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Existence of Riemannian metrics with positive biorthogonal curvature on simply connected 5-manifolds

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Abstract

We show that the closed simply connected 5-manifold $S^3 \times S^2$ admits Riemannian metrics with strictly positive averages of sectional curvatures of any 2-planes tangent at a given point and which are separated by the smallest distance in the Grassmanian of 2-planes. These metrics have positive Ricci curvature yet there are 2-planes of negative sectional curvature. We use these metrics to show that every closed connected simply connected 5-manifold with vanishing second Stiefel-Whitney class and torsion-free homology admits a Riemannian metric with strictly positive average of sectional curvatures of any pair of orthogonal 2-planes. We show that the symmetric space metric on the Wu manifold satisfies such lower curvature bound.

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Chapter 1

Introduction

Riemannian manifolds of non-negative and positive sectional curvature have been extensively studied, essentially from the beginning of Riemannian geometry. Non-negatively curved examples are fairly plentiful: they are closed under products and include all biquotients (as a consequence of O'Neill's formula [24]), and many cohomogeneity one manifolds, see [33] for an extensive survey. Contrasting this, manifolds with positive sectional curvature seem to be quite rare. For example, apart from spheres and projective spaces, there are no known examples above dimension 24. Further, apart from dimensions 7 and 13, in each dimension there are only finitely many known examples, up to diffeomorphism. This suggests that there should be obstructions to equipping a non-negatively curved Riemannian manifold with a new positively curved metric. However, for closed simply connected manifolds, no such obstructions are known. Almost 100 years ago, Hopf conjectured that $S^2 \times S^2$ (whose standard product metric is non-negatively curved) should not admit a metric of positive sectional curvature. While many partial results are known, the full conjecture has not been resolved. As such, in [4], Bettiol introduces a new notion of curvature called distance curvature as well as the special case of biorthogonal curvature. Positive distance curvature is a weaker property than positive sectional curvature, so one hopes that constructions of such metrics will be more abundant. In particlar, Bettiol shows $S^2 \times S^2$ admits a metric of positive distance curvature. Later, using a surgery theoretic result due to Hoezel [16], Bettiol [2] classifies closed simply connected 4-manifolds admitting metrics of biorthogonal curvature.

The purpose of this thesis is to study the existence of Riemannian metrics on 5-manifolds that satisfy a lower bound on their distance curvature and biorthogonal curvature. The distance curvature is the minimum of the average between sectional curvatures of two 2-planes that are at least some distance apart in the Grassmanian; see Definition 2.14. The biorthogonal curvature is a particular case of the distance curvature, where we use a distance function on the Grasmannian called symmetric space distance and a maximal distance between the planes, so that we are taking averages of two 2-planes that are orthogonal to each other, see Definition 2.15. Our studies build upon work of Bettiol [4], [3], [2] who constructed a family of metrics of positive distance curvature on the product of two 2-spheres $S^2 \times S^2$ and determined the homeomorphism classes of closed simply connected smooth 4-manifolds that admit such metrics.

The main contribution of the thesis is the following theorem.

Theorem 1.1. *For every $\theta > 0$, there is a Riemannian manifold $(S^3 \times S^2, g^\theta)$ such* *that:*

1. $\sec^{\theta}_{g^{\theta}} > 0$.
2. *There is a metric* g^0 *such that* $g^{\theta} \to g^0$ *in the* C^k*-topology as* $\theta \to 0$ *for* $k \geq 0$. *The metric* g^0 *is Wilking's metric* g_W *of almost-positive curvature.*
3. *There is a 2-plane* $\sigma \in \mathrm{Gr}_2\left(T_m(S^3 \times S^2)\right)$ *with* $\sec^{\theta}_{g^{\theta}}(\sigma) < 0$.
4. $\mathrm{Ric}_{g^{\theta}} > 0$.

In particular, there is a Riemannian metric of positive biorthogonal curvature on $S^3 \times S^2$.

Theorem 1.1 is an extension to $S^3 \times S^2$ of the construction of metrics on $S^2 \times S^2$ due to Bettiol.

By using

1) Positivity of biorthogonal curvature is preserved under connected sums, see [3, Proposition 7.11];

2) Bettiol's construction, which under certain conditions can be used to deform metrics with almost positive curvature into metrics with positive biorthogonal curvature, see [4, Section 3];

3) Wilking's construction of a metric with almost positive curvature on $\mathbb{R}P^3 \times \mathbb{R}P^2$, see [30, Section 5]; Ziller's proof [33, Section 5]; and

4) Smale's classification of simply connected 5-manifolds with torsion-free second homology and trivial second Stiefel-Whitney class [27, Main Theorem], we obtained a following result.

Theorem 1.2. *Every closed connected simply connected 5-manifold with zero second Stiefel-Whitney class and torsion-free homology admits a Riemannian metric of positive biorthogonal curvature.*

In Section 3.1, we show that the symmetric space structure on the Wu manifold has positive biorthogonal curvature. In particular, the hypothesis on the second Stiefel-Whitney class and homology of Theorem 1.2 are merely technical in nature. We expect that they can be removed.

A recollection of background notions and results is included in Chapter 2. Chapter 3 starts by showing that the Wu manifold with the symmetric space structure has positive biorthogonal curvature. Next, Bettiol's construction of metrics with positive distance curvature on $S^2 \times S^2$ is recalled. Finally, connected sums and their relation to biorthogonal curvature is considered. In Chapter 4 we first prove Theorem 1.1, and then in the final section we prove Theorem 1.2. Appendix A introduces the Gell-Mann matrices that are used in the calculations on the Wu manifold.

Chapter 2

Background

2.1 Sectional, Ricci, and scalar curvature

We begin by recalling several basic definitions.

Definition 2.1. A **Riemannian manifold** (M, g) is a pair where M is a smooth manifold and g is a symmetric $(0,2)$-tensor field on M, i.e., a section of the S_2T^*M-bundle over M such that its restriction $\mathrm{g}|_m$ to each point $m \in M$ is a positive definite scalar product on T_mM.

Each Riemannian manifold has associated to it a unique connection on the tangent bundle, called the Levi-Civita connection satisfying what Peterson calls Fundamental Theorem of Riemannian geometry, [25, Chapter 2, Theorem 2.2.2]

Theorem 2.2. *For a pair of vector fields (X, Y) on a Riemannian manifold (M, g), an assignment*

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \tag{2.1}$$

$$\nabla_X Y := \nabla(X, Y) \tag{2.2}$$

is uniquely defined by the following properties:

1. *$X \mapsto \nabla_X Y$ is a (1,1)-tensor, i.e., it is well defined for all tangent vectors $Y_m \in T_mM$ and linear*

$$\nabla_{\alpha X_1 + \beta X_2} Y = \alpha \nabla_{X_1} X + \beta \nabla_{X_2} Y \,. \tag{2.3}$$

2. *$Y \mapsto \nabla_X Y$ is a derivation:*

$$\begin{aligned} \nabla_X (Y_1 + Y_2) &= \nabla_X Y_1 + \nabla_X Y_2 \,, \\ \nabla_X (\phi Y) &= X(\phi) Y + \phi \nabla_X Y \,, \end{aligned} \tag{2.4}$$

   for $\phi \in C^\infty(M)$.

3. *Covariant differentiation ∇ is torsion free:*

$$\nabla_X Y - \nabla_Y X = [X, Y] \tag{2.5}$$

4. *Covariant differentiation is metric:*

$$Z\left(\mathrm{g}(X,Y)\right) = \mathrm{g}(\nabla_Z X, Y) + \mathrm{g}(X, \nabla_Z Y)\,. \tag{2.6}$$

The symbol ∇_X has its usual meaning from Riemannian geometry, i.e, ∇_X is the covariant derivative in the direction of X that corresponds to the Levi-Civita connection. See [25, Chapter 2, Chapter 3] for conventions we are using and more details.

In what follows, we adapt standard definitions of sectional, Ricci, and scalar curvature in Riemannian geometry from [25]. Riemann curvature tensor is in the principal fibre bundle language just a curvature of the Levi-Civita connection, see [13, Chapter 5, Chapter 9] for the details of this aproach. For our purposes, Riemann curvature tensor is a (1,3)-tensor defined for all locally defined vector fields X, Y, Z on (M, g) as

Definition 2.3. The **Riemann curvature tensor** is a $(1,3)$-tensor field given by

$$\mathrm{Riem}_{\mathrm{g}}(X,Y)Z := \left(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\right)Z\,, \tag{2.7}$$

for vector fields $X, Y,$ and Z. Using the metric, we can lower the index, or turn $\mathrm{Riem}_{\mathrm{g}}$ from a $(1,3)$-tensor into a $(0,4)$-tensor

$$\mathrm{Riem}_{\mathrm{g}}\left(X,Y,Z,W\right) := \mathrm{g}\left(\mathrm{Riem}_{\mathrm{g}}(X,Y)Z, W\right)\,. \tag{2.8}$$

The symbol $\mathrm{Riem}_{\mathrm{g}}$ in (2.8) is overloaded, but whether we are working with the $(1,3)$ or $(0,4)$ version will always be clear from the context. The following proposition gives the symmetries of $\mathrm{Riem}_{\mathrm{g}}$

Proposition 2.4. *[25, Proposition 3.1.1] The Riemann curvature tensor* $\mathrm{Riem}_{\mathrm{g}}$ *satisfies:*

1. $\mathrm{Riem}_{\mathrm{g}}$ *is skew-symmetric in the first two and the last two entries:*

$$\mathrm{Riem}_{\mathrm{g}}\left(X,Y,Z,W\right) = -\mathrm{Riem}_{\mathrm{g}}\left(Y,X,Z,W\right) = \mathrm{Riem}_{\mathrm{g}}\left(Y,X,W,Z\right) \tag{2.9}$$

2. $\mathrm{Riem}_{\mathrm{g}}$ *is symmetric between the first two and last two entries:*

$$\mathrm{Riem}_{\mathrm{g}}\left(X,Y,Z,W\right) = \mathrm{Riem}_{\mathrm{g}}\left(W,Z,X,Y\right) \tag{2.10}$$

3. $\mathrm{Riem}_{\mathrm{g}}$ *R satisfies a cyclic permutation property called* Bianchi's first identity*:*

$$\mathrm{Riem}_{\mathrm{g}}\left(X,Y\right)Z + \mathrm{Riem}_{\mathrm{g}}\left(Z,X\right)Y + \mathrm{Riem}_{\mathrm{g}}\left(Y,Z\right)X = 0\,. \tag{2.11}$$

In the following definition $\mathrm{Gr}_2(TM)$ is the Grassmanian bundle of 2-planes over M and σ is a 2-plane, i.e. $\sigma \in \mathrm{Gr}_2(T_mM)$

Definition 2.5. The **Sectional curvature** of (M, g) is a map

$$\sec_{\mathrm{g}} : \mathrm{Gr}_2(TM) \to \mathbb{R}\,, \tag{2.12}$$

defined by

$$\sec_{\mathrm{g}}(\sigma) := \frac{\mathrm{Riem}_{\mathrm{g}}(X,Y,Y,X)}{\mathrm{g}(X,X)\mathrm{g}(Y,Y) - \mathrm{g}(X,Y)^2}\,, \tag{2.13}$$

where $\sigma \in \mathrm{Gr}_2(T_mM)$, and X and Y a basis of σ. We call the real number $\sec_{\mathrm{g}}(\sigma)$ **sectional curvature of the 2-plane σ.**

Lemma 2.6. *Sectional curvature* (2.13) *doesn't depend on the choice of basis of* σ*.*

Having in mind Lemma 2.6, we will interchangeably use $X \wedge Y$ and σ to denote the 2-plane $\sigma = \mathrm{span}\{X, Y\}$.

In terms of Riemann curvature tensor, Ricci curvature (0,2)-tensor, is defined as the following trace.

Definition 2.7. The **Ricci curvature tensor** is a (0,2)-tensor defined by

$$\mathrm{Ric_g}(X, Y) := \mathrm{Trace_g}\left(\cdot \mapsto \mathrm{Riem_g}\left(\cdot, X\right) Y\right) . \tag{2.14}$$

One should think of the dot in the previous expression as defining a function.

$$Z \mapsto \mathrm{Riem_g}\left(Z, X\right) Y \tag{2.15}$$

In other words, in (2.15) we keep the variables X and Y fixed and vary Z over its allowed set of values. With notation (2.15) in mind $\cdot \mapsto \mathrm{Riem_g}\left(\cdot, X\right) Y$ is an $\mathbb{R} - linear$ function from $T_m M$ to itself, i.e, $\mathbb{R}$-linear operator. Since the metric tensor g induces a scalar product on $T_m M$, we have enough data to define a unique trace operation that features in the equation (2.14).

Lemma 2.8. *Ricci tensor is symmetric, i.e,*

$$\mathrm{Ric_g}(X, Y) = \mathrm{Ric_g}(Y, X) . \tag{2.16}$$

Definition 2.9. The **Ricci curvature** of (M, g) is a map

$$\mathrm{Ric_g} : T^1 M \to \mathbb{R} , \tag{2.17}$$

defined by

$$\mathrm{Ric_g}(X) := \mathrm{Ric_g}(X, X) , \tag{2.18}$$

for a unit vector X.

Since Ricci tensor is symmetric, there is one and only one way to take its trace. Scalar curvature is the unique trace of the Ricci curvature tensor.

Definition 2.10. The **scalar curvature** of (M, g) is a map

$$\mathrm{scal_g} : M \to \mathbb{R} , \tag{2.19}$$

defined by

$$\mathrm{scal_g} := \mathrm{Trace_g}\left(\mathrm{Ric_g}\right) . \tag{2.20}$$

Scalar curvature can be obtained as a sum of sectional curvatures in a following way

Lemma 2.11. *Let* $\{e_i\}_{i=1\ldots\mathrm{dim(M)}}$ *be an orthonormal basis of* $T_m M$*. Then the scalar curvature of* (M, g) *is given by*

$$\mathrm{scal_g} = \sum_{i,j=1}^{\mathrm{dim(M)}} \mathrm{sec_g}(e_i \wedge e_j) . \tag{2.21}$$

2.2 Distance curvature

In this section, we introduce a notion of curvature that will be our main interest in this thesis. Distance curvature is an average of sectional curvatures of two 2-planes that are some distance apart on the Grassmanian. We follow Bettiol [3, Chapter 5]. First, we introduce a distance on the Grassmanian of 2-planes of an Euclidean vector space V. Then, considering tangent space at each point of a Riemannian manifold, we define distance curvature. Bettiol discusses different distance functions, but we will only consider symmetric space distance, since this distance function leads to the biorthogonal curvature. Let $P, P' \in \mathrm{Gr}_2(V)$ and let S_σ and $S_{\sigma'}$ be the intersections of the unit sphere in V, $S_V = \{v \in V : \|v\|^2 = 1\}$ with σ and σ', respectively.

