Isomonodromy Deformations with Coalescent Eigenvalues and Applications - Bedlewo 2018

Davide Guzzetti, SISSA, Via Bonomea, 265
34136 Triese – Italy
guzzetti@sissa.it

Abstract: I review my talk at Bedlewo, dealing with a differential system depending on deformation parametrs, with a Fuchsian singularity at $z = 0$, and an irregular one at $z = \infty$ of Poincaré rank 1. The eigenvalues of the leading matrix at $z = \infty$ may coalesce along a coalescence locus. In the recent paper [8], the isomonodromic deformation theory has been extended to this non-generic case, which was not considered by Jimbo, Miwa and Ueno. We explain this extension, with applications.

Keywords: Isomonodromy deformations, Stokes phenomenon, Coalescence of eigenvalues, Monodromy data, Quantum cohomology

1 Introduction

This paper reviews my talk at Bedlewo, in September 2018, and is based on [8], [9] and [21] (see also [11], [12]). We consider a linear differential system of dimension $n \times n$

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right)Y, \qquad \Lambda(u) = \mathrm{diag}(u_1, ..., u_n), \tag{1.1}$$

depending on complex parameters $u = (u_1, ..., u_n)$, with matrix coefficient $A(u)$ depending holomorphically on u is a polydisc of $\mathbb{C}^n$. Here, $z = 0$ is a Fuchsian singularity, $z = \infty$ is an irregular one of Poincaré rank 1. The polydisc can be of two kinds: centered at u^0, such that $u_j^0 \neq u_k^0$ for any $1 \leqslant j \neq k \leqslant n$; or centered at a *coalescence point* u^c such that

$$u_j^c = u_k^c \quad \text{for some } j \neq k,$$

so that some eigenvalues of $\Lambda(u)$ coalesce at u^c.

System (1.1) is of a very specific form, but is important for several reasons. First of all, it is related to a Fuschsian system by Laplace transform [3], [16], [20], [17]. Secondly, if A is skew-symmetric, it describes a deformed flat connection on a semisimple Frobenius manifold [13] [14] [15] (and [31] [32] [33] [27] [29]). Its monodromy data play the role of local moduli. If system (1.1) has coalescing eigenvalues, it gives the isomonodromic description of Frobenius manifolds *remaining semisimple at the locus of coalescent canonical coordinates* [9],[12] such as the quantum cohomology of Grassmannians (see [7] [9],[10],[17]). For $n = 3$, a special case of (1.1) gives an isomonodromic description of the general sixth Painlevé equation [28].

For future use, let $\mathbb{D}(w)$ denote a polydisc centered at w. Let

$$\Delta := \mathbb{D}(u^c) \cap \Big(\bigcup_{j \neq k}\{u_j - u_k = 0\}\Big),$$

be the *coalescence locus* in $\mathbb{D}(u^c)$ ($\Delta = \{u^c\}$ if $n = 1$). We consider the following domains for (z, u):

$$\mathcal{D}^0 := (\mathbb{C}\backslash\{z = 0\}) \times \mathbb{D}(u^0), \qquad \mathcal{D}^c := (\mathbb{C}\backslash\{z = 0\}) \times (\mathbb{D}(u^c))\backslash\Delta).$$

If X is a topological space, we denote by $\mathcal{R}(X)$ its universal covering.

If $\Lambda(u)$ has distinct eigenvalues in a sufficiently small $\mathbb{D}(u^0)$, and certain fundamental matrix solutions ("fundamental" means that the matrix solution is invertible) with a canonical structure satisfy a total differential system (see (2.11) below), then the monodromy data of (1.1) are constant over $\mathbb{D}(u^0)$ and can be computed fixing $u = u^0$. This is the well known result of the isomonodromic theory of Jimbo, Miwa and Ueno [23]. Notice that this theory relies on the fact that there exist fundamental matrix solutions of (1.1), in "canonical form", which are holomorphic on $\mathcal{R}(\mathcal{D}^0) = \mathcal{R}(\mathbb{C}_z\backslash\{0\}) \times \mathbb{D}(u^0)$ (see [22], [34], [35]). Moreover, the Stokes phenomenon can be described based on sectors which are independent of u.

