Periodic solutions of nearly integrable Hamiltonian systems bifurcating from infinite-dimensional tori

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Dedicated to Shair Ahmad, on the occasion of his 85th birthday

Abstract

We prove the existence of periodic solutions of some infinite-dimensional nearly integrable Hamiltonian systems, bifurcating from infinite-dimensional tori, by the use of a generalization of the Poincaré–Birkhoff Theorem.

1 Introduction

The aim of this paper is to provide the existence of periodic solutions bifurcating from an infinite-dimensional invariant torus for a nearly integrable Hamiltonian system.

The finite-dimensional case was treated in [1, 2, 4, 5, 6] by assuming the existence of an invariant torus made of periodic solutions all sharing the same period, under some non-degeneracy conditions. Let us briefly describe the main result in this setting. Denoting by $H(I, \varphi) = \mathcal{K}(I)$ the Hamiltonian of a completely integrable system in $\mathbb{R}^{2N}$ (as usual, we denote by φ and I the angle and the action variables, respectively), we can write the corresponding system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) \\ -\dot I = 0\,. \end{cases}$$

Assume that there is a $I^0 \in \mathbb{R}^N$ such that

$$\det \mathcal{K}''(I^0) \neq 0\,. \tag{1.1}$$

Consider now the perturbed system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t,\varphi,I) \\ -\dot I = \varepsilon\nabla_\varphi P(t,\varphi,I)\,, \end{cases}$$

where $P(\cdot,\varphi,I)$ is T-periodic, and $P(t,\cdot,I)$ is τ_k-periodic in φ_k, for every $k = 1,\dots,N$. Assume that there exist some integers $m_1,\dots,m_N$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1,\dots,m_N\tau_N)\,. \tag{1.2}$$

Then, for $|\varepsilon|$ small enough, there are at least $N+1$ solutions $(\varphi(t), I(t))$ satisfying

$$\varphi(t+T) = \varphi(t) + T\,\nabla\mathcal{K}(I^0)\,, \quad I(t+T) = I(t)\,, \quad \text{for every } t \in \mathbb{R}\,, \tag{1.3}$$

and these solutions are near to some solutions of the unperturbed problem, i.e., briefly,

$$\varphi(t) \approx \varphi(0) + t\nabla\mathcal{K}(I^0)\,, \quad I(t) \approx I^0.$$

Notice that, by (1.2) and (1.3), $\varphi_k(t+T) = \varphi_k(t) + m_k\tau_k$, for every $k = 1, \dots, N$. Since usually φ_k is interpreted as an angle, with $\tau_k = 2\pi$, we may consider these as "periodic solutions" having period T. However, in the following, it will be better to keep more freedom in the choice of the periods τ_k.

Clearly enough, being $P(\cdot, \varphi, I)$ also mT-periodic for every positive integer m, one could search "periodic solutions" having period mT, as well (the so-called "subharmonic solutions"). We refer to [6] for a complete description of the problem, and for a more general statement, obtained by the use of the Poincaré–Birkhoff theorem.

The above result was recently extended in [7] for systems of the type

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t, \varphi, I, z) \\ -\dot I = \varepsilon\nabla_\varphi P(t, \varphi, I, z) \\ J\dot z = \mathcal{A}z + \varepsilon\nabla_z P(t, \varphi, I, z)\,, \end{cases} \tag{1.4}$$

where $J = \begin{pmatrix} 0 & -I_M \\ I_M & 0 \end{pmatrix}$ denotes the standard $2M \times 2M$ symplectic matrix and $\mathcal{A}$ is a symmetric non-resonant matrix, meaning that the only T-periodic solution of the unperturbed equation $J\dot z = \mathcal{A}z$ is the constant $z = 0$. Assuming (1.1), (1.2) and that ∇P, the gradient of P with respect to (φ, I, z), is uniformly bounded, the existence of at least $N+1$ solutions $(\varphi(t), I(t), z(t))$ satisfying (1.3) and $z(t+T) = z(t)$ was proved, when $|\varepsilon|$ is small enough.

The aim of this paper is to extend the above results to an infinite-dimensional setting. Let X and Z be the separable Hilbert spaces which will replace $\mathbb{R}^N$ and $\mathbb{R}^{2M}$, respectively. So, when looking at system (1.4), the functions $\varphi(t)$ and $I(t)$ will vary in X, while $z(t)$ will belong to Z. The spaces X and Z may be infinite-dimensional, finite-dimensional, or even reduced to $\{0\}$. If X is finite-dimensional, the cases $Z = \{0\}$ and Z finite-dimensional correspond to the settings in [6] and [7], respectively. However, if X or Z are infinite-dimensional, we will be able to prove the bifurcation of *at least one* periodic orbit from an invariant torus, which can also be infinite-dimensional. The multiplicity problem remains open.

In order to obtain our existence result in infinite-dimensions, we ask all the functions to be Lipschitz continuous on bounded sets, and the perturbing term ∇P to be uniformly bounded. Moreover, we need a special structure of the autonomous Hamiltonian function: in our assumptions **(Dec1)** and **(Dec2)** below, roughly speaking, the functions involved must be decomposable in a sequence of finite-dimensional blocks. This allows us to tackle the problem by a finite-dimensional approximating process, applying a version of the Poincaré–Birkhoff theorem for the reduced systems, and carefully estimating the so-found periodic solutions in order to guarantee their convergence to a periodic solution of the infinite-dimensional system.

2 The main result

We want to treat a system of the type (1.4) in an infinite-dimensional setting. To this aim, let X and E be two separable Hilbert spaces, and set $\mathcal{X} = X^2 \times E^2$. We will use the notation $\omega = (\varphi, I, z)$ for the elements of $\mathcal{X}$, with $\varphi, I \in X$ and $z = (x, y) \in E^2$. For simplicity, we will write $Z = E^2$, and we define $J : Z \to Z$ as $J(x, y) = (-y, x)$. (The same notation J will also be used with the same meaning in similar settings.) Let us introduce all the assumptions we need.

The continuous functions $\mathcal{K} : X \to \mathbb{R}$ and $P : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ are assumed to be continuously differentiable with respect to I and ω, respectively. The function $t \mapsto P(t, \omega)$ is T-periodic, for some $T > 0$. Moreover, we assume the following Lipschitz condition on bounded sets.

(L) For every $R > 0$ there exist two positive constants $L_R, \mathcal{L}_R$ such that

$$\|\nabla\mathcal{K}(I') - \nabla\mathcal{K}(I'')\| \le L_R\|I' - I''\| ,$$

for every $I', I'' \in X$ with $\|I'\| < R$, $\|I''\| < R$, and

$$\|\nabla_\omega P(t, \omega') - \nabla_\omega P(t, \omega'')\| \le \mathcal{L}_R\|\omega' - \omega''\| ,$$

for every $t \in [0, T]$ and $\omega', \omega'' \in \mathcal{X}$ with $\|\omega'\| < R$ and $\|\omega''\| < R$.

Introducing some Hilbert bases of X and E, we can identify these spaces either with some $\mathbb{R}^n$, if they are finite-dimensional, or with ℓ^2, the space of real sequences $(\alpha_k)_k$ which satisfy $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Each of the vectors φ, I in X and z in Z will then be written in their coordinates, e.g., $\varphi = (\varphi_1, \varphi_2, \dots)$, or $\varphi = (\varphi_k)_k$, with $\varphi_k \in \mathbb{R}$, while $I = (I_k)_k$ and $z = (z_l)_l$, with $z_l = (x_l, y_l) \in \mathbb{R}^2$. Notice that these sequences may be finite.

We also ask P to be periodic in the φ-variables, as follows.

($\mathbf{P}_\tau$) The function $P(t, \varphi, I, z)$ is τ_k-periodic in each φ_k, i.e., for $k = 1, 2, \dots$,

$$P(t, \dots, \varphi_k + \tau_k, \dots, I, z) = P(t, \dots, \varphi_k, \dots, I, z) , \quad \text{for every } (t, \varphi, I, z) \in [0, T] \times \mathcal{X} ;$$

moreover, if $\dim X = \infty$, then the sequence $(\tau_k)_k$ belongs to ℓ^2.

Concerning $\nabla_\omega P$, we assume it to be bounded and precompact, in the following sense.

