname{div}\left(M(\nabla f)\right) = G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2). \tag{4.61}$$

where

$$M(\xi) := \frac{\xi}{\sqrt{1 + |\xi|^2}}, \quad \forall \xi \in \mathbb{R}^n.$$

By [23, Theorem 27.1] we can take the derivatives of (4.61). Then,

$$-\operatorname{div}\left(\nabla M(\nabla f) \, \nabla \partial_i f\right) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2) \tag{4.62}$$

for every $i = 1, \ldots, n$. By arguing as in the proof of Theorem 4.1.10 we can see that the matrix $\nabla M(\nabla f)$ is uniformly elliptic.

Step 1:

$$f \ C^2\text{-Hölder continuous} \implies u^+, u^- \ C^2\text{-Hölder continuous resp. on } \overline{H}^c \cap \mathbf{C}_{r/2}(x_0) \text{ and } \overline{H} \cap \mathbf{C}_{r/2}(x_0).$$

Moreover, there exists a universal constant $C = C(n, A) > 0$ and $\eta \in (0, \frac{1}{2})$ such that

$$\|Q \, u^+\|_{C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}(x_0))} \leq C \quad \text{and} \quad \|Q \, u^-\|_{C^{2,\eta}(\overline{H} \cap \mathbf{C}_{r/2}(x_0))} \leq C. \tag{4.63}$$

Proof of Step 1. Assume $x_0 = 0$. Let $H := \{x \in \mathbb{R}^n : x_n = x \cdot e_n \le 0\}$ be the half space in $\mathbb{R}^n$. By choosing normal coordinates around the graph of f (note that this is possible since f is C^2 and by Lemma 4.2.1), we can assume that

$$\Gamma_f \cap \mathbf{C}_{r/2} = \partial H \cap \mathbf{C}_{r/2},$$

where $\Gamma_f \cap \mathbf{C}_{r/2} := \{(x', f(x')) : x' \in \mathbf{D}_{r/2}\}$ and that u solves the following equation

$$-\operatorname{div}(a_H A \nabla u) = \rho\, \mathbf{1}_H, \tag{4.64}$$

where by (4.58), A is a Hölder continuous elliptic matrix (hence $A_{nn} \neq 0$) such that $A_{jn} = 0$ for every $j \neq n$.

By taking the derivative with respect to the tangential coordinates $j \neq n$ on (4.64) we deduce

$$\begin{aligned} -\operatorname{div}\big(a_H A \nabla \partial_j u\big) &= \partial_j(\rho\, \mathbf{1}_H) + \operatorname{div}\big(\partial_j(a_H A) \nabla u\big) \\ &= \operatorname{div}((\rho\, \mathbf{1}_H)\, e_j + \partial_j(a_H A) \nabla u) \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \end{aligned} \tag{4.65}$$

Notice that a_H is constant along tangential directions and that $(a_H\, A)^+$, $(a_H\, A)^-$ have coefficients respectively in $C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2})$ and $C^{0,\eta}(\overline{H} \cap C_{r/2})$ with a norm which is bounded by a universal constant uniform with respect to Q. Furthermore,

$$((\rho\, \mathbf{1}_H)\, e_j + \partial_j(a_H A) \nabla u)^+ \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}) \;\text{ and }\; ((\rho\, \mathbf{1}_H)\, e_j + \partial_j(a_H A) \nabla u)^- \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_{r/2}).$$

Hence by exploiting Lemma 4.1.3 we deduce

$$\partial_j u^+ \in C^{1,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}) \quad \text{and} \quad \partial_j u^- \in C^{1,\eta}(\overline{H} \cap \mathbf{C}_{r/2}) \qquad \forall j \neq n. \tag{4.66}$$

Furthermore, by (4.64) and multiplying by Q we have

$$-Q \sum_{i,j=1}^{n} \{a_H\, A_{ij} \partial_{ij} u + \partial_i(a_H\, A_{ij}) \partial_j u\} = Q\rho\, \mathbf{1}_H. \tag{4.67}$$

Thanks to the form of the matrix A we obtain

$$-a_H\, A_{nn} \partial_{nn}(Qu) = \sum_{i,j \neq n} \{a_H\, A_{ij} \partial_{ij}(Qu) + \partial_i(a_H\, A_{ij})\, \partial_j(Qu)\} + Q\rho\, \mathbf{1}_H. \tag{4.68}$$