In $\mathbb{D}(u^c)$ the situation is different. First of all, clearly $\mathcal{R}(\mathcal{D}^c) \neq \mathcal{R}(\mathbb{C}_z\backslash\{0\}) \times (\mathbb{D}(u^c))\backslash\Delta)$. Hence, a fundamental matrix solution $Y(z,u)$ holomorphic on $\mathcal{R}(\mathcal{D}^c)$, may have a singularity locus at Δ, namely the limit for $u \to u^* \in \Delta$ along any direction may diverge, and Δ is in general a *branching* locus [26]. Moreover, if $\mathring{Y}(z)$ is a fundamental matrix solution of

$$\frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y, \tag{1.2}$$

its monodromy data differ from those of a fundamental solution $Y(z,u)$ of (1.1), for $u \notin \Delta$ (see [1], [2], [8]). Other problems, with asymptotic behaviour and Stokes phenomenon will be discussed later.

In spite of all these problems, we would like to extend the isomonodromy deformation theory of [23] in order to include coalescence points, namely determine if and when locally constant monodromy data of (1.1) can be well defined on the whole $\mathbb{D}(u^c)$, and if they coincide with the data of (1.2) obtained by fixing $u = u^c$. Besides its theoretical importance, a positive answer also has computational consequences, because the monodromy data of (1.2) turn out to be *explicitly computable* in terms of special functions in important cases (on the contrary, in general they are not computable if we start from (1.1), being the computation a very transcendental problem). Computability relies on the fact that (1.2) is simpler than (1.1), thanks to the coalescence phenomenon itself (see below). The result so obtained for (1.2) would provide the data of (1.1) on the whole $\mathbb{D}(u^c)$. Moreover, if such an approach is justified, it can be applied in cases when the system has a geometrical meaning, like the applications to the quantum cohomology of Grassmannians in [9], [10], in which cases we *only know* (1.2) explicitly, but we do not know $A(u)$ away from a coalescent point u^c.

These issues were not addressed in the literature, until a first partial approach to the problem appeared in [17]. A complete theory has been developed in [8] and applied to quantum cohomology in [9], [10]. The conclusions of [8] have then been reconsidered in [30] in algebro-geometric terms, adding the important result of extending of the inverse point of view of Malgrange [24], namely the existence of integrable deformations of the solution of a Riemann-Hilbert problem at a coalescence point. In [21], the results of [8] have been reformulated in the language of Pfaffian systems.

2 The case of $\mathbb{D}(u^0)$

We start from the generic case, when the deformation parameters u (i.e. the eigenvalues of Λ) vary in $\mathbb{D}(u^0) \subset \mathbb{C}^n\backslash\bigcup_{j\neq k}\{u_j - u_k = 0\}$; no coalescence of eigenvalues occurs in this case. This is the assumption made by Jimbo, Miwa and Ueno in [23]. However, differently form [23], we will not assume that A is non-resonant.

There are two approaches in order to describe isomonodromy deformations of (1.1). One starts by proving the existence of fundamental solutions of (1.1) in "canonical" form and *holomorphic* on $\mathcal{R}(\mathcal{D}^0)$, and then proceeds by showing that they satisfy a Pfaffian system $dY = \omega Y$, with a specific ω, if and only if the deformation is isomonodromic. This is the approach in [23] in the generic case, and in [8] when coalescences occur (and also in [5],[6] for Fuchsian systems). The other approach, as in [21], starts by assuming that $dY = \omega Y$ exists, and then proceeds by deriving the structure of ω which guarantees that a set of essential monodromy data are constant.

Let us start from the second approach, and then we will come back to the first later. Suppose that (1.1) is the z-equation of a Frobenius-integrable Pfaffian system

$$dY = \omega Y, \qquad \omega = \omega_0(z,u)dz + \sum_{j=1}^{n} \omega_j(z,u)du_j \tag{2.1}$$

$$d\omega = \omega \wedge \omega \tag{2.2}$$

$$\omega_0(z,u) = \Lambda(u) + \frac{A(u)}{z}, \qquad \omega_j(z,u) \text{ holomorphic on } \mathbb{C} \times \mathbb{D}(u^0). \tag{2.3}$$

Here $dY = \partial_z Y\, dz + \sum_{j=1}^{n} \partial_{u_j} Y\, du_j$. With the assumption (2.3), there exists a **weakly isomonodromic** fundamental matrix solution $Y(z,u)$ of (1.1), which is holomorphic on $\mathcal{R}(\mathcal{D}^0) = \mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^0)$. If M is its monodromy matrix associated with a loop γ around $z = 0$

$$Y(\gamma z, u) = Y(z,u)M,$$

weakly isomonodromic means that

$$dM = 0.$$

Other monodromy data of system (1.1), such as the Stokes matrices introduced below, may depend on u. From the results of [36] and [4] , we obtain fundamental solutions in "canonical" form at the Fuchsian singularity $z = 0$, according to the following

