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*Original*

Periodic solutions of nearly integrable Hamiltonian systems bifurcating from infinite-dimensional tori / Fonda, Alessandro; Klun, Giuliano; Sfecci, Andrea. - In: NONLINEAR ANALYSIS. - ISSN 0362-546X. - 201:12(2020). [10.1016/j.na.2019.111720]

*Availability:*

This version is available at: 20.500.11767/106503 since: 2020-01-08T11:44:57Z

*Publisher:*

*Published*

DOI:10.1016/j.na.2019.111720

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Elsevier

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# Periodic solutions of nearly integrable Hamiltonian systems bifurcating from infinite-dimensional tori

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*Dedicated to Shair Ahmad, on the occasion of his 85th birthday*

## Abstract

We prove the existence of periodic solutions of some infinite-dimensional nearly integrable Hamiltonian systems, bifurcating from infinite-dimensional tori, by the use of a generalization of the Poincaré–Birkhoff Theorem.

## 1 Introduction

The aim of this paper is to provide the existence of periodic solutions bifurcating from an infinite-dimensional invariant torus for a nearly integrable Hamiltonian system.

The finite-dimensional case was treated in [1, 2, 4, 5, 6] by assuming the existence of an invariant torus made of periodic solutions all sharing the same period, under some non-degeneracy conditions. Let us briefly describe the main result in this setting. Denoting by  $H(I, \varphi) = \mathcal{K}(I)$  the Hamiltonian of a completely integrable system in  $\mathbb{R}^{2N}$  (as usual, we denote by  $\varphi$  and  $I$  the angle and the action variables, respectively), we can write the corresponding system

$$\begin{cases} \dot{\varphi} = \nabla \mathcal{K}(I) \\ -\dot{I} = 0. \end{cases}$$

Assume that there is a  $I^0 \in \mathbb{R}^N$  such that

$$\det \mathcal{K}''(I^0) \neq 0. \quad (1.1)$$

Consider now the perturbed system

$$\begin{cases} \dot{\varphi} = \nabla \mathcal{K}(I) + \varepsilon \nabla_I P(t, \varphi, I) \\ -\dot{I} = \varepsilon \nabla_\varphi P(t, \varphi, I), \end{cases}$$

where  $P(\cdot, \varphi, I)$  is  $T$ -periodic, and  $P(t, \cdot, I)$  is  $\tau_k$ -periodic in  $\varphi_k$ , for every  $k = 1, \dots, N$ . Assume that there exist some integers  $m_1, \dots, m_N$  for which

$$T \nabla \mathcal{K}(I^0) = (m_1 \tau_1, \dots, m_N \tau_N). \quad (1.2)$$

Then, for  $|\varepsilon|$  small enough, there are at least  $N + 1$  solutions  $(\varphi(t), I(t))$  satisfying

$$\varphi(t + T) = \varphi(t) + T \nabla \mathcal{K}(I^0), \quad I(t + T) = I(t), \quad \text{for every } t \in \mathbb{R}, \quad (1.3)$$

and these solutions are near to some solutions of the unperturbed problem, i.e., briefly,

$$\varphi(t) \approx \varphi(0) + t\nabla\mathcal{K}(I^0), \quad I(t) \approx I^0.$$

Notice that, by (1.2) and (1.3),  $\varphi_k(t+T) = \varphi_k(t) + m_k\tau_k$ , for every  $k = 1, \dots, N$ . Since usually  $\varphi_k$  is interpreted as an angle, with  $\tau_k = 2\pi$ , we may consider these as “periodic solutions” having period  $T$ . However, in the following, it will be better to keep more freedom in the choice of the periods  $\tau_k$ .

Clearly enough, being  $P(\cdot, \varphi, I)$  also  $mT$ -periodic for every positive integer  $m$ , one could search “periodic solutions” having period  $mT$ , as well (the so-called “subharmonic solutions”). We refer to [6] for a complete description of the problem, and for a more general statement, obtained by the use of the Poincaré–Birkhoff theorem.

The above result was recently extended in [7] for systems of the type

$$\begin{cases} \dot{\varphi} = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t, \varphi, I, z) \\ -\dot{I} = \varepsilon\nabla_\varphi P(t, \varphi, I, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z P(t, \varphi, I, z), \end{cases} \quad (1.4)$$

where  $J = \begin{pmatrix} 0 & -I_M \\ I_M & 0 \end{pmatrix}$  denotes the standard  $2M \times 2M$  symplectic matrix and  $\mathcal{A}$  is a symmetric non-resonant matrix, meaning that the only  $T$ -periodic solution of the unperturbed equation  $J\dot{z} = \mathcal{A}z$  is the constant  $z = 0$ . Assuming (1.1), (1.2) and that  $\nabla P$ , the gradient of  $P$  with respect to  $(\varphi, I, z)$ , is uniformly bounded, the existence of at least  $N+1$  solutions  $(\varphi(t), I(t), z(t))$  satisfying (1.3) and  $z(t+T) = z(t)$  was proved, when  $|\varepsilon|$  is small enough.

The aim of this paper is to extend the above results to an infinite-dimensional setting. Let  $X$  and  $Z$  be the separable Hilbert spaces which will replace  $\mathbb{R}^N$  and  $\mathbb{R}^{2M}$ , respectively. So, when looking at system (1.4), the functions  $\varphi(t)$  and  $I(t)$  will vary in  $X$ , while  $z(t)$  will belong to  $Z$ . The spaces  $X$  and  $Z$  may be infinite-dimensional, finite-dimensional, or even reduced to  $\{0\}$ . If  $X$  is finite-dimensional, the cases  $Z = \{0\}$  and  $Z$  finite-dimensional correspond to the settings in [6] and [7], respectively. However, if  $X$  or  $Z$  are infinite-dimensional, we will be able to prove the bifurcation of *at least one* periodic orbit from an invariant torus, which can also be infinite-dimensional. The multiplicity problem remains open.

In order to obtain our existence result in infinite-dimensions, we ask all the functions to be Lipschitz continuous on bounded sets, and the perturbing term  $\nabla P$  to be uniformly bounded. Moreover, we need a special structure of the autonomous Hamiltonian function: in our assumptions **(Dec1)** and **(Dec2)** below, roughly speaking, the functions involved must be decomposable in a sequence of finite-dimensional blocks. This allows us to tackle the problem by a finite-dimensional approximating process, applying a version of the Poincaré–Birkhoff theorem for the reduced systems, and carefully estimating the so-found periodic solutions in order to guarantee their convergence to a periodic solution of the infinite-dimensional system.

## 2 The main result

We want to treat a system of the type (1.4) in an infinite-dimensional setting. To this aim, let  $X$  and  $E$  be two separable Hilbert spaces, and set  $\mathcal{X} = X^2 \times E^2$ . We will use the notation  $\omega = (\varphi, I, z)$  for the elements of  $\mathcal{X}$ , with  $\varphi, I \in X$  and  $z = (x, y) \in E^2$ . For simplicity, we will write  $Z = E^2$ , and we define  $J : Z \rightarrow Z$  as  $J(x, y) = (-y, x)$ . (The same notation  $J$  will also be used with the same meaning in similar settings.) Let us introduce all the assumptions we need.

The continuous functions  $\mathcal{K} : X \rightarrow \mathbb{R}$  and  $P : \mathbb{R} \times \mathcal{X} \rightarrow \mathbb{R}$  are assumed to be continuously differentiable with respect to  $I$  and  $\omega$ , respectively. The function  $t \mapsto P(t, \omega)$  is  $T$ -periodic, for some  $T > 0$ . Moreover, we assume the following Lipschitz condition on bounded sets.

**(L)** For every  $R > 0$  there exist two positive constants  $L_R, \mathcal{L}_R$  such that

$$\|\nabla \mathcal{K}(I') - \nabla \mathcal{K}(I'')\| \leq L_R \|I' - I''\|,$$

for every  $I', I'' \in X$  with  $\|I'\| < R, \|I''\| < R$ , and

$$\|\nabla_\omega P(t, \omega') - \nabla_\omega P(t, \omega'')\| \leq \mathcal{L}_R \|\omega' - \omega''\|,$$

for every  $t \in [0, T]$  and  $\omega', \omega'' \in \mathcal{X}$  with  $\|\omega'\| < R$  and  $\|\omega''\| < R$ .