Since the right hand side of the previous equation is Hölder continuous then

$$\partial_{nn}(Qu^+) \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}) \quad \text{and} \quad \partial_{nn}(Qu^-) \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_{r/2}).$$

Moreover (4.66) implies

$$\partial_{nj}(Qu^+) \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}) \quad \text{and} \quad \partial_{nj}(Qu^-) \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_{r/2}),$$

for every $j \neq n$. Therefore,

$$Qu^+ \in C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}) \quad \text{and} \quad Qu^- \in C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}).$$

By (4.2) in Lemma 4.1.3 we deduce also that

$$\|Q\, \nabla u^+\|_{C^{1,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2})} \quad \text{and} \quad \|Q\, \nabla u^-\|_{C^{1,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2})},$$

are bounded by a constant which depends on the Hölder norms of ∇Qu^+, ∇Qu^-, $Q\rho$ (hence by Proposition 2.1.3, Qu^-), the coefficients of $(a_H A)^+$ and $(a_H A)^-$. As mentioned before, the Hölder norms of $(a_H A)^+$ and $(a_H A)^-$ in turn are bounded by a universal constant. Notice that by (4.35) we can bound also

$$\|Q\,\nabla u^+\|_{C^{0,\eta}(\overline{H}^c\cap\mathbf{C}_{r/2})} \quad \text{and} \quad \|Q\,\nabla u^-\|_{C^{0,\eta}(\overline{H}^c\cap\mathbf{C}_{r/2})},$$

by a universal constant. Then we obtain (4.63).

Thus, Step 1 is proved.

Step 2:

$$f\ C^2\text{-Hölder continuous} \implies f\ C^3\text{-Hölder continuous.}$$

Proof of Step 2: By definition of M, since f is C^2-Hölder continuous, we have also that $\nabla M(\nabla f)$ is C^1-Hölder continuous. Step 1 and (4.63) imply that the function G is C^1-Hölder continuous with $[G]_{C^{1,\eta}}$ uniformly bounded . Then by (4.62) we have that Schauder estimates hold in this case and f is C^3-Hölder continuous, thus Step 2 is proved.

Step 3:

$$f\ C^3\text{-Hölder continuous} \implies u^+, u^-\ C^3\text{-Hölder continuous resp. on } \overline{H}^c\cap\mathbf{C}_{r/2}(x_0) \text{ and } \overline{H}\cap\mathbf{C}_{r/2}(x_0).$$

Moreover, there exists a universal constant $C = C(n,A) > 0$ and $\eta\in(0,\frac{1}{2})$ such that

$$\|Q\,u^+\|_{C^{3,\eta}(\overline{H}^c\cap\mathbf{C}_{r/2}(x_0))} \le C \quad \text{and} \quad \|Q\,u^-\|_{C^{3,\eta}(\overline{H}\cap\mathbf{C}_{r/2}(x_0))} \le C. \tag{4.69}$$

Proof of Step 3. We argue as in the proof of Step 1. By taking the derivatives with respect to $i\neq n$ in (4.65) we obtain

$$-\operatorname{div}(a_H A\nabla\partial_{ij}u) = \partial_{ij}\rho\,\mathbf{1}_H + \operatorname{div}(\partial_{ij}(a_H A)\nabla u) + 2\operatorname{div}(\partial_i(a_H\,A)\nabla\partial_j u)\,, \tag{4.70}$$

for every $i,j\neq n$. By Lemma 4.1.3 we deduce

$$\partial_{ij}u^+\in C^{2,\eta}(\overline{H}^c\cap\mathbf{C}_{r/2}) \quad \text{and} \quad \partial_{ij}u^-\in C^{2,\eta}(\overline{H}\cap\mathbf{C}_{r/2}),$$

for every $i,j\neq n$. Furthermore, multiplying (4.65) by Q and thanks to the form of the matrix A we have

$$\begin{aligned} -a_H A_{nn}\partial_{nn}(\partial_j Qu) = \sum_{i,j\neq n}\{a_H A_{ij}\partial_{ij}(\partial_j Qu) + \partial_i(a_H A_{ij})\partial_j(\partial_j Qu)\} + \partial_j Q\rho\,\mathbf{1}_H \\ + \operatorname{div}\big(\partial_j(a_H A)\,\nabla Qu\big) \quad \forall j\neq n. \end{aligned} \tag{4.71}$$