Proposition 2.1 ([21]). *System (2.1) admits a weakly isomonodromic fundamental matrix solution $Y(z,u)$, holomorphic on $\mathcal{R}(\mathcal{D}^0)$, in Levelt form*

$$Y(z,u) = G(u)\left(I + \sum_{j=1}^{\infty} \Psi_j(u) z^j\right) z^D z^L, \tag{2.4}$$

where the series is convergent uniformly w.r.t. $u \in \mathbb{D}(u^0)$ with holomorphic matrices $G(u)$ and $\Psi_j(u)$. D is diagonal with integer entries (called valuations), $L = \log M/2\pi i$ with eigenvalues having real part lying in $[0,1)$, and $D + \lim_{z\to 0} z^D L z^{-D} =: J$ is a Jordan form of A, such that $G(u)^{-1}A(u)G(u) = J$. Moreover, A satisfies

$$dA = [\sum_{j=1}^{n} \omega_j(0,u)du_j, A].$$

$G(u)$ is a fundamental matrix solution of the Frobenius-integrable system

$$dG = (\sum_{j=1}^{n} \omega_j(0,u)du_j)G.$$

Finally,

$$dJ = dL = dD = 0.$$

In order to give fundamental solutions with "canonical" form at $z = \infty$, we introduce the Stokes rays of $\Lambda(u^0)$, which are rays in $\mathcal{R}(\mathbb{C}\backslash\{0\})$ defined by

$$\Re((u_j^0 - u_k^0)z) = 0, \quad \Im((u_j^0 - u_k^0)z) < 0, \quad 1 \leqslant j \neq k \leqslant n.$$

Similarly, we consider the Stokes rays of $\Lambda(u)$. Let us consider a direction $\arg z = \tau$ which does not coincide with any of the Stokes rays of $\Lambda(u^0)$, called *admissible at* u^0. If $\mathbb{D}(u^0)$ is sufficiently small, when u varies the Stokes rays of $\Lambda(u)$ rotate without crossing $\arg z = \tau \mod \pi$. For any $r \in \mathbb{Z}$, we consider sector $\mathcal{S}_r(u)$ in $\mathcal{R}(\mathbb{C}\backslash\{0\})$, containing the "half-plane" $\tau - (r-2)\pi < \arg z < \tau - (r-1)\pi$ and extending up to the nearest Stokes rays outside. Then, $\mathcal{S}_r(\mathbb{D}(u^0)) = \bigcap_{u \in \mathcal{S}(u^0)} \mathcal{S}_r(u)$ has central angular opening greater than π. Such an amplitude assures uniqueness of actual solutions with a given asymptotics, as in the following

Proposition 2.2 (Sibuya [35], [34], [22]; see also [23], [8], [21])**.** *Consider the* unique *formal solution*

$$Y_F(z,u) = F(z,u) z^{B(u)} \exp\{z\Lambda(u)\}$$

of (1.1), where $B(u) = \mathrm{diag}(A_{11}(u), ..., A_{nn}(u))$ *and* $F(z,u) = I + \sum_{k=1}^{\infty} F_k(u) z^{-k}$ *is a formal series, with holomorphic matrix coefficients* $F_k(u)$*. Then, there exist* unique *fundamental matrix solutions*

$$Y_r(z,u) = \widehat{Y}_r(z,u) z^{B(u)} \exp\{z\Lambda(u)\} \tag{2.5}$$

of (1.1), holomorphic on $\mathcal{R}(\mathcal{D}^0)$*, such that uniformly in* $u \in \mathbb{D}(u^0)$ *the following asymptotic behaviours hold*

$$\widehat{Y}_r(z,u) \sim F(z,u) \quad \text{for } z \to \infty \text{ in } \mathcal{S}_r(\mathbb{D}(u^0)). \tag{2.6}$$