($\mathbf{P}_{bd}$) There exist $(\alpha_k^\star)_k$ and $(\alpha_l^\sharp)_l$ such that, for every $k, l = 1, 2, \dots$,

$$\left|\frac{\partial P}{\partial \varphi_k}(t, \omega)\right| + \left|\frac{\partial P}{\partial I_k}(t, \omega)\right| \le \alpha_k^\star , \qquad \left|\frac{\partial P}{\partial x_l}(t, \omega)\right| + \left|\frac{\partial P}{\partial y_l}(t, \omega)\right| \le \alpha_l^\sharp ,$$

for every $(t, \omega) \in [0, T] \times \mathcal{X}$. If $\dim X = \infty$ or $\dim Z = \infty$, then $(\alpha_k^\star)_k$ or $(\alpha_l^\sharp)_l$ belong to ℓ^2, respectively.

Notice that the sets $\prod_{k=1}^{\infty}[-\alpha_k^\star, \alpha_k^\star]$ and $\prod_{l=1}^{\infty}[-\alpha_l^\sharp, \alpha_l^\sharp]$ are Hilbert cubes, hence compact sets in ℓ^2.

Let $\mathcal{A} : Z \to Z$ be a linear *bounded selfadjoint* operator. We need the following non-resonance assumption.

(NR) Denoting by

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset L^2([0, T], Z) \to L^2([0, T], Z) , \quad \mathcal{L}z = J\dot{z} ,$$

the unbounded selfadjoint operator with domain

$$\mathcal{D}(\mathcal{L}) = \{z \in H^1([0, T], Z) : z(0) = z(T)\} ,$$

we assume that $0 \notin \sigma(\mathcal{L} - \mathcal{A})$.

In the case when Z is infinite-dimensional, we need to assume a particular structure for the function $\mathcal{A}$.

(Dec1) If $\dim Z = \infty$, there exists a sequence of positive integers $(N_m^\sharp)_{m\geq 1}$ and functions $\mathcal{A}_m : \mathbb{R}^{2N_m^\sharp} \to \mathbb{R}^{2N_m^\sharp}$ such that, writing any vector $z \in Z$ as $z = (\vec{z}_1, \dots, \vec{z}_m, \dots)$, with $\vec{z}_m = (\vec{x}_m, \vec{y}_m) \in \mathbb{R}^{2N_m^\sharp}$, we have that

$$\mathcal{A}z = (\mathcal{A}_1\vec{z}_1, \dots, \mathcal{A}_m\vec{z}_m, \dots) .$$

Concerning the function $\mathcal{K}$, its gradient will be "guided" by some linear *bounded selfadjoint invertible* operator $\mathcal{B} : X \to X$, with bounded inverse, as we now specify. First of all, similarly as before, in the case when X is infinite-dimensional, we need to assume a particular structure for the functions $\mathcal{B}$ and $\mathcal{K}$.

(Dec2) If $\dim X = \infty$, there exists a sequence of positive integers $(N_j^\star)_{j\geq 1}$ and functions $\mathcal{B}_j : \mathbb{R}^{N_j^\star} \to \mathbb{R}^{N_j^\star}$, $\mathcal{K}_j : \mathbb{R}^{N_j^\star} \to \mathbb{R}$ such that, writing any vector $I \in X$ as $I = (\vec{I}_1, \dots, \vec{I}_j, \dots)$, with $\vec{I}_j \in \mathbb{R}^{N_j^\star}$, we have that

$$\mathcal{B}\, I = (\mathcal{B}_1\vec{I}_1, \dots, \mathcal{B}_j\vec{I}_j, \dots) , \qquad \mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j) .$$

We now fix $I^0 \in X$, and introduce our *twist condition*.

(Tw) There exist two positive constants $\bar{c}, \bar{\rho}$ such that, for every $j = 1, 2, \dots$,

$$\|\vec{I}_j - \vec{I}_j^0\| \leq \bar{\rho} \quad \Rightarrow \quad \Big\langle \nabla\mathcal{K}_j(\vec{I}_j) - \nabla\mathcal{K}_j(\vec{I}_j^0) \,,\, \mathcal{B}_j(\vec{I}_j - \vec{I}_j^0) \Big\rangle \geq \bar{c}\, \|\vec{I}_j - \vec{I}_j^0\|^2 .$$

Finally, we assume a compatibility condition between T and the periods introduced in **(P$_\tau$)**.

(C$_\tau$) There exist some integers $m_1, m_2, \dots$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1, m_2\tau_2, \dots) .$$

We are now ready to state our main result.

Theorem 2.1. *Let the above assumptions hold. Then, for every $\sigma > 0$ there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, there is a solution of system*

$$\begin{cases} \dot{\varphi} = \nabla\mathcal{K}(I) + \varepsilon\nabla_I P(t, \varphi, I, z) \\ -\dot{I} = \varepsilon\nabla_\varphi P(t, \varphi, I, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z P(t, \varphi, I, z) , \end{cases} \tag{2.1}$$

satisfying

$$\varphi(t+T) = \varphi(t) + T\nabla\mathcal{K}(I^0) , \quad I(t+T) = I(t) , \quad z(t+T) = z(t) , \tag{2.2}$$

and such that

$$\|\varphi(t) - \varphi(0) - t\nabla\mathcal{K}(I^0)\| + \|I(t) - I^0\| + \|z(t)\| < \sigma , \quad \textit{for every } t \in \mathbb{R} . \tag{2.3}$$

Remark 2.2. When X is finite-dimensional, we will see that condition **(Tw)** can be generalized to

(Tw′) There exists a positive constant $\bar{\rho}$ such that

$$\|I - I^0\| \leq \bar{\rho} \quad \Rightarrow \quad \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) \,,\, \mathcal{B}(I - I^0) \big\rangle > 0 ;$$

a still more general condition, adopted in [6], is the following:

$$0 \in \mathrm{cl}\Big\{ \rho \in]0, +\infty[\,:\, \min_{\|I - I^0\| = \rho} \big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) \,,\, \mathcal{B}(I - I^0) \big\rangle > 0 \Big\} ,$$

where $\mathrm{cl}\,\mathcal{S}$ denotes the closure of a set $\mathcal{S}$.

3 Preliminaries for the proof

We will carry out the proof of Theorem 2.1 in the case $\dim X = \infty$ and $\dim Z = \infty$, with some specific remarks on the finite-dimensional cases. By the change of variables

$$(\xi(t), I(t), z(t)) = (\varphi(t) - t\nabla\mathcal{K}(I^0), I(t), z(t)), \tag{3.1}$$

system (2.1) becomes

$$\begin{cases} \dot{\xi} = \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0) + \varepsilon\nabla_I \widehat{P}(t,\xi,I,z) \\ -\dot{I} = \varepsilon\nabla_\xi \widehat{P}(t,\xi,I,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z \widehat{P}(t,\xi,I,z), \end{cases} \tag{3.2}$$

where

$$\widehat{P}(t,\xi,I,z) = P(t,\xi + t\nabla\mathcal{K}(I^0), I, z).$$

We use the notation $\zeta = (\xi, I, z)$; the Hamiltonian function is thus

$$\widehat{H}(t,\zeta) = \mathcal{K}(I) - \big\langle \nabla\mathcal{K}(I^0), I \big\rangle + \tfrac{1}{2}\left\langle \mathcal{A}z, z\right\rangle + \varepsilon\widehat{P}(t,\zeta).$$

Combining **(P$_\tau$)** with **(C$_\tau$)**, we see that the function $\widehat{P}(\cdot,\xi,I,z)$ is T-periodic, and $\widehat{P}(t,\cdot,I,z)$ is τ_k-periodic in ξ_k, for every $k = 1,2,\ldots$

Some additional notations are now necessary. By assumption **(Dec2)**, the vectors $\xi, I \in X$ decompose in vectors $\vec{\xi}_j, \vec{I}_j \in \mathbb{R}^{N_j^\star}$. Setting

$$S_0^\star = 0\,, \quad S_j^\star = \sum_{i=1}^{j} N_i^\star \quad \text{for } j \geq 1\,,$$

we can explicitly write the components of $\vec{\xi}_j, \vec{I}_j$ as

$$\vec{\xi}_j = (\xi_{S_{j-1}^\star+1}, \xi_{S_{j-1}^\star+2}, \ldots, \xi_{S_j^\star}), \qquad \vec{I}_j = (I_{S_{j-1}^\star+1}, I_{S_{j-1}^\star+2}, \ldots, I_{S_j^\star}).$$