Since the right hand side of the previous equation is Hölder continuous, then

$$\partial_{nnj}(Qu^+)\in C^{0,\eta}(\overline{H}^c\cap\mathbf{C}_{r/2}) \quad \text{and} \quad \partial_{nnj}(Qu^-)\in C^{0,\eta}(\overline{H}\cap\mathbf{C}_{r/2}),$$

for every $j\neq n$. By Lemma 4.1.3 we deduce

$$\partial_{ijn}(Qu^+)\in C^{0,\eta}(\overline{H}^c\cap\mathbf{C}_{r/2}) \quad \text{and} \quad \partial_{ijn}(Qu^-)\in C^{0,\eta}(\overline{H}\cap\mathbf{C}_{r/2}),$$

for every $i,j\neq n$. Notice that dividing (4.68) by $A_{nn}\neq 0$ and deriving along the normal direction then $\partial_{nnn}(Qu^+)$ and $\partial_{nnn}(Qu^-)$ are Hölder continuous respectively in $\overline{H}^c$ and $\overline{H}$.

Therefore,

$$Qu^+ \in C^{3,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2}) \quad \text{and} \quad Qu^- \in C^{3,\eta}(\overline{H} \cap \mathbf{C}_{r/2}).$$

In order to obtain (4.69) we argue similarly to Step 1: by (4.2) in Lemma 4.1.3 we deduce also that

$$\|Q\,\nabla u^+\|_{C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2})} \quad \text{and} \quad \|Q\,\nabla u^-\|_{C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2})},$$

are bounded by a constant which depends on the C^1 Hölder norms of ∇Qu^+, ∇Qu^-, Qu^-, the coefficients of $(a_H A)^+$ and $(a_H A)^-$. Since f is C^3, by definition of A one can see that the C^1 Hölder norms of $(a_H A)^+$ and $(a_H A)^-$ in turn are bounded by a universal constant. Notice that by (4.63) we can bound also

$$\|Q\,\nabla u^+\|_{C^{1,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2})} \quad \text{and} \quad \|Q\,\nabla u^-\|_{C^{1,\eta}(\overline{H}^c \cap \mathbf{C}_{r/2})},$$

by a universal constant. Then we obtain (4.69).

Thus, Step 3 is proved.

Step 4:

$$f\ C^3\text{-Hölder continuous} \implies f\ C^4\text{-Hölder continuous.}$$

Proof of Step 4: By definition of M, since f is C^3-Hölder continuous, we have also that $\nabla M(\nabla f)$ in (4.62) is C^2-Hölder continuous. By Step 3 and (4.69) we deduce that G is C^2-Hölder continuous with its norm uniformly bounded. Then by (4.62) Schauder estimates apply and f is C^4-Hölder continuous.

Iterating this argument we obtain the smoothness of f. □

5

Minimality of the ball in the small charge regime

Contents

In this chapter we investigate the shape of charged liquid droplets minimizing ($\mathcal{P}_{\beta,K,Q,R}$) whenever the total charge $Q > 0$ is small. The final aim in [24] is to deduce the following theorem.

Theorem 5.0.1. *Fix $K > 0$, $\beta > 1$. Then there exists $Q_0 > 0$ such that for all $Q < Q_0$ and any $R \geq 1$ the only minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) are the balls of radius* 1*.*

We mention that, once we get the above result, we solve questions (ii) and (partially) (iii) in the Introduction of the thesis. In order to prove Theorem 5.0.1 we will follow these main steps:

(1) Theorem 5.0.1 holds true for the so-called *nearly spherical sets*. Those sets can be described as subgraphs of smooth functions defined over the boundary of the unitary ball, see Definition 5.1.1. Precisely, the only minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) among all nearly spherical sets are balls whenever its total charge Q is small enough.

(2) If E_Q is a minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) for fixed total charge Q, then E_Q is nearly spherical when $Q > 0$ is small enough.

In Section 5.1 we give the strategy to prove (1). In Section 5.2 we show how to reduce our problem to nearly spherical sets (point (2)).

5.1 Proof for nearly-spherical sets

The fundamental step in order to prove (1) is to deduce a Taylor expansion of the energy $\mathcal{F}$ near the ball B_1, for the class of nearly-spherical sets.

We start this section with some preliminary definitions.