Notice that

$$(F_1)_{ij} = \frac{A_{ij}}{u_j - u_i}, \ i \neq j, \qquad (F_1)_{ii} = -\sum_{j \neq i} A_{ij} F_{ji}, \tag{2.7}$$

$$(F_k)_{ij} = \frac{1}{u_j - u_i} \left\{ \Big(A_{ii} - A_{jj} + k - 1 \Big) (F_{k-1})_{ij} + \sum_{p \neq i} A_{ip} (F_{k-1})_{pj} \right\}, \quad i \neq j; \tag{2.8}$$

$$k(F_k)_{ii} = -\sum_{j \neq i} A_{ij} (F_k)_{ji}. \tag{2.9}$$

Holomorphic Stokes matrices $S_r(u)$ are defined by

$$Y_{r+1}(z,u) = Y_r(z,u) S_r(u).$$

The fundamental matrix solutions $Y_r(z,u)$ of the differential system (1.1) are not solutions of the Pfaffian system (2.1) in general. There exists a *central connection matrix* $C_r(u)$ such that

$$Y_r(z,u) = Y(z,u) C_r(u).$$

Substituting $Y(z,u) = Y_r(z,u) C_r(u)^{-1}$ into (2.1), and using (2.2) and the structure (2.5), we obtain the following

Proposition 2.3. *([21]) For $1 \leqslant j \leqslant n$, let $D_j(u)$ be an arbitrary holomorphic diagonal matrix, Then*

$$\omega_j(z,u) = zE_j + [F_1(u), E_j] + D_j(u),$$
$$dB = 0,$$
$$d(C_r^{-1}) = (\sum D_j(u) du_j) C_r^{-1}.$$

Here $(E_j)_{kl} = \delta_{jk}\delta_{jl}$.

Definition 2.1. *The system (1.1) is* **strongly isomonodromic** *if $dC_r = 0$, or equivalently if all the $Y_r(z,u)$ satisfy the Pfaffian system (2.1). In this case, all the matrices $D_j = 0$ and*

$$\omega_j(z,u) = zE_j + [F_1(u), E_j]. \tag{2.10}$$

We recall that the Stokes matrices S_0, S_1, together with B, C_0 and L are sufficient to calculate all the other S_r and C_r [8], and to compute the monodromy at $z = 0$ of both Y (which is $M = e^{2\pi i L}$) and Y_0 (which is $M_0 := e^{2\pi i B}(S_0S_1)^{-1} = C_1^{-1}MC_0$). Hence, it makes sense to give the following

Definition 2.2. *We call* **essential monodromy data** *the data S_0, S_1, B, C_0, L, D.*

Notice that the eigenvalues of $A(u)$ are the eigenvalues of $D + L$.

Proposition 2.4 ([21])**.** *The system (1.1) is strongly isomonodromic if and only if*

$$dS_0 = dS_1 = dC_0.$$

Notice that also $dL = dB = dD = 0$, by Propositions 2.1 and 2.3.

Above, we have started from a Pfaffian system. As mentioned at the beginning of the section, the other approach starts from fundamental matrix solutions of (1.1) in canonical form, namely the $Y_r(z,u)$ of Proposition 2.2, and a fundamental solution in Levelt form $Y(z,u)$, with structure as in (2.4). Notice that $Y(z,u)$ and the $Y_r(z,u)$ are not assumed to satisfy a Pfaffian system, which still has to be constructed. If some eigenvalues of $A(u)$ differ by integers (some eigenvalues coincide if the integer is zero, or we have a resonance in case of non-zero integer), then $Y(z,u)$ is not necessarily holomorphic on $\mathcal{R}(\mathcal{D}^0)$. A Levelt fundamental solution $Y(z,u)$ can be chosen holomorphic on $\mathcal{R}(\mathcal{D}^0)$ if we make additional assumptions (see [8]): if A has multiple eigenvalues, we must assume that $A(u)$ is *holomorphically similar* to a Jordan form $J(u)$, through a holomorphic invertible $G(u)$ such that $G(u)^{-1}A(u)G(u) = J(u)$. If moreover a resonance occurs, we assume that *it is independent of* u (namely, given eigenvalues $\mu_i(u)$ and $\mu_j(u)$ of A, if $\mu_i(u^0) - \mu_j(u^0) = m_{ij} \in \mathbb{Z}\backslash\{0\}$, then $\mu_i(u) - \mu_j(u) = m_{ij}$ all over $\mathbb{D}(u^0)$, and moreover, representing $\mu_j = \rho_j(u) + d_j$, $0 \leqslant \Re\rho_j < 1$, $d_j \in \mathbb{Z}$, then d_j is constant. Notice that $D = \text{diag}(d_1, ..., d_n)$ in (2.4)). Then, we can show that $Y(z,u)$ and the $Y_r(z,u)$ satisfy a Pfaffian system if and only if the deformation is strongly isomonodromic, according to the following

Proposition 2.5 ([8])**.** *With the above assumptions, system (1.1) is strongly isomonodromic if and only if there exists an integrable Pfaffian system satisfied by the $Y_r(z,u)$ and $Y(z,u)$, of the form*

$$dY = \omega(z,u)Y, \qquad \omega(z,u) = \left(\Lambda(u) + \frac{A(u)}{z}\right) dz + \sum_{j=1}^{n} (zE_j + [F_1(u), E_j]) du_j. \tag{2.11}$$

If the deformation is strongly isomonodromic, then $G(u)$ in (2.4) is a holomorphic fundamental solution of

$$dG = (\sum_{j=1}^{n}[F_1(u), E_j]du_j)G, \tag{2.12}$$

and $dA = [\sum_{j=1}^{n}[F_1(u), E_j]du_j, A]$.