Introducing some Hilbert bases of  $X$  and  $E$ , we can identify these spaces either with some  $\mathbb{R}^n$ , if they are finite-dimensional, or with  $\ell^2$ , the space of real sequences  $(\alpha_k)_k$  which satisfy  $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$ . Each of the vectors  $\varphi, I$  in  $X$  and  $z$  in  $Z$  will then be written in their coordinates, e.g.,  $\varphi = (\varphi_1, \varphi_2, \dots)$ , or  $\varphi = (\varphi_k)_k$ , with  $\varphi_k \in \mathbb{R}$ , while  $I = (I_k)_k$  and  $z = (z_l)_l$ , with  $z_l = (x_l, y_l) \in \mathbb{R}^2$ . Notice that these sequences may be finite.

We also ask  $P$  to be periodic in the  $\varphi$ -variables, as follows.

**(P<sub>τ</sub>)** The function  $P(t, \varphi, I, z)$  is  $\tau_k$ -periodic in each  $\varphi_k$ , i.e., for  $k = 1, 2, \dots$ ,

$$P(t, \dots, \varphi_k + \tau_k, \dots, I, z) = P(t, \dots, \varphi_k, \dots, I, z), \quad \text{for every } (t, \varphi, I, z) \in [0, T] \times \mathcal{X};$$

moreover, if  $\dim X = \infty$ , then the sequence  $(\tau_k)_k$  belongs to  $\ell^2$ .

Concerning  $\nabla_\omega P$ , we assume it to be bounded and precompact, in the following sense.

**(P<sub>bd</sub>)** There exist  $(\alpha_k^*)_k$  and  $(\alpha_l^\#)_l$  such that, for every  $k, l = 1, 2, \dots$ ,

$$\left| \frac{\partial P}{\partial \varphi_k}(t, \omega) \right| + \left| \frac{\partial P}{\partial I_k}(t, \omega) \right| \leq \alpha_k^*, \quad \left| \frac{\partial P}{\partial x_l}(t, \omega) \right| + \left| \frac{\partial P}{\partial y_l}(t, \omega) \right| \leq \alpha_l^\#,$$

for every  $(t, \omega) \in [0, T] \times \mathcal{X}$ . If  $\dim X = \infty$  or  $\dim Z = \infty$ , then  $(\alpha_k^*)_k$  or  $(\alpha_l^\#)_l$  belong to  $\ell^2$ , respectively.

Notice that the sets  $\prod_{k=1}^{\infty} [-\alpha_k^*, \alpha_k^*]$  and  $\prod_{l=1}^{\infty} [-\alpha_l^\#, \alpha_l^\#]$  are Hilbert cubes, hence compact sets in  $\ell^2$ .

Let  $\mathcal{A} : Z \rightarrow Z$  be a linear *bounded selfadjoint* operator. We need the following non-resonance assumption.

**(NR)** Denoting by

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2([0, T], Z) \rightarrow L^2([0, T], Z), \quad \mathcal{L}z = J\dot{z},$$

the unbounded selfadjoint operator with domain

$$\mathcal{D}(\mathcal{L}) = \{z \in H^1([0, T], Z) : z(0) = z(T)\},$$

we assume that  $0 \notin \sigma(\mathcal{L} - \mathcal{A})$ .

In the case when  $Z$  is infinite-dimensional, we need to assume a particular structure for the function  $\mathcal{A}$ .

**(Dec1)** If  $\dim Z = \infty$ , there exists a sequence of positive integers  $(N_m^\sharp)_{m \geq 1}$  and functions  $\mathcal{A}_m : \mathbb{R}^{2N_m^\sharp} \rightarrow \mathbb{R}^{2N_m^\sharp}$  such that, writing any vector  $z \in Z$  as  $z = (\vec{z}_1, \dots, \vec{z}_m, \dots)$ , with  $\vec{z}_m = (\vec{x}_m, \vec{y}_m) \in \mathbb{R}^{2N_m^\sharp}$ , we have that

$$\mathcal{A}z = (\mathcal{A}_1 \vec{z}_1, \dots, \mathcal{A}_m \vec{z}_m, \dots).$$

Concerning the function  $\mathcal{K}$ , its gradient will be “guided” by some linear *bounded selfadjoint invertible* operator  $\mathcal{B} : X \rightarrow X$ , with bounded inverse, as we now specify. First of all, similarly as before, in the case when  $X$  is infinite-dimensional, we need to assume a particular structure for the functions  $\mathcal{B}$  and  $\mathcal{K}$ .

**(Dec2)** If  $\dim X = \infty$ , there exists a sequence of positive integers  $(N_j^*)_{j \geq 1}$  and functions  $\mathcal{B}_j : \mathbb{R}^{N_j^*} \rightarrow \mathbb{R}^{N_j^*}$ ,  $\mathcal{K}_j : \mathbb{R}^{N_j^*} \rightarrow \mathbb{R}$  such that, writing any vector  $I \in X$  as  $I = (\vec{I}_1, \dots, \vec{I}_j, \dots)$ , with  $\vec{I}_j \in \mathbb{R}^{N_j^*}$ , we have that

$$\mathcal{B}I = (\mathcal{B}_1 \vec{I}_1, \dots, \mathcal{B}_j \vec{I}_j, \dots), \quad \mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j).$$

We now fix  $I^0 \in X$ , and introduce our *twist condition*.

**(Tw)** There exist two positive constants  $\bar{c}, \bar{\rho}$  such that, for every  $j = 1, 2, \dots$ ,

$$\|\vec{I}_j - \vec{I}_j^0\| \leq \bar{\rho} \quad \Rightarrow \quad \left\langle \nabla \mathcal{K}_j(\vec{I}_j) - \nabla \mathcal{K}_j(\vec{I}_j^0), \mathcal{B}_j(\vec{I}_j - \vec{I}_j^0) \right\rangle \geq \bar{c} \|\vec{I}_j - \vec{I}_j^0\|^2.$$

Finally, we assume a compatibility condition between  $T$  and the periods introduced in **(P<sub>τ</sub>)**.

**(C<sub>τ</sub>)** There exist some integers  $m_1, m_2, \dots$  for which

$$T \nabla \mathcal{K}(I^0) = (m_1 \tau_1, m_2 \tau_2, \dots).$$

We are now ready to state our main result.

**Theorem 2.1.** *Let the above assumptions hold. Then, for every  $\sigma > 0$  there exists  $\bar{\varepsilon} > 0$  such that, if  $|\varepsilon| \leq \bar{\varepsilon}$ , there is a solution of system*

$$\begin{cases} \dot{\varphi} = \nabla \mathcal{K}(I) + \varepsilon \nabla_I P(t, \varphi, I, z) \\ -\dot{I} = \varepsilon \nabla_\varphi P(t, \varphi, I, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z P(t, \varphi, I, z), \end{cases} \quad (2.1)$$

satisfying

$$\varphi(t+T) = \varphi(t) + T \nabla \mathcal{K}(I^0), \quad I(t+T) = I(t), \quad z(t+T) = z(t), \quad (2.2)$$

and such that

$$\|\varphi(t) - \varphi(0) - t \nabla \mathcal{K}(I^0)\| + \|I(t) - I^0\| + \|z(t)\| < \sigma, \quad \text{for every } t \in \mathbb{R}. \quad (2.3)$$

**Remark 2.2.** When  $X$  is finite-dimensional, we will see that condition **(Tw)** can be generalized to

**(Tw')** There exists a positive constant  $\bar{\rho}$  such that

$$\|I - I^0\| \leq \bar{\rho} \quad \Rightarrow \quad \langle \nabla \mathcal{K}(I) - \nabla \mathcal{K}(I^0), \mathcal{B}(I - I^0) \rangle > 0;$$

a still more general condition, adopted in [6], is the following:

$$0 \in \text{cl} \left\{ \rho \in ]0, +\infty[ : \min_{\|I - I^0\| = \rho} \langle \nabla \mathcal{K}(I) - \nabla \mathcal{K}(I^0), \mathcal{B}(I - I^0) \rangle > 0 \right\},$$

where  $\text{cl } \mathcal{S}$  denotes the closure of a set  $\mathcal{S}$ .