Definition 5.1.1 (nearly spherical set)**.** An open bounded set $\Omega \subset \mathbb{R}^n$ is called *nearly spherical* of class $C^{k,\gamma}$ parametrized by φ, if there exists $\varphi \in C^{k,\gamma}$ with $\|\varphi\|_\infty \le \frac{1}{2}$ such that

$$\partial\Omega = \{(1+\varphi(x))x : x \in \partial B_1\}.$$

We also define the barycenter of an open set.

Definition 5.1.2. For an open set $\Omega \subset \mathbb{R}^n$, x_Ω denotes the *barycenter* of Ω, namely

$$x_\Omega = \frac{1}{|\Omega|}\int_\Omega x\,dx.$$

We will use the following theorem due to Fuglede, [14, Theorem 1.2].

Theorem 5.1.3 (Fuglede Theorem)**.** *There exist two constants $\delta = \delta(n) > 0$ and $C_F = C_F(n) > 0$ such that for any Ω nearly spherical set of class $C^{1,\gamma}$ parametrized by φ with $|\Omega| = |B_1|$, $x_\Omega = 0$ and $\|\varphi\|_{C^{1,\gamma}} \le \delta$ then the following inequality holds*

$$P(\Omega) - P(B_1) \ge C_F\,\|\varphi\|^2_{H^1(\partial B_1)}.$$

The proof of the following theorem will be addressed in [24].

Theorem 5.1.4. *There exist $C_T = C_T(n,A) > 0$ and $\delta = \delta(n,A) > 0$ with the following property: if $\Omega \subset \mathbb{R}^n$ is nearly spherical of class $C^{3,\gamma}$ with $\|\varphi\|_{C^{3,\gamma}} \le \delta$ then*

$$\mathcal{G}_{\beta,K}(\Omega) - \mathcal{G}_{\beta,K}(B_1) \le C_T\,\|\varphi\|^2_{H^1(\partial B_1)}.$$

We conclude this section by showing how Theorem 5.1.3 and Theorem 5.1.4 imply the following main theorem for nearly-spherical sets.

Theorem 5.1.5. *Let $E_Q \subset \mathbb{R}^n$ be a nearly spherical of class $C^{3,\gamma}$ minimizer of ($\mathcal{P}_{\beta,K,Q,R}$) such that $\|\varphi\|_{C^{3,\gamma}} \le \delta$ and*

$$0 \le Q < \left(\frac{C_F}{C_T}\right)^{1/2},$$

where δ, C_F and C_T are as in Theorem 5.1.3 and Theorem 5.1.4. Then $E = B_1$.

Proof. Assume that $E_Q \subset \mathbb{R}^n$ is nearly spherical parametrized by $\varphi \in C^{3,\gamma}$. Without loss of generality we may assume $x_{E_Q} = 0$. Notice that $|E_Q| = |B_1|$. Then by the minimality of E_Q we have

$$P(E_Q) - P(B_1) \le Q^2\,\{\mathcal{G}_{\beta,K}(B_1) - \mathcal{G}_{\beta,K}(\Omega)\}\,. \tag{5.1}$$

By exploiting the minimality of B_1 for the perimeter, Theorem 5.1.4 and (5.1) we obtain

$$0 \le P(E_Q) - P(B_1) \le Q^2\,\{\mathcal{G}_{\beta,K}(B_1) - \mathcal{G}_{\beta,K}(\Omega)\} \le C_T\,Q^2\|\varphi\|^2_{H^1(\partial B_1)}, \tag{5.2}$$

where $C_T = C_T(n, A) > 0$ is as in Theorem 5.1.4. Inequality (5.2) together with Fuglede theorem yield

$$
\begin{aligned}
0 &\le P(E_Q) - P(B_1) \le Q^2 \left\{ \mathcal{G}_{\beta,K}(B_1) - \mathcal{G}_{\beta,K}(\Omega) \right\} \le \frac{C_T}{C_F}\, Q^2\, C_F\, \|\varphi\|^2_{H^1(\partial B_1)} \\
&\le \frac{C_T}{C_F}\, Q^2 \left(P(E_Q) - P(B_1) \right).
\end{aligned} \tag{5.3}
$$

Therefore if $Q < (C_F/C_T)^{1/2}$ then $P(E_Q) - P(B_1) = 0$. By the isoperimetric inequality $P(E_Q) = P(B_1)$ implies $E_Q = B_1$. □

5.2 Reduction to nearly-spherical sets

The reduction of problem ($\mathcal{P}_{\beta,K,Q,R}$) to nearly-spherical sets is based on the higher regularity results obtained in Chapter 4.