Notice that if the deformation is strongly isomonodromic, then the eigenvalues of $A(u)$ are constant, so if a resonance occurs, it is independent of u. Moreover, if $A(u)$ is holomorphic on $\mathbb{D}(u^0)$, so that also $[F_1(u), E_j]$ is, then $G(u)$ exists holomorphic satisfying (2.12), such that $G^{-1}AG = J$, so that $A(u)$ is holomorphically similar to J.

In conclusion, in this section we have reviewed the "standard" isomonodromic deformation theory of Jimbo-Miwa-Ueno, with the addition that we have included a possibly resonant $A(u)$. This has been a preparation for the discussion of the non-generic case occurring when some eigenvalues of Λ coalesce.

3 The case of $\mathbb{D}(u^c)$ – Coalescent Eigenvalues

In presence of coalescences, the situation is more problematic. In $\mathcal{R}(\mathbb{C}\backslash\{0\})$, we introduce the Stokes rays of $\Lambda(u^c)$

$$\Re((u_i^c - u_k^c)z) = 0, \quad \Im((u_i^c - u_k^c)z) < 0, \quad u_i \neq u_k,$$

and an admissible direction $\arg z = \tilde{\tau}$ at u^c (namely, a direction not assumed by the Stokes rays above). Analogously, at any $u \in \mathbb{D}(u^c)$ we define Stokes rays $\Re((u_i - u_j)z) = 0$, $\Im((u_i - u_j)z) < 0$ of $\Lambda(u)$.

This time, if u varies in $\mathbb{D}(u^c)$, no matter how the polydisc is small, some Stokes rays may cross the admissible directions $\arg z = \tilde{\tau} \mod \pi$. Indeed, let i, j, k be labels such that $u_i^c = u_j^c \neq u_k^c$. Then, as u moves away from u^c, a Stokes ray of $\Lambda(u^c)$ characterized by $\Re((u_i^c - u_k^c)z) = 0$ generates three rays. Two of them are $\Re((u_i - u_k)z) = 0$ and $\Re((u_j - u_k)z) = 0$. If $\mathbb{D}(u^c)$ is sufficiently small, they do not cross $\arg z = \tilde{\tau} \mod \pi$ as u varies. The third ray is $\Re((u_i - u_j)z) = 0$. If u varies in such a way that u makes a complete loop $(u_i - u_j) \mapsto (u_i - u_j)e^{2\pi i}$ around the locus $\{u \in \mathbb{D}(u^c) \mid u_i - u_j = 0\} \subset \Delta$, then the ray crosses $\arg z = \tilde{\tau} \mod \pi$ and $\arg z = \tilde{\tau} - \pi \mod \pi$. This implies that Proposition 2.2 does not hold.

The choice of $\tilde{\tau}$ induces a **cell decomposition** of $\mathbb{D}(u^c)$. Let $X(\tilde{\tau})$ be the locus of points u in $\mathbb{D}(u^c)$ such that there exists a Stokes ray of $\Lambda(u)$ (so infinitely many in $\mathcal{R}(\mathbb{C}\backslash\{0\})$) with direction $\tilde{\tau}$.

Proposition 3.1 ([8]). *Each connected component of $\mathbb{D}(u^c)\backslash(\Delta \cup X(\tilde{\tau}))$ is simply connected and homeomorphic to a ball, namely it is a topological cell, called* $\boldsymbol{\tilde{\tau}}$**-cell**.

The cells above have the important meaning that, if u varies in a cell, then no Stokes rays cross the admissible directions $\arg z = \tilde{\tau} \mod \pi$.

