Contact interaction in Quantum Mechanics and Gamma convergence

G.F.Dell'Antonio

Math, Dept University Sapienza Roma

and

Mathematics area Sissa

This talk will be about problems in Q.M. that belong mathematically to Functional Analysis (extensions of symmetric operators).

It is dedicated to Gianni dal Maso on the occasion of his 65th birthday.

Gianni is a very good scientist and a very good friend.

He is also a very good teacher and mentor, judging from the affection of all his Ph.D. students I know.

The Math sector in Sissa owes much to the time and efforts Gianni puts in promoting the standards of excellence.

One more reason to praise Gianni is that in his book on Gamma convergence it is clearly pointed out that Gamma convergence implies resolvent convergence (but not quadratic form convergence). This in turn implies convergence of spectra of self-adjoint operators and of Wave Operators (Scattering matrices), a daily bread in Functional Analysis and Mathematical Physics.

!!!HAPPY BIRTHDAY GIANNI !!!

This talk will be about problems in Q.M. that belong mathematically to Functional Analysis (extensions of symmetric operators) and make essential use of Gamma convergence.

We shall describe contact interactions, as representatives of interactions of extremely short range. We work in R^3.

The mathematical tools we use, a part from Gamma convergence, are rather elementary, the standard tools used in the mathematics of Quantum Mechanics.

Recall that in Quantum Mechanics the the interaction is described by a Schrödinger equation and the hamiltonian is the sum of a kinetic part H_0 (the free hamiltonian, usually a second order partial differential operator,) and a potential part, usually a negative function with various regularity properties.

We introduce two different types of ”zero range” (contact) interactions, weak and strong, through the use of of boundary conditions (to be defined soon). Formally they correspond to placing on the boundaries $x_i - x_j =$ distributions of different orders.

They can also be defined, with some care, as distributional limit of suitable sequences of regular potentials.

In this case this ”interaction” must be properly defined; we shall see the role in this of Gamma convergence.

Weak contact requires the presence of a zero energy resonance (a solution of the two-body problem $(H_0 + V)\phi = 0$ which has the asymptotic behavior $\frac{1}{|x|}$ (and therefore it is not in $L^2(R^3)$).

Making use of Gamma convergence we prove that a system if three identical particles in interaction through *weak contact* is described by the Gross-Pitayeswii equation (a cubic focusing p.d.e)

We will prove that this system has infinitely many bound states with energies that scale as $-\frac{c}{\sqrt{n}}, \;\; c > 0$

We will also remark, without giving full details, that the system of three bodies which mutually interact through a weak contact has as semiclassical limit a three-body newtonian system; the scaling of energies matches.

We will briefly compare our analysis with that of a Bose-Einstein condensate as presented e.g. in [B,O,S] since the Gross-Pitayeskii equation appears in both formulations.

In the latter case one considers the limit as $N \to \infty$ of a system of N particles interacting through a potential of range N.

A normalization factor $\frac{1}{N}$ is added to have a single particle (average) result.

One looks at "marginals" (reduced density matrices).

Setting $\epsilon = \frac{1}{N}$ this interaction in the limit $N \to \infty$ would represents "weak contact" (to be defined soon) *it there were a zero energy resonance*.

Since in experiments the presence of a zero energy resonance is essential (and obtained through a Feshbach effect by coupling the system to a suitably tuned e.m. field) one expects that in the proof the presence of a zero energy resonance plays a major role.

We shall call "weak contact" the zero range interaction that requires the presence of a zero-energy resonance.

We will see that joint weak contact between three particles produces an infinite number of bound states.

In our proof we make essential use of Gamma convergence, a variational approach introduced 60 years ago by E. de Giorgi in the context of homogenization.

We mention that Gamma convergence has also a role in the description of the structure of the Fermi sea in Solid State Physics, leading to a discrete spectrum (an infinite number of bound states for "conduction electrons in a crystal).

And also in finding the ground state in the Nelson model of the Polaron (interaction of a quantum particle with quantized zero mass field).

We will not discuss these topics here.

The common feature is that Gamma convergence allows to "make sense" of interactions *formally* described by delta functions.

Contact interactions in R^3 are often defined by imposing that the wave function in the domain of the hamiltonian satisfies at the *coincidence manyfold* Γ

$$\Gamma \equiv \cup_{i,j}\Gamma_{i,j} \qquad \Gamma_{i,j} \equiv \{x_i = x_j\},\ \ i \neq j \quad x_i \in R^3. \tag{1}$$

the boundary conditions $\phi(X) = \{ \frac{C_{i,j}}{|x_i - x_j|} + D_{i,j} \quad i \neq j \}$

These conditions were used already in 1935 by H.Bethe and R.Peierls [B,P] (and before them by Fermi) in the description of the interaction between proton and neutron. They were later used by Skorniakov and Ter-Martirosian [T,S] in the analysis of three body scattering within the Faddaev formalism.