### 3 Preliminaries for the proof

We will carry out the proof of Theorem 2.1 in the case  $\dim X = \infty$  and  $\dim Z = \infty$ , with some specific remarks on the finite-dimensional cases. By the change of variables

$$(\xi(t), I(t), z(t)) = (\varphi(t) - t\nabla\mathcal{K}(I^0), I(t), z(t)), \quad (3.1)$$

system (2.1) becomes

$$\begin{cases} \dot{\xi} = \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) + \varepsilon\nabla_I\widehat{P}(t, \xi, I, z) \\ -\dot{I} = \varepsilon\nabla_\xi\widehat{P}(t, \xi, I, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z\widehat{P}(t, \xi, I, z), \end{cases} \quad (3.2)$$

where

$$\widehat{P}(t, \xi, I, z) = P(t, \xi + t\nabla\mathcal{K}(I^0), I, z).$$

We use the notation  $\zeta = (\xi, I, z)$ ; the Hamiltonian function is thus

$$\widehat{H}(t, \zeta) = \mathcal{K}(I) - \langle \nabla\mathcal{K}(I^0), I \rangle + \frac{1}{2} \langle \mathcal{A}z, z \rangle + \varepsilon\widehat{P}(t, \zeta).$$

Combining **(P<sub>τ</sub>)** with **(C<sub>τ</sub>)**, we see that the function  $\widehat{P}(\cdot, \xi, I, z)$  is  $T$ -periodic, and  $\widehat{P}(t, \cdot, I, z)$  is  $\tau_k$ -periodic in  $\xi_k$ , for every  $k = 1, 2, \dots$

Some additional notations are now necessary. By assumption **(Dec2)**, the vectors  $\xi, I \in X$  decompose in vectors  $\vec{\xi}_j, \vec{I}_j \in \mathbb{R}^{N_j^*}$ . Setting

$$S_0^* = 0, \quad S_j^* = \sum_{i=1}^j N_i^* \quad \text{for } j \geq 1,$$

we can explicitly write the components of  $\vec{\xi}_j, \vec{I}_j$  as

$$\vec{\xi}_j = (\xi_{S_{j-1}^*+1}, \xi_{S_{j-1}^*+2}, \dots, \xi_{S_j^*}), \quad \vec{I}_j = (I_{S_{j-1}^*+1}, I_{S_{j-1}^*+2}, \dots, I_{S_j^*}).$$

Similarly, by assumption **(Dec1)**, the vector  $z \in Z$  decomposes in vectors  $\vec{z}_m \in \mathbb{R}^{2N_m^\#}$ . Setting

$$S_0^\# = 0, \quad S_m^\# = \sum_{i=1}^m N_i^\# \quad \text{for } m \geq 1,$$

we can explicitly write the components of  $\vec{z}_m$  as

$$\vec{z}_m = (z_{S_{m-1}^\#+1}, z_{S_{m-1}^\#+2}, \dots, z_{S_m^\#}).$$

We define the sequences  $(a_j^*)_j, (a_m^\#)_m$  in  $\ell^2$  by

$$a_j^* = \left( \sum_{i=1}^{N_j^*} (\alpha_{S_{j-1}^*+i}^*)^2 \right)^{1/2}, \quad a_m^\# = \left( \sum_{i=1}^{N_m^\#} (\alpha_{S_{m-1}^\#+i}^\#)^2 \right)^{1/2}.$$

Notice that  $\|a^*\|_{\ell^2} = \|\alpha^*\|_{\ell^2}$  and  $\|a^\#\|_{\ell^2} = \|\alpha^\#\|_{\ell^2}$ .

**Remark 3.1.** When  $X$  has a finite dimension  $d_X$ , we can define the sequence  $(N_j^*)_j$  taking  $N_1^* = d_X$  and  $N_j^* = 0$  for  $j \geq 2$ . Similarly when  $Z$  is finite-dimensional.

Without loss of generality, from now on we will assume that  $I^0 = 0$ , a situation which can be recovered by a simple translation. The strategy of the proof of Theorem 2.1 will be to construct a finite-dimensional approximation of system (3.2), and then pass to the limit on the dimension. Precisely, we define the projections  $\Pi_{S_{\mathcal{J}}^*} : X \rightarrow X$  and  $\Pi_{S_{\mathcal{J}}^\#} : Z \rightarrow Z$  as

$$\Pi_{S_{\mathcal{J}}^*} v = (\vec{v}_1, \dots, \vec{v}_{\mathcal{J}}, 0, 0, \dots), \quad \Pi_{S_{\mathcal{J}}^\#} z = (\vec{z}_1, \dots, \vec{z}_{\mathcal{J}}, 0, 0, \dots),$$

and consider the truncated system

$$\begin{cases} \dot{\xi} = \Pi_{S_{\mathcal{J}}^*} [\nabla \mathcal{K}(I) - \nabla \mathcal{K}(0) + \varepsilon \nabla_I \widehat{P}(t, \xi, I, z)] \\ -\dot{I} = \Pi_{S_{\mathcal{J}}^*} [\varepsilon \nabla_{\xi} \widehat{P}(t, \xi, I, z)] \\ J\dot{z} = \Pi_{S_{\mathcal{J}}^\#} [\mathcal{A}z + \varepsilon \nabla_z \widehat{P}(t, \xi, I, z)]. \end{cases} \quad (3.3)$$

We thus have the Hamiltonian function

$$\widehat{H}_{\mathcal{J}}(t, \zeta) = \mathcal{K}(\Pi_{S_{\mathcal{J}}^*} I) - \langle \nabla \mathcal{K}(0), \Pi_{S_{\mathcal{J}}^*} I \rangle + \frac{1}{2} \langle \mathcal{A} \Pi_{S_{\mathcal{J}}^\#} z, \Pi_{S_{\mathcal{J}}^\#} z \rangle + \varepsilon \widehat{P}(t, \Pi_{S_{\mathcal{J}}^*} \xi, \Pi_{S_{\mathcal{J}}^*} I, \Pi_{S_{\mathcal{J}}^\#} z).$$

Notice that the function

$$\widehat{P}_{\mathcal{J}}(t, \xi, I, z) = \widehat{P}(t, \Pi_{S_{\mathcal{J}}^*} \xi, \Pi_{S_{\mathcal{J}}^*} I, \Pi_{S_{\mathcal{J}}^\#} z)$$

satisfies both **(L)** and **(P <sub>$\tau$</sub> )** with the same constants, for every index  $\mathcal{J} \geq 1$ , and observe that system (3.3) is equivalent to

$$\begin{cases} \dot{\vec{\xi}}_j = \nabla \mathcal{K}_j(\vec{I}_j) - \nabla \mathcal{K}_j(0) + \varepsilon \nabla_{\vec{I}_j} \widehat{P}_{\mathcal{J}}(t, \xi, I, z) \\ -\dot{\vec{I}}_j = \varepsilon \nabla_{\vec{\xi}_j} \widehat{P}_{\mathcal{J}}(t, \xi, I, z) \\ J\dot{\vec{z}}_j = \mathcal{A}_j \vec{z}_j + \varepsilon \nabla_{\vec{z}_j} \widehat{P}_{\mathcal{J}}(t, \xi, I, z) \\ \dot{\xi}_i = 0 \\ -\dot{\vec{I}}_i = 0 \\ J\dot{\vec{z}}_i = 0 \end{cases} \quad \begin{matrix} j \leq \mathcal{J}, \\ \\ \\ i > \mathcal{J}. \end{matrix} \quad (3.4)$$

It can be viewed as two uncoupled systems, the first one in a finite-dimensional space (the “approximating system”), and the second one, infinite-dimensional, having only constant solutions. From now on, we will take  $\vec{\xi}_i(t), \vec{I}_i(t), \vec{z}_i(t)$  identically equal to zero when  $i \geq \mathcal{J}$ .

Concerning the “approximating system”, we will need the following slight modification of [7, Corollary 2.3]. Let us consider the finite-dimensional Hamiltonian system

$$J\dot{\zeta} = \nabla_{\zeta} H(t, \zeta), \quad (3.5)$$

with  $\zeta = (\xi, I, z) \in \mathbb{R}^{N+N+2M}$ , where the Hamiltonian function is  $T$ -periodic in  $t$ . Here we use the notation  $\xi = (\vec{\xi}_1, \dots, \vec{\xi}_{\mathcal{J}}), I = (\vec{I}_1, \dots, \vec{I}_{\mathcal{J}})$ .

**Theorem 3.2.** *Assume that  $H(t, \zeta) = \frac{1}{2} \langle \mathbb{A}z, z \rangle + G(t, \zeta)$ , where  $\mathbb{A}$  is a symmetric  $2M \times 2M$  matrix such that  $z \equiv 0$  is the unique  $T$ -periodic solution of equation  $J\dot{z} = \mathbb{A}z$ , and there exists a constant  $c_1$  such that*

$$|\nabla_{\zeta} G(t, \zeta)| \leq c_1, \quad \text{for every } (t, \zeta) \in \mathbb{R} \times \mathbb{R}^{2(M+N)}.$$

Let  $G(t, \xi, I, z)$  be periodic in the variables  $\xi_1, \dots, \xi_N$ . Assume moreover the existence of some positive constants  $r'_j < r''_j$  and symmetric invertible matrices  $\mathcal{B}_j$ , with  $j = 1, \dots, \mathcal{J}$ , such that, for any solution  $\zeta(t) = (\xi(t), I(t), z(t))$  of (3.5), if

$$r'_j \leq \|\vec{I}_j(0) - \vec{I}_j^0\| \leq r''_j \quad \text{and} \quad \|\vec{I}_i(0) - \vec{I}_i^0\| \leq r''_i \quad \text{for every } i \neq j,$$

then

$$\left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0), \mathcal{B}_j(\vec{I}_j(0) - \vec{I}_j^0) \right\rangle > 0.$$

Then, there are at least  $N+1$  geometrically distinct  $T$ -periodic solutions  $\zeta(t) = (\xi(t), I(t), z(t))$  of (3.5), such that

$$\|\vec{I}_j(0) - \vec{I}_j^0\| < r'_j, \quad \text{for every } j = 1, \dots, \mathcal{J}.$$

## 4 Proof of Theorem 2.1

In what follows, we always assume that  $|\varepsilon| \leq 1$ , and we denote by  $\bar{\rho}$  the constant introduced in assumption **(Tw)**. Moreover, as in the previous section, we assume  $I^0 = 0$ .