We pick up a point u^0 in a $\tilde{\tau}$-cell, and we consider a polydisc $\mathbb{D}(u^0)$ *all contained* in the cell. Assume that $A(u)$ is holomorphic on $\mathbb{D}(u^c)$, so it is on $\mathbb{D}(u^0)$. As in Section 2, for $u \in \mathbb{D}(u^0)$ we can define $Y_r(z, u)$, $Y(z, u)$ and strong isomonodromic deformations on $\mathbb{D}(u^0)$, if sufficiently small, with the associated Pfaffian system $dY = \omega Y$, where ω_j is as in (2.10). Notice that

$$\omega_k(z, u) = zE_k + \left(\frac{A_{ij}(u)(\delta_{ik} - \delta_{jk})}{u_i - u_j}\right)_{i,j=1}^{n}, \tag{3.1}$$

is holomorphic not only on $\mathbb{D}(u^0)$, but also on $\mathbb{D}(u^c)\backslash\Delta$. Thus, $Y_r(z,u)$, $Y(z,u)$ extends analytically on $\mathcal{R}(\mathcal{D}^c)$, and Δ may become a branching locus for them (multi-valuedness w.r.t. loops in $\mathbb{D}(u^c)\backslash\Delta$). From (2.7)-(2.9), we see that the $F_k(u)$ have poles at Δ.

A non-trivial result is the following

Proposition 3.2 ([8]). *The asymptotic behaviours (2.6) hold uniformly not only on $\mathbb{D}(u^0)$, but also on every compact subset K of the $\tilde{\tau}$-cell containing u^0, for $z \to \infty$ in $\mathcal{S}_r(K) := \bigcap_{u\in K} \mathcal{S}_r(u)$ (the sector having central opening angle greater than π).*

Nevertheless, the asymptotics fails if u moves outside the cell, because some Stokes rays cross the admissible directions $\tilde{\tau}$ mod(π).

Fortunately, as proved in [8], the isomonodromic deformation theory can be extended to the whole $\mathbb{D}(u^c)$ if and only if the entries $A_{ij}(u)$ satisfy the vanishing conditions (3.2) below. The two following propositions are the first step of the extension.

Proposition 3.3 ([8],[21]). *Let $A(u)$ be holomorphic on $\mathbb{D}(u^c)$. The form ω as in (3.1) is holomorphic on $\mathbb{D}(u^c)$ if and only if*

$$A_{ij}(u) = \mathcal{O}(u_i - u_j) \to 0 \quad \text{whenever } (u_i - u_j) \to 0 \text{ for } u \text{ approaching } \Delta. \tag{3.2}$$

In this case, $Y(z,u)$ and the $Y_r(z,u)$, $r \in \mathbb{Z}$, have analytic continuation on $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^c)$, so they are holomorphic of $u \in \mathbb{D}(u^c)$, and Δ is neither a singularity locus nor a branching locus.

From (2.12), it follows that $A(u)$ is holomorphically similar to a Jordan form (through $G(u)$) if and only if (3.2) holds.

Notice that the vanishing conditions (3.2) imply that $A_{ij}(u^c) = 0$ if $u_i^c = u_j^c$.

Proposition 3.4 ([8],[21]). *If (3.2) holds, all the matrix coefficients $F_k(u)$ are holomorphic on $\mathbb{D}(u^c)$.*

The second step is the following fundamental and non-trivial result of [8].

Theorem 3.1 ([8]). *Consider a system (1.1), not necessarily isomonodromic, with $A(u)$ holomorphic on a sufficiently small $\mathbb{D}(u^c)$, or the more general system*

$$\frac{dY}{dz} = \left(\Lambda(u) + \sum_{j=1}^{\infty} \frac{A_j(u)}{z}\right) Y,$$

where the series is uniformly convergent on $\mathbb{D}(u^c)$, with holomorphic coefficients. Assume that the coefficients $F_k(u)$ of the formal solution $Y_F(z,u) = F(z,u)z^{B(u)}e^{\Lambda(u)z}$, and that the fundamental solutions $Y_r(z,u) = \widehat{Y}_r(z,u)z^{B(u)}e^{\Lambda(u)z}$, initially defined for $u \in \mathbb{D}(u^0)$, have analytic continuation in u on the whole $\mathbb{D}(u^c)$ (this happens for the isomonodromic system (1.1) if and only if the vanishing conditions (3.2) hold). Then

1) *For any $r \in \mathbb{Z}$, the asymptotics $\widehat{Y}_r(z,u) \sim F(z,u)$ holds for $z \to \infty$ is a sector $\widehat{\mathcal{S}}_r$ independent of u, of central opening angle greater than π, unifromly in every compact subset of $\mathbb{D}(u^c)$.*