First of all in this section we deduce the L^∞-closeness of minimizers to the unitary ball in the small charge regime. Let us start with the following proposition.

Proposition 5.2.1 (L^1-closeness to the ball)**.** *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of $(\mathcal{P}_{\beta,K,Q_h,R})$. Then, up to translations, $E_h \to B_1$ in L^1 and $P(E_h) \to P(B_1)$ when $h \to \infty$.*

Proof. By the quantitative isoperimetric inequality, [16, Theorem 1.1] for every $h \in \mathbb{N}$ there exists x_h such that

$$|E_h \Delta B_1(x_h)|^2 \le C \left(P(E_h) - P(B_1) \right),$$

for some constant $C = C(n) > 0$ which depends only on n. By translating each set E_h of x_h we can assume without loss of generality the following inequality

$$|E_h \Delta B_1|^2 \le C \left(P(E_h) - P(B_1) \right). \tag{5.4}$$

By the minimality of E_h we have

$$
\begin{aligned}
\mathcal{F}_{\beta,K,Q_h,R}(E_h) &= P(E_h) + Q_h^2\, \mathcal{G}_{\beta,K}(E_h) \\
&\le P(B_1) + Q_h^2\, \mathcal{G}_{\beta,K}(B_1) = \mathcal{F}_{\beta,K,Q_h,R}(B_1), \quad \forall\, h \in \mathbb{N}.
\end{aligned}
$$

Hence (5.4) yields

$$|E_h \Delta B_1|^2 \le C \left(P(E_h) - P(B_1) \right) \le C\, Q_h^2\, \mathcal{G}_{\beta,K}(B_1) \quad \forall\, h \in \mathbb{N},$$

for some constant $C = C(n) > 0$ which depends only on the dimension n.

Then $Q_h \to 0$ implies $E_h \to B_1$ in L^1 and $P(E_h) \to P(B_1)$ when $h \to \infty$. □

Thanks to the density estimates (see Theorems 2.5.5 and 2.5.7), we can improve the convergence of Proposition 5.2.1.

Proposition 5.2.2 (L^∞-closeness to the ball)**.** *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of $(\mathcal{P}_{\beta,K,Q_h,R})$. Then, up to translations, $E_h \to \overline{B}_1$ in the Kuratowski sense and $\partial E_h \to \partial B_1$ in the Hausdorff sense.*

Proof. First, we prove the Kuratowski convergence of E_h to the ball B_1, i.e.

(i) $x_h \to x$, $x_h \in E_h \Rightarrow x \in \overline{B}_1$,

(ii) $x \in \overline{B}_1 \Rightarrow \exists x_h \in E_h$ such that $x_h \to x$.

In order to prove (i) let $x_h \to x$ and $x_h \in E_h$. Assume by contradiction that $x \notin \overline{B}_1$. Then there exits $B_s(x) \subset \mathbb{R}^n$ such that $B_s(x) \cap B_1 = \emptyset$. By Theorem 2.5.7, density estimates for the volume of each minimizer E_h hold true. Moreover for Q_h small enough there exist a radius $\bar{r} > 0$ and a constant $C > 0$, both independent of Q_h, such that

$$|B_r(x_h) \cap E_h| \geq C\, r^n \quad \forall r \leq \bar{r}. \tag{5.5}$$

Hence we can choose $r \leq \min\{\bar{r}, s\}$ and $h(r) \in \mathbb{N}$ such that $B_{\frac{r}{2}}(x_h) \subset B_r(x)$ for every $h \geq h(r)$. Then

$$|B_r(x) \cap E_h| \geq |B_{\frac{r}{2}}(x_h) \cap E_h| \geq C\, r^n. \tag{5.6}$$

By the L^1-convergence of E_h to B_1 and (5.6) we deduce $|B_r(x) \cap B_1| > 0$ a contradiction with $B_s(x) \cap B_1 = \emptyset$.

The proof of (ii) follows by arguing similarly as above by exploiting the L^1-convergence. Analogously, by using density estimates for the perimeter of E_h and the convergence of perimeters $P(E_h) \to P(B_1)$, one can prove that $\partial E_h \to \partial B_1$ in the Hausdorff sense. □

By combining Proposition 5.2.2 and Theorem 4.1.10, we conclude this section by showing that the minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) are nearly-spherical in the small charge regime.