**Lemma 4.1.** *There is a constant  $C > 0$  with the following property: if  $\zeta(t) = (\xi(t), I(t), z(t))$  is a solution of (3.2) with  $\|\vec{I}_j(0)\| \leq \bar{\rho}$ , for some  $j \geq 1$ , then*

$$\|\vec{\xi}_j(t) - \vec{\xi}_j(0) - t[\nabla \mathcal{K}_j(\vec{I}_j(0)) - \nabla \mathcal{K}_j(0)]\| + \|\vec{I}_j(t) - \vec{I}_j(0)\| \leq C|\varepsilon|a_j^*, \quad \text{for every } t \in [0, T].$$

The same property holds for the solutions of (3.4), when  $j = 1, \dots, \mathcal{J}$ .

*Proof.* Let us start computing, for every  $t \in [0, T]$  and every  $k \in \{S_{j-1}^* + 1, \dots, S_{j-1}^* + N_j^* = S_j^*\}$ ,

$$|I_k(t) - I_k(0)| \leq \int_0^t |\dot{I}_k(s)| ds \leq |\varepsilon| \int_0^T \left| \frac{\partial \widehat{P}}{\partial \xi_k}(s, \zeta(s)) \right| ds \leq |\varepsilon| T \alpha_k^*.$$

Then we easily get

$$\|\vec{I}_j(t) - \vec{I}_j(0)\| \leq |\varepsilon| T \left( \sum_{i=1}^{N_j^*} (\alpha_{S_{j-1}^* + i}^*)^2 \right)^{1/2} = |\varepsilon| T a_j^*.$$

Moreover,

$$\begin{aligned} \|\vec{\xi}_j(t) - \vec{\xi}_j(0) - t[\nabla \mathcal{K}_j(\vec{I}_j(0)) - \nabla \mathcal{K}_j(0)]\| &\leq \int_0^t \|\dot{\vec{\xi}}_j(s) - [\nabla \mathcal{K}_j(\vec{I}_j(0)) - \nabla \mathcal{K}_j(0)]\| ds \\ &\leq \int_0^T \|\nabla \mathcal{K}_j(\vec{I}_j(s)) - \nabla \mathcal{K}_j(\vec{I}_j(0))\| ds + |\varepsilon| \int_0^T \|\nabla_{\vec{I}_j} \widehat{P}(s, \zeta(s))\| ds \\ &\leq \int_0^T L \|\vec{I}_j(s) - \vec{I}_j(0)\| ds + |\varepsilon| T a_j^* \\ &\leq |\varepsilon| T(1 + LT) a_j^*, \end{aligned}$$

where  $L$  is a suitable Lipschitz constant provided by **(L)**. The proof is thus completed.  $\square$

**Lemma 4.2.** *There exist  $\bar{\varepsilon} > 0$  and a sequence  $(\delta_j)_j$  in  $\ell^2$ , with  $\delta_j \in ]0, \bar{\rho}]$ , satisfying the following property: if  $\zeta(t) = (\xi(t), I(t), z(t))$  is a solution of (3.2), with  $|\varepsilon| < \bar{\varepsilon}$  and  $\delta_j \leq \|\vec{I}_j(0)\| \leq \bar{\rho}$ , for some  $j \geq 1$ , then*

$$\left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0), \mathcal{B}_j \vec{I}_j(0) \right\rangle > 0.$$

The same property holds for the solutions of (3.4), when  $j = 1, \dots, \mathcal{J}$ .



*Proof.* If  $\|\vec{I}_j(0)\| \leq \bar{\rho}$  for some  $j \geq 1$ , then, by Lemma 4.1 and **(Tw)**,

$$\begin{aligned} \left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0), \mathcal{B}_j \vec{I}_j(0) \right\rangle &= \left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0) - T[\nabla \mathcal{K}_j(\vec{I}_j(0)) - \nabla \mathcal{K}_j(0)], \mathcal{B}_j \vec{I}_j(0) \right\rangle + \\ &\quad + T \left\langle \nabla \mathcal{K}_j(\vec{I}_j(0)) - \nabla \mathcal{K}_j(0), \mathcal{B}_j \vec{I}_j(0) \right\rangle \\ &\geq -C|\varepsilon|a_j^* \|\mathcal{B}_j\| \|\vec{I}_j(0)\| + T\bar{c}\|\vec{I}_j(0)\|^2 \\ &= \left( -C|\varepsilon|a_j^* \|\mathcal{B}_j\| + T\bar{c}\|\vec{I}_j(0)\| \right) \|\vec{I}_j(0)\|. \end{aligned}$$

Setting

$$\delta_j := \min \left\{ \bar{\rho}, \frac{2C}{\bar{c}T} a_j^* \|\mathcal{B}_j\| \right\},$$

we easily verify that  $(\delta_j)_j \in \ell^2$ , since  $(\|\mathcal{B}_j\|)_j$  is bounded by  $\|\mathcal{B}\|$  and  $(a_j^*)_j \in \ell^2$ ; in particular, there exists an integer  $j_0$  such that

$$\delta_j = \frac{2C}{\bar{c}T} a_j^* \|\mathcal{B}_j\|, \quad \text{for every } j \geq j_0.$$

So, we see that, since  $|\varepsilon| \leq 1$  and  $\|\vec{I}_j(0)\| \geq \delta_j$ ,

$$-C|\varepsilon|a_j^* \|\mathcal{B}_j\| + T\bar{c}\|\vec{I}_j(0)\| > 0,$$

for every  $j \geq j_0$ . For the remaining finite number of integers  $j \in \{1, \dots, j_0 - 1\}$  we simply need to choose  $|\varepsilon|$  sufficiently small, thus finishing the proof.  $\square$

**Remark 4.3.** When  $X$  is finite-dimensional, the above estimate simplifies, in view of the compactness of the closed balls centered at the origin, so the first condition in **(Tw')** is sufficient in this case. Concerning the second condition in **(Tw')**, we see that it guarantees the existence of a sequence of balls, with smaller and smaller radii, over which the twist condition still holds.

Notice that the set

$$\Xi_I = \prod_{j=1}^{\infty} B^{N_j^*}[0, \delta_j + Ca_j^*],$$

where  $B^n[0, R]$  denotes the closed ball  $\{v \in \mathbb{R}^n : \|v\| \leq R\}$ , is compact, being homeomorphic to a Hilbert cube. We now modify the function  $\mathcal{K}$  outside  $\Xi_I$ , in order that the gradient of the modified function be bounded. Let  $R_I > 0$  be such that  $\Xi_I \subseteq \{v \in X : \|v\| \leq R_I\}$ , and  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth decreasing function such that

$$\psi(s) = 1 \text{ if } s \leq R_I, \quad \psi(s) = 0 \text{ if } s \geq 2R_I.$$

Define  $\tilde{\mathcal{K}} : X \rightarrow \mathbb{R}$  as  $\tilde{\mathcal{K}}(I) = \psi(\|I\|)\mathcal{K}(I)$ . Then, when  $I \neq 0$ ,

$$\|\nabla \tilde{\mathcal{K}}(I)\| = \left\| \psi'(\|I\|)\mathcal{K}(I) \frac{I}{\|I\|} + \psi(\|I\|)\nabla \mathcal{K}(I) \right\| \leq c_1|K(I)| + \|\nabla \mathcal{K}(I)\|,$$

for some  $c_1 > 0$ . By assumption **(L)**, we can find a Lipschitz constant  $L$  such that, for every  $s \in [0, 1]$ , if  $\|I\| \leq 2R_I$ ,

$$\|\nabla \mathcal{K}(sI)\| \leq \|\nabla \mathcal{K}(sI) - \nabla \mathcal{K}(0)\| + \|\nabla \mathcal{K}(0)\| \leq L\|I\| + \|\nabla \mathcal{K}(0)\|.$$