2) *The entries (i,j) of the Stokes matrices corresponding to $u_i^c = u_j^c$ vanish:*

$$(S_r(u))_{ij} = (S_r(u))_{ji} = 0 \qquad \forall\ 1 \leqslant i \neq j \leqslant n \text{ such that } u_i^c = u_j^c.$$

3) *The limits*

$$\mathring{Y}_F(z) = \lim_{u\to u^c} Y_F(z,u) = Y_F(z,u^c),$$

$$\mathring{Y}_r(z) = \lim_{u\to u^c} Y_r(z,u) = Y_r(z,u^c)$$

are respectively formal and actual fundamental solutions of (1.2), or more generally of

$$\frac{dY}{dz} = \left(\Lambda(u^c) + \sum_{j=1}^{\infty} \frac{A_j(u^c)}{z}\right) Y, \tag{3.3}$$

satisfying $\mathring{Y}_r(z) \sim \mathring{Y}_F(z)$ for $z \to \infty$ in $\widehat{\mathcal{S}}_r$ (actually, in a bigger sector). The $\mathring{Y}_r$ are related by Stokes matrices

$$\mathring{S}_r = \lim_{u\to u^c} S_r(u) = S_r(u^c).$$

4) *In general, (1.2) or (3.3) does not have a unique formal solution. However, if $A_{ii} - A_{jj} \notin \mathbb{Z}\backslash\{0\}$, then (3.3) has a unique formal solution, which coincides with $Y_F(z,u^c)$ above.*

As a corollary of Theorem 3.1 and of Propositions 3.3 and 3.4 (and some more work), in the isomonodromic case we obtain the following

Theorem 3.2 ([8])**.** *Let (1.1) be (strongly) isomonodromic with $A(u)$ holomorphic on $\mathbb{D}(u^c)$. Then, the essential monodromy data S_0, S_1, $B = \mathrm{diag}(A(u^c))$, C_0, L, D, initially defined on $\mathbb{D}(u^0)$ contained in a $\widetilde{\tau}$-cell of $\mathbb{D}(u^c)$, are well defined and constant on the whole $\mathbb{D}(u^c)$. They satisfy*

$$S_0 = \mathring{S}_0, \quad S_1 = \mathring{S}_1, \quad L = \mathring{L}, \quad C_0 = \mathring{C}_0, \quad D = \mathring{D},$$

where

- *$\mathring{S}_0$, $\mathring{S}_1$ are the Stokes matrices of fundamental solutions $\mathring{Y}_0(z)$, $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$ of (1.2) having asymptotic behaviour $\mathring{Y}_F(z) = Y_F(z,u^c)$;*
- *$\mathring{L}$, $\mathring{D}$ are the exponents of a fundamental solution of (1.2) in Levelt form $\mathring{Y}(z) = \mathring{G}\left(I + \sum_{j=1}^{\infty} \mathring{\Psi}_j z^j\right) z^{\mathring{D}} z^{\mathring{L}}$, which one can prove to always correspond to the value at $u = u^c$ of a Level form (2.4) of (1.1) (see [8], [21]);*
- *$\mathring{C}_0$ connects $\mathring{Y}_0(z) = \mathring{Y}(z)\mathring{C}_0$.*

Notice that

$$(S_r)_{ij} = (S_r)_{ji} = 0 \quad \forall\ 1 \leqslant i \neq j \leqslant n \text{ such that } u_i^c = u_j^c.$$

Corollary 3.1 ([8])**.** *If $A_{ii} - A_{jj} \notin \mathbb{Z}\backslash\{0\}$, then necessarily there exists only the formal solution $\mathring{Y}_F(z) = Y_F(z,u^c)$ of (1.2), which is unique. Hence, Theorem 3.2 implies that, in order to obtain the essential monodromy data of (1.1), it sufficies to compute the essential monodromy data of (1.2).*

Notice that (1.2) is simper than (1.1), because $A_{ij}(u^c) = 0$ if $u_i^c = u_j^c$. In important cases, this simplification allows to actually compute $\mathring{Y}(z)$ and the $\mathring{Y}_r(z)$ in terms of special functions, and the corresponding monodromy data.

In conclusion, we have reached our goal of extending the isomonodromy deformation theory to the case of coalescing eigenvalues, when $A(u)$ is holomorphically similar to a Jordan form, and the goal of computing the constant monodromy data of (1.1) using only (1.2). This result is particularly useful in some geometric applications (see next section), when A is only known at $u = u^c$.

It is worth mentioning that there is a weak converse of Theorem 3.2.