We call *strong contact* the case $C_{i,j} \neq 0$, $D_{i,j} = 0$ and weak contact the case $C_{i,j} = 0$, $D_{i,j} \neq 0$.

These two types of interactions give *complementary and independent results* .

Both are independent and complementary to those due potential of Rollnick class.

Notice that the above definition *is formal*.

Functions that satisfy strong boundary conditions *are not in the Hilbert space* $L^2(R^3)$ *(and therefore not in the domain of the free Schrödinger hamiltonian* H_0).

Integration by parts produces a potential term proportional to $\delta(x_i - x_j)$. The corresponding quadratic form is not weakly continuous.

Therefore the first task is to describe the hamiltonian as self-adjoint operator.

We are looking for extensions of $\hat{H}_0$, the symmetric (but not self-adjoint) operator defined as the free Hamiltonian restricted to functions that vanish in a neighborhood of Γ.

In Theoretical Physics the interest in the subject was renewed by recent advances in low energy physics and by the flourishing of research on ultra-cold atoms interacting through potentials of very short range.

In what follows we analyze first the case of three particles one of which is in strong contact with the other two and after the case of joint weak contact among three particles.

If the interaction is sufficiently strong both lead to the *Efimov effect* [E] i.e. the presence of an infinite number of bound states with eigenvalues that converge to zero as $\frac{-C}{\sqrt{n}}$.

This effect is also present in dimension one for particles (electrons) which satisfy the Pauli equation and move on a lattice with Y-shaped vertices where the interaction takes place, a model suggested by images taken with electron microscopes. This Efimov effect originates the *Fermi sea*.

We will prove

Theorem

In the three-body case weak mutual contact of three identical (quantum) particles is described by a self-adjoint operator with an infinite number of negative eigenvalues which scale as $-\frac{c}{\sqrt{n}}$ $c > 0$.

The system satsfies the (cubic, focusing) Gross-Pitaievskii equation $i\frac{\partial\phi}{\partial t}(t.x) = -\Delta\phi(t,x) - C|\phi(t,x)|^2\phi(t,x)$.

Each partial two-body system has a zero-energy resonance and the two-body potential is the limit when $\epsilon \to 0$ of potentials that scale as $V^\epsilon(|x_1 - x_j) = \frac{1}{\epsilon^2}V(\frac{|x_i-x_j|}{\epsilon})$

....

To make connection with physics we will prove that strong contact interactions are limits *in strong resolvent sense* when $\epsilon \to 0$ of interactions through two-body potentials of class $C^1 \cap L^1(R^3)$ which scale according to $V_\epsilon(|y|) = \epsilon^{-3} V(\frac{|y}{\epsilon})$.

For weak contact the scaling is and $V_\epsilon(|y|) = \epsilon^{-2} V(\frac{|y|}{\epsilon})$ *and there must be a zero energy resonance*.

We will see that this last condition is of topological origin (compactness)

Strong contact interaction is not defined in R^3 for a two-particle system.

Weak contact of two pairs of particles has a bound state (in the center of mass and in momentum space the resolvent at the origin is the inverse of a two-by-two matrix with zeroes on the diagonal).

If one of the masses of the particles "is infinite" the interaction of the remaining particle is called *point interaction* [A]. .

The study of resolvent in the case of point interaction is difficult because in momentum space the zero energy resonances "interferes" with the singularity of the free resolvent.

This difficulty is overcome by the procedure that we shall introduce.

We now describe the hamiltonian of our system in the case of strong contact.

The quadratic form of the free hamiltonian is a strictly positive form with domain the space of absolutely continuous functions.

The delta distribution defines a bounded negative bilinear form in this space.

Therefore *if the sum defines a self-adjoint operator*, this operator is bounded below.

It remains to be proven that this self-adjoint operator exists.

Consider a system of three particles in which one is in strong contact separately with the other two (for simplicity all particles are identical). We choose the reference system in which the barycenter is at rest.

We can take advantage of the regularity of the wave function of the non interacting particle and integrate by parts with respect to its coordinates.

The concrete formulation of this operation is realized through the introduction of an invertible map, which we call *the Krein map* $\mathcal{K}$, to a space of more singular functions, called Minlos space $\mathcal{M}$ (the idea of introducing this space came from reading [M1])

This space is obtained from $L^2(R^9)$ acting with $(H_0)^{-\frac{1}{2}}$ (this action is on the domain of this operator; since we going to invert the map, we can assume that on the complement it acts as identity).

The map acts differently on operators and on quadratic forms.

On the potential part (which is only defied as quadratic form) it acts as $\delta(x_i - x_j) \to (H_0)^{-\frac{1}{2}}\delta(x_i - x_j)(H_0)^{-\frac{1}{2}}$.