Similarly, by assumption **(Dec1)**, the vector $z \in Z$ decomposes in vectors $\vec{z}_m \in \mathbb{R}^{2N_m^\sharp}$. Setting

$$S_0^\sharp = 0\,, \quad S_m^\sharp = \sum_{i=1}^{m} N_i^\sharp \quad \text{for } m \geq 1\,,$$

we can explicitly write the components of $\vec{z}_m$ as

$$\vec{z}_m = (z_{S_{m-1}^\sharp+1}, z_{S_{m-1}^\sharp+2}, \ldots, z_{S_m^\sharp}).$$

We define the sequences $(a_j^\star)_j$, $(a_m^\sharp)_m$ in ℓ^2 by

$$a_j^\star = \Big(\sum_{i=1}^{N_j^\star} (\alpha^\star_{S_{j-1}^\star+i})^2\Big)^{1/2}, \quad a_m^\sharp = \Big(\sum_{i=1}^{N_m^\sharp} (\alpha^\sharp_{S_{m-1}^\sharp+i})^2\Big)^{1/2}.$$

Notice that $\|a^\star\|_{\ell^2} = \|\alpha^\star\|_{\ell^2}$ and $\|a^\sharp\|_{\ell^2} = \|\alpha^\sharp\|_{\ell^2}$.

Remark 3.1. When X has a finite dimension d_X, we can define the sequence $(N_j^\star)_j$ taking $N_1^\star = d_X$ and $N_j^\star = 0$ for $j \geq 2$. Similarly when Z is finite-dimensional.

Without loss of generality, from now on we will assume that $I^0 = 0$, a situation which can be recovered by a simple translation. The strategy of the proof of Theorem 2.1 will be to construct a finite-dimensional approximation of system (3.2), and then pass to the limit on the dimension. Precisely, we define the projections $\Pi_{S^\star_{\mathcal{J}}} : X \to X$ and $\Pi_{S^\sharp_{\mathcal{J}}} : Z \to Z$ as

$$\Pi_{S^\star_{\mathcal{J}}} v = (\vec{v}_1, \dots, \vec{v}_{\mathcal{J}}, 0, 0, \dots)\,, \qquad \Pi_{S^\sharp_{\mathcal{J}}} z = (\vec{z}_1, \dots, \vec{z}_{\mathcal{J}}, 0, 0, \dots)\,,$$

and consider the truncated system

$$\begin{cases} \dot{\xi} = \Pi_{S^\star_{\mathcal{J}}}[\nabla\mathcal{K}(I) - \nabla\mathcal{K}(0) + \varepsilon\nabla_I\widehat{P}(t,\xi,I,z)] \\ -\dot{I} = \Pi_{S^\star_{\mathcal{J}}}[\varepsilon\nabla_\xi\widehat{P}(t,\xi,I,z)] \\ J\dot{z} = \Pi_{S^\sharp_{\mathcal{J}}}[\mathcal{A}z + \varepsilon\nabla_z\widehat{P}(t,\xi,I,z)]\,. \end{cases} \tag{3.3}$$

We thus have the Hamiltonian function

$$\widehat{H}_{\mathcal{J}}(t,\zeta) = \mathcal{K}(\Pi_{S^\star_{\mathcal{J}}} I) - \big\langle \nabla\mathcal{K}(0)\,,\, \Pi_{S^\star_{\mathcal{J}}} I\big\rangle + \tfrac{1}{2}\big\langle \mathcal{A}\Pi_{S^\sharp_{\mathcal{J}}} z\,,\, \Pi_{S^\sharp_{\mathcal{J}}} z\big\rangle + \varepsilon\widehat{P}(t, \Pi_{S^\star_{\mathcal{J}}}\xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z)\,.$$

Notice that the function

$$\widehat{P}_{\mathcal{J}}(t,\xi,I,z) = \widehat{P}(t, \Pi_{S^\star_{\mathcal{J}}}\xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z)$$

satisfies both **(L)** and **($\mathbf{P}_\tau$)** with the same constants, for every index $\mathcal{J} \geq 1$, and observe that system (3.3) is equivalent to

$$\begin{cases} \dot{\vec{\xi}}_j = \nabla\mathcal{K}_j(\vec{I}_j) - \nabla\mathcal{K}_j(0) + \varepsilon\nabla_{\vec{I}_j}\widehat{P}_{\mathcal{J}}(t,\xi,I,z) & \\ -\dot{\vec{I}}_j = \varepsilon\nabla_{\vec{\xi}_j}\widehat{P}_{\mathcal{J}}(t,\xi,I,z) & j \leq \mathcal{J}\,, \\ J\dot{\vec{z}}_j = \mathcal{A}_j\vec{z}_j + \varepsilon\nabla_{\vec{z}_j}\widehat{P}_{\mathcal{J}}(t,\xi,I,z) & \\ & \\ \dot{\vec{\xi}}_i = 0 & \\ -\dot{\vec{I}}_i = 0 & i > \mathcal{J}\,. \\ J\dot{\vec{z}}_i = 0 & \end{cases} \tag{3.4}$$

It can be viewed as two uncoupled systems, the first one in a finite-dimensional space (the "approximating system"), and the second one, infinite-dimensional, having only constant solutions. From now on, we will take $\vec{\xi}_i(t), \vec{I}_i(t), \vec{z}_i(t)$ identically equal to zero when $i \geq \mathcal{J}$.

Concerning the "approximating system", we will need the following slight modification of [7, Corollary 2.3]. Let us consider the finite-dimensional Hamiltonian system

$$J\dot{\zeta} = \nabla_\zeta H(t,\zeta)\,, \tag{3.5}$$

with $\zeta = (\xi, I, z) \in \mathbb{R}^{N+N+2M}$, where the Hamiltonian function is T-periodic in t. Here we use the notation $\xi = (\vec{\xi}_1, \dots, \vec{\xi}_{\mathcal{J}})$, $I = (\vec{I}_1, \dots, \vec{I}_{\mathcal{J}})$.

Theorem 3.2. *Assume that $H(t,\zeta) = \frac{1}{2}\langle \mathbb{A}z, z\rangle + G(t,\zeta)$, where $\mathbb{A}$ is a symmetric $2M \times 2M$ matrix such that $z \equiv 0$ is the unique T-periodic solution of equation $J\dot{z} = \mathbb{A}z$, and there exists a constant c_1 such that*

$$|\nabla_\zeta G(t,\zeta)| \leq c_1\,, \qquad \textit{for every } (t,\zeta) \in \mathbb{R} \times \mathbb{R}^{2(M+N)}\,.$$

Let $G(t,\xi,I,z)$ be periodic in the variables $\xi_1,\dots,\xi_N$. Assume moreover the existence of some positive constants $r'_j < r''_j$ and symmetric invertible matrices $\mathcal{B}_j$, with $j=1,\dots,\mathcal{J}$, such that, for any solution $\zeta(t)=(\xi(t),I(t),z(t))$ of (3.5), *if*

$$r'_j \le \|\vec{I}_j(0)-\vec{I}^0_j\| \le r''_j \quad \text{and} \quad \|\vec{I}_i(0)-\vec{I}^0_i\| \le r''_i \ \text{ for every } i\neq j\,,$$

then

$$\Big\langle \vec{\xi}_j(T)-\vec{\xi}_j(0)\,,\,\mathcal{B}_j(\vec{I}_j(0)-\vec{I}^0_j)\Big\rangle > 0\,.$$

Then, there are at least $N+1$ geometrically distinct T-periodic solutions $\zeta(t)=(\xi(t),I(t),z(t))$ of (3.5), *such that*

$$\|\vec{I}_j(0)-\vec{I}^0_j\| < r'_j\,, \quad \text{for every } j=1,\dots,\mathcal{J}\,.$$

4 Proof of Theorem 2.1

In what follows, we always assume that $|\varepsilon|\le 1$, and we denote by $\bar\rho$ the constant introduced in assumption **(Tw)**. Moreover, as in the previous section, we assume $I^0=0$.