Definition 2.12. Let $(V, \langle \cdot, \cdot \rangle)$ be an Eucledian vector space and let $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ be two 2-planes in V. The **principal angles** $0 \leq \theta_1 \leq \theta_2 \leq \frac{\pi}{2}$ between σ and σ' are, respectively, the smallest and the largest angle that a line in σ makes with a 2-plane σ', i.e.,

$$\theta_1 = \min_{v \in S_\sigma} \arccos \left(\max_{w \in S_{\sigma'}} \langle v, w \rangle \right) \tag{2.22}$$

$$\theta_2 = \max_{v \in S_\sigma} \arccos \left(\max_{w \in S_{\sigma'}} \langle v, w \rangle \right) . \tag{2.23}$$

Definition 2.13. The **Symmetric space distance** between two 2-planes $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ is defined as:

$$\mathrm{dist}(\sigma, \sigma') := \sqrt{\theta_1^2 + \theta_2^2}\,, \tag{2.24}$$

where θ_1 and θ_2 are principal angles between σ and σ'.

For a Riemannian manifold (M, g), one gets a fiberwise distance function dist on $\mathrm{Gr}_2 TM$, that is, a distance function on each $\mathrm{Gr}_2(T_m M)$ that varies continuously with $m \in M$, by taking the Euclidean space V to be $T_m M$ in the previous definitions.

Definition 2.14. The **distance curvature** of (M, g) for $\theta > 0$ is a map

$$\sec_{\mathrm{g}}^{\theta} : \mathrm{Gr}_2(TM)\,, \to \mathbb{R} \tag{2.25}$$

defined by

$$\sec_{\mathrm{g}}^{\theta}(\sigma) := \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_m M) \\ \mathrm{dist}(\sigma, \sigma') \geq \theta}} \frac{1}{2} \left(\sec_{\mathrm{g}}(\sigma) + \sec_{\mathrm{g}}(\sigma') \right) , \tag{2.26}$$

where $\sigma \in \mathrm{Gr}_2(T_m M)$. We call the real number $\sec_{\mathrm{g}}^{\theta}(\sigma)$ **the distance curvature of the 2-plane σ.**

For maximal value of θ, $\theta = \frac{\pi}{\sqrt{2}}$ every vector from σ is orthogonal to every vector from σ' and we say that the 2-planes σ and σ' are orthogonal. We call distance curvature for $\theta = \frac{\pi}{\sqrt{2}}$ biorthogonal curvature.

Definition 2.15. The **biorthogonal curvature** of (M, g) is a map

$$\sec_{\mathrm{g}}^{\perp} : \mathrm{Gr}_2(TM) \to \mathbb{R}\,, \tag{2.27}$$

defined by

$$\sec_{\mathrm{g}}^{\perp}(\sigma) := \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_m M) \\ \sigma' \subset \sigma^{\perp}}} \frac{1}{2}\left(\sec_{\mathrm{g}}(\sigma) + \sec_{\mathrm{g}}(\sigma')\right). \tag{2.28}$$

where $\sigma \in \mathrm{Gr}_2(T_m M)$. We call the real number $\sec_{\mathrm{g}}^{\perp}(\sigma)$ **the biorthogonal curvature of the 2-plane σ.**

Note that, biorthogonal curvature is defined for manifolds of dimension four or higher. In dimension four, the orthogonal subspace of 2-plane is a unique 2-plane and taking the minimum in (2.28) can be omitted, i.e. in four dimensions

$$\sec_{\mathrm{g}}^{\perp}(\sigma) = \frac{1}{2}\left(\sec_{\mathrm{g}}(\sigma) + \sec_{\mathrm{g}}(\sigma^{\perp})\right). \tag{2.29}$$

2.3 Notions of positivity of curvature

In this section we introduce some lower bounds on curvatures defined in the last two sections, i.e, notions of positivity of curvature, and explore relationships between them.

Definition 2.16. A Riemannian manifold (M, g) **has positive sectional curvature** if its sectional curvature is a strictly positive function. We denote this as $\sec_{\mathrm{g}} > 0$. Similarly, (M, g) has **non-negative sectional curvature** if its sectional curvature is a non-negative function. We denote this as $\sec_{\mathrm{g}} \geq 0$.

A weaker notion than positivity of sectional curvature is the following.

Definition 2.17. A Riemannian manifold (M, g) **has almost-positive curvature** if its sectional curvature is strictly positive everywhere except at points in a subset of measure zero $L \subset M$.

By the continuity of sectional curvature, a manifold with almost positive curvature has non-negative sectional curvature.

Positivity of Ricci, Scalar, Distance, and Biorthogonal curvature is defined in a similar fashion to Definition 2.16. We state the definitions for completeness.

Definition 2.18. A Riemannian manifold (M, g) **has positive Ricci curvature** if its Ricci curvature is a strictly positive function. We denote this as $\mathrm{Ric}_{\mathrm{g}} > 0$.

Definition 2.19. A Riemannian manifold (M, g) **has positive scalar curvature** if its scalar curvature is a strictly positive function. We denote this as $\mathrm{scal}_{\mathrm{g}} > 0$.

Lemma 2.20. *If (M, g) has positive Ricci curvature then (M, g) has positive scalar curvature.*

Proof. Let $\{e_i\}_{i=1\ldots\dim(M)}$ be an orthonormal basis of M, then

$$\mathrm{scal_g} = \sum_{i=1}^{\dim(M)} \mathrm{Ric_g}(e_i) > 0 \tag{2.30}$$

since every term in the sum is positive by assumption. □

Note that converse of Lemma 2.20 does not hold, as a following counter-example shows.

Example 2.20.1. The product of a 2-sphere and 2-torus, $(S^2 \times T^2, \mathrm{g}_{S^2} + \mathrm{g}_{T^2})$ where g_{S^2} is the radius one round metric and g_{T^2} is a flat metric the 2-torus, and plus denotes a product metric., has $\mathrm{scal_g} > 0$, but not $\mathrm{Ric_g} > 0$. To see this, let $\{e_1, e_2, e_3, e_4\}$ denote an orthonormal basis with vectors e_1 and e_2 tangent to S^2 and vectors e_3 and e_4 tangent to T^2, then

$$\mathrm{Ric_g}(e_1) = \mathrm{Ric_g}(e_2) = 1\,, \tag{2.31}$$

and

$$\mathrm{Ric_g}(e_3) = \mathrm{Ric_g}(e_4) = 0\,. \tag{2.32}$$

It follows that scalar curvature is constant and positive

$$\mathrm{scal_g} = 2\,, \tag{2.33}$$

while Ricci curvature is not positive, because of (2.32). Furthermore, since the fundamental group of $S^2 \times T^2$ is $\mathbb{Z}^2$, by Bonnet–Myers Theorem, [25, Theorem 6.3.3], $S^2 \times T^2$ does not admit a metric of positive Ricci curvature.

Definition 2.21. A Riemannian manifold (M, g) **has positive distance curvature** if its distance curvature is a strictly positive function. We denote this as $\sec_{\mathrm{g}}^{\theta} > 0$.

Definition 2.22. A Riemannian manifold (M, g) **has positive biorthogonal curvature** if its biorthogonal curvature is a strictly positive function. We denote this as $\sec_{\mathrm{g}}^{\perp} > 0$.

By definition, $\sec_{\mathrm{g}}^{\theta_1} > 0$ implies $\sec_{\mathrm{g}}^{\theta_2} > 0$, for $\theta_2 > \theta_1$. In particular, since biorthogonal curvature is the distance curvature for maximal value of θ, positive distance curvature implies positive biorthogonal curvature. In what follows we will show that, similarly to Ricci curvature, positive biorthogonal curvature implies positive scalar curvature. To this end we cite the following Lemma

Lemma 2.23. *[11, Lemma 3.100] For any symmetric bilinear ϕ form on $\mathbb{R}^n$*

$$\int_{S^{n-1}} \phi(V,V)\mathrm{d}S_V^{n-1} = \frac{1}{n}\mathrm{vol_g}(S^{n-1})\mathrm{Trace_g}(\phi)\,. \tag{2.34}$$

Using 2.23 one can prove the following.

Lemma 2.24. *If the biorthogonal curvature of (M, g) is positive at a point $m \in M$, then the scalar curvature is positive at that point.*

Proof. Fix a point $m \in M$. Scalar curvature at that point can be written as a following average

$$\mathrm{scal}_{\mathrm{g}}(m) = \frac{n(n-1)}{\mathrm{vol}_{\mathrm{g}}(S^{n-1})\mathrm{vol}_{\mathrm{g}}(S^{n-2})} \int_{S^{n-2}} \int_{S^{n-1}} \sec_{\mathrm{g}}(U \wedge V) \mathrm{d}S_U^{n-1} \mathrm{d}S_V^{n-2} \tag{2.35}$$

by applying Lemma 2.23 two times. First on the trace that defines Ricci curvature, (2.14) and then on the trace that defines scalar curvature (2.20). More geometrically

$$\mathrm{scal}_{\mathrm{g}}(m) = \frac{\int_{\mathrm{Gr}_2(T_m M)} \star \sec_{\mathrm{g}}(\sigma)}{\int_{\mathrm{Gr}_2(T_m M)} \star 1}, \tag{2.36}$$

where, $\star$ denotes the Hodge star operator, see [13, Chapter 3], possibly up to a positive constant factor that is irrelevant for the proof. We proceed by contradiction and assume that $\mathrm{scal}_{\mathrm{g}}(m) \leq 0$ and $\sec_{\mathrm{g}}^{\perp}(\sigma) > 0$ for all 2-planes $\sigma \in \mathrm{Gr}_2(T_m M)$. Then we have the following chain of inequalities

$$\begin{aligned} 0 \geq \mathrm{scal}_{\mathrm{g}}(m) &= \lambda \int_{\sigma \in \mathrm{Gr}_2(T_m M)} \star \sec_{\mathrm{g}}(\sigma) \\ &= \lambda \int_{\sigma \in \mathrm{Gr}_2(T_m M)} \star \frac{1}{2} \left(\sec_{\mathrm{g}}(\sigma) + \sec_{\mathrm{g}}(\sigma)\right) \\ &\geq \lambda \int_{\sigma \in \mathrm{Gr}_2(T_m M)} \star \frac{1}{2} \left(\sec_{\mathrm{g}}(\sigma) + \min_{\sigma' \subset \sigma^{\perp}} \sec_{\mathrm{g}}(\sigma')\right) \\ &= \lambda \int_{\sigma \in \mathrm{Gr}_2(T_m M)} \star \sec_{\mathrm{g}}^{\perp}(\sigma) . \end{aligned} \tag{2.37}$$

In conclusion, $\int_{\sigma \in \mathrm{Gr}_2(T_m M)} \star \sec_{\mathrm{g}}^{\perp}(\sigma) \leq 0$, but this means that there exists a 2-plane σ'' at point $m \in M$ with $\sec_{\mathrm{g}}^{\perp}(\sigma'') \leq 0$, contradicting the initial assumption. □

Applying Lemma 2.24 to all points of M gives the following Corollary.

Corollary 2.24.1. *If (M, g) has positive biorthogonal curvature, then (M, g) has positive scalar curvature.*

Despite both positive Ricci curvature and positive biorthogonal curvature implying positive scalar curvature, positive Ricci curvature does not imply positive biorthogonal curvature, nor does positive biorthogonal curvature imply positive Ricci curvature.

Example 2.24.1. The Riemannian manifold $(S^2 \times S^2, \mathrm{g})$, with g a product of round metrics on a 2-sphere of radius one S^2 has positive Ricci curvature, but doesn't have positive biorthogonal curvature. The manifold $(S^2 \times S^2, \mathrm{g})$ is an Einstein manifold with

$$\mathrm{Ric}_{\mathrm{g}} = \mathrm{g}, \tag{2.38}$$

and thus has positive Ricci curvature. However, sectional curvature of any mixed 2-plane is zero and an orthogonal 2-plane to a mixed 2-plane is again a mixed 2-plane, and so biorthogonal curvature of any mixed 2-plane on $(S^2 \times S^2, \mathrm{g})$ is zero. By mixed 2-plane we mean a 2-plane a 2-plane is mixed if and only if its projection to each factor has 1-dimensional image. Note that in [4], Bettiol deforms the metric g to a metric of positive distance curvature that is arbitrarily close to g in the C^k-topology. In this thesis, we apply a similar deformation to obtain a metric of positive distance curvature on $S^3 \times S^2$, see Section 4.

The following is an example of a manifold that has $\sec^\perp > 0$, but doesn't have $\mathrm{Ric}_{\mathrm{g}} > 0$.

Example 2.24.2. The product Riemannian manifold $(M^3 \times S^1, \mathrm{g}_M + d\theta^2)$, where g_M is a metric with positive sectional curvature, has positive biorthogonal curvature, but does not have positive Ricci curvature. A 2-plane is flat if and only if it is mixed, while 2-planes that are not mixed have positive sectional curvature and in particular positive biorthogonal curvature. Since S^1 is 1-dimensional, a plane orthogonal to a mixed plane cannot be mixed. Thus the biorthogonal curvature of a mixed 2-plane is positive. On the other hand, the fundamental group of $M^3 \times S^1$ is $\pi_1(M) \times \mathbb{Z}$, and so by Bonnet-Myers Theorem, $M^3 \times S^1$ does not admit a metric of positive Ricci curvature.

2.4 Riemannian submersions

In this section we define Riemannian submersions and homogeneous spaces. Finally we recall result by O'Neill [24, Corollary 1.3] and a Theorem by Tapp [29, Theorem 1.1] that we will need in later Chapters. See [25, Chapter 1] and [34, Section 1] for more details.

Definition 2.25. Let (M, g_M) and (N, g_N) be Riemannian manifolds and let ϕ be a smooth submersion from M to N, i.e.,

$$\pi \in \{\phi \in C^\infty(M, N) : (\forall m \in M)(\phi_*|_m \text{ is surjective})\} . \tag{2.39}$$

If in addition π is such that at every point $m \in M$ the following holds

$$\mathrm{g}_M(X, Y)(m) = \mathrm{g}_N(\pi_* X, \pi_* Y)(\pi(m)) , \tag{2.40}$$

for all $X, Y \in T_m M$, we call ϕ a **Riemannian submersion**.

Remark 2.25.1. Let π be a Riemannian submersion from (M, g_M) to (N, g_N). Then at every point $m \in M$, we call null-space of $\pi_*|_m$ the **vertical space** of submersion π.

Remark 2.25.2. The **horizontal space** is the orthogonal complement of the vertical space in $T_m M$.

While vertical space can be defined in the same manner for any submersion, notion of the horizontal space requires M and N to be equipped with Riemannian metrics and condition (2.40).

Definition 2.26. For a Riemannian submersion

$$\pi : (M, \mathrm{g}_M) \to (N, \mathrm{g}_N) \tag{2.41}$$

to each locally defined vector field on N, X we can associate a unique locally defined horizontal vector field $\bar{X}$, i.e. $\bar{X}(m) \in \mathrm{Hor}_m M$, such that

$$\pi_* \bar{X} = X . \tag{2.42}$$

The vector field $\bar{X}$ is called the **horizontal lift** of X.