Regarded as bilinear forms H_0 and the delta commute; they both are singular elements of an abelian algebra $\mathcal{A}$.

Therefore this map can be written $\delta \to \delta(H_0)^{-1}$.

Our approach is therefore in the path followed by Birman, Krein and Visik [B][K] for the study of self-adjoint extensions of positive operators. The main difference is that we work with quadratic forms [A,S] [K,S]

On the free Hamiltonian (an operator) the Krein map acts as $H_0 \to (H_0)^{\frac{1}{2}}$.

This map can be written $H_0 \to (H_0)^{-\frac{1}{4}} H_0 (H_0)^{-\frac{1}{4}}$.

Therefore both the operators and the "generators" of the maps belong to the algebra $\mathcal{A}$.

In $\mathcal{M}$ the "potential" has in position space the form $-\frac{C}{|x|} + B$ where B is a *positive* bounded operator with kernel that vanishes on the diagonal.

Therefore in $\mathcal{M}$ the potential term is a bona-fide operator.

In $\mathcal{M}$ the kinetic energy is a pseudo-differential operator of order one

The singularities at the origin (in position space) of the kinetic and the potential terms are *homogeneous* .

Therefore [D,R][I,O,R] there are values C_1 and C_2 of the positive constant C (that depend on the masses and on the coupling constant) such that for $C < C_1$ their sum is a positive weakly closed quadratic form (and represents therefore [Ka] a positive self-adjoint operator), for $C_1 \leq C < C_2$ it decomposes into a continuous family of self-adjoint operators $H_{C,\alpha}$ with one negative eigenvalue $\lambda(C,\alpha)$.

For $C \geq C_2$ it decomposes into a continuous family (parametrized by α of self–adjoint operators with a sequence $\lambda_n(C,\alpha)$ of negative eigenvalues with the asymptotic $\lambda_n(C,\alpha) \simeq -b(C.\alpha)n$

These results are derived [D,R][I,O,R] using the Mellin transform and properties of the Bessel functions.

The Mellin transform "diagonalizes" the sum of the kinetic and potential terms.

This can be proved by explicit computations but can be seen also as a consequence of the fact that both forms belong to the algebra $\mathcal{A}$ and the Mellin transform corresponds to a change of coordinates.

Results of this type in $\mathcal{M}$ are also obtained in [M1], [M2] and in [C] in another context. .

We must now come back to the original physical space.

This is done inverting the Krein map.

Notice that in $\mathcal{M}$ both the kinetic term and the potential term are self-adjoint operators.

Due to the change in metric topology, after inversion their sum defines is in "physical space" a one parameter family of *weakly closed* quadratic forms $Q(C, \alpha)$ which are well ordered and uniformly bounded below.

Since the interaction potential is rotational invariant, only the s wave is affected so that all forms are *strictly convex*.

The Krein map $\mathcal{K}$ is *fractioning* (the target space is a space of less regular functions) and *mixing* (the square root is not diagonal in the channels).

This suggests the use of Gamma convergence [Dal] , a variational method pintroduced sixty years ago by E. de Giorgi and originally used for the study of "homogenization" of finely structured materials.

Gamma convergence selects the infimum of an ordered sequence of strictly convex quadratic forms bounded below in a compact domain of a topological space Y . The Gamma limit is the quadratic form characterized by the following relations

$$\forall y \in Y, y_n \to y, F(y) = liminf F(y_n) \ , \ \forall x \in Y_n \forall \{x_n\} : F(x) \leq limsup_n F_n(x_n) \tag{2}$$

,

The first condition implies that F is a common lower bound for the F_n , the second implies that the bound is optimal.

The condition for existence of the Gamma limit is that the sequence be contained in a compact set for the topology of Y (so that a Palais-Smale converging sequence exists).

In our case the topology is the Frechet topology defined by the Sobolev seminorms and compactness is assured by the absence of zero energy resonances.

Therefore in our case the Gamma limit exists.

By a theorem of Kato [K] the limit form admits strong closure and this defines a self-adjoint hamiltonian.

If $C_1 \leq C < C_2$ this Hamitonian has a bound state, if $C \geq C_2$ it has an Efimov sequence of bound states with eigenvalues that scale as $-a\sqrt{n}$..

It is possible to prove [D,R] that the moduli of the eigenfunctions have the form $\frac{1}{|x_i - x_j| log(n|x_i - x_j|)}$. The orthogonality is due the fact that the phases increase linearly with n.

The generalized eigenvectors can also be given and therefore the model is in this sense *completely solvable*

One may wonder what happens to the quadratic forms that do not correspond to a minimum and therefore are not strongly closable.

They correspond to different boundary conditions at contact.