Moreover,

$$\begin{aligned} |K(I)| &= \left| K(0) + \int_0^1 \langle \nabla \mathcal{K}(sI), I \rangle ds \right| \leq |K(0)| + \sup_{s \in [0,1]} \|\nabla \mathcal{K}(sI)\| \|I\| \\ &\leq |K(0)| + (L\|I\| + \|\nabla \mathcal{K}(0)\|) \|I\|. \end{aligned}$$

Hence,

$$\|\nabla \tilde{\mathcal{K}}(I)\| \leq c_1 |K(0)| + (2R_I c_1 + 1)(2R_I L + \|\nabla \mathcal{K}(0)\|), \quad \text{for every } I \in X.$$

We define  $\mathbb{A} = \text{diag}(\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{J}})$  as a *block-diagonal* matrix having a diagonal formed by the matrices  $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{J}}$  introduced in **(Dec1)**, i.e. such that

$$\mathbb{A}(\vec{z}_1, \dots, \vec{z}_{\mathcal{J}}) = (\mathcal{A}_1 \vec{z}_1, \dots, \mathcal{A}_{\mathcal{J}} \vec{z}_{\mathcal{J}}).$$

It is easy to verify, using **(NR)**, that  $z \equiv 0$  is the unique  $T$ -periodic solution of equation  $J\dot{z} = \mathbb{A}z$ . Then, by Theorem 3.2, for every  $\mathcal{J}$  there is a  $T$ -periodic solution

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t))$$

of (3.4), with

$$\|\vec{I}_{\mathcal{J}_j}(0)\| < \delta_j, \quad \text{for every } j \geq 1. \quad (4.1)$$

(Recall that we have chosen the last constant components of the solutions of (3.4) to be equal to zero.) By Lemma 4.1, these solutions satisfy

$$\|\vec{I}_{\mathcal{J}_j}(t)\| \leq \delta_j + Ca_j^*, \quad \text{for every } t \in [0, T],$$

i.e.,

$$I_{\mathcal{J}}(t) \in \Xi_I, \quad \text{for every } t \in [0, T]. \quad (4.2)$$

Let us now consider the component  $\xi_{\mathcal{J}}(t)$  of the solution. By the periodicity assumption **(P<sub>τ</sub>)**, we can assume without loss of generality that  $\xi_k(0) \in [0, \tau_k]$ , for every  $k \geq 1$ . From Lemma 4.1, property **(L)** and (4.1), we have

$$|\xi_k(t) - \xi_k(0)| \leq \|\vec{\xi}_j(t) - \vec{\xi}_j(0)\| \leq Ca_j^* + TL\delta_j, \quad \text{for every } t \in [0, T],$$

for a suitable Lipschitz constant  $L$ . Setting  $b_k := Ca_j^* + TL\delta_j$ , where  $j$  is the index such that  $S_{j-1}^* < k \leq S_j^*$ , and defining

$$\Xi_{\xi} = \prod_{k=1}^{\infty} [-b_k, \tau_k + b_k],$$

we have that

$$\xi_{\mathcal{J}}(t) \in \Xi_{\xi}, \quad \text{for every } t \in [0, T]. \quad (4.3)$$

We now need an a priori estimate on  $z_{\mathcal{J}}(t)$ .

**Lemma 4.4.** *There exists a sequence  $(R_j)_j \in \ell^2$  of positive constants such that, for every  $T$ -periodic solution  $\zeta(t) = (\xi(t), I(t), z(t))$  of (3.2), we have*

$$\|\vec{z}_j\|_{C([0, T], \mathbb{R}^{2N_j^{\#}})} \leq |\varepsilon| R_j,$$

for every  $j \geq 1$ . The same property holds for every  $T$ -periodic solution of (3.4), when  $j = 1, \dots, \mathcal{J}$ .

*Proof.* Fix  $j \geq 1$  and consider the  $j$ -th block of the third equation in (3.2), i.e.

$$\mathcal{L}_j \bar{z}_j = \mathcal{A}_j \bar{z}_j + \varepsilon \nabla_{\bar{z}_j} \widehat{P}(t, \zeta), \quad (4.4)$$

where  $\mathcal{L}_j$  denotes the  $j$ -th block of the linear operator  $\mathcal{L}$  introduced in (NR), i.e.

$$\mathcal{L}_j \bar{z}_j = \mathcal{L}_j(z_{S_{j-1}^\#}^\#, \dots, z_{S_j^\#}^\#) = (J\dot{z}_{S_{j-1}^\#}^\#, \dots, J\dot{z}_{S_j^\#}^\#). \quad (4.5)$$

From hypothesis (Dec1), we have  $\sigma(\mathcal{L}_j - \mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \mathcal{A})$ . Hence, using (NR),  $0 \notin \sigma(\mathcal{L}_j - \mathcal{A}_j)$  and (4.4) is equivalent to

$$\bar{z}_j = \varepsilon (\mathcal{L}_j - \mathcal{A}_j)^{-1} \nabla_{\bar{z}_j} \widehat{P}(t, \zeta).$$

Moreover,

$$\|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| = \frac{1}{\text{dist}(0, \sigma(\mathcal{L}_j - \mathcal{A}_j))} \leq \frac{1}{\text{dist}(0, \sigma(\mathcal{L} - \mathcal{A}))} = \|(\mathcal{L} - \mathcal{A})^{-1}\|,$$

and consequently, setting  $r_j := \sqrt{T} a_j^\# \|(\mathcal{L} - \mathcal{A})^{-1}\|$ , we have that

$$\|\bar{z}_j\|_{L^2([0, T], \mathbb{R}^{2N_j^\#})} \leq |\varepsilon| \|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\bar{z}_j} \widehat{P}\|_{L^2([0, T], \mathbb{R}^{2N_j^\#})} \leq |\varepsilon| r_j.$$

Since  $\bar{z}_j$  solves (4.4), we have that  $\dot{\bar{z}}_j \in L^2([0, T], \mathbb{R}^{2N_j^\#})$ , and

$$\|\dot{\bar{z}}_j\|_{L^2([0, T], \mathbb{R}^{2N_j^\#})} \leq \|\mathcal{A}_j\| \|\bar{z}_j\|_{L^2([0, T], \mathbb{R}^{2N_j^\#})} + |\varepsilon| \sqrt{T} a_j^\# \leq |\varepsilon| (\|\mathcal{A}_j\| r_j + \sqrt{T} a_j^\#).$$

So, setting  $C_j = (1 + \|\mathcal{A}_j\|) r_j + \sqrt{T} a_j^\#$ ,

$$\|\bar{z}_j\|_{H^1([0, T], \mathbb{R}^{2N_j^\#})} \leq |\varepsilon| C_j. \quad (4.6)$$

By the continuous immersion of  $H^1([0, T], Z)$  in  $\mathcal{C}([0, T], Z)$ , cf. [14, §23.6], we can find a constant  $\chi > 0$  such that

$$\|z\|_{\mathcal{C}([0, T], Z)} \leq \chi \|z\|_{H^1([0, T], Z)},$$

for every  $z \in H^1([0, T], Z)$ . Since  $\mathcal{C}([0, T], \mathbb{R}^{2N_j^\#})$  and  $H^1([0, T], \mathbb{R}^{2N_j^\#})$  can be seen as subsets of  $\mathcal{C}([0, T], Z)$  and  $H^1([0, T], Z)$ , respectively, simply adding an infinite number of null components, we obtain

$$\|\bar{z}_j\|_{\mathcal{C}([0, T], \mathbb{R}^{2N_j^\#})} \leq \chi \|\bar{z}_j\|_{H^1([0, T], \mathbb{R}^{2N_j^\#})} \leq |\varepsilon| \chi C_j.$$

The proof is thus completed, taking  $R_j = \chi C_j$ .  $\square$

Defining

$$\Xi_z = \prod_{j=1}^{\infty} B^{2N_j^\#}[0, R_j],$$

we have thus proved that

$$z_{\mathcal{J}}(t) \in \Xi_z, \quad \text{for every } t \in [0, T]. \quad (4.7)$$

Summing up, by (4.2), (4.3), (4.7), we have that, setting  $\Xi = \Xi_\xi \times \Xi_I \times \Xi_z$ , the  $T$ -periodic solutions we found satisfy

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t)) \in \Xi, \quad \text{for every } t \in [0, T].$$

Notice that  $\Xi$  is compact, being the product of three compact sets. We will now prove that there is a subsequence of  $(\zeta_{\mathcal{J}})_{\mathcal{J}}$  which uniformly converges to a solution of (3.2).