Theorem 3.3 ([8])**.** *Let the deformation be strongly isomonodromic on a polydisc $\mathbb{D}(u^0)$ contained in a $\widetilde{\tau}$-cell of $\mathbb{D}(u^c)$. If $(S_0)_{ij} = (S_0)_{ji} = 0$ and $(S_1)_{ij} = (S_1)_{ji} = 0$ $\forall\ 1 \leqslant i \neq j \leqslant n$ such that $u_i^c = u_j^c$, then the canonical solutions $Y_r(z,u)$ and $A(u)$ extend as meromorphic functions on $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times (\mathbb{D}(u^c)\backslash\Delta)$, namely as meromorphic functions of $u \in \mathbb{D}(u^c)\backslash\Delta$, so that Δ is not a branching locus. For any u which is not a pole, the asymptotics $Y_r(z,u) \sim Y_F(z,u)$ holds as $z \to \infty$ in a suitable sector containing $\widehat{\mathcal{S}}_r$.*

4 Applications

If $n = 3$ and $A(u)$ has a specific form, then we can use $x = (u_3-u_1)/(u_2-u_1)$ as a deformation parameter, and parametrize $A = A(x, y(x), dy(x)/dx, k(x))$ using two functions y and k, and the derivative dy/dx. Then, one can prove that $d\omega = \omega \wedge \omega$ is equivalent to the fact that $y(x)$ satisfies the sixth Painlevé equation [28] (and $k(x)$ is given by quadratures). The condition (3.2) corresponds to Painlevé functions $y(x)$ holomorphic at a fixed singularity $x = 0$, 1 or ∞. Theorem 3.2 and Corollary 3.1 allow to compute the monodormy data associated with such functions (see [8] for an example).

If $A(u)$ is skew-symmetric, then our $dY = \omega Y$ locally describes the flatness condition of a certain z-deformed connection on a semisimple Frobenius manifold [13],[15]. The eigenvalues $u = (u_1, ..., u_n)$ are local coordinates, called *canonical*. The locally constant monodromy data of the system (1.1) associated with a local "chart" of the manifold, called *chamber* [9], play the role of local moduli, because a solution (2.4) of (1.1) can be obtained by solving a Riemann-Hilbert problem staring from the monodromy data. From a Levelt form (2.4), the local structure of the manifold can be explicitly constructed (formulae are in [15], with a misprint corrected in [19], where they are applied to some relevant examples of 3-dimensional Frobenius manifolds). If the monodromy data associated with a chamber are known, a simple algebraic operation on them, given by an action of the braid group, transforms them into the data associated with another chambers. Then, by the Riemann-Hilbert approach mentioned before, starting from the transformed data, the local Frobenius structure is obtained for other chambers. This allows to construct, in principle, the *analytic continuation* of a Frobenius structure, starting from the local one.

The computation of locally constant monodormy data of system (1.1) associated with a chamber, from which we want to start the procedure of analytic continuation explained above, is a highly transcendental problem, which rarely can be solved analytically. This is the reason why B. Dubrovin conjectured (at the 1998 ICM in Berlin [14]) that for the Frobenius manifold given by the quantum cohomology of projective Fano varieties X, the monodromy data can be computed in algebro-geometric terms, by suitable algebraic objects in the derived category of coherent sheaves on X. If the conjecture is verified, it would provide an algebraic tool to compute monodromy data as local moduli, and then proceed with the analytic continuation of the (globally unknown) structure of the quantum cohomology of X. As far as Stokes matrices are concerned, the conjecture was prove in [18] if $X = \mathbb{P}^n$ is a complex projective space. No coalescence phenomenon occurs in this case. On the other hand, already for the next simple case when $X = \mathrm{Gr}(k, n)$ is a Grassmannian of k-dimensinal vector spaces in $\mathbb{C}^n$, a coalescence phenomenon occurs for almost all (k, n) (precisely, for $P_1(n) \leqslant k \leqslant n - P_1(n)$, where $P_1(n)$ =minimum prime number dividing n; see [7]). Moreover, the Frobenius structure is explicitly known *only at a coalescence point* u^c. Hence, Theorem 3.2 and Corollary 3.1 are essential results to justify any computation of monodromy data. In [9] and [10], the conjecture has been clarified, refined and proved for the quantum cohomology of $\mathrm{Gr}(k, n)$. In particular, in [9], Theorem 3.2 and Corollary 3.1 have been used for the explicit analytic calculation of monodromy data in the example of $\mathrm{Gr}(2, 4)$.

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