Lemma 4.1. *There is a constant $C>0$ with the following property: if $\zeta(t)=(\xi(t),I(t),z(t))$ is a solution of* (3.2) *with $\|\vec{I}_j(0)\|\le\bar\rho$, for some $j\ge 1$, then*

$$\|\vec{\xi}_j(t)-\vec{\xi}_j(0)-t[\nabla\mathcal{K}_j(\vec{I}_j(0))-\nabla\mathcal{K}_j(0)]\| + \|\vec{I}_j(t)-\vec{I}_j(0)\| \le C|\varepsilon|a^\star_j\,, \ \text{ for every } t\in[0,T]\,.$$

The same property holds for the solutions of (3.4), *when $j=1,\dots,\mathcal{J}$.*

Proof. Let us start computing, for every $t\in[0,T]$ and every $k\in\{S^\star_{j-1}+1,\dots,S^\star_{j-1}+N^\star_j=S^\star_j\}$,

$$|I_k(t)-I_k(0)| \le \int_0^t |\dot I_k(s)|\,ds \le |\varepsilon|\int_0^T \Big|\,\frac{\partial\widehat{P}}{\partial\xi_k}(s,\zeta(s))\Big|\,ds \le |\varepsilon|\,T\alpha^\star_k\,.$$

Then we easily get

$$\|\vec{I}_j(t)-\vec{I}_j(0)\| \le |\varepsilon|\,T\Big(\sum_{i=1}^{N^\star_j}(\alpha^\star_{S^\star_{j-1}+i})^2\Big)^{1/2} = |\varepsilon|\,Ta^\star_j\,.$$

Moreover,

$$\begin{aligned}
\|\vec{\xi}_j(t)-\vec{\xi}_j(0)-t[\nabla\mathcal{K}_j(\vec{I}_j(0))-\nabla\mathcal{K}_j(0)]\| &\le \int_0^t \|\dot{\vec{\xi}}_j(s)-[\nabla\mathcal{K}_j(\vec{I}_j(0))-\nabla\mathcal{K}_j(0)]\|\,ds\\
&\le \int_0^T \|\nabla\mathcal{K}_j(\vec{I}_j(s))-\nabla\mathcal{K}_j(\vec{I}_j(0))\|\,ds + |\varepsilon|\int_0^T \|\nabla_{\vec{I}_j}\widehat{P}(s,\zeta(s))\|\,ds\\
&\le \int_0^T L\|\vec{I}_j(s)-\vec{I}_j(0)\|\,ds + |\varepsilon|\,Ta^\star_j\\
&\le |\varepsilon|\,T(1+LT)a^\star_j\,,
\end{aligned}$$

where L is a suitable Lipschitz constant provided by **(L)**. The proof is thus completed. □

Lemma 4.2. *There exist $\bar\varepsilon>0$ and a sequence $(\delta_j)_j$ in ℓ^2, with $\delta_j\in\,]0,\bar\rho]$, satisfying the following property: if $\zeta(t)=(\xi(t),I(t),z(t))$ is a solution of* (3.2), *with $|\varepsilon|<\bar\varepsilon$ and $\delta_j\le\|\vec{I}_j(0)\|\le\bar\rho$, for some $j\ge 1$, then*

$$\Big\langle \vec{\xi}_j(T)-\vec{\xi}_j(0)\,,\,\mathcal{B}_j\,\vec{I}_j(0)\Big\rangle > 0\,.$$

The same property holds for the solutions of (3.4), *when $j=1,\dots,\mathcal{J}$.*

Proof. If $\|\vec{I}_j(0)\| \leq \bar{\rho}$ for some $j \geq 1$, then, by Lemma 4.1 and **(Tw)**,

$$\begin{aligned}\left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0)\,,\, \mathcal{B}_j\,\vec{I}_j(0)\right\rangle &= \left\langle \vec{\xi}_j(T) - \vec{\xi}_j(0) - T[\nabla\mathcal{K}_j(\vec{I}_j(0)) - \nabla\mathcal{K}_j(0)]\,,\, \mathcal{B}_j\,\vec{I}_j(0)\right\rangle + \\ &\quad + T\left\langle \nabla\mathcal{K}_j(\vec{I}_j(0)) - \nabla\mathcal{K}_j(0)\,,\, \mathcal{B}_j\,\vec{I}_j(0)\right\rangle \\ &\geq -C|\varepsilon| a_j^\star\,\|\mathcal{B}_j\|\,\|\vec{I}_j(0)\| + T\bar{c}\|\vec{I}_j(0)\|^2 \\ &= \left(-C|\varepsilon| a_j^\star\,\|\mathcal{B}_j\| + T\bar{c}\|\vec{I}_j(0)\|\right)\|\vec{I}_j(0)\|\,.\end{aligned}$$

Setting

$$\delta_j := \min\left\{\bar{\rho}\,,\, \frac{2C}{\bar{c}T}\,a_j^\star\,\|\mathcal{B}_j\|\right\}\,,$$

we easily verify that $(\delta_j)_j \in \ell^2$, since $(\|\mathcal{B}_j\|)_j$ is bounded by $\|\mathcal{B}\|$ and $(a_j^\star)_j \in \ell^2$; in particular, there exists an integer j_0 such that

$$\delta_j = \frac{2C}{\bar{c}T}\,a_j^\star\,\|\mathcal{B}_j\|\,, \qquad \text{for every } j \geq j_0\,.$$

So, we see that, since $|\varepsilon| \leq 1$ and $\|\vec{I}_j(0)\| \geq \delta_j$,

$$-C|\varepsilon| a_j^\star\,\|\mathcal{B}_j\| + T\bar{c}\|\vec{I}_j(0)\| > 0\,,$$

for every $j \geq j_0$. For the remaining finite number of integers $j \in \{1, \ldots, j_0 - 1\}$ we simply need to choose $|\varepsilon|$ sufficiently small, thus finishing the proof. $\square$

Remark 4.3. When X is finite-dimensional, the above estimate simplifies, in view of the compactness of the closed balls centered at the origin, so the first condition in **(Tw')** is sufficient in this case. Concerning the second condition in **(Tw')**, we see that it guarantees the existence of a sequence of balls, with smaller and smaller radii, over which the twist condition still holds.

Notice that the set

$$\Xi_I = \prod_{j=1}^{\infty} B^{N_j^\star}[0, \delta_j + C a_j^\star]\,,$$

where $B^n[0,R]$ denotes the closed ball $\{v \in \mathbb{R}^n : \|v\| \leq R\}$, is compact, being homeomorphic to a Hilbert cube. We now modify the function $\mathcal{K}$ outside Ξ_I, in order that the gradient of the modified function be bounded. Let $R_I > 0$ be such that $\Xi_I \subseteq \{v \in X : \|v\| \leq R_I\}$, and $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function such that

$$\psi(s) = 1 \;\text{ if } s \leq R_I\,, \qquad \psi(s) = 0 \;\text{ if } s \geq 2R_I\,.$$

Define $\widetilde{\mathcal{K}} : X \to \mathbb{R}$ as $\widetilde{\mathcal{K}}(I) = \psi(\|I\|)\mathcal{K}(I)$. Then, when $I \neq 0$,

$$\|\nabla\widetilde{\mathcal{K}}(I)\| = \left\|\psi'(\|I\|)\mathcal{K}(I)\frac{I}{\|I\|} + \psi(\|I\|)\nabla\mathcal{K}(I)\right\| \leq c_1|K(I)| + \|\nabla\mathcal{K}(I)\|\,,$$

for some $c_1 > 0$. By assumption **(L)**, we can find a Lipschitz constant L such that, for every $s \in [0,1]$, if $\|I\| \leq 2R_I$,

$$\|\nabla\mathcal{K}(sI)\| \leq \|\nabla\mathcal{K}(sI) - \nabla\mathcal{K}(0)\| + \|\nabla\mathcal{K}(0)\| \leq L\|I\| + \|\nabla\mathcal{K}(0)\|\,.$$

Moreover,

$$|K(I)| = \left| K(0) + \int_0^1 \langle \nabla\mathcal{K}(sI)\,,\, I\rangle \; ds \right| \le |K(0)| + \sup_{s\in[0,1]} \|\nabla\mathcal{K}(sI)\|\, \|I\|$$
$$\le |K(0)| + \big(L\|I\| + \|\nabla\mathcal{K}(0)\|\big)\, \|I\|\,.$$

Hence,

$$\|\nabla\widetilde{\mathcal{K}}(I)\| \le c_1|K(0)| + (2R_Ic_1+1)(2R_IL + \|\nabla\mathcal{K}(0)\|)\,, \qquad \text{for every } I\in X.$$

