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Non-collapsed spaces with Ricci curvature bounded from below

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Abstract

We propose a definition of *non-collapsed* space with Ricci curvature bounded from below and we prove the versions of Colding’s volume convergence theorem and of Cheeger-Colding dimension gap estimate for RCD spaces.

In particular this establishes the stability of non-collapsed spaces under non-collapsed Gromov-Hausdorff convergence.

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1 Introduction

Lott-Villani in [43] and Sturm in [49, 50] introduced a synthetic notion of lower Ricci curvature bounds for metric measure spaces: their approach is based on suitable convexity properties for entropy-like functionals over the space of probability measures equipped with the quadratic Kantorovich distance W_2 . The classes of spaces that they introduced are called $\text{CD}(K, N)$, standing for lower Curvature bound by $K \in \mathbb{R}$ and upper Dimension bound by $N \in [1, \infty]$ (in [43] only the cases $K = 0$ and $N = \infty$ have been considered).

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Since then the study of these classes of spaces has been a very flourishing research area, see for instance the surveys [53, 51] and references therein. Among the various fine tunings of Lott-Sturm-Villani's proposal, we mention the definition of $\text{CD}^*(K, N)$ spaces proposed by Bacher-Sturm in [11]: under minor technical conditions, this is locally equivalent to the $\text{CD}(K, N)$, has better local-to-global properties but a priori leads to slightly suboptimal constants in various geometric and functional inequalities (but see (1.2) below).

Since the very beginning, one of the main research lines has been, and still is, that of understanding the geometric properties of such spaces. Here fundamental ideas come from the theory of Ricci-limit spaces developed in the nineties by Cheeger and Colding [16, 17, 18, 19, 22]: one would like at least to replicate all their results in the synthetic framework, and then hopefully to obtain, thanks to the new point of view, new insights about both smooth and non-smooth objects having Ricci curvature bounded from below. In this direction it has been soon realized that the classes of $\text{CD}/\text{CD}^*(K, N)$ spaces are not really suitable for the development of this program: the problem is that Finlser structures are included (see the last theorem in [52]) and for these Cheeger-Colding's results are not valid. For instance, the Cheeger-Colding-Gromoll splitting theorem fails in finite dimensional Banach spaces.

Motivated by this problem the second author proposed in [31] to reinforce the Lott-Sturm-Villani condition with the functional-analytic notion of infinitesimal Hilbertianity:

$$(X, d, \mathfrak{m}) \text{ is infinitesimal Hilbertian provided } W^{1,2}(X, d, \mathfrak{m}) \text{ is an Hilbert space.} \quad (1.1)$$

This definition is the result of a research program devoted to the understanding of the heat flow [27, 32, 3] on $\text{CD}(K, N)$ spaces, and in particular of the introduction of the class of $\text{RCD}(K, \infty)$ spaces - R standing for Riemannian - in a collaboration with Ambrosio and Savaré [4].

In (1.1), $W^{1,2}(X, d, \mathfrak{m})$ is the Sobolev space of real valued functions on X as introduced by Cheeger in [15] (see also the alternative, but equivalent, descriptions provided in [48] and [3]). It is a priori non-trivial, but nevertheless true, that infinitesimal Hilbertianity is stable under mGH -convergence when coupled with a $\text{CD}(K, N)$ condition. Moreover, as proved by the second author in [28] (see also [30]), the splitting theorem holds in the class of infinitesimally Hilbertian $\text{CD}(0, N)$ spaces.

In a different direction, in another collaboration [5] of the second author with Ambrosio and Savaré it has been introduced the class of $\text{BE}(K, N)$ spaces: these are spaces in which, in a suitable sense, the Bochner inequality with parameters K, N holds (BE stands for Bakry-Émery). The key points of [5] are the proof that the class of $\text{BE}(K, N)$ spaces is stable under mGH -convergence (and thus provides another reasonable synthetic notion of spaces having a curvature-dimension bound) and that for $N = \infty$ it coincides with that of $\text{RCD}(K, \infty)$ spaces.

This circle of ideas has been closed in [25] (and later in [9]) where it has been proved that

$$\text{BE}(K, N) = \text{CD}^*(K, N) + \text{infinitesimal Hilbertianity}.$$

More recently, Cavalletti-Milman in [14] proved in high generality, and in particular without relying on infinitesimal Hilbertianity, that it holds

$$\text{CD}(K, N) = \text{CD}^*(K, N) \quad \left(\begin{array}{l} \text{under some kind of non-branching assumption} \\ \text{which always holds in } \text{RCD}(K, \infty) \text{ spaces} \end{array} \right) \quad (1.2)$$

The results in [14] are stated for spaces with finite reference measure but the kind of arguments used seems to indicate that the same also holds without this restriction. For this reason in

this manuscript we shall work with $\text{RCD}(K, N) := \text{CD}(K, N) + \text{Inf.Hilb.}$ spaces, rather than with $\text{RCD}^*(K, N) := \text{CD}^*(K, N) + \text{Inf.Hilb.}$ ones which have been recently more popular. In any case, all our arguments are local in nature and since the local versions of $\text{CD}(K, N)$ and $\text{CD}^*(K, N)$ are known to be equivalent from the very first paper [11] where CD^* has been introduced, our results are independent from [14].

We now turn to the description of the content of this manuscript.

Thanks to the celebrated volume convergence result by Colding [22], and to its generalization to Ricci-limit spaces by Cheeger-Colding [17], we know that for a pointed-Gromov-Hausdorff-converging sequence of pointed Riemannian manifolds (M_n, p_n) with the same dimension and Ricci curvature uniformly bounded from below, the volume of the unit ball around p_n either stays away from 0 (i.e. $\inf_n \text{Vol}_n(B_1^{M_n}(p_n)) > 0$) or it converges to 0. Limit spaces are called *non-collapsed* or *collapsed* according to whether they are obtained as limits of sequences of the former or latter kind respectively.

As it turned out from the analysis done in [16, 17, 18, 19], non-collapsed spaces are more regular than collapsed ones and it is therefore natural to look for a synthetic counterpart of this class of spaces. To do so we should look for an *intrinsic* characterization of non-collapsed Ricci-limit spaces, i.e. for one which does not rely on the existence of a converging sequence having suitable properties. A work in this direction has also been done by Kitabeppu in [42] (see also Remark 1.14).

Let us observe that the aforementioned volume convergence result grants, as noticed in [17], that: a pGH-limit space (X, d) of a sequence of N -dimensional manifolds with Ricci curvature uniformly bounded from below is non-collapsed if and only if

$$\text{the volume measures weakly converge to the measure } \mathcal{H}^N \text{ on } X \text{ and } \mathcal{H}^N(X) > 0,$$

where here and in the following \mathcal{H}^N is the N -dimensional Hausdorff measure.

Since for a $\text{CD}(K, N)$ space (X, d, \mathbf{m}) the requirement $\mathbf{m}(X) > 0$ is part of the definition, the above motivates the following:

Definition 1.1 (Non-collapsed RCD spaces). *Let $K \in \mathbb{R}$ and $N \geq 1$. We say that (X, d, \mathbf{m}) is a non-collapsed $\text{RCD}(K, N)$ space, $\text{ncRCD}(K, N)$ space in short, provided it is an $\text{RCD}(K, N)$ space and $\mathbf{m} = \mathcal{H}^N$.*

From the known structural properties of $\text{RCD}(K, N)$ spaces it is not hard to show that if (X, d, \mathbf{m}) is a $\text{ncRCD}(K, N)$, then N must be an integer. This follows for instance from the rectifiability results proved in [44], [41], [35] (see Theorem 2.13). Alternatively, this can be proved by blow-up arguments, see Theorem 1.12 and in particular the implication (iii) \Rightarrow (iv).

Imitating the arguments in [17] we shall prove that $\text{ncRCD}(K, N)$ spaces are stable under Gromov-Hausdorff (thus a priori not necessarily *measured*-GH) convergence in the sense made precise by the following theorem:

Theorem 1.2 (Non-collapsed and collapsed convergence). *Let $(X_n, d_n, \mathcal{H}^N, x_n)$ be a sequence of pointed $\text{ncRCD}(K, N)$ spaces. Assume that (X_n, d_n, x_n) converges to (X_∞, d_∞, x) in the pointed-Gromov-Hausdorff topology. Then precisely one of the following happens:*

- i) $\overline{\lim}_{n \rightarrow \infty} \mathcal{H}^N(B_1(x_n)) > 0$. In this case the $\overline{\lim}$ is actually a limit and $(X_n, d_n, \mathcal{H}^N, x_n)$ converges in the pointed-measured-Gromov-Hausdorff topology to $(X_\infty, d_\infty, \mathcal{H}^N, x)$. In particular $(X_\infty, d_\infty, \mathcal{H}^N)$ is a $\text{ncRCD}(K, N)$ space.*

ii) $\lim_{n \rightarrow \infty} \mathcal{H}^N(B_1(x_n)) = 0$. In this case $\dim_{\mathcal{H}}(X_\infty) \leq N - 1$.

Here and in what follows $\dim_{\mathcal{H}}(X)$ is the Hausdorff dimension of the metric space X . Notice that in particular

non-collapsed limits of Riemannian manifolds in the sense of Cheeger-Colding are non-collapsed spaces in our sense,

explaining our choice of terminology.

Theorem 1.2 is strictly related to the following two results. The first generalizes the already mentioned volume convergence theorem to the RCD setting. Notice that there is no non-collapsing assumption.

Theorem 1.3 (Continuity of \mathcal{H}^N). *For $K \in \mathbb{R}$, $N \in [1, \infty)$ and $R \geq 0$ let $\mathbb{B}_{K,N,R}$ be the collection of all (equivalence classes up to isometry of) closed balls of radius R in $\text{RCD}(K, N)$ spaces equipped with the Gromov-Hausdorff distance.*

Then the map $\mathbb{B}_{K,N,R} \ni Z \mapsto \mathcal{H}^N(Z)$ is real valued and continuous.

Such theorem is true even for open balls, see equation (2.20). Notice also that Gromov precompactness theorem for RCD spaces and the stability of the RCD condition grant that $\mathbb{B}_{K,N,R}$ is compact w.r.t. the Gromov-Hausdorff topology (see also the proof of Theorem 1.3 given at the end of Section 3.1).

The second result, analogous to [17, Theorem 3.1], concerns the Hausdorff dimension of an $\text{RCD}(K, N)$ space; again there is not an assumption about non-collapsing, but on the other hand N is assumed to be integer.

Theorem 1.4 (Dimension gap). *Let $K \in \mathbb{R}$, $N \in \mathbb{N}$, $N \geq 1$, and X an $\text{RCD}(K, N)$ space. Then either $\dim_{\mathcal{H}}(X) = N$ or $\dim_{\mathcal{H}}(X) \leq N - 1$.*

Since \mathbb{R} is $\text{RCD}(0, 1 + \varepsilon)$, we see that the assumption $N \in \mathbb{N}$ is necessary in the above. For non-integer N 's this last result easily implies the following:

Corollary 1.5. *Let $K, N \in \mathbb{R}$, $N \geq 1$, and X an $\text{RCD}(K, N)$ space. Then $\dim_{\mathcal{H}}(X) \leq [N]$, where $[\cdot]$ denotes the integer part.*

Notice that this is sharp because for every $N \in (1, 2)$ the space $([0, \pi], d_E, \sin^{N-1}(t) dt)$ is an $\text{RCD}(N - 1, N)$ space whose Hausdorff dimension is 1.

As a quite direct consequence of Theorem 1.2 and its proof we also obtain the following volume (almost) rigidity result:

Theorem 1.6 (Volume rigidity). *For every $\varepsilon > 0$ and $N \in \mathbb{N}$, $N \geq 1$ there is $\delta = \delta(\varepsilon, N)$ such that the following holds. Let (X, d, \mathcal{H}^N) be a $\text{ncRCD}(-\delta, N)$ space and $\bar{x} \in X$ such that*

$$\mathcal{H}^N(B_1^X(\bar{x})) \geq \mathcal{H}^N(B_1^{\mathbb{R}^N}(0))(1 - \delta).$$

Then

$$d_{\text{GH}}(\overline{B}_{1/2}^X(\bar{x}), \overline{B}_{1/2}^{\mathbb{R}^N}(0)) \leq \varepsilon.$$

The example of a unit ball in a cylinder shows that we cannot replace 1/2 with 1 in the conclusion, see also the discussion in [24]. A simple consequence of the Bishop-Gromov inequality combined with Corollary 2.14 and the above Theorem is the following:

Corollary 1.7. *Let (X, d, \mathcal{H}^N) be a $\text{ncRCD}(0, N)$ space, then for all $x \in X$ and $r > 0$,*

$$\mathcal{H}^N(B_r^X(x)) \leq \omega_N r^N. \quad (1.3)$$

Moreover, if there exists $\bar{x} \in X$ and $\bar{r} > 0$ achieving equality in (1.3), then $B_{\bar{r}/2}^X(\bar{x})$ is isometric to $B_{\bar{r}/2}^{\mathbb{R}^N}(0)$. As a consequence, a point $x \in X$ is regular (i.e. all tangent cones are isometric to \mathbb{R}^N) if and only if

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^N(B_r^X(x))}{\omega_N r^N} = 1.$$

We now pass to the description of the main properties of non-collapsed spaces. A first result is about the stratification of their singular set: denote by $S_k(X)$ the set of points $x \in X$ such that no tangent space splits off a factor \mathbb{R}^{k+1} (see (3.48) for the precise definition). In the same spirit of classical stratification results in geometric measure theory first established in [1] and axiomatized in [54] we have the following result, compare with [17, Theorem 4.7].

Theorem 1.8 (Stratification). *Let $K \in \mathbb{R}$, $N \in \mathbb{N}$, $N \geq 1$ and let X be a $\text{ncRCD}(K, N)$ space.*

Then $\dim_{\mathcal{H}}(\mathcal{S}_k(X)) \leq k$ for every $k \in \mathbb{N}$.

Beside these, all the other properties of ncRCD spaces that we are able to prove hold in the a priori larger class of *weakly* non-collapsed spaces, which we now introduce.

For $K \in \mathbb{R}$, $N \in [1, \infty)$ and $r \geq 0$ let us consider the volume of the ball of radius r in the reference ‘space form’ defined by

$$v_{K,N}(r) := \omega_N \int_0^r |s_{K,N}(t)|^{N-1} dt,$$

where $\omega_N := \frac{\pi^{N/2}}{\int_0^\infty t^{N/2} e^{-t} dt}$ coincides for integer N with the volume of the unit ball in \mathbb{R}^N and

$$s_{K,N}(r) := \begin{cases} \sqrt{\frac{N-1}{K}} \sin(r\sqrt{\frac{K}{N-1}}), & \text{if } K > 0, \\ r, & \text{if } K = 0, \\ \sqrt{\frac{N-1}{|K|}} \sinh(r\sqrt{\frac{|K|}{N-1}}), & \text{if } K < 0. \end{cases}$$

Then the Bishop-Gromov inequality, which is valid in the class of $\text{MCP}(K, N)$ spaces (see [45] and [50] and recall that an $\text{RCD}(K, N)$ space is also $\text{MCP}(K, N)$), states that

$$r \mapsto \frac{\mathbf{m}(B_r(x))}{v_{K,N}(r)} \quad \text{is decreasing} \quad (1.4)$$

for any $x \in \text{supp}(\mathbf{m})$. Therefore the following definition is meaningful:

Definition 1.9 (Bishop-Gromov density). *Let $K \in \mathbb{R}$, $N \in [1, \infty)$ and let (X, d, \mathbf{m}) be a $\text{MCP}(K, N)$ space with $\text{supp}(\mathbf{m}) = X$. For $x \in X$ we define the Bishop-Gromov density at x as*

$$\vartheta_N[X, d, \mathbf{m}](x) := \lim_{r \rightarrow 0} \frac{\mathbf{m}(B_r(x))}{v_{K,N}(r)} = \sup_{r > 0} \frac{\mathbf{m}(B_r(x))}{v_{K,N}(r)}. \quad (1.5)$$

Notice that by the very definition of $v_{K,N}(r)$ we have

$$\lim_{r \rightarrow 0} \frac{v_{K,N}(r)}{\omega_N r^N} = 1, \quad (1.6)$$

hence

$$\vartheta_N[\mathbf{X}, \mathbf{d}, \mathbf{m}](x) = \lim_{r \rightarrow 0} \frac{\mathbf{m}(B_r(x))}{\omega_N r^N}, \quad (1.7)$$

whence the choice of omitting the K in the notation of the Bishop-Gromov density. Still, the definition (1.5) allows to directly exploit (1.4) and this simplifies some proofs.

We note that for an $\text{RCD}(K, N)$ space the Bishop-Gromov density can be equal to ∞ at almost every point, a simple example being the $\text{RCD}(0, N)$ space $([0, \infty), \mathbf{d}_E, x^{N-1} d\mathcal{L}^1)$, where here and in the sequel \mathbf{d}_E will denote the euclidean distance. In a sense what is happening in this example is that there is a gap between the ‘functional analytic’ upper bound on the dimension N of the space and its ‘geometric’ dimension. This motivates the following:

Definition 1.10 (Weakly non-collapsed RCD spaces). *Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. We say that $(X, \mathbf{d}, \mathbf{m})$ is a weakly non-collapsed $\text{RCD}(K, N)$, $\text{wncRCD}(K, N)$ in short, space provided it is $\text{RCD}(K, N)$, it holds $\text{supp}(\mathbf{m}) = X$ and*

$$\vartheta_N[\mathbf{X}, \mathbf{d}, \mathbf{m}](x) < +\infty \quad \text{for } \mathbf{m} - \text{a.e. } x.$$

Notice that by classical results about differentiation of measures (see e.g. Lemma 2.11), if \mathcal{H}^N is a Radon measure on X we know that

$$\overline{\lim}_{r \downarrow 0} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} \leq 1 \quad \mathcal{H}^N - \text{a.e. } x \in X$$

and thus in particular

a non-collapsed $\text{RCD}(K, N)$ space is also weakly non-collapsed,

see Corollary 2.14.