Remark 2.26.1. By (2.40) and (2.42), the vector $X(\pi(m))$ and its horizontal lift $\bar{X}(m)$ have the same length.

Example 2.26.2. Homogeneous spaces, for details see [26, Appendix 2], and section 2.5. Suppose that G is a compact Lie group that acts from the left, transitively, and isometrically on a compact Riemannian manifold (M, g) and call $H < G$ a Lie subgroup of G that is smoothly isomorphic to isotropy groups of every point in M. Under these assumptions, the canonical projection:

$$\pi : G \to G/H \tag{2.43}$$

is a submersion, and there is a diffeomorphism:

$$\phi : G/H \xrightarrow{\sim} M\,. \tag{2.44}$$

Diffeomorphism ϕ, precomposed with the canonical projection π, induces an anti-homomorphism from the Lie algebra of right invariant vector fields on G to the Lie algebra of Killing vector fields on (M, g), i.e.,

$$(\phi \circ \pi)_* : [X, Y]_{\mathfrak{g}} \mapsto (\phi \circ \pi)_* \left([X, Y]_{\mathfrak{g}}\right) = -[(\phi \circ \pi)_* X, (\phi \circ \pi)_* Y]_M\,. \tag{2.45}$$

Kernel of this anti-homomorphism is precisely the Lie algebra of H and there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp} \tag{2.46}$$

that is orthogonal with respect to some fixed left-G and $\mathrm{Ad}(H)$ invariant metric Q on G. Suppose that $(\phi \circ \pi)(\mathrm{e}) = m_{\mathrm{e}} \in M$, then

$$(\phi \circ \pi)_*(\mathrm{e}) : \mathfrak{h}^{\perp} \xrightarrow{\sim} T_{m_{\mathrm{e}}} M \tag{2.47}$$

is an isomorphism of vector spaces, and one can choose the metric Q in such a way to promote the isomorphism $(\phi \circ \pi)_*(\mathrm{e})$ into an isometry. Thus, making submersion $\phi \circ \pi$ into a Riemannian submersion

$$(\phi \circ \pi) : (G, \mathrm{Q}) \to (M, \mathrm{g})\,. \tag{2.48}$$

Vertical space and horizontal space of Riemannian submersion 2.48 at the identity element e are $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$, while the orthogonal decomposition

$$T_{\mathrm{e}} G = \mathrm{Ver}_{\mathrm{e}} G \oplus \mathrm{Hor}_{\mathrm{e}} G \tag{2.49}$$

is precisely the decomposition (2.46). Differential of left translation on G, L_{g*}, preserves the splitting (2.49) because Q is left invariant, and we get vertical and horizontal sub-spaces at points other than the identity by left translating tangent space at the identity $T_{\mathrm{e}} G$.

The following result of O'Neill describes how sectional curvature behaves under Riemannian submersions.

Theorem 2.27. *[24, Corollary 1.3] For a Riemannian submersion*

$$\pi : (M, \mathrm{g}_M) \to (N, \mathrm{g}_N)\,, \tag{2.50}$$

2-plane $X \wedge Y$ and its horizontal lift $\bar{X} \wedge \bar{Y}$ the following holds

$$\sec_{\mathrm{g}_N}(X \wedge Y) = \sec_{\mathrm{g}_M}(\bar{X} \wedge \bar{Y}) + \frac{3}{4}\|[\bar{X}, \bar{Y}]^{\mathrm{Ver}}\|^2\,, \tag{2.51}$$

where superscript Ver *denotes projection to the vertical subspace.*

A consequence of (2.51) is

$$\sec_{\mathrm{g}_N}(X \wedge Y) \geq \sec_{\mathrm{g}_M}(\bar{X} \wedge \bar{Y})\,. \tag{2.52}$$

In particular, if (M, g_M) has non-negative sectional curvature so does (N, g_N). For Riemannian submersions from Lie groups with bi-invariant metric the second term on the right hand side of (2.51) is zero on flat horizontal planes by a result of Tapp [29, Theorem 1.1]

Theorem 2.28. *[29, Theorem 1.1] If*

$$\pi : (G, \mathrm{Q}) \to (B, \mathrm{g})\,, \tag{2.53}$$

is a Riemannian submersion from a Lie group with a bi-invariant metric, then

1. *Every horizontal flat 2-plane in G projects to a flat 2-plane in B.*
2. *Every flat 2-plane in B exponentiates to a totally geodesic immersion of $\mathbb{R}^2$ with a flat metric.*

Inequality (2.52) and Theorem 2.28 imply that in the case of a Riemannian submersion (2.53) flat 2-planes in B are in one to one correspondence with horizontal flat 2-planes in G. We will find this result useful in later chapters. Another useful Corollaryx of Theorem 2.28 is the following.

Corollary 2.28.1. *Let (G, Q) be Lie group equipped with a bi-invariant metric. If*

$$\pi : (G, \mathrm{Q}) \to (M, \mathrm{g}_M) \tag{2.54}$$

and

$$\rho : (M, \mathrm{g}_M) \to (B, \mathrm{g}_B) \tag{2.55}$$

are Riemannian submersions, then any horizontal flat 2-plane in M projects to a flat 2-plane in B.

2.5 Lie groups, symmetric spaces, and Cartan decomposition

In this section we review classical results about compact Lie groups and symmetric spaces. We follow [1], [26], [23], and Eschenburg's notes on symmetric spaces [9]. First, we adapt a part of [1, Chapter 2, Proposition 2.26] to our conventions and notation.

Proposition 2.29. *Let G be a Lie group equipped with a bi-invariant metric Q, and $X, Y \in \mathfrak{g} = T_eG$. Then*

$$\mathrm{Riem}_{\mathrm{Q}}(X,Y,Y,X) = \frac{1}{4}\|[X,Y]\|^2 \,. \tag{2.56}$$

It follows that if X and Y are an orthonormal basis of a 2-plane $\sigma \in \mathrm{Gr}_2(T_eG)$, then

$$\sec_{\mathrm{Q}}(\sigma) = \frac{1}{4}\|[X,Y]\|^2 \,. \tag{2.57}$$

Definition 2.30. A **Geodesic symmetry** at a point m of a connected Riemannian manifold (M, g) is an isometry

$$s_m : M \to M \,, \tag{2.58}$$

such that

$$s_m(m) = m \,, \tag{2.59}$$

and

$$(s_m)_*|_m = -\mathrm{Id}_{T_mM} \,. \tag{2.60}$$

It can be shown that $s_m^2 = \mathrm{id}_M$.

Definition 2.31. A Riemannian manifold (M, g) is called a **symmetric space** if for every $m \in M$ there exists a geodesic symmetry at m.

A symmetric space is geodesically complete because any geodesic can be extended indefinitely via symmetries about its endpoints. Furthermore, every symmetric space is a homogeneous space. To see this, take two points of a symmetric space (M, g), $m_1, m_2 \in M$ and connect them by a unique length minimizing geodesic γ. The geodesic symmetry about the midpoint of γ is an isometry that sends m_1 to m_2. Since m_1 and m_2 are arbitrary (M, g) is a homogeneous space. This means that, as with any other homogeneous space, we can pick an arbitrary point $m_e \in M$ and realize M as a coset space of G/H, see example 2.48. Here, G is a Lie group of midpoint geodesic symmetries and H is the isotropy group of m_e. We can also pull back the metric g from M to G/H by the identifying diffeomorphism, and extend it to a left invariant, $\mathrm{Ad}_{\mathfrak{g}}(H)$ invariant metric on G, which we will denote by Q.

Next, we define an involutive automorphism of G by conjugating by the geodesic symmetry $s_{m_e} \in G$

$$\Theta : g \mapsto s_{m_e} g s_{m_e}^{-1} \,. \tag{2.61}$$

If we denote the fixed point set of Θ by F, i.e., $F := \{k \in G; \Theta(k) = k\}$ and by F_e the identity component of F, then

$$F_e \subset H \subset F \,. \tag{2.62}$$

Let θ be differential of Θ at the identity of G, i.e.,

$$\theta := \Theta_*|_e : \mathfrak{g} \to \mathfrak{g} \,. \tag{2.63}$$

Differential θ is an involutive automorphism of $\mathfrak{g}$ with eigenvalues ± 1. Corresponding eigenspace decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\,, \tag{2.64}$$

where

$$\mathfrak{m} := \{X \in \mathfrak{g}; \theta(X) = -X\}\,, \tag{2.65}$$

$$\mathfrak{h} := \{X \in \mathfrak{g}; \theta(X) = X\}\,. \tag{2.66}$$

Because of (2.62), $\mathfrak{h}$ is the Lie algebra of H. Furthermore, because of the way Q is constructed, decomposition (2.64) is Q-orthogonal. Since θ is an automorphism of $\mathfrak{g}$ we have

$$[\theta(X), \theta(Y)] = \theta([X, Y])\,. \tag{2.67}$$

From (2.67) and definitions (2.65), (2.66) it follows that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\,,\ [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\,,\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,. \tag{2.68}$$

Definition 2.32. An involutive automorphism

$$\theta : \mathfrak{g} \to \mathfrak{g} \tag{2.69}$$

of a Lie algebra $\mathfrak{g}$ is called a **Cartan involution** if $\mathrm{ad}(\mathfrak{h})|_{\mathfrak{m}}$ is a Lie algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{m})$, where $\mathfrak{h}$ is $+1$ eigenspace of θ, and $\mathfrak{m}$ is the -1 eigenspace of θ.

Definition 2.33. Direct sum decompostion

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{2.70}$$

of a Lie algebra $\mathfrak{g}$ is called **Cartan decomposition** if $\mathrm{ad}(\mathfrak{h})|_{\mathfrak{m}}$ is a Lie algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{m})$ and

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\,,\ [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\,,\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,. \tag{2.71}$$

It is easy to see that each Cartan involution corresponds to a unique Cartan decomposition and vice versa. Equation (2.64) is a Cartan decomposition of the Lie algebra of the midpoint geodesic symmetries of a symmetric space (M, g), thus we can associate a Cartan decomposition to a symmetric space. The compactness assumption is satisfied for compact M. Converse is also true. Given a Cartan decomposition of a Lie algebra $\mathfrak{g}$ we can associate to it a unique simply connected symmetric space.

Example 2.33.1. Every Cartan decomposition of $\mathfrak{su}(n)$ is conjugate to one of the types **AI**, **AII**, and **AIII** [6, Chapter 2]:

1. Type **AI** corresponds to a decomposition into purely real and purely imaginary subspaces

$$\mathfrak{su}(n) = \mathfrak{so}(n) \oplus \mathfrak{so}(n)^{\perp}\,. \tag{2.72}$$

Associated Cartan involution is

$$\theta : X \mapsto -X^T\,. \tag{2.73}$$

2. Type **AII** decomposition is defined for even n. It is given by

$$\mathfrak{su}(n) = \mathfrak{sp}(\frac{n}{2}) \oplus \mathfrak{sp}(\frac{n}{2})^{\perp}\,. \tag{2.74}$$

Associated Cartan involution is

$$\theta : X \mapsto -JX^T J\,, \tag{2.75}$$

where

$$J = \begin{bmatrix} 0 & I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & 0 \end{bmatrix}\,. \tag{2.76}$$

3. Type **AIII** is given in term of two positive integers such that $p + q = n$. It is of the form

$$\mathfrak{su}(n) = \mathfrak{h} \oplus \mathfrak{m}\,, \tag{2.77}$$

where

$$\mathfrak{h} := \operatorname{span}\left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} ; A \in \mathfrak{u}(p)\,, B \in \mathfrak{u}(q)\,, \operatorname{Tr}(A) + \operatorname{Tr}(B) = 0 \right\}\,, \tag{2.78}$$

$$\mathfrak{m} := \operatorname{span}\left\{ \begin{bmatrix} 0 & C \\ -C^* & 0 \end{bmatrix} ; C \in \operatorname{Mat}_{p\times q}(\mathbb{C}) \right\}\,. \tag{2.79}$$

Corresponding Cartan involution is

$$\theta : X \mapsto I_{p,q} X I_{p,q}\,, \tag{2.80}$$

where

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}\,. \tag{2.81}$$

Simply connected symmetric space corresponding to a type AI decomposition of $\mathfrak{su}(3)$ is called the Wu manifold and in section 3.1 we will show that this manifold has positive biorthogonal curvature. To do this, we will need the following result.

Proposition 2.34. *[23, Proposition 7.29] Let G be a compact Lie group. If $\mathfrak{a}$ and $\mathfrak{a}'$ are two maximal abelian subalgebras of $\mathfrak{m}$ then there is a member $h \in H$ with* $\operatorname{Ad}(h)\mathfrak{a}' = \mathfrak{a}$.

2.6 Cheeger deformations on Lie groups

In this section, we introduce Cheeger deformations in the special case of bi-invariant metrics on Lie groups, see [5], [10].

Let G be a Lie group, and let $K \subset G$ be closed subgroup. Equip G with a bi-invariant metric (G, g_0). Consider the right diagonal action of K on $G \times K$,

$$(g,k)k' = (gk', kk')\,, \tag{2.82}$$

for $g \in G$ and $k, k' \in K$. Orbit space of (2.82) is

$$G \cong (G \times K)/\Delta K\,, \tag{2.83}$$

with the quotient map given by

$$\begin{aligned} \rho &: G \times K \to G \\ \rho(g,k) &= gk^{-1}\,. \end{aligned} \tag{2.84}$$

Equip the $G \times K$ with a product of bi-invariant metrics $(G \times K, \mathrm{g}_0 + t\mathrm{g}_0|_K)$, where $t > 0$. Because action (2.82) is by isometries, there is an induced metric g_1 making the quotient map ρ into a Riemannian submersion

$$\rho : (G \times K, \mathrm{g}_0 \oplus t\mathrm{g}_0|_K) \to (G, \mathrm{g}_1)\,. \tag{2.85}$$

Induced metric g_1 is the Cheeger deformation of g_0. Since $(G \times K, \mathrm{g}_0 + t\mathrm{g}_0|_K)$ has $\sec \geq 0$ and Riemannian submersions don't decrease the curvature it follows that $\sec_{\mathrm{g}_1} \geq 0$. Next, consider two actions on $(G \times K, \mathrm{g}_0 + t\mathrm{g}_0|_K)$,

$$g' \star (g,k) = (g'g, k)\,, \tag{2.86}$$

$$k' * (g,k) = (g, k'k)\,, \tag{2.87}$$

for $g' \in G$, $k' \in K$, $(g,k) \in G \times K$. Action (2.86) and (2.87) are by isometries and commute with the action (2.82), so they descend to actions by isometries on (G, g_1). One has

$$\rho(g' \star (g,k)) = \rho(g'g, k) = g'gk^{-1} = g'\rho(g,k)\,, \tag{2.88}$$

$$\rho(k' * (g,k)) = \rho((g, k'k)) = g(k'k)^{-1} = \rho(g,k)k'^{-1}\,, \tag{2.89}$$

so (2.86) descends to left multiplication by elements of G, and (2.87) descends to the right multiplication by elements of K. It follows that the metric g_1 is G-left invariant and K-right invariant. However, right multiplication by an arbitrary element of G is not an isometry of (G, g_1).