We define $\mathbb{A} = \mathrm{diag}(\mathcal{A}_1,\dots,\mathcal{A}_{\mathcal{J}})$ as a *block-diagonal* matrix having a diagonal formed by the matrices $\mathcal{A}_1,\dots,\mathcal{A}_{\mathcal{J}}$ introduced in **(Dec1)**, i.e. such that

$$\mathbb{A}(\vec{z}_1,\dots,\vec{z}_{\mathcal{J}}) = (\mathcal{A}_1\vec{z}_1,\dots,\mathcal{A}_{\mathcal{J}}\vec{z}_{\mathcal{J}})\,.$$

It is easy to verify, using **(NR)**, that $z\equiv 0$ is the unique T-periodic solution of equation $J\dot{z} = \mathbb{A}z$. Then, by Theorem 3.2, for every $\mathcal{J}$ there is a T-periodic solution

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t))$$

of (3.4), with

$$\|\vec{I}_{\mathcal{J}_j}(0)\| < \delta_j\,, \quad \text{for every } j\ge 1\,. \tag{4.1}$$

(Recall that we have chosen the last constant components of the solutions of (3.4) to be equal to zero.) By Lemma 4.1, these solutions satisfy

$$\|\vec{I}_{\mathcal{J}_j}(t)\| \le \delta_j + Ca_j^\star\,, \quad \text{for every } t\in[0,T]\,,$$

i.e.,

$$I_{\mathcal{J}}(t) \in \Xi_I\,, \quad \text{for every } t\in[0,T]\,. \tag{4.2}$$

Let us now consider the component $\xi_{\mathcal{J}}(t)$ of the solution. By the periodicity assumption **($\mathbf{P}_\tau$)**, we can assume without loss of generality that $\xi_k(0)\in[0,\tau_k]$, for every $k\ge 1$. From Lemma 4.1, property **(L)** and (4.1), we have

$$|\xi_k(t)-\xi_k(0)| \le \|\vec{\xi}_j(t)-\vec{\xi}_j(0)\| \le Ca_j^\star + TL\delta_j\,, \qquad \text{for every } t\in[0,T]\,,$$

for a suitable Lipschitz constant L. Setting $b_k := Ca_j^\star + TL\delta_j$, where j is the index such that $S_{j-1}^\star < k \le S_j^\star$, and defining

$$\Xi_\xi = \prod_{k=1}^{\infty} [-b_k, \tau_k + b_k]\,,$$

we have that

$$\xi_{\mathcal{J}}(t) \in \Xi_\xi\,, \quad \text{for every } t\in[0,T]\,. \tag{4.3}$$

We now need an a priori estimate on $z_{\mathcal{J}}(t)$.

Lemma 4.4. *There exists a sequence $(R_j)_j\in\ell^2$ of positive constants such that, for every T-periodic solution $\zeta(t) = (\xi(t), I(t), z(t))$ of* (3.2)*, we have*

$$\|\vec{z}_j\|_{\mathcal{C}([0,T],\mathbb{R}^{2N_j^\sharp})} \le |\varepsilon| R_j\,,$$

for every $j\ge 1$. The same property holds for every T-periodic solution of (3.4)*, when $j = 1,\dots,\mathcal{J}$.*

Proof. Fix $j \geq 1$ and consider the j-th block of the third equation in (3.2), i.e.

$$\mathcal{L}_j \vec{z}_j = \mathcal{A}_j \vec{z}_j + \varepsilon \nabla_{\vec{z}_j} \widehat{P}(t, \zeta)\,, \tag{4.4}$$

where $\mathcal{L}_j$ denotes the j-th block of the linear operator $\mathcal{L}$ introduced in **(NR)**, i.e.

$$\mathcal{L}_j \vec{z}_j = \mathcal{L}_j(z_{S^\sharp_{j-1}+1}, \dots, z_{S^\sharp_j}) = (J\dot{z}_{S^\sharp_{j-1}+1}, \dots, J\dot{z}_{S^\sharp_j})\,. \tag{4.5}$$

From hypothesis **(Dec1)**, we have $\sigma(\mathcal{L}_j - \mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \mathcal{A})$. Hence, using **(NR)**, $0 \notin \sigma(\mathcal{L}_j - \mathcal{A}_j)$ and (4.4) is equivalent to

$$\vec{z}_j = \varepsilon(\mathcal{L}_j - \mathcal{A}_j)^{-1} \nabla_{\vec{z}_j} \widehat{P}(t, \zeta)\,.$$

Moreover,

$$\|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| = \frac{1}{\mathrm{dist}(0, \sigma(\mathcal{L}_j - \mathcal{A}_j))} \leq \frac{1}{\mathrm{dist}(0, \sigma(\mathcal{L} - \mathcal{A}))} = \|(\mathcal{L} - \mathcal{A})^{-1}\|\,,$$

and consequently, setting $r_j := \sqrt{T} a^\sharp_j \|(\mathcal{L} - \mathcal{A})^{-1}\|$, we have that

$$\|\vec{z}_j\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon|\, \|(\mathcal{L}_j - \mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\vec{z}_j} \widehat{P}\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon| r_j\,.$$

Since $\vec{z}_j$ solves (4.4), we have that $\dot{\vec{z}}_j \in L^2([0,T], \mathbb{R}^{2N^\sharp_j})$, and

$$\|\dot{\vec{z}}_j\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} \leq \|\mathcal{A}_j\| \|\vec{z}_j\|_{L^2([0,T], \mathbb{R}^{2N^\sharp_j})} + |\varepsilon| \sqrt{T} a^\sharp_j \leq |\varepsilon| \Big(\|\mathcal{A}_j\| r_j + \sqrt{T} a^\sharp_j \Big)\,.$$

So, setting $C_j = (1 + \|\mathcal{A}_j\|) r_j + \sqrt{T} a^\sharp_j$,

$$\|\vec{z}_j\|_{H^1([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon| C_j\,. \tag{4.6}$$

By the continuous immersion of $H^1([0,T], Z)$ in $\mathcal{C}([0,T], Z)$, cf. [14, §23.6], we can find a constant $\chi > 0$ such that

$$\|z\|_{\mathcal{C}([0,T],Z)} \leq \chi \|z\|_{H^1([0,T],Z)}\,,$$

for every $z \in H^1([0,T], Z)$. Since $\mathcal{C}([0,T], \mathbb{R}^{2N^\sharp_j})$ and $H^1([0,T], \mathbb{R}^{2N^\sharp_j})$ can be seen as a subsets of $\mathcal{C}([0,T], Z)$ and $H^1([0,T], Z)$, respectively, simply adding an infinite number of null components, we obtain

$$\|\vec{z}_j\|_{\mathcal{C}([0,T], \mathbb{R}^{2N^\sharp_j})} \leq \chi \|\vec{z}_j\|_{H^1([0,T], \mathbb{R}^{2N^\sharp_j})} \leq |\varepsilon| \chi C_j\,.$$

The proof is thus completed, taking $R_j = \chi C_j$. □

Defining

$$\Xi_z = \prod_{j=1}^{\infty} B^{2N^\sharp_j}[0, R_j]\,,$$

we have thus proved that

$$z_{\mathcal{J}}(t) \in \Xi_z\,, \quad \text{for every } t \in [0,T]\,. \tag{4.7}$$

Summing up, by (4.2), (4.3), (4.7), we have that, setting $\Xi = \Xi_\xi \times \Xi_I \times \Xi_z$, the T-periodic solutions we found satisfy

$$\zeta_{\mathcal{J}}(t) = (\xi_{\mathcal{J}}(t), I_{\mathcal{J}}(t), z_{\mathcal{J}}(t)) \in \Xi\,, \quad \text{for every } t \in [0,T]\,.$$

Notice that Ξ is compact, being the product of three compact sets. We will now prove that there is a subsequence of $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ which uniformly converges to a solution of (3.2).