This alters the order of the quadratic forms and the Gamma limit is different.

Remark that we have used the free Schrödinger Hamiltonian to construct the Krein map; we could instead use the magnetic Schrödinger Hamiltonian and we would reach other extensions.

REMARK

The procedure we have followed is such that at the end of the process the kinetic energy is not changed while the potential term is regularized; it is therefore a sort of *renormalization of the interaction* but notice that the method is variational and non perturbative.

We shall now prove that the strong contact hamiltonian is limit *in strong resolvent sense* of Hamiltonians of three body systems in which one particle interacts with the other two through potentials of class $L^1(R^3)$ which scale as $V^{\epsilon}(x_i - x_j) = \frac{C}{\epsilon^3} V(\frac{|x_i - x_j|}{\epsilon})$.

Such Hamiltonians do not have zero energy resonances; therefore the quadratic forms belong to a compact set in the topology given by Sobolev semi-norms.

The sequence is strictly decreasing as a function of ϵ and is bounded below by the quadratic form of the strong contact interaction.

Every strictly decreasing sequence in a compact space has a unique limit if it is uniformly bounded below. The limit is contained in the image under the inverse of the Krein map of the limit set; the sequence has therefore the same Gamma limit. Therefore the quadratic forms of the ϵ-hamiltonians Gamma converge to the quadratic form of the hamiltonian of contact interaction.

Gamma convergence implies strong resolvent convergence; therefore the hamiltonians $H_0 + V^\epsilon(x_0 - x_1) + V^\epsilon(x_0 - x_2)$ converge *in the strong resolvent sense* to the Hamiltonian of separate strong contact of a particle with two particles.

Notice that strong resolvent convergence implies convergence of spectra and of Wave Operators but *does not imply convergence of quadratic forms*: it only implies convergence for sequences that remain uniformly bounded [Dal]..

Therefore the result cannot be obtained in perturbation theory.

Remark also the method we follow is variational and therefore no estimates are given of the convergence as a function of the parameter ϵ.

Gamma convergence is stable under the addition of regular potentials; the proof makes use of a formalism due to Kato and improved by Konno e Kuroda [K,K] If C is large enough this leads to an Efimov sequence of bound states.

For a three-particle system with strong contact the ground state is unique and its wave function can be take real.

The other bound states have complex-valued wave functions.

They cannot be found by a traditional Morse analysis and requires the use of Bott's index theory [Ek].

We have seen that in order to have a self-adjoint operator describing the strong contact of two particles, a third non interacting particle is needed.

This is the case for a system of two species of particle (e,g, protons and neutrons) in which members of each specie don't interact among themselves and interact through strong contact (a delta function) pairwise with members of the other specie.

If the members of each specie are identical particles, a three-body problem is sufficient to describe the system.

In the case of Bose-Einstein condensate the "bosons" are not elementary particles; they are composites that are described *effectively* by wave functions that are symmetric under permutation of two variables; in this sense they represent "bosons"

We describe the condensate as originated by mutual weak contact of three particle systems.

Two-body weak contact has a zero energy resonance. The presence of a zero energy resonance spoils compactness.

To recover compactness we must require that also the approximating two-body potentials lead to a Hamiltonian with a zero energy resonance, so that this resonance "can be subtracted away" before taking the limit $\epsilon \to 0$.

To produce a Bose-Einstein condensate one needs potentials which in two body systems lead a zero energy resonance.

This is a major difficulty in producing a Bose-Einstrein condensate; in actual experiments the zero energy resonance is obtained introducing a suitable external e.m. field that generate a zero energy resonance through a Feshbach effect.

The other difficulty is in "stabilizing" the system and making it dense. This requires the introduction of a *confining potential* and it is essential that this confinement does not spoil the effect of weak contact.

It is therefore important a result of Konno-Kuroda [K,K] according to which potentials of Rollnik class and weak contact "potentials" lead to *independent* and complemantary effect.

Any description of the condensate that does not introduce a zero energy two-body resonance does not catch the physics of the problem

Mathematically we can proceed as in the case of strong separate contact and introduce in the same way the auxiliary space $\mathcal{M}$.

To see the mathematical role of the zero energy resonance recall that, in the case of strong separate contact, in the space $\mathcal{M}$ the potential term contains a *positive* term that we have called B.

We will prove that in the case of mutual weak contact *this term is not present*.

Therefore the kinetic energy term and the potential energy term have the same behavior under scaling.

Not surprisingly since the two particle Hamiltonian of weak contact is covariant under scaling.

We will use the Birman-Schwinger formula for the difference of the resolvent of the hamiltonian for *simultaneous weak contact* of three particles and the resolvent of the strong separate case.