From (4.6), recalling that  $|\varepsilon| \leq 1$ , we have

$$\|z_{\mathcal{J}}(t_1) - z_{\mathcal{J}}(t_2)\| \leq |t_1 - t_2|^{1/2} \left( \int_0^T \|\dot{z}_{\mathcal{J}}(s)\|^2 ds \right)^{1/2} \leq |t_1 - t_2|^{1/2} \left( \sum_{j=1}^{\infty} C_j^2 \right)^{1/2}.$$

Looking at the variables  $I_{\mathcal{J}}(t)$ , by  $(\mathbf{P}_{bd})$  we have that

$$\|I_{\mathcal{J}}(t_1) - I_{\mathcal{J}}(t_2)\| \leq |t_2 - t_1|^{1/2} \left( \int_0^T \|\dot{I}_{\mathcal{J}}(s)\|^2 ds \right)^{1/2} \leq |t_2 - t_1|^{1/2} \sqrt{T} \|a^*\|_{\ell^2}.$$

Concerning the variables  $\xi_{\mathcal{J}}(t)$ , we first observe that

$$\begin{aligned} \|\dot{\xi}_{\mathcal{J}}(s)\| &\leq \|\nabla \mathcal{K}(I_{\mathcal{J}}(s)) - \nabla \mathcal{K}(0)\| + \|a^*\|_{\ell^2} \\ &\leq L \|I_{\mathcal{J}}(s)\| + \|a^*\|_{\ell^2} \leq L \left( \sum_{j=1}^{\infty} (\delta_j + C a_j^*)^2 \right)^{1/2} + \|a^*\|_{\ell^2} := \widehat{C}, \end{aligned}$$

where  $L$  is a suitable Lipschitz constant provided by  $(\mathbf{L})$ . Then,

$$\|\xi_{\mathcal{J}}(t_1) - \xi_{\mathcal{J}}(t_2)\| \leq |t_2 - t_1|^{1/2} \left( \int_0^T \|\dot{\xi}_{\mathcal{J}}(s)\|^2 ds \right)^{1/2} \leq |t_2 - t_1|^{1/2} \sqrt{T} \widehat{C}.$$

Hence, the sequence  $(\zeta_{\mathcal{J}})_{\mathcal{J}}$  is equi-uniformly continuous on  $[0, T]$  and takes its values in a compact subset of  $\mathcal{X}$ . By the Ascoli–Arzelà Theorem, we find a subsequence, still denoted by  $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ , which uniformly converges to a certain continuous function  $\zeta^{\natural} : [0, T] \rightarrow \mathcal{X}$ , such that  $\zeta^{\natural}(t) \in \Xi$  for every  $t \in [0, T]$ , and  $\zeta^{\natural}(0) = \zeta^{\natural}(T)$ . We are going to prove that  $\zeta^{\natural}$  solves (3.2), following the lines of the proof of [3, Theorem 3].

Let us consider the solution  $\zeta_{\infty}$  of system (3.2) such that  $\zeta_{\infty}(0) = \zeta^{\natural}(0)$  which, by the boundedness of  $\nabla \mathcal{K}$  and  $\nabla_{\zeta} \widehat{P}$ , is certainly defined on  $[0, T]$ . We will prove that the sequence  $(\zeta_{\mathcal{J}})_{\mathcal{J}}$  converges uniformly to  $\zeta_{\infty}$ , thus obtaining that  $\zeta_{\infty} = \zeta^{\natural}$ . To this aim, we write the integral formulation of systems (3.2) and (3.3), for  $\mathcal{J} \geq 1$ :

$$\zeta_{\infty}(t) = \zeta_{\infty}(0) - \int_0^t J \nabla_{\zeta} \widehat{H}(s, \zeta_{\infty}(s)) ds, \quad (4.8)$$

$$\zeta_{\mathcal{J}}(t) = \zeta_{\mathcal{J}}(0) - \int_0^t J \nabla_{\zeta} \widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s)) ds. \quad (4.9)$$

In order to simplify the notations, we introduce the projection

$$\mathcal{P}_{\mathcal{J}}(\zeta) = \mathcal{P}_{\mathcal{J}}(\xi, I, z) = (\Pi_{S_{\mathcal{J}}^*} \xi, \Pi_{S_{\mathcal{J}}^*} I, \Pi_{S_{\mathcal{J}}^{\natural}} z).$$

Let us write

$$\|\zeta_{\mathcal{J}}(t) - \zeta_{\infty}(t)\| \leq \|\zeta_{\mathcal{J}}(t) - \mathcal{P}_{\mathcal{J}} \zeta_{\infty}(t)\| + \|\mathcal{P}_{\mathcal{J}} \zeta_{\infty}(t) - \zeta_{\infty}(t)\|.$$

By an elementary argument,

$$\|\mathcal{P}_{\mathcal{J}} \zeta_{\infty}(t) - \zeta_{\infty}(t)\| \rightarrow 0, \quad \text{as } \mathcal{J} \rightarrow \infty, \quad (4.10)$$

uniformly with respect to  $t \in [0, T]$ . From (4.8) and (4.9), since  $\mathcal{P}_{\mathcal{J}}J = J\mathcal{P}_{\mathcal{J}}$ , we have

$$\begin{aligned} \|\zeta_{\mathcal{J}}(t) - \mathcal{P}_{\mathcal{J}}\zeta_{\infty}(t)\| &\leq \|\zeta_{\mathcal{J}}(0) - \mathcal{P}_{\mathcal{J}}\zeta_{\infty}(0)\| + \\ &+ \int_0^t \|J\nabla_{\zeta}\widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s)) - J\mathcal{P}_{\mathcal{J}}\nabla_{\zeta}\widehat{H}(s, \zeta_{\infty}(s))\| ds. \end{aligned} \quad (4.11)$$

Notice that

$$\|\zeta_{\mathcal{J}}(0) - \mathcal{P}_{\mathcal{J}}\zeta_{\infty}(0)\| \leq \|\zeta_{\mathcal{J}}(0) - \zeta_{\infty}(0)\| = \|\zeta_{\mathcal{J}}(0) - \zeta^{\natural}(0)\| \rightarrow 0, \quad \text{as } \mathcal{J} \rightarrow \infty. \quad (4.12)$$

Since  $\nabla_{\zeta}\widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s)) = \mathcal{P}_{\mathcal{J}}\nabla_{\zeta}\widehat{H}(s, \zeta_{\mathcal{J}}(s))$ , the integral term in (4.11) satisfies

$$\int_0^t \left\| J\mathcal{P}_{\mathcal{J}}\left(\nabla_{\zeta}\widehat{H}(s, \zeta_{\mathcal{J}}(s)) - \nabla_{\zeta}\widehat{H}(s, \zeta_{\infty}(s))\right) \right\| ds \leq L \int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_{\infty}(s)\| ds,$$

where  $L$  is a suitable Lipschitz constant. Summing up, we have

$$\|\zeta_{\mathcal{J}}(t) - \zeta_{\infty}(t)\| \leq c_{\mathcal{J}} + L \int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_{\infty}(s)\| ds,$$

where  $(c_{\mathcal{J}})_{\mathcal{J}}$  is a sequence, provided by the limits in (4.10) and (4.12), such that  $\lim_{\mathcal{J}} c_{\mathcal{J}} = 0$ . Hence, by Gronwall's Lemma,

$$\|\zeta_{\mathcal{J}}(t) - \zeta_{\infty}(t)\| \leq c_{\mathcal{J}}e^{Lt}, \quad \text{for every } t \in [0, T],$$

implying that  $\zeta_{\mathcal{J}} \rightarrow \zeta_{\infty}$  uniformly on  $[0, T]$ . We conclude that  $\zeta_{\infty} = \zeta^{\natural}$  on  $[0, T]$ , thus showing that  $\zeta_{\infty}(0) = \zeta_{\infty}(T)$ , so that  $\zeta_{\infty}$  is a  $T$ -periodic solution of (3.2).

By the inverse change of variables

$$(\varphi(t), I(t), z(t)) = (\xi(t) + t\nabla\mathcal{K}(I^0), I(t), z(t)),$$

cf. (3.1), we have a solution of (2.1), satisfying (2.2). Moreover, condition (2.3) holds true, by Lemmas 4.1 and 4.4, suitably reducing, if necessary, the value of  $\bar{\varepsilon}$ . The proof of Theorem 2.1 is thus completed.  $\square$

## 5 Applications

### 5.1 Coupling second order with linear systems

We first state a simple lemma, which may be useful for the verification of the twist condition.