From (4.6), recalling that $|\varepsilon| \le 1$, we have

$$\|z_{\mathcal{J}}(t_1) - z_{\mathcal{J}}(t_2)\| \le |t_1 - t_2|^{1/2} \left(\int_0^T \|\dot z_{\mathcal{J}}(s)\|^2 \, ds \right)^{1/2} \le |t_1 - t_2|^{1/2} \Big(\sum_{j=1}^{\infty} C_j^2 \Big)^{1/2}.$$

Looking at the variables $I_{\mathcal{J}}(t)$, by **(P$_{bd}$)** we have that

$$\|I_{\mathcal{J}}(t_1) - I_{\mathcal{J}}(t_2)\| \le |t_2 - t_1|^{1/2} \left(\int_0^T \|\dot I_{\mathcal{J}}(s)\|^2 \, ds \right)^{1/2} \le |t_2 - t_1|^{1/2} \sqrt{T}\, \|a^\star\|_{\ell^2} .$$

Concerning the variables $\xi_{\mathcal{J}}(t)$, we first observe that

$$\begin{aligned} \|\dot\xi_{\mathcal{J}}(s)\| &\le \|\nabla \mathcal{K}(I_{\mathcal{J}}(s)) - \nabla\mathcal{K}(0)\| + \|a^\star\|_{\ell^2} \\ &\le L \|I_{\mathcal{J}}(s)\| + \|a^\star\|_{\ell^2} \le L \Big(\sum_{j=1}^{\infty} (\delta_j + C a_j^\star)^2 \Big)^{1/2} + \|a^\star\|_{\ell^2} := \widehat{C}\,, \end{aligned}$$

where L is a suitable Lipschitz constant provided by **(L)**. Then,

$$\|\xi_{\mathcal{J}}(t_1) - \xi_{\mathcal{J}}(t_2)\| \le |t_2 - t_1|^{1/2} \left(\int_0^T \|\dot\xi_{\mathcal{J}}(s)\|^2 \, ds \right)^{1/2} \le |t_2 - t_1|^{1/2} \sqrt{T}\, \widehat{C}\,.$$

Hence, the sequence $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ is equi-uniformly continuous on $[0,T]$ and takes its values in a compact subset of $\mathcal{X}$. By the Ascoli–Arzelà Theorem, we find a subsequence, still denoted by $(\zeta_{\mathcal{J}})_{\mathcal{J}}$, which uniformly converges to a certain continuous function $\zeta^\natural : [0,T] \to \mathcal{X}$, such that $\zeta^\natural(t) \in \Xi$ for every $t \in [0,T]$, and $\zeta^\natural(0) = \zeta^\natural(T)$. We are going to prove that $\zeta^\natural$ solves (3.2), following the lines of the proof of [3, Theorem 3].

Let us consider the solution ζ_∞ of system (3.2) such that $\zeta_\infty(0) = \zeta^\natural(0)$ which, by the boundedness of $\nabla\mathcal{K}$ and $\nabla_\zeta \widehat{P}$, is certainly defined on $[0,T]$. We will prove that the sequence $(\zeta_{\mathcal{J}})_{\mathcal{J}}$ converges uniformly to ζ_∞, thus obtaining that $\zeta_\infty = \zeta^\natural$. To this aim, we write the integral formulation of systems (3.2) and (3.3), for $\mathcal{J} \ge 1$:

$$\zeta_\infty(t) = \zeta_\infty(0) - \int_0^t J \nabla_\zeta \widehat{H}(s, \zeta_\infty(s)) \, ds\,, \tag{4.8}$$

$$\zeta_{\mathcal{J}}(t) = \zeta_{\mathcal{J}}(0) - \int_0^t J \nabla_\zeta \widehat{H}_{\mathcal{J}}(s, \zeta_{\mathcal{J}}(s)) \, ds\,. \tag{4.9}$$

In order to simplify the notations, we introduce the projection

$$\mathscr{P}_{\mathcal{J}}(\zeta) = \mathscr{P}_{\mathcal{J}}(\xi, I, z) = (\Pi_{S^\star_{\mathcal{J}}} \xi, \Pi_{S^\star_{\mathcal{J}}} I, \Pi_{S^\sharp_{\mathcal{J}}} z)\,.$$

Let us write

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le \|\zeta_{\mathcal{J}}(t) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(t)\| + \|\mathscr{P}_{\mathcal{J}}\zeta_\infty(t) - \zeta_\infty(t)\|\,.$$

By an elementary argument,

$$\|\mathscr{P}_{\mathcal{J}}\zeta_\infty(t) - \zeta_\infty(t)\| \to 0\,, \quad \text{as } \mathcal{J} \to \infty\,, \tag{4.10}$$

uniformly with respect to $t \in [0,T]$. From (4.8) and (4.9), since $\mathscr{P}_{\mathcal{J}} J = J \mathscr{P}_{\mathcal{J}}$, we have

$$\|\zeta_{\mathcal{J}}(t) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(t)\| \le \|\zeta_{\mathcal{J}}(0) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(0)\| + \\ + \int_0^t \|J\nabla_\zeta \widehat{H}_{\mathcal{J}}(s,\zeta_{\mathcal{J}}(s)) - J\mathscr{P}_{\mathcal{J}}\nabla_\zeta \widehat{H}(s,\zeta_\infty(s))\|\,ds\,. \tag{4.11}$$

Notice that

$$\|\zeta_{\mathcal{J}}(0) - \mathscr{P}_{\mathcal{J}}\zeta_\infty(0)\| \le \|\zeta_{\mathcal{J}}(0) - \zeta_\infty(0)\| = \|\zeta_{\mathcal{J}}(0) - \zeta^\natural(0)\| \to 0\,, \quad \text{as } \mathcal{J} \to \infty\,. \tag{4.12}$$

Since $\nabla_\zeta \widehat{H}_{\mathcal{J}}(s,\zeta_{\mathcal{J}}(s)) = \mathscr{P}_{\mathcal{J}}\nabla_\zeta \widehat{H}(s,\zeta_{\mathcal{J}}(s))$, the integral term in (4.11) satisfies

$$\int_0^t \left\| J\mathscr{P}_{\mathcal{J}}\Big(\nabla_\zeta \widehat{H}(s,\zeta_{\mathcal{J}}(s)) - \nabla_\zeta \widehat{H}(s,\zeta_\infty(s))\Big)\right\| ds \le L\int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_\infty(s)\|\,ds\,,$$

where L is a suitable Lipschitz constant. Summing up, we have

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le c_{\mathcal{J}} + L\int_0^t \|\zeta_{\mathcal{J}}(s) - \zeta_\infty(s)\|\,ds\,,$$

where $(c_{\mathcal{J}})_{\mathcal{J}}$ is a sequence, provided by the limits in (4.10) and (4.12), such that $\lim_{\mathcal{J}} c_{\mathcal{J}} = 0$. Hence, by Gronwall's Lemma,

$$\|\zeta_{\mathcal{J}}(t) - \zeta_\infty(t)\| \le c_{\mathcal{J}} e^{Lt}\,, \quad \text{for every } t \in [0,T]\,,$$

implying that $\zeta_{\mathcal{J}} \to \zeta_\infty$ uniformly on $[0,T]$. We conclude that $\zeta_\infty = \zeta^\natural$ on $[0,T]$, thus showing that $\zeta_\infty(0) = \zeta_\infty(T)$, so that ζ_∞ is a T-periodic solution of (3.2).

By the inverse change of variables

$$(\varphi(t), I(t), z(t)) = (\xi(t) + t\nabla\mathcal{K}(I^0), I(t), z(t))\,,$$

cf. (3.1), we have a solution of (2.1), satisfying (2.2). Moreover, condition (2.3) holds true, by Lemmas 4.1 and 4.4, suitably reducing, if necessary, the value of $\bar\varepsilon$. The proof of Theorem 2.1 is thus completed. □

5 Applications

5.1 Coupling second order with linear systems

We first state a simple lemma, which may be useful for the verification of the twist condition.