Also, from (1.4) it follows that ϑ_N is lower-semicontinuous both as a function on the fixed $\text{RCD}(K, N)$ space $(X, \mathbf{d}, \mathbf{m})$ and along a pmGH-converging sequence (see Lemma 2.2). This easily implies the stability of the weakly non-collapsed condition w.r.t. pmGH-convergence, see Theorem 2.3.

Remark 1.11. By analogy with the properties of Ricci-limit spaces obtained in [16, 17, 18, 19, 23] we believe that

if $(X, \mathbf{d}, \mathbf{m})$ is $\text{RCD}(K, N)$ and $\vartheta_N[\mathbf{X}, \mathbf{d}, \mathbf{m}] < \infty$ on a set of positive \mathbf{m} -measure, then up to multiply \mathbf{m} by a positive constant the space is $\text{ncRCD}(K, N)$

and in particular that any weakly non-collapsed space is, up to multiply the measure by a positive constant, non-collapsed. Note in particular that a consequence of the above property would be the constancy of the dimension of $\text{RCD}(K, N)$ spaces in the case when there is at least a N -dimensional piece¹. This fact is proved, in full generality, for Ricci limit spaces by Colding and Naber in [23]. ■

¹During the revision process of this manuscript, Bruè and Semola proved in [12] that finite dimensional RCD spaces have constant dimension regardless of such ‘maximality’ condition.

The geometric significance of the finiteness of ϑ_N is mostly based on the fact that

$$\text{if } \vartheta_N[X, \mathbf{d}, \mathbf{m}](x) < \infty \text{ then every tangent space at } x \text{ is a metric cone} \quad (1.8)$$

which in turn follows directly from the ‘volume cone to metric cone’ property of RCD spaces obtained by the authors in [24, Theorem 1.1] (see Proposition 2.8 for the proof of (1.8)).

With this said, we have the following equivalent characterizations of weakly non-collapsed spaces:

Theorem 1.12. *Let $K \in \mathbb{R}$, $N \in [1, \infty)$ and let $(X, \mathbf{d}, \mathbf{m})$ be a $\text{RCD}(K, N)$ space with $\text{supp } \mathbf{m} = X$. Then the following are equivalent:*

(i) X is a $\text{wncRCD}(K, N)$ space.

(ii) $\mathbf{m} \ll \mathcal{H}^N$.

(iii) There exists a function $\vartheta_1 \in L^1_{\text{loc}}(\mathcal{H}^N)$ such that $\mathbf{m} = \vartheta_1 \mathcal{H}^N$.

(iv) N is integer and for \mathbf{m} -a.e. $x \in X$ there exists a constant $\vartheta_2(x)$ such that

$$(X, \mathbf{d}/r, \mathbf{m}/r^N, x) \xrightarrow{pmGH} (\mathbb{R}^N, \mathbf{d}_E, \vartheta_2(x) \mathcal{L}^N, 0) \quad \text{as } r \downarrow 0.$$

(v) N is integer and for \mathbf{m} -a.e. $x \in X$ it holds

$$(X, \mathbf{d}/r, \mathbf{m}/c_r, x) \xrightarrow{pmGH} (\mathbb{R}^N, \mathbf{d}_E, \mathcal{L}^N/c(N), 0) \quad \text{as } r \downarrow 0,$$

where

$$c_r := \int_{B_r^X(x)} \left(1 - \frac{\mathbf{d}(y, x)}{r}\right) d\mathbf{m}(y) \quad c(N) := \int_{B_1^{\mathbb{R}^N}(0)} (1 - |y|) d\mathcal{L}^N(y).$$

(vi) N is integer and for \mathbf{m} -a.e. $x \in X$ it holds

$$(X, \mathbf{d}/r, x) \xrightarrow{pGH} (\mathbb{R}^N, \mathbf{d}_E, 0) \quad \text{as } r \downarrow 0.$$

(vii) The tangent module $L^2(TX)$ has constant dimension equal to N .

Moreover in the above statements

$$\vartheta_1(x) = \vartheta_2(x) = \vartheta_N[X, \mathbf{d}, \mathbf{m}](x) < +\infty \quad \text{for } \mathbf{m}\text{-a.e. } x. \quad (1.9)$$

Finally, if any of these holds then (referring to [29] for the necessary definitions) it holds

$$H^{2,2}(X) = D(\Delta) \quad \text{and} \quad \text{tr Hess}(f) = \Delta f \quad \forall f \in H^{2,2}(X). \quad (1.10)$$

Remark 1.13. We believe that if $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, N)$ space for which (1.10) holds, then there exists $n \in \mathbb{N}$, $n \in [1, N]$, such that $(X, \mathbf{d}, \mathbf{m})$ is a weakly non-collapsed $\text{RCD}(K, n)$ space. Notice that according to Han’s results in [40], this would be true if one knew that the tangent module has constant dimension, in which case one should pick n to be such dimension². ■

²As already mentioned, Bruè and Semola recently proved in [12] that indeed finite dimensional RCD spaces have constant dimension. As a consequence of their result, the conjecture in this remark holds.

Remark 1.14. The definition proposed by Kitabeppu in [42] in our formalism reads as: $\vartheta_N[X](x) < \infty$ for every $x \in X$ (in particular such spaces are weakly non-collapsed in our sense). Then in [42] it has been proved that such spaces have many of the properties stated in Theorem 1.12, see [42, Theorem 1.4], and it has also been noticed that (1.8) holds. Our proofs of these facts are essentially the same as those in [42]. ■

We conclude mentioning that the characterization of non-collapsed spaces via blow-ups allows to deduce that ‘products’ and ‘factorizations’ of (weakly) non-collapsed spaces are still (weakly) non-collapsed, see Proposition 2.15 and compare it with the non-trivial behaviour - even on \mathbb{R}^n - of products of Hausdorff measures, see e.g. [26, 2.10.29].

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2 Weakly non-collapsed spaces

2.1 Stability

In this section we prove the stability of the class of $\text{wncRCD}(K, N)$ spaces w.r.t. pointed-measured-Gromov-Hausdorff convergence.

In all the upcoming discussion, a *metric space* is always a complete and separable space (sometimes we will consider convergence of open balls in such spaces, but this creates no problems in the definition of Gromov-Hausdorff convergence) and a *metric measure space* is a metric space equipped with a non-negative and *non-zero* Radon measure which is finite on bounded sets. Moreover, by $C(\alpha, \beta, \gamma, \dots)$ we will always intend a constant whose value depends on the parameters $\alpha, \beta, \gamma, \dots$ and nothing else.

Let us begin recalling some basic definitions that will be used throughout the text. The Hausdorff (semi-)distance between two subsets A, B of a metric space Y is given by

$$d_H(A, B) := \inf\{\varepsilon \geq 0 : B \subset A^\varepsilon \text{ and } A \subset B^\varepsilon\}$$

where A^ε denotes the ε -neighbourhood of A , i.e. the set of points at distance $< \varepsilon$ from A .

With this said, we now recall the definitions of the various kind of Gromov-Hausdorff convergences that we shall use. Notice that for the case of pointed and pointed-measured convergences our definitions are not really the correct ones in the general case, but given that we will always deal with geodesic metrics and uniformly locally doubling measures, our (simplified) approach is equivalent to the correct definitions, see for instance the discussions in [39, Chapter 3], [13, Section 8.1], [33, Section 3.5].

Definition 2.1 (Gromov-Hausdorff convergences). *Let (X_n, d_n) , $n \in \mathbb{N} \cup \{\infty\}$ be metric spaces. We say that (X_n, d_n) converges to (X_∞, d_∞) in the Gromov-Hausdorff (GH in short) sense provided there exist a metric spaces (Y, d_Y) and isometric embeddings $\iota_n : X_n \rightarrow Y$, $n \in \mathbb{N} \cup \{\infty\}$, such that*

$$d_H(\iota_n(X_n), \iota_\infty(X_\infty)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

If the spaces are pointed, i.e. selected points $x_n \in X_n$ are given, we say that (X_n, \mathbf{d}_n, x_n) converges to $(X_\infty, \mathbf{d}_\infty, x_\infty)$ in the pointed-Gromov-Hausdorff (pGH in short) sense provided there exist a metric spaces (Y, \mathbf{d}_Y) and isometric embeddings $\iota_n : X_n \rightarrow Y$, $n \in \mathbb{N} \cup \{\infty\}$, such that:

- i) $\iota_n(x_n) \rightarrow \iota_\infty(x_\infty)$ in Y ,
- ii) for every $R > 0$ we have

$$\mathbf{d}_H(\iota_n(B_R(x_n)), \iota_\infty(B_R(x_\infty))) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

If moreover the spaces X_n are endowed with Radon measures \mathbf{m}_n finite on bounded sets, we say that $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n)$ converges to $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, x_\infty)$ in the pointed-measured-Gromov-Hausdorff (pmGH in short) sense provided there are Y and (ι_n) satisfying (i), (ii) above and moreover it holds:

- iii) $(\iota_n)_* \mathbf{m}_n$ weakly converges to $(\iota_\infty)_* \mathbf{m}_\infty$, i.e. for every $\varphi \in C_b(Y)$ with bounded support we have

$$\int \varphi d(\iota_n)_* \mathbf{m}_n \rightarrow \int \varphi d(\iota_\infty)_* \mathbf{m}_\infty \quad \text{as } n \rightarrow \infty.$$

In any of these cases, the collection of the space Y and isometric embeddings (ι_n) is called realization of the convergence and in any of these cases, given $z_n \in X_n$, $n \in \mathbb{N} \cup \{\infty\}$, we say that (z_n) converges to z_∞ , and write $z_n \xrightarrow{GH} z_\infty$ provided there exists a realization such that

$$\lim_{n \rightarrow \infty} \mathbf{d}_Y(\iota_n(z_n), \iota_\infty(z_\infty)) = 0.$$

Notice that in presence of non-trivial automorphism of the limits space X_∞ it might be that the same sequence (z_n) converges to two different points $z_\infty, z'_\infty \in X_\infty$. This creates no issues in the foregoing discussion.

We shall frequently use, without further reference, the fact that the class of $\text{RCD}(K, N)$ spaces is closed w.r.t. pmGH-convergence (see [43], [49], [50], [4], [31], [33]).

Since the Bishop-Gromov inequality (1.4) implies that the measure is locally doubling, we can use Gromov's compactness theorem (see [39, Section 5.A]) to deduce that

$$\begin{aligned} & \text{if } (X_n, \mathbf{d}_n, \mathbf{m}_n, x_n), n \in \mathbb{N}, \text{ are } \text{RCD}(K_n, N) \text{ spaces with } N \in [1, \infty), \text{supp}(\mathbf{m}_n) = X_n, \\ & \mathbf{m}_n(B_1(x_n)) \in [v, v^{-1}] \text{ for every } n \in \mathbb{N} \text{ and some } v \in (0, 1) \text{ and } K_n \rightarrow K \in \mathbb{R}, \\ & \text{then there is a subsequence pmGH-converging to some } \text{RCD}(K, N) \text{ space} \\ & (X, \mathbf{d}, \mathbf{m}, x) \text{ with } \text{supp}(\mathbf{m}) = X \text{ and a realization with } Y \text{ proper.} \end{aligned} \tag{2.1}$$

Recall that a metric space is *proper* provided closed bounded sets are compact. Notice that a direct consequence of the definitions is that

$$\forall y_\infty \in X_\infty \quad \text{there exists } y_n \in X_n, n \in \mathbb{N} \text{ such that } y_n \xrightarrow{GH} y_\infty. \tag{2.2}$$

We also recall that on $\text{MCP}(K, N)$ spaces (see [45] and [50]) $(X, \mathbf{d}, \mathbf{m})$ with $\text{supp}(\mathbf{m}) = X$ and $N < \infty$, from the spherical version of Bishop-Gromov inequality - see [50, Inequality (2.4)] - it holds

$$\mathbf{m}(B_r(x)) = \mathbf{m}(\bar{B}_r(x)) \quad \forall x \in X, r > 0 \tag{2.3}$$

and in turn this easily implies that if $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n) \xrightarrow{pmGH} (X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, x_\infty)$ and all such spaces are MCP(K, N) with measures of full support, then

$$y_n \xrightarrow{GH} y_\infty \quad \Rightarrow \quad \mathbf{m}_n(B_r(y_n)) \rightarrow \mathbf{m}_\infty(B_r(y_\infty)) \quad \forall r > 0, \quad (2.4)$$

which easily follows by the weak convergence of the measures and by the fact that $\mathbf{m}(\partial B_r(y_\infty)) = 0$ under our assumptions (from (2.3)). Let us now collect some basic simple properties of the Bishop-Gromov density:

Lemma 2.2 (Basic properties of the Bishop-Gromov density). *Let $K \in \mathbb{R}$, $N \in [1, \infty)$. Then:*

(i) *Let $(X_j, \mathbf{d}_j, \mathbf{m}_j, \bar{x}_j)$ be a sequence of pointed MCP(K, N) spaces pmGH-converging to a limit MCP(K, N) space $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, \bar{x}_\infty)$. Then*

$$x_j \xrightarrow{GH} x_\infty \quad \Rightarrow \quad \liminf_{j \rightarrow \infty} \vartheta_N[X_j, \mathbf{d}_j, \mathbf{m}_j](x_j) \geq \vartheta_N[X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty](x_\infty).$$

In particular, on a given MCP(K, N) space $(X, \mathbf{d}, \mathbf{m})$, the function $\vartheta_N : X \rightarrow [0, \infty]$ is lower-semicontinuous (and thus Borel measurable).

(ii) *Let $(X, \mathbf{d}, \mathbf{m})$ be a MCP(K, N) space. Then \mathbf{m} -a.e. point $x \in \{\vartheta_N < \infty\}$ is an approximate continuity point for ϑ_N , i.e.*

$$\lim_{r \rightarrow 0} \frac{\mathbf{m}\left(\{y \in B_r(x) : |\vartheta_N(y) - \vartheta_N(x)| > \varepsilon\}\right)}{\mathbf{m}(B_r(x))} = 0 \quad (2.5)$$

for every $\varepsilon > 0$.

(iii) *Let $(X, \mathbf{d}, \mathbf{m})$ be a MCP(K, N) space and, for $r > 0$, put $(X_r, \mathbf{d}_r, \mathbf{m}_r) = (X, \mathbf{d}/r, \mathbf{m}/r^N)$. Then for every $x \in X$ we have $\vartheta_N[X_r, \mathbf{d}_r, \mathbf{m}_r](x) = \vartheta_N[X, \mathbf{d}, \mathbf{m}](x)$.*

Proof Point (iii) trivially follows from (1.7) and point (i) is a direct consequence of the definitions, of (2.4) and of the monotonicity granted by the Bishop Gromov inequality (1.4).

For point (ii) note that the Bishop-Gromov inequality (1.4) grants that \mathbf{m} is locally doubling, hence the Lebesgue differentiation Theorem applies to every function $f \in L^1_{\text{loc}}(X)$:

$$\lim_{r \rightarrow 0} \frac{1}{\mathbf{m}(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| d\mathbf{m}(y) = 0 \quad \mathbf{m}\text{-a.e. } x. \quad (2.6)$$

By applying (2.6) to, for instance, $f(x) := \arctan \vartheta_N(x)$ one easily gets (ii). \square

A stability result for the class of wncRCD spaces now easily follows, see Remark 2.4 below for some comments on the statement:

Theorem 2.3 (Stability of weakly non-collapsed spaces). *Let $K \in \mathbb{R}$, $N \in \mathbb{R}$ and let $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n)$ be a sequence of wncRCD(K, N) spaces pmGH-converging to some limit space $(X, \mathbf{d}, \mathbf{m}, x)$. Assume that for every $R > 0$ there is an increasing function $f_R : [0, +\infty] \rightarrow [0, +\infty]$ with $f_R(+\infty) = +\infty$ such that*

$$\overline{\lim}_{n \rightarrow \infty} \int_{B_R^{X_n}(x_n)} f_R \circ \vartheta_N[X_n, \mathbf{d}_n, \mathbf{m}_n] d\mathbf{m}_n < \infty. \quad (2.7)$$

Then $(X, \mathbf{d}, \mathbf{m})$ is a wncRCD(K, N) space and for every $R > 0$ it holds

$$\int_{B_R^X(x)} f_R \circ \vartheta_N[X, \mathbf{d}, \mathbf{m}] d\mathbf{m} \leq \underline{\lim}_{n \rightarrow \infty} \int_{B_R^{X_n}(x_n)} f_R \circ \vartheta_N[X_n, \mathbf{d}_n, \mathbf{m}_n] d\mathbf{m}_n. \quad (2.8)$$

Proof From the stability of the RCD condition we know that $(X, \mathbf{d}, \mathbf{m})$ is RCD(K, N). Let $(Y, \mathbf{d}_Y, (\iota_n))$ be a realization of the pmGH-convergence and define $\tilde{\vartheta}_n : Y \rightarrow [0, \infty]$ as

$$\tilde{\vartheta}_n(y) := \begin{cases} \vartheta_N[X_n, \mathbf{d}_n, \mathbf{m}_n](\iota_n^{-1}(y)), & \text{if } y \in \iota(X_n), \\ +\infty, & \text{otherwise.} \end{cases}$$

and similarly $\tilde{\vartheta}$. Then from point (i) of Lemma 2.2 and the monotonicity of f_R we deduce that

$$y_n \rightarrow y \quad \Rightarrow \quad \chi_{B_R^Y(\iota(x))}(y) f_R(\tilde{\vartheta}(y)) \leq \liminf_{n \rightarrow \infty} \chi_{B_R^Y(\iota_n(x_n))}(y_n) f_R(\tilde{\vartheta}_n(y_n))$$

having also used the fact that if $y \in B_R^Y(\iota(x))$ then eventually $y_n \in B_R^Y(\iota_n(x_n))$. By the simple Lemma 2.5 below this inequality and the weak convergence of $(\iota_n)_* \mathbf{m}_n$ to $\iota_* \mathbf{m}$ give

$$\int \chi_{B_R^Y(\iota(x))} f_R \circ \tilde{\vartheta} \, d\iota_* \mathbf{m} \leq \liminf_{n \rightarrow \infty} \int \chi_{B_R^Y(\iota_n(x_n))} f_R \circ \tilde{\vartheta}_n \, d(\iota_n)_* \mathbf{m}_n,$$

which is (2.8). In particular, taking into account (2.7) we deduce that

$$\int_{B_R^X(x)} f_R \circ \vartheta_N[X, \mathbf{d}, \mathbf{m}] \, d\mathbf{m} < \infty$$

and since $f_R(+\infty) = +\infty$, this forces $\vartheta_N[X, \mathbf{d}, \mathbf{m}]$ to be finite \mathbf{m} -a.e.. \square

Remark 2.4. It is not hard to check that, in this last theorem, if all the spaces X_n are Riemannian manifolds of the same dimension k converging to a smooth Riemannian manifold X , then necessarily $k = N$ and X has dimension k . In particular, the convergence is non-collapsed in the sense of Cheeger and Colding, [17].