Let $\mathfrak{k}$ denote the Lie algebra of subgroup K. Lie algebra of G, $\mathfrak{g}$ splits into an orthogonal sum

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}\,, \tag{2.90}$$

Differential of the submersion ρ is

$$\rho_*|_{(e,e)}(X,Y) = \frac{d}{dt}|_{t=0}\rho(\exp(tX),\exp(tY)) = \\ = \frac{d}{dt}|_{t=0}\left(\exp(tX)\exp(-tY)\right) = X - Y\,. \tag{2.91}$$

It follows that the vertical subspace of $T_{(e,e)}(G\times K)$ is

$$\mathrm{Ver}_{(e,e)} = \{(X,X) : X\in\mathfrak{k}\} \tag{2.92}$$

Given a vector $X\in\mathfrak{g}$, one can see that its horizontal lift is

$$\bar{X} = (X_{\mathfrak{p}} + \frac{t}{1+t}X_{\mathfrak{k}}, -\frac{1}{1+t}X_{\mathfrak{k}})\,. \tag{2.93}$$

Let Φ be a symmetric, positive linear map of $\mathfrak{g}$, defined as

$$\mathrm{g}_1(X,Y) = \mathrm{g}_0(\Phi X, Y)\,. \tag{2.94}$$

Then one has

$$\Phi X = X_{\mathfrak{p}} + \frac{t}{1+t}X_{\mathfrak{k}}\,, \tag{2.95}$$

$$\Phi^{-1}X = X_{\mathfrak{p}} + \frac{1+t}{t}X_{\mathfrak{k}}\,, \tag{2.96}$$

and one can write the horizontal lift (2.93) as

$$\bar{X} = (\Phi X, -t^{-1}(\Phi X)_{\mathfrak{k}})\,. \tag{2.97}$$

By iterating the preceding construction one arrives at the following Lemma. See [8] for a similar construction.

Lemma 2.35. *Let*

$$K_n \subset K_{n-1} \subset \ldots K_1 \subset G = K_0 \tag{2.98}$$

be a chain of closed subgroups of a compact Lie group (G,g_0)*. Denote the Lie algebra of* K_i *by* $\mathfrak{k}_i$ *and by* $\mathfrak{p}_i$ *the* $\mathrm{g}_0|_{K_{i-1}}$*-orthogonal complement of* $\mathfrak{k}_i$ *in* $\mathfrak{k}_{i-1}$*, i.e,*

$$\mathfrak{k}_{i-1} = \mathfrak{p}_i \oplus \mathfrak{k}_i\,, \tag{2.99}$$

for $i = 1,2\ldots n$*. Then, the metric* g_n *on* G *defined by*

$$\mathrm{g}_n(X,Y) = \mathrm{g}_0(\Phi X, Y)\,,\ X,Y\in\mathfrak{g}\,, \tag{2.100}$$

where Φ *is* g_0*-symmetric, positive linear map given by*

$$\Phi X = X_{\mathfrak{p}_1} + \frac{1}{1+t_1^{-1}}X_{\mathfrak{p}_2} + \frac{1}{1+t_1^{-1}+t_2^{-1}}X_{\mathfrak{p}_3} + \ldots + \\ + \frac{1}{1+\sum_{i=1}^{n-1}t_i^{-1}}X_{\mathfrak{p}_n} + \frac{1}{1+\sum_{i=1}^{n}t_i^{-1}}X_{\mathfrak{k}_n}\,, \tag{2.101}$$

for positive real numbers $t_1, t_2\ldots t_n$*, has the following properties:*

1. g_n *is G-left invariant.*

2. g_n *is* K_n*-right invariant.*

3. (G, g_n) *has* $\sec_{\mathrm{g}_n} \geq 0$.

Proof. Proof consists of successively applying Cheeger deformation, starting from

$$(G \times K_1 \times K_2 \times ... \times K_n, \mathrm{g}_0 + t_1 \mathrm{g}_0|_{K_1} + t_2 \mathrm{g}_0|_{K_2} + ... + t_n \mathrm{g}_0|_{K_n}) . \tag{2.102}$$

Consider the following submersions

$$\begin{aligned}
\pi_1 : G \times K_1 \times K_2 \times ... \times K_n &\to G \times K_2 \times ... \times K_n \\
\pi_1(g, k_1, k_2, ..., k_n) &= (g k_1^{-1}, k_2, ..., k_n) , \\
\pi_2 : G \times K_2 \times K_3 \times ... \times K_n &\to G \times K_3 \times ... \times K_n \\
\pi_2(g, k_2, k_3, ..., k_n) &= (g k_2^{-1}, k_3, ..., k_n) , \\
&\vdots \\
\pi_i : G \times K_i \times K_{i+1} \times ... \times K_n &\to G \times K_{i+1} \times ... \times K_n \\
\pi_2(g, k_i, k_{i+1}, ..., k_n) &= (g k_i^{-1}, k_{i+1}, ..., k_n) , \\
&\vdots \\
\pi_n : G \times K_n &\to G \\
\pi_n(g, k_n) &= g k_n^{-1} ,
\end{aligned} \tag{2.103}$$

and let $\rho = \pi_n \circ \pi_{n-1} \circ ... \circ \pi_2 \circ \pi_1$, metric g_n is the metric that makes ρ into a Riemannian submersion. Routine calculations show that horizontal lift all the way up of $X \in \mathfrak{g}$ is

$$\bar{X} = \left(\Phi X, -t_1^{-1}(\Phi X)_{\mathfrak{k}_1}, -t_2^{-1}(\Phi X)_{\mathfrak{k}_2}, ..., -t_n^{-1}(\Phi X)_{\mathfrak{k}_n}\right) , \tag{2.104}$$

where Φ is given by the expression (2.101). Using (2.104) and the fact that ρ is a Riemannian submersion to (G, g_n), one finds that (2.100) holds. Group G acts by isometries on (2.102) by left multiplication on the first factor. This action descends to an action by isometries on (G, g_n) given by left multiplication. The group K_n acts by isometries on (2.102) by left multiplication of the last factor and this action descends to an action by isometries on (G, g_n) that is the right multiplication by the inverse. The product (2.102) has non-negative sectional curvature, and since Riemannian submersions don't decrease curvature, it follows that (G, g_n) has $\sec_{\mathrm{g}_n} \geq 0$. □

Necessary and sufficient conditions for a 2-plane on (G, g_n) to be flat are given by the following Lemma.

Lemma 2.36. *Let* (G, g_n) *be a compact Lie group with a metric obtained by iterated Cheeger deformations as in Lemma 2.35. Then a 2-plane* $X \wedge Y \in Gr_2(\mathfrak{g})$ *is flat if and only if*

$$[(\Phi X)_{\mathfrak{k}_i}, (\Phi Y)_{\mathfrak{k}_i}] = 0 , \tag{2.105}$$

for all $i = 0, 1, ...n$, *where* $\mathfrak{k}_i$ *is the Lie algebra of* K_i *and* Φ *is the isomorphism* (2.101).

Proof. Denote the product metric (2.102) by

$$\mathrm{g}_K = \mathrm{g}_0 + t_1 \mathrm{g}_0|_{K_1} + t_2 \mathrm{g}_0|_{K_2} + \ldots + t_n \mathrm{g}_0|_{K_n} \,. \tag{2.106}$$

By Theorem 2.28, $\sec_{\mathrm{g}_n}(X \wedge Y) = 0$ if and only if $\sec_{\mathrm{g}_K}(\bar{X} \wedge \bar{Y}) = 0$. Sectional curvature is zero, $\sec_{\mathrm{g}_K}(\bar{X} \wedge \bar{Y}) = 0$, if and only if unnormalized sectional curvature is zero, $\mathrm{Riem}_{\mathrm{g}}(\bar{X}, \bar{Y}, \bar{Y}, \bar{X}) = 0$. Using (2.104) and (2.106) we have

$$\begin{aligned} \mathrm{Riem}_{\mathrm{g}}(\bar{X}, \bar{Y}, \bar{Y}, \bar{X}) = {} & \mathrm{Riem}_{\mathrm{g}_0}(\Phi X, \Phi Y, \Phi Y, \Phi X) + \\ & + t_1^{-3} \mathrm{Riem}_{\mathrm{g}_0}((\Phi X)_{\mathfrak{k}_1}, (\Phi Y)_{\mathfrak{k}_1}, (\Phi Y)_{\mathfrak{k}_1}, (\Phi X)_{\mathfrak{k}_1}) + \\ & + t_2^{-3} \mathrm{Riem}_{\mathrm{g}_0}((\Phi X)_{\mathfrak{k}_2}, (\Phi Y)_{\mathfrak{k}_2}, (\Phi Y)_{\mathfrak{k}_2}, (\Phi X)_{\mathfrak{k}_2}) + \\ & \vdots \\ & + t_n^{-3} \mathrm{Riem}_{\mathrm{g}_0}((\Phi X)_{\mathfrak{k}_n}, (\Phi Y)_{\mathfrak{k}_n}, (\Phi Y)_{\mathfrak{k}_n}, (\Phi X)_{\mathfrak{k}_n}) \,. \end{aligned} \tag{2.107}$$

Using expression (2.56) we have

$$\begin{aligned} \mathrm{Riem}_{\mathrm{g}}(\bar{X}, \bar{Y}, \bar{Y}, \bar{X}) = \frac{1}{4} \|[\Phi X, \Phi Y]\|^2_{\mathrm{g}_0} + \frac{t_1^{-3}}{4} \|[(\Phi X)_{\mathfrak{k}_1}, (\Phi Y)_{\mathfrak{k}_1}]\|^2_{\mathrm{g}_0} + \\ + \frac{t_2^{-3}}{4} \|[(\Phi X)_{\mathfrak{k}_2}, (\Phi Y)_{\mathfrak{k}_2}]\|^2_{\mathrm{g}_0} + \ldots + \frac{t_n^{-3}}{4} \|[(\Phi X)_{\mathfrak{k}_n}, (\Phi Y)_{\mathfrak{k}_n}]\|^2_{\mathrm{g}_0} \,. \end{aligned} \tag{2.108}$$

Since (2.108) is a sum of non-negative terms, it is zero if and only if all of the terms on the right-hand side are zero. This is condition (2.105), completing the proof. □

2.7 Biquotients and Wilking's doubling trick

In this section, we discuss biquotients following [10], [21]. We proceed to describe Wilking's doubling trick as in [31]. Finally, we characterise flat 2-planes on biquotients equipped with metrics obtained by Wilking's doubling trick following [7].

We start with a following definition.

Definition 2.37. Let G be a compact Lie group and let $H \subset G \times G$ be a closed subgroup such that the action of H on G given by

$$(h_1, h_2) \star g = h_1 g h_2^{-1} \,, \tag{2.109}$$

for $(h_1, h_2) \in H$ and $g \in G$, is effectively free, i.e., an element $h \in H$ has a fixed point if and only if h is in the kernel of action (2.109). In this case we call the orbit space of the action (2.109) a **biquotient** and denote it $G /\!\!/ H$.

If $H = \{e\} \times H'$, where $H' \subset G$ is a closed subgroup, biquotient $G /\!\!/ H$ is the homogeneous space G/H'.

A metric g on G that is invariant under the action (2.109) induces a metric $\tilde{\mathrm{g}}$ on $G /\!\!/ H$ making the projection

$$\pi : (G, \mathrm{g}) \to (G /\!\!/ H, \bar{\mathrm{g}}) \tag{2.110}$$

into a Riemannian submersion. In what follows we will mostly be concerned with biquotients equipped with metrics induced in such a way. There are two natural families of metrics on $G /\!\!/ H$; the family of metrics induced by left invariant metrics on G, and the family induced by the right invariant metrics on G. Wilking's doubling trick is a construction that gives an even larger family of natural metrics on $G /\!\!/ H$.

Lemma 2.38. *Let $H \subset G \times G$ as in definition 2.37, and let $\Delta G \subset G \times G$ denote the diagonal subgroup. Then the action of $\Delta G \times H$ on $G \times G$ given by*

$$(a, h) \star (c, d) = a \cdot (c, d) \cdot h^{-1}, \tag{2.111}$$

for $a \in \Delta G$, and $h \in H$, is effectively free, the biquotient $\Delta G \backslash G \times G/H$ is canonically diffeomorphic to $G/\!\!/H$, and the class of left invariant Ad_H-invariant on $G \times G$ induces a cone of metrics on the quotient containing the two original families.

Proof. Straightforward calculation shows that action (2.111) is effectively free if and only if action (2.109) is effectively free. The canonical diffeomorphism is induced by the map

$$G \times G \to G, \ (a, b) \mapsto a^{-1}b. \tag{2.112}$$

Finally, consider all left invariant Ad_H-invariant product metrics $\mathrm{g}_1 + \mathrm{g}_2$ on $G \times G$. Subfamily of metrics for which g_1 is a bi-invariant metric on G corresponds to the family of metrics on $G/\!\!/H$ induced from left invariant metrics on G, while subfamily of metrics for which g_2 is a bi-invariant metric corresponds to the family of metrics on $G/\!\!/H$ induced by the right invariant metrics. □

Next, we describe vertical and horizontal distributions of the Riemannian submersion

$$\pi : (G \times G, \mathrm{g}_1 \oplus \mathrm{g}_2) \to (\Delta G \backslash G \times G/U, \tilde{\mathrm{g}}), \tag{2.113}$$

where g_1 and g_2 are left invariant metrics on G, and $\mathrm{g}_1 \oplus \mathrm{g}_2$ is invariant under the right action of U. Note that we have changed notation for the closed subgroup to $U \subset G \times G$, because we will later use H for groups along which we will Cheeger deform. Since every orbit of the $\Delta G \times U$ passes through a point of the form $(e, g) \in G \times G$ it is enough to consider only points of this form. An orbit through (e, g) is given by

$$F_{(e,g)} = \{(g' u_1^{-1}, g' g u_2^{-1}) : g' \in G, \ (u_1, u_2) \in U\}. \tag{2.114}$$

An arbitrary curve contained in $F_{(e,g)}$ is given by

$$\gamma(t) = (\exp(tX)\exp(-tU_1), \exp(tX)g\exp(-tU_2)), \tag{2.115}$$

where $X \in \mathfrak{g}$ and $(U_1, U_2) \in \mathfrak{u}$. By differentiating we get that the vertical subspace of Riemannian submersion (2.113) at the point (e, g) is

$$\mathrm{Ver}_{(e,g)}(G \times G) = \{(X - U_1, R_{g*}X - L_{g*}U_2) : X \in \mathfrak{g}, (U_1, U_2) \in \mathfrak{u}\}. \tag{2.116}$$

Let Φ_1 and Φ_2 be isomorphisms of $\mathfrak{g}$ such that

$$\mathrm{g}_1(X, Y) = \mathrm{g}_0(\Phi_1 X, Y) \tag{2.117}$$

and

$$\mathrm{g}_2(X, Y) = \mathrm{g}_0(\Phi_2 X, Y), \tag{2.118}$$

for $X, Y \in \mathfrak{g}$, where g_0 is a bi-invariant metric on G. We look for the horizontal vectors at (e, g) in the form $(\Phi_1^{-1} H_1, L_{g*}\Phi_2^{-1} H_2) \in T_{(e,g)}(G \times G)$, where $H_1, H_2 \in$

$\mathfrak{g}$. Since Φ_1 and Φ_2 are isomorphisms of $\mathfrak{g}$ and (Id, L_{g*}) is an isomorphism from $T_{(e,e)}(G\times G)$ to $T_{(e,g)}(G\times G)$, there is no loss of generality. A straightforward calculation shows that the horizontal subspace at (e,g) is

$$\begin{aligned}\mathrm{Hor}_{(e,g)}(G\times G) = \{(-\Phi_1^{-1}\mathrm{Ad}_g X, L_{g*}\Phi_2^{-1}X) :\\ X\in\mathfrak{g}\,,\ \mathrm{g}_0(X, \mathrm{Ad}_{g^{-1}}U_1 - U_2) = 0 \text{ for all } (U_1,U_2)\in\mathfrak{u}\}\,.\end{aligned} \tag{2.119}$$

Note that the map

$$X\mapsto(-\Phi_1^{-1}\mathrm{Ad}_g X, L_{g*}\Phi_2^{-1}X) \tag{2.120}$$

is an isomorphism of linear subspace $\{X\in\mathfrak{g} : \mathrm{g}_0(X,\mathrm{Ad}_{g^{-1}}U_1 - U_2) = 0\}\subset\mathfrak{g}$ and $\mathrm{Hor}_{(e,g)}(G\times G)$.