**Lemma 5.1.** *If there exists  $I^0 \in X$  such that  $\mathcal{K} : X \rightarrow \mathbb{R}$  is twice continuously differentiable at  $I^0$  and  $\mathcal{K}''(I^0) : X \rightarrow X$  is invertible, with bounded inverse, then there exist two positive constants  $\bar{c}, \bar{\rho}$  such that*

$$\|I - I^0\| \leq \bar{\rho} \quad \Rightarrow \quad \langle \nabla\mathcal{K}(y) - \nabla\mathcal{K}(I^0), \mathcal{K}''(I^0)(y - I^0) \rangle \geq \bar{c}\|y - I^0\|^2.$$

Moreover, if  $\dim X = \infty$  and, with the usual notation,  $\mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j)$ , then condition **(Tw)** holds.

*Proof.* Since  $\mathcal{B} := \mathcal{K}''(I^0) : X \rightarrow X$  is invertible with bounded inverse, there exists  $\gamma > 0$  such that  $\|\mathcal{B}I\| \geq \gamma\|I\|$  for every  $I \in X$ . Then,

$$\begin{aligned} \langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0), \mathcal{B}(I - I^0) \rangle &= \\ &= \int_0^1 \langle \mathcal{K}''(I^0 + s(I - I^0))(I - I^0), \mathcal{B}(I - I^0) \rangle ds \\ &= \|\mathcal{B}(I - I^0)\|^2 + \int_0^1 \langle [\mathcal{K}''(I^0 + s(I - I^0)) - \mathcal{B}](I - I^0), \mathcal{B}(I - I^0) \rangle ds \\ &\geq (\gamma^2 - \|\mathcal{B}\| \cdot \|\mathcal{K}''(I^0 + s(I - I^0)) - \mathcal{B}\|) \|I - I^0\|^2. \end{aligned}$$

Since  $\mathcal{K}''$  is continuous at  $I^0$ , there exists  $\bar{\rho} > 0$  such that, if  $I \in X$  satisfies  $\|I - I^0\| \leq \bar{\rho}$ , then

$$\|\mathcal{K}''(I) - \mathcal{B}\| = \|\mathcal{K}''(I) - \mathcal{K}''(I^0)\| \leq \frac{\gamma^2}{2\|\mathcal{B}\|},$$

so

$$\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0), \mathcal{B}(I - I^0) \rangle \geq \frac{\gamma^2}{2} \|I - I^0\|^2, \quad (5.1)$$

and the first part of the lemma is thus proved.

Assume now that  $\mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j)$ . We have that

$$\mathcal{B}I = (\mathcal{B}_1\vec{I}_1, \dots, \mathcal{B}_j\vec{I}_j, \dots),$$

where  $\mathcal{B}_j = \mathcal{K}_j''(\vec{I}_j^0)$ . Then, **(Tw)** is verified directly from (5.1) defining, for every  $j \in \{1, 2, \dots\}$ , the vector  $I$  as  $\vec{I}_i = \vec{I}_i^0$  if  $i \neq j$ , once  $\vec{I}_j$  has been chosen.  $\square$

We thus have the following.

**Corollary 5.2.** *Assume **(L)**, **(P<sub>τ</sub>)**, **(P<sub>bd</sub>)**, **(NR)**, **(Dec1)**, **(Dec2)** and **(C<sub>τ</sub>)** hold. If  $\mathcal{K} : X \rightarrow \mathbb{R}$  is twice continuously differentiable at  $I^0$  and  $\mathcal{K}''(I^0) : X \rightarrow X$  is invertible, with bounded inverse, then there exists  $\bar{\varepsilon} > 0$  such that, if  $|\varepsilon| \leq \bar{\varepsilon}$ , system (2.1) has a  $T$ -periodic solution.*

Let us now consider an equation in an infinite-dimensional space of the type

$$\begin{cases} \frac{d}{dt} (\nabla\Phi \circ \dot{x}) = \varepsilon \nabla_x F(t, x, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z F(t, x, z). \end{cases} \quad (5.2)$$

Let, for definiteness,  $\dim X = \infty$  and  $\dim Z = \infty$ . Concerning the bounded selfadjoint operator  $\mathcal{A}$ , we require the nonresonance assumption **(NR)** and that it decomposes as in **(Dec1)**. For the differential operator in the first equation, we suppose that there exists a sequence of positive integers  $(N_j)_{j \geq 1}$  such that, writing any vector  $y \in X$  as  $y = (\vec{y}_1, \dots, \vec{y}_j, \dots)$ , with  $\vec{y}_j \in \mathbb{R}^{N_j}$ ,

$$\Phi(y) = \sum_{j=1}^{\infty} \Phi_j(\vec{y}_j),$$

where each  $\Phi_j$  is a continuous real valued strictly convex function defined on a closed ball  $\bar{B}(0, a_j)$  in  $\mathbb{R}^{N_j}$ , continuously differentiable in the open ball  $B(0, a_j)$ , with  $\nabla\Phi_j : B(0, a_j) \rightarrow X$  being a homeomorphism, and  $\nabla\Phi_j(0) = 0$ .

Denoting by  $\Phi_j^*$  the Legendre–Fenchel transform of  $\Phi_j$ , we have that  $\Phi_j^* : X \rightarrow \mathbb{R}$  is strictly convex and coercive, with  $\nabla\Phi^* = (\nabla\Phi)^{-1} : X \rightarrow B(0, a)$ , cf. [11, Chapter 2]. We can define

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec{y}_j),$$

so that system (5.2) can be written as a Hamiltonian system

$$\begin{cases} \dot{x} = \nabla\Phi^*(y) \\ \dot{y} = \varepsilon \nabla_x F(t, x, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z F(t, x, z). \end{cases}$$

So, we are in the situation of system (2.1), taking  $\mathcal{K}(I) = \Phi^*(I)$  and  $P(t, \varphi, I, z) = F(t, \varphi, z)$ .

An example is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \left(1 - \sqrt{1 - \|\vec{y}_j\|^2}\right),$$

for which, writing  $x = (\vec{x}_1, \dots, \vec{x}_j, \dots)$ , system (5.2) becomes

$$\begin{cases} \frac{d}{dt} \frac{\dot{\vec{x}}_j}{\sqrt{1 - \|\dot{\vec{x}}_j\|^2}} = \varepsilon \nabla_{\vec{x}_j} F(t, x, z), & j = 1, 2, \dots \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z F(t, x, z), \end{cases} \quad (5.3)$$

so that, in the first equation, we can see a kind of *relativistic operator*. We then have the following.

**Corollary 5.3.** *In the above setting, assume moreover the following conditions:*

**(L)** *for every  $R > 0$  there exists a positive constant  $L_R$  such that*

$$\|\nabla_u F(t, u') - \nabla_u F(t, u'')\| \leq L_R \|u' - u''\|,$$

*for every  $t \in [0, T]$  and  $u' = (x', z'), u'' = (x'', z'') \in X \times Z$  with  $\|u'\| < R$  and  $\|u''\| < R$ ;*

**(F <sub>$\tau$</sub> )** *the function  $F(t, x, z)$  is  $\tau_k$ -periodic in each  $x_k$ , and the sequence  $(\tau_k)_k$  belongs to  $\ell^2$ ;*

**(F<sub>bd</sub>)** *there exist  $(\alpha_k^*)_k$  and  $(\alpha_l^\sharp)_l$  in  $\ell^2$  such that, for every  $k, l = 1, 2, \dots$ ,*

$$\left| \frac{\partial F}{\partial x_k}(t, x, z) \right| \leq \alpha_k^*, \quad \|\nabla_{z_l} F(t, x, z)\| \leq \alpha_l^\sharp,$$

*for every  $(t, x, z) \in [0, T] \times X \times Z$ .*

*Then, there exists  $\bar{\varepsilon} > 0$  such that, if  $|\varepsilon| \leq \bar{\varepsilon}$ , system (5.3) has a  $T$ -periodic solution.*

*Proof.* Taking  $I^0 = 0$ , we have that  $\nabla\Phi^*(0) = 0$  and  $(\Phi^*)''(0) = \text{Id}$ . So, assumption **(C <sub>$\tau$</sub> )** is fulfilled taking  $m_1 = m_2 = \dots = 0$  and, in view of Lemma 5.1, we can apply Theorem 2.1 to conclude.  $\square$

We have thus obtained an extension to infinite-dimensional systems of a result in [10].

Another possible situation where Theorem 2.1 applies is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \left( \sqrt{1 + \|\vec{y}_j\|^2} - 1 \right).$$

In this case, we find

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec{y}_j) = \sum_{j=1}^{\infty} \left( 1 - \sqrt{1 - \|\vec{y}_j\|^2} \right),$$

and the first equation in system (5.2) becomes

$$\frac{d}{dt} \frac{\dot{\vec{x}}_j}{\sqrt{1 + \|\dot{\vec{x}}_j\|^2}} = \varepsilon \nabla_{\vec{x}_j} F(t, x, z), \quad j = 1, 2, \dots$$

involving a kind of *mean curvature operator*.