In this respect the following example might be explanatory: Let S_r^1 be the 1-dimensional sphere of radius r and consider the cylinder $M_n := \mathbb{R} \times S_{1/n}^1$ equipped with its natural product distance \mathbf{d}_n and volume measure vol_n . It is clear that as $n \rightarrow \infty$ the metric spaces (M_n, \mathbf{d}_n) converge in the pGH-topology to the real line, which trivially has smaller dimension (since in all these manifolds the isometry group acts transitively, the choice of reference point is irrelevant and thus omitted).

Let us now consider convergence of the metric *measure* spaces $(M_n, \mathbf{d}_n, \text{vol}_n)$. Notice that for any $r > 1/n$ and $p_n \in M_n$ we have that

$$\text{vol}_n(B_r(p_n)) \sim \frac{r}{n} \quad \text{as} \quad n \rightarrow \infty \quad (2.9)$$

and thus the measures \mathbf{m}_n weakly converge, in any realisation of the pGH-convergence, to 0. However, the choice of the null measure is excluded by the definition of metric measure space - see the beginning of Section 2.1 -, so that $(\mathbb{R}, \mathbf{d}_E, 0)$ is not a legitimate metric measure space and the spaces $(M_n, \mathbf{d}_n, \text{vol}_n)$ do not satisfy the assumptions of Theorem 2.3 above, because they do not converge anywhere in the pmGH-topology. We emphasize that the choice of excluding null reference measures is customary in this research field, see for instance [33] and references therein for a discussion of this topic in relation to convergence of mm-structures.

The typical way to avoid measures disappearing in the limit is to renormalise them via the multiplication by an appropriate constant: this is precisely what Cheeger-Colding do in [17], [18], [19] when defining renormalised limit measure. In our case, by (2.9) we are led to

consider the measures $\mathbf{m}_n := c_n \text{vol}_n$ with $c_n \sim n$, so that the spaces $(M_n, \mathbf{d}_n, \mathbf{m}_n)$ converge in the pmGH-sense to $(\mathbb{R}, \mathbf{d}_E, c\mathcal{L}^1)$ for some $c > 0$. Thus we have $\vartheta_2[M_n, \mathbf{d}_n, \mathbf{m}_n] \equiv c_n \rightarrow +\infty$ and $\mathbf{m}_n(B_r(p_n)) \rightarrow c\mathcal{L}^1(B_r(0)) = 2cr > 0$ as $n \rightarrow \infty$. Hence for any function $z \mapsto f(z)$ going to $+\infty$ as $z \rightarrow +\infty$ we have

$$\int_{B_r(p_n)} f \circ \vartheta_2[M_n, \mathbf{d}_n, \mathbf{m}_n] d\mathbf{m}_n = f(c_n) \mathbf{m}_n(B_r(p_n)) \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty,$$

so that the assumption (2.7) does not hold in this case.

We conclude pointing out that the notion of (weakly) non-collapsed space makes sense only when coupled with the dimension which is being considered, so that it can very well be that a sequence of 2-dimensional non-collapsed spaces converges, with collapsing, to a 1-dimensional non-collapsed space. This is precisely what happens in the example we discussed here. \blacksquare

In the proof of Theorem 2.3 we used the following known simple variant of the classical Fatou lemma:

Lemma 2.5 (A variant of Fatou's lemma). *Let (Y, \mathbf{d}_Y) be a complete and separable metric space, $\{\mu_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ be Radon measures finite on bounded sets such that*

$$\lim_{n \rightarrow \infty} \int \varphi d\mu_n = \int \varphi d\mu_\infty$$

for every $\varphi \in C_b(Y)$ with bounded support. Also, let $f_n : Y \rightarrow \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N} \cup \{\infty\}$ be such that

$$y_n \rightarrow y \quad \Rightarrow \quad f_\infty(y) \leq \varliminf_{n \rightarrow \infty} f_n(y_n) \quad (2.10)$$

and $f_n \geq g$ for every $n \in \mathbb{N} \cup \{\infty\}$ for some $g \in C_b(Y)$ with bounded support.

Then

$$\int f_\infty d\mu_\infty \leq \varliminf_{n \rightarrow \infty} \int f_n d\mu_n.$$

Proof Replacing f_n with $f_n - g$ we can assume that the f_n 's are non-negative. Then we follow verbatim the proof in [2, Lemma 8.2] which, although presented on \mathbb{R}^d , actually holds also in our context. \square

2.2 Tangent spaces

In this section we study the tangent spaces of weakly non-collapsed spaces, here is the definition that we will adopt (notice the chosen scaling of the measure):

Definition 2.6 ((metric) tangent space). *Let $K \in \mathbb{R}$, $N \in [1, \infty)$, $(X, \mathbf{d}, \mathbf{m})$ an $\text{RCD}(K, N)$ space with $\text{supp}(\mathbf{m}) = X$ and $x \in X$.*

We say that (Y, \mathbf{d}_Y, o) is a metric tangent space of X at x if there exists a sequence $r_n \downarrow 0$ such that

$$(X_{r_n}, \mathbf{d}_{r_n}, x) := (X, \mathbf{d}/r_n, x) \xrightarrow{pGH} (Y, \mathbf{d}_Y, o) \quad \text{as } n \uparrow \infty.$$

Similarly, we say that $(Y, \mathbf{d}_Y, \mathbf{m}_Y, o)$ is a tangent space of X at x if there exists a sequence $r_n \downarrow 0$ such that

$$(X_{r_n}, \mathbf{d}_{r_n}, \mathbf{m}_{r_n}, x) := (X, \mathbf{d}/r_n, \mathbf{m}/r_n^N, x) \xrightarrow{pmGH} (Y, \mathbf{d}_Y, \mathbf{m}_Y, o) \quad \text{as } n \uparrow \infty.$$

Notice that the Bishop-Gromov inequality (1.4) gives that $\inf_{r \in (0,1)} \mathbf{m}_r(B_1(x)) > 0$ for every $x \in \text{supp}(\mathbf{m})$ and if $\vartheta_N(x) < \infty$ then by (1.7) we also have that $\sup_{r \in (0,1)} \mathbf{m}_r(B_1(x)) < \infty$. Hence recalling (2.1) we see that given an $\text{RCD}(K, N)$ space $(X, \mathbf{d}, \mathbf{m})$ and a point $x \in X$ with $\vartheta_N(x) < \infty$, the family $(X_r, \mathbf{d}_r, \mathbf{m}_r, x)$, $r \in (0, 1)$, is precompact and, by scaling, any limit space as $r \downarrow 0$ is $\text{RCD}(0, N)$.

For the definition of cone built over a metric space see for instance [13, Definition 3.6.16]. We then give the following:

Definition 2.7 (Metric (measure) cones). *We say that (X, \mathbf{d}) is a metric cone with vertex $x \in X$ provided there is a metric space (Z, \mathbf{d}_Z) and an isometry ι between X and the cone over Z sending x to the vertex.*

If X is also endowed with a Radon measure \mathbf{m} we say that it is a metric measure cone provided there are Z, ι as before and moreover there are a Radon measure \mathbf{m}_Z on Z and $\alpha \geq 1$ such that

$$\mathbf{d}(\iota_*\mathbf{m})(r, z) = \mathbf{d}r \otimes r^{\alpha-1} \mathbf{d}\mathbf{m}_Z(z).$$

In this case we say that X is an α -metric measure cone.

A crucial regularity property of weakly non-collapsed spaces is contained in the following statement, which in turn is a direct consequence of the ‘volume cone to metric cone’ for RCD spaces obtained in [24, Theorem 1.1]:

Proposition 2.8 (Tangent spaces are cones). *Let $K \in \mathbb{R}$, $N \in [1, \infty)$, $(X, \mathbf{d}, \mathbf{m})$ an $\text{RCD}(K, N)$ space and $\bar{x} \in X$ such that $\vartheta_N(\bar{x}) < \infty$. Then every tangent space $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, o)$ at \bar{x} is an N -metric measure cone based in o and it holds*

$$\vartheta_N[X, \mathbf{d}, \mathbf{m}](\bar{x}) = \vartheta_N[X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty](o) = \frac{\mathbf{m}_\infty(B_\varrho(o))}{\omega_N \varrho^N} \quad \forall \varrho > 0. \quad (2.11)$$

Proof Let $r_n \downarrow 0$ be such that the rescaled spaces $(X_{r_n}, \mathbf{d}/r_n, \mathbf{m}/r_n^N, \bar{x})$ pmGH-converge to the $\text{RCD}(0, N)$ space $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, o)$. We shall apply [24, Theorem 1.1] to the space X_∞ . From the very definition of pmGH-convergence and recalling (2.4), for any $\varrho > 0$ we have

$$\frac{\mathbf{m}_\infty(B_\varrho(o))}{\omega_N \varrho^N} = \lim_{n \rightarrow \infty} \frac{\mathbf{m}_\infty(B_{r_n \varrho}(\bar{x}))}{\omega_N (r_n \varrho)^N} = \vartheta_N[X, \mathbf{d}, \mathbf{m}](\bar{x}). \quad (2.12)$$

Hence $\varrho \mapsto \frac{\mathbf{m}_\infty(B_\varrho(o))}{\varrho^N}$ is constant and according to [24, Theorem 1.1] this is enough to deduce that X_∞ is a N -metric measure cone based in o . Also, letting $\varrho \downarrow 0$ in (2.12) we deduce (2.11). \square

The fact that tangent cones of wncRCD spaces are in fact a.e. Euclidean spaces is based on the following simple lemma. Notice that the first part of the statement only assumes the space to be a metric cone, and not a metric measure cone: the rigidity is possible because the splitting theorem for $\text{RCD}(0, N)$ spaces only requires the existence of a straight line on the given space and this is a metric requirement (as opposed to a metric-measure requirement).

Lemma 2.9. *Let $N \in [1, \infty)$ and let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(0, N)$ space which, for every $x \in X$, is a metric cone with vertex in x . Then there exists $m \in \mathbb{N}$ and $c_m > 0$ such that $(X, \mathbf{d}, \mathbf{m}) = (\mathbb{R}^m, \mathbf{d}_E, c_m \mathcal{L}^m)$.*

If we also know a priori that X is an N -metric measure cone with vertex x for every $x \in X$, then $N \in \mathbb{N}$ and $m = N$ in the above.

Proof By the very definition of metric cone with vertex \bar{x} any point $x \in X \setminus \{\bar{x}\}$ lies in the interior of a half line (i.e. an isometric embedding of $[0, +\infty)$). Moreover, by assumption, for every $x \in X \setminus \{\bar{x}\}$ and $r > 0$ the pointed spaces $(X, \mathbf{d}/r, x)$ and (X, \mathbf{d}, x) are isometric and therefore any metric tangent space at x must coincide with X itself. Given that x lies in the interior of a length minimising geodesic, the tangent space, and hence X itself, must contain a line through x , see for instance [?, Proof of Theorem 1.1] for a similar argument. Thus the splitting theorem for RCD spaces [28], [30] grants that $(X, \mathbf{d}, \mathbf{m})$ splits off a line, i.e. it is isomorphic to the product of the Euclidean line \mathbb{R} and a metric measure space $(X', \mathbf{d}', \mathbf{m}')$. Moreover, such X' is a point if $N \in [1, 2)$ and a RCD(0, $N - 1$) space if $N \geq 2$. By iterating this fact finitely many times we obtain the desired conclusion.

The last statement is now obvious. \square

We then have the following:

Proposition 2.10. *Let $K \in \mathbb{R}$, $N \in [1, \infty)$, $(X, \mathbf{d}, \mathbf{m})$ an RCD(K, N) space with $\text{supp}(\mathbf{m}) = X$ and $\bar{x} \in X$. Assume that \bar{x} is a point of approximate continuity of $\vartheta_N[X]$, i.e. $\vartheta_N[X](\bar{x}) < \infty$ and (2.5) holds.*

Then $N \in \mathbb{N}$ and $(\mathbb{R}^N, \mathbf{d}_E, \vartheta_N[X](\bar{x})\mathcal{L}^N, 0)$ is the only tangent space of X at \bar{x} .

Proof Let $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, o)$ be a tangent space at \bar{x} , let $r_n \downarrow 0$ be a sequence that realises it and pick $y \in X_\infty$. We claim that there exists a sequence $n \mapsto y_n \in X_{r_n}$ such that $y_n \xrightarrow{\text{GH}} y$ and

$$\vartheta_N[X, \mathbf{d}, \mathbf{m}](y_n) \rightarrow \vartheta_N[X, \mathbf{d}, \mathbf{m}](\bar{x}). \quad (2.13)$$

Indeed, let $n \mapsto \tilde{y}_n \in X_{r_n} = X$ be arbitrary such that $\tilde{y}_n \xrightarrow{\text{GH}} y$ (recall (2.2)), notice that $\mathbf{d}(\tilde{y}_n, \bar{x}) \rightarrow 0$ and that the choice of \bar{x} and the fact that \mathbf{m} is doubling grant that for every $r, \varepsilon > 0$ the balls $B_{rr_n}(\tilde{y}_n) \subset X$ must eventually intersect the set $\{x : |\vartheta(x) - \vartheta(\bar{x})| < \varepsilon\}$. Hence with a perturbation and diagonalization argument, starting from (\tilde{y}_n) we can produce the desired (y_n) .

With this said, for any $\varrho > 0$ we have

$$\frac{\mathbf{m}_\infty(B_\varrho(y))}{\omega_N \varrho^N} \stackrel{(2.4)}{=} \lim_{n \rightarrow \infty} \frac{\mathbf{m}(B_{\varrho r_n}(y_n))}{\omega_N (\varrho r_n)^N} \stackrel{(1.6), (1.4)}{\leq} \lim_{n \rightarrow \infty} \vartheta_N[X, \mathbf{d}, \mathbf{m}](y_n) \stackrel{(2.13)}{=} \vartheta_N[X, \mathbf{d}, \mathbf{m}](\bar{x}). \quad (2.14)$$

On the other hand, putting $R := \mathbf{d}_\infty(y, o)$ and using again the Bishop-Gromov inequality (1.4) (recall that X_∞ is RCD(0, N)) we have

$$\begin{aligned} \frac{\mathbf{m}_\infty(B_\varrho(y))}{\omega_N \varrho^N} &\stackrel{(1.4)}{\geq} \lim_{r \rightarrow \infty} \frac{\mathbf{m}_\infty(B_r(y))}{\omega_N r^N} = \lim_{r \rightarrow \infty} \frac{\mathbf{m}_\infty(B_{r+R}(y))}{\omega_N (r+R)^N} \\ &\geq \lim_{r \rightarrow \infty} \frac{\mathbf{m}_\infty(B_r(o))}{\omega_N r^N} \frac{r^N}{(r+R)^N} \stackrel{(2.11)}{=} \vartheta_N[X, \mathbf{d}, \mathbf{m}](\bar{x}). \end{aligned} \quad (2.15)$$

From (2.14) and (2.15) we deduce that

$$\varrho \mapsto \frac{\mathbf{m}_\infty(B_\varrho(y))}{\omega_N \varrho^N} \text{ is constantly equal to } \vartheta_N[X, \mathbf{d}, \mathbf{m}](\bar{x}) \quad (2.16)$$

and from [24] we can then deduce that $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, y)$ is a N -metric measure cone. Then arbitrariness of $y \in X_\infty$ and the simple Lemma 2.9 above give the conclusion. \square

2.3 Equivalent characterizations of weakly non-collapsed spaces

Here we shall prove Theorem 1.12 about different equivalent characterizations of weakly non-collapsed spaces.

We shall make use of the following classical result about differentiation of measures, see e.g. [10, Theorem 2.4.3] for the proof.