The following Lemma will be used to locate flat 2-planes in Section 4.1.

Lemma 2.39. *[7, Lemma 6.1.3] Let g_1 and g_2 be metrics on compact Lie group G obtained by the iterated Cheeger deformations of the bi-invariant metric g_0 along the chains of subgroups*

$$H_n\subset H_{n-1}\subset ...H_1\subset H_0 = G\,, \tag{2.121}$$

and

$$K_m\subset K_{m-1}\subset ...K_1\subset K_0 = G\,, \tag{2.122}$$

respectively, and let $U\subset H_n\times K_m$ be a closed subgroup. Let

$$\pi : (G\times G, \mathrm{g}_1\oplus\mathrm{g}_2)\to(\Delta G\backslash G\times G/U, \mathrm{g})\,, \tag{2.123}$$

denote the Riemannian submersion to $\Delta G\backslash G\times G/U$ with the induced metric g. The biquotient $(\Delta G\backslash G\times G/U,\mathrm{g})$ has a flat 2-plane at a point $\pi(e,g)$ if there exists a pair of linearly independent vectors $X,Y\in\mathfrak{g}$ such that for all $(U_1,U_2)\in\mathfrak{u}$ we have

$$\mathrm{g}_0(X,\mathrm{Ad}_{g^{-1}}U_1 - U_2) = \mathrm{g}_0(Y,\mathrm{Ad}_{g^{-1}}U_1 - U_2) = 0\,, \tag{2.124}$$

and

$$[(\mathrm{Ad}_g X)_{\mathfrak{h}_i}, (\mathrm{Ad}_g Y)_{\mathfrak{h}_j}] = 0\,, \tag{2.125}$$

$$[X_{\mathfrak{k}_i}, Y_{\mathfrak{k}_j}] = 0\,, \tag{2.126}$$

hold for all $i = 0,1,...n$ and $j = 0,1,...m$. Moreover, any flat 2-plane at $\pi(e,g)$ arises in this fashion.

Proof. First, construct the product of Cheeger deformed metrics on $G\times G$ as in Lemma 2.35 and denote the corresponding Riemannian submersion by

$$\rho_1\times\rho_2 : ((G\times H_1\times...\times H_n)\times(G\times K_1\times...\times K_m), \mathrm{g}_H\oplus\mathrm{g}_K)\to(G\times G, \mathrm{g}_1\oplus\mathrm{g}_2) \tag{2.127}$$

, where g_H and g_K are metrics as in (2.102) corresponding to the two chains of subgroups. Next, given $X,Y\in\mathfrak{g}$ such that for all $(U_1,U_2)\in\mathfrak{u}$, $\mathrm{g}_0(X,\mathrm{Ad}_{g^{-1}}U_1 -$

$U_2) = \mathrm{g}_0(Y, \mathrm{Ad}_{g^{-1}}U_1 - U_2) = 0$ holds, we have the corresponding horizontal vectors at $(e, g) \in G \times G$, as in (2.119)

$$\bar{X} = (\bar{X}_1, \bar{X}_2) = (-\Phi_1^{-1}\mathrm{Ad}_g X, L_{g*}\Phi_2^{-1}X) \tag{2.128}$$

$$\bar{Y} = (\bar{Y}_1, \bar{Y}_2) = (-\Phi_1^{-1}\mathrm{Ad}_g Y, L_{g*}\Phi_2^{-1}Y)\,, \tag{2.129}$$

and $\bar{X}$ and $\bar{Y}$ are linearly independent if and only if X and Y are linearly independent. Because $\mathrm{g}_1 + \mathrm{g}_2$ is the product of two metrics with non-negative sectional curvature, $\sec_{\mathrm{g}_1+\mathrm{g}_2}(\bar{X} \wedge \bar{Y}) = 0$, if and only if both $\sec_{\mathrm{g}_1}(\bar{X}_1 \wedge \bar{Y}_1) = 0$ and $\sec_{\mathrm{g}_2}(\bar{X}_2 \wedge \bar{Y}_2) = 0$ hold. By Lemma 2.36 this is the case if and only if conditions (2.125) and (2.126) hold. It follows that the horizontal 2-plane $\bar{X} \wedge \bar{Y}$ is flat and by Corollary 2.28.1 it projects to a flat 2-plane on $(\Delta G\backslash G \times G/U, \mathrm{g})$. In the other direction, because the horizontal lift of a flat 2-plane at $\pi(e, g)$ is flat, it is of the form $\bar{X} \wedge \bar{Y}$, where $\bar{X}$ and $\bar{Y}$ are given by (2.128) and (2.129), with X and Y satisfying (2.124), (2.125), and (2.126). □

2.8 First order conformal deformations

In this section we discuss first-order conformal deformations of Riemannian metrics and some related results following [3, Chapter 3].

Definition 2.40. For a compact Riemannian manifold (M, g), a function $\phi : M \to \mathbb{R}$, and a small enough $s > 0$, the following is also a Riemannian metric on M

$$\mathrm{g}_s = (1 + s\phi)\,\mathrm{g}\,. \tag{2.130}$$

The metric (2.130) is called the **first-order conformal deformation** of g.

We will use the following Lemmas from [3, Chapter 3] in Section 4.2 to construct a metric of positive distance curvature on $S^3 \times S^2$.

Lemma 2.41. *[3, Corollary 3.4,] Let (M, g) be a Riemannian manifold with $\sec_{\mathrm{g}} \geq 0$, and let $X, Y \in T_mM$ be an orthonormal basis of a flat 2-plane σ $\sec_{\mathrm{g}}(\sigma) = 0$. Then, for a first order conformal deformation of* g

$$\mathrm{g}_s = (1 + s\phi)\mathrm{g}\,, \tag{2.131}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\sec_{\mathrm{g}_s}(\sigma)\,|_{s=0} = -\frac{1}{2}\mathrm{Hess}\,\phi(X, X) - \frac{1}{2}\mathrm{Hess}\,\phi(Y, Y)\,. \tag{2.132}$$

Lemma 2.42. *[3, Lemma 3.5] Let $f : [0, S] \times K \to \mathbb{R}$ be a smooth function, where $S > 0$ and K is a compact subset of a manifold. Assume that $f(0, x) \geq 0$ for all $x \in K$, and $\frac{\partial f}{\partial s} > 0$ if $f(0, x) = 0$. Then there exists $s_* > 0$ such that $f(s, x) > 0$ for all $x \in K$ and $0 < s < s_*$.*

An important difference between conformal deformations and Cheeger deformations from Section 2.6 is that, while Cheeger deformations preserve lower curvature bounds on the sectional curvature, in general, this is not the case for conformal deformations.

Chapter 3

Positive biorthogonal curvature

3.1 Wu manifold

The main result in this section is a proof that the symmetric space metric on the Wu manifold has positive biorthogonal curvature. The proof relies on a result that was presented in section 2.5. More precisely, we use the fact that flat 2-planes of $\mathrm{SU}(3)/\mathrm{SO}(3)$ correspond to horizontal flat 2-planes of $\mathrm{SU}(3)$ and characterize horizontal flat 2-planes of $\mathrm{SU}(3)$ as conjugates of a maximal abelian subalgebra of $\mathfrak{su}(3)$ by elements of $\mathrm{SO}(3)$. Finally, we introduce a basis for $\mathfrak{su}(3)$ and use this characterization to show that no two flat 2-planes can be orthogonal, hence, proving the result. The contents of this section, in a more condensed form, can be found in [28].

The Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$ is a rational homology 5-sphere with second homotopy group of order two [32]. When equipped with a metric $(\mathrm{SU}(3)/\mathrm{SO}(3), \mathrm{g})$, that makes the canonical submersion

$$\begin{aligned} \pi : (\mathrm{SU}(3), \mathrm{Q}) &\to (\mathrm{SU}(3)/\mathrm{SO}(3), \mathrm{g})\,, \\ u &\mapsto u\mathrm{SO}(3)\,, \end{aligned} \tag{3.1}$$

into a Riemannian submersion, the Wu manifold is a symmetric space. In (3.1) Q is a bi-invariant metric on $\mathrm{SU}(3)$. As a comparison to the main result of this section, we note that the metric g has positive Ricci curvature. Now we can state the main result of this section.

Proposition 3.1. *The symmetric space* $(\mathrm{SU}(3)/\mathrm{SO}(3), \mathrm{g})$ *has positive biorthogonal curvature.*

Proof. The left action of $\mathrm{SU}(3)$ on $\mathrm{SU}(3)/\mathrm{SO}(3)$ induced from left multiplication on $\mathrm{SU}(3)$ by (3.1) is transitive and isometric for the symmetric space metric. This means that we can study curvature at one point of $\mathrm{SU}(3)/\mathrm{SO}(3)$ and isometrically translate the results to any other point. The Cartan decomposition that corresponds to $\mathrm{SU}(3)/\mathrm{SO}(3)$ is of type AIII, see 2.33.1

$$T_e\mathrm{SU}(3) \simeq \mathfrak{su}(3) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)^{\perp}\,. \tag{3.2}$$

Decomposition (3.2) is orthogonal with respect to the bi-invariant metric and is precisely the decomposition of $T_e\mathrm{SU}(3)$ into vertical and horizontal subspaces of the Riemannian submersion (3.1). Hence, we have

$$T_{\mathrm{SO}(3)}(\mathrm{SU}(3)/\mathrm{SO}(3)) \simeq \mathfrak{so}(3)^{\perp}\,. \tag{3.3}$$

Riemannian manifold $(\mathrm{SU}(3)/\mathrm{SO}(3), \mathrm{g})$ has non-negative sectional curvature because it is the image of a Riemannian submersion from a manifold with non-negative sectional curvature. Hence, to conclude that $\mathrm{SU}(3)/\mathrm{SO}(3)$ has positive biorthogonal curvature, we need to show that no two flat 2-planes are orthogonal to each other. A result of Tapp, stated in the Theorem 2.28, implies that a 2-plane on $\mathrm{SU}(3)/\mathrm{SO}(3)$ is flat if and only if its horizontal lift is flat. Thus, it is enough to consider horizontal flat 2-planes at the identity of $\mathrm{SU}(3)$.

A horizontal 2-plane $X \wedge Y \subset \mathfrak{so}(3)^{\perp}$ at the identity of $\mathrm{SU}(3)$ is flat if and only if $[X, Y] = 0$. Since the maximal number of linearly independent commuting matrices in $\mathfrak{su}(3)$ is two, every horizontal flat 2-plane corresponds to a maximal abelian subalgebra of $\mathfrak{so}(3)^{\perp}$

$$\operatorname{span}\{X, Y\} = \mathfrak{a} \subset \mathfrak{so}(3)^{\perp}\,. \tag{3.4}$$

By a fundamental fact about Cartan decomposition, see proposition 2.34 for the precise statement, any two maximal abelian subalgebras of $\mathfrak{so}(3)^{\perp}$ are conjugate by an element of $\mathrm{SO}(3)$. This means that by fixing one maximal abelian subalgebra, or one horizontal flat 2-plane we can parametrize all horizontal flat 2-planes by $\mathrm{SO}(3)$. In what follows we will obtain an explicit parametrization of horizontal flat 2-planes at the identity of $\mathrm{SU}(3)$, and so a parametrization of flat 2-planes at a point of $\mathrm{SU}(3)/\mathrm{SO}(3)$ by choosing a basis for $\mathfrak{su}(3)$, fixing a horizontal flat 2-plane and parametrizing $\mathrm{SO}(3)$ by Euler angles. We use this explicit parametrization to show that no two flat 2-planes can be orthogonal. For the basis of $\mathfrak{su}(3)$, we choose $\{-i\lambda_i\}_{i=1,\dots,8}$, where the λ_i's are traceless, self-adjoint 3 by 3 matrices known as the Gell-Mann matrices, see Appendix A. The scalar product on $\mathfrak{su}(3)$ that corresponds to the bi-invariant metric is

$$\langle X, Y\rangle = -\frac{1}{2}\operatorname{Tr}(XY)\,, \tag{3.5}$$

for $X, Y \in \mathfrak{su}(3)$ and the basis $\{-i\lambda_i\}_{i=1,\dots,8}$ is orthonormal with respect to (3.5). The Cartan decomposition (3.2) in this basis is

$$\mathfrak{so}(3) = \operatorname{span}\{-i\lambda_2, -i\lambda_5, -i\lambda_7\} \tag{3.6}$$

and

$$\mathfrak{so}(3)^{\perp} = \operatorname{span}\{-i\lambda_1, -i\lambda_3, -i\lambda_4, -i\lambda_6, -i\lambda_8\}\,. \tag{3.7}$$