Since each  $\nabla \Phi_j^*$  is defined only on the open ball  $B(0, 1)$ , we must first modify and extend the Hamiltonian function outside a ball  $B(0, r)$ , with  $r \in ]0, 1[$ , and then be careful that the  $\vec{y}_j$  component of the  $T$ -periodic solution we find remains in  $B(0, r)$ . We omit the details, for brevity. Stating the analogue of Corollary 5.3, we thus obtain an infinite-dimensional version of some results obtained in [8, 9] (see also [13], where bounded variation solutions are considered).

## 5.2 Perturbations of “superintegrable” systems

In this section we study a slightly different situation with respect to system (2.1). We are going to consider the Hamiltonian system

$$\begin{cases} \dot{\varphi} = \nabla \mathcal{K}(I) + \eta^2 \nabla_I P(t, \varphi, I, z) \\ -\dot{I} = \eta^2 \nabla_{\varphi} P(t, \varphi, I, z) \\ J\dot{z} = \eta \mathcal{A}z + \eta^2 \nabla_z P(t, \varphi, I, z), \end{cases} \quad (5.4)$$

with Hamiltonian function

$$H(t, \varphi, I, z) = \mathcal{K}(I) + \frac{\eta}{2} \langle \mathcal{A}z, z \rangle + \eta^2 P(t, \varphi, I, z).$$

The following result extends to an infinite-dimensional setting [7, Theorem 4.1], which was motivated by the study of perturbations of superintegrable systems, cf. [12].

**Theorem 5.4.** *Assume **(L)**, **(P<sub>τ</sub>)**, **(P<sub>bd</sub>)**, **(Dec1)**, **(Dec2)**, **(Tw)** and **(C<sub>τ</sub>)**. Moreover let the operator  $\mathcal{A}$  be invertible with a bounded inverse. Then, for every  $\sigma > 0$  there exists  $\bar{\eta} > 0$  such that, if  $|\eta| \leq \bar{\eta}$ , system (5.4) has a solution satisfying (2.2) and (2.3).*

Notice that the nonresonance assumption **(NR)** is not required here.

*Proof.* Arguing as above we can perform the change of variable (3.1) and set without loss of generality  $I^0 = 0$ , so to obtain

$$\begin{cases} \dot{\xi} = \nabla \mathcal{K}(I) - \nabla \mathcal{K}(0) + \eta^2 \nabla_I \widehat{P}(t, \xi, I, z) \\ -\dot{I} = \eta^2 \nabla_{\xi} \widehat{P}(t, \xi, I, z) \\ J\dot{z} = \eta \mathcal{A}z + \eta^2 \nabla_z \widehat{P}(t, \xi, I, z), \end{cases} \quad (5.5)$$



and, for every index  $\mathcal{J} \geq 1$ , its approximation

$$\begin{cases} \dot{\xi} = \Pi_{S_{\mathcal{J}}}[\nabla\mathcal{K}(I) - \nabla\mathcal{K}(0) + \eta^2\nabla_I\widehat{P}(t, \xi, I, z)] \\ -\dot{I} = \Pi_{S_{\mathcal{J}}}[\eta^2\nabla_{\xi}\widehat{P}(t, \xi, I, z)] \\ J\dot{z} = \Pi_{S_{\mathcal{J}}}[\eta\mathcal{A}z + \eta^2\nabla_z\widehat{P}(t, \xi, I, z)]. \end{cases} \quad (5.6)$$

Lemmas 4.1 and 4.2 holds again, simply replacing  $|\varepsilon|$  with  $\eta^2$  and  $\bar{\varepsilon}$  with  $\bar{\eta}^2$ . The statement and the proof of Lemma 4.4, however, must be modified as follows.

**Lemma 5.5.** *There exists a sequence  $(r_j)_j \in \ell^2$  of positive constants such that, for every  $T$ -periodic solution  $\zeta(t) = (\xi(t), I(t), z(t))$  of (5.5) we have*

$$\|\bar{z}_j\|_{L^2([0, T], \mathbb{R}^{2N_j^\sharp})} \leq |\eta|r_j,$$

for every  $j \geq 1$ . The same conclusion holds for every solution of (5.6), when  $j = 1, \dots, \mathcal{J}$ .

*Proof.* Fix  $j \geq 1$  and consider the  $j$ -th block of the third equation in (5.6), i.e.

$$\mathcal{L}_j \bar{z}_j = \eta\mathcal{A}_j \bar{z}_j + \eta^2 \nabla_{\bar{z}_j} \widehat{P}(t, \zeta), \quad (5.7)$$

where  $\mathcal{L}_j$  denotes the  $j$ -th block of the linear operator  $\mathcal{L}$ , cf. (4.5). From hypothesis **(Dec1)**, we have that  $\sigma(\mathcal{L}_j - \eta\mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \eta\mathcal{A})$ . We set  $\eta_0 = \min\{1, \frac{\pi}{T\|\mathcal{A}\|}\}$  and, recalling that  $0 \notin \sigma(\mathcal{A})$ , we choose  $\delta \in (0, \frac{\pi}{T})$  such that  $\sigma(\mathcal{A}) \cap [-\delta, \delta] = \emptyset$ .

**Claim.** When  $|\eta| < \eta_0$ , every  $\lambda \in \sigma(\mathcal{L} - \eta\mathcal{A})$  satisfies  $|\lambda| > \delta|\eta|$ .

In order to prove this Claim, notice that, if  $\lambda \in \sigma(\mathcal{L} - \eta\mathcal{A})$ , there exists a non-trivial  $T$ -periodic solution  $z$  of  $Jz' = (\eta\mathcal{A} - \lambda I)z$ , so

$$\sigma(J(\eta\mathcal{A} - \lambda I)) \cap \frac{2\pi}{T}i\mathbb{Z} \neq \emptyset. \quad (5.8)$$

If  $|\lambda| \geq \pi/T$ , then  $|\lambda| > \delta > \delta|\eta|$ . So, we can assume  $|\lambda| < \pi/T$ . In this case, we have

$$\|J(\eta\mathcal{A} - \lambda I)\| \leq |\eta|\|\mathcal{A}\| + |\lambda| < \frac{2\pi}{T},$$

so,

$$\mu \in \sigma(J(\eta\mathcal{A} - \lambda I)) \Rightarrow |\mu| \leq \|J(\eta\mathcal{A} - \lambda I)\| < \frac{2\pi}{T}.$$

By (5.8), we have that  $0 \in \sigma(J(\eta\mathcal{A} - \lambda I))$  and, since  $J$  is invertible,  $0 \in \sigma(\eta\mathcal{A} - \lambda I)$ . Hence,  $\frac{\lambda}{\eta} \in \sigma(\mathcal{A})$  and so  $|\frac{\lambda}{\eta}| > \delta$ , thus proving the Claim.

From now on we assume  $|\eta| < \eta_0$ . By the Claim, in particular,  $0 \notin \sigma(\mathcal{L} - \eta\mathcal{A})$  and so  $\mathcal{L} - \eta\mathcal{A}$  is invertible, as well as  $\mathcal{L}_j - \eta\mathcal{A}_j$ , with bounded inverses. Hence, (5.7) is equivalent to

$$\bar{z}_j = \eta^2(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1} \nabla_{\bar{z}_j} \widehat{P}(t, \zeta).$$

Moreover,

$$\|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| = \frac{1}{\text{dist}(0, \sigma(\mathcal{L}_j - \eta\mathcal{A}_j))} \leq \frac{1}{\text{dist}(0, \sigma(\mathcal{L} - \eta\mathcal{A}))} \leq \frac{1}{\delta|\eta|},$$

and consequently

$$\|\bar{z}_j\|_{L^2([0, T], \mathbb{R}^{2N_j^\sharp})} \leq \eta^2 \|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\bar{z}_j} \widehat{P}\|_{L^2([0, T], \mathbb{R}^{2N_j^\sharp})} \leq \frac{\eta^2 \sqrt{T} a_j^\sharp}{\delta|\eta|} = |\eta| \frac{\sqrt{T} a_j^\sharp}{\delta},$$

thus concluding the proof of the lemma.  $\square$

The proof of Theorem 5.4 can now be completed following again the lines of the proof of Theorem 2.1.  $\square$

**Acknowledgement.** The authors have been partially supported by an Italian research association GNAMPA-INdAM.

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Mathematics Subject Classification: 34C25, 47H15

Keywords: periodic solutions; bifurcation; infinite-dimensional dynamical systems; superintegrable systems.