Lemma 2.11 (Density w.r.t. Hausdorff measures). *Let (X, \mathbf{d}) be a complete and separable metric space, \mathbf{m} a Radon measure on it and for $\alpha \geq 0$ define the α -upper density function as:*

$$\bar{\vartheta}_\alpha(\mathbf{m}, x) := \overline{\lim}_{r \downarrow 0} \frac{\mathbf{m}(B_r(x))}{\omega_\alpha r^\alpha}.$$

Then for every Borel $B \subset X$ and $c > 0$ it holds

$$\bar{\vartheta}_\alpha(\mathbf{m}, x) \geq c \quad \forall x \in B \quad \Rightarrow \quad \mathbf{m}(B) \geq c \mathcal{H}^\alpha(B), \quad (2.17)$$

$$\bar{\vartheta}_\alpha(\mathbf{m}, x) \leq c \quad \forall x \in B \quad \Rightarrow \quad \mathbf{m}(B) \leq c 2^\alpha \mathcal{H}^\alpha(B), \quad (2.18)$$

Let us point out a direct consequence of the above which, being based on the Bishop-Gromov inequality only, is valid on general MCP(K, N) spaces:

Proposition 2.12. *Let $K \in \mathbb{R}$, $N \in [1, \infty)$ and $(X, \mathbf{d}, \mathbf{m})$ a MCP(K, N) space with $\text{supp}(\mathbf{m}) = X$. Then for every $R > 0$ there is $C = C(K, N, R)$ such that for every $x \in X$ it holds*

$$\mathcal{H}^N|_{B_R(x)} \leq \frac{C(K, N, R)}{\mathbf{m}(B_1(x))} \mathbf{m}|_{B_R(x)}. \quad (2.19)$$

In particular, \mathcal{H}^N is a Radon measure on X , is absolutely continuous w.r.t. \mathbf{m} and it holds

$$\mathcal{H}^N(B_r(x)) = \mathcal{H}^N(\bar{B}_r(x)) \quad \forall x \in X, r > 0. \quad (2.20)$$

Proof The Bishop-Gromov inequality (1.4) implies that $\vartheta_N[X, \mathbf{d}, \mathbf{m}](y) \geq \frac{\mathbf{m}(B_{2R}(y))}{v_{K,N}(2R)} \geq \frac{\mathbf{m}(B_1(x))}{v_{K,N}(2R)}$ for every $y \in B_R(x)$ and $R > 1$. Also, from (1.7) we know that $\bar{\vartheta}_N(\mathbf{m}, y) = \vartheta_N[X, \mathbf{d}, \mathbf{m}](y)$ for every $y \in X$. Hence (2.19) comes from (2.17) and then (2.20) follows from (2.3). \square

Before coming to the proof of Theorem 1.12 let us collect in the following statement the known rectifiability properties of RCD spaces:

Theorem 2.13 (Rectifiability of RCD spaces). *Let $K \in \mathbb{R}$, $N \in [1, \infty)$ and $(X, \mathbf{d}, \mathbf{m})$ be an RCD(K, N) space. Then we can write*

$$X = \mathcal{N} \cup \bigcup_{k=1}^M \bigcup_{j \in \mathbb{N}} U_j^k \quad (2.21)$$

for Borel sets \mathcal{N}, U_j^k where $\mathbf{m}(\mathcal{N}) = 0$, $M \in \mathbb{N}$, $M \leq N$, each U_j^k is bi-Lipschitz to a subset of \mathbb{R}^k , and for \mathbf{m} -a.e. $x \in U_j^k$ the metric tangent space at x is the Euclidean space \mathbb{R}^k . Moreover for any j, k it holds

$$\mathbf{m}|_{U_j^k} = \vartheta_j^k \mathcal{H}^k|_{U_j^k} \quad (2.22)$$

for some Borel function $\vartheta_j^k : X \rightarrow \mathbb{R}$ which also satisfies

$$\vartheta_j^k(x) = \lim_{r \downarrow 0} \frac{\mathbf{m}(B_r(x) \cap U_j^k)}{\omega_k r^k} = \lim_{r \downarrow 0} \frac{\mathbf{m}(B_r(x))}{\omega_k r^k} \quad \mathcal{H}^k|_{U_j^k} - \text{a.e. } x. \quad (2.23)$$

Proof The existence of the partition (2.21), of bi-Lipschitz charts and the fact that metric tangent spaces are Euclidean have all been proved in [44]. Property (2.22) has been proved in [41], [35]. These informations together grant that $\mathbf{m}|_{U_j^k}$ is a k -rectifiable measure according to [7, Definition 5.3], hence the first equality in (2.23) follows from [7, Theorem 5.4]. To conclude, notice that if

$$\vartheta_j^k(x) < \overline{\lim}_{r \downarrow 0} \frac{\mathbf{m}(B_r(x))}{\omega_k r^k}$$

holds in a Borel set $A \subset U_j^k$ of positive \mathcal{H}^k -measure, then we can find $b > a \geq 0$ and a Borel set $A' \subset A$ such that $\mathcal{H}^N(A') > 0$, $\vartheta_j^k \leq a$ \mathcal{H}^k -a.e. on A' and $\overline{\lim}_{r \downarrow 0} \frac{\mathbf{m}(B_r(x))}{\omega_k r^k} \geq b$ for $x \in A'$.

This would lead to

$$\mathbf{m}(A') \stackrel{(2.17)}{\geq} b \mathcal{H}^k(A') > a \mathcal{H}^N(A') \stackrel{(2.22)}{\geq} \mathbf{m}(A'),$$

which is impossible. This proves the second equality in (2.23) and concludes the proof (see also [6] for similar arguments). \square

We are now ready to prove Theorem 1.12.

Proof of Theorem 1.12

(i) \Rightarrow (ii) By (2.18) we know that $\mathbf{m}|_{\{\theta < +\infty\}} \ll \mathcal{H}^N$ and since by hypothesis we have that $\mathbf{m}(\{\vartheta_N = +\infty\}) = 0$, the claim follows.

(ii) \Rightarrow (iii) Proposition 2.12 grants that \mathcal{H}^N is σ -finite, hence the claim follows by the Radon-Nikodym theorem.

(iii) \Rightarrow (i) We consider the decomposition (2.21) and notice that the assumption $\mathbf{m} = \vartheta_1 \mathcal{H}^N$ and (2.22) forces $\mathbf{m}(U_j^k) = 0$ for every $k < N$ and $j \in \mathbb{N}$ and, since $\mathbf{m}(X) > 0$, N to be an integer. Hence for every j we have

$$\vartheta_N[X](x) \stackrel{(1.7)}{=} \lim_{r \rightarrow 0} \frac{\mathbf{m}(B_r(x))}{\omega_N r^N} \stackrel{(2.23)}{=} \vartheta_1(x) < +\infty \quad \text{for } \mathbf{m}|_{U_j^N}\text{-a.e. } x. \quad (2.24)$$

(i) \Rightarrow (iv) Consequence of the assumptions, point (ii) of Lemma 2.2 and Proposition 2.10, which also grant that

$$\vartheta_N = \vartheta_2 \quad \mathbf{m}\text{-a.e.} \quad (2.25)$$

(iv) \Rightarrow (v) This is immediate, since one can easily check that

$$c_r/r^N = \int_{B_1^{X_r}} (1 - \mathbf{d}_r(y, x)) d\mathbf{m}_r(y) \quad \rightarrow \quad \vartheta_2(x) \int_{B_1^{\mathbb{R}^N}} (1 - |y|) d\mathcal{L}^N(y).$$

(v) \Rightarrow (vi) Trivial by definitions.

(vi) \Rightarrow (ii) By Theorem 2.13 we know that for every k, j , for \mathbf{m} -a.e. $x \in U_j^k$ the metric tangent space at x is \mathbb{R}^k . Thus our assumption forces \mathbf{m} to be concentrated on $\cup_j U_j^N$ and the conclusion follows recalling (2.22).

(vi) \Leftrightarrow (vii) This is an immediate consequence of [36, Theorem 5.1].

Proof of xxx Consequence of (2.24) and (2.25).

(vii) \Rightarrow (1.10) This follows from [40, Proposition 4.1]. \square

An easy consequence of the above is:

Corollary 2.14 (ncRCD \Rightarrow wncRCD). *Let (X, d, \mathcal{H}^N) be a ncRCD(K, N) space. Then*

$$\vartheta_N(x) \leq 1 \quad \forall x \in X. \quad (2.26)$$

In particular, $(\tilde{X}, d, \mathcal{H}^N)$ is wncRCD(K, N), where $\tilde{X} \subset X$ is the support of \mathcal{H}^N .

Proof By point (iii) of Theorem 1.12 and (1.9) we see that $\vartheta_N \leq 1$ \mathcal{H}^N -a.e.. Then (2.26) follows by the lower semicontinuity of ϑ_N established in point (i) of Lemma 2.2. \square

2.4 Tensorization and factorization

Given two metric measure spaces (X_1, d_1, \mathbf{m}_1) and (X_2, d_2, \mathbf{m}_2) , by their product we mean the product $X_1 \times X_2$ equipped with the distance $d_1 \otimes d_2$ defined by

$$(d_1 \otimes d_2)^2((x_1, x_2), (x'_1, x'_2)) := d_1^2(x_1, x'_1) + d_2^2(x_2, x'_2) \quad \forall x_1, x'_1 \in X_1, x_2, x'_2 \in X_2$$

and the product measure $\mathbf{m}_1 \times \mathbf{m}_2$.

Recall that the product of an RCD(K, N_1) and an RCD(K, N_2) space is RCD($K, N_1 + N_2$) (see [50], [4], [5]).

With this said, thanks to characterization of wncRCD spaces via blow-ups obtained in Theorem 1.12 we can easily prove that products and factors of wncRCD (resp. ncRCD) are wncRCD (resp. ncRCD):

Proposition 2.15 (Tensorization and factorization of non-collapsed spaces). *Let (X_i, d_i, \mathbf{m}_i) be RCD(K, N_i) spaces, $i = 1, 2$, with $K \in \mathbb{R}$ and $N_i \in [1, \infty)$ and consider the product space $(X_1 \times X_2, d_1 \otimes d_2, \mathbf{m}_1 \times \mathbf{m}_2)$.*

Then $X_1 \times X_2$ is wncRCD($K, N_1 + N_2$) if and only if X_1 is wncRCD(K, N_1) and X_2 is wncRCD(K, N_2).

Similarly, $X_1 \times X_2$ is ncRCD($K, N_1 + N_2$) if and only if for some constant $c > 0$ $(X_1, d_1, c\mathbf{m}_1)$ is ncRCD(K, N_1) and $(X_2, d_2, c^{-1}\mathbf{m}_2)$ is ncRCD(K, N_2).

Proof From Theorem 2.13 we know that for \mathbf{m}_1 -a.e. x_1 the metric tangent space of X_1 at x_1 is $\mathbb{R}^{n_1(x_1)}$ with $n_1(x_1) \leq N_1$. Similarly for X_2 . Then from the very definition of pGH-convergence and Fubini's theorem it is readily checked that $\mathbb{R}^{n_1(x_1)+n_2(x_2)}$ is the metric tangent space of $X_1 \times X_2$ at (x_1, x_2) for $\mathbf{m}_1 \times \mathbf{m}_2$ -a.e. (x_1, x_2) .

Thus the claims about wncRCD spaces follows by the characterization given in point (vi) of Theorem 1.12.

For the case of ncRCD spaces we can assume, by what just proved, that $X_1, X_2, X_1 \times X_2$ are all wncRCD spaces. Then we notice that, much like in the metric case just considered, if $(\mathbb{R}^{N_1}, d_E, \vartheta_{N_1}[X_1](x_1)\mathcal{L}^{N_1}, 0)$ (resp. $(\mathbb{R}^{N_2}, d_E, \vartheta_{N_2}[X_2](x_2)\mathcal{L}^{N_2}, 0)$) is the tangent space of X_1 (resp. X_2) at x_1 (resp. x_2), then $(\mathbb{R}^{N_1+N_2}, d_E, \vartheta_{N_1}(x_1)\vartheta_{N_2}(x_2)\mathcal{L}^{N_1+N_2}, 0)$ is the tangent space of $X_1 \times X_2$ at (x_1, x_2) . Hence taking into account the characterization of wncRCD spaces in point (iv) of Theorem 1.12 we deduce that

$$\vartheta_{N_1+N_2}[X_1 \times X_2](x_1, x_2) = \vartheta_{N_1}[X_1](x_1)\vartheta_{N_2}[X_2](x_2) \quad (\mathbf{m}_1 \times \mathbf{m}_2) - a.e. (x_1, x_2). \quad (2.27)$$

Hence if $\vartheta_{N_1}[X_1](x_1) = c > 0$ \mathbf{m}_1 -a.e. and $\vartheta_{N_2}[X_2](x_2) = c^{-1} > 0$ \mathbf{m}_2 -a.e. it trivially follows that $\vartheta_{N_1+N_2}[X_1 \times X_2] = 1$ a.e., thus showing that $X_1 \times X_2$ is ncRCD (by (iii) of Theorem

1.12 and (2.11)). Conversely, if the left-hand-side of (2.27) is a.e. equal to 1, then the identity (2.27) forces $\vartheta_{N_1}[X_1]$ and $\vartheta_{N_2}[X_2]$ to be a.e. constant and since the product of these constants must be 1 we must have $\vartheta_{N_1}[X_1](x_1) = c$ \mathbf{m}_1 -a.e. and $\vartheta_{N_2}[X_2](x_2) = c^{-1} > 0$ \mathbf{m}_2 -a.e. for some $c > 0$, which is the claim. \square

3 Non-collapsed spaces

3.1 Continuity of \mathcal{H}^N

In this section we prove the continuity of \mathcal{H}^N as stated in Theorem 1.3.

A key ingredient that we shall need is the ‘almost splitting via excess theorem’ proved by Mondino and Naber in [44, Theorem 5.1]: we shall present such result in the simplified form that we need referring to [44] for the more general statement.

Here and in the following for $p \in X$ we put $d_p(\cdot) := d(\cdot, p)$ and for $p, q \in X$ we put $e_{p,q} := d_p + d_q - d(p, q)$.

Theorem 3.1. *For every $k \in \mathbb{N}$, $N \in \mathbb{R}$, $1 \leq k \leq N$ and $\varepsilon \in (0, 1)$ there is $\delta_1 = \delta_1(\varepsilon, k, N) \leq 1$ such that the following holds.*

Assume that (X, d, \mathbf{m}) is an $\text{RCD}(-\delta_1, N)$ space with $\text{supp}(\mathbf{m}) = X$ and that there are points $\bar{x}, \{p_i, q_i\}_{1 \leq i \leq k}, \{p_i + p_j\}_{1 \leq i < j \leq k}$ in X with $d(p_i, \bar{x}), d(q_i, \bar{x}), d(p_i + p_j, \bar{x}) \geq 1/\delta_1$ such that

$$\sum_{1 \leq i \leq k} \int_{B_R^X(\bar{x})} |De_{p_i, q_i}|^2 d\mathbf{m} + \sum_{1 \leq i < j \leq k} \int_{B_R^X(\bar{x})} \left| D \left(\frac{d_{p_i} + d_{p_j}}{\sqrt{2}} - d_{p_i + p_j} \right) \right|^2 d\mathbf{m} \leq \delta_1 \quad (3.1)$$

for all $1 \leq R \leq 1/\delta_1$.

Then there exists a metric space Y and a map $\phi : X \rightarrow Y$ such that if we define

$$u := (d_{p_1} - d_{p_1}(\bar{x}), \dots, d_{p_k} - d_{p_k}(\bar{x})) : X \rightarrow \mathbb{R}^k,$$

the map $U := (u, \phi) : X \rightarrow \mathbb{R}^k \times Y$ provides an ε -isometry of $B_1^X(\bar{x})$ to $B_1^{\mathbb{R}^k \times Y}((0, \phi(x)))$, i.e.:

$$\begin{aligned} \forall x, y \in B_1^X(\bar{x}) \text{ it holds} & \quad |d_X(x, y) - d_{\mathbb{R}^k \times Y}(U(x), U(y))| \leq \varepsilon, \\ \forall z \in B_1^{\mathbb{R}^k \times Y}((0, \phi(x))) \text{ there is } x \in B_1^X(\bar{x}) \text{ such that} & \quad d_{\mathbb{R}^k \times Y}(U(x), z) \leq \varepsilon. \end{aligned}$$

Furthermore if $k = N$ we can take Y to be a single point.

Very shortly and roughly said, the idea of the proof is the following: For given $\varepsilon > 0$ one picks a sequence $\delta_{1,n} \downarrow 0$ and a corresponding sequence of spaces X_n satisfying the assumptions for $\delta_1 = \delta_{1,n}$ pmGH-converging to a limit X . Then by an Ascoli-Arzelà-type argument, up to subsequences the corresponding functions $u_n : X_n \rightarrow \mathbb{R}^k$ converge to a limit $u : X \rightarrow \mathbb{R}^k$ and, this is the key point of the proof, thanks to (3.1) such limit map u is a metric submersion. The compactness (2.1) of the class of $\text{RCD}(-1, N)$ spaces then gives the conclusion (see [44, Theorem 5.1] for the details).

An important consequence of the above theorem is the following sort of ‘ ε -regularity’ result (see also [44, Theorem 6.8]) that we shall state for the case $k = N \in \mathbb{N}$ only; notice that, as discussed in [44], the map u_ε is $(1 + \varepsilon)$ -biLipschitz for arbitrary values of k , but in order to obtain the key inequality (3.3) the ‘maximality’ assumption $k = N \in \mathbb{N}$ is necessary (see in particular inequalities (3.14) and (3.15)).