The B-S formula is

$$\frac{1}{H_2 - z} - \frac{1}{H_1 - z} = \frac{1}{H_2 - z} K_{1,2} \frac{1}{H_1 - z}, K = U^{\frac{1}{2}} \frac{1}{H_1 - z} U^{\frac{1}{2}} \qquad U = -(H_2 - H_1) \tag{3}$$

where $H2$, H_1 are self-adjoint operators, and z is chosen out of the spectrum of both operators and K is the Birman-Schwinger kernel.

We can take for H_2 the Hamiltonian of joint weak contact and for H_1 the hamiltonian of strong separate contact.

We already proved strong resolvent convergence as $\epsilon \to 0$ for H_2 and H_1.

Take ϵ finite and consider the contribution to $\frac{1}{H-z}$ coming from those terms in the perturbative in which all three potentials contribute.

In these terms we can "take away" the factor ϵ^{-2} from one of the potentials and attribute a factor ϵ^{-1} to each of the other two.

Together with the kinetic energy the limit $\epsilon \to 0$ provides in physical space the resolvent of H_1 .

The remaining terms are *negative* and in $\mathcal{M}$ in the limit $\epsilon \to 0$ cancel in the quadratic form the term B that was present in the case of separate strong contact.

Therefore In $\mathcal{M}$ in the weak joint contact case the kinetic and potential energies scale in the same way under dilation

This is not a surprise since the hamiltonian of weak contact is scale covariant.

We can now invert the Krein map and take the limit $\epsilon \to 0$ in the strong resolvent sense.

Since B is positive, the minimum of the energy functional is provided by the simultaneous weak contact of three particles *AND NOT BY STRONG CONTACT*.

As before Gamma convergence gives a unique self-adjoint operator bounded below with an Efimov sequence of bound states that scale as $-c\frac{1}{\sqrt{n}}$.

Each of the infinitely many bound states in the case of joint weak contact is lower that the corresponding state of separate strong contact.

The lowest energy state of the system is the ground state for weak simultaneous contact interactions.

In the case of simultaneous weak contact of three particles the hamiltonian is the limit in strong resolvent sense of the sum of the free Hamiltonian and the product of three attractive two-body potential that scale as $V^\epsilon(x_i - x_j) = \frac{1}{\epsilon^2} V(\frac{x_i - x_j}{\epsilon})$ *and have a zero energy resonance*.

There is an Efimov sequence of bound states with energies that scale as $-C\frac{1}{\sqrt{n}}$.

The three-body system satisfies !he Gross-Pitaiewsii equation $i\frac{\partial}{\partial t}\phi = -\Delta\phi(x) - g|\phi(x)|^2\phi(x)$

To see the origin of the cubic term notice that now there are three weak contacts and this is "equivalent" to $\delta(x_i - x_j)\delta(x_j - x_k)$: In fact the three potentials scale with a factor ϵ^{-2} and in the center of mass frame when acting on continuous functions one can write $(\epsilon^2)^3 = (\epsilon^3)^2$ take advantage of the fact that the distribution δ "commutes" with H_0.

The energy functional

$$E(\phi) = \int |\nabla \phi(x)|^2 - C \int |\phi(x)|^4 dx \tag{4}$$

is obtained by taking the scalar product with a wave function and integrating by parts the kinetic term.

It has infinitely many critical points; only the lowest corresponds to a wave function that is real. The functions that describe the other bound states are complex-valued and therefore the other critical points of the functional and can be only obtained by a generalization of morse Theory [E] i.e introducing an index or through mountain-pass techniques..

Since the particles are identical one obtains an energy form which is the sum of a free part and the integral over the product of the two densities.

The bound states of the three particle system are critical points of this Gross-Pitaieskii functional.

Notice that only the ground state is obtained by standard Morse theory;

We have so far described the bound states of a three particle system.

If one neglects the interaction between triples, the ground state of system of $3N$ identical particles any three of which are in *joint* weak contact is approximately $\otimes_{k=1}^{N}\Omega_s^k$ where k is the index of distinct triplets in joint weak contact and Ω_s is the ground state of a three particle system.

Since the particles are identical bosons the totally symmetric product is understood.

Remember that the wave functions of the bound states are extended therefore in order to find the ground state of the system one must take into account the contribution of the *tails*

The presence of a strong confining potential (usually provided by the electric potential of strong laser beams) has the effect of "compressing " the system.

This effect is *independent and complementary* to that of the internal interactions.

This permits to have condensate of different forms, even disk-like or cigar-like (but never two-dimensional)

The analysis can now be done using perturbation theory.

This operation changes slightly the wave function of the ground state and may "destroy" part of the point spectrum.

The regime we have described may be called *high density regime* because the three particles are bound together by three mutual weak contacts and we have seen that interaction is stronger that strong separate of one particle with the other two and therefore the *ground state* of this system has a very small support

If the confining potential is strong the gas is dense.