Matrices λ_3 and λ_8 are diagonal, so we use $-\lambda_3 \wedge \lambda_8$ for the reference horizontal flat 2-plane. Every horizontal flat 2-plane, $X \wedge Y$, with $X, Y \in \mathfrak{so}(3)^{\perp}$ such that $[X, Y] = 0$, can now be written as

$$X \wedge Y = -\mathrm{Ad}_r(\lambda_3 \wedge \lambda_8)\,, \tag{3.8}$$

for some $r \in \mathrm{SO}(3)$. Suppose that $X \wedge Y$ and $X' \wedge Y'$ are two such 2-planes with, $X \wedge Y$ given by (3.8), and $X' \wedge Y'$ by

$$X' \wedge Y' = -\mathrm{Ad}_{r'}(\lambda_3 \wedge \lambda_8)\,, \tag{3.9}$$

for some $r' \in \mathrm{SO}(3)$. For the 2-planes (3.8) and (3.9) to be orthogonal it is necessary and sufficient that the equations

$$\langle \mathrm{Ad}_r\lambda_3, \mathrm{Ad}_{r'}\lambda_3\rangle = 0\,, \tag{3.10}$$

$$\langle \mathrm{Ad}_r \lambda_3, \mathrm{Ad}_{r'} \lambda_8 \rangle = 0\,, \tag{3.11}$$

$$\langle \mathrm{Ad}_r \lambda_8, \mathrm{Ad}_{r'} \lambda_3 \rangle = 0\,, \tag{3.12}$$

and

$$\langle \mathrm{Ad}_r \lambda_8, \mathrm{Ad}_{r'} \lambda_8 \rangle = 0 \tag{3.13}$$

hold. Using the Ad-invariance of the bi-invariant metric, equations (3.10), (3.11), (3.12), and (3.13) can be rewritten as

$$\langle \lambda_3, \mathrm{Ad}_{r^{-1}r'} \lambda_3 \rangle = 0\,, \tag{3.14}$$

$$\langle \lambda_3, \mathrm{Ad}_{r^{-1}r'} \lambda_8 \rangle = 0\,, \tag{3.15}$$

$$\langle \lambda_8, \mathrm{Ad}_{r^{-1}r'} \lambda_3 \rangle = 0\,, \tag{3.16}$$

and

$$\langle \lambda_8, \mathrm{Ad}_{r^{-1}r'} \lambda_8 \rangle = 0\,. \tag{3.17}$$

We now use the Euler angle parametrization of SO(3) to write $r^{-1}r' \in \mathrm{SO}(3)$ as

$$r^{-1}r' = \exp(-i\lambda_2 x)\exp(-i\lambda_5 y)\exp(-i\lambda_2 z)\,, \tag{3.18}$$

where $x, y, z \in \mathbb{R}$. Plugging (3.18) into equations (3.14), (3.15), (3.16), and (3.17) and calculating the traces explicitly, we find

$$0 = \langle \lambda_3, \mathrm{Ad}_{r^{-1}r'} \lambda_3 \rangle = \frac{1}{4}\cos(2x)\,(3 + \cos(2y))\cos(2z) - \sin(2x)\cos(y)\sin(2z)\,, \tag{3.19}$$

$$0 = \langle \lambda_3, \mathrm{Ad}_{r^{-1}r'} \lambda_8 \rangle = -\frac{\sqrt{3}}{2}\cos(2x)\sin^2(y)\,, \tag{3.20}$$

$$0 = \langle \lambda_8, \mathrm{Ad}_{r^{-1}r'} \lambda_3 \rangle = -\frac{\sqrt{3}}{2}\cos(2z)\sin^2(y)\,, \tag{3.21}$$

and

$$0 = \langle \lambda_8, \mathrm{Ad}_{r^{-1}r'} \lambda_8 \rangle = \frac{1}{4}(1 + 3\cos(2y))\,. \tag{3.22}$$

Equations (3.20), (3.21), and (3.22) imply $\cos^2(y) = 1/3$ and $\cos(2x) = \cos(2z) = 0$. Plugging this into equation (3.19), we obtain

$$\langle \lambda_3, \mathrm{Ad}_{r^{-1}r'} \lambda_3 \rangle \neq 0, \tag{3.23}$$

and conclude that there is no solution to the system given by equations (3.19), (3.20), (3.21), and (3.22). This shows that no two flat 2-planes are orthogonal.

□

In section 4.3 we will show that all closed simply connected with torsion-free homology and zero second Stiefel-Whitney class admit a metric of positive biorthogonal curvature. The Wu manifold doesn't satisfy these conditions on the homology and the second Stiefel-Whitney class, suggesting that they are technical in nature.

3.2 Bettiol's construction

In this section, we will review Bettiol's constuction of metric with positive distance curvature for any $\theta > 0$ on $S^2 \times S^2$ given in [3, Chapter 6] and [4]. Our construction of a metric with the same property on $S^3 \times S^2$ in Sections 4.1 and 4.2 closely parallels Bettiol's construction. The construction is carried out in two steps. First, the product of the round metrics on $S^2 \times S^2$ is deformed to a metric where almost all points have a unique flat curvature plane, and then this metric is conformaly deformed to a metric of positive distance curvature. Finally, we show that Bettiol's construction can be made to commute with taking certain discrete quotients, and thus gives metrics of positive distance curvature on $S^2 \times \mathbb{PR}^2 = S^2 \times S^2/\mathbb{Z}_2$ and $L_2' = S^2 \times S^2/\mathbb{Z}_2$ as well.

The first step of the construction is carried out by a general version of Cheeger deformation, whose particular case was discussed in Section 2.6. Given a Riemannian manifold (M, g) and a Lie group G that acts freely and by isometries on (M, g). One considers a following Riemannian submersion

$$\pi : (M \times G, \mathrm{g} + \frac{1}{t}\mathrm{Q}) \to (M, \mathrm{g}_t)\,, \tag{3.24}$$

where t is a positive real number and Q is a bi-invariant metric on G. For $(m, g) \in M \times G$, π is given by $\pi(m, g) = g^{-1}m$ and M is obtained as the orbit space of the action

$$g'\dot{(}m, g) = (g'm, g'g)\,, \tag{3.25}$$

for $g, g' \in G$ and $m \in M$, of G on $G \times M$. Action (3.25) is by isometries on the product $(M \times G, \mathrm{g} + \frac{1}{t}\mathrm{Q})$, and thus the metric g_t is well defined. The family of metrics g_t, for $t > 0$ is called the Cheeger deformation of g. Bettiol's construction starts with the

$$S^2 \times S^2 = \{(p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|p_1\|^2 = \|p_2\|^2 = 1\} \subset \mathbb{R}^3 \times \mathbb{R}^3\,, \tag{3.26}$$

where the metric g on $S^2 \times S^2$ is induced from the product of canonical metrics on $\mathbb{R}^3$. Next, the diagonal action of $SO(3)$ on $S^2 \times S^2$ given by

$$A(p_1, p_2) = (Ap_1, Ap_2)\,, \tag{3.27}$$

for $A \in SO(3)$ and $(p_1, P_2) \in S^2 \times S^2$, is used to obtain the Cheeger deformation of g. The Cheeger deformed metric g_t has non-negative sectional curvature, and Bettiol shows that at points away from submanifolds

$$\Delta^{\pm} = \{(p_1, \pm p_1) \in S^2 \times S^2\} \simeq S^2\,, \tag{3.28}$$

there is exactly one flat 2-plane. So, at these points the distance curvature is positive. However, at each point of (3.28) there is 1-parameter family of flat 2-planes. Furthermore at these points of (3.28) even the Biorthogonal curvature can be zero.

Second step of the constuction involves a first order deformation of the metric g_t. The conformal factor is given by

$$f = -\chi_+\psi_+ - \chi_-\psi_-\,, \tag{3.29}$$

where χ_+ is the bump function of Δ^+, i.e., function that is identically one in a tubular neighborhood of Δ^+ and identically zero outside of a larger tubular neighborhood of Δ^+, while ψ^+ is the square of the Riemannian distance function form Δ_+

$$\psi_+(m) = \mathrm{dist}_{\mathrm{g}_t}\left(m, \Delta^+\right)^2 . \tag{3.30}$$

Functions χ_- and ψ^- are similarly defined for Δ_-. Bettiol then proceeds to show that the first order conformally deformed metric

$$\mathrm{g}_{s,t} = (1 + sf)\mathrm{g}_t \,, \tag{3.31}$$

has positive distance curvature for any $\theta > 0$. We mimic this construction precisely in Section 4.2, and give the details there.

Important thing to note here is that while Cheeger deformation preserves the non-negativity sectional cutvature of the starting metric g, first order conformal deformation does not, i.e., there is a 2-plane $\sigma \in \mathrm{Gr}_2(T_m(S^2 \times S^2))$ with $\sec_{\mathrm{g}_{s,t}} < 0$.

Now we will show that the construction can be used to obtain discrete quotients

$$S^2 \times \mathbb{RP}^2 = S^2 \times S^2/\mathbb{Z}_2 \,, \tag{3.32}$$

and

$$L_2' = S^2 \times S^2/\mathbb{Z}_2 \,. \tag{3.33}$$

We start with (3.32). First observe that the involution

$$I : S^2 \times S^2 \to S^2 \times S^2 \tag{3.34}$$

given by

$$(p_1, p_2) \mapsto (p_1, -p_2) \,, \tag{3.35}$$

for $(p_1, p_2) \in S^2 \times S^2$ is an isometry of the metric g. Furthermore, it is easy to see that the involution (3.34) commutes with the action (3.27). It follows that I is also an isometry of the Cheeger deformed metric. This means that there is a well defined metric on the quotient $S^2 \times S^2/\ I = S^2 \times \mathbb{RP}^2$ such that the quotient map is a Riemannian submersion. Since Riemannian submersions don't decrese the curvature, the lower curvature bound is preserved. If we could show that I is also an isometry of conformaly deformed metric we would have obtained a metric of positive distance curvature on $S^2 \times \mathbb{RP}^2$. A necessary and sufficient condition for I to be an isometry of $\mathrm{g}_{s,t}$ is for the conformal factor to be invariant, i.e.,

$$f \circ I = f \,. \tag{3.36}$$

Note that because I interchanges Δ^+ and Δ^- and it is an isometry of g_t. We have that

$$\psi_\pm \circ I = \psi_\mp \,. \tag{3.37}$$

Next we choose the bump functions in such a way to satisfy

$$\chi_\pm \circ I = \chi_\mp \,. \tag{3.38}$$

From (3.29), (3.37), and (3.38) is clear that (3.36) holds. Thus metric of positive distance curvature on $S^2 \times S^2$ desends to $S^2 \times \mathbb{RP}^3$.

Following a similar line of reasoning for the involution

$$J : S^2 \times S^2 \to S^2 \times S^2 \tag{3.39}$$

defined by

$$J : (p_1, p_2) \mapsto (-p_1, -p_2)\,, \tag{3.40}$$

we obtain a metric of positive distance curvature on L'_2.

3.3 Connected sums

We first state the well-known definition of connected sum following Kervaire-Milnor [22, Section 2]; cf. [14, Definition 1.3.4].

Definition 3.2. Let M_1 and M_2 be closed connected oriented n-manifolds and let

$$i_i : D^n \hookrightarrow M_i \tag{3.41}$$

be embeddings of the n-disk for $i = 1, 2$. Suppose that the embedding i_1 preserves orientation, while i_2 reverses it. The connected sum of M_1 and M_2 is the n-manifold defined as

$$M_1 \# M_2 := \frac{(M_1 \backslash i_1(0)) \sqcup (M_2 \backslash i_2(0))}{\sim} \tag{3.42}$$

where the equivalence relation identifies $i_1(tu)$ with $i_2((1-t)u)$ for each unit vector $u \in S^{n-1} = \partial D^n$ and $0 < t < 1$.

A key ingredient in the proofs in this section and those in Section 4.3, is the following surgery stability result regarding Riemannian metrics of positive biorthogonal curvature.

Proposition 3.3. *Bettiol [3, Proposition 7.11]. Let (M_1, g_1) and (M_2, g_2) be closed smooth n-manifolds. Suppose that $\sec^{\perp}_{g_i} > 0$ for $i = 1, 2$. There is a Riemannian metric $(M_1 \# M_2, g)$ such that $\sec^{\perp}_{g} > 0$.*

Proof of Proposition 3.3 relies on the work by Hoelzel [16].

Bettiol classified up to homeomorphism the closed simply connected 4-manifolds that admit a metric of positive biorthogonal curvature in [2].

Theorem 3.4. *[2, Section 1, Theorem]*

Let M be a closed smooth simply connected 4-manifold. Up to homeomorphism, the following are equivalent:

1. *M admits a metric with $\sec^{\perp} > 0$;*
2. *M admits a metric with $\mathrm{Ric} > 0$;*
3. *M admits a metric with $\mathrm{scal} > 0$.*

Remark 3.4.1. *Homeomorphism classes of manifolds from Theorem 3.4 are*

$$m\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}\,, \tag{3.43}$$

if $w_2(M)\neq 0$ *and to*

$$n(S^2\times S^2)\#S^4 \tag{3.44}$$

if $w_2(M)=0$ *for* $m,n\in\mathbb{N}_0$.

In this section, we show that Bettiol's work yields further classification results on closed 4-manifolds with non-trivial fundamental group.

Lemma 3.5. *A closed smooth orientable 4-manifold with infinite cyclic fundamental group is homeomorphic to a 4-manifold that admits a Riemannian metric of positive biorthogonal curvature if and only if it is homeomorphic to a 4-manifold that admits a Riemannian metric of positive scalar curvature.*

Proof. By [19, Collorary 1.2], see also [20] and [17], every closed smooth orientable 4-manifold with infinite cyclic fundamental group is TOP-split, i.e., it can be written as a connected sum $S^1\times S^3\#M_1$ where M_1 is a simply connected 4-manifold. Since $S^1\times S^3$ with the product metric has positive scalar curvature and positive biorthogonal curvature, and positivity of scalar curvature and positivity of biorthogonal curvature, Proposition 3.3, being closed under connected sum operation. The connected sum $S^1\times S^3\#M_1$ admits a metric of positive scalar(biorthogonal) curvature if and only if M_1 admits a metric of positive scalar(biorthogonal) curvature. Since M_1 is simply connected, by Theorem 3.4, M_1 admits a metric of positive scalar curvature if and only if it admits a metric of positive biorthogonal curvature and the claim follows. □

Lemma 3.6. *Let M be a closed connected nonorientable 4-manifold with fundamental group of order two such that $w_1^2(M)+w_2(M)=0$. Then M is homeomorphic to a manifold that admits a Riemannian metric of positive biorthogonal curvature.*

Proof. According to Hambleton-Kreck-Teichner [18, Theorem 1 and Theorem 3], such a 4-manifold is homeomorphic to

$$S^2\times\mathbb{RP}^2\#(n-1)(S^2\times S^2) \tag{3.45}$$

for a given $n\in\mathbb{N}$. The results of Bettiol stated in Theorem 3.4, the fact that $S^2\times\mathbb{RP}^2$ admits a metric of positive biorthogonal curvature as was shown in Section 3.2, and Proposition 3.3 imply that the 4-manifolds (3.45) admit a Riemannian metric of positive biorthogonal curvature for every $n\in\mathbb{N}$. □

We say that a 4-manifold M has a w_2-type (I) if the second Stiefel-Whitney class of its universal cover is non-zero $w_2(\tilde{M})\neq 0$, w_2-type (II) if its second Stiefel-Whitney class is zero $w_2(M)=0$, and w_2-type (III) if $w_2(M)\neq 0$, but $w_2(\tilde{M})=0$.