Proposition 3.2. *For every $N \in \mathbb{N}$, $N \geq 1$ and $\varepsilon \in (0, 1)$ there is $\delta_2 = \delta_2(\varepsilon, N) > 0$ such that the following holds. Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(-\delta_2, N)$ space with $\text{supp}(\mathbf{m}) = X$ and $\bar{x} \in X$ such that*

$$\mathbf{d}_{\text{GH}}\left(\left(B_{1/\delta_2}^X(\bar{x}), \mathbf{d}\right), \left(B_{1/\delta_2}^{\mathbb{R}^N}(0), \mathbf{d}_{\mathbb{E}}\right)\right) < \delta_2. \quad (3.2)$$

Then there exists a Borel set $U_\varepsilon \subset B_1^X(\bar{x})$ and a $(1+\varepsilon)$ -biLipschitz map $u_\varepsilon : U_\varepsilon \rightarrow u_\varepsilon(U_\varepsilon) \subset \mathbb{R}^N$ such that

$$\mathcal{L}^N(u_\varepsilon(U_\varepsilon) \cap B_1^{\mathbb{R}^N}(0)) \geq (1 - \varepsilon) \mathcal{L}^N(B_1^{\mathbb{R}^N}(0)). \quad (3.3)$$

Proof We divide the proof in two steps:

Step 1: construction of U_ε , u_ε and $(1 + \varepsilon)$ -biLipschitz estimate This is the content of [44, Theorem 6.8], however since some aspects of the proof will be needed to obtain (3.3) we briefly recall the argument.

Step 1.1: basic ingredients We start observing that for any $R \geq 1$ and any $f \in \text{Lip}(X)$, it holds the simple inequality

$$\int_{B_R(\bar{x})} |\text{D}f|^2 \, \mathbf{d}\mathbf{m} \leq \|f\|_{L^\infty(B_{2R}(\bar{x}))} (\|\Delta f\|_{L^1(B_{2R}(\bar{x}))} + \mathbf{m}(B_{2R}(\bar{x})) \text{Lip}(f)), \quad (3.4)$$

as can be proved with an integration by parts (see e.g. [31] for all the relevant definitions and properties of integration by parts and Laplacian) starting from $\int_{B_R(\bar{x})} |\text{D}f|^2 \, \mathbf{d}\mathbf{m} \leq \int_X |\text{D}f|^2 \varphi \, \mathbf{d}\mathbf{m}$ for $\varphi := (1 - \mathbf{d}(\cdot, B_R(\bar{x})))^+$. (In fact one can easily get rid of the term $\text{Lip}(f)$ in the right hand side provided φ has bounded Laplacian. The existence of such cut-off functions - i.e. Lipschitz and with bounded Laplacian - in the context of Ricci-limit spaces has been proved in [16] and frequently used as important technical tool in [17, 18, 19, 20, 21, 22]; their existence in the RCD setting has been proved in [8, Lemma 6.7] and [34, Theorem 3.12], see also [44, Lemma 3.1].).

A second ingredient is the Laplacian comparison estimate for the distance function (see [31]) which ensures that on an $\text{RCD}(-1, N)$ space, if $\mathbf{d}(p, \bar{x}) \geq 8R \geq 8$ then $\Delta \mathbf{d}_p \leq C(N, R)$ on $B_{4R}(\bar{x})$ (this should be understood as an inequality between measures, but for the purpose of this outline let us think at $\Delta \mathbf{d}_p$ as a function). From this bound it is not hard to get the estimate

$$\|\Delta \mathbf{d}_p\|_{L^1(B_{2R}(\bar{x}))} \leq C(N, R) \mathbf{m}(B_{4R}(\bar{x})) \quad \forall p \notin B_{8R}(\bar{x}) \quad (3.5)$$

(this is in fact reverse engineering: in [31] the fact that $\Delta \mathbf{d}_p$ is a measure is obtained by proving an inequality like (3.5) with the total variation norm in place of the L^1 one).

Step 1.2: geometric argument Let $\tilde{\eta}, \delta_1 \in (0, 1)$ and notice - by direct simple computation - that there exists $R \geq \frac{32}{\delta_1}$ such that

$$\sup_{x \in B_{8/\delta_1}^{\mathbb{R}^N}(0)} (|x - Re_i| + |x + Re_i| - 2R) \leq \tilde{\eta} \quad \forall i = 1, \dots, N.$$

Hence if (3.2) is satisfied for some $\delta_2 \leq \min\{\tilde{\eta}, \frac{1}{R}\}$, letting $p_i, q_i \in X$ be points corresponding to $Re_i, -Re_i$ respectively in the δ_2 -isometry we obtain

$$\|e_{p_i, q_i}\|_{L^\infty(B_{4/\delta_1}^X(\bar{x}))} \leq 3\delta_2 + \tilde{\eta} \leq 4\tilde{\eta} \quad \forall i = 1, \dots, N.$$

Noticing that $\mathbf{d}(p_i, \bar{x}), \mathbf{d}(q_i, \bar{x}) \geq R - \delta_2 \geq \frac{16}{\delta_1}$, from (3.5) we deduce that

$$\|\Delta e_{p_i, q_i}\|_{L^1(B_{4/\delta_1}(\bar{x}))} \leq C(\delta_1, N) \mathbf{m}(B_{8/\delta_1}(\bar{x})) \quad \forall i = 1, \dots, N$$

and since e_{p_i, q_i} is 2-Lipschitz, these last two bounds, (3.4) and (1.4) imply that

$$\int_{B_{2/\delta_1}} |De_{p_i, q_i}|^2 \, \mathbf{d}\mathbf{m} \leq 4\tilde{\eta} C(\delta_1, N).$$

The same line of thoughts yields the analogous inequality for the function $\frac{\mathbf{d}_{p_i + p_j}}{\sqrt{2}} - \mathbf{d}_{p_i + p_j}$ for properly chosen points $p_i + p_j \in X$ with $\mathbf{d}(p_i + p_j, \bar{x}) \geq 16/\delta_1$.

Now we fix $\varepsilon \in (0, 1)$, pick $\delta_1 = \delta_1(\varepsilon, N)$ given by Theorem 3.1, let $\eta \ll \varepsilon$ be a small parameter to be fixed later and notice that we can rephrase what we just proved as: there exists $\delta_2 = \delta_2(\delta_1, \eta, N) \leq 1$ such that if (3.2) is satisfied for such δ_2 - which we shall hereafter assume -, then we can find points $\{p_i, q_i\}_{1 \leq i \leq N}$, $\{p_i + p_j\}_{1 \leq i < j \leq N}$ with $\mathbf{d}(p_i, \bar{x}), \mathbf{d}(q_i, \bar{x}), \mathbf{d}(p_i + p_j, \bar{x}) \geq 2/\delta_1$ and such that

$$\sum_{1 \leq i \leq N} \int_{B_{2/\delta_1}^X(\bar{x})} |De_{p_i, q_i}|^2 \, \mathbf{d}\mathbf{m} + \sum_{1 \leq i < j \leq N} \int_{B_{2/\delta_1}^X(\bar{x})} \left| \mathbf{D} \left(\frac{\mathbf{d}_{p_i} + \mathbf{d}_{p_j}}{\sqrt{2}} - \mathbf{d}_{p_i + p_j} \right) \right|^2 \, \mathbf{d}\mathbf{m} \leq \eta^2. \quad (3.6)$$

Step 1.3: use of the maximal function Consider the function $f : X \rightarrow \mathbb{R}$ defined as

$$f(x) := \sum_{1 \leq i \leq N} |De_{p_i, q_i}|^2 + \sum_{1 \leq i < j \leq N} \left| \mathbf{D} \left(\frac{\mathbf{d}_{p_i} + \mathbf{d}_{p_j}}{\sqrt{2}} - \mathbf{d}_{p_i + p_j} \right) \right|^2 \quad (3.7)$$

and its maximal function $M : B_1^X(\bar{x}) \rightarrow \mathbb{R}$ given by

$$M(x) = \sup_{0 < R < 1/\delta_1} \int_{B_R^X(x)} f(x) \, \mathbf{d}\mathbf{m}.$$

We put

$$U := \{x \in B_1^X(\bar{x}) : M(x) \leq \eta\}$$

and note that the left hand side of (3.1) is invariant under rescaling of the distance (essentially because it holds $|\mathbf{D}_r \mathbf{d}_p^r| = |\mathbf{D} \mathbf{d}_p|$, where $\mathbf{d}_p^r = \mathbf{d}_p/r$ and $|\mathbf{D}_r \cdot|$ is the weak upper gradient computed with respect to the metric measure space $(X, \mathbf{d}/r, \mathbf{m})$). Hence for $x \in U$ we can apply Theorem 3.1 to the scaled space $(X, \mathbf{d}/r, \mathbf{m})$ for $r \in (0, 1)$ (notice that since we multiplied the distance by a factor > 1 , the space is ‘flatter’ than the original one and in any case still $\text{RCD}(-\delta_2, N)$, in particular 3.1 is applicable) - to infer that, provided η , and hence δ_2 , are sufficiently small, the map

$$u := (\mathbf{d}_{p_1} - \mathbf{d}_{p_1}(\bar{x}), \dots, \mathbf{d}_{p_N} - \mathbf{d}_{p_N}(\bar{x}))$$

is an ε -isometry at every scale in the range $(0, 1)$ around points on U : this is sufficient to prove that it is $(1 + \varepsilon)$ bi-Lipschitz when restricted to U , see the proof of [44, Theorem 6.8] for the details.

Step 2: proof of estimate (3.3) From the trivial set identity

$$B_1^{\mathbb{R}^N}(0) \cap u(U) = \left(B_1^{\mathbb{R}^N}(0) \setminus (B_1^{\mathbb{R}^N}(0) \setminus u(B_1^X(\bar{x}))) \right) \setminus (u(B_1^X(\bar{x})) \setminus u(U))$$

we deduce that

$$\mathcal{L}^N(B_1^{\mathbb{R}^N}(0) \cap u(U)) \geq \mathcal{L}^N(B_1^{\mathbb{R}^N}(0)) - \mathcal{L}^N(B_1^{\mathbb{R}^N}(0) \setminus u(B_1^X(\bar{x}))) - \mathcal{L}^N(u(B_1^X(\bar{x})) \setminus u(U)). \quad (3.8)$$

Step 2.1: estimate of the size of $u(B_1^X(\bar{x})) \setminus u(U)$ Recall that since $\delta_2 \leq 1$, the space X is $\text{RCD}(-1, N)$ and thus $\mathfrak{m}|_{B_{2/\delta_1}^X(\bar{x})}$ is doubling with a constant depending only on δ_1 and N . Hence according to the weak L^1 estimates for the maximal function we have

$$\frac{\mathfrak{m}(B_1^X(\bar{x}) \setminus U)}{\mathfrak{m}(B_1^X(\bar{x}))} \leq \frac{C(\delta_1, N)}{\eta \mathfrak{m}(B_1^X(\bar{x}))} \int_{B_{2/\delta_1}^X(\bar{x})} f(x) \, \mathrm{d}\mathfrak{m} \stackrel{(3.6)}{\leq} C(\delta_1, N) \eta \frac{\mathfrak{m}(B_{2/\delta_1}^X(\bar{x}))}{\mathfrak{m}(B_1^X(\bar{x}))} \leq C(\delta_1, N) \eta. \quad (3.9)$$

Now notice that u is \sqrt{N} -Lipschitz so that since $u(B_1^X(\bar{x})) \setminus u(U) \subset u(B_1^X(\bar{x}) \setminus U)$ we have

$$\mathcal{L}^N(u(B_1^X(\bar{x})) \setminus u(U)) \leq \mathcal{L}^N(u(B_1^X(\bar{x}) \setminus U)) \leq (\sqrt{N})^N \mathcal{H}^N(B_1^X(\bar{x}) \setminus U)$$

and therefore using (2.19) with $R = 1$ we get

$$\mathcal{L}^N(u(B_1^X(\bar{x})) \setminus u(U)) \leq C(K, N) \frac{\mathfrak{m}(B_1^X(\bar{x}) \setminus U)}{\mathfrak{m}(B_1^X(\bar{x}))} \stackrel{(3.9)}{\leq} C(\delta_1, K, N) \eta. \quad (3.10)$$

Step 2.2: estimate of the size of $B_1^{\mathbb{R}^N}(0) \setminus u(B_1^X(\bar{x}))$ Since $B_{1/2}^{\mathbb{R}^N}(-\frac{1}{2}e_1) \subset B_1^{\mathbb{R}^N}(0) \setminus B_1^{\mathbb{R}^N}(e_1)$ we have that

$$\text{if } S \subset B_{\sqrt{N}}^{\mathbb{R}^N}(0) \text{ is } \frac{1}{2}\text{-dense in } B_1^{\mathbb{R}^N}(0), \text{ then } S \cap B_1^{\mathbb{R}^N}(e_1) \neq \emptyset. \quad (3.11)$$

Then let $\bar{\delta} := \delta_1(\frac{1}{2}, N, N)$ be given by Theorem 3.1 and put

$$\lambda = \lambda(N) := \frac{\sqrt{N} + 1}{\bar{\delta}}.$$

Apply Lemma 3.3 below to such λ and to the open set $A := B_1^{\mathbb{R}^N}(0) \setminus u(\bar{B}_1^X(\bar{x}))$ (notice that since $u(\bar{x}) = 0$ we have $A \neq B_1^{\mathbb{R}^N}(0)$) to find balls $\{B_{r_k}^{\mathbb{R}^N}(y_k)\}_{k=1}^M$ satisfying the properties (i), ..., (iv) stated in the lemma. By property (ii), for every $k = 1, \dots, M$ there exists $z_k \in \bar{B}_1^X(\bar{x})$ such that $u(z_k) \in \partial B_{r_k}^{\mathbb{R}^N}(y_k)$. Moreover by point (i) and the \sqrt{N} -Lipschitzianity of u ,

$$u(B_{r_k}^X(z_k)) \subset B_{\sqrt{N}r_k}^{\mathbb{R}^N}(u(z_k)) \setminus B_{r_k}^{\mathbb{R}^N}(y_k). \quad (3.12)$$

We now claim that there exists a radius $\rho_k \in (r_k, r_k/\bar{\delta})$ such that

$$\int_{B_{\rho_k}(z_k)} f(x) \, \mathrm{d}\mathfrak{m} \geq \bar{\delta} \quad (3.13)$$

where f is the function defined in (3.7) (recall that the points $p_i, q_i, p_i + p_j$ have been already fixed in Step 1.2). Indeed if (3.13) fails we can apply the scaled version of Theorem 3.1 to deduce that $u(B_{r_k}^X(z_k))$ is $\frac{1}{2}r_k$ -dense in $B_{r_k}^{\mathbb{R}^N}(u(z_k))$, a contradiction with (the scaled version of) (3.11) and (3.12). Using again that $\mathfrak{m}|_{B_{2/\delta_1}^X(\bar{x})}$ is doubling with a constant depending only on δ_1 and N , and that $\rho_k \leq r_k/\bar{\delta} \leq 1/\bar{\delta}$ we see that

$$\mathfrak{m}(B_{2/\delta_1}^X(\bar{x})) \leq C(\delta_1, N) \frac{\mathfrak{m}(B_{\rho_k}^X(z_k))}{\rho_k^N} \quad \text{for every } k = 1, \dots, M \quad (3.14)$$

and therefore from the fact that $r_k \leq \rho_k$ we obtain

$$\sum_{k=1}^M r_k^N \leq \sum_{k=1}^M \rho_k^N \stackrel{(3.14)}{\leq} \frac{C(\delta_1, N)}{\mathfrak{m}(B_{2/\delta_1}^X(\bar{x}))} \sum_{k=1}^M \mathfrak{m}(B_{\rho_k}^X(z_k)) \stackrel{(3.13)}{\leq} \frac{C(\delta_1, N)}{\bar{\delta} \mathfrak{m}(B_{2/\delta_1}^X(\bar{x}))} \sum_{k=1}^M \int_{B_{\rho_k}^X(z_k)} f \, \mathfrak{d}\mathfrak{m}. \quad (3.15)$$

Since u is \sqrt{N} -Lipschitz and $u(z_k) \in \partial B_{r_k}^{\mathbb{R}^N}(y_k)$ we have

$$u(B_{\rho_k}^X(z_k)) \subset u(B_{r_k/\bar{\delta}}^X(z_k)) \subset B_{\sqrt{N}r_k/\bar{\delta}}^{\mathbb{R}^N}(u(z_k)) \subset B_{(\sqrt{N}+1)r_k/\bar{\delta}}^{\mathbb{R}^N}(y_k) = B_{\lambda(N)r_k}^{\mathbb{R}^N}(y_k).$$

This inclusion and the fact that, by property (iii) in Lemma 3.3, the balls $B_{\lambda(N)r_k}^{\mathbb{R}^N}(y_k)$ are disjoint, grant that the balls $B_{\rho_k}^X(z_k) \subset B_{2/\delta_1}^X(\bar{x})$ are disjoint as well. Hence from (3.15) we get

$$\sum_{k=1}^M r_k^N \leq \frac{C(\delta_1, N)}{\bar{\delta} \mathfrak{m}(B_{2/\delta_1}^X(\bar{x}))} \int_{B_{2/\delta_1}^X(\bar{x})} f \, \mathfrak{d}\mathfrak{m} \stackrel{(3.6)}{\leq} \frac{C(\delta_1, N)}{\bar{\delta}} \eta^2$$

which together with property (iv) in Lemma 3.3 gives

$$\mathcal{L}^N(B_1^{\mathbb{R}^N} \setminus u(\overline{B_1^X(\bar{x})})) \leq C(\lambda(N), N) \frac{C(\delta_1, N)}{\bar{\delta}} \eta^2.$$

The conclusion comes plugging this bound and (3.10) into (3.8) and by picking η , and thus δ_2 , sufficiently small w.r.t. ε . \square

In the proof of the above proposition we have used the following covering Lemma from [22], we report here its simple proof for the sake of completeness.