There is another regime of Bose-Einstein condensateIt may be called *low density regime*

Any triple of particles has now only two weak contacts; if the particles are identical there is only one bound state.

The bound state has now a much larger support and even in presence of strong constraining potential this results in a lower density of the gas. .

If the constraining potential is sufficiently strong there can be now chains of arbitrary number $N \geq 3$ of identical bosons in which each particle is in separate weak contact with two other particles and *there is no three-particles joint weak contact*.

If the triples are well separated one has a unique bound state of the separate three body systems and the triple does not satisfy the Gross-Pitaiewskii equation but rather an equation in which the non linear part is $\int \phi^*(x_1)\Xi(x_1 - x_2)\phi(x_2)\Xi(x_1 - x)|\phi(x)|dx_1dx_2$ where $\Xi(y)$ a (not tempered) distribution limit of which defines weak contact.

For any N this system is obtained as strong resolvent limit as $\epsilon \to 0$ of a system of N triples of particles interacting through two-body potentials that scale as $V^\epsilon(|x_i - x_j|) = \epsilon^{-2}\frac{V(|x_i - x_j}{\epsilon})$ and have a zero energy resonance.

REMARK

We have seen that weak contact requires the presence of a zero energy resonance and this is necessary for having a Bose-Einstein condensate. This explains why there are no two-dimensional condensates.

In two dimension there is only one type of contact (the delta function and the laplacian are homogeneous at short distances);

This contact does not lead to a resonance but rather to a bound state.

The absence of the zero energy resonance implies the impossibility to have a condensate.

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THE UNITARY GAS

Since it belongs to the same class of problems, we mention hare briefly the Unitary Gas.

This is a gas of spin $\frac{1}{2}$ (massive) non relativistic fermions that interact through weak contact.

Since particles "with the same spin orientation" cannot have weak contact (due to the antisymmetry of the wave function) the system can be equivalently described as a collection of systems consisting of couples of spinors with different spin orientation in weak contact.

This is commonly described in Solid State Physics as the formation of *Cooper pairs* and the mechanism for their formation has been described by Bardeen-Cooper-Schrieffer.

We recall that weak contact requires the presence of a zero energy resonance and this in turn requires a Feshbach adjustment of the e.m. field.

Cooper pairs have the statistics of bosons and therefore one can consider weak contact of three Cooper pairs.

The presence of a Feshbach mechanism is important also in this step.

In Solid State Physics this is often called B.C.S to Bose-Einstein transition.

Since the systems with potentials V^ϵ provide resolvent convergence when $\epsilon \to 0$ we can use this ϵ-approximation to find the spectrum of the hamiltonian of the system.

In the ϵ approximation due to antisymmetry of the wave function (the particles are fermions) all terms in the perturbative expansion that give a negative contribution to the energy cancel with corresponding terms with a positive contribution.

Therefore in the ϵ approximation the Hamiltonian (energy) is non negative and by strong resolvent convergence the Hamiltonian of the limit system is positive.

Therefore the Unitary Gas has a non negative hamiltonian; in the Physics Literature this result is known as "Stability of the Unitary Gas".

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SEMICLASSICAL LIMIT

One verifies easily that weak contact provides Coulomb interaction between the barycenters of coherent states. In this sense the Newtonian three body system can be regarded as semiclassical limit of the quantum three body problem with a simultaneous weak contact.

This system has an Efimov sequence of bound states that are the counterpart of the orbits of the classical three-body problem. Both systems have an infinite number of periodic solution (bound states) ad the discrete spectra correspond (bound states on one side, energy of periodic orbits on the other side)

Therefore this "weak form" of the semiclassical limit is satisfied and mutual weak contact among three particles should be compared with the three-body problem in Newton mechanics.

For a WKB analysis of the cubic defocusing cubic equation see [C] [C,W] [G]

For the focusing cubic equation (related to weak contact of three bodies) there are no zero energy resonances and the analysis in [C] holds (the WKB semiclassical limit can be taken for initial data in $\mathcal{H}^1$).

For the discrete part of the spectrum the weak contact among three particles is in some sense "completely solvable" : it allows a complete description of the eigenvalues and of the (generalized) eigenfunctions.

A more direct W,K,B approach to the semi-classical limit is the following.

Without changing the dynamics we can use the Hamiltonian $H_0 + \lambda, \ \lambda \in R$

For $\lambda \to \infty$ the Krein map can now be related to the semiclassical limit. The space $\mathcal{M}_\lambda$ for $\lambda \simeq \hbar^{-1}$ can be regarded as semiclassical space. Setting $\frac{1}{\sqrt{\lambda}} = \hbar$, a part from an irrelevant constant one has in $\mathcal{M}_\lambda$ to first order in $\frac{1}{\lambda}$ the hamiltonian of the three-body newtonian system.