Lemma 3.7. *Every closed smooth orientable 4-manifold with fundamental group of order two and w_2-type (I) and (III) is homeomorphic to a 4-manifold that admits a metric of positive biorthogonal curvature.*

Proof. According to Hambleton-Kreck-Teichner [18, Theorem 1 and Theorem 3], such a 4-manifold is homeomorphic to

$$n\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}\#L_2' \,, \tag{3.46}$$

for w_2-type (I), and

$$(k-1)(S^2\times S^2)\#L_2' \,, \tag{3.47}$$

for w_2-type (III). By Proposition 3.3 and the fact that L_2' admits a metric of positive biorthogonal curvature as was shown in Section 3.2, each of them admits a metric of positive biorthogonal curvature. □

Chapter 4

Positive Distance Curvature on $S^3 \times S^2$

We proceed to prove the main result of this thesis.

Theorem 4.1. *For every $\theta > 0$, there is a Riemannian manifold $(S^3 \times S^2, g^\theta)$ such that:*

1. $\sec^\theta_{\mathrm{g}^\theta} > 0$.
2. *There is a metric g^0 such that $\mathrm{g}^\theta \to \mathrm{g}^0$ in the C^k-topology as $\theta \to 0$ for $k \geq 0$. The metric g^0 is Wilking's metric g_W of almost-positive curvature.*
3. *There is a 2-plane $\sigma \in \mathrm{Gr}_2\left(T_m(S^3 \times S^2)\right)$ with $\sec^\theta_{\mathrm{g}^\theta}(\sigma) < 0$.*
4. $\mathrm{Ric}_{\mathrm{g}^\theta} > 0$.

In particular, there is a Riemannian metric of positive biorthogonal curvature on $S^3 \times S^2$.

The proof of Theorem 4.1 consists of two steps and it builds upon Bettiol's construction of a metric with positive distance curvature for any $\theta > 0$ on $S^2 \times S^2$ given in [3, Chapter 6] and [4]. We described Bettiol's construction in Section 3.2. Theorem 4.1 should be compared [4, Theorem].

The Chapter is structured as follows. In Section 4.1 we review Wilking's metric of almost positive curvature on $S^3 \times S^2$. In [30], also see [33, Section 5]. This is the first step of the construction and it involves Cheeger deformation. The second step of the construction involves a first order conformal deformation of Wilking's metric and is given in Section 4.2.

4.1 Metric of almost positive curvature on $S^3 \times S^2$

In [30], Wilking constructed a metric of almost positive curvature on $\mathbb{R}P^3 \times \mathbb{R}P^2$; see Definition 2.17. Since $\mathbb{R}P^3 \times \mathbb{R}P^2$ an odd-dimensional and non-orientable manifold, Synge's Theorem implies it does not admit a metric of positive sectional curvature. Hence, Wilking's result is a counterexample to the deformation conjecture. In what follows we will be interested in a metric with almost positive curvature on $S^3 \times S^2$, as it was described in [33, Section 5]. These two metrics are related in the following way. A metric on $S^3 \times S^2$ arises as the pullback of a metric on $\mathbb{R}P^3 \times \mathbb{R}P^2$ by the universal covering map. The following construction is essentially the same as Willking's construction from [30].

Since S^3 is parallelizable, its unit tangent sphere bundle is

$$T_1 S^3 = S^3 \times S^2 \tag{4.1}$$

which can be embedded into $\mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{H} \times \mathbb{H}$ in the following way

$$S^3 \times S^2 = \{(p,v) \in \mathbb{H} \times \mathbb{H}; |p| = |v| = 1, \langle p, v\rangle = 0\} \subset \mathbb{H} \times \mathbb{H}\,. \tag{4.2}$$

Here $\langle x, y\rangle = \mathrm{Re}\,(\bar{x}y)$ and $|x|^2 = \langle x, x\rangle$. The group $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1) = S^3 \times S^3$, acts on $S^3 \times S^2$ by

$$(q_1, q_2) \star (p, v) = (q_1 p \bar{q}_2, q_1 v \bar{q}_2)\ , \tag{4.3}$$

for all $(q_1, q_2) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. This action is effective and transitive. The isotropy group of $(1, i) \in S^3 \times S^2$ is $H = \left\{\left(e^{i\phi}, e^{i\phi}\right) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)\right\} < G$. Note that $H \simeq S^1$. Thus, $S^3 \times S^2 \simeq G/H$ is a homogeneous space. Discrete subgroup $L < G$ generated by $(1, -1)$ and (j, j) normalizes H, so it follows that $\mathbb{RP}^3 \times \mathbb{RP}^2 \simeq G/\left(LH\right)$. Wilking considered this homogeneous space as a biquotient. We now will do the same but for $S^3 \times S^2 = G/H$, as it is done in [33, Section 5], but in more detail.

We start by defining a left invariant metric Q_t, on $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ by Cheeger deforming a bi-invariant metric on G along the diagonal subgroup $\Delta\mathrm{Sp}(1) = \{(a, a) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)\} < \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. This metric can we written as (3.12)

$$\mathrm{Q}_t\left(\left(X, Y\right), \left(X', Y'\right)\right) = \mathrm{Q}\left(\Phi_t\left(X, Y\right), \left(X', Y'\right)\right)\ , \tag{4.4}$$

where we take $t = \frac{1}{2}$ for the deformation parameter in Φ_t, i.e.,

$$\Phi_t\left(X, Y\right) = \left(X, Y\right) - \frac{1}{2} P\left(X, Y\right)\ , \tag{4.5}$$

denoting by P the projection onto the diagonal subalgebra $\Delta\mathfrak{sp}(1) \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Explicitly:

$$\begin{aligned} P\left(X, Y\right) &= \frac{1}{2}\left(X + Y, \quad X + Y\right) \\ \left(1 - P\right)\left(X, Y\right) &= \frac{1}{2}\left(X - Y, -X + Y\right)\ . \end{aligned} \tag{4.6}$$

The induced metric g_t on

$$S^3 \times S^2 \simeq G/H \cong \Delta G \backslash G \times G/\left\{(1,1)\right\} \times H \tag{4.7}$$

is the one that makes the biquotient submersion

$$\pi : \left(G \times G, \mathrm{Q}_t \oplus \mathrm{Q}_t\right) \to \left(S^3 \times S^2, \mathrm{g}_t\right) \tag{4.8}$$

into a Riemannian submersion. Note that π is explicitly

$$\pi\left(\left(a, b\right), \left(c, d\right)\right) = \left(\left(a, b\right)^{-1}\left(c, d\right)\right) \star (1, i) = \left(\bar{a}c\bar{d}b, \bar{a}ci\bar{d}b\right) \in S^3 \times S^2\,. \tag{4.9}$$

Next we locate flat 2-planes on $(S^3 \times S^2, \mathrm{g}_t)$ using Lemma 2.39. Every flat 2-plane at $\pi((1,1),(a,b))$ is a projection of a horizontal flat 2-plane $H_1 \wedge H_2$ at $((1,1),(a,b))$ spanned by (from now on we drop the index t from Φ)

$$(4.10) \qquad \begin{aligned} H_1 &= \left(-\Phi^{-1}\mathrm{Ad}_{(a,b)}(V\ ,\ W), L_{(a,b)*}\Phi^{-1}(V\ ,\ W)\right) \\ H_2 &= \left(-\Phi^{-1}\mathrm{Ad}_{(a,b)}(V', W'), L_{(a,b)*}\Phi^{-1}(V', W')\right)\ , \end{aligned}$$

where $(V, W) \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ is such that

$$(4.11) \qquad \mathrm{Q}\left(\left(V, W\right), \left(i, i\right)\right) = \mathrm{Q}\left(\left(V', W'\right), \left(i, i\right)\right) = 0\,,$$

and

$$(4.12) \qquad \begin{aligned} \left[\mathrm{Ad}_{(a,b)}\left(V, W\right), \mathrm{Ad}_{(a,b)}\left(V', W'\right)\right] &= 0 \\ \left[P\mathrm{Ad}_{(a,b)}\left(V, W\right), P\mathrm{Ad}_{(a,b)}\left(V', W'\right)\right] &= 0 \\ \left[\left(V, W\right), \left(V', W'\right)\right] &= 0 \\ \left[P\left(V, W\right), P\left(V', W'\right)\right] &= 0\,. \end{aligned}$$

Since the Lie algebra of $\{(1,1)\} \times H$ is spanned by $((0,0),(i,i))$ condition (4.11) is necessary and sufficient for H_1 and H_2 to be horizontal. Conditions (4.12) are necessary and sufficient for the 2-plane $H_1 \wedge H_2$ to be flat. When solving (4.11) and (4.12) one should note that they are linear in (V, W) and (V', W'), so that the space of solutions corresponding to the flat 2-plane $H_1 \wedge H_2$ is span$\{(V, W), (V', W')\}$. Solutions to (4.11) and (4.12) are of the form span$\{(V, 0), (0, V)\}$, where $V \in \mathrm{Im}(\mathbb{H})$ is nonzero, satisfies $\langle V, i\rangle = 0$, and $[\mathrm{Ad}_a V, \mathrm{Ad}_b V] = 0$. For every such V we get a flat 2-plane at $\pi((1,1),(a,b))$ that is a projection of a flat 2-plane spanned by

$$(4.13) \qquad \begin{aligned} H_1 &= \left(-\Phi^{-1}\mathrm{Ad}_{(a,b)}(V, 0), L_{(a,b)*}\Phi^{-1}(V, 0)\right) \\ H_2 &= \left(-\Phi^{-1}\mathrm{Ad}_{(a,b)}(0, V), L_{(a,b)*}\Phi^{-1}(0, V)\right)\ . \end{aligned}$$

The condition $[\mathrm{Ad}_a V, \mathrm{Ad}_b V] = 0$ is equivalent to $\mathrm{Ad}_{\bar{a}b} V = \pm V$. For the plus sign we have $\bar{a}bV = V\bar{a}b$, so that $\bar{a}b$ has to be proportional to V, thus because $\langle V, i\rangle = 0$ we have $\langle \bar{a}b, i\rangle = 0$. For the minus sign, real part of $\bar{a}b$ must be zero, i.e., $\langle \bar{a}b, 1\rangle = 0$. Flat 2-planes on $S^3 \times S^2$ are located on submanifolds $\pi(S_1)$ and $\pi(S_i)$, with

$$(4.14) \qquad \begin{aligned} S_1 &= \{((1,1),(a,b)) \in G \times G : \langle \bar{a}b, 1\rangle = 0\}\ , \\ S_i &= \{((1,1),(a,b)) \in G \times G : \langle \bar{a}b, i\rangle = 0\}\ . \end{aligned}$$

Now that we have located the flat 2-planes, a natural question is how many of them are there at each point. Let us fix the length of V by $\langle V, V\rangle = 1$ and note that V and $-V$ give the same 2-plane. If $\langle \bar{a}b, 1\rangle = 0$, condition $\mathrm{Ad}_{\bar{a}b} V = -V$ is equivalent to $\langle \bar{a}b, V\rangle = 0$, thus if $\bar{a}b \neq \pm i$ then there are three independent conditions on V: $\langle V, V\rangle = 1$, $\langle V, i\rangle = 0$, and $\langle V, \bar{a}b\rangle = 0$. These conditions determine a unique V, thus they determine a unique flat 2-plane. However, if $\bar{a}b = \pm i$ then conditions $\langle V, i\rangle = 0$ and $\langle V, \bar{a}b\rangle = 0$ are the same, so there is a one-parameter family of flat 2-planes at these points. Similarly, for $\langle \bar{a}b, i\rangle = 0$ and $\bar{a}b \neq \pm 1$, condition $\mathrm{Ad}_{\bar{a}b} V = V$, provided that $\langle V, V\rangle = 1$, is equivalent to $\langle \bar{a}b, V\rangle = \pm 1$, so we get three independent conditions determining a unique V. However, if $\bar{a}b = \pm 1$ then $\mathrm{Ad}_{\bar{a}b} V = V$ is satisfied for every V, so we have two conditions giving a one-parameter family of flat 2-planes. Let $S_{1,\pm i}$ and $S_{i,\pm 1}$ be submanifolds of S_1 and S_i defined by

$$
\begin{aligned}
S_{1,\pm i} &= \{((1,1),(a,b)) \in G \times G : \bar{a}b = \pm i\} \subset S_1\,, \\
S_{i,\pm 1} &= \{((1,1),(a,b)) \in G \times G : \bar{a}b = \pm 1\} \subset S_i\,.
\end{aligned}
\tag{4.15}
$$

At each point of $\pi\left(S_1 \backslash \left(S_{1,i} \cup S_{1,-i}\right)\right)$ and $\pi\left(S_i \backslash \left(S_{i,1} \cup S_{i,-1}\right)\right)$ there is exactly one flat 2-plane, while at each point of $\pi\left(S_{1,i}\right)$, $\pi\left(S_{1,-i}\right)$, $\pi\left(S_{i,1}\right)$, and $\pi\left(S_{i,-1}\right)$ there is a one-parameter family of flat 2-planes. Both $\pi(S_1)$ and $\pi(S_i)$ are diffeomorphic to $S^2 \times S^2$, with $\pi(S_1) \cap \pi(S_i)$ being diffeomorphic to $SO(3)$, while $\pi(S_{1,i})$, $\pi(S_{1,-i})$, $\pi(S_{i,1})$, and $\pi(S_{i,-1})$ are all diffeomorphic to S^2. For example:

$$
\begin{aligned}
\pi(S_{1,\pm i}) &= \{(\mp a i \bar{a}, \pm 1); a \in \mathrm{Sp}(1)\} = S^2 \times \{\pm 1\} \subset S^3 \times S^2\,, \\
\pi(S_{i,\pm 1}) &= \{(\pm 1, \pm a i \bar{a}); a \in \mathrm{Sp}(1)\} = \{\pm 1\} \times S^2 \subset S^3 \times S^2\,.
\end{aligned}
\tag{4.16}
$$