Lemma 5.1. *If there exists $I^0 \in X$ such that $\mathcal{K} : X \to \mathbb{R}$ is twice continuously differentiable at I^0 and $\mathcal{K}''(I^0) : X \to X$ is invertible, with bounded inverse, then there exist two positive constants $\bar c$, $\bar\rho$ such that*

$$\|I - I^0\| \le \bar\rho \quad \Rightarrow \quad \big\langle \nabla\mathcal{K}(y) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{K}''(I^0)(y - I^0)\big\rangle \ge \bar c\,\|y - I^0\|^2.$$

Moreover, if $\dim X = \infty$ and, with the usual notation, $\mathcal{K}(I) = \sum_{j=1}^\infty \mathcal{K}_j(\vec{I}_j)$, then condition **(Tw)** *holds.*

Proof. Since $\mathcal{B} := \mathcal{K}''(I^0) : X \to X$ is invertible with bounded inverse, there exists $\gamma > 0$ such that $\|\mathcal{B}\, I\| \geq \gamma \|I\|$ for every $I \in X$. Then,

$$\begin{aligned}
&\big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{B}(I - I^0) \big\rangle = \\
&\quad = \int_0^1 \big\langle \mathcal{K}''\big(I^0 + s(I - I^0)\big)(I - I^0)\,,\, \mathcal{B}(I - I^0) \big\rangle \, ds \\
&\quad = \|\mathcal{B}(I - I^0)\|^2 + \int_0^1 \big\langle \big[\mathcal{K}''\big(I^0 + s(I - I^0)\big) - \mathcal{B}\big]\,(I - I^0)\,,\, \mathcal{B}(I - I^0) \big\rangle \, ds \\
&\quad \geq \big(\gamma^2 - \|\mathcal{B}\| \cdot \|\mathcal{K}''\big(I^0 + s(I - I^0)\big) - \mathcal{B}\|\big)\, \|I - I^0\|^2 \,.
\end{aligned}$$

Since $\mathcal{K}''$ is continuous at I^0, there exists $\bar{\rho} > 0$ such that, if $I \in X$ satisfies $\|I - I^0\| \leq \bar{\rho}$, then

$$\|\mathcal{K}''(I) - \mathcal{B}\| = \|\mathcal{K}''(I) - \mathcal{K}''(I^0)\| \leq \frac{\gamma^2}{2\|\mathcal{B}\|},$$

so

$$\big\langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0)\,,\, \mathcal{B}(I - I^0) \big\rangle \geq \frac{\gamma^2}{2} \|I - I^0\|^2, \tag{5.1}$$

and the first part of the lemma is thus proved.

Assume now that $\mathcal{K}(I) = \sum_{j=1}^{\infty} \mathcal{K}_j(\vec{I}_j)$. We have that

$$\mathcal{B}\, I = (\mathcal{B}_1 \vec{I}_1, \ldots, \mathcal{B}_j \vec{I}_j, \ldots)\,,$$

where $\mathcal{B}_j = \mathcal{K}_j''(\vec{I}_j^{\,0})$. Then, **(Tw)** is verified directly from (5.1) defining, for every $j \in \{1, 2, \ldots\}$, the vector I as $\vec{I}_i = \vec{I}_i^{\,0}$ if $i \neq j$, once $\vec{I}_j$ has been chosen. $\square$

We thus have the following.

Corollary 5.2. *Assume* **(L)**, **(P$_\tau$)**, **(P$_{bd}$)**, **(NR)**, **(Dec1)**, **(Dec2)** *and* **(C$_\tau$)** *hold. If $\mathcal{K} : X \to \mathbb{R}$ is twice continuously differentiable at I^0 and $\mathcal{K}''(I^0) : X \to X$ is invertible, with bounded inverse, then there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, system* (2.1) *has a T-periodic solution.*

Let us now consider an equation in an infinite-dimensional space of the type

$$\begin{cases} \dfrac{d}{dt}\,(\nabla\Phi \circ \dot{x}) = \varepsilon\, \nabla_x F(t, x, z) \\ J\dot{z} = \mathcal{A}z + \varepsilon \nabla_z F(t, x, z)\,. \end{cases} \tag{5.2}$$

Let, for definiteness, $\dim X = \infty$ and $\dim Z = \infty$. Concerning the bounded selfadjoint operator $\mathcal{A}$, we require the nonresonance assumption **(NR)** and that it decomposes as in **(Dec1)**. For the differential operator in the first equation, we suppose that there exists a sequence of positive integers $(N_j)_{j\geq 1}$ such that, writing any vector $y \in X$ as $y = (\vec{y}_1, \ldots, \vec{y}_j, \ldots)$, with $\vec{y}_j \in \mathbb{R}^{N_j}$,

$$\Phi(y) = \sum_{j=1}^{\infty} \Phi_j(\vec{y}_j)\,,$$

where each Φ_j is a continuous real valued strictly convex function defined on a closed ball $\overline{B}(0, a_j)$ in $\mathbb{R}^{N_j}$, continuously differentiable in the open ball $B(0, a_j)$, with $\nabla\Phi_j : B(0, a_j) \to X$ being a homeomorphism, and $\nabla\Phi_j(0) = 0$.

Denoting by Φ_j^* the Legendre–Fenchel transform of Φ_j, we have that $\Phi_j^* : X \to \mathbb{R}$ is strictly convex and coercive, with $\nabla\Phi^* = (\nabla\Phi)^{-1} : X \to B(0,a)$, cf. [11, Chapter 2]. We can define

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec{y}_j)\,,$$

so that system (5.2) can be written as a Hamiltonian system

$$\begin{cases} \dot{x} = \nabla\Phi^*(y) \\ \dot{y} = \varepsilon\,\nabla_x F(t,x,z) \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z F(t,x,z)\,. \end{cases}$$

So, we are in the situation of system (2.1), taking $\mathcal{K}(I) = \Phi^*(I)$ and $P(t,\varphi,I,z) = F(t,\varphi,z)$.

An example is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \Big(1 - \sqrt{1 - \|\vec{y}_j\|^2}\,\Big)\,,$$

for which, writing $x = (\vec{x}_1, \dots, \vec{x}_j, \dots)$, system (5.2) becomes

$$\begin{cases} \dfrac{d}{dt}\,\dfrac{\dot{\vec{x}}_j}{\sqrt{1 - \|\dot{\vec{x}}_j\|^2}} = \varepsilon\,\nabla_{\vec{x}_j} F(t,x,z)\,, \qquad j = 1,2,\dots \\ J\dot{z} = \mathcal{A}z + \varepsilon\nabla_z F(t,x,z)\,, \end{cases} \tag{5.3}$$

so that, in the first equation, we can see a kind of *relativistic operator*. We then have the following.

Corollary 5.3. *In the above setting, assume moreover the following conditions:*

(L) *for every $R > 0$ there exists a positive constant L_R such that*

$$\|\nabla_u F(t,u') - \nabla_u F(t,u'')\| \le L_R \|u' - u''\|\,,$$

for every $t \in [0,T]$ and $u' = (x',z'), u'' = (x'',z'') \in X \times Z$ with $\|u'\| < R$ and $\|u''\| < R$;

($\mathbf{F}_\tau$) *the function $F(t,x,z)$ is τ_k-periodic in each x_k, and the sequence $(\tau_k)_k$ belongs to ℓ^2;*

($\mathbf{F}_{bd}$) *there exist $(\alpha_k^\star)_k$ and $(\alpha_l^\sharp)_l$ in ℓ^2 such that, for every $k,l = 1,2,\dots$,*

$$\left|\frac{\partial F}{\partial x_k}(t,x,z)\right| \le \alpha_k^\star\,, \qquad \|\nabla_{z_l} F(t,x,z)\| \le \alpha_l^\sharp\,,$$

for every $(t,x,z) \in [0,T] \times X \times Z$.

Then, there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \le \bar{\varepsilon}$, system (5.3) *has a T-periodic solution.*

Proof. Taking $I^0 = 0$, we have that $\nabla\Phi^*(0) = 0$ and $(\Phi^*)''(0) = \mathrm{Id}$. So, assumption **($\mathbf{C}_\tau$)** is fulfilled taking $m_1 = m_2 = \cdots = 0$ and, in view of Lemma 5.1, we can apply Theorem 2.1 to conclude. □

We have thus obtained an extension to infinite-dimensional systems of a result in [10].

Another possible situation where Theorem 2.1 applies is provided by the choice

$$\Phi(y) = \sum_{j=1}^{\infty} \left(\sqrt{1 + \|\vec{y}_j\|^2} - 1 \right).$$

In this case, we find

$$\Phi^*(y) = \sum_{j=1}^{\infty} \Phi_j^*(\vec{y}_j) = \sum_{j=1}^{\infty} \left(1 - \sqrt{1 - \|\vec{y}_j\|^2} \right),$$

and the first equation in system (5.2) becomes

$$\frac{d}{dt} \frac{\dot{\vec{x}}_j}{\sqrt{1 + \|\dot{\vec{x}}_j\|^2}} = \varepsilon \nabla_{\vec{x}_j} F(t, x, z), \qquad j = 1, 2, \dots$$

involving a kind of *mean curvature operator*.