Lemma 3.3. *Let $N \in \mathbb{N}$, $N \geq 1$, and $\lambda \geq 1$. Then there exists a constant $C = C(\lambda, N)$ such that the following holds.*

For every $A \subsetneq B_1^{\mathbb{R}^N}(0)$ open there exists a finite family of balls $\{B_{r_k}^{\mathbb{R}^N}(y_k)\}_{k=1}^M$, such that

- (i) $B_{r_k}^{\mathbb{R}^N}(y_k) \subset A$ for every k ,
- (ii) $(\partial B_{r_k}^{\mathbb{R}^N}(y_k) \cap \partial A) \setminus \partial B_1^{\mathbb{R}^N}(0) \neq \emptyset$ for every k ,
- (iii) The family $\{B_{\lambda r_k}^{\mathbb{R}^N}(y_k)\}_{k=1}^M$ is disjoint,
- (iv) It holds

$$\mathcal{L}^N(A) \leq C(\lambda, N) \sum_{k=1}^M r_k^N. \quad (3.16)$$

Proof We claim that if $B \subset A$ is a ball with $\bar{B} \subset B_1(0)$, then there exists another ball B' such that

$$B \subset B' \subset A \quad \text{and} \quad (\partial B' \cap \partial A) \setminus \partial B_1^{\mathbb{R}^N}(0) \neq \emptyset. \quad (3.17)$$

Indeed, let $B = B_r(v)$, put $B_t := B_{(1-t)r+t}(tv)$, notice that the family $(0, 1) \ni t \mapsto B_t$ is increasing and that $\bar{B}_t \subset B_1(0)$ for every $t \in [0, 1)$. Since $\cup_{t \in [0, 1)} B_t = B_1(0)$ and A is strictly included in $B_1(0)$, by a simple compactness argument we find a least $t_0 \in [0, 1)$ such that $\bar{B}_{t_0} \cap (\mathbb{R}^N \setminus A) \neq \emptyset$ and since $\bar{B}_{t_0} \subset B_1(0)$ we must also have $\bar{B}_{t_0} \cap (B_1(0) \setminus A) \neq \emptyset$. To conclude the proof of the claim simply notice that from the trivial identity $B_{t_0} = \cup_{t \in [0, t_0)} \bar{B}_t$ and the minimality of t_0 we have $B_{t_0} \subset A$, hence $B' := B_{t_0}$ does the job.

Now let $K \subset A$ be compact so that $\mathcal{L}^N(A) \leq 2\mathcal{L}^N(K)$ and, by compactness, find a finite family of balls $\{B_{r_j}^{\mathbb{R}^N}(y_j)\}_{j=1}^L$ with closure included in A and covering K . Up to replace each of these balls with the corresponding one B' as in (3.17), we can assume that this family satisfies (i), (ii). We shall now build a subfamily for which (iii), (iv) also hold. Up to reordering we can assume that $r_1 \geq \dots \geq r_L$. Then put $j_1 := 1$ and define recursively

$$j_n := \text{least index } j \text{ such that } B_{\lambda r_j}^{\mathbb{R}^N}(y_j) \cap \bigcup_{i=1}^{n-1} B_{\lambda r_{j_i}}^{\mathbb{R}^N}(y_{j_i}) = \emptyset.$$

Since the original family was finite, this process ends at some step M and, by construction, the family $\{B_{r_{j_i}}^{\mathbb{R}^N}(y_{j_i})\}_{i=1}^M$ fulfils (i), (ii), (iii). Also, by construction for every $j > 1$ there is $i \in \{1, \dots, M\}$ such that $r_{j_i} \geq r_j$ and $B_{\lambda r_j}^{\mathbb{R}^N}(y_j) \cap B_{\lambda r_{j_i}}^{\mathbb{R}^N}(y_{j_i}) \neq \emptyset$, hence $B_{\lambda r_j}^{\mathbb{R}^N}(y_j) \subset B_{3\lambda r_{j_i}}^{\mathbb{R}^N}(y_{j_i})$ and thus

$$K \subset \bigcup_{j=1}^L B_{r_j}^{\mathbb{R}^N}(y_j) \subset \bigcup_{j=1}^L B_{\lambda r_j}^{\mathbb{R}^N}(y_j) \subset \bigcup_{i=1}^M B_{3\lambda r_{j_i}}^{\mathbb{R}^N}(y_{j_i}),$$

so that (3.16) holds with $C(\lambda, N) := 2\omega_N(3\lambda)^N$. \square

We can now prove the continuity of \mathcal{H}^N under a uniform Riemannian-curvature-dimension condition:

Proof of Theorem 1.3

Set up If $N \notin \mathbb{N}$, Theorem 2.13 implies that $\mathcal{H}^N(X) = 0$ for any RCD(K, N) space X , hence in this case there is nothing to prove. We shall therefore assume $N \in \mathbb{N}$.

Let $(Z_n) \subset \mathbb{B}_{K, N, R}$ be GH-converging to some limit $Z_\infty \in \mathbb{B}_{K, N, R}$ and for each $n \in \mathbb{N}$, let $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n)$ be an RCD(K, N) space such that $Z_n = \bar{B}_R^{X_n}(x_n)$. Up to replace \mathbf{m}_n with $\mathbf{m}_n/\mathbf{m}_n(Z_n)$ we can, and will, assume that $\mathbf{m}_n(Z_n) = 1$ for every $n \in \mathbb{N}$. Then by the compactness of the class of RCD(K, N) spaces (2.1), up to pass to a subsequence, not relabeled, we have $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n) \xrightarrow{pmGH} (X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, x_\infty)$ for some pointed RCD(K, N) space X_∞ . It is then clear that $Z_\infty = \bar{B}_R^{X_\infty}(x_\infty)$. To conclude it is now sufficient to prove that $\mathcal{H}^N(Z_n) \rightarrow \mathcal{H}^N(Z_\infty)$, because such continuity property is independent on the subsequence chosen.

Upper semicontinuity Let $(X, \mathbf{d}, \mathbf{m})$ be a generic RCD(K, N) space with $\text{supp}(\mathbf{m}) = X$. We claim that

$$\mathcal{H}^N = \frac{1}{\vartheta_N[X]} \mathbf{m} \tag{3.18}$$

where it is intended that $\frac{1}{\vartheta_N[X]}(x) = 0$ if $\vartheta_N[X](x) = \infty$. To see this start observing that the Bishop-Gromov inequality (1.4) grants that

$$\vartheta_N[X](x) \geq C(K, N)\mathbf{m}(B_1(x)), \tag{3.19}$$

so that $\frac{1}{\vartheta_N[X]} \in L_{loc}^1(X, \mathbf{m})$ and the right hand side of (3.18) defines a Radon measure. Then, in the notation of Theorem 2.13, by (2.23) for $k = N$ it follows that equality holds in (3.18) if we restrict both sides to $\cup_j U_j^N$, so that to conclude it is sufficient to show that on $X \setminus (\cup_j U_j^N)$ both sides of (3.18) are zero. The fact that $\mathcal{H}^N(X \setminus (\cup_j U_j^N)) = 0$ is a trivial consequence of Proposition 2.12 and Theorem 2.13, while (2.22) and (2.23) imply that for $k < N$, $k \in \mathbb{N}$ we have $\vartheta_N[X] = \infty$ \mathbf{m} -a.e. on $\cup_j U_j^k$. Hence our claim (3.18) is proved.

Now we apply Lemma 2.5 to the functions

$$f_n := -\frac{1}{\vartheta_N[\mathbb{X}_n]} \chi_{\bar{B}_R^{X_n}(x_n)}, \quad n \in \mathbb{N} \cup \{\infty\},$$

point (i) of Lemma 2.2 grants that (2.10) is satisfied, while from (3.19) and the assumption $\mathfrak{m}_n(Z_n) = 1$ it easily follows that the f_n 's are uniformly bounded from below by a continuous function with bounded support. Hence the conclusion of Lemma 2.5 gives

$$-\mathcal{H}^N(Z_\infty) \stackrel{(3.18)}{=} \int f_\infty \, d\mathfrak{m}_\infty \leq \varliminf_{n \rightarrow \infty} \int f_n \, d\mathfrak{m}_n \stackrel{(3.18)}{=} \varliminf_{n \rightarrow \infty} -\mathcal{H}^N(Z_n)$$

and thus the desired upper semicontinuity:

$$\overline{\lim}_{n \rightarrow \infty} \mathcal{H}^N(Z_n) \leq \mathcal{H}^N(Z_\infty).$$

Lower semicontinuity Theorem 2.13 ensures that for \mathcal{H}^N -a.e. $x \in Z_\infty$ the metric tangent space of \mathbb{X}_∞ at x is \mathbb{R}^N .

Now fix $\varepsilon > 0$, let $\delta_2 := \delta_2(\varepsilon, N)$ be given by Proposition 3.2 and notice that what we just said grants that for \mathcal{H}^N -a.e. $x \in Z_\infty$ there exists $\bar{r} = \bar{r}(x)$ such that for every $r \in (0, \bar{r})$ we have

$$d_{\text{GH}}(B_{r/\delta_2}^{X_\infty}(x), B_{r/\delta_2}^{\mathbb{R}^N}(0)) \leq r\delta_2,$$

thus, since Proposition 2.12 grants that $\mathcal{H}^N(Z_\infty) < \infty$, we can use Vitali's covering lemma (see e.g. [10, Theorem 2.2.2]) to find a finite number of points $y_{\infty,i} \in B_1^{X_\infty}(x_\infty)$ and radii $r_i > 0$, $i = 1, \dots, M$, such that

$$d_{\text{GH}}(B_{r_i/\delta_2}^{X_\infty}(y_{\infty,i}), B_{r_i/\delta_2}^{\mathbb{R}^N}(0)) \leq r_i\delta_2 \quad \forall i, \quad (3.20a)$$

$$\bar{B}_{r_i}^{X_\infty}(y_{\infty,i}) \cap \bar{B}_{r_{i'}}^{X_\infty}(y_{\infty,i'}) = \emptyset \quad \forall i \neq i', \quad (3.20b)$$

$$d_\infty(y_{\infty,i}, x_\infty) + r_i < 1 \quad \forall i, \quad (3.20c)$$

$$\mathcal{H}^N(B_1^{X_\infty}(x_\infty)) \leq \varepsilon + \omega_N \sum_i r_i^N. \quad (3.20d)$$

For each i find a sequence $y_{n,i} \xrightarrow{GH} y_{\infty,i}$ (recall (2.2)) and notice that there is $\bar{n} \in \mathbb{N}$ such that for every $n \geq \bar{n}$ properties (3.20a), (3.20b), (3.20c) hold with $y_{n,i}$ and x_n in place of $y_{\infty,i}$, x_∞ respectively for any i . In particular, from (3.20a) for $y_{n,i}$ we can apply the scaled version of Proposition 3.2 to deduce that

$$\mathcal{H}^N(B_{r_i}^{X_n}(y_{n,i})) \geq (1 - \varepsilon)\omega_N r_i^N \quad \forall i \quad (3.21)$$

and since (3.20b), (3.20c) for the $y_{n,i}$'s ensure that the balls $B_{r_i}^{X_n}(y_{n,i})$, $i = 1, \dots, M$, are disjoint and contained in $B_1^{X_n}(x_n)$ for every $n \geq \bar{n}$ we deduce that

$$\mathcal{H}^N(B_1^{X_n}(x_n)) \geq \sum_i \mathcal{H}^N(B_{r_i}^{X_n}(y_{n,i})) \stackrel{(3.21)}{\geq} (1 - \varepsilon)\omega_N \sum_i r_i^N \quad \forall n \geq \bar{n}.$$

Hence from (3.20d) we obtain

$$\varliminf_{n \rightarrow \infty} \mathcal{H}^N(B_1^{X_n}(x_n)) \geq (1 - \varepsilon)(\mathcal{H}^N(B_1^{X_\infty}(x_\infty)) - \varepsilon)$$

and, recalling (2.20), we conclude by the arbitrariness of $\varepsilon > 0$. \square

3.2 Dimension gap

In this section we shall prove the ‘dimension gap’ Theorem 1.4 and in doing so we will closely follow the arguments in [17, Section 3].

The crucial part of argument, provided in Proposition 3.5, is the proof that the Hausdorff dimension of the set of points for which no tangent space splits off a line is at most $N - 1$. To clarify the structure of the proof, we isolate in the following lemma the measure-theoretic argument which will ultimately lead to such estimate on the dimension (see [17, Proposition 3.2]):

Lemma 3.4. *Let X be an $\text{RCD}(K, N)$ space and $\Omega_\eta \subset X$ Borel subsets indexed by a parameter $\eta > 0$ such that it holds*

$$\mathbf{m}(\Omega_\eta) \leq C\eta \quad \forall \eta \in (0, c) \quad (3.22)$$

for some $C, c > 0$. For $\tau \in (0, 1)$ consider the sets

$$\Omega_{\tau, \eta} := \{x \in \Omega_\eta : \mathbf{d}(x, X \setminus \Omega_\eta) > \tau\eta\}. \quad (3.23)$$

Then

$$\dim_{\mathcal{H}} \left(\bigcap_{j \in \mathbb{N}} \bigcup_{k \geq j} \Omega_{\tau, 2^{-k}} \right) \leq N - 1.$$

Proof For given $\bar{x} \in X$, $R > 0$ let $\Omega_{R, \eta} := \Omega_\eta \cap B_R(\bar{x})$ and $\Omega_{\tau, R, \eta}$ be defined as in (3.23) with $\Omega_{R, \eta}$ in place of Ω_η . Then for every $\tau, \eta \in (0, 1)$ the definition easily gives $\Omega_{\tau, \eta} \cap B_{R-1}(\bar{x}) \subset \Omega_{\tau, R, \eta}$ and therefore

$$\dim_{\mathcal{H}} \left(\bigcap_{j \in \mathbb{N}} \bigcup_{k \geq j} \Omega_{\tau, 2^{-k}} \right) = \lim_{R \rightarrow \infty} \dim_{\mathcal{H}} \left(\bigcap_{j \in \mathbb{N}} \bigcup_{k \geq j} \Omega_{\tau, 2^{-k}} \cap B_{R-1}(\bar{x}) \right) \leq \lim_{R \rightarrow \infty} \dim_{\mathcal{H}} \left(\bigcap_{j \in \mathbb{N}} \bigcup_{k \geq j} \Omega_{\tau, R, 2^{-k}} \right).$$

Hence up to replacing Ω_η with $\Omega_\eta \cap B_R(\bar{x})$ and then sending $R \uparrow \infty$ we can, and will, assume that $\Omega_\eta \subset B_R(\bar{x})$ for every $\eta \in (0, 1)$.

Now let $x_1, \dots, x_n \in \Omega_{\tau, \eta}$ be with $\mathbf{d}(x_i, x_j) \geq \tau\eta$ and denote by $C' = C(K, N, R, \bar{x})$ a constant depending only on K, N, R, \bar{x} (and thus independent on τ, η) whose values might change in the various instances it appears: since the balls $B_{\tau\eta/2}(x_i)$ are disjoint and contained in Ω_η we have

$$n \stackrel{(1.4)}{\leq} C' \sum_{i=1}^n \mathbf{m}(B_1(x_i)) \stackrel{(1.4)}{\leq} C'(\tau\eta)^{-N} \mathbf{m} \left(\bigcup_{i=1}^n B_{\frac{\tau\eta}{2}}(x_i) \right) \leq C'(\tau\eta)^{-N} \mathbf{m}(\Omega_\eta) \stackrel{(3.22)}{\leq} CC'(\tau\eta)^{-N} \eta.$$

If the family $\{x_1, \dots, x_n\}$ is maximal we have $\Omega_{\tau, \eta} \subset \bigcup_{i=1}^n B_{\tau\eta}(x_i)$ and thus for any $\varepsilon > 0$ the above implies

$$\mathcal{H}_{2\tau\eta}^{N-1+\varepsilon}(\Omega_{\tau, \eta}) \leq CC'(\tau\eta)^{-N} \eta (2\tau\eta)^{N-1+\varepsilon} \leq CC' \tau^{\varepsilon-1} \eta^\varepsilon. \quad (3.24)$$

Hence for any $j \in \mathbb{N}$ we have

$$\begin{aligned} \mathcal{H}_{2\tau 2^{-j}}^{N-1+\varepsilon} \left(\bigcap_{j' \in \mathbb{N}} \bigcup_{k \geq j'} \Omega_{\tau, 2^{-k}} \right) &\leq \mathcal{H}_{2\tau 2^{-j}}^{N-1+\varepsilon} \left(\bigcup_{k \geq j} \Omega_{\tau, 2^{-k}} \right) \leq \sum_{k \geq j} \mathcal{H}_{2\tau 2^{-k}}^{N-1+\varepsilon}(\Omega_{\tau, 2^{-k}}) \\ &\stackrel{\text{by (3.24)}}{\leq} \sum_{k \geq j} CC' \tau^{\varepsilon-1} 2^{-\varepsilon k} = CC' \tau^{\varepsilon-1} 2^{\varepsilon(1-j)} \end{aligned}$$

and letting $j \uparrow \infty$ we conclude. \square

Let us now give few definitions, following [17, Definition 2.10]. For a given $\text{RCD}(K, N)$ space X with measure having full support we define

$$\mathcal{E}_1(X) := \{x \in X : \text{every metric tangent space at } x \text{ splits off a line}\}$$

where we say that a metric measure space (X, d, \mathbf{m}) splits off a line provided X is isomorphic to the product of another metric measure space (X', d', \mathbf{m}') and the real line, i.e. there is a measure preserving isometry $\Phi : X' \times \mathbb{R} \rightarrow X$, where the measure on $X' \times \mathbb{R}$ is the product of \mathbf{m}' and the Lebesgue measure and the distance is given by

$$d_{X' \times \mathbb{R}}^2((x, t), (y, s)) := d'(x, y)^2 + |t - s|^2, \quad \forall x, y \in X', t, s \in \mathbb{R}.$$

Also, for given $\bar{x} \in X$ we put

$$\mathcal{E}_{1, \bar{x}}(X) := \left\{ x \neq \bar{x} : \begin{array}{l} \forall \varepsilon, \varepsilon' > 0 \text{ there exists } \bar{r} = \bar{r}(\varepsilon, \varepsilon', x) \text{ such that for every } r \in (0, \bar{r}) \\ \text{there is } y \in B_{\varepsilon r}(x) \text{ and a unit speed geodesic} \\ \gamma : [0, d(\bar{x}, y) + \frac{r}{\varepsilon'}] \rightarrow X \text{ such that } \gamma_0 = \bar{x} \text{ and } \gamma_{d(\bar{x}, y)} = y \end{array} \right\}.$$

We claim that

$$\mathcal{E}_{1, \bar{x}}(X) \subset \mathcal{E}_1(X) \quad \forall \bar{x} \in X. \quad (3.25)$$

Indeed, let $x \in \mathcal{E}_{1, \bar{x}}(X)$, $\varepsilon, \varepsilon' > 0$ and $\bar{r} = \bar{r}(\varepsilon, \varepsilon', x)$ as above. Then for $r \in (0, \bar{r})$ let y, γ as above and notice that the appropriate restriction of γ is a geodesic of length $\geq 2 \min\{\frac{r}{\varepsilon'}, d(x, \bar{x}) - \varepsilon r\}$ whose midpoint y has distance $\leq \varepsilon r$ from x . Hence in the rescaled space $(X_r, d_r) := (X, d/r)$ there is a geodesic of length $\geq 2 \min\{\frac{1}{\varepsilon'}, \frac{d(x, \bar{x})}{r} - \varepsilon\}$ whose midpoint has distance $\leq \varepsilon$ from x . Letting $r \downarrow 0$ we conclude by a compactness argument that on every tangent space at x there is a geodesic of length $\geq \frac{2}{\varepsilon}$ whose midpoint has distance $\leq \varepsilon$ from the origin, thus the arbitrariness of $\varepsilon, \varepsilon'$ and again a compactness argument ensure the existence of a line through the origin. Since every tangent space to an $\text{RCD}(K, N)$ space is a $\text{RCD}(0, N)$ space, the splitting theorem [28], [30] gives (3.25).