Remark 5.2.3. A minimizer E_Q of the problem ($\mathcal{P}_{\beta,K,Q,R}$) satisfies the hypothesis of Theorems 4.1.10 and 4.2.2 whenever $Q > 0$ is small enough. Indeed, assume $x_0 \in \partial B_1$. Then, by the regularity of ∂B_1, there exists a radius $r = r(\beta) > 0$ such that

$$r + \mathbf{e}_{B_1}(x_0, 2r) \leq \frac{\varepsilon_{\mathrm{reg}}}{2}, \tag{5.7}$$

where $\varepsilon_{\mathrm{reg}}$ is as in Theorem 4.2.2. On the other hand by Proposition 5.2.2 we have that E_Q converges to B_1 in the Kuratowski sense when $Q \to 0$. Hence, by properties of the excess function, $\mathbf{e}_{E_Q}(x_0, 2r) \to \mathbf{e}_{B_1}(x_0, 2r)$ when $Q \to 0$. By Proposition 2.4.1 we also have $Q^2\, D_{E_Q}(x_0, 2r) \to 0$ when $Q \to 0$. Therefore,

$$r + \mathbf{e}_{E_Q}(x_0, 2r) + Q^2\, D_{E_Q}(x_0, 2r) \leq \varepsilon_{\mathrm{reg}}, \tag{5.8}$$

when $Q > 0$ is small enough.

Theorem 5.2.4. *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of ($\mathcal{P}_{\beta,K,Q_h,R}$). Then for h big enough E_h is nearly spherical of class C^∞, i.e. there exists $\varphi_h \in C^\infty$ with uniform bounds and $\|\varphi_h\|_{L^\infty} < \frac{1}{2}$ such that*

$$\partial E_h = \{(1 + \varphi_h(x))x : x \in \partial B_1\}.$$

Moreover, $\|\varphi_h\|_{C^k} \to 0$ when $h \to \infty$, for every $k \in \mathbb{N}$.

Proof. Fix a point $\bar{x} \in \partial B_1$. By Remark 5.2.3 there exists $\bar{r} > 0$ and a smooth function g such that

$$\partial B_1 \cap \mathbf{C}\,(\bar{x}, r, \nu_{B_1}(\bar{x})) = \partial \left\{ x \in \mathbb{R}^n \,:\, \mathbf{q}^{\nu_{B_1}(\bar{x})}(x - \bar{x}) < g(\mathbf{p}^{\nu_{B_1}(\bar{x})}(x - \bar{x})) \right\} \cap \mathbf{C}\,(\bar{x}, r, \nu_{B_1}(\bar{x})) \tag{5.9}$$

for every $0 < r \leq \bar{r}$. Furthermore, there exists $r_0 \leq \bar{r}$ small enough and $f_h \in C^\infty\left(\mathbf{D}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right)\right)$ such that

(5.10)

$$\partial E_h \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right) = \partial\left\{x \in \mathbb{R}^n \,:\, \mathbf{q}^{\nu_{B_1}(\bar{x})}(x - \bar{x}) < f_h(\mathbf{p}^{\nu_{B_1}(\bar{x})}(x - \bar{x}))\right\} \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right)$$

for every h big enough and $r \leq r_0$. Define $\varphi_h^{\bar{x}}(x) := f_h(g^{-1}(x))$ for every $x \in \partial B_1$. Then $\{\varphi_h^{\bar{x}}\}_{h\in\mathbb{N}}$ is a family of C^∞ functions with $\|\varphi_h^{\bar{x}}\|_{C^k}$ uniformly bounded (by Theorem 4.2.2) such that

$$\partial E_h \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right) = \{(1 + \varphi_h^{\bar{x}}(x))x : x \in \partial B_1\}.$$

Hence, by a covering argument we obtain a family $\{\varphi_h\}_{h\in\mathbb{N}}$ of C^∞ functions with $\|\varphi_h\|_{C^k}$ uniformly bounded such that

$$\partial E_h = \{(1 + \varphi_h(x))x : x \in \partial B_1\}.$$

By Ascoli-Arzelà and the convergence of ∂E_h to ∂B_1 in the sense of Kuratowski we obtain that $\varphi_h \to 0$ in $C^{k-1}(\partial B_1)$ for every $k \in \mathbb{N}$. □

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