Notice that in the semiclassical limit the free hamiltonian is scaled by a factor to $\hbar^{-2}$ and the Coulomb potential is scaled by a factor $\hbar^{-1}$ If we identify the radius of the potential (the parameter ϵ) with $\hbar$ (both have the dimension of a length) the limit $\hbar = \epsilon \to 0$ gives contact interaction at a quantum scale, Coulomb interaction at a semiclassical scale.

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EFIMOV EFFECT IN QUANTUM MECHANICS

The analysis of the case of strong separate contact of a particle with two identical particles can be used to give one more proof of the Efimov effect in Quantum Mechanics [S]: a system of three quantum (Schrödinger) particles that has two zero energy resonances has an infinite number of bound states (we could add : with energies that scale as $\frac{C}{\sqrt{n}}$)

Indeed a special conformal transformation leaves the free hamiltonian invariant and turns zero energy resonances into function that have a $\frac{C}{|x_i - x_j}$ behavior at contact

Both are in the weak closure of the domains of the corresponding operators.

Since unitary maps leave the spectrum invariant the result follows for our analysis of strong contact.

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STRONG AND WEAK INTERACTION HAVE COMPLEMENTARY EFFECTS.

Theorem 1

In three dimensions for $N \geq 3$ contact interactions and weak-contact interactions contribute *separately* and *independently* to the spectral properties and to the boundary conditions at the contact manyfold.

Contact interaction contribute to the Efimov part of the spectrum and to the T-M boundary condition $\frac{c_{i,j}}{|x_j - x_i|}$ at the boundary $\Gamma \equiv \cup_{i,j} \Gamma_{i,j}$. Weak-contact interactions contribute to the constant terms at the boundary and may contribute to the (finite) negative part spectrum.

$\diamondsuit$

For an unified presentation (which includes also the proof that the addition of a regular potential does not change the picture) it is convenient to use a symmetric presentation due to Kato and Konno-Kuroda [KK] (who generalize previous work by Krein and Birman) for hamiltonians that can be written in the form

$$H = H_0 + H_{int} \qquad H_{int} = B^* A \tag{5}$$

where B, A are densely defined closed operators with $D(A) \cap D(B) \subset D(H_0)$ and such that, for every z in the resolvent set of H_0, the operator $A\frac{1}{H_0+z}B^*$ has a bounded extension, denoted by $Q(z)$.

We give details in the case $N = 3$.

Since we consider the case of attractive forces, and therefore negative potentials it is convenient to denote by $-V_k(|y|)$ the two body potentials. The particle's coordinates are $x_k \in R^3, k = 1,2,3$. We take the interaction potential to be of class C^1 and set

$$V^\epsilon(X) = \sum_{i \neq j} [V_1^\epsilon(|x_i - x_j|) + V_2^\epsilon(|x_i - x_j|) + V_3^\epsilon(|x_i - x_j|)] \qquad (6)$$

where V_1 and V_2 are negative and $V_3(|y)$ is a regular potential.

For each pair of indices i,j we define $V_1^\epsilon(|y|)) = \frac{1}{\epsilon^3} V_1(\frac{|y|}{\epsilon})$ and $V_2^\epsilon(|y|) = \frac{1}{\epsilon^2} V_2(\frac{|y|}{\epsilon})$.We leave V_3 unscaled.

We define $B^\epsilon = A^\epsilon = \sqrt{-V^\epsilon}$. For $\epsilon > 0$ using Krein resolvent formula one can give explicitly the operator B^ϵ as *convergent power series* of products of the free resolvent $R_0(z), Rez > 0$ and the square roots of the sum of potentials $V_k^\epsilon \;\; k = 1, 2, 3$.

One has then for the resolvent $R(z) \equiv \frac{1}{H+z}$ the following form [K,K]

$$R(z) - R_0(z) = [R_0(z)B^\epsilon][1 - Q^\epsilon(z)]^{-1}[B^\epsilon R_0(z)] \qquad z > 0 \quad (7)$$

with

$$R_0(z) = \frac{1}{H_0 + z} \qquad Q^\epsilon(z) = B^\epsilon \frac{1}{H_0 + z} B^\epsilon \quad (8)$$

Proof of Theorem 1

We approximate the zero range hamiltonian with the one parameter family of hamiltonians

$$H_\epsilon = H_0 + + \sum_{m,n} V^\epsilon(|x_n - x_m|) \;\; n \neq m, x_m \in R^3 \tag{9}$$

The potential is the sum of three terms

$$V^\epsilon(|y|) = \sum_{i=1}^{3} V_i^\epsilon, \quad V_1^\epsilon(|y|) = \frac{1}{\epsilon^3} V_1(\frac{|y|}{\epsilon}), \;\; V_2^\epsilon(|y|) = \frac{1}{\epsilon^2} V_2(\frac{|y|}{\epsilon}) \tag{10}$$

(we omit the index m, n) . All potentials are of class C^1. The potential V_3 is unscaled.