We then have the following:

Proposition 3.5. *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, N)$ space and $\bar{x} \in X$. Then*

$$\dim_{\mathcal{H}}(X \setminus \mathcal{E}_{1, \bar{x}}(X)) \leq N - 1 \quad (3.26)$$

and thus also

$$\dim_{\mathcal{H}}(X \setminus \mathcal{E}_1(X)) \leq N - 1. \quad (3.27)$$

Proof

Step 1: structure of the argument. From (3.25) and (3.26) the estimate (3.27) follows, hence we focus in proving (3.26). Since trivially $\dim_{\mathcal{H}}(\{\bar{x}\}) = 0 \leq N - 1$, to conclude it is sufficient to prove that for any $R > 2$ we have

$$\dim_{\mathcal{H}}\left(\text{Ann}_{R/2}(\bar{x}) \cap (X \setminus \mathcal{E}_{1, \bar{x}}(X))\right) \leq N - 1 \quad \text{where} \quad \text{Ann}_R(\bar{x}) := B_R(\bar{x}) \setminus B_{1/R}(\bar{x}). \quad (3.28)$$

Fix $\bar{x} \in X$ and for $\eta > 0$ define

$$\Omega_{\eta}^R := \{x \in \text{Ann}_R(\bar{x}) : \text{there is no unit speed geodesic } \gamma : [0, d(x, \bar{x}) + \eta] \rightarrow X \\ \text{such that } \gamma_0 = \bar{x}, \gamma_{d(x, \bar{x})} = x\}$$

and, for $\tau \in (0, 1)$, define $\Omega_{\tau, \eta}^R$ as in (3.23). We shall prove that for any $R > 2$ we have

$$\mathbf{m}(\Omega_{\eta}^R) \leq C\eta \quad \forall \eta \in (0, \frac{1}{R}) \quad (3.29)$$

for some $C = C(K, N, R, \bar{x})$ and

$$\text{Ann}_{R/2}(\bar{x}) \cap (\mathbf{X} \setminus \mathcal{E}_{1, \bar{x}}) \subset \bigcup_i \left(\text{Ann}_{R/2}(\bar{x}) \cap \bigcap_{j \geq i} \bigcup_{k \geq j} \Omega_{2^{-i}, 2^{-k}}^R \right). \quad (3.30)$$

Thanks to Lemma 3.4, these are sufficient to get (3.28) and the conclusion.

Step 2: proof of (3.29) We assume $\mathbf{m}(\text{Ann}_R(\bar{x})) > 0$ or there is nothing to prove, then we put $\mu_0 := \mathbf{m}(\text{Ann}_R(\bar{x}))^{-1} \mathbf{m}|_{\text{Ann}_R(\bar{x})}$, $\mu_1 := \delta_{\bar{x}}$ and let $\pi \in \mathcal{P}(C([0, 1], \mathbf{X}))$ be the only optimal geodesic plan from μ_0 to μ_1 (see [37]). Then from [37, Theorem 3.4] we know that $(e_t)_* \pi \ll \mathbf{m}$ for every $t \in [0, 1)$ and that for its density ρ_t it holds

$$\rho_t^{-\frac{1}{N}}(\gamma_t) \geq (\mathbf{m}(\text{Ann}_R(\bar{x})))^{\frac{1}{N}} \sigma_{K, N}^{(1-t)}(\mathbf{d}(\gamma_0, \gamma_1)) \quad \text{where} \quad \sigma_{K, N}^{(t)}(d) := \frac{\sinh(td\sqrt{|K|N})}{\sinh(d\sqrt{|K|N})}$$

π -a.e. γ , hence using the fact that $\sigma_{K, N}^{(t)}(d)$ is decreasing in d we deduce that

$$(e_t)_* \pi \leq \frac{1}{\mathbf{m}(\text{Ann}_R(\bar{x})) (\sigma_{K, N}^{(1-t)}(R))^N} \mathbf{m} \quad (3.31)$$

while the construction ensures that

$$(e_t)_* \pi \text{ is concentrated on } B_R(\bar{x}) \setminus \Omega_{\frac{t}{R}}^R \quad \forall t \in (0, 1). \quad (3.32)$$

Therefore for $\eta < \frac{1}{R}$ using the above with $t := \eta R$ we have

$$\begin{aligned} \mathbf{m}(B_R(\bar{x}) \setminus \Omega_{\eta}^R) &\stackrel{(3.31)}{\geq} \mathbf{m}(\text{Ann}_R(\bar{x})) (\sigma_{K, N}^{(1-\eta R)}(R))^N \pi \left(e_{\eta R}^{-1}(B_R(\bar{x}) \setminus \Omega_{\eta}^R) \right) \\ &\stackrel{(3.32)}{=} \mathbf{m}(\text{Ann}_R(\bar{x})) (\sigma_{K, N}^{(1-\eta R)}(R))^N \end{aligned} \quad (3.33)$$

and since $\Omega_{\eta}^R \subset \text{Ann}_R(\bar{x}) \subset B_R(\bar{x})$ yields $\Omega_{\eta}^R = \text{Ann}_R(\bar{x}) \setminus (B_R(\bar{x}) \setminus \Omega_{\eta}^R)$ in turn this gives

$$\mathbf{m}(\Omega_{\eta}^R) \stackrel{(3.33)}{\leq} \mathbf{m}(\text{Ann}_R(\bar{x})) \left(1 - (\sigma_{K, N}^{(1-\eta R)}(R))^N \right)$$

which, using the explicit expression of $\sigma_{K, N}^{(1-\eta R)}(R)$, gives our claim (3.29).

Step 3: proof of (3.30) We shall prove the equivalent inclusion

$$\text{Ann}_{R/2}(\bar{x}) \cap \mathcal{E}_{1, \bar{x}} \supset \text{Ann}_{R/2}(\bar{x}) \cap \bigcap_i \bigcup_j \bigcap_{k \geq j} (\mathbf{X} \setminus \Omega_{2^{-i}, 2^{-k}}^R). \quad (3.34)$$

Let x belonging to the right hand side of (3.34) and $\varepsilon, \varepsilon' > 0$. Pick $i \in \mathbb{N}$ such that $2^{-i} \leq \varepsilon \varepsilon'$ and find j such that $x \in \mathbf{X} \setminus \Omega_{2^{-i}, 2^{-k}}^R$ for every $k \geq j$. Up to increase j we can also assume that $2^{-j} < \frac{1}{R}$, then we put $\bar{r} := \varepsilon' 2^{-j}$ and for given $r \in (0, \bar{r})$ we let $k \geq j$ be such that $\varepsilon' 2^{-(k+1)} < r \leq \varepsilon' 2^{-k}$.

By definition of $\Omega_{2^{-i}, 2^{-k}}^R$ we know that there is $y \in \mathbf{X} \setminus \Omega_{2^{-k}}^R$ with $\mathbf{d}(x, y) \leq 2^{-i-k}$ and since $x \in \text{Ann}_{R/2}(\bar{x})$ the bound $2^{-i-k} \leq 2^{-j} < \frac{1}{R}$ grants that $y \in \text{Ann}_R(\bar{x}) \setminus \Omega_{2^{-k}}^R$. By definition of $\Omega_{2^{-k}}^R$ this means that there exists a unit speed geodesic starting from \bar{x} passing through y of length $\mathbf{d}(\bar{x}, y) + 2^{-k} \geq \mathbf{d}(\bar{x}, y) + \frac{r}{\varepsilon'}$ and since $\mathbf{d}(x, y) \leq 2^{-i-k} \leq \varepsilon \varepsilon'$, taking into account the arbitrariness of $\varepsilon, \varepsilon'$ we just showed that $x \in \mathcal{E}_{1, \bar{x}}$, which was our claim. \square

To get the proof of the dimension gap and, later, of the stratification result we shall use some facts about the $\mathcal{H}_\infty^\alpha$ pre-measure. Two direct consequences of the definitions are

$$\mathcal{H}_\infty^\alpha(A) = 0 \quad \Leftrightarrow \quad \mathcal{H}^\alpha(A) = 0 \quad (3.35)$$

and

$$\mathbf{d}_H(A_n, A) \rightarrow 0, \quad A \text{ compact} \quad \Rightarrow \quad \mathcal{H}_\infty^\alpha(A) \geq \overline{\lim}_{n \rightarrow \infty} \mathcal{H}_\infty^\alpha(A_n). \quad (3.36)$$

A subtler result relates the Hausdorff measure and the density of the ∞ -Hausdorff premeasure, see [38, Proposition 11.3] and [26, Theorem 2.10.17] for the proof:

Lemma 3.6 (Density of ∞ -Hausdorff premeasure). *Let (X, d) be a metric space, $\alpha \geq 0$ and $E \subset X$ a Borel set. Then for \mathcal{H}^α -a.e. $x \in E$ we have*

$$\overline{\lim}_{r \downarrow 0} \frac{\mathcal{H}_\infty^\alpha(E \cap B_r(x))}{r^\alpha} \geq 2^{-\alpha} \omega_\alpha.$$

A last property of Hausdorff measures that we shall use is the following, see [26, Theorem 2.10.45] for the proof:

$$\mathcal{H}^\alpha(X) = 0 \quad \Leftrightarrow \quad \mathcal{H}^{\alpha+1}(X \times \mathbb{R}) = 0, \quad (3.37)$$

valid for any $\alpha \geq 0$ and metric space X .

We can now prove Theorem 1.4:

Proof of Theorem 1.4 We shall assume that $\dim_{\mathcal{H}}(X) > N - 1$ and prove that $\mathcal{H}^N(X) > 0$, thanks to Proposition 2.12 this is sufficient to conclude. We start claiming that

there exists an iterated tangent space of X which is the Euclidean space \mathbb{R}^N . (3.38)

To prove this, let $\varepsilon > 0$ be so that $\mathcal{H}^{N-1+\varepsilon}(X) > 0$, then by Proposition 3.5 we also have $\mathcal{H}^{N-1+\varepsilon}(\mathcal{E}_1(X)) > 0$ and we can apply Lemma 3.6 to $E := \mathcal{E}_1(X)$ to find $x \in \mathcal{E}_1(X)$ and $r_n \downarrow 0$ such that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{H}_\infty^{N-1+\varepsilon}(\mathcal{E}_1(X) \cap B_{r_n}(x))}{r_n^{N-1+\varepsilon}} \geq 2^{-\alpha} \omega_\alpha. \quad (3.39)$$

Recalling (2.1), up to pass to a not relabeled subsequence we can assume that the spaces $(X_n, d_n, \mathbf{m}_n, x_n) := (X, d/r_n, \mathbf{m}/\mathbf{m}(B_{r_n}(x)), x)$ pmGH-converge to some $\text{RCD}(0, N)$ space $(Y^1, d_{Y^1}, \mathbf{m}_{Y^1}, o)$ as $n \uparrow \infty$. It is clear that after embedding all these spaces into a realization of such convergence we have that $\overline{B}_1^{d_n}(x_n) \rightarrow \overline{B}_1^{d_{Y^1}}(o)$ w.r.t. the Hausdorff distance and thus we have

$$\mathcal{H}_\infty^{N-1+\varepsilon}(\overline{B}_1^{Y^1}(o)) \stackrel{(3.36)}{\geq} \overline{\lim}_{n \rightarrow \infty} \mathcal{H}_\infty^{N-1+\varepsilon}(\overline{B}_1^{X_n}(x_n)) = \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{H}_\infty^{N-1+\varepsilon}(\overline{B}_{r_n}^X(x))}{r_n^{N-1+\varepsilon}} \stackrel{(3.39)}{>} 0$$

which, by (3.35), forces

$$\mathcal{H}^{N-1+\varepsilon}(Y^1) > 0. \quad (3.40)$$

By definition of $\mathcal{E}_1(X)$ the fact that $x \in \mathcal{E}_1(X)$ grants that $Y^1 = \mathbb{R} \times X^1$ and since Y^1 is $\text{RCD}(0, N)$, the splitting ensures that X^1 is either a point or $N \geq 2$ and X^1 is $\text{RCD}(0, N - 1)$.

If X^1 is a point we have $Y^1 = \mathbb{R}$ and (3.40) forces $N + \varepsilon \leq 2$. Since $N \in \mathbb{N}$ and $N \geq 1$, this implies $N = 1$.

If instead $N \geq 2$ we use (3.40) and (3.37) to deduce that $\mathcal{H}^{N-2+\varepsilon}(X^1) > 0$ and repeat the argument with the $\text{RCD}(0, N-1)$ space X^1 in place of X and $N-1$ in place of N .

Iterating this procedure after exactly N steps we arrive at a tangent space Y^N to the $\text{RCD}(0, 1)$ space X^{N-1} of the form $Y^N = \mathbb{R} \times X^N$, and since Y^N is itself $\text{RCD}(0, 1)$ this forces X^N to be a point.

In summary, we proved claim (3.38). Therefore by a diagonalization argument there is $\tilde{r}_n \downarrow 0$ and $(\tilde{x}_n) \subset X$ such that for the rescaled spaces $(\tilde{X}_n, \tilde{\mathbf{d}}_n) := (X, \mathbf{d}/\tilde{r}_n)$ it holds

$$\lim_{n \rightarrow \infty} \mathbf{d}_{\text{GH}}\left(\bar{B}_R^{\tilde{X}_n}(x_n), \bar{B}_R^{\mathbb{R}^N}(0)\right) = 0 \quad \forall R > 0. \quad (3.41)$$

Now we consider $\delta_2 = \delta_2(1/2, N)$ be given by Proposition 3.2 and pick $R := \frac{1}{\delta_2}$ in (3.41) above to conclude that for n sufficiently big we have $\tilde{r}_n^2 K \geq -\delta_2$ and (3.2) is satisfied for the $\text{RCD}(\tilde{r}_n^2 K, N)$ space $(\tilde{X}_n, \tilde{\mathbf{d}}_n, \mathbf{m})$. Fix such n and let $U \subset \tilde{X}_n$, $u : U \rightarrow u(U) \subset \mathbb{R}^N$ be given by Proposition 3.2 with $\varepsilon = \frac{1}{2}$. Notice that (3.3) forces $\mathcal{H}^N(u(U)) = \mathcal{L}^N(u(U)) > 0$ and since u is biLipschitz we also have that $U \subset \tilde{X}_n$ has positive \mathcal{H}^N measure in the space $(\tilde{X}_n, \tilde{\mathbf{d}}_n)$; given that \tilde{X}_n is obtained by rescaling of X , we see that $U \subset X$ also has positive \mathcal{H}^N measure in the space X , which gives the conclusion. \square

The proof of Corollary 1.5 can now be easily obtained:

Proof of Corollary 1.5 If N is integer the claim is a direct consequence of Proposition 2.12 (see also [50, Corollary 2.3]). Otherwise let $[N]_+ := \min\{n \in \mathbb{N} : N \leq n\}$ and notice that $N < [N]_+$ and that $(X, \mathbf{d}, \mathbf{m})$ is an $\text{RCD}(K, [N]_+)$ space. Thus by Theorem 1.4 and again Proposition 2.12 we conclude that $\dim_{\mathcal{H}}(X) \leq [N]_+ - 1 = [N]$. \square

3.3 Non-collapsed and collapsed convergence

Having at disposal the ‘continuity of volume’ granted by Theorem 1.3 and the ‘dimension gap’ Theorem 1.4 we can now easily obtain the stability of the class of $\text{ncRCD}(K, N)$ spaces as stated in Theorem 1.2:

Proof of Theorem 1.2

(i) The fact that the $\bar{\text{lim}}$ is actually a lim is a direct consequence of Theorem 1.3 (recall also (2.20)). This and the compactness of the class of $\text{RCD}(K, N)$ spaces (see (2.1)) ensure that up to pass to a subsequence, not relabeled, we can assume that there is a Radon measure \mathbf{m}_∞ on X_∞ such that $(X_n, \mathbf{d}_n, \mathcal{H}^N, x_n)$ pmGH-converge to $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty, x)$ and to conclude it is sufficient to prove that $\mathbf{m}_\infty = \mathcal{H}^N$, as this will in particular imply that the limit metric measure space does not depend on the particular converging subsequence chosen.