Since each $\nabla \Phi_j^*$ is defined only on the open ball $B(0,1)$, we must first modify and extend the Hamiltonian function outside a ball $B(0,r)$, with $r \in]0,1[$, and then be careful that the $\vec{y}_j$ component of the T-periodic solution we find remains in $B(0,r)$. We omit the details, for briefness. Stating the analogue of Corollary 5.3, we thus obtain an infinite-dimensional version of some results obtained in [8, 9] (see also [13], where bounded variation solutions are considered).

5.2 Perturbations of "superintegrable" systems

In this section we study a slightly different situation with respect to system (2.1). We are going to consider the Hamiltonian system

$$\begin{cases} \dot{\varphi} = \nabla \mathcal{K}(I) + \eta^2 \nabla_I P(t, \varphi, I, z) \\ -\dot{I} = \eta^2 \nabla_\varphi P(t, \varphi, I, z) \\ J\dot{z} = \eta \mathcal{A} z + \eta^2 \nabla_z P(t, \varphi, I, z), \end{cases} \tag{5.4}$$

with Hamiltonian function

$$H(t, \varphi, I, z) = \mathcal{K}(I) + \frac{\eta}{2} \langle \mathcal{A} z \,,\, z \rangle + \eta^2 P(t, \varphi, I, z).$$

The following result extends to an infinite-dimensional setting [7, Theorem 4.1], which was motivated by the study of perturbations of superintegrable systems, cf. [12].

Theorem 5.4. *Assume* **(L)**, **(P_τ)**, **(P_{bd})**, **(Dec1)**, **(Dec2)**, **(Tw)** *and* **(C_τ)**. *Moreover let the operator $\mathcal{A}$ be invertible with a bounded inverse. Then, for every $\sigma > 0$ there exists $\bar{\eta} > 0$ such that, if $|\eta| \le \bar{\eta}$, system* (5.4) *has a solution satisfying* (2.2) *and* (2.3).

Notice that the nonresonance assumption **(NR)** is not required here.

Proof. Arguing as above we can perform the change of variable (3.1) and set without loss of generality $I^0 = 0$, so to obtain

$$\begin{cases} \dot{\xi} = \nabla \mathcal{K}(I) - \nabla \mathcal{K}(0) + \eta^2 \nabla_I \widehat{P}(t, \xi, I, z) \\ -\dot{I} = \eta^2 \nabla_\xi \widehat{P}(t, \xi, I, z) \\ J\dot{z} = \eta \mathcal{A} z + \eta^2 \nabla_z \widehat{P}(t, \xi, I, z), \end{cases} \tag{5.5}$$

and, for every index $\mathcal{J} \geq 1$, its approximation

$$\begin{cases} \dot{\xi} = \Pi_{S^\star_{\mathcal{J}}}[\nabla\mathcal{K}(I) - \nabla\mathcal{K}(0) + \eta^2\nabla_I\widehat{P}(t,\xi,I,z)] \\ -\dot{I} = \Pi_{S^\star_{\mathcal{J}}}[\eta^2\nabla_\xi\widehat{P}(t,\xi,I,z)] \\ J\dot{z} = \Pi_{S^\sharp_{\mathcal{J}}}[\eta\mathcal{A}z + \eta^2\nabla_z\widehat{P}(t,\xi,I,z)] \,. \end{cases} \tag{5.6}$$

Lemmas 4.1 and 4.2 holds again, simply replacing $|\varepsilon|$ with η^2 and $\bar{\varepsilon}$ with $\bar{\eta}^2$. The statement and the proof of Lemma 4.4, however, must be modified as follows.

Lemma 5.5. *There exists a sequence $(r_j)_j \in \ell^2$ of positive constants such that, for every T-periodic solution $\zeta(t) = (\xi(t), I(t), z(t))$ of* (5.5) *we have*

$$\|\vec{z}_j\|_{L^2([0,T],\mathbb{R}^{2N^\sharp_j})} \leq |\eta| r_j \,,$$

for every $j \geq 1$. The same conclusion holds for every solution of (5.6)*, when $j = 1, \ldots, \mathcal{J}$.*

Proof. Fix $j \geq 1$ and consider the j-th block of the third equation in (5.6), i.e.

$$\mathcal{L}_j\vec{z}_j = \eta\mathcal{A}_j\vec{z}_j + \eta^2\nabla_{\vec{z}_j}\widehat{P}(t,\zeta) \,, \tag{5.7}$$

where $\mathcal{L}_j$ denotes the j-th block of the linear operator $\mathcal{L}$, cf. (4.5). From hypothesis **(Dec1)**, we have that $\sigma(\mathcal{L}_j - \eta\mathcal{A}_j) \subseteq \sigma(\mathcal{L} - \eta\mathcal{A})$. We set $\eta_0 = \min\{1, \frac{\pi}{T\|\mathcal{A}\|}\}$ and, recalling that $0 \notin \sigma(\mathcal{A})$, we choose $\delta \in (0, \frac{\pi}{T})$ such that $\sigma(\mathcal{A}) \cap [-\delta, \delta] = \varnothing$.

Claim. When $|\eta| < \eta_0$, every $\lambda \in \sigma(\mathcal{L} - \eta\mathcal{A})$ satisfies $|\lambda| > \delta|\eta|$.

In order to prove this Claim, notice that, if $\lambda \in \sigma(\mathcal{L}-\eta\mathcal{A})$, there exists a non-trivial T-periodic solution z of $Jz' = (\eta\mathcal{A} - \lambda I)z$, so

$$\sigma\big(J(\eta\mathcal{A} - \lambda I)\big) \cap \frac{2\pi}{T}i\mathbb{Z} \neq \varnothing \,. \tag{5.8}$$

If $|\lambda| \geq \pi/T$, then $|\lambda| > \delta > \delta|\eta|$. So, we can assume $|\lambda| < \pi/T$. In this case, we have

$$\|J(\eta\mathcal{A} - \lambda I)\| \leq |\eta|\,\|\mathcal{A}\| + |\lambda| < \frac{2\pi}{T} \,,$$

so,

$$\mu \in \sigma\big(J(\eta\mathcal{A} - \lambda I)\big) \quad \Rightarrow \quad |\mu| \leq \|J(\eta\mathcal{A} - \lambda I)\| < \frac{2\pi}{T} \,.$$

By (5.8), we have that $0 \in \sigma(J(\eta\mathcal{A} - \lambda I))$ and, since J is invertible, $0 \in \sigma(\eta\mathcal{A} - \lambda I)$. Hence, $\frac{\lambda}{\eta} \in \sigma(\mathcal{A})$ and so $|\frac{\lambda}{\eta}| > \delta$, thus proving the Claim.

From now on we assume $|\eta| < \eta_0$. By the Claim, in particular, $0 \notin \sigma(\mathcal{L} - \eta\mathcal{A})$ and so $\mathcal{L} - \eta\mathcal{A}$ is invertible, as well as $\mathcal{L}_j - \eta\mathcal{A}_j$, with bounded inverses. Hence, (5.7) is equivalent to

$$\vec{z}_j = \eta^2(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\nabla_{\vec{z}_j}\widehat{P}(t,\zeta) \,.$$

Moreover,

$$\|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| = \frac{1}{\operatorname{dist}(0, \sigma(\mathcal{L}_j - \eta\mathcal{A}_j))} \leq \frac{1}{\operatorname{dist}(0, \sigma(\mathcal{L} - \eta\mathcal{A}))} \leq \frac{1}{\delta|\eta|} \,,$$

and consequently

$$\|\vec{z}_j\|_{L^2([0,T],\mathbb{R}^{2N^\sharp_j})} \leq \eta^2\,\|(\mathcal{L}_j - \eta\mathcal{A}_j)^{-1}\| \cdot \|\nabla_{\vec{z}_j}\widehat{P}\|_{L^2([0,T],\mathbb{R}^{2N^\sharp_j})} \leq \frac{\eta^2\sqrt{T}a^\sharp_j}{\delta|\eta|} = |\eta|\frac{\sqrt{T}a^\sharp_j}{\delta} \,,$$

thus concluding the proof of the lemma. □

The proof of Theorem 5.4 can now be completed following again the lines of the proof of Theorem 2.1. □

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