Define

$$U^{\epsilon}(|y|) = V_2^{\epsilon} + V_3 \tag{11}$$

If $\epsilon > 0$ the Born series converges and the resolvent can be cast in the Konno-Kuroda form, [K,K] where the operator B is given as (convergent) power series of convolutions of the potential U^{ϵ} and V_1^{ϵ} with the resolvent of H_0. In general

$$\sqrt{V_1^{\epsilon}(|y|) + U^{\epsilon}(|y|)} \neq \sqrt{V_1^{\epsilon}(|y|)} + \sqrt{U^{\epsilon}(|y|)} \tag{12}$$

and in the Konno-Kuroda formula for the resolvent of the operator H_{ϵ} one loses separation between the two potentials V_1^{ϵ} and U^{ϵ}.

Notice however that, if V_1^ϵ and U^ϵ are of class C^1 , the L^1 norm of U^ϵ vanishes as $\epsilon \to 0$ uniformly on the support of V_1^ϵ. By the Cauchy inequality one has

$$lim_{\epsilon \to 0} \| \sqrt{V_1^\epsilon(y)} . \sqrt{U^\epsilon(y)} \|^1 = 0 \tag{13}$$

Therefore if the limit exists the strong and weak contact interactions act independently.

♡

Weak-contact case: separation of the regular part

Consider now *separate* weak-contact interaction of a particle with a pair of indentical particles.

We allow for the presence of a "regular part" represented by a smooth two body L^1 potential of finite range and call *singular part* the quasi contact interaction and the resonance.

Following the same steps that led to the proof of Theorem 1 on proves that for a weak-contact interaction of a particle with two identical particles the singular term (pure weak-contact) and the regular term in the two-body part of the interaction *contribute separately* to the spectral structure of the hamiltonian.

BOUNDARY CHARGES

An important aspect of contact interactions is that they are extension of H_0 that are *entirely due to "charges at the boundary"*. In the present case the boundary are *internal* i.e. they are the *contact manifolds*.

Compare with electrostatics: in that case the boundary has co-dimension one and the Krein map can be identified with the Weyl map from potentials to charges. It is therefore natural to refer to Minlos space as the space of charges [D,F,T].

Also in the strong contact case the distribution of "charges at the boundary" determine uniquely the self-adjoint extension and each function in the domain can be written as the sum of a part in the domain of the Krein map and a "regular part in the domain of H_0.

We sketch here the proof.

Let H be the self-adjoint extension that represent the contact interaction.

Choose λ in such a way that $H + \lambda I$ is invertible and define (as for smooth potentials) the Krein kernel W_λ by

$$\frac{1}{H+\lambda} = \frac{1}{H_0+\lambda} + \frac{1}{H_0+\lambda} W_\lambda \frac{1}{H_0+\lambda} \tag{14}$$

We want to prove that the elements in the domain of the contact hamiltonian H are of the form $\psi = \phi + \zeta$ where $\phi \in D(H_0)$ and ζ is in Krein space.

The action of H on elements in its domain is

$$(H+\lambda)\psi = (H_0+\lambda)\phi \qquad \psi = \phi + K_\lambda \psi \tag{15}$$

so that the action of $H+\lambda$ is completely determined by the action of $H_0+\lambda$ on *a regular part* ϕ and by a singular part *in Minlos space* by obtained acting on ψ by K_λ.

Notice the analogy with electrostatics; the singular part is determined by the charges. The Weyl map takes the role of the Krein map.

The formal proof (modulo control of the domains) is as follows

$$((H+\lambda)\psi, \frac{1}{H_0+\lambda}(H+\lambda)\psi) = ((H_0+\lambda)\phi, \frac{1}{H+\lambda}(H_0+\lambda)\phi)$$
$$= (\phi, (H_0+\lambda)\phi) + (K_\lambda\psi, W_\lambda K_\lambda\psi) \quad (16)$$

Gamma convergence substantiates this formal argumentt. Therefore only "the space of charges" enters in the description of the domain.

It is worth stressing the connection with the *theory of boundary triples* [B,M,N].

This a generalization of the Weyl map in electrostatics from potential in a bounded set Ω in R^3 with regular boundaries to charges at the boundary $\partial\Omega$.

In this context the Krein map may be regarded as a Weyl map between "potentials" and "charges" (the charges belong to a space of more singular functions).

But in the present setting the "boundary charges" are placed on a boundary of co-dimension three (the contact manyfold) and not on an external boundary of co-dimension one as in electrostatics (and in most of the papers on boundary triples).

For contact interactions the boundary *is internal*[L,S]

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