By Theorems 2.3, 1.12 and point (i) of Lemma 2.2 we know that $\mathbf{m}_\infty = \vartheta \mathcal{H}^N$ for some $\vartheta \leq 1$, so that our aim is to prove that $\vartheta = 1$ \mathbf{m}_∞ -a.e.. If not, there would exist $y_\infty \in X_\infty$ and $r > 0$ such that

$$\mathbf{m}_\infty(B_r^{X_\infty}(y_\infty)) < \mathcal{H}^N(B_r^{X_\infty}(y_\infty)). \quad (3.42)$$

Now find a sequence $y_n \xrightarrow{GH} y_\infty$ (recall (2.2)), notice that $\mathbf{d}_{\text{GH}}(\bar{B}_r^{X_n}(y_n), \bar{B}_r^{X_\infty}(y_\infty)) \rightarrow 0$ as $n \rightarrow \infty$ and use Theorem 1.3 to obtain

$$\lim_{n \rightarrow \infty} \mathcal{H}^N(B_r^{X_n}(y_n)) \stackrel{(2.20)}{=} \lim_{n \rightarrow \infty} \mathcal{H}^N(\bar{B}_r^{X_n}(y_n)) = \mathcal{H}^N(\bar{B}_r^{X_\infty}(y_\infty)) \stackrel{(3.42)}{>} \mathbf{m}_\infty(B_r^{X_\infty}(y_\infty)),$$

contradicting (2.4).

(ii) We argue by contradiction and assume

$$\dim_{\mathcal{H}}(X_{\infty}) > N - 1. \quad (3.43)$$

By the compactness of the class of $\text{RCD}(K, N)$ spaces we know that there exists a Radon measure \mathbf{m}_{∞} on X_{∞} and a subsequence, not relabeled, such that the normalized spaces $(X_n, \mathbf{d}_n, \mathcal{H}^N/\mathcal{H}^N(B_1(x_n)), x_n)$ pmGH-converge to $(X_{\infty}, \mathbf{d}_{\infty}, \mathbf{m}_{\infty}, x)$ (recall (2.1)). In particular, this grants that $(X_{\infty}, \mathbf{d}_{\infty}, \mathbf{m}_{\infty})$ is an $\text{RCD}(K, N)$ space, so that our assumption (3.43) and Theorem 1.4 yield that $\mathcal{H}^N(X_{\infty}) > 0$ and thus there is $x' \in X$ such that $\mathcal{H}^N(B_1(x')) > 0$. Now find a sequence $x'_n \xrightarrow{GH} x'$ (recall (2.2)) and use Theorem 1.3 (and (2.20)) to obtain that

$$\mathcal{H}^N(B_1^{X_n}(x'_n)) \rightarrow \mathcal{H}^N(B_1^{X_{\infty}}(x')) > 0.$$

Taking into account that $\lim_{n \rightarrow \infty} \mathbf{d}_n(x_n, x'_n) = \mathbf{d}_{\infty}(x, x') < \infty$, such convergence and the uniform local doubling property granted by the Bishop-Gromov inequality give that

$$\varliminf_{n \rightarrow \infty} \mathcal{H}^N(B_1^{X_n}(x_n)) > 0,$$

which contradicts our assumption $\lim_{n \rightarrow \infty} \mathcal{H}^N(B_1^{X_n}(x_n)) = 0$ and thus yields the thesis. \square

3.4 Volume rigidity

Collecting what proved so far it is now easy to establish the volume rigidity result, Theorem 1.6, and its Corollary 1.7.

Proof of Theorem 1.6

Step 1: set up Let $(X, \mathbf{d}, \mathcal{H}^N)$ be a $\text{ncRCD}(0, N)$ space and $\bar{x} \in X$ such that

$$\mathcal{H}^N(B_1^X(\bar{x})) \geq \mathcal{H}^N(B_1^{\mathbb{R}^N}(0)). \quad (3.44)$$

We shall prove that this implies that $\overline{B}_{1/2}^X(\bar{x})$ is isometric to $\overline{B}_{1/2}^{\mathbb{R}^N}(0)$. Thanks to Gromov compactness theorem (2.1) and to the stability of the ncRCD condition under non-collapsed convergence established in Theorem 1.2, this is sufficient to conclude.

Step 2: the cone Y Consider the function $(0, 1] \ni r \mapsto \frac{\mathcal{H}^N(B_r(\bar{x}))}{\omega_N r^N}$ and notice that the Bishop-Gromov inequality grants that it is non-increasing, that by (3.44) its value at $r = 1$ is 1 and, recalling the definition of ϑ_N and Corollary 2.14, that it converges to $\vartheta_N[X](\bar{x}) \leq 1$ as $r \downarrow 0$. Hence

$$\vartheta_N[X](\bar{x}) = \frac{\mathcal{H}^N(B_r(\bar{x}))}{\omega_N r^N} = 1 \quad \forall r \in (0, 1] \quad (3.45)$$

and by the ‘volume cone to metric cone’ [24, Theorem 1.1] we get the existence of an N -cone $(Y, \mathbf{d}_Y, \mathbf{m}_Y, o)$ and a measure preserving isometry $\iota : B_{1/2}^X(\bar{x}) \rightarrow B_{1/2}^Y(o)$.

It follows that $\mathbf{m}_Y = \mathcal{H}^N$ and thus that Y is $\text{ncRCD}(0, N)$. Also, the very definition of ϑ_N and the properties of ι give that

$$\vartheta_N[X](x) = \vartheta_N[Y](\iota(x)) \quad \forall x \in B_1^X(\bar{x}). \quad (3.46)$$

Now observe that by a simple scaling argument it is easy to see that $\vartheta_N[\mathbf{Y}]$ is constant along rays, and this fact together with the lower semicontinuity of $\vartheta_N[\mathbf{Y}]$ given by point (i) in Lemma 2.2 shows that $\vartheta_N[\mathbf{Y}](o) \leq \vartheta_N[\mathbf{Y}](y)$ for every $y \in \mathbf{Y}$. Therefore

$$1 \stackrel{(3.45)}{=} \vartheta_N[\mathbf{X}](\bar{x}) \stackrel{(3.46)}{=} \vartheta_N[\mathbf{Y}](o) \leq \vartheta_N[\mathbf{Y}](y) \stackrel{(2.26)}{\leq} 1 \quad \forall y \in \mathbf{Y}. \quad (3.47)$$

Step 3: $\mathbf{Y} = \mathbb{R}^N$ According to Lemma 2.9 it is sufficient to prove that \mathbf{Y} is an N -metric measure cone centered at any $y \in \mathbf{Y}$ and by the ‘volume cone to metric cone’ [24, Theorem 1.1] in order to prove this it is sufficient to show that $r \mapsto \frac{\mathcal{H}^N(B_r(y))}{\omega_N r^N}$ is constant for every $y \in \mathbf{Y}$. By the very definition of N -cone this is true for $y = o$, then for general $y \in \mathbf{Y}$ put $R := d_{\mathbf{Y}}(y, o)$ and notice that

$$\lim_{r \rightarrow \infty} \frac{\mathcal{H}^N(B_r(y))}{\omega_N r^N} = \lim_{r \rightarrow \infty} \frac{\mathcal{H}^N(B_{r+R}(y))}{\omega_N (r+R)^N} \geq \lim_{r \rightarrow \infty} \frac{\mathcal{H}^N(B_r(o))}{\omega_N r^N} \frac{r^N}{(r+R)^N} = \vartheta_N[\mathbf{Y}](o) \stackrel{(3.47)}{=} 1,$$

so that the conclusion follows from (3.47) and the monotonicity granted by Bishop-Gromov inequality (1.4). \square

Proof of Corollary 1.7 Inequality (1.3) immediately follows from Corollary 2.14 and the Bishop-Gromov inequality (1.4). The scaled version of Theorem 1.6 ensures the desired rigidity for the equality case. Moreover, by the second step in the proof of Theorem (1.6) we see that if

$$\vartheta(x) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^N(B_r^{\mathbf{X}}(x))}{\omega_N r^N} = 1,$$

then the spaces $(X, d/r, \mathcal{H}^n)$ converges to $(\mathbb{R}^N, d_E, \mathcal{H}^N)$. The converse being an easy consequence of Theorem 1.2, this concludes the proof. \square

3.5 Stratification

Here we prove the stratification result stated in Theorem 1.8; notice the similarity with the proof of Theorem 1.4.

Let us begin by giving the definition of the k -singular set $S_k(\mathbf{X})$:

$$S_k(\mathbf{X}) := \left\{ x \in \mathbf{X} : \text{for every tangent space } (\mathbf{Y}, d_{\mathbf{Y}}, \mathfrak{m}_{\mathbf{Y}}, y) \text{ of } \mathbf{X} \text{ at } x \text{ we have} \right. \\ \left. d_{\text{GH}}(\bar{B}_1^{\mathbf{Y}}(y), \bar{B}_1^{\mathbb{R}^{k+1} \times \mathbf{Z}}((0, z))) > 0 \text{ for all pointed spaces } (\mathbf{Z}, d_{\mathbf{Z}}, z) \right\} \quad (3.48)$$

We can now prove Theorem 1.8:

Proof of Theorem 1.8 We argue by contradiction, thus we assume that for some $k \in \mathbb{N}$ and $\text{ncRCD}(K, N)$ space \mathbf{X} we have $\dim_{\mathcal{H}} S_k(\mathbf{X}) > k$. Hence for some $k' > k$ it holds

$$\mathcal{H}^{k'}(S_k(\mathbf{X})) > 0. \quad (3.49)$$

Then for $\varepsilon > 0$ define

$$S_k^\varepsilon(\mathbf{X}) = \left\{ x \in \mathbf{X} : d_{\text{GH}}(B_r^{\mathbf{X}}(x), B_r^{\mathbb{R}^{k+1} \times \mathbf{Z}}((0, z))) \geq \varepsilon r \quad \forall r \in (0, \varepsilon), \text{ pointed } \mathbf{Z} \right\}$$

and note that $S_k^\varepsilon(X)$ is closed and that $S_k(X) = \bigcup_{i \in \mathbb{N}} S_k^{2^{-i}}(X)$. From this and (3.49) it follows that there exists $\bar{\varepsilon} > 0$ such that $\mathcal{H}^{k'}(S_k^{\bar{\varepsilon}}(X)) > 0$. We now apply Lemma 3.6 to $E := S_k^{\bar{\varepsilon}}(X)$ to deduce that there exists $x \in S_k^{\bar{\varepsilon}}(X)$ and $r_n \downarrow 0$ such that

$$\lim_{n \rightarrow \infty} \frac{\mathcal{H}_\infty^{k'}(S_k^{\bar{\varepsilon}}(X) \cap B_{r_n}^X(x))}{r_n^{k'}} \geq 2^{-k'} \omega_{k'}. \quad (3.50)$$

By Corollary 2.14 we have $\vartheta_N[X, \mathbf{d}, \mathbf{m}](x) \leq 1$ and thus by Proposition 2.8, up to pass to a non-re-labeled subsequence, we can assume that the rescaled spaces $(X_n, \mathbf{d}_n, \mathbf{m}_n, x_n) := (X, \mathbf{d}/r_n, \mathbf{m}/r_n^N, x)$ pmGH-converge to a pointed metric measure cone $(Y, \mathbf{d}_Y, \mathbf{m}_Y, y)$ which, by Theorem 1.2, is ncRCD(0, N).

Embedding all these spaces into a proper realization of this pmGH-convergence (recall (2.1)) and using the metric version of Blaschke's theorem (see [13, Theorem 7.3.8]) we see that, extracting if necessary a further subsequence, we can assume that the compact sets $S_k^{\bar{\varepsilon}}(X_n) \cap B_1^{\mathbf{d}_n}(x_n)$ converge to some compact set $A \subset Y$ w.r.t. the Hausdorff distance. A simple diagonal argument based on the very definition of $S_k^{\bar{\varepsilon}}(Y)$ then shows that $A \subset S_k^{\bar{\varepsilon}}(Y)$ and thus

$$\mathcal{H}_\infty^{k'}(S_k^{\bar{\varepsilon}}(Y)) \geq \mathcal{H}_\infty^{k'}(A) \stackrel{(3.36)}{\geq} \overline{\lim}_{n \rightarrow \infty} \mathcal{H}_\infty^{k'}(S_k^{\bar{\varepsilon}}(X_n) \cap B_1^{\mathbf{d}_n}(x_n)) = \overline{\lim}_{n \rightarrow \infty} \frac{\mathcal{H}_\infty^{k'}(S_k^{\bar{\varepsilon}}(X) \cap B_{r_n}^{\mathbf{d}}(x))}{r_n^{k'}} \stackrel{(3.50)}{>} 0.$$

Hence by (3.35) we also have $\mathcal{H}^{k'}(S_k^{\bar{\varepsilon}}(Y) \setminus \{y\}) > 0$ and we can repeat the argument to find $z \in S_k^{\bar{\varepsilon}}(Y)$, $z \neq y$, and a tangent cone $(Y', \mathbf{d}_{Y'}, \mathbf{m}_{Y'}, y')$ at z , which is ncRCD(0, N), such that

$$\mathcal{H}^{k'}(S_k^{\bar{\varepsilon}}(Y')) > 0. \quad (3.51)$$

Since $z \neq y$ the cone Y' contains a line passing through its origin y' and thus by the splitting theorem for RCD spaces we deduce that $Y' = \mathbb{R} \times X^1$ for some metric measure space $(X^1, \mathbf{d}^1, \mathbf{m}^1)$.

If $k = 0$ this is enough to conclude, because such splitting contradicts the choice $z \in S_k^{\bar{\varepsilon}}(Y)$. Otherwise $k \geq 1$, hence $k' > 1$ and (3.51) and $N \in \mathbb{N}$ force $N \geq 2$. Then the splitting grants that X^1 is an RCD(0, $N - 1$) space and from the fact that Y' is ncRCD(0, N) and Proposition 2.15 we deduce that in fact X^1 is ncRCD(0, $N - 1$). Taking into account the trivial implication

$$(r, x^1) \in S_k(\mathbb{R} \times X^1) \iff x^1 \in S_{k-1}(X^1)$$

and (3.37), from (3.51) we deduce that

$$\dim_{\mathcal{H}}(S_{k-1}(X^1)) > k - 1.$$

We can therefore repeat the whole argument with X^1 and $k - 1$ in place of X and k : iterating we eventually find a contradiction and achieve the proof. \square

Remark 3.7 (Polar spaces). This theorem is also valid, with the same proof, in the a priori larger class of RCD(K, N) spaces X such that every iterated tangent cone is a metric cone (notice that the analogue of Proposition 2.15 holds, rather trivially, for this class of spaces). Spaces with this property have been called *polar* in [17].

Notice that wncRCD(K, N) spaces such that $\vartheta_N[X]$ is locally bounded from above are polar, and that Theorem 2.3 grants that this class of spaces is stable w.r.t. pmGH-convergence provided we impose a uniform local upper bound on the ϑ 's. \blacksquare

Remark 3.8 (Boundary of ncRCD spaces). In the case of non-collapsed Ricci limit spaces it has been shown in [17] that

$$S_{N-1}(X) \setminus S_{N-2}(X) = \emptyset. \quad (3.52)$$

This is however false in the present situation, because, for instance, the closed unit ball $\bar{B}_1(0) \subset \mathbb{R}^N$ is a perfectly legitimate ncRCD(0, N) space and every point in the boundary belongs to $S_{N-1}(X) \setminus S_{N-2}(X)$.

The problem is the presence of the boundary: looking for a moment just at smooth objects, compact manifolds with (convex) boundary are always RCD(K, N) spaces for suitable K, N but not considered in [17] as objects whose limits define Ricci-limit spaces. Then in [17] it has been proved (with an argument also linked to topology) that in the non-collapsing situation boundary of balls converge to boundary of balls, a fact which quite easily implies (3.52).

This line of thoughts suggests to define the boundary ∂X of a ncRCD(K, N) space X as

$$\partial X := \text{closure of } (S_{N-1}(X) \setminus S_{N-2}(X)).$$

Then, mostly by analogy with the theory of Ricci-limit and Alexandrov spaces, a number of natural non-trivial questions arise:

- Given a non-collapsing sequence $X_n \rightarrow X$ of ncRCD spaces, is it true that ∂X_n converge to ∂X ?
- Is it true that either $\partial X = \emptyset$ or ∂X is $N - 1$ -rectifiable with $\mathcal{H}^{N-1}|_{\partial X}$ locally finite?
- Is $X \setminus \partial X$ a convex subset of X ? That is, is it true that for any $x, y \in X \setminus \partial X$ there is a (or perhaps, is any) geodesic connecting them entirely contained in $X \setminus \partial X$?
- Let Y be a connected component of ∂X . Is Y connected by Lipschitz paths? If so:
 - let d_Y be the intrinsic distance on Y induced by the distance on X : is (Y, d_Y) an Alexandrov space of non-negative curvature? (notice that the analogous of this latter question for Alexandrov spaces is open - see [47, Section 9]).
 - Let X' be another ncRCD(K, N) space and assume that Y_1, \dots, Y_n and Y'_1, \dots, Y'_n are the connected components of ∂X and $\partial X'$ respectively. Assume also that for any $i = 1, \dots, n$ the spaces Y_i and Y'_i with the induced length metrics are isometric and glue X and X' along their boundaries via such isometries. Is the resulting space ncRCD(K, N)? (the analogous statement for Alexandrov spaces holds, see [46]).

■

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