Next, consider the diagonal action of $\mathrm{Sp}(1)$ from the right on the first factor of $G \times G$,

$$
g * ((a,b),(c,d)) = ((ag, bg),(c,d))\,. \tag{4.17}
$$

Action (4.17) is isometric with respect to $\mathrm{g}_t \oplus \mathrm{g}_t$ and commutes with the actions of ΔG and $\{(1,1)\} \times H$, thus it induces the following isometric action of $\mathrm{Sp}(1)$ on $S^3 \times S^2$:

$$
g * (p, v) = (\bar{g}pg, \bar{g}vg)\,. \tag{4.18}
$$

Kernel of action (4.18) is $\{1,-1\} \subset \mathrm{Sp}(1)$, thus the action (4.18) is an effective action of $\mathrm{SO}(3) = \mathrm{Sp}(1)/\{1,-1\}$. It is easy to check that for all $g \in \mathrm{Sp}(1)$, $g*\pi(S_1) \subset \pi(S_1)$ and $g*\pi(S_i) \subset \pi(S_i)$ hold, thus action (4.18) restricts to actions on $\pi(S_1)$ and $\pi(S_i)$. The diffeomorphism

$$
\begin{aligned}
\phi_1 : \pi(S_1) &\to S^2 \times S^2 = (\mathrm{Im}(\mathbb{H}) \cap Sp(1)) \times (\mathrm{Im}(\mathbb{H}) \cap Sp(1)) \\
(p, v) &\mapsto (p, \bar{p}v)
\end{aligned}
\tag{4.19}
$$

intertwines restriction of the action (4.18) to $\pi(S_1)$ and the diagonal action $\mathrm{SO}(3)$ on $S^2 \times S^2$ given by the usual rotation action of $\mathrm{SO}(3)$ on each of the factors realized via conjugations by unit quaternions. Similarly, the diffeomorphism

$$
\begin{aligned}
\phi_i : \pi(S_i) &\to S^2 \times S^2 = (\mathrm{Im}(\mathbb{H}) \cap Sp(1)) \times (\mathrm{Im}(\mathbb{H}) \cap Sp(1)) \\
(p, v) &\mapsto (p\bar{v}, v)
\end{aligned}
\tag{4.20}
$$

intertwines restriction of (4.18) to $\pi(S_i)$ with the diagonal action of $\mathrm{SO}(3)$ on $S^2 \times S^2$. Thus the restriction of the action (4.18) to $\pi(S_1) = S^2 \times S^2$ is a cohomogeneity one action equivalent to the diagonal action of $\mathrm{SO}(3)$ and similarly for restriction to $\pi(S_i)$. Singular orbits on $\pi(S_1)$ are given by $\pi(S_{1,\pm i}) = S^2$ while singular orbits on $\pi(S_i)$ are given by $\pi(S_{i,\pm 1}) = S^2$. While $\mathrm{SO}(3)$ acts on $\pi(S_1) \cap \pi(S_i)$ freely and transitively, so $\pi(S_1) \cap \pi(S_i) = \mathrm{SO}(3)$.

Flat 2-planes on $\pi(S_1)$ are tangent to $\pi(S_1)$ and are vertizontal with respect to the projection $\mathrm{pr} : \pi(S_1) \to \pi(S_1)/\mathrm{SO}(3)$. Similarly, flat 2-planes on $\pi(S_i)$ are tangent to $\pi(S_i)$ and are vertizontal with respect to the projection $\mathrm{pr} : \pi(S_i) \to \pi(S_i)/\mathrm{SO}(3)$.

The SO(3)-actions on $\pi(S_1)$ and $\pi(S_i)$ dictate the number of flat 2-planes on $\pi(S_1)$ and $\pi(S_i)$. SO(3)-action is isometric, so its induced action on the Grassmanian preserves curvatures and, in particular, maps flat 2-planes to flat 2-planes. There is no nontrivial element of SO(3) that fixes a point in $\pi(S_1 \backslash (S_{1,i} \cup S_{1,-i}))$, so flat 2-plane at a point of $\pi(S_1 \backslash (S_{1,i} \cup S_{1,-i}))$ can only be mapped to a flat 2-plane at some other point of $\pi(S_1 \backslash (S_{1,i} \cup S_{1,-i}))$. However, a point of $\pi(S_{1,i})$, for example, is fixed by a subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3)$, so the action of SO(2) on a flat 2-plane at a point of $\pi(S_{1,i})$ gives a one-parameter family of flat 2-planes at that point.

4.2 Metric of positive distance curvature on the 5-manifold $S^3 \times S^2$

The next step in our construction is to apply conformal deformations to the metric of almost positive curvature g_t on $S^3 \times S^2$ from the previous section in order to obtain metric a with positive distance curvature 2.14 curvature on $S^3 \times S^2$. Actually, the construction yields a metric that satisfies a stronger condition $\sec^\theta > 0$ for all $\theta > 0$, see [3, Chapter 5] . We will use a deformation similar to the one Bettiol uses to construct a metric of $\sec^\theta_{\mathrm{g}} > 0$ on $S^2 \times S^2$ in [4].

Analogously to Bettiol's construction of a metric with $\sec^\theta_{\mathrm{g}} > 0$ for any $\theta > 0$ on $S^2 \times S^2$ [4], [3, Proposition 6.5], we prove the following

Theorem 4.2. *Manifold $S^3 \times S^2$ admits a metric of positive distance curvature, $\sec^\theta > 0$, for any $\theta > 0$. In particular, $S^3 \times S^2$ admits a metric of positive biorthogonal curvature.*

Proof. Start with the metric of almost positive curvature g_t on $S^3 \times S^2$ from previous section and consider its first-order conformal deformation. Following submanifolds of $S^3 \times S^2$, $\pi(S_{1,i})$, $\pi(S_{1,-i})$, $\pi(S_{i,1})$, and $\pi(S_{i,-1})$ are compact and pairwise disjoint, as it can be seen from (4.16). This means that they admit pairwise disjoint tubular neighborhoods and by using partitions of unity one can construct a function $\chi_{1,i} : S^3 \times S^2 \to \mathbb{R}$ that is identically zero outside of a tubular neighborhood of $\pi(S_{1,i})$ and identically one inside a smaller tubular neighborhood of $\pi(S_{1,i})$. Similarly, one constructs functions $\chi_{1,-i}$, $\chi_{i,1}$ and $\chi_{i,-1}$, with the same property, but for the submanifolds $\pi(S_{1,-i})$, $\pi(S_{i,1})$ and $\pi(S_{i,-1})$. Next, consider functions from $S^3 \times S^2$ to the reals given by

$$\begin{aligned} \psi_{1,i}(m) &= \mathrm{dist}_{\mathrm{g}_t}\left(m, \pi(S_{1,i})\right)^2 \\ \psi_{1,-i}(m) &= \mathrm{dist}_{\mathrm{g}_t}\left(m, \pi(S_{1,-i})\right)^2 \ , \\ \psi_{i,1}(m) &= \mathrm{dist}_{\mathrm{g}_t}\left(m, \pi(S_{i,1})\right)^2 \ , \\ \psi_{i,-1}(m) &= \mathrm{dist}_{\mathrm{g}_t}\left(m, \pi(S_{i,-1})\right)^2 \ , \end{aligned} \tag{4.21}$$

for $p \in S^3 \times S^2$. Here $\mathrm{dist}_{\mathrm{g}_t}$ is the distance function on $(S^3 \times S^2, \mathrm{g}_t)$ considered as a complete metric space. Now define a function $\phi : S^3 \times S^2 \to \mathbb{R}$ as

$$\phi := -\chi_{1,i}\psi_{1,i} - \chi_{1,-i}\psi_{1,-i} - \chi_{i,1}\psi_{i,1} - \chi_{i,-1}\psi_{i,-1} \ , \tag{4.22}$$

and use it to first-order conformaly deform g_t,

$$\mathrm{g}_{s,t} = (1 + s\phi)\mathrm{g}_t \, . \tag{4.23}$$

At a point $m \in \pi(S_{1,i})$ we have

$$\text{Hess}\,\phi(X,X) = -\text{Hess}\,\psi_{1,i}(X,X) = -2\mathrm{g}_{s,t}(X_\perp, X_\perp)^2 = -2\|X_\perp\|^2_{\mathrm{g}_{s,t}}\,, \tag{4.24}$$

where $X_\perp$ denotes the component of X perpendicular to $\pi(S_{1,i})$. At points of $\pi(S_{1,-i})$, $\pi(S_{i,1})$, and $\pi(S_{i,-1})$ equations similar to (4.24) are true.

For any $\theta > 0$ consider the compact subset of

$$\left(S^3 \times S^2\right) \times \mathrm{Gr}_2\left(T\left(S^3 \times S^2\right)\right) \times \mathrm{Gr}_2\left(T\left(S^3 \times S^2\right)\right) \tag{4.25}$$

given by

$$K_\theta := \left\{(m,\sigma,\sigma') : \sigma,\sigma' \in \mathrm{Gr}_2\left(T_m\left(S^3 \times S^2\right)\right), \mathrm{dist}\left(\sigma,\sigma'\right) \geq \theta\right\}\,, \tag{4.26}$$

and define

$$\begin{gathered} f : [0,S] \times K_\theta \to \mathbb{R} \\ f\left(s,(m,\sigma,\sigma')\right) := \frac{1}{2}\left(\sec_{\mathrm{g}_{s,t}}(\sigma) + \sec_{\mathrm{g}_{s,t}}(\sigma')\right)\,. \end{gathered} \tag{4.27}$$

Now, $f\left(0,(m,\sigma,\sigma')\right) \geq 0$, because $\sec_{\mathrm{g}_{s,t}} \geq 0$. Furthermore, $f\left(0,(m,\sigma,\sigma')\right) = 0$ only for $m \in \pi(S_{1,i}) \cup \pi(S_{1,-i}) \cup \pi(S_{i,-1}) \cup \pi(S_{i,-1})$, because these are the only points of $S^3 \times S^2$ that contain more that one flat 2-plane. Let (m,σ,σ') be such that $f\left(0,(m,\sigma,\sigma')\right) = 0$, and let $\sigma = X \wedge Y$ and $\sigma' = Z \wedge W$, with X, Y g_t-orthonormal and Z, W g_t-orthonormal. Then by, Lemma 2.41 and equation (4.24) at these points of K_θ we have

$$\begin{aligned} &\frac{\partial f}{\partial s}|_{s=0} = \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\sec_{\mathrm{g}_{s,t}}(X \wedge Y) + \sec_{\mathrm{g}_{s,t}}(Z \wedge W)\right)|_{s=0} \\ &= -\frac{1}{2}\text{Hess}\,\phi(X,X) - \frac{1}{2}\text{Hess}\phi(Y,Y) - \frac{1}{2}\text{Hess}\phi(Z,Z) - \frac{1}{2}\text{Hess}\phi(W,W) \\ &= \|X_\perp\|^2_{\mathrm{g}_t} + \|Y_\perp\|^2_{\mathrm{g}_t} + \|Z_\perp\|^2_{\mathrm{g}_t} + \|W_\perp\|^2_{\mathrm{g}_t} > 0\,. \end{aligned} \tag{4.28}$$

The previous expression is strictly greater that zero because $\mathrm{span}\,\{X,Y,Z,W\}$ is at least 3-dimensional, since $X \wedge Y$ and $Z \wedge W$ are different 2-planes, and $\pi(S_{1,i})$,$\pi(S_{1,-i})$,$\pi(S_{i,1})$, and $\pi(S_{i,-1})$ are two dimensional, meaning that at least one of the perpendicular components $X_\perp$, $Y_\perp$, $Z_\perp$, or $W_\perp$ is nonzero. Thus, assumptions of Lemma 2.42 for the function f are satisfied, so there is an s_* such that $f\left(s,(m,\sigma,\sigma')\right) > 0$ for all $(m,\sigma,\sigma') \in K_\theta$ and $0 < s < s_*$. This is precisely the condition $\sec^\theta_{\mathrm{g}_{s,t}} > 0$ which proves the Theorem. □

The claims of Items 2. - 5. of Theorem 1.1 follow from our construction verbatim to Bettiol's work.

4.3 Metrics of positive biorthogonal curvature on 5-manifolds

In this section, we use Smale's classification of closed spin simply connected 5-manifolds with no torsion in homology [27], see also[15, Corollary 7.30]

Theorem 4.3. *A closed connected simply connected 5-manifold M with zero second Stiefel-Whitney class $w_2(M) = 0$ and torsion-free homology $H_2(M) \simeq \mathbb{Z}^k$ is determined up to diffeomorphism by its second Betti number $b_2(M) = \operatorname{rank}(H_2(M)) = k \in \mathbb{N}_0$. In particular, there is a diffeomorphism $M \simeq_{C^\infty} S^5 \# k\,(S^2 \times S^3)$*

Theorem 4.1 along with the stability of positive biorthogonal curvature under connected sums that was stated in Proposition 3.3 allow us to prove the following result.

Theorem 4.4. *If M is a closed connected simply-connected 5-manifold with second Stiefel-Whitney class $w_2(M) \equiv 0$ and second homology group $H_2(M;\mathbb{Z}) \simeq \mathbb{Z}^r$ then M admits a Riemannian metric g such that $\sec^{\perp}_{\mathrm{g}} > 0$.*

Proof. By Theorem 4.3, every such manifold is ether S^5 for $H_2(M;\mathbb{Z}) = 0$, or a connected sum of $k \in \mathbb{N}$ copies of $S^3 \times S^2$ for $H_2(M;\mathbb{Z}) = \mathbb{Z}^k$. The sphere S^5 with the round metric has positive sectional curvature, so it admits a metric of positive biorthogonal curvature. By Theorem 4.1, $S^3 \times S^2$ admits a metric of positive biorthogonal curvature. By Proposition 3.3 the connected sum $\# k(S^3 \times S^2)$, also admits a metric of positive biorthogonal curvature, completing the proof. □

Appendix A

Gell-Mann matrices

The following matrices λ_l are traceless self-adjoint 3 by 3 matrices, the Gell-Mann matrices [12]:

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{A.1}$$
$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The set $\{\lambda_l\}_{l=1,2,...,8}$ is a basis of the real vector space of 3 by 3 traceless selfadjoint matrices. In physics notation, $\{\lambda_l\}_{l=1,2,...,8}$ is a complete set of generators for the real Lie algebra $\mathfrak{su}(3)$ with structure coefficients ${f_{lm}}^n \in \mathbb{R}$ defined as:

$$[\lambda_l, \lambda_n] = i{f_{lm}}^n \lambda_n \,. \tag{A.2}$$

In mathematics notation, elements of the real Lie algebra $\mathfrak{su}(3)$ are antiselfdjoint, with a corresponding basis:

$$E_l := -i\lambda_l \,,\; l = 1, 2, ..., 8 \,, \tag{A.3}$$

and structure coeficients defined by the equation:

$$[E_l, E_m] = {f_{lm}}^n E_n \,. \tag{A.4}$$

Note that structure coeficients ${f_{lm}}^n \in \mathbb{R}$ in equations A.2 and A.4 are the same real numbers. The group elements in physics notation and mathematics notation are the same. This is because in physics notation $-i$ in the exponential is assumed:

$$u = \exp\left(-i\alpha^l \lambda_l\right) = \exp\left(\alpha^l E_l\right) \,, \tag{A.5}$$

For real numbers $\alpha_l \in \mathbb{R}$, $l = 1, 2, ..., 8$. Furthermore, note that the terms structure coefficients and structure constants are used interchangeably in